DOI 10.1007/s10778-016-0776-0
International Applied Mechanics, Vol. 52, No. 5, September, 2016

MODELING THE COLLISION WITH FRICTION OF RIGID BODIES

A. G. Zabuga

Different models of a perfectly inelastic collision of rigid bodies in plane motion are compared. Formulas for the impact impulses are derived for the Kane–Levinson–Whittaker model based on the kinematic restitution factor, the Routh model based on the kinetic restitution factor, and the Stronge model based on the energy restitution factor. It is shown that these formulas coincide if the collision of rough rigid bodies in plane motion is perfectly inelastic

Keywords: perfectly inelastic collision, restitution factor, rigid body, limiting friction factor, Amontons–Coloumb law

Introduction. Mechanical/mathematical models of collision of perfectly rigid bodies are based on the following assumptions [4, 6–8, 14, 15, 17]: (i) the impact strains are negligibly small compared with the size of the mechanical system; (ii) the impact forces are infinitely strong compared with the other forces acting in the mechanical system; (iii) the impact time is negligible compared with the analysis time.

It follows from these assumptions that the collision instantaneously changes the velocities, but not the coordinates of points of the system. In this connection, it is more convenient to study a collision in terms of not forces, but impulses, which are the integrals of the reaction forces of the bodies at the contact point over the impact time. In view of the third assumption, the concept of impact time makes sense only when the impact process is analyzed; if the motion of the system is analyzed, then the collision is considered to occur at some instant of time.

The above assumptions appear insufficient to describe a collision. The properties of the colliding bodies should also be known. All the models addressed here assume that the Amontons–Coulomb law of friction holds for the normal and tangential components of the impact impulse at the contact point between the colliding bodies. The models differ only in the last assumption where the restitution coefficient is introduced. For example, the Kane–Levinson–Whittaker model [9, 11, 12, 17] employs the kinematic restitution coefficient, which, according to Newton's hypothesis, is equal to the inverse ratio of the normal velocities of the centers of mass of the bodies after and before the collision. The Routh model [4, 5, 8, 14] uses the kinetic restitution coefficient, which, according to Poisson's hypothesis, is equal to the ratio of the normal component of the impact impulse during the period from the maximum compression to the end of the collision process to the normal component of the impact impulse during the period from the beginning of the collision to the maximum compression. The Stronge model [15, 16] uses the energy restitution coefficient, whose square, according to Stronge's hypothesis, is equal to the inverse ratio of the work done by the normal component of the impact impulse during the period from the maximum compression to the end of the collision to the work done by the normal impact impulse during the period from the beginning of the collision to the maximum compression.

In [7, 15], it was proved that the three models lead to the same results in certain cases. Such cases are a collision without friction, a central collision (the line connecting the centers of mass of the colliding bodies passes through the contact point), and a collision at which the velocity of one of the colliding bodies relative to the other body at the contact point does not reverse sign and does not vanish. Note that since the result of modeling a collision without friction does not depend on which of the restitution

S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, 3 Nesterova St., Kyiv, Ukraine 03057, e-mail: model@inmech.kiev.ua. Translated from Prikladnaya Mekhanika, Vol. 52, No. 5, pp. 137–144, September–October, 2016. Original article submitted December 28, 2015.

1063-7095/16/5205-0557 ©2016 Springer Science+Business Media New York

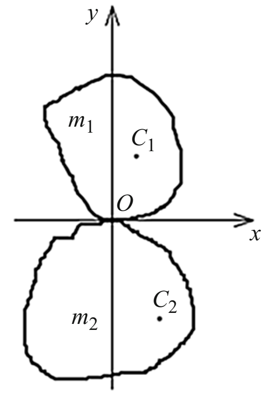

Fig. 1

coefficients has been chosen, many textbooks on theoretical mechanics, such as [1–3], do not distinguish these coefficients and accept Newton's hypothesis.

Our goal here is to compare the three models in describing an absolutely inelastic collision of rigid bodies in plane motion.

1. Problem Formulation. Consider a material system of two colliding rigid bodies with masses m_1 and m_2 and centers of mass C_1 and C_2 (Fig. 1). The origin of the Cartesian coordinate system Oxy is at the contact point. The Ox-axis is directed along the common tangent, and the Oy-axis along the common normal to the surface of the bodies m_1 and m_2. The plane motion of the bodies m_1 and m_2 occurs in the plane of the figure. The coordinate system Oxy is in the same plane.

