Asset Allocation methods in python

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1 Introduction

This project aims to understand and discuss the most known methodologies used in the field of asset allocation. We will deepen, respectively, the Mean Variance optimization, the Black Litterman approach, and the Bayesian approach. For each of this approach, we also found the GMV portfolio and computed the relative statistics.
Moreover, for every of these methodologies, we are going to compute all the classical descriptive statistics, i.e., mean, variance, standard deviation, skew, kurtosis, and Sharpe Ratio, in order to, at the end, make a comparison of the final results.

2 Asset Allocation

2.1 Analysis of the Returns - Statistics

First of all, we removed from our dataset all the years with the unavailable value becauses certain data were missing and, moreover, we removed all the companies that in the last years were delisted, reducing the sample to 83 equities. Once computed the classical statistics, we decided to check whether one of the main stylized facts for financial returns is satisfied or not, the gaussianity of our returns. To do so, we runned a test of normality, which combines skew and kurtosis. For the daily data, for all the 83 equities, we rejected the null hyphotesis of the fact that returns follow a Gaussian distribution at every canonical significance level.

On the other hand, for the monthly data, we reject the null only for 44 equities. In conclusion, it's worthwhile to say that we must treat the data and the result carefully.

Daily sample statistics						
	Mean	**STD**	**Variance**	**Skew**	**Kurtosis**	**p-value**
I:LDO	0,000458	0,025004	0,000625	0,052659	11,34313	2,57E-49
I:ECK	0,001684	0,030759	0,000946	2,200217	14,04682	7,4E-137
I:LRZ	-0,0003	0,030082	0,000905	0,865584	11,6788	1,49E-73
I:PIRL	2,76E-05	0,02376	0,000565	-0,05223	6,000976	1,85E-33
I:STL	0,000694	0,023657	0,00056	-0,3633	6,868366	1,09E-41
I:PINF	-0,0006	0,024446	0,000598	1,743496	21,38274	1,4E-128
I:BRE	0,000463	0,021093	0,000445	-0,01779	3,614252	1,79E-22
I:ISP	0,00031	0,020344	0,000414	-0,87133	9,539712	1,41E-68
I:ILLB	6,63E-05	0,019296	0,000372	0,042383	6,822686	1,81E-36
I:UCG	0,000873	0,026713	0,000714	-0,29193	5,690216	1,69E-35
I:BANC	0,000505	0,019575	0,000383	0,169953	12,79826	1,57E-53
I:BPE	0,000462	0,027482	0,000755	0,34563	8,551804	1,23E-46
I:FCBK	0,000294	0,021826	0,000476	-0,13051	3,734601	8,49E-24
I:CPR	0,000556	0,017105	0,000293	-0,37852	11,87248	6,17E-56
I:SPAC	-0,00042	0,027747	0,00077	0,370675	5,417072	2,52E-36
I:CALT	0,000577	0,018528	0,000343	-0,12685	4,21276	3,43E-26
I:ENEL	0,000298	0,01682	0,000283	-1,6661	19,22954	2E-122
I:ARN	0,002484	0,029791	0,000888	1,246349	7,626323	1,12E-79
I:A2A	0,000225	0,017501	0,000306	-1,31525	14,55198	2,1E-99
I:TRN	0,00045	0,015309	0,000234	-0,9193	10,05287	5,25E-72
I:ACE	-3E-05	0,01626	0,000264	-0,92241	8,022378	1,99E-66
I:BMED	0,000453	0,020103	0,000404	-0,32973	6,223663	1,79E-38
I:TIPS	0,000504	0,016845	0,000284	0,498009	16,19156	9,76E-68
I:MB	0,000422	0,020178	0,000407	-0,87564	13,23231	2,97E-77
I:EQUI	0,000301	0,01722	0,000297	0,911738	11,17179	2,11E-74
I:ANI	0,000118	0,022198	0,000493	-0,13091	5,503966	4,68E-32
I:TITR	-0,00023	0,026143	0,000683	0,333598	21,72559	1,27E-70
I:TIT	-0,0003	0,026779	0,000717	0,94617	19,86731	5,09E-91
I:ENV	6,69E-05	0,017822	0,000318	0,838189	8,59104	1,7E-64
I:VAL	-0,00012	0,01677	0,000281	0,275788	6,876999	1,29E-39
I:CLT	0,000109	0,017637	0,000311	1,870145	17,15702	2,7E-128
I:HER	8,51E-05	0,017789	0,000316	-0,63711	14,13371	3,58E-69
I:IRE	-9E-05	0,017392	0,000302	-0,73867	7,981013	1,74E-58
I:IG	0,000146	0,015504	0,00024	-0,6389	7,915517	2,58E-54
I:ELN	0,001098	0,025098	0,00063	0,394759	4,548917	3,8E-33
I:AMP	0,000853	0,022912	0,000525	-0,4436	6,289522	6,68E-42
I:DLG	7,95E-05	0,022318	0,000498	0,424613	3,482848	1,27E-28
I:CNHI	0,000705	0,024376	0,000594	-0,76051	5,67024	3,5E-51
I:FD	-0,00071	0,030898	0,000955	1,675458	13,49774	1,6E-113
I:IP	0,000707	0,020949	0,000439	-0,25547	2,456987	2,5E-18
I:IKG	0,001222	0,023211	0,000539	1,176659	13,85453	7,51E-92
I:ENAV	1,95E-05	0,017289	0,000299	1,299761	17,69365	7,5E-104
I:PST	0,000384	0,019213	0,000369	-1,5801	19,1367	1,7E-118
I:RCS	-0,0004	0,021204	0,00045	-0,02032	6,337043	1,2E-34
I:CAI	-0,00051	0,02296	0,000527	0,564298	6,044593	5,07E-45

	Mean	STD	Variance	Skew	Kurtosis	p-value
I:MON	-0,00061	0,025496	0,00065	1,958561	20,66258	6,1E-137
I:GAMB	0,000116	0,028698	0,000824	1,775071	11,77749	2,6E-114
I:UNI	0,000374	0,020451	0,000418	-0,24266	11,83836	1,11E-52
I:G	0,000276	0,014709	0,000216	-0,78614	11,21898	3,93E-69
I:SRG	0,000221	0,015766	0,000249	-1,80891	23,27606	1,2E-133
I:ENI	0,000105	0,020364	0,000415	-1,22904	19,41743	4,2E-103
I:TOD	0,0003	0,026347	0,000694	0,892648	11,44115	3,44E-74
I:REC	0,000384	0,01818	0,000331	0,476451	17,0493	2,19E-68
I:RN	0,002057	0,036491	0,001332	1,30446	8,757764	9,26E-86
I:BRI	0,0003	0,021427	0,000459	0,131498	4,849566	3,09E-29
I:FUL	0,000438	0,036304	0,001318	2,517014	15,92006	8,2E-153
I:AISW	0,000639	0,031512	0,000993	2,2288	16,49333	3E-142
I:AGL	0,000357	0,027361	0,000749	1,423903	28,95169	3E-122
I:JUVE	-0,00048	0,025832	0,000667	-0,15651	9,980818	6,78E-47
I:SSL	0,000136	0,022695	0,000515	-0,67448	15,93166	8,88E-74
I:CLE	-0,00047	0,030791	0,000948	1,790507	19,05622	1,3E-127
I:B	-0,00017	0,02091	0,000437	1,053151	11,1949	1,02E-80
I:CEM	0,000336	0,019544	0,000382	-0,04338	3,178191	3,97E-20
I:US	0,000118	0,014015	0,000196	-0,30511	4,666312	2,24E-31
I:BZU	0,000427	0,020021	0,000401	-0,04475	7,144046	1,37E-37
I:CE	0,000432	0,01722	0,000297	-0,12488	3,040096	6,8E-20
I:DAN	0,000421	0,0223	0,000497	0,396541	4,850481	1,39E-34
I:ITM	0,000324	0,015365	0,000236	-0,06962	4,3252	2,8E-26
I:ZUC	0,00057	0,026734	0,000715	1,188621	10,25084	1,54E-84
I:IPG	0,0002	0,025959	0,000674	-0,04436	12,4803	8,94E-52
I:VIN	0,000315	0,020425	0,000417	0,372346	3,116109	4,51E-25
I:EDNR	0,000424	0,015527	0,000241	-0,74984	16,26257	2,33E-77
I:RAT	7,31E-05	0,023321	0,000544	1,047422	11,90108	4,7E-82
I:GAB	0,001452	0,031864	0,001015	1,301222	6,650707	7,89E-79
I:MS	-0,00042	0,022686	0,000515	-0,10708	5,555007	4,91E-32
I:ERG	0,000599	0,019372	0,000375	-0,27637	14,49873	1,13E-58
I:CMB	0,000521	0,020023	0,000401	0,030136	3,37725	3,31E-21
I:SAB	0,000131	0,020628	0,000426	0,514908	4,334795	6,73E-36
I:BE	0,000239	0,026592	0,000707	2,12184	12,00058	1,7E-129
I:SOL	0,000901	0,016236	0,000264	0,209216	1,387875	2,84E-10
I:DAL	-0,00073	0,026863	0,000722	0,914037	7,847665	1,63E-65
I:BSS	-6,6E-06	0,029475	0,000869	-0,24843	7,201552	3,51E-40
I:SAFI	0,001016	0,032535	0,001059	0,661356	9,276027	3,55E-59

Monthly sample statistics						
	Mean	**STD**	**Variance**	**Skew**	**Kurtosis**	**p-value**
I:LDO	0,006403	0,114737	0,013165	0,65385	4,495046	0,000606
I:ECK	0,03514	0,137693	0,018959	1,656689	3,40555	3,7E-06
I:LRZ	-0,00623	0,126853	0,016092	0,827604	4,726823	0,000182
I:PIRL	0,000777	0,104084	0,010833	-0,45258	0,584982	0,229038
I:STL	0,015456	0,117094	0,013711	-0,658	2,568012	0,006007
I:PINF	-0,01308	0,093965	0,008829	-0,70609	2,420186	0,005678
I:BRE	0,011272	0,109831	0,012063	0,107054	-0,45823	0,747869
I:ISP	0,007578	0,104875	0,010999	-0,31791	2,801299	0,018244
I:ILLB	-0,00182	0,094501	0,00893	-0,99607	2,691812	0,000719
I:UCG	0,019645	0,125468	0,015742	-0,31359	1,7788	0,073903
I:BANC	0,011049	0,0874	0,007639	-0,95324	2,34292	0,001495
I:BPE	0,010749	0,13328	0,017764	0,531705	0,677702	0,146542
I:FCBK	0,004405	0,086855	0,007544	-0,01466	0,375038	0,746693
I:CPR	0,009735	0,070803	0,005013	-0,67287	0,601527	0,080633
I:SPAC	-0,01317	0,112165	0,012581	0,19962	0,798474	0,376842
I:CALT	0,011855	0,075736	0,005736	-0,07925	1,21124	0,254846
I:ENEL	0,004643	0,074429	0,00554	0,040165	0,717214	0,50358
I:ARN	0,054716	0,143537	0,020603	1,690522	3,730743	2E-06
I:A2A	0,004301	0,08123	0,006598	-1,10471	2,336208	0,000578
I:TRN	0,007845	0,050123	0,002512	-0,12558	-0,55881	0,597779
I:ACE	-0,00189	0,070825	0,005016	-0,49452	1,512186	0,055812
I:BMED	0,010377	0,099981	0,009996	-0,75745	3,186796	0,001585
I:TIPS	0,009739	0,066689	0,004447	-0,13171	0,038493	0,882632
I:MB	0,008408	0,096879	0,009386	-0,96434	3,696303	0,000251
I:EQUI	0,006041	0,074269	0,005516	0,018084	1,446173	0,188986
I:ANI	0,004401	0,110441	0,012197	-0,33788	1,666186	0,080418
I:TITR	-0,00692	0,11618	0,013498	0,995357	1,931762	0,002023
I:TIT	-0,00717	0,126583	0,016023	1,522633	4,425471	2,82E-06
I:ENV	-0,00139	0,041988	0,001763	0,887331	2,789934	0,001235
I:VAL	-0,00495	0,056733	0,003219	0,362815	0,680916	0,283583
I:CLT	0,00144	0,066607	0,004437	2,256988	9,143897	3,8E-10
I:HER	-0,00108	0,06537	0,004273	-0,86086	0,937356	0,018358
I:IRE	-0,00244	0,078376	0,006143	-0,60044	0,407745	0,14409
I:IG	0,001504	0,054059	0,002922	-0,20887	0,185964	0,711262
I:ELN	0,026078	0,139543	0,019472	0,034736	0,366301	0,749415
I:AMP	0,017012	0,098503	0,009703	-0,52723	0,96423	0,103349
I:DLG	0,001359	0,104746	0,010972	0,253876	-0,08094	0,72865
I:CNHI	0,015799	0,109712	0,012037	-0,32809	2,807799	0,017545
I:FD	-0,01803	0,127846	0,016345	2,17897	9,32932	5,54E-10
I:IP	0,015491	0,102467	0,0105	-0,37956	-0,3102	0,473886
I:IKG	0,027308	0,106859	0,011419	0,759526	1,896688	0,00868
I:ENAV	-0,00148	0,075233	0,00566	-0,09222	1,531284	0,161025
I:PST	0,005649	0,077971	0,00608	-0,20309	0,892381	0,331253
I:RCS	-0,00613	0,096431	0,009299	0,055184	1,723925	0,126181
I:CAI	-0,01011	0,113349	0,012848	0,366276	0,40629	0,38253

