BASIC CONCEPTS OF MODAL SCALING

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IOMAC'19
8th International Operational Modal Analysis Conference
2019 May12-14 Copenhagen

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ABSTRACT

Mode shapes can be normalized in many different ways, the most common techniques being mass normalization, normalization to the unit length of the mode shape (length scaling) and normalization to a component (usually to the largest component) equal to unity (DOF scaling). For this reason, the modal mass of a mode shape is not unique but it depends on the normalization method used to define the mode shape. On one hand, the modal mass can be a mass or an inertia. On the other hand, the units of the modal mass depend on the scaling technique and its magnitude depends on the number of degree of freedoms (DOFs) used to discretize the model. In this paper, all these drawbacks are studied and a new and better definition of the modal mass is presented, which leads to a better engineering interpretation of this modal parameter

Keywords: Modal mass, Structural Dynamics, Modal Analysis

1. INTRODUCTION

When the modal model is used to define the dynamic behavior of a mechanical system, the modal parameters (natural frequencies, mode shapes and damping ratios) are needed [1,2,3,4]. Mass normalized mode shapes (hereafter denoted as $\boldsymbol{\phi}$), contain information of both the mode shape and the modal mass. A mode shape is defined un-scaled if it is not mass normalized. If un-scaled mode shapes, hereafter denoted as $\boldsymbol{\psi}$**,** are used, a new modal parameter for each mode, known as modal mass, is needed to define the dynamic behavior of a mechanical system. When classical modal analysis (CMA), also known as experimental modal analysis (EMA), is used to estimate the modal parameters of a system, mass normalized mode shapes are obtained in the estimation process [1,2,3]. On the contrary, in operational modal analysis (OMA) the force is not measured and only un-scaled mode shapes can be estimated [4].

The equation of motion of a structural dynamic system with no damping subjected to a force $\boldsymbol{p}$ is given by [4,5,6]:

$$\boldsymbol{K}\boldsymbol{u} + \boldsymbol{M}\ddot{\boldsymbol{u}} = \boldsymbol{p} \tag{1}$$

Where $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{M}$ the mass matrix and $\boldsymbol{u}$ and $\ddot{\boldsymbol{u}}$ are the displacement and the acceleration vector, respectively. Eq. (1) provides the eigenvalue equation:

$$(\boldsymbol{K}-\boldsymbol{M}\omega^2)\boldsymbol{\psi}=\boldsymbol{0} \tag{2}$$

Where $\boldsymbol{\psi}$ and ω are the unscaled mode shape and the natural frequency, respectively.

Mode shapes can be normalized in many different ways but when working with experimental mode shapes, the most common types of normalization are:

- Mass normalization (or normalization to the mass matrix of the system).
- Normalization to the unit length of the vector
- Normalization to a component equal to unity (usually the largest component).

On the other hand, numerical mode shapes are generally mass normalized or normalized to a component equal to unity.

The un-scaled ($\boldsymbol{\psi}$) and the scaled or mass normalized ($\boldsymbol{\phi}$) mode-shape vectors are related by the expression:

$$\boldsymbol{\phi}=\boldsymbol{\psi}\frac{1}{\sqrt{m}} \tag{3}$$

Where m is the modal mass and which can be obtained from the expression [4,5,6]:

$$m=\boldsymbol{\psi}^T\boldsymbol{M}\boldsymbol{\psi} \tag{4}$$

The length of a mode shape $\boldsymbol{\psi}$ (also denoted in algebra as Euclidean norm, Euclidean length or L^2 norm) is given by:

$$L_\psi=\sqrt{\boldsymbol{\psi}^T\cdot\boldsymbol{\psi}} \tag{5}$$

A mode shape is said to be mass normalized if the modal mass is dimensionless unity, i..e:

$$m_\phi=\boldsymbol{\phi}^{\boldsymbol{T}}\boldsymbol{M}\boldsymbol{\phi}=1 \tag{6}$$

Mass normalized mode shapes $\boldsymbol{\phi}$ can be obtained from the unscaled mode shapes $\boldsymbol{\psi}$ by:

$$\{\boldsymbol{\phi}\}=\frac{\boldsymbol{\psi}}{\sqrt{\boldsymbol{\psi}^T\boldsymbol{M}\boldsymbol{\psi}}} \tag{7}$$

