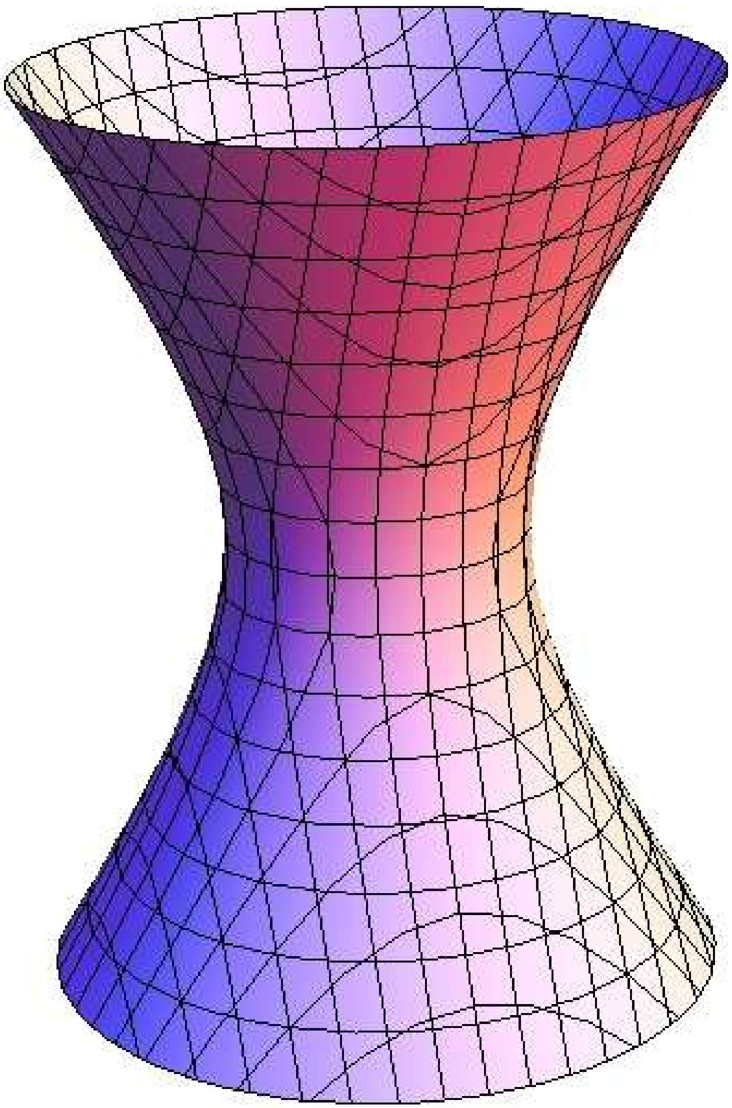

Properties of Symbols on a Hyperboloid

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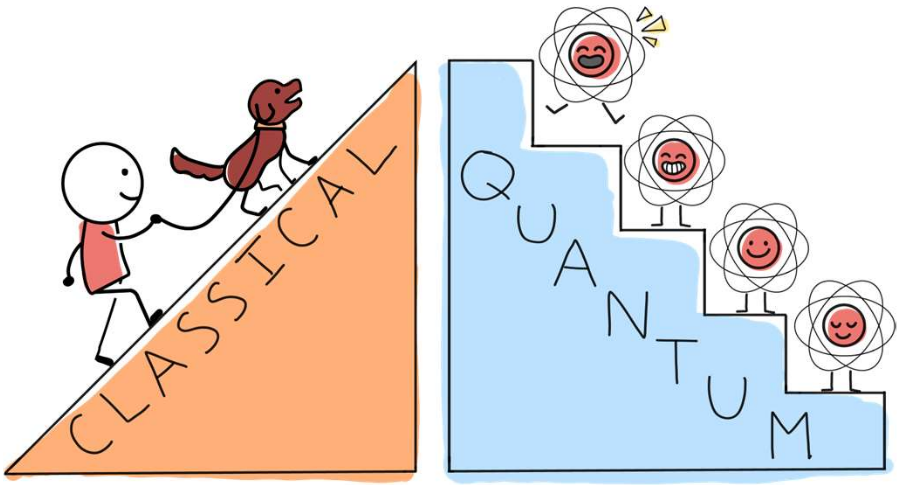
Classical Vs. Quantum Mechanics

- Classical Mechanics (CM) – What we experience on a daily basis.
- Quantum Mechanics (QM) – Applies at the subatomic level.
- Fundamentally distinct – QM only allows physical properties of a system, such as momentum and energy, to exist in discrete quantities.



The Quantum Atlas

Flat Phase Space

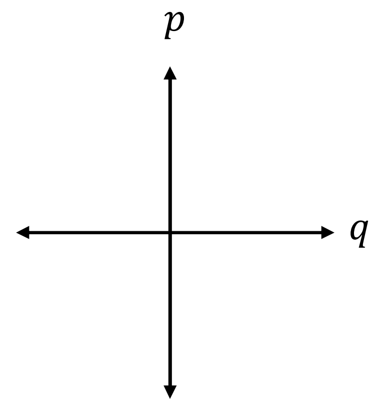
- Describes physical systems in terms of position and momentum.
- The canonical coordinates p and q can be used to generate the Heisenberg Lie algebra satisfying the commutation relations: $[p,q]=1$, $[p,1]=[q,1]=0$. This is a symmetry Lie algebra.
- p and q also satisfy the Poisson bracket relation: $\{p,q\}=1$.

Position: $\hat{q}\psi(x) = x\psi(x)$

Momentum: $\hat{p}\psi(x) = i\hbar\frac{\partial}{\partial x}\psi(x)$

<u>Poisson bracket</u>

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}$$



Differential Operators and Symbols

- QM commonly uses differential operators (DOs), e.g. the momentum operator.
- Each DO has an associated symbol that does not contain derivatives.
- Symbols allow us to deal with multiplication of DOs without so much messy math.
- Composition of Operators: $\hat{p}\hat{p}\psi(x)$
- Symbol Product: Obtained from transferring from composition of operators
- Leads to equivariance: If the same operation is applied to a DO and its symbol, then the new symbol must be the symbol of the new DO.

Research Question

- Mathematically combining CM and QM is difficult.
- QM uses a small number called Planck's constant, denoted by h, to define the discrete units of physical quantities.
- Since CM has no discrete units of physical quantities, any consistent mathematical framework of QM must also apply to CM as $h \to 0$. This is known as the Correspondence Principle (CP).
- Various ways of showing the CP on a flat phase space already exist.

<u>Hypothesis</u>

Can we define a symbol product on the curved phase space of a hyperboloid such that it satisfies the Correspondence Principle?

Bokulich et al.

The Correspondence Principle

- The Correspondence Principle is trickier to show in curved phase spaces.
- As it applies to our study of DOs and their symbols, the Correspondence Principle has two