	Mean	**STD**	**Variance**	**Skew**	**Kurtosis**	**p-value**
I:MON	-0,01757	0,084835	0,007197	0,542453	5,163952	0,00052
I:GAMB	-0,00231	0,115975	0,01345	0,529941	2,412539	0,013737
I:UNI	0,006075	0,091346	0,008344	0,011742	1,733147	0,126366
I:G	0,005928	0,079404	0,006305	0,053054	2,320089	0,055387
I:SRG	0,002394	0,051732	0,002676	-0,24474	-0,43758	0,610558
I:ENI	0,000688	0,09072	0,00823	0,717894	2,644685	0,003951
I:TOD	0,006524	0,133333	0,017778	1,077807	2,397986	0,000633
I:REC	0,005978	0,059358	0,003523	-0,19832	-0,00581	0,80706
I:RN	0,048105	0,184916	0,034194	1,184659	3,221553	0,000109
I:BRI	0,00262	0,084815	0,007194	-0,2571	0,99535	0,256304
I:FUL	0,01256	0,210121	0,044151	5,136234	32,38528	1,3E-20
I:AISW	0,013249	0,135124	0,018258	0,642049	0,887069	0,066075
I:AGL	0,006297	0,140144	0,01964	1,468895	8,045411	1,54E-07
I:JUVE	-0,01668	0,10853	0,011779	0,369528	0,601407	0,305101
I:SSL	0,004137	0,122025	0,01489	0,146525	6,062927	0,000757
I:CLE	-0,01416	0,112467	0,012649	0,254454	1,505215	0,127184
I:B	-0,00725	0,051753	0,002678	0,747817	1,876677	0,009528
I:CEM	0,006528	0,083935	0,007045	0,74295	0,551393	0,058585
I:US	0,00066	0,061217	0,003747	0,377577	0,005592	0,495197
I:BZU	0,006512	0,072923	0,005318	-0,45244	-0,13281	0,384773
I:CE	0,011155	0,092415	0,008541	0,709556	3,641343	0,001187
I:DAN	0,008644	0,097822	0,009569	-0,0087	-0,21495	0,986262
I:ITM	0,00556	0,053771	0,002891	0,518683	0,446364	0,202483
I:ZUC	0,007738	0,107156	0,011482	2,217409	5,863692	6,66E-09
I:IPG	0,002058	0,108428	0,011757	1,005246	1,07985	0,006254
I:VIN	0,003625	0,057332	0,003287	0,651152	0,164852	0,1381
I:EDNR	0,008746	0,062427	0,003897	-0,46695	1,989238	0,032099
I:RAT	-0,00297	0,065746	0,004323	-0,26433	0,952392	0,266698
I:GAB	0,040428	0,219012	0,047966	1,759254	4,605852	4,74E-07
I:MS	-0,01036	0,099801	0,00996	0,931533	2,795589	0,000937
I:ERG	0,012032	0,075812	0,005747	0,236918	2,371994	0,039828
I:CMB	0,010193	0,086647	0,007508	-0,83778	4,126863	0,000329
I:SAB	0,003394	0,105335	0,011095	0,032867	0,030924	0,959524
I:BE	0,004629	0,11443	0,013094	1,269209	2,985252	8,38E-05
I:SOL	0,019083	0,067277	0,004526	0,228902	-0,67264	0,35435
I:DAL	-0,01388	0,126544	0,016013	0,655652	0,057755	0,143846
I:BSS	-0,00152	0,130442	0,017015	-0,39766	-0,67288	0,216458
I:SAFI	0,023875	0,158366	0,02508	1,308797	1,989239	0,000243

To test whether the returns of each stock are normally distributed or not, we used one of the most known test provided by the scipy library, the D'Agostino and Pearson one. This function tests the null hypothesis that a sample comes from a normal distribution. It is based on D'Agostino and Pearson's [1], [2] test that combines skew and kurtosis to produce an omnibus test of normality.
Below, we provided a pictorial example on just one stock and we repetead this test for every stock in the sample.

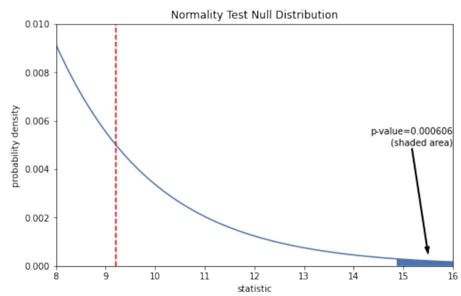

Figure 1: Normal null test distribution

2.2 Variance, Covariance and Correlation Matrices

Throughout the whole job, for each section, we will report the same results for both daily and monthly data. Here, we plotted, respectively, the correlation matrix and the variance covariance matrix for the whole sample. The same, but with data rather than colour, can be found in the code file.

2.2.1 Daily matrices

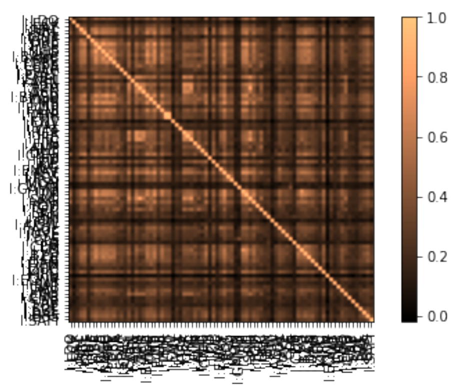

Figure 2: Daily correlation matrix

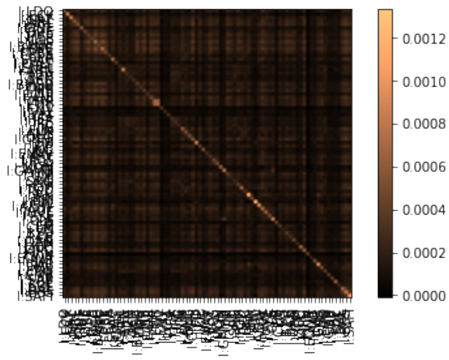

Figure 3: Daily variance-covariance matrix

2.2.2 Monthly matrices

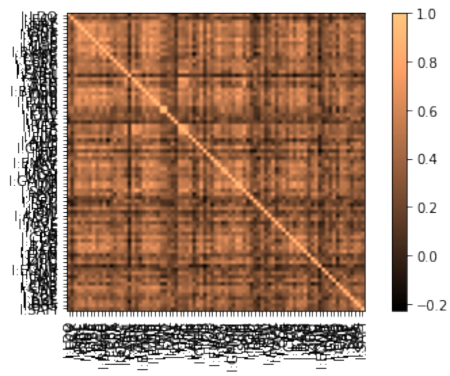

Figure 4: Monthly correlation matrix

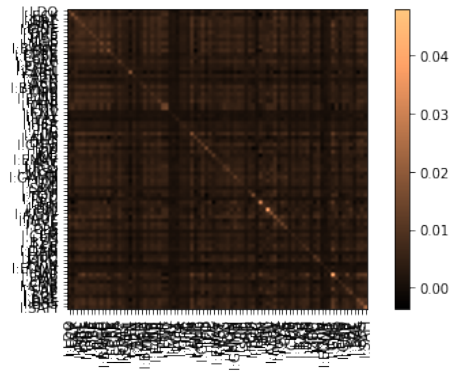

Figure 5: Monthly variance-covariance matrix

2.3 Securities sample selection

Our sample selecting process is based on the correlation matrix, both for daily and monthly data. Once got the correlation matrix, it consists in:

- Get all possible combinations of stock pairs: The code uses the combinations function from itertools to generate all possible combinations of stock pairs based on the columns of the correlation matrix.
- Calculate the mean correlation for each stock: The code calculates the mean correlation for each stock by taking the mean along axis=1 of the correlation matrix using .mean(axis=1). This computes the average correlation of each stock with other stocks.
- Sort the stocks based on mean correlation and original order: The code sorts the stocks based on their mean correlation values in ascending order using sort_values. The resulting sorted stocks are stored in sorted_stocks while maintaining the original order using the .index property.
- The code selects the top 12 stocks with the lowest mean correlation.

In conclusion, we have that the top 12 stocks with lowest mean correlation are:

Ticker	**Corr**
I:VIN	0,061264
I:FUL	0,106446
I:ENV	0,117296
I:ECK	0,123312
I:MON	0,128166
I:B	0,136845
I:FD	0,137097
I:GAMB	0,143569
I:BE	0,17055
I:IKG	0,173696
I:RAT	0,176864
I:ZUC	0,182786

2.4 Behavior of security prices

Below you can find the cumulative returns of each security price, both for daily and monthly data.

2.4.1 Daily prices

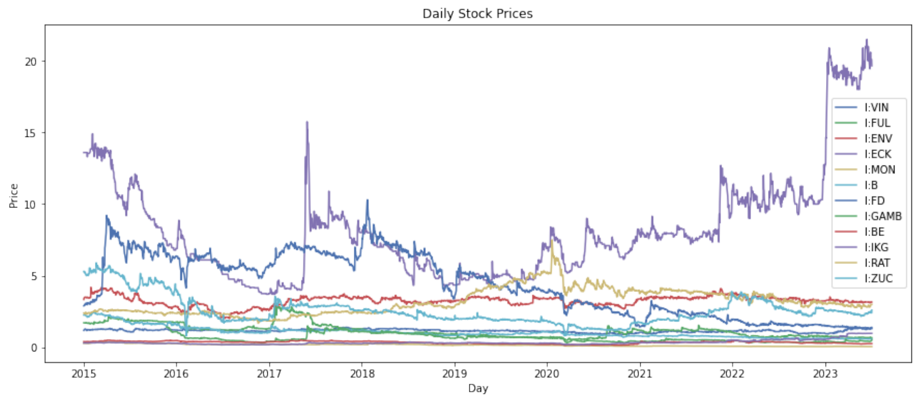

Figure 6: Daily behaviour of selected securities

2.4.2 Monthly prices

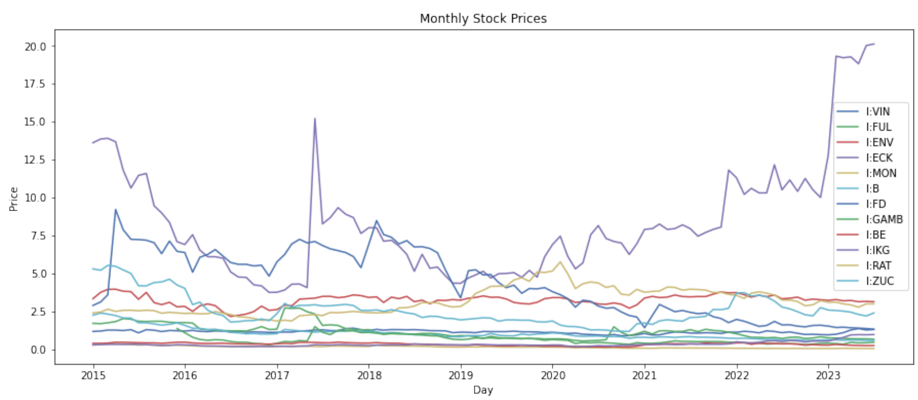

Figure 7: Monthly behaviour of selected securities

3 Mean Variance Optimization

As required, we implemented two versions of the Mean Variance Optimization:

- Short selling allowed with no bounds, i.e. weights may assume negative values.
- Short selling not allowed, i.e. all weights must be positive.