A mode shape $\{\boldsymbol{\psi}\}$ is normalized to the unit length, hereafter denoted $\boldsymbol{\psi_L}$, if its length is unity, and it can be obtained by:

$$\boldsymbol{\psi_L}=\frac{\boldsymbol{\psi}}{\sqrt{\boldsymbol{\psi}^T\cdot\boldsymbol{\psi}}}=\frac{\boldsymbol{\psi}}{L_\psi} \tag{8}$$

Where the length of the vector $\boldsymbol{\psi_L}$ is unity, i.e:

$$L_{\psi_L}=\sqrt{\boldsymbol{\psi}_{\boldsymbol{L}}^T\cdot\boldsymbol{\psi_L}}=1 \tag{9}$$

Another common type of normalization consists of assigning a magnitude equal to one to a component of the vector (usually to the largest component). A mode shape normalized in this way is hereafter denoted as $\boldsymbol{\psi_U}$. If the r-th DOF is chosen to be equal to unity, the mode shape $\boldsymbol{\psi_U}$ is related to $\boldsymbol{\psi}$ by:

$$\boldsymbol{\psi_U} = \frac{\boldsymbol{\psi}}{\psi_r} \tag{10}$$

Where ψ_r is an scalar corresponding to the r-th component of the vector $\boldsymbol{\psi}$.

If eq. (10) is particularized to mode shapes normalized to the unit length and to mode shapes normalized to a component equal to unity, it results in:

$$\boldsymbol{\phi} = \boldsymbol{\psi_L} \frac{\mathbf{1}}{\sqrt{m_{\psi_L}}} = \boldsymbol{\psi_U} \frac{\mathbf{1}}{\sqrt{m_{\psi_U}}} \tag{11}$$

Where m_{ψ_L} and m_{ψ_U} are the modal masses corresponding to the mode shapes $\boldsymbol{\psi_L}$ and $\boldsymbol{\psi_U}$, respectively.

Pre-multiplication of eq. (11) by $\boldsymbol{\phi^T}$, gives:

$$\frac{m_\phi = 1}{L_\phi^2} = \frac{m_{\psi_L}}{L_{\psi_L}^2 = 1} = \frac{m_{\psi_U}}{L_{\psi_U}^2} \tag{12}$$

2. UNITS

The modal mass m_ϕ corresponding to the mass normalized mode shape $\boldsymbol{\phi}$ is dimensionless unity. From eq. (6) it is easily inferred that the translational components $\boldsymbol{\phi_T}$ of the mode shape $\boldsymbol{\phi}$ have the units $\frac{1}{\sqrt{kg}}$ in the international system, whereas the units of the rotational components $\boldsymbol{\phi_T}$ are $\frac{1}{m\sqrt{kg}}$. On the other hand, the translational components of the mode shapes $\boldsymbol{\psi_L}$ and $\boldsymbol{\psi_U}$ are dimensionless and the modal masses m_{ψ_L} and m_{ψ_U} are given in kg. This has been summarized in Table 1.

Table 1. Units of modal parameters in the international system (SI). ----- indicates dimensionless.

Mass normalization		Normalization to unit length		Normalization to a component equal to unity	
Variable	Unit	Variable	Unit	Variable	unit
$\boldsymbol{\phi_T}$	$\frac{1}{\sqrt{kg}}$	$\boldsymbol{\psi_{L_T}}$	-----	$\boldsymbol{\psi_{U_T}}$	------
$\boldsymbol{\phi_R}$	$\frac{1}{m\sqrt{kg}}$	$\boldsymbol{\psi_{L_R}}$	$\frac{1}{m}$	$\boldsymbol{\psi_{U_R}}$	$\frac{1}{m}$
$m_\phi = 1$	-----	m_{ψ_L}	kg	m_{ψ_U}	kg
$L_\phi = \sqrt{\{\phi\}^T \cdot \{\phi\}}$	$\frac{1}{\sqrt{kg}}$	$L_{\psi_L} = 1$ $L_{\psi_L} = \sqrt{\{\psi_L\}^T \cdot \{\psi_L\}}$	-----	$L_{\psi_U} = \sqrt{\{\psi_U\}^T \cdot \{\psi_U\}}$	-----
q_ϕ	$m\sqrt{kg}$	q_{ψ_L}	m	q_{ψ_U}	m

With respect to the length of mode shapes, L_{ψ_L} and L_{ψ_U} are dimensionless whereas L_ϕ have units of $\frac{1}{\sqrt{kg}}$.