Using the momentum and moment-of-momentum theorems and Newton's third law, we derive the following relations for the velocities of translation and rotation of the bodies m_1 and m_2:

$$\begin{cases} m_1(v_{1x} - v_{1x0}) = -I_T, \\ m_1(v_{1y} - v_{1y0}) = I_N, \\ G_1(\omega_1 - \omega_{10}) = -I_N x_1 - I_T y_1, \\ m_2(v_{2x} - v_{2x0}) = I_T, \\ m_2(v_{2y} - v_{2y0}) = -I_N, \\ G_2(\omega_2 - \omega_{20}) = I_N x_2 + I_T y_2, \end{cases} \tag{1.1}$$

where v_{1x} and v_{1x0} are the projections of the velocity of the center of mass of the body m_1 onto the Ox-axis after and before the collision, respectively; v_{1y} and v_{1y0} are the projections of the velocity of the center of mass of the body m_1 onto the Oy-axis after and before the collision, respectively; G_1 is the moment of inertia of the body m_1 about the axis that passes through its center of mass and is perpendicular to the plane of the figure; ω_1 and ω_{10} are the angular speed of rotation of the body m_1 around this axis after and before the collision, respectively; x_1 and y_1 are the coordinates of the center of mass of the body m_1 in the coordinate system Oxy; I_T is the tangential (directed along the Ox-axis) component of the impact impulse; I_N is the normal (directed along the Oy-axis) component of the impact impulse; v_{2x} and v_{2x0} are the projections of the velocity of the center of mass of the body m_2 onto the Ox-axis after and before the collision, respectively; v_{2y} and v_{2y0} are the projections of the velocity of the center of mass of the body m_2 onto the Oy-axis after and before the impact, respectively; G_2 is the moment of inertia of the body m_2 about the axis that passes through its center of mass and is perpendicular to the plane of the figure; ω_2 and ω_{20} are the angular speed of rotation of the body m_2 around this axis after and before the collision, respectively; x_2 and y_2 are the coordinates of the center of mass of the body m_2 in the coordinate system Oxy.

The basic task of the theory of collision of bodies in plane motion is to find v_{1x}, v_{1y}, ω_1, v_{2x}, v_{2y}, and ω_2, given v_{1x0}, v_{1y0}, ω_{10}, v_{2x0}, v_{2y0}, and ω_{20}. To this end, Eqs. (1.1) are used to find the values of I_N and I_T, which generally depend on how the collision is modeled.

Before comparing the collision models, it is convenient to introduce the projections v_N and v_{N0} of the relative velocity of the bodies m_1 and m_2 onto the Oy-axis at the contact point (the normal component of the velocity) after and before the collision and the projections v_T and v_{T0} of the relative velocity of the bodies m_1 and m_2 onto the Ox-axis at the contact point (the tangential component of the velocity) after and before the collision. Proceeding from the kinematics of the system, we obtain the following relations for v_N, v_{N0}, v_T, and v_{T0}:

$$v_N = v_{2y} - \omega_2 x_2 - v_{1y} + \omega_1 x_1, \quad v_{N0} = v_{2y0} - \omega_{20} x_2 - v_{1y0} + \omega_{10} x_1,$$

$$v_T = v_{1x} + \omega_1 y_1 - v_{2x} - \omega_2 y_2, \quad v_{T0} = v_{1x0} + \omega_{10} y_1 - v_{2x0} - \omega_2 y_2. \tag{1.2}$$

Eliminating v_{1x}, v_{1x0}, v_{1y}, v_{1y0}, ω_1, ω_{10}, v_{2x}, v_{2x0}, v_{2y}, v_{2y0}, ω_2, and ω_{20} from Eqs. (1.1) and (1.2), we get

$$v_N = v_{N0} - aI_N - bI_T, \quad v_T = v_{T0} - bI_N - \tilde{a}I_T, \tag{1.3}$$

where $a = 1/m_1 + 1/m_2 + x_1^2/G_1 + x_2^2/G_2$, $\tilde{a} = 1/m_1 + 1/m_2 + y_1^2/G_1 + y_2^2/G_2$, $b = x_1 y_1/G_1 + x_2 y_2/G_2$.

Note that since the masses and the moments of inertia are always positive, the following inequalities always hold: $a > 0$, $\tilde{a} > 0$, and $a\tilde{a} > b^2$.

All the models of collision of rough rigid bodies assume that the Amontons–Coulomb law of friction holds for the normal and tangential components of the impact impulse [6–10, 13–15, 17]. This means that it is necessary to examine three cases. In one case, the tangential velocity does not vanish and does not reverse sign during the collision, the Amontons–Coulomb law having the form

$$I_T = -\mu I_N \operatorname{sign} v_{T0}, \tag{1.4}$$

where μ is the coefficient of sliding friction.

