Introduction to Copulas in Finance

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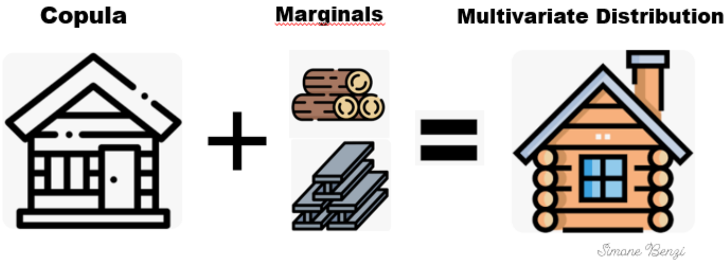
Intuition behind Copulas

Imagine you want to build a house.
Marginal Distribution are as the foundational pillars of the house, whereas the **Copula** is the blueprint that dictates how these pillars are connected to create the final house **Multivariate Distribution**.



Intuition behind Copulas

- Copula is a function that ties different marginals to create a Multivariate distribution
- Copula can be seen as the dependence structure since it dictates how marginal are interconnected
- Copula structure (blueprint house) remains the same, albeit we change of marginals (materials)

(Inverse) Probability integral transform

Definition: **Probability integral transform**. Let X be a random variable with cumulative distribution function $F_X(x)$, then the random variable $Y = F_X(X)$ follows a uniform distribution on the interval $(0, 1)$. In other words: $Y = F_X(X) \sim U(0, 1)$.

Definition: **Inverse Probability integral transform**. Let X be a random variable, then $X = F_X^{-1}(U)$ maps from a uniform distribution to the original random variable X through the inverse CDF. where U is a random variable uniformly distributed on the interval $(0, 1)$.

(Inverse) Probability integral transform

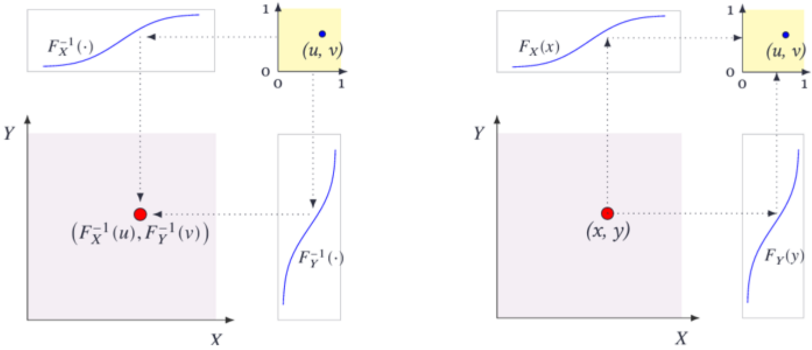


Figure: From the left: We can use the marginal inverse CDFs to map from (u, v) to $(F_X^{-1}(u), F_Y^{-1}(v))$. Conversely, the second picture show how can use the marginal CDFs to map from (x, y) to a point (u, v) on the unit square.

Copula definition

Definition: Copula is a multivariate cumulative distribution function for which the marginal probability distribution of each variable is uniform on the interval [0, 1].
It is a function from $[0,1]^d \to [0,1]$.

Sklar Theorem: Let F be a multivariate distribution function with margins $F_1, \dots, F_d$. Then, there exists a copula C such that:

$$F_{XY}(x_1, x_2, \dots, x_d) = C\{F_1(x_1), F_2(x_2), \dots, F_d(x_d)\}$$

Decomposition of Multivariate Distribution

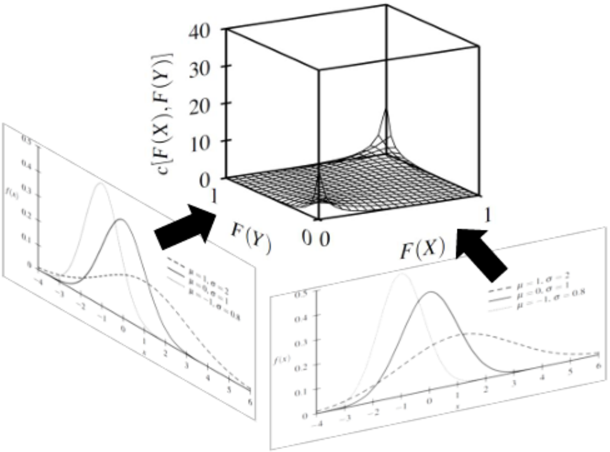


Figure: Decomposition of Multivariate Distribution as Copula (box) and univariate Marginals distribution

Why Copula is Useful?

- Several combinations of marginals do not have a closed form to generate the desired multivariate distribution.
- For instance, it is feasible to generate random samples from a joint normal distribution carrying out Cholesky decomposition.
- It's not straightforward to do the same when marginals are different. In this case, the joint distribution has to take into account manifold parameters, and this usually leads to complex dependence structures.

Why Copula is useful ?

- **Example**: Assume $X \sim$ **Gumbel and** $Y \sim$ **Gamma**
- **Problem**: **We don't know a priori their joint distribution** $F_{XY}(x, y)$, since it is challenging to gauge because the latter has to take into account manifold parameters and this usually lead to a complex dependence structure.
- **Solution**: Yet, we **know their individual CDFs**. Therefore, Copula is useful since it models the correlation structure and simultaneously gathers the univariate CDFs $\rightarrow C\{F_1(x_1), F_2(x_2)\}$ to obtain a joint distribution.

Why Copula is useful ?

- In financial markets, is well known that returns are neither normally distributed nor symmetric, and is not seldom to have dependent assets that exhibit tail dependence and outliers.