Experimental mode shapes usually only contain translational DOF's and they can be scaled with any of the methods presented in section 1. On the other hand, some numerical mode shapes only contain translational DOF's (3D models, 2D planar elasticity models), whereas beams, plates and shells contain both translational and rotational DOF's. When a mode shape contains both translational and rotational DOF's, it cannot be normalized to the unit length (the length cannot be calculated because the units are not consistent).

The modal mass can also be a modal inertia. If the mode shape $\boldsymbol{\psi} = \begin{Bmatrix} \boldsymbol{\psi}_T \\ \boldsymbol{\psi}_R \end{Bmatrix}$, containing both translational and rotational components, is normalized to a translational component equal to unity, the modal mass, is given in kg (see table 1). However, if we normalize the vector $\boldsymbol{\psi}$ to a rotational component equal to unity, the translational DOF's are given in meters and the rotational DOF's are dimensionless. The modal mass in this case is given in kgm^2,i.e. it is a modal inertia.

In analytical models both modal mass and modal inertia can be modal parameters of a mode shape. However, mass normalization and normalization to a component equal to unity (usually the largest translational component) are the normalization methods commonly available in numerical programs, i.e. the modal mass (kg) is the parameter normally consider in finite element programs.

If the vector of displacements $\boldsymbol{u}$ is decomposed in modal coordinates using mass-normalized mode shapes:

$$\boldsymbol{u} = \boldsymbol{\phi} \cdot \boldsymbol{q}_{\boldsymbol{\phi}} \tag{13}$$

the modal coordinates $\boldsymbol{q}_{\boldsymbol{\phi}}$ have the units $m\sqrt{kg}$.

If mode shapes $\boldsymbol{\psi}_{\boldsymbol{L}}$ are used in the modal decomposition:

$$\boldsymbol{u} = \boldsymbol{\phi} \cdot \boldsymbol{q}_{\psi_L} \tag{14}$$

the translational components of the mode shapes are dimensionless, and the units of the modal coordinates $\boldsymbol{q}_{\psi_L}$coincide with the units of vector $\boldsymbol{u}$. The modal coordinates $\boldsymbol{q}_{\boldsymbol{\phi}}$and $\boldsymbol{q}_{\psi_L}$are related by:

$$\boldsymbol{q}_{\boldsymbol{\phi}} = \boldsymbol{q}_{\psi_L} \cdot \sqrt{m_{\psi_L}} \tag{15}$$

The same can be said for mode shapes $\boldsymbol{\psi}_{\boldsymbol{U}}$.

3. EXPANSION AND REDUCTION OF MODELS

In structural dynamics model reduction means to reduce a finite element model to one with fewer degrees of freedom while the dynamic characteristics of the system are maintained [7]. In order that the reduced model have the same modal parameters as the full model, the modal masses must remain unchanged with all kinds of normalization, i.e.:

$$m_{\phi} = m_{\phi a} = 1, \qquad m_{\psi_L} = m_{\psi_{La}} \qquad m_{\psi_U} = m_{\psi_{Ua}} \tag{16}$$

Where the subindex 'a' indicates model with active DOF's.

However, the following properties are modified:

- The length of mode shapes $\boldsymbol{\phi}_{\boldsymbol{a}}, \boldsymbol{\psi}_{\boldsymbol{L}_{\boldsymbol{a}}}$ ***and*** $\boldsymbol{\psi}_{\boldsymbol{U}_{\boldsymbol{a}}}$ are different to that of the corresponding full mode shapes (see table 2).
- the reduced mode shape $\boldsymbol{\psi}_{\boldsymbol{L}_{\boldsymbol{a}}}$ is no longer normalized to the unit length. If the reduced mode shape is normalized to the unit length($\boldsymbol{\psi}^*_{\boldsymbol{L}_{\boldsymbol{a}}}$ in table 2), the modal mass has to be modified accordingly.

- With mode shapes $\boldsymbol{\psi}_{\boldsymbol{U}}$ and $\boldsymbol{\psi}_{\boldsymbol{U_a}}$, the DOF with the component equal to unity must be the same in both the full and the reduced models. Otherwise, the modal mass has to be modified accordingly.