In the second case, the tangential velocity vanishes and remains zero until the end of the collision:

$$I_T = I_{T1} + I_{T2}, \quad I_N = I_{N1} + I_{N2}, \quad I_{T1} = -\mu I_{N1} \operatorname{sign} v_{T0}, \quad I_{T2} = -\frac{b}{\tilde{a}} I_{N2}, \tag{1.5}$$

where I_{T1} and I_{N1} are the tangential and normal components of the impact impulse, respectively, during the period from the beginning of the collision to the vanishing of the tangential velocity; I_{T2} and I_{N2} are the tangential and normal components of the impact impulse, respectively, during the period from the vanishing of the tangential velocity to the end of the collision.

In the third case, the tangential velocity reverses sign during the collision, the Amontons–Coulomb law having the form

$$I_T = I_{T1} + I_{T2}, \quad I_N = I_{N1} + I_{N2}, \quad I_{T1} = -\mu I_{N1} \operatorname{sign} v_{T0}, \quad I_{T2} = \mu I_{N2} \operatorname{sign} v_{T0}. \tag{1.6}$$

Which of the cases takes place can be ascertained as follows. First, assume that the first case holds. If the calculated v_T has the same sign as v_{T0}, or $v_T = 0$, then the assumption is correct and formula (1.4) is valid. If v_T and v_{T0} have opposite signs, then either the second or the third case takes place. It is the second case, i.e., formulas (1.5), if $|b|/\tilde{a} \le \mu$, and the third case, i.e., formulas (1.6), if $|b|/\tilde{a} > \mu$.

2. Absolutely Inelastic Collision of Bodies in Plane Motion by the Kane–Levinson–Whittaker Model. The Kane–Levinson–Whittaker model [9, 17] is based on Newton's hypothesis

$$e_K = -v_N / v_{N0}, \tag{2.1}$$

where e_K is the kinematic restitution coefficient assumed known. The collision is absolutely inelastic, i.e., $e_K = 0$. Substituting $e_K = 0$ into (2.1), we get $v_N = 0$.

In the first case where the tangential velocity does not vanish and does not reverse sign during the collision, the components of the impact impulse can be found from Eqs. (1.3), (1.4), and (2.1) with $e_K = 0$:

$$I_N = \frac{v_{N0}}{a - \mu b \operatorname{sign} v_{T0}}, \quad I_T = -\mu I_N \operatorname{sign} v_{T0}. \tag{2.2}$$

Formulas (2.2) and Eqs. (1.1) represent the solution of the basic problem of collision with friction in the first case. Note that formulas (2.2) will also hold for the Routh and Stronge models because, as indicated in the introduction, all the models lead to the same results if the tangential velocity does not reverse sign and does not vanish during the collision.

Let us now consider the second case of collision with friction where the tangential velocity vanishes and remains zero until the end of the collision. The second equation in (1.3) with $v_T = 0$ yields

$$I_T = (v_{T0} - bI_N)/\tilde{a}. \tag{2.3}$$

Substituting (2.3) into the first equation in (1.3) and considering that $v_N = 0$ after an absolutely inelastic collision, we obtain

$$I_N = \frac{\tilde{a}v_{N0} - bv_{T0}}{a\tilde{a} - b^2}. \tag{2.4}$$

Formulas (1.1), (2.3), and (2.4) represent the solution of the basic problem of collision with friction in the second case.

To apply the Kane–Levinson–Whittaker model in the third case, it is convenient to consider two stages of the collision process. At the first stage, the tangential velocity decreases from the initial value v_{T0} to zero. At the second stage, the tangential velocity increases in magnitude from zero to a finite value v_T and has direction and sign opposite to those of v_{T0}. Proceeding in the same way as in deriving Eqs. (1.3), we obtain

$$\begin{cases} v_{N1} = v_{N0} - aI_{N1} - bI_{T1}, \\ 0 = v_{T0} - bI_{N1} - \tilde{a}I_{T1}, \end{cases} \qquad \begin{cases} v_N = v_{N1} - aI_{N2} - bI_{T2}, \\ v_T = -bI_{N2} - \tilde{a}I_{T2}, \end{cases} \tag{2.5}$$

where v_{N1} is the normal velocity after the first stage of the collision. It is clear that the first system of equations (2.5) corresponds to the first stage, while the second system to the second stage, the components I_{N1}, I_{T1}, I_{N2}, and I_{T2} having the same sense as in (1.6). Summing the equations of (2.5) and taking the first two relations in (1.6) into account, we arrive at Eqs. (1.3).