- **How can I measure the dependence between two financial asset?**

Pitfalls of Pearson Correlation

- What do you think of when you hear the word **'Correlation'**?
- Arguably is Pearson correlation coefficient, given by the formula:

$$\rho(X, Y) = \frac{\mathrm{cov}(X, Y)}{\sigma_X \sigma_Y}$$

  It's a measure of linear dependence and it assumes that:
  - Data is normally distributed;
  - The relation among the variables is linear;
  - There are no outliers.

- Furthermore, it is not invariant under monotone transformations, hence when apply Copula the correlation between the variable can change

Rank Correlation

- Alternative dependence measures that do not suffer from the preceding assumptions are **Kendall tau** and **Spearman rho**.
- They are measures of concordance based on ranks, which do not depend marginal distributions. This is the crucial connection with Copulas
- Moreover, they can be employed to gauge Copula with one parameter. The relation of Kendall's tau and Copula is :

$$\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u_1, u_2)\, dC(u_1, u_2) - 1$$

Copula as dependence structure

- Copula are very flexible in modelling the dependence structure of random variable. It can capture comonotonic and countermonotonic relationships.

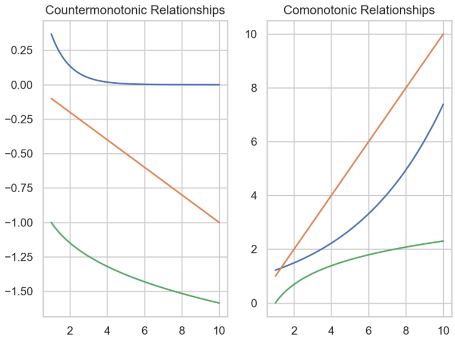


Figure: Type of dependency that Copula can model

The main two families of Copula are:

- **Elliptical Copulas**

✓ Gaussian Copula, T-Copula

- **Archimedean Copulas**

✓ Gumbel, Frank, Clayton Copula

Simulation Elliptical and Archimedean Copula

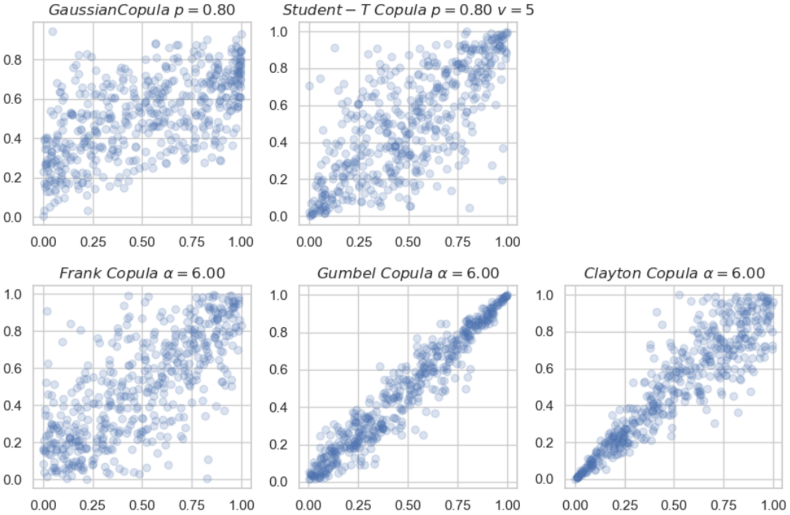


Figure: Simulation of different kind of Copulas

- **Given the above correlation structures of Copulas, what is left to create a simulation of a multivariate distribution ?**

Gaussian and Clayton Copula with normal marginals

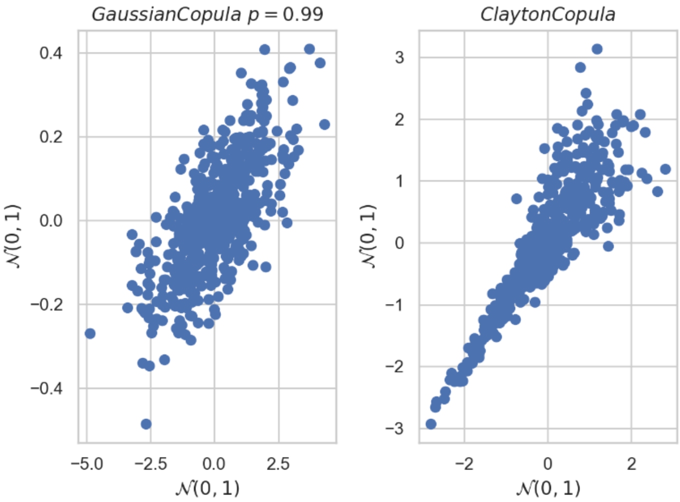


Figure: Gaussian vs Clayton Copula both with normal marginals

Application to Copulas in Finance

- Let's move on R for the real code session !
- The aim is to estimate the **joint distribution** of Apple and Exxon returns through a **Survival Gumbel and T student Copulas** employing **Maximum Likelihood**
- In addition, we **simulated synthetic returns** to test whether our models are able to capture the behaviour of historical data.

Conclusion - Key Insights

- Copula is a function that ties different marginals to create a Multivariate distribution
- Copula can be seen as the dependence structure since it dictates how marginal are interconnected
- Copula structure remains the same, albeit we change of marginals
- The two main families of Copula are Elliptical and Archimedean, and each Copula can model a different type of correlation
- Copula parameters can be estimated by Maximum Likelihood Estimation and by Kendall's Tau
- Copula can be employed to simulate stock returns and model tail risk. It has many application in risk management and CDOs pricing.