All the aforementioned properties are showed in table 2 by means an example.

Table 2. Example of mode shapes and modal masses in reduction

Mass normalization	Normalization to unit length	Normalization to a component equal to unity
$\boldsymbol{\phi} = \begin{Bmatrix} \boldsymbol{\phi_a} \\ \boldsymbol{\phi_d} \end{Bmatrix} = \begin{Bmatrix} \begin{Bmatrix} 0.9 \\ 0.72 \end{Bmatrix}_a \\ \{0.45\}_d \end{Bmatrix}$ $m_\phi = 1$ $L_\phi = 1.1597$	$\boldsymbol{\psi_L} = \begin{Bmatrix} \begin{Bmatrix} 0.7272 \\ 0.5818 \end{Bmatrix}_a \\ \{0.3636\}_d \end{Bmatrix}$ $m_{\psi_L} = 0.6535$ L_{ψ_L}=1	$\boldsymbol{\psi_U} = \begin{Bmatrix} \begin{Bmatrix} 1 \\ 0.8 \end{Bmatrix}_a \\ \{0.5\}_d \end{Bmatrix}$ $m_{\psi_U} = 1.234$ $L_{\psi_U} = 1.3748$
$Reduction$ $\Downarrow$	$Reduction$ $\Downarrow$	$Reduction$ $\Downarrow$

$\boldsymbol{\phi_a} = \begin{Bmatrix} 0.9 \\ 0.72 \end{Bmatrix}$ $m_{\phi_a} = 1$ $L_{\phi a} = 1.0689$	$\boldsymbol{\psi_{L_a}} = \begin{Bmatrix} 0.7272 \\ 0.5818 \end{Bmatrix}$ $m_{\psi_{La}} = 0.6535$ $L_{\psi_{La}} = 0.9313$	$\boldsymbol{\psi}^*_{\boldsymbol{L_a}} = \begin{Bmatrix} 0.7808 \\ 0.6247 \end{Bmatrix}$ $m^*_{\psi_{La}} = 0.7535$ $L^*_{\psi_{La}} = 1$	$\boldsymbol{\psi_{U_a}} = \begin{Bmatrix} 1 \\ 0.8 \end{Bmatrix}$ $m_{\psi_{Ua}} = 1.234$ $L_{\psi_{Ua}}$=1.2806

The modal masses $m_{\psi_L}$and $m^*_{\psi_{La}}$are related by means of the equation:

$$m_{\psi_L} = m^*_{\psi_{La}} \cdot L^2_{\psi_{L_a}} \tag{17}$$

Where $L^2_{\psi_{L_a}}$ is the square length of the vector $\boldsymbol{\psi_{L_a}}$.

The same procedure must be followed with expansion of mode shapes.

4. MODAL ASSURANCE CRITERIA (MAC)

The modal assurance criteria (MAC) is the most used criteria to compare two mode shapes. We consider by convenience the mode shapes $\boldsymbol{\phi_x}$ and $\boldsymbol{\phi_{FE}}$, the subindices 'x' and'FE' denoting experimental and numerical models, respectively. The modal assurance criteria (MAC) is given as the square of the dot product between two vectors with unit length [1,2,3,4], i.e.:

$$MAC = \left|\boldsymbol{\psi}^T_{\boldsymbol{x_L}} \cdot \boldsymbol{\psi}_{\boldsymbol{FE_L}}\right|^2 = \cos^2\theta \tag{18}$$

From eq. (18) it is inferred that we cannot calculate the MAC between two vectors containing both translational and rotational DOF's, because these vectors cannot be normalized to the unit length.

In case of other kinds of normalization, the following expression must be used [1,2,3,4]:

$$MAC = \frac{\left|\boldsymbol{\phi}^T_{\boldsymbol{x}} \cdot \boldsymbol{\phi}_{\boldsymbol{FE}}\right|^2}{\left|\boldsymbol{\phi}^T_{\boldsymbol{x}} \cdot \boldsymbol{\phi}_{\boldsymbol{x}}\right|\left|\boldsymbol{\phi}^T_{\boldsymbol{FE}} \cdot \boldsymbol{\phi}_{\boldsymbol{FE}}\right|} = \frac{\left|\boldsymbol{\psi}^T_{\boldsymbol{x_U}} \cdot \boldsymbol{\psi}_{\boldsymbol{FE_U}}\right|^2}{\left|\boldsymbol{\psi}^T_{\boldsymbol{x_U}} \cdot \boldsymbol{\psi}_{\boldsymbol{x_U}}\right|\left|\boldsymbol{\psi}^T_{\boldsymbol{FE_U}} \cdot \boldsymbol{\psi}_{\boldsymbol{FE_U}}\right|} \tag{19}$$