Eliminating the unknowns $v_{N1}, I_{N1}, I_{T1}, I_{N2}$, and I_{T2} from Eqs. (1.6) and (2.5) and considering that $v_N = 0$ for an absolutely inelastic collision, we obtain

$$I_N = \frac{v_{N0}(b - \mu\tilde{a}\,\mathrm{sign}v_{T0}) + 2\mu b v_{T0}\mathrm{sign}v_{T0}}{(a + \mu b\,\mathrm{sign}v_{T0})(b - \mu\tilde{a}\,\mathrm{sign}v_{T0})}, \qquad I_T = \mu\left(I_N - \frac{2v_{T0}}{b - \mu\tilde{a}\,\mathrm{sign}v_{T0}}\right)\mathrm{sign}v_{T0}. \tag{2.6}$$

Formulas (1.1) and (2.6) represent the solution of the basic problem of collision with friction in the third case.

3. Absolutely Inelastic Collision of Bodies in Plane Motion by the Routh and Stronge Models. The Routh model [8, 14] is based on Poisson's hypothesis

$$e_D = I_{NA}/I_{NB}, \quad I_{NB} + I_{NA} = I_N, \tag{3.1}$$

where e_D is the kinetic (dynamic) restitution coefficient assumed known; I_{NB} is the normal component of the impact impulse during the period from the beginning of the collision to the maximum compression (i.e., the normal component of the velocity vanishes); I_{NA} is the normal component of the impact impulse during the period from the maximum compression to the end of the collision. Note that the normal components I_{NB} and I_{NA} appearing in (3.1) are generally not equal to the normal components I_{N1} and I_{N2} appearing in (1.5) and (1.6). Certainly, the equality $I_{NB} + I_{NA} = I_{N1} + I_{N2} = I_N$ always holds.

If the collision is absolutely inelastic, then $e_D = 0$. Then $I_{NA} = 0$ and $I_{NB} = I_N$ follow from Eqs. (3.1). The equality $I_{NA} = 0$ means that an absolutely inelastic collision ends once the compression of the colliding bodies has reached its maximum. Thus, the Routh model leads to the equality $v_N = 0$, which is the same result as that obtained with the Kane–Levinson–Whittaker model. Moreover, as indicated above, Eqs. (1.1)–(1.6), and (2.2) are valid for all the three models. The same applies to Eqs. (2.3) and (2.5) because they have been derived with no assumptions on the restitution coefficient. Then relations (2.4) and (2.6) for, respectively, the second and third cases of absolutely inelastic collision with friction are valid as well because they follow from the equality $v_N = 0$ and formulas (1.1)–(1.6), (2.3), and (2.5). We can now conclude that the Kane–Levinson–Whittaker and Routh models produce identical results in all the three cases of absolutely inelastic collision with friction.

Let us finally consider the Stronge model [15, 16]. It is based on Stronge's hypothesis

$$e_E^2 = -W_{NA}/W_{NB}, \tag{3.2}$$

where e_E is the energy restitution coefficient assumed known; W_{NB} is the work done by the normal reaction force at the contact point during the period from the beginning of the collision to the maximum compression; W_{NA} is the work done by the normal reaction force at the contact point during the period from the maximum compression to the end of the collision.

To use the Stronge model, it is necessary to determine the work done by the normal reaction force at the contact point. When a material point moves along a straight line, the work done by the force is defined by $W = \int_{s_1}^{s_2} F(s)ds$, where $F(s)$ is the projection of the force acting on the material point onto the straight line along which the point moves, s is the distance traveled by the point, $[s_1, s_2]$ is the segment on which the work done by the force $F(s)$ is calculated. However, this definition of the work is inconvenient because the reaction forces acting during the collision are infinite, and the path of integration is infinitesimal because the strains are assumed negligible. Therefore, it is convenient to use the formula $ds = v(t)dt$, where $v(t)$ is the velocity of the material point as a function of time t, to replace the path integral by the time integral and then to use the formula $F(t)dt = I_F$, where I_F is the impulse of the force, to replace the time integral by the impulse integral. Doing so gives

$$W = \int_{s_1}^{s_2} F(s)ds = \int_{t(s_1)}^{t(s_2)} F(t)v(t)dt = \int_{I_F(s_1)}^{I_F(s_2)} v(I_F)dI_F,$$

where $t(s_1)$ and $I_F(s_1)$ are the values of time and impulse at the point s_1 of the path, $t(s_2)$ and $I_F(s_2)$ are the values of time and impulse at the point s_2, and $v(I_F)$ is the velocity of the material point as a function of I_F. The works W_{NB} and W_{NA} appearing in (3.2) can be determined in a similar way (the example of a material point moving along a straight line was chosen for the sake of simplicity):

$$W_{NB} = \int_0^{I_{NB}} v_N(I)dI, \quad W_{NA} = \int_{I_{NB}}^{I_N} v_N(I)dI \quad (I_{NB} + I_{NA} = I_N), \tag{3.3}$$

where $v_N(I)$ is the normal velocity as a function of the impact impulse I. The explicit expression of the function $v_N(I)$ can be derived in the same way as Eqs. (1.3) and (2.5), which include the normal velocity. The impact impulses I_{NB} and I_{NA} appearing in (3.1) and (3.3) have the same sense, but their values predicted by the Routh and Stronge models are generally different.