For what concerns the expected return of our securities, we used the CAPM. However, it's important to recognise that this is not the optimal strategy, since it requires several strong assumptions.
CAPM is widely used in financial theory, yet, it is paramount to be aware of its limitations and the simplifying assumptions on which it is based. It assumes:

1. Efficient markets: All investors have the same information and expectations about the future. However, we know real markets is constantly subject to inefficiencies, and irrational investor behavior.
2. Perfect diversification: The CAPM assumes that investors have access to an unlimited set of diversified investments. In practice investors have restricted access to certain assets.
3. Absence of transaction costs, taxes on capital gains and short-selling restrictions.
4. Linear relationship between return and risk: The CAPM assumes a linear relationship between the expected return of an asset and its systemic risk, measured by the beta coefficient.

Furthemore, as Richard Roll, American Economist, shown, the market portfolio cannot be unique, since it is always possible to identify any other portfolio on the frontier as the risk-free security changes, and this aspect becomes even more relevant if we refer to the notion of the orthogonal portfolio. Anytime we employ CAPM, we have to ask to ourselves if we can identify a market portfolio has efficient, namely it has to be a correct specification of the entire universe of investable securities. Therefore the analyst has to asses that in order to fulfill the requirement of market efficiency and achieve a ”perfect” Security Market Line (SML), we must rely on a widely representative index.

3.1 Daily data (GMV and Max Sharpe)

3.1.1 Short Selling is Allowed

Considering now the case in which short selling is allowed, we firsly represent in a table the weights we should allocate to each stock, then we plot the efficient frontier on it to represent both the Max Sharpe ratio portfolio (indicated by the red dot), and the GMV portfolio (indicated by the red star).

	Max Sharpe	GMV
I:VIN	0.067366	0.180990
I:FUL	0.036292	0.029842
I:ENV	0.137845	0.214476
I:ECK	0.072498	0.061627
I:MON	0.061037	0.070206
I:B	0.101759	0.117664
I:FD	0.046668	0.027780
I:GAMB	0.066655	0.048447
I:BE	0.088800	0.049536
I:IKG	0.128600	0.079256
I:RAT	0.089582	0.071836
I:ZUC	0.102897	0.048340

This table represents the weights must be assigned to each stock, based on the Max Sharpe portfolio (2nd column) or the GMV allocation (3rd column). For the first portoflio can be seen that the preferred stocks should be I:ENV and I:IKG, respectively with a weight of 13.78% and 12.86%. Result which is partially confirmed on the GMV allocation where I:IKG is still the preferred stock with a 21.44%, while the second one is I:VIN with a 18.09%, followed by I:B with a share of 11.76%.

In terms of annualized statistics, the MS portfolio shows a mean of 6.07% with a standard deviation of 16.33% and a Sharpe ratio of 0,25, while the GMV portfolio has a mean of 5.55%, a standard deviation of 15,24% and a Sharpe Ratio of 0.23.

Below, the plot of the efficient frontier representing both portfolios:

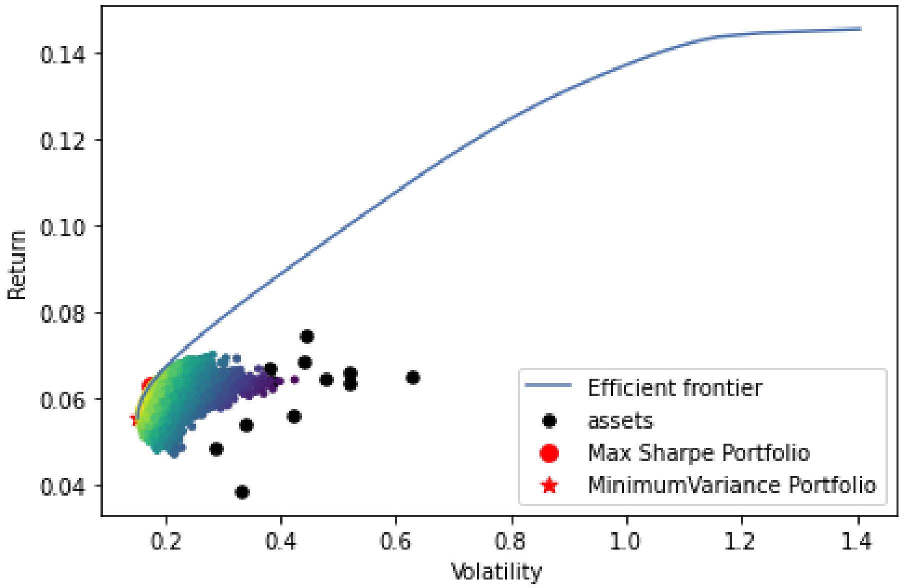

Figure 8: EF daily Short Selling allowed

As suggested at the beginning, even in our restricted sample, we have evidence of a non-gaussian distribution of the returns. Indeed, the Global Minimum Variance portfolio and Max Sharpe portfolio returns do not follow a Gaussian distribution and present "fatter" tails, highlighting the presence of outliers, a few moments, and that the probability to meet extreme values is high.

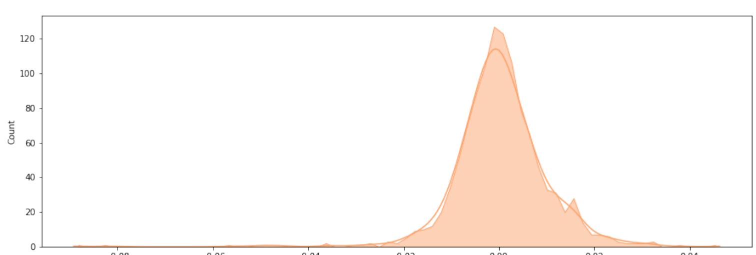

Figure 9: Daily Distribution of the Max Sharpe Portoflio

Mean	0.00033
STD	0.0100
Variance	0.000101
Skew	-1.336
Kurtosis	12.415

While, for what concerns the Global Mean Variance Portfolio:

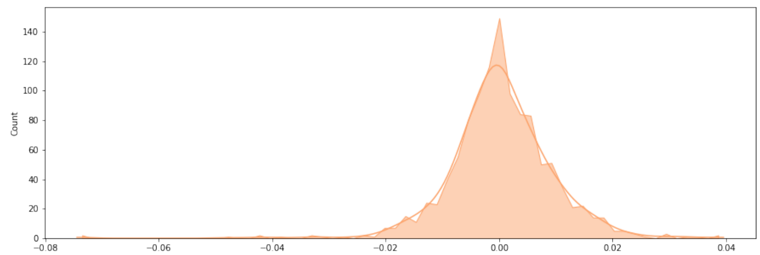

Figure 10: Daily distribution of the GMV Portoflio

Mean	0.0002
STD	0.0093
Variance	8.8223
Skew	-0.9791
Kurtosis	8.8821

3.1.2 Short selling is not allowed

In our case, applying a constraint on the weights is useless. As can be seen in the tables above, even if we allow for short selling (i.e., negative weights), the model is not suggesting any negative value.
Therefore, the two cases coincide, and there's no difference between them, at least for the daily settings.

3.2 Monthly data (GMV and Max Sharpe)

On the other hand, when dealing with monthly data, things are different. Let's start again by showing the ideal portoflio composition for both the Max Sharpe and GMV, followed by the statistics of the portfolios and, at the end, the distribution of the returns in order to understand whether they're normal or not.

3.2.1 Short Selling is Allowed

	Max Sharpe	GMV
I:VIN	0.085500	0.296966
I:FUL	-0.007545	-0.021032
I:ENV	0.229343	0.515082
I:ECK	0.044353	-0.012214
I:MON	0.036589	-0.028877
I:B	0.181019	0.215604
I:FD	0.054117	0.065301
I:GAMB	0.079109	-0.053710
I:BE	0.020043	-0.089615
I:IKG	0.040843	-0.012561
I:RAT	0.169927	0.104057
I:ZUC	0.066701	0.020999

This table represents the weights must be assigned to each stock, based on the Max Sharpe portfolio (2nd column) or the GMV allocation (3rd column), but this time on a monthly basis. For the first portfolio the preferred stocks are again I:ENV, with a 22.93% and I:B, with a 18.1%, but this time it's important to notice that, differently from the daily situation, we must go short on I:FUL, with a 0.75% weight. In the GMV case, instead, the situation is mixed. We should go long on 6 stocks and short on the other six. The most important stocks in our portfolio will be I:ENV with a seize of 51.5% and, for the short side, I:BE, with a 8.96%.

For what concerns the annualized statistics, the MS portfolio shows a mean of 7%, a standard deviation of 13,98% and a Sharpe Ratio of 0.35, while the GMV one shows a 4,4% mean, 9,7% standard deviation and a lower Sharpe Ratio, 0.25.

While, the efficient frontier:

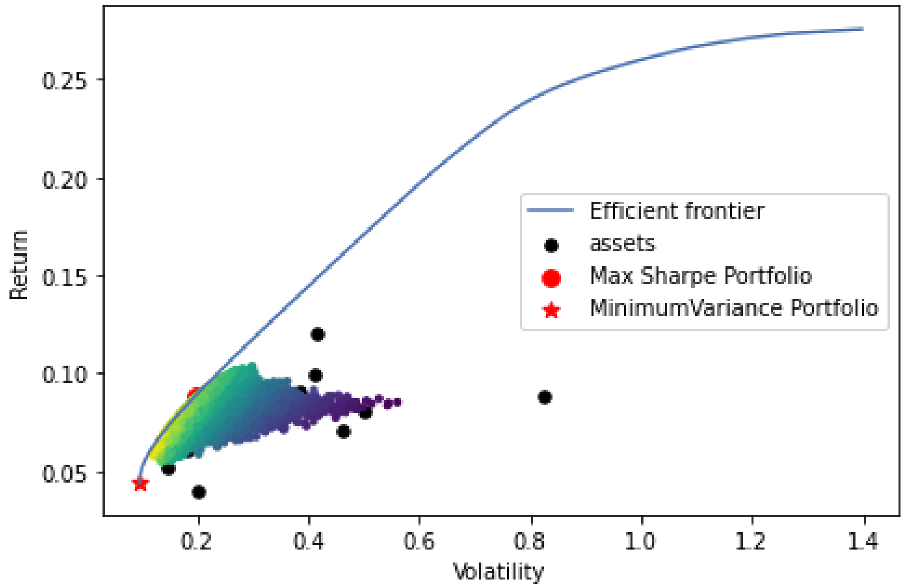

Figure 11: EF monthly short selling allowed

Let's now plot both the monthly statistics and the distribution of the returns.

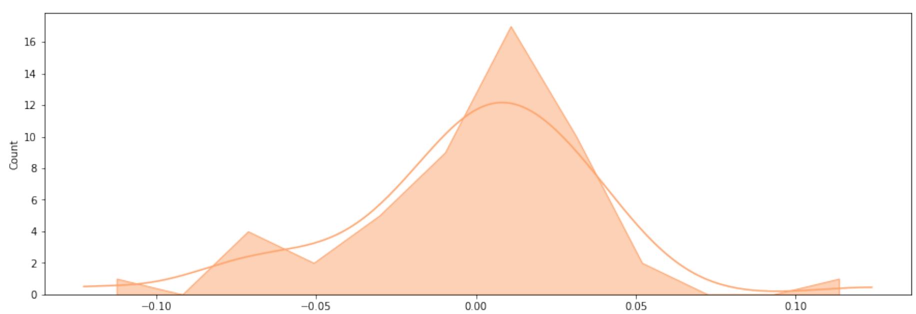

Figure 12: Monthly distribution of the Max Sharpe portoflio - Short Selling

Mean	0.0004
STD	0.0398
Variance	0.0015
Skew	-0.3633
Kurtosis	2.288

And the relative one for the GMV portfolio:

Mean	0.0033
STD	0.0274
Variance	0.0007
Skew	-0.3952
Kurtosis	0.5333

3.2.2 Short selling is not allowed

In this situation, computing the weights is worthwhile since, when not applying any constraints, we had some negative weights.Now we put a floor on each weight with a zero, meaning no stock can be short-sold.

	Max Sharpe	GMV
I:VIN	0.082358	0.268084
I:FUL	0.000000	0.000000
I:ENV	0.226605	0.417001
I:ECK	0.046484	0.000000
I:MON	0.030911	0.000000
I:B	0.178548	0.183099
I:FD	0.054559	0.065143
I:GAMB	0.082252	0.000000
I:BE	0.016786	0.000000
I:IKG	0.041833	0.000000
I:RAT	0.170847	0.066673
I:ZUC	0.068818	0.000000

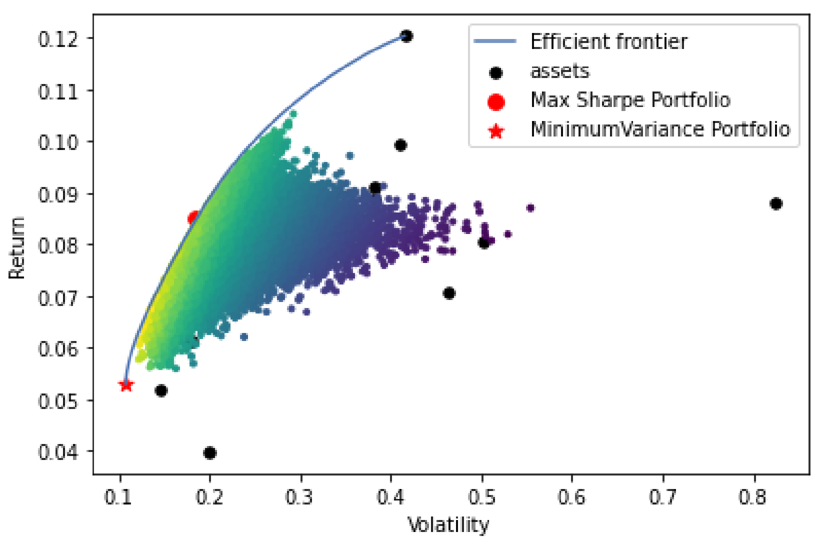

Figure 13: EF monthly short selling NOT allowed

Again, let's compute the daily statistics and check whether the returns are normaly distributed or not.

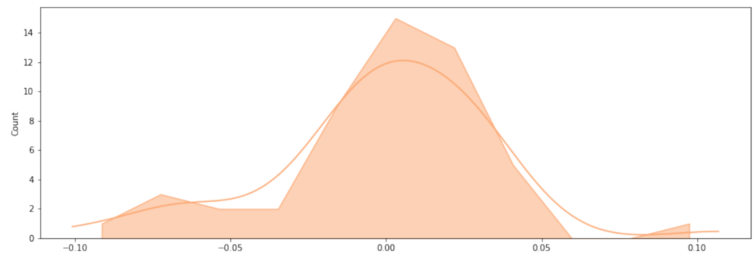

Figure 14: Max Sharpe Monthly returns distribution

Mean	0.0001
STD	0.040
Variance	0.0016
Skew	-0.3573
Kurtosis	2.2682

Now, the relative statics for the GMVP:

Mean	0.0023
STD	0.0030
Variance	0.0009
Skew	-0.2412
Kurtosis	0.3371

As before, we have shown that the returns do not follow a Gaussian Distribution, so they're not normal and must be treated carefully.

4 FTSE ITALIA All Share TR - Statistics

The FTSE Italia All Share Total Return index represents the italian stock market, including big, mid and small cap italian societies. We computed the main statistics.

4.0.1 Daily data

Mean	0.0004
STD	0.0138
Variance	0.0002
Skew	-1.3297
Kurtosis	15.1817

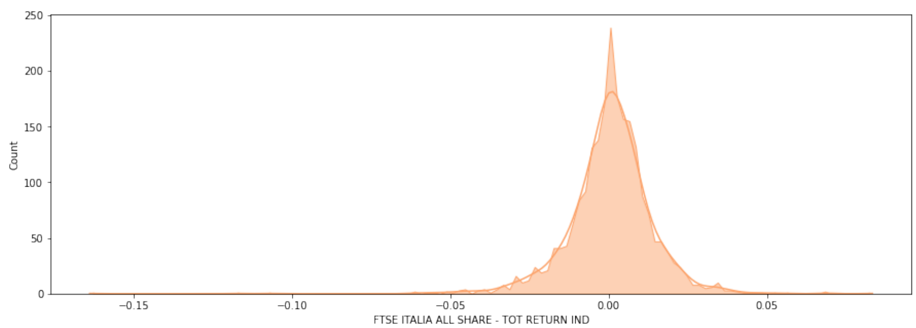

Figure 15: FTSE Italia All Share Total Return Daily returns

4.0.2 Monthly data

	Values
Mean	0.0087
STD	0.0581
Variance	0.0033
Skew	-0.559
Kurtosis	2.9924

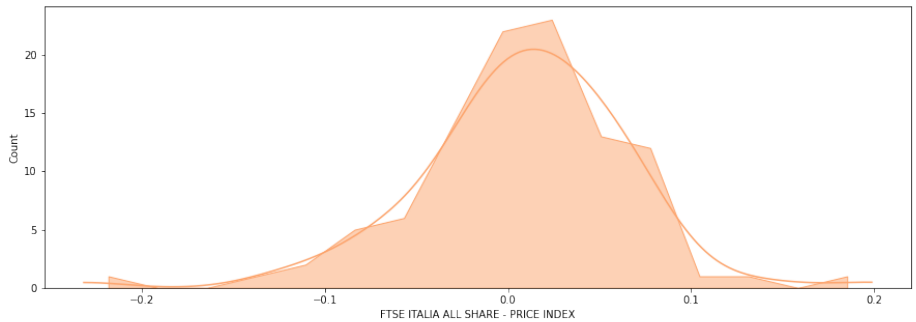

Figure 16: FTSE Italia All Share Total Return Monthly returns

In both cases, we tested whether the returns are normally distributed or not. As before, we reject H0.
If we want to make a comparison, just by looking at the daily data, some differences can be found from our portfolio (We are considering the Max Sharpe Portfolio computed through MVO). The mean of the daily distribution of the FTSE Italia All Share is slightly higher if compared to the one of our portfolio. The same result can be highlighted also for the other statistics. In both cases we have a negative skew, indicating that the distribution has a longer left tail or is "skewed to the left." This means that the distribution is characterized by a concentration of values towards the right side of the distribution, with fewer values towards the left side.
Another similarity can be found in the Kurtosis, even if the one of the FTSE Italia All Share is higher than the one in our portfolio. In both cases, the distributions are leptokurtic,meaning the distributions have heavier tails and a higher peak compared to a normal distribution. This indicates the presence of outliers or extreme values.

5 Beta

As required, we used as benchmark the FTSE Italia All Share Total Return index. However, since it represents the whole Italian stock market, it would have been the best choice anyway.

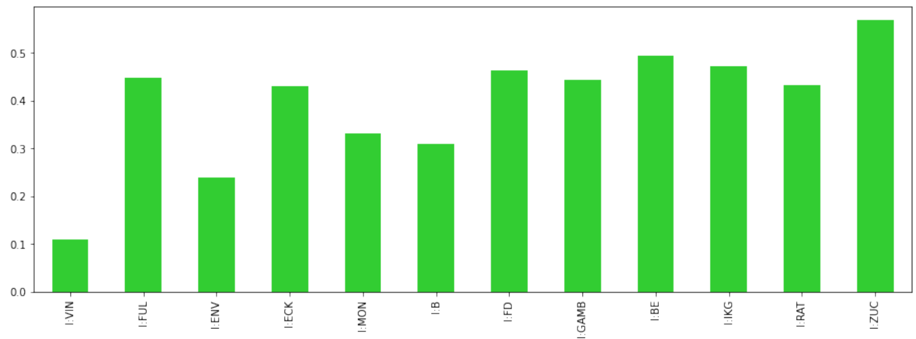

Figure 17: Daily beta of portfolio components

In the daily case, all the stocks have a beta lower than one, meaning they're "defensive". When the benchmark goes up by one, they will go up to, but by a smaller amount. The same reasoning, however, is applied for losses.
The largest beta is the one of ZUC, which is 0.56. The lowest beta is the one of VIN, which is 0.11.

The beta of the daily portfolio is 0.3927.

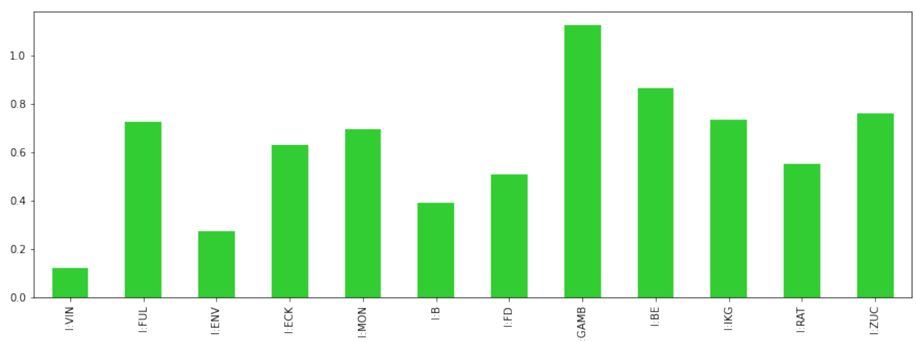

Figure 18: Monthly beta of portfolio components

In the monthly case, the situation is quite similar. Eleven out of 12 stocks have a beta lower than one, leading to an overall defensive portfolio. The stock with the largest beta is I:GAMB, with a value of 1.12. The stock with the lowest, instead, as in the daily case, is I:VIN.

The beta of the monthly portfolio is 0.4991.

The latter is higher than the daily one, meaning that it is more aggressive. However, they're both well below one, so they can be considered defensive.

6 Security Market Line

The Security Market Line (SML) represents the relationship between expected returns and systematic risk of a security or portfolio.
The SML states that the expected return of a security or portfolio depends on its system beta, which measures the sensitivity of a security or portfolio to market movements.
The SML is graphically represented as a straight line starting from the risk-free rate (typically represented by the risk-free interest rate) and extending upward with a slope equal to the market risk premium.

We set a risk free rate of 3%.

6.0.1 Daily Data

As can be seen, only two of the stocks considered for our portfolio when represented by the SML turn out to be undervalued, two stocks are correctly valued (i.e., they are on the SML), and the majority are overvalued (there are eight stocks that are below the Security Market Line).

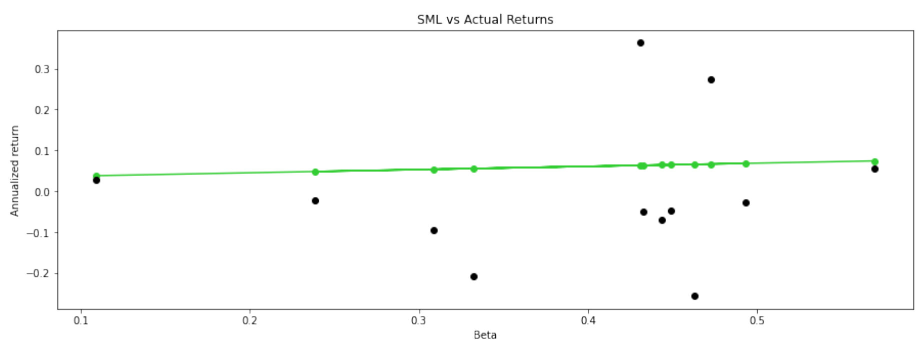

Figure 19: SML Daily

If we consider only two stocks, in this case VIN and BUL, the result still holds, since the SML plotted with only two securities is equivalent to the SML with all the stocks.

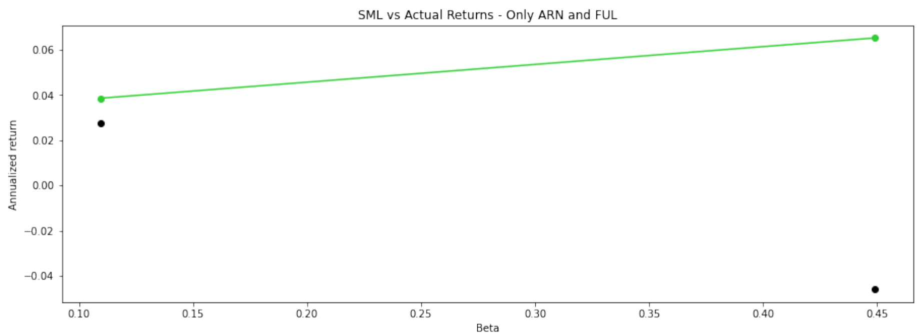

Figure 20: SML Daily

6.0.2 Monthly Data

On the other hand, when dealing with monthly data, two stocks can be considered undervalued since are placed above the SML and only one can be considered priced at the fair value, while the remaining 9 stocks are all overvalued.

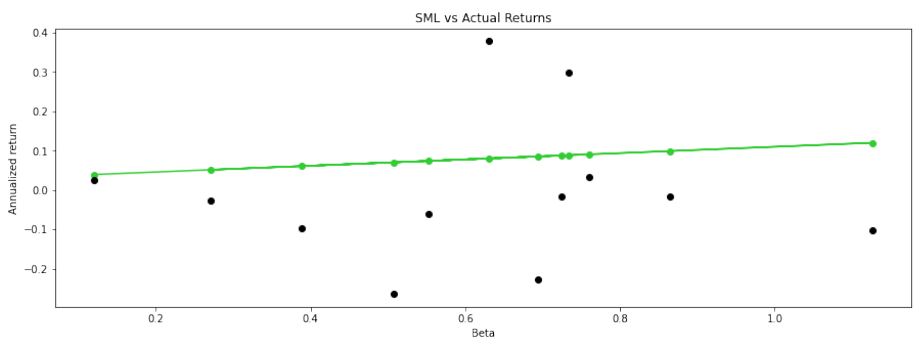

Figure 21: SML Monthly

And even in this case, when considering only two stocks, the result holds.

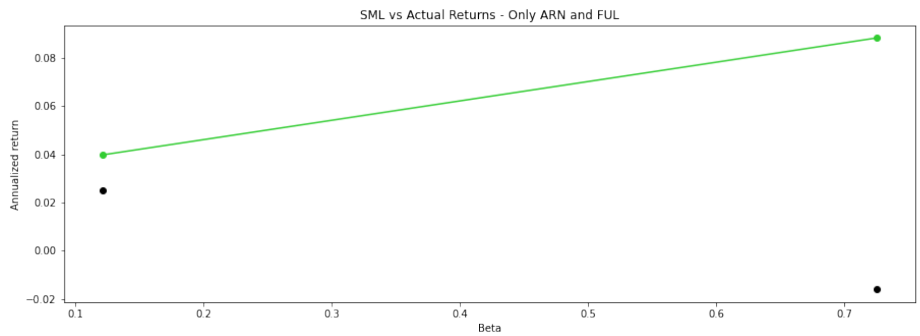

Figure 22: SML Monthly on two stocks

7 Black Litterman

7.0.1 Theory behind the model

The Black-Litterman asset allocation model, is a sophisticated portfolio construction method based on Bayesian analysis that overcomes the problem of highly-concentrated portfolios, input-sensitivity, and estimation error maximization. These three related and well-documented problems with mean-variance optimization are the most likely reasons that more practitioners do not use the Markowitz paradigm. The Black-Litterman model uses a Bayesian approach to combine the subjective views of an investor regarding the expected returns of one or more assets with the market equilibrium vector of expected returns (the prior distribution) to form a new, mixed estimate of expected returns.

Basically we need to carry out three main steps:

1. A prior equilibrium distribution employing ”Implied equilibrium return”;
2. An investor’s view (similar to likelihood, both relative and absolute);
3. A posterior distribution;

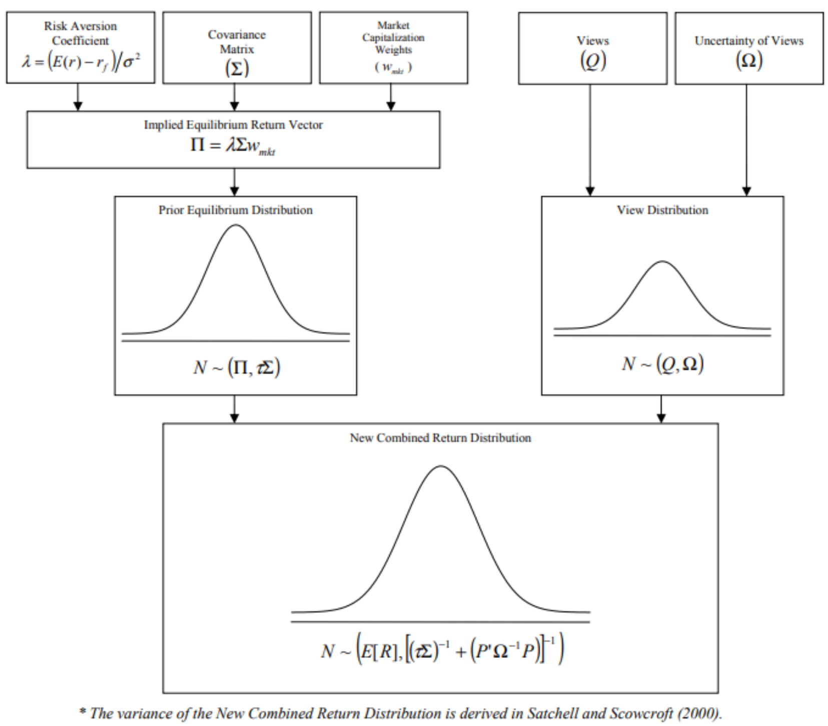

Figure 23: Black Litterman pictorially explanation from Idzorek 2004

Let's introduce the Black-Litterman formula and provide a brief description of each of its elements following Idzorek (2004).

7.0.2 Priors

Following Black and Litterman's original approach, we should have used the market capitalization of the assets to estimate the implied expected returns, called 'equilibrium return' as well.
Equilibrium returns are the set of returns that clear the market. The latter are derived using a reverse optimization method in which the vector of implied excess equilibrium returns is extracted from known information using:

$$\Pi = \lambda \Sigma w_{mkt}$$

where:

- Π is the Implied Excess Equilibrium Return Vector (N x 1 column vector);
- λ is the risk aversion coefficient;
- Σ is the covariance matrix of excess returns (N x N matrix);
- w_{mkt} is the market capitalization weight (N x 1 column vector) of the assets.

However, since we didn't have sufficient data to compute the market-implied returns, we used as a proxy the CAPM expected returns, as shown in the Izdorak's paper, where the CAPM Return Vector results to be quite similar to the Implied Equilibrium Return Vector, with a correlation coefficient of 99.8%.

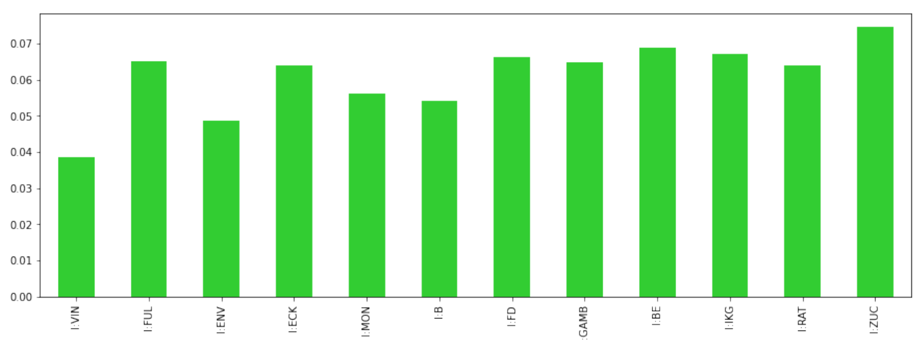

Figure 24: BL Daily Prior estimates for mu - CAPM

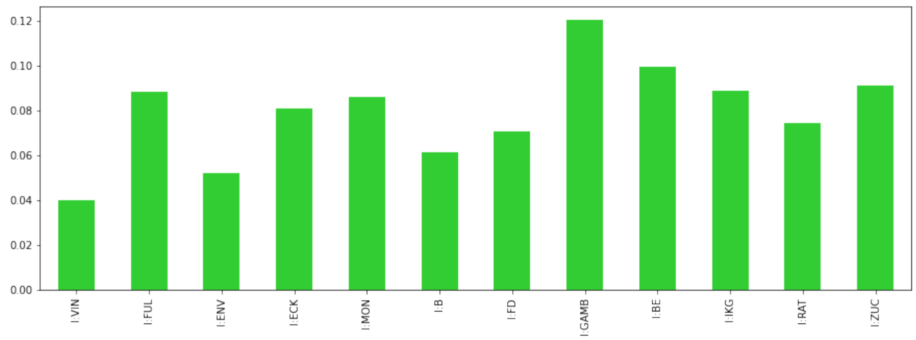

Figure 25: BL Monthly Prior estimates for mu - CAPM

7.1 Q and P - Views

Assuming the investor has k different views on a total of n assets, the views are represented as linear combinations of expected returns:

$$P * \mu = Q + \epsilon$$

where:

- P is a matrix (k $\times$ n) containing the weight of each investor view, each row of P represents a view, where the value is different from 0 if the corresponding asset is subject to a view, otherwise it is equal to 0. If the view is relative then the sum of the weights will equal 0, if the view is absolute then the sum of the weights will equal 1.
- μ is the vector (n $\times$ 1) containing the average of the expected returns;
- Q is a vector (k $\times$ 1) containing the expected returns for each view;
- ϵ is a vector (k $\times$ 1) containing the random errors committed in the views.

In our case the number of views (k) is 4; thus, the View Vector (Q) is a $4x1$ column vector. The uncertainty of the views results in a random, unknown, independent, normally-distributed Error Term Vector (ε) with a mean of 0 and covariance matrix Ω. Thus, a view has the form $Q+\varepsilon$. General Case: Example:

$$Q+\varepsilon=\begin{bmatrix} Q_1 \\ \vdots \\ Q_k \end{bmatrix}+\begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix} \quad Q+\varepsilon=\begin{bmatrix} 0.02 \\ 0.02 \\ -0.05 \\ 0.05 \end{bmatrix}+\begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix}$$

Except in the hypothetical case in which a clairvoyant investor is 100% confident in the expressed view, the error term (ε) is a positive or negative value other than 0 . The Error Term Vector (ε) does not directly enter the Black-Litterman formula. However, the variance of each error term (ω), which is the absolute difference from the error term's (ε) expected value of 0 , does enter the formula. The variances of the error terms (ω) form Ω, where Ω is a diagonal covariance matrix with 0 's in all of the off-diagonal positions. The off-diagonal elements of Ω are 0 's because the model assumes that the views are **independent** of one another. The variances of the error terms (ω) represent the uncertainty of the views. The larger the variance of the error term (ω), the greater the uncertainty of the view.

$$\Omega = \begin{bmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \omega_k \end{bmatrix}$$

The expressed views in column vector Q are matched to specific assets by Matrix P. Each expressed view results in a $1xN$ row vector. Thus, K views result in a KxN matrix. In our case P is a $4x12$ matrix.

We formulated **4 views**, two absolutes and two relatives. For what concerns the two **absolute** views:

1. **Vianini** (I:VIN) will do worse than -5%. The reason is linked to the monetary policy conducted by the ECB in the latest period and its implication on the real estate sector. Interest rates are rising and the peak has yet not been reached, putting more pressure on this sector and, for us, this would lead to a period of turmoil.

2. **EcoSuntek** (I:ECK) predicted return is set equal to 5%; This estimate is based on the policies that the world is currently pursuing towards a shift from the use of fossil fuels, from which companies whose business is based on the renewables, could benefit.

Furthermore, we made two **relative** views:

1. **Beewize** (I:FUL) will outperform **Monfir** (I:MON) by at least a 2%. The reason for that estimation is based on the businesses of the two societies, for which we believe the former will benefit the most in the years to come.

2. **EcoSuntek** (I:ICK) will outperform **Zucchi** (I:ZUC) by at least a 2%. In this case, there are two main reasons: the first one is based on the business, and we explained before why we like the former one. The second reasoning is based on the fact that a significant amount of Zucchi's capital has been acquired by a private fund, and there are several studies that show the negative impact of this kind of event on stock performance.

Indeed, this is our P - Matrix:

P Matrix - 4 x 12

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
0	0.0	1.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0
2	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	-0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

7.1.1 Omega (error terms from views) - Confidence Matrix

For computing the confidence matrix we will follow Idzorek (2004).

The author proposed that "the diagonal elements of Ω be derived in a manner that is based on the user-specified confidence levels and that results in portfolio tilts, which approximate $w_{100\%} - w_{mkt}$ multiplied by the user-specified confidence level (C).

$$Tilt_k \approx (w_{100\%} - w_{mkt}) * C_k$$

where $Tilt_k$ is the approximate caused by the k_{th} view (NxI column vector) and, C_k is the confidence in the k_{th} view.
Furthermore, in the absence of other views, the approximate recommended weight vector resulting from the view is:

$$w_{k,\%} \approx w_{mkt} + Tilt_k$$

where $w_{k,\%}$ is the target weight vector based on the tilt caused by the k_{th} view (Nxl column vector).
The steps of the procedure are as follows:

1. For each view (k), calculate the New Combined Return Vector ($E\left[R_{100\%}\right]$) using the Black-Litterman formula under 100% certainty, treating each view as if it was the only view.

$$E\left[R_{k,100\%}\right] = \Pi + \tau\Sigma p_k' \left(p_k \tau \Sigma p_k'\right)^{-1} \left(Q_k - p_k \Pi\right)$$

where $E\left[R_{k,100\%}\right]$ is the Expected Return Vector based on 100% confidence in the k th view (Nx I column vector); p_k identifies the assets involved in the k_{th} view ($1xN$ row vector); and, Q_k is the k th View (IxI).

2. Calculate $w_{k,100\%}$, the weight vector based on 100% confidence in the k th view, using the unconstrained maximization formula.

$$w_{k,100\%} = (\lambda\Sigma)^{-1} E\left[R_{k,100\%}\right]$$

3. Calculate (pair-wise subtraction) the maximum departures from the market capitalization weights caused by 100% confidence in the k th view.

$$D_{k,100\%} = w_{k,100\%} - w_{mkt}$$

where $D_{k,100\%}$ is the departure from market capitalization weight based on 100% confidence in k th view (Nxl column vector). Note: The asset classes of $w_{k,100\%}$ that are not part of the k th view retain their original weight leading to a value of 0 for the elements of $D_{k,100\%}$ that are not part of the k th view.

4. Multiply (pair-wise multiplication) the N elements of $D_{k,100\%}$ by the user-specified confidence (C_k) in the k th view to estimate the desired tilt caused by the k th view.

$$Tilt_k = D_{k,100\%} * C_k$$

where $Tilt_k$ is the desired tilt (active weights) caused by the k_{th} view (NxI column vector); and, C_k is an Nxl column vector where the assets that are part of the view receive the user-specified confidence level of the k th view and the assets that are not part of the view are set to 0.

5. Estimate (pair-wise addition) the target weight vector $\left(w_{k,\%}\right)$ based on the tilt.

$$w_{k,\%} = w_{mkt} + Tilt_k$$

6. Find the value of ω_k (the k th diagonal element of Ω), representing the uncertainty in the k th view, that minimizes the sum of the squared differences between $w_{k,\%}$ and w_k

$$\min \sum \left(w_{k,\%} - w_k\right)^2$$

subject to $\omega_k > 0$ where

$$w_k = [\lambda\Sigma]^{-1}\left[(\tau\Sigma)^{-1} + p_k'\omega_k^{-1}p_k\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + p_k'\omega_k^{-1}Q_k\right]$$

7. Repeat steps $1-6$ for the K views, build a KxK diagonal Ω matrix in which the diagonal elements of Ω are the ω_k values calculated in step 6 , and solve for the New Combined Return Vector ($E[R]$) using the formula for the new combined return vector:

$$E[R] = \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q\right]$$

Throughout this process, the value of scalar (τ) is held constant and does not affect the new Combined Return Vector ($E[R]$), which eliminates the difficulties associated with specifying it. Despite the relative complexities of the steps for specifying the diagonal elements of Ω, the key advantage of this new method is that it enables the user to determine the values of Ω based on an intuitive 0% to 100% confidence scale." Izdzorek (2004).

This methodology is implemented in PyPortfolioOpt, an open-source python library created by R. A. Martin [15].

We set as follow the confidence ranges for our views:

1. 20% that BeeWize will outperform by 5% Monfir;
2. 15% that EcoSuntek will outperform by 2% Zucchi;
3. 60% that Vianini will do worst than -5
4. 50% that EcoSuntek will do perform better than 5

From this views and the confidence level we decided to set, we computed the confidence matrix, from which is it possible to see what we underlined before, so that to the views with low confidence level is associated an higher value of error and viceversa, implying that the larger the variance of the error term (ω), the greater the uncertainty of the view.

Confidence Matrix - Daily Data

	0	**1**	**2**	**3**
0	0.108794	0.000000	0.000000	0.000000
1	0.000000	0.127923	0.000000	0.000000
2	0.000000	0.000000	0.003694	0.000000
3	0.000000	0.000000	0.000000	0.013455

While, when dealing with monthly data, it becomes:

Confidence Matrix - Monthly Data

	0	**1**	**2**	**3**
0	0.121869	0.000000	0.000000	0.00000
1	0.000000	0.116651	0.000000	0.00000
2	0.000000	0.000000	0.001339	0.00000
3	0.000000	0.000000	0.000000	0.01264

7.1.2 Posterior

K is used to represent the number of views and N is used to express the number of assets in the formula. The formula for the new Combined Return Vector ($E[R]$) is

$$E[R] = \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q\right]$$

where:

- $E[R]$ is the new (posterior) Combined Return Vector (Nxl column vector);

- τ is a scalar;
- Σ is the covariance matrix of excess returns (NxN matrix)
- P is a matrix that identifies the assets involved in the views (KxN matrix or $1xN$ row vector in the special case of 1 view);
- Ω is a diagonal covariance matrix of error terms from the expressed views representing the uncertainty in each view (KxK matrix);
- Π is the Implied Equilibrium Return Vector (Nxl column vector);
- Q is the View Vector (KxI column vector).

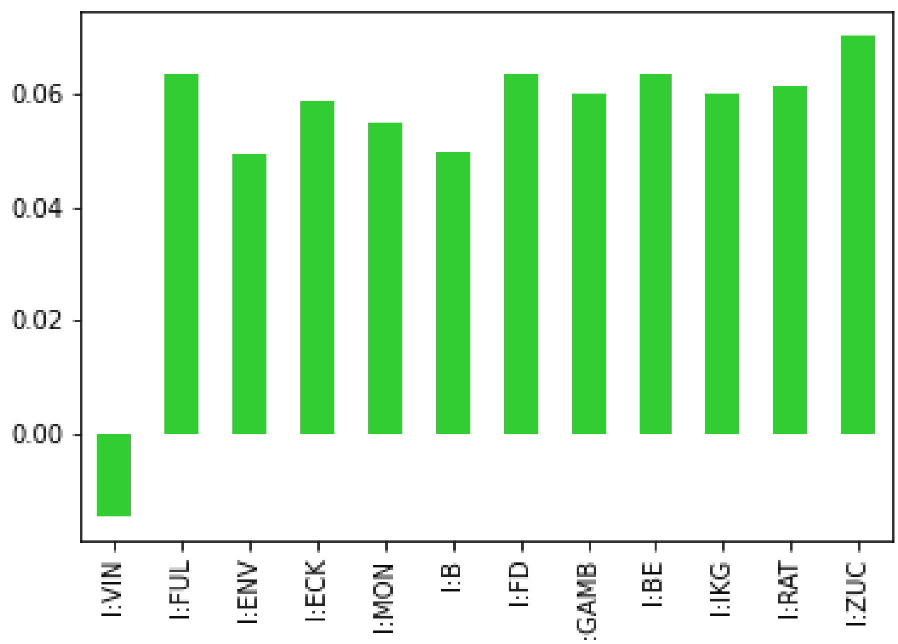

Figure 26: BL Posterior estimates for mu (annualized) - Computed on **daily** returns

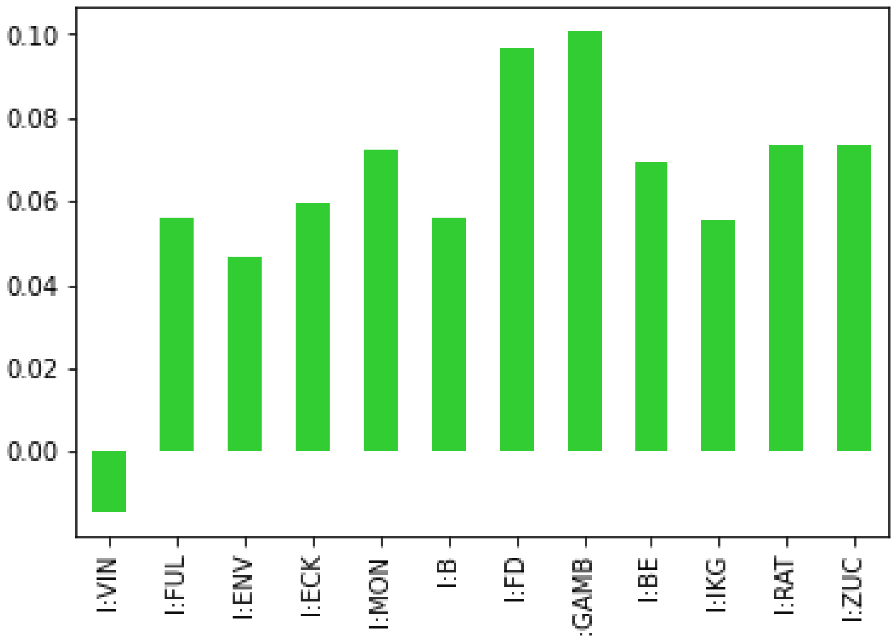

Figure 27: BL Posterior estimates for mu (annualized) - Computed on **monthly** returns

7.1.3 Predictive - Efficient Frontier

7.2 Daily (GMV and Max Sharpe)

Following that, we begin a daily analysis by reporting all the weights and right away take into account the efficient frontier produced from an optimization using the Black and Litterman model, where The portfolio with the highest Sharpe ratio and the star in the GMV portfolio are denoted with the pointer and star, respectively.

7.2.1 Short Selling Allowed

	Max Sharpe	GMV
I:VIN	-0.381055	0.185605
I:FUL	0.060128	0.029570
I:ENV	0.208273	0.212698
I:ECK	0.093238	0.063117
I:MON	0.086929	0.069648
I:B	0.153749	0.116689
I:FD	0.070512	0.027549
I:GAMB	0.100711	0.048046
I:BE	0.134170	0.049126
I:IKG	0.194305	0.078599
I:RAT	0.135351	0.071240
I:ZUC	0.143688	0.048113

The implementation of our views led to a significant change in the capital that should to allocate to each stock.
About the Max Sharpe Portfolio, when dealing with the Mean Variance Optimization we had no negative values, while here, I:VIN, even though is only one, must be shorted by allocating 38.1%. When dealing with the long side, instead, the stocks with the highest weights are: I:ENV, I:IKG, I:B, I:ZUC, I:RAT, I:BE. The main thing that should be underlined is that the model is not suggesting to short I:VIN randomly. Indeed, this results is implied by our negative absolute view on this society. In terms of annualized statistics, this portfolio shows a return of 8.68%, a standard deviation of 26,34% and a Sharpe Ratio of 0.25.
For what concerns the GMV portfolio, we have no stocks that must be shorted and the most of the capital should be invested in I:ENV(21%).
In terms of annualized statistics, this portfolio shows a return of 4.33%, a standard deviation of 15.55% and a significantly lower Sharpe Ratio of 0.15.
A comparison with the portfolios computed throught the MVO can be made. Indeed, the portfolios obtained with the BL methodology have higher mean, standard deviation and sharpe ratio, while the kurtosis is lower, but still higher than 3 (leptokurtic) and the skew is still negative, but also in this case, lower.(negative skew).
Below, you can find the tables reporting the daily statistics:

Mean	0.0003
STD	0.0158
Variance	0.0002
Skew	-0.6891
Kurtosis	6.8426
Sharpe	0.25

And the one for the GMV:

Mean	0.0002
STD	0.0093
Variance	8.8220
Skew	-0.969
Kurtosis	8.7519
Sharpe	0.15

While, the corresponding plot of the efficient frontier is:

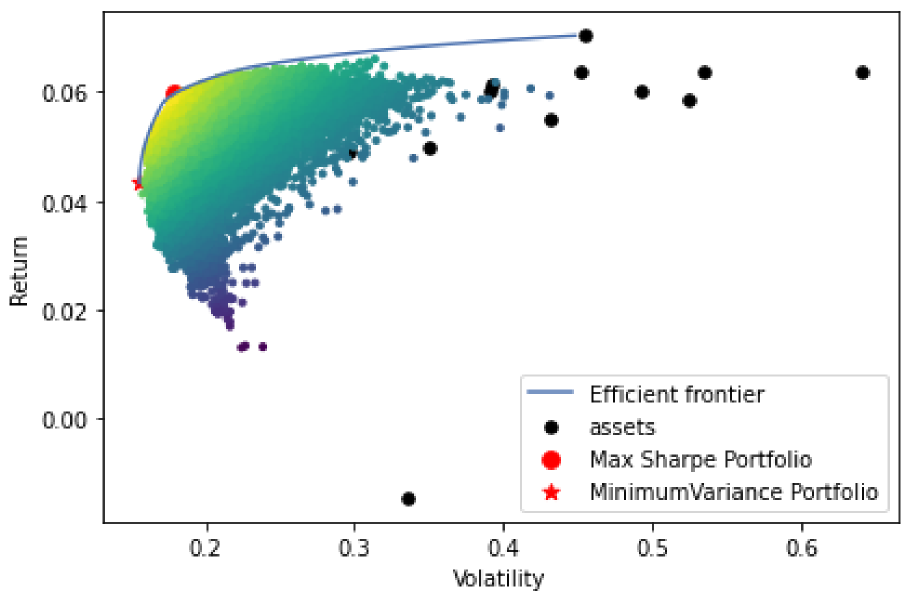

Figure 28: **Daily** Black Litterman frontier - Short selling

7.2.2 Short Selling not allowed

	Max Sharpe	GMV
I:VIN	0.000000	0.185605
I:FUL	0.043706	0.029570
I:ENV	0.168895	0.212698
I:ECK	0.073325	0.063117
I:MON	0.070102	0.069648
I:B	0.098779	0.116689
I:FD	0.051719	0.027549
I:GAMB	0.068417	0.048046
I:BE	0.090775	0.049126
I:IKG	0.124002	0.078599
I:RAT	0.101187	0.071240
I:ZUC	0.109094	0.048113

Clearly I:VIN, to which previously was assigned a negative value, now reports a null value, i.e. no investments must be made on this society. However, as before, even if with a lower weight, the portfolio is mostly composed by I:ENV, I:IKG, I:B, I:ZUC, I:RAT, I:BE. The annualized statistics for the Max Sharpe portfolio are a return of 5.8%, a standard deviation of 17.43% and a Sharpe of 0.22.
Moreover, what can be noted is that any of this portfolio presents a leptokurtic distribution, meaning that the value of the kurtosis is higher than 3 and this is a stylized fact of the financial returns. It means that the distribution has heavier tails and a higher peak compared to a normal distribution. This indicates the presence of outliers or extreme values.
If we want to make a comparison between the two approach, for the Max sharpe portfolio where the short selling is not allowed the kurtosis is almost the double of the corresponding portfolio when the short selling is allowed.

Mean	0.0003
STD	0.0104
Variance	0.0001
Skew	-1.2989
Kurtosis	12.44
Sharpe	0.22

While the one the GMV portfolio, in the case in which the short selling is not allowed, is the same as the one computed before, since there were no sign of stocks to short.

7.3 Monthly (GMV and Max Sharpe)

7.3.1 Short Selling Allowed

	Max Sharpe	GMV
I:VIN	-0.984978	0.301456
I:FUL	-0.003819	-0.020911
I:ENV	0.504722	0.511216
I:ECK	0.082020	-0.011407
I:MON	0.067738	-0.028623
I:B	0.398373	0.213986
I:FD	0.119097	0.064811
I:GAMB	0.174098	-0.053307
I:BE	0.044109	-0.088943
I:IKG	0.089884	-0.012467
I:RAT	0.373963	0.103276
I:ZUC	0.134794	0.020912

I:VIN's shares in the portfolio should be -0.98%. The reasoning, as said already, is in the view we implemented towards this society. Moreover, other than Vianini, even I:FUL should be shorted. In both occasion could be highlighted that in the monthly case, the kurtosis is smaller than 3, meaning that the distribution is platikurtic. It means that the distribution has lighter tails and a flatter peak compared to a normal distribution. This indicates that the dataset has fewer outliers or extreme values than a normal distribution.

Again, as made before, a comparison with the portfolios computed throught the MVO can be made. In this situation, the portfolios obtained with the BL methodology have higher mean, by far an higher standard deviation and slightly an higher sharpe ratio, while the kurtosis is lower, but still smaller than 3 (mesokurtic) and the skew are quite similar and negative (negative skew).

Mean	0.005
STD	0.0935
Variance	0.0087
Skew	-0.3802
Kurtosis	0.55
Sharpe	0.36

Mean	0.0032
STD	0.0274
Variance	0.0007
Skew	-0.3943
Kurtosis	0.51
Sharpe	0.3

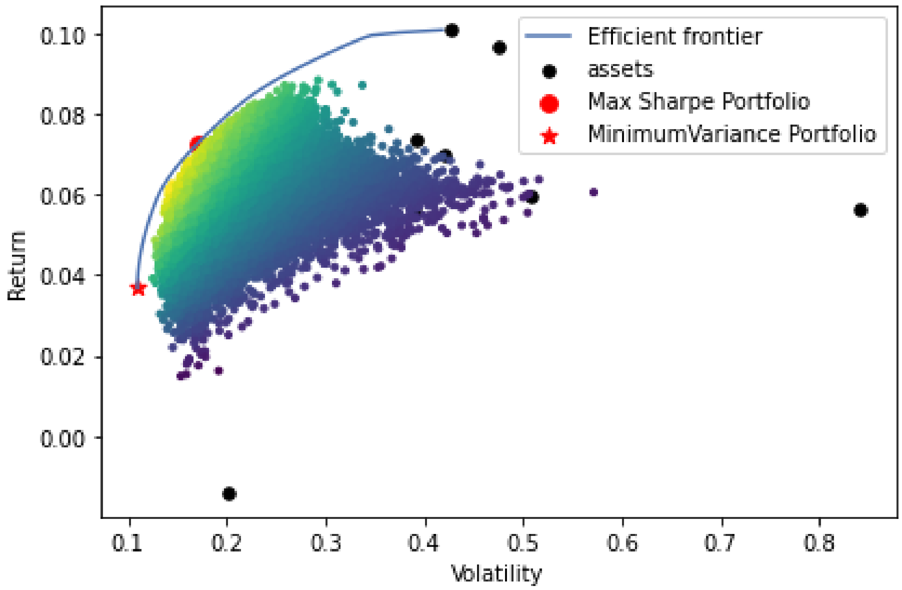

Figure 29: **Monthly** Black Litterman frontier - short selling

7.3.2 Short Selling Not Allowed

	Max Sharpe	GMV
I:VIN	0.000000	0.273995
I:FUL	0.000000	0.000000
I:ENV	0.285180	0.414043
I:ECK	0.035238	0.000000
I:MON	0.000000	0.000000
I:B	0.197007	0.181303
I:FD	0.119435	0.064511
I:GAMB	0.059396	0.000000
I:BE	0.000000	0.000000
I:IKG	0.000000	0.000000
I:RAT	0.235447	0.066147
I:ZUC	0.068297	0.000000

As evidenced from the table above, the application of the constraints has a significant impact. Indeed, for the Max Sharpe portfolio, to five stocks is associated a zero weight, while the most weighted are I:ENV, I:RAT and I:B.
Instead, for the GMV one, six stocks have zero weight.

Below, you can find the monthly statistics both for Max Sharpe and GMV portfolios.

Mean	0.003
STD	0.0412
Variance	0.0016
Skew	-0.36
Kurtosis	1.60
Sharpe	0.31

And the one for the GMV:

Mean	0.0022
STD	0.0308
Variance	0.0009
Skew	-0.2292
Kurtosis	0.3304
Sharpe	0.15

8 Bayesian

8.1 Theory behind the model

From an holistic point of view the main differences existing between the classical and bayesian asset allocation that we analysed in this paper, arise from the choice of the density to be considered in performing the asset allocation. The density condenses whether the investor accounts for parameter uncertainty or focuses on predictability in asset returns.
Therefore, true advantage of Bayesian approach lies in the possibility to specify a set of subjective views depicted by prior PDF and the data.
Our goal is to derive the posterior useful to make inference about parameters mu by the means of a specification of a prior PDF.
We consider Informative priors, which are a way to include personal views on the parameters maintaining an analytical tractability in order to get posterior density.
We posit conjugate prior normally distributed, namely a prior PDF that shares the same distribution as the conditional likelihood and for which the posterior densities are member of the same distribution family as the priors.
Moreover, we assume a multivariate normal prior hypothesis about μ, given by:

$$f_{pr}(\mu) = N(\mu_0, \Lambda_0)$$

where:

$$\mu_0 = \hat{\mu} + \hat{\sigma}$$

$$\Lambda_0 = 2 * \hat{\Sigma}$$

The posterior density, from Bayes' Rule is given by:

$$f_{p0}(\mu|Y, \Sigma) \sim f(Y|\mu, \Sigma) f_{pr}(\mu)$$

The posterior distribution is $N(\mu_1, \Sigma_1)$ i.e:

$$f_{p0}(\mu|Y, \Sigma) \sim N(\mu_1, \Sigma_1)$$

$$\mu_1 = [T\Sigma^{-1} + \Lambda_0^{-1}]^{-1}[T\Sigma^{-1}\hat{\mu} + \Lambda_0^{-1}\mu_0]$$

$$\Sigma_1 = [T\Sigma^{-1} + \Lambda_0^{-1}]^{-1}$$

To compute the predictive density, we need to integrate out the unknown µ from the product of the sample density with the posterior:

$$f(r_{t+1}|Y\Sigma) = \int_{\Theta} f(r_{t+1}|\mu, \Sigma) f_{p0}(u|Y, \Sigma) d\mu$$

Which is:

$$f(r_{t+1}|Y\Sigma) = N(\mu_1, \Sigma_1 + \Sigma)$$

We could plug the predictive density parameters into the classical formula:

$$W_{portfolio} = \frac{1}{\lambda} \Sigma_1^{-1} \mu_1$$

Where λ represents the risk adversion coefficient.

8.1.1 Bayesian Daily

Below you can find the table with all the weights associated to each stocks deriving from the application of the Bayesian approach on the daily data:

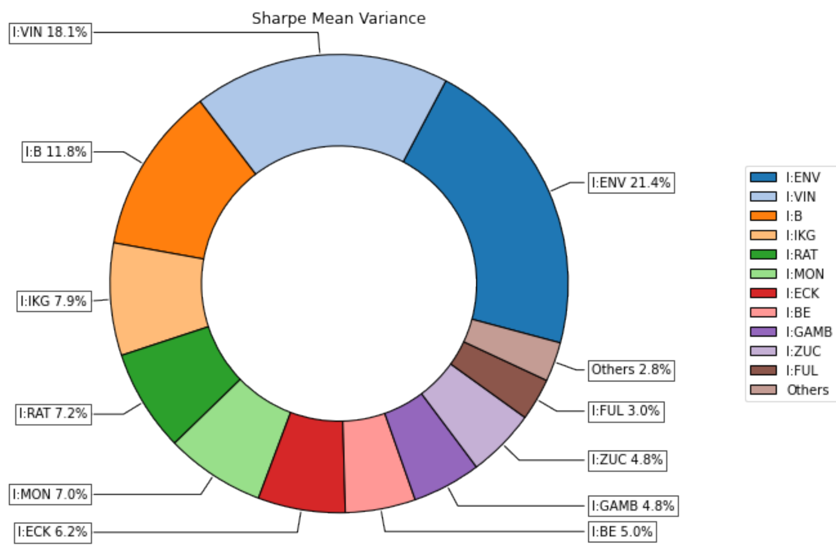

Figure 30: Weights allocation - Bayesian

This are the relative **daily** statistics of the portfolio:

Mean	0,0002
STD	0.0093
Variance	0.00008
Skew	-0.9791
Kurtosis	8.8821
Sharpe	0.23

By following this methodology, the most weighted stocks in our portfolio should be I:ENV, I:VIN and I:B, which are the only three stocks that reports a two digit weight.

In this case, this portfolio is reporting a lower mean than the one computed throught the MVO and, consequently, the one computed through BL. At the same manner, also the sharpe ratio and the standard deviation will be lower. The distribution is slightly less negative skewed and has a lower kurtosis, so is less leptokurtic.

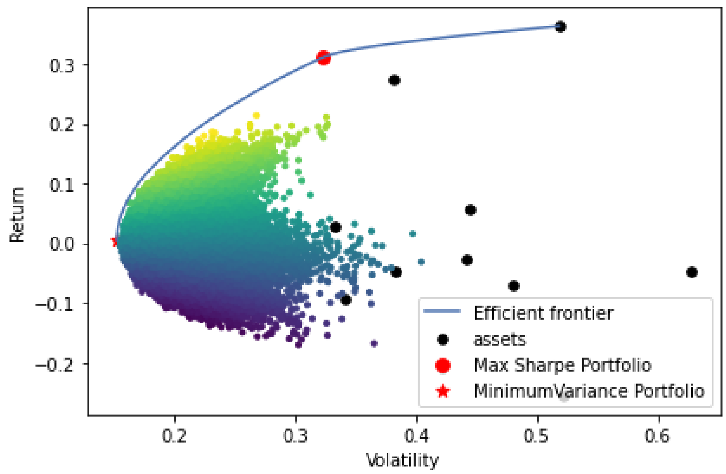

Figure 31: **Daily** Bayesian Efficient Frontier - No Short Selling

8.1.2 Bayesian Monthly

Table with the weights for each stock:

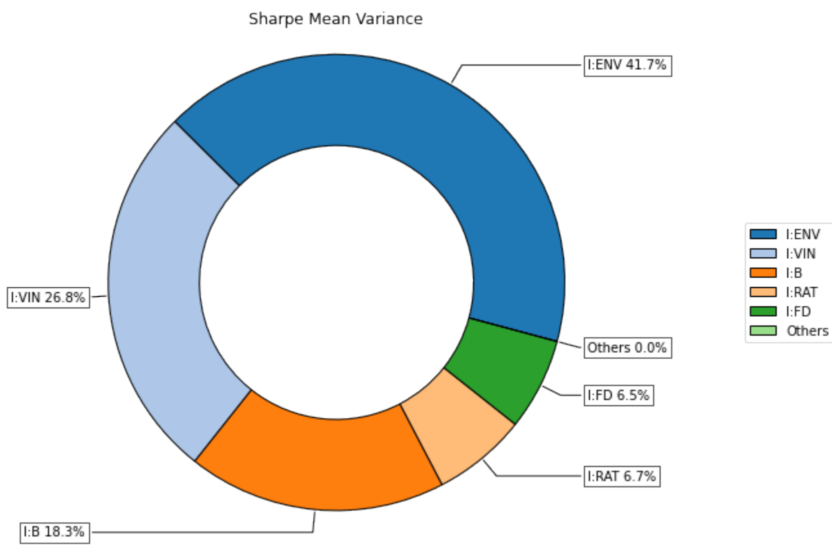

Figure 32: Weights allocation - Bayesian

And relative statistics:

Mean	0.0023
STD	0.0308
Variance	0.0009
Skew	-0.2412
Kurtosis	0.3371
Sharpe	0.30

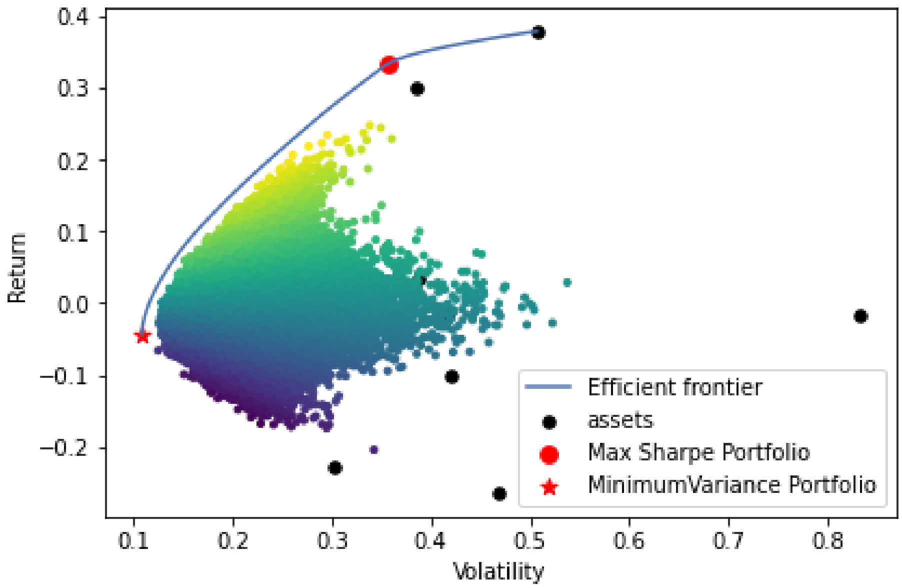

Figure 33: **Monthly** Bayesian Efficient Frontier - No Short Selling

9 Comparison between the methodologies

9.1 Differences and potential improvement

The disparities in asset allocation observed among the portfolios derived from different assumptions can be attributed to various factors. In terms of assumptions, each portfolio strategy relies on distinct suppositions and methodologies. For instance, the Mean Variance Portfolio employs historical return and covariance data and is based on the strong assumption that returns are normally distributed, which is something we have shown is almost every time not true and that asset allocations are determined based on a single period of investment. The Black Litterman approach extends the MVO approach and is reliant on the subjective views of investors about expected returns on different assets or asset classes, which are combined with a prior estimate of expected returns based on a benchmark equilibrium distribution. However, the latter should be computed using the implied market capitalization-weight of the asset, but in our case this is replaced by an estimation obtained from the CAPM, for the reasons we already discussed during the paper. The GMV, which is an optimization method, computed starting from the application of the already cited models, relies on the fact that the investor choices the portfolio with the minimum variance, regardless the expected return. The bayesian approach is mainly based on the posterior function, which is the respective density function of the estimate that produces a random variable that gets value within a given range of the risk, that allows the blend between the prior density, which represents the investor's experience, and the likelihood.

These divergent assumptions give rise to discrepancies in asset allocation decisions, which can be influenced also by the risk preferences of each strategy.
The GMV, for instance, as stated before, places a premium on risk reduction, while others may seek higher returns.
Differences can be found also in the inputs and constraints, where the availability and quality of data, investment constraints (such as minimum/maximum allocations), and investor preferences influence asset allocation decisions (example: our absolute negative view on Vianini led the Max Sharpe model computed throught the Black Litterman approach to suggest negative weights on this stock, while for the GMV, since it doesn't consider the investor's view, Vianini should have had a long position). Differences in these inputs and constraints across portfolio strategies lead to divergent asset allocations.
Another difference that must be underlined is the one that the Black-Litterman model overcomes the most-often cited weaknesses of mean-variance optimization (unintuitive, highly concentrated portfolios, input-sensitivity, and estimation error-maximization), leading to significant changes in the weights allocation.
Indeed, MVO has some several drawback, which as a waterfall, impact on the weights allocation:

- It can generate unintuitive, highly-concentrated portfolios.
- It makes the massive presumption that all the assets will continue to behave and perform just as they have done in the past. Not only does it assume the returns and volatility will remain the same, it also assumes that correlations between all the assets in question will remain stable through time. We know these assumptions are just not realistic.
- It makes the massive presumption that all the assets will continue to behave and perform just as they have done in the past. Not only does it assume the returns and volatility will remain the same, it also assumes that correlations between all the assets in question will remain stable through time. We know these assumptions are just not realistic.
- The model happens to be extremely sensitive to variations in these input values. If the input values are changed, even by relatively small amounts, the optimal portfolio weightings created by the model can swing and vary wildly.

In conclusion, the Bayesian approach is more robust compared to the frequentist approach, as the latter is entirely sample dependent. A small change in the sample will lead to a substantial change in the estimates. But the bayesian algorithm is robust to the sample proportion.

9.1.1 Mixed Portfolio

A possible way to improve upon all such results is to create a mixed portfolio, in our case an equal weighted portfolio, which is capable to take advantage of all the benefits of each model, reducing the drawbacks that would arise if only one methodology were used.
Mixed Portfolio = 0.25 * GMVMeanVarPF + 0.25 * PFBayesian + 0.25 * BlackLitPF + 0.25* MeanVarPF

9.1.2 Daily

The new weights are:

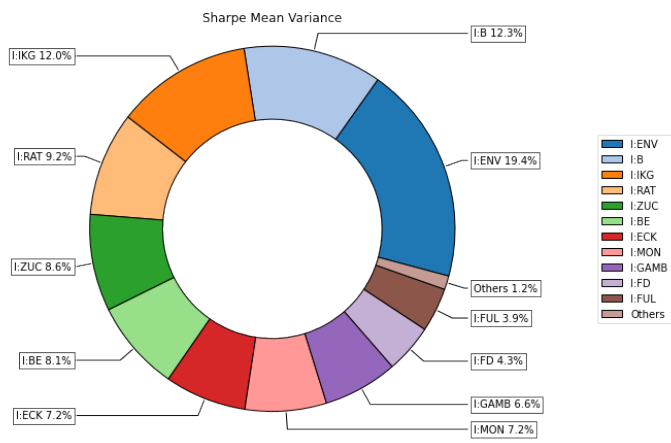

Figure 34: Weights allocation - Mixed Portfolio

As required, we computed all the statistics for this portoflio, both for daily and monthly data, respectively:

Mean	0.0002
STD	0.0101
Variance	0.0001
Skew	-1.2391
Kurtosis	11.8884
Sharpe	0.23

Now, we compare all the portfolio we obtained:

Daily Data Portfolios Comparison

	Mean	**STD**	**Variance**	**Skew**	**Kurtosis**	**Sharpe**
MeanVar	0.0003	0.0100	0.0001	-1.3359	12.4054	0.25
GMV	0.0002	0.009	0.00008	-0.9991	9.8821	0.21
BlackLit	0.0003	0.0158	0.0002	-0.6891	6.8426	0.25
Bayesian	0.00025	0.0093	0.00008	-0.9791	8.8821	0.23
Mixed	0.00028	0.0101	0.0001	-1.2391	11.88	0.23

9.1.3 Monthly

On the other hand, for the monthly data, the new weights are:

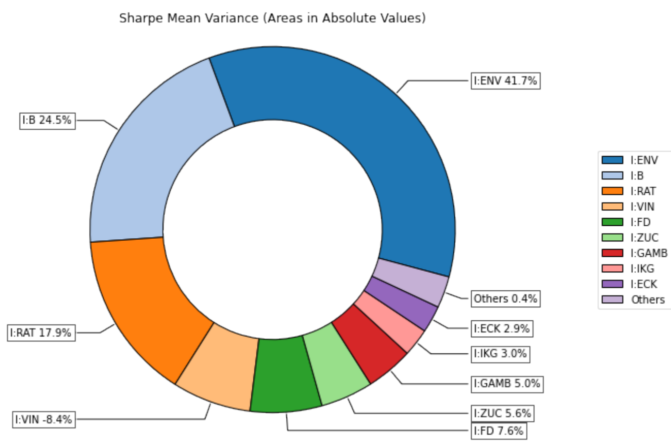

Figure 35: Weights allocation - Mixed Portfolio

Mean	0.0028
STD	0.0408
Variance	0.0016
Skew	-0.4678
Kurtosis	1.4918
Sharpe	0.23

Monthly Data Portfolios Comparison

	Mean	**STD**	**Variance**	**Skew**	**Kurtosis**	**Sharpe**
MeanVar	0.004	0.0398	0.0015	-0.3633	2.288	0.36
GMV	0.003	0.0274	0.0007	-0.3952	0.5333	0.25
BlackLit	0.005	0.0935	0.0087	-0.3802	0.55	0.37
Bayesian	0.0035	0.0308	0.0009	-0.2412	0.3371	0.30
Mixed	0.0038	0.0408	0.00166	-0.4678	1.4918	0.32

However, this is only a possible solution, not the unique one. Indeed, we could create different mixed portfolios just by changing the weights. Is worthwhile to say that the statistics of the blended portfolio are proportionally related to the statistics of each asset allocation method, since each asset allocation assigns weights to the same 12 stocks.
The mixed portfolio statistics will be closer to a certain methodology as the latter's weight increases.
In conclusion, the weights that could be assigned to each allocation may arise from the investor's experience, the awareness of the assumption every model relies on, the quality of the data and by the implementation of some quantitative methodologies as the maximization of the Sharpe Portfolio.

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