5. COMPARISON OF MODAL MASSES

Let's consider two mode shapes $\boldsymbol{\psi}_{\boldsymbol{x_U}}$and $\boldsymbol{\psi}_{\boldsymbol{FE_U}}$ with modal masses $m_{x_U}$and $m_{\psi_{FE_U}}$, respectively. The modal assurance criteria (MAC) can be used to compare both mode shapes and the MAC is unique because both mode shapes have to be normalized to the unit length of the vector. However, as it can be seen in the next example, if we compare the modal masses of two modes, the discrepancies are different depending on the normalization used for the mode shapes.

Example:

$$\boldsymbol{\phi}_x = \begin{Bmatrix} 0.32 \\ 0.56 \\ 0.27 \end{Bmatrix} \rightarrow \boldsymbol{\psi}_{x_L} = \begin{Bmatrix} 0.4577 \\ 0.8009 \\ 0.3861 \end{Bmatrix} \rightarrow \boldsymbol{\psi}_{x_U} = \begin{Bmatrix} 0.5714 \\ 1.0 \\ 0.4821 \end{Bmatrix}$$

$$m_{xL} = 2.045 \qquad m_{xU} = 3.189$$

$$\boldsymbol{\phi}_{FE} = \begin{Bmatrix} 0.30 \\ 0.59 \\ 0.30 \end{Bmatrix} \rightarrow \boldsymbol{\psi}_{FE_L} = \begin{Bmatrix} 0.4128 \\ 0.8119 \\ 0.4128 \end{Bmatrix} \rightarrow \boldsymbol{\psi}_{FE_U} = \begin{Bmatrix} 0.5085 \\ 1.0 \\ 0.5085 \end{Bmatrix}$$

$$m_{FE_L} = 1.894 \qquad m_{FE_U} = 2.873$$

$$\epsilon_L = 100 \cdot \left(1 - \frac{m_{FEL}}{m_{xL}}\right) = 7.38\% \quad \epsilon_U = 100 \cdot \left(1 - \frac{m_{FEU}}{m_{xL}}\right) = 9.909\%$$

We do not obtain the same error comparing the modal masses of mode shapes normalized to the length or comparing those corresponding to mode shapes normalized with the maximum component equal to unity. This is because the mode shapes $\boldsymbol{\psi}_{xU}$ and $\boldsymbol{\psi}_{FE_U}$ do not have the same length (figure 1).

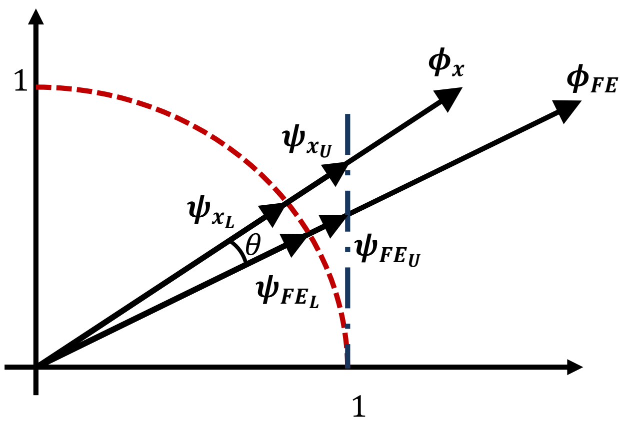

Figure 1. Representation of two-dimensional vectors with different kinds of normalization

In operational modal analysis, the force is not measured and the modal mass cannot be estimated [4]. Some authors [8] have proposed to normalize experimental un-scaled mode shapes using the modal masses of a finite element model. However, two aspects must be considered:

- Mode shapes $\boldsymbol{\psi}_{xU}$ and $\boldsymbol{\psi}_{FE_U}$ have different length (see figure 1), which means that the modal mass m_{FE_U} must not be used to scale $\boldsymbol{\psi}_{xU}$.
- The vector $\boldsymbol{\psi}_{FE_U}$ cannot be normalized to the unit length if both rotational and translational DOF's are present in the mode shape.

If the experimental mode shape only contains translational components (as it is common in modal analysis) the following procedure is recommended:

- The modal mass m_{FE_U} and the mode shape $\boldsymbol{\psi}_{FE_U}$ are known from the FE model.
- The mode shape $\boldsymbol{\psi}_{FE_U}$ is reduced to the active DOF's (vector $\boldsymbol{\psi}_{FE_{Ua}}$) whose modal mass is $m_{FE_{Ua}} = m_{FE_U}$
- The modal mass $m_{FE_{La}}$ of the vector $\boldsymbol{\psi}_{FE_{La}}$ can be obtained by:

$$m_{FE_{La}} = \frac{m_{FE_{Ua}}}{L^2_{\psi_{FE_{Ua}}}} \tag{1}$$

- The experimental mode shape $\boldsymbol{\psi}_{x_{La}}$is normalized by:

$$\widehat{\boldsymbol{\Phi}}_{x_a} = \frac{\boldsymbol{\psi}_{x_{La}}}{\sqrt{m_{FE_{La}}}} \tag{1}$$

Where '$\widehat{\ }$'indicates approximation.

Example:

$$\boldsymbol{\psi}_{x_{La}} = \begin{Bmatrix} 0.4577 \\ 0.8009 \\ \ \end{Bmatrix} \qquad \boldsymbol{\psi}_{FE_U} = \begin{Bmatrix} 0.5085 \\ 1.0 \\ 0.5085 \end{Bmatrix} \rightarrow \boldsymbol{\psi}_{FE_{Ua}} = \begin{Bmatrix} 0.5085 \\ 1.0 \\ \ \end{Bmatrix}$$

$$m_{FE_U} = 2.873 \qquad m_{FE_{Ua}} = 2.873$$

$$m_{FE_{La}} = \frac{m_{FE_{Ua}}}{L^2_{\psi_{FE_{Ua}}}} = \frac{2.873}{1.2586} = 2.287$$

$$\widehat{\boldsymbol{\Phi}}_{x_a} = \frac{\begin{Bmatrix} 0.4577 \\ 0.800 \end{Bmatrix}}{\sqrt{2.287}} = \begin{Bmatrix} 0.303 \\ 0.530 \end{Bmatrix} \qquad \boldsymbol{\phi}_x = \begin{Bmatrix} 0.32 \\ 0.56 \\ 0.27 \end{Bmatrix}$$

6. CONCLUSIONS

Modal mass is one of the modal parameters needed to define the dynamic behavior of a structure. The values that the modal mass can take are dependent on the type of normalization. In this paper, the most common types of normalization have been considered: mass normalization, normalization to the unit length and normalization to the largest component equal to unit.

The units of the modal mass are also dependent on the normalization. The modal mass of mass normalized mode shape is dimensionless unity. If the modal masses of two modes are compared, the mode shapes have to be normalized to the unit length.

Although the length of a vector given by eq. (5) is usually considered in algebra and also in the field of structural dynamics and modal analysis, it presents two main inconveniences:

- Mode shapes with translational and rotational components cannot be normalized to length. As a result, the modal assurance criteria (MAC) cannot be calculated when a mode shape have both translational and rotational DOF's.
- The length of the mode shapes is dependent on the number of DOF's consider in the mode shape vector.

For the aforementioned reasons, mass normalization and normalization to a component equal to unity (usually the largest translational component) are the normalization methods commonly available in numerical programs.

Both modal mass and modal inertia can be modal parameters of a mode shape. The modal mass of a mode shape normalized to a displacement equal to unity, is given in kg. On the other hand, if the normalization is to a rotation equal to unity, then modal mass is given in kgm^2, i.e. it is a modal inertia. The same can be said for mode shapes normalized to the unit length: modal mass in kg for a mode shape with translational components and modal inertia in kgm^2 for mode shapes with rotational components.

ACKNOWLEDGEMENTS

The financing support given by Banco Santander through the Mobility Grants for Researchers of Professors of the University of Oviedo, is gratefully appreciated. The authors acknowledge the funding received from Centre for Oil and Gas - DTU/Danish Hydrocarbon Research and Technology Centre (DHRTC).

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