If the collision is absolutely inelastic, then $e_E = 0$. Hence, $W_{NA} = 0$ follows from formulas (3.2). This means that after the compression reaches its maximum, the normal impact impulse performs no work. Thus, $I_{NA} = 0$ and $I_{NB} = I_N$ follow from (3.3). Moreover, by definition, the impact impulse reaches the value I_{NB} once the compression has become maximum, which means that $v(I_{NB}) = 0$. Thus, if $I_N = I_{NB}$, then $v_N = v(I_N) = v(I_{NB}) = 0$. We can now conclude that, in the case of absolutely inelastic collision with friction, the Stronge model, as well as the Kane–Levinson–Whittaker and Routh models, predicts that $v_N = 0$. Then, following the same line of reasoning as with the Routh model, we see that formulas (2.2)–(2.6) are also valid for the Stronge model.

The final conclusion is that the solution of the problem of collision with friction of bodies in plane motion does not depend on which of the models (Kane–Levinson–Whittaker, Routh, or Stronge) is used. This solution is represented by formulas (2.2)–(2.6).

Conclusions. Three different models of absolutely inelastic collision of rough rigid bodies in plane motion have been compared. Expressions for the impact impulses have been derived for the Kane–Levinson–Whittaker model based on the kinematic restitution factor, the Routh model based on the kinetic restitution factor, and the Stronge model based on the energy restitution factor. It has been proved that these expressions will be the same if the collision of rough rigid bodies in plane motion is absolutely inelastic.

REFERENCES

1. N. V. Butenin, Ya. L. Lunts, and D. R. Merkin, *Course of Theoretical Mechanics* [in Russian], Vol. 2, Nauka, Moscow (1979).
2. N. N. Bukhgol'ts, *Basic Course in Theoretical Mechanics* [in Russian], Part 2, Nauka, Moscow (1966).

3. V. F. Zhuravlev, *Fundamentals of Theoretical Mechanics* [in Russian], Fizmatlit, Moscow (2001).
4. A. P. Ivanov, *Dynamics of Systems Undergoing Mechanical Impacts* [in Russian], Mezhd. Progr. Obraz., Moscow (1997).
5. V. V. Kosarchuk and A. V. Agarkov, "Predicting the life of rails based on a contact fatigue crack initiation criterion," *DETUT*, *Ser. Tekhn. Tekhnol.*, **20**, 77–90 (2012).
6. Ya. G. Panovko, *An Introduction to the Theory of Mechanical Shock* [in Russian], Nauka, Moscow (1977).
7. G. Gilardi and I. Sharf, "Literature survey of contact dynamics modeling," *Mechanism and Machine Theory*, **37**, 1213–1239 (2002).
8. W. Goldsmith, *Impact: The Theory and Physical Behavior of Colliding Solids*, Edward Arnold London (1960).
9. T. T. Kane and D. A. Levinson, *Dynamics. Theory and Applications*, McGraw-Hill New York (2005).
10. V. D. Kubenko, "Stress state of an elastic half-plane under nonstationary loading," *Int. Appl. Mech.*, **51**, No. 2, 121–129 (2015).
11. V. B. Larin, "Algorithms for solving a unilateral quadratic matrix equation and the model updating problem," *Int. Appl. Mech.*, **50**, No. 3, 321–334 (2014).
12. A. A. Martynyuk and N. V. Nikitina, "On periodic motion in three-dimensional systems," *Int. Appl. Mech.*, **51**, No. 4, 369–379 (2015).
13. N. P. Plakhtienko, "Translation of a rigid body with gravity–friction seismic dampers," *Int. Appl. Mech.*, **45**, No. 7, 786–796 (2009).
14. E. J. Routh, *Dynamics of a System of Rigid Bodies*, Edward Arnold London (1860).
15. W. J. Stronge, *Impact Mechanics*, Cambridge University Press, Cambridge (2000).
16. W. J. Stronge, "Unraveling paradoxical theories for rigid body collisions," *ASME J. Appl. Mech.*, **58**, No. 4, 1049–1055 (1991).
17. E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, Cambridge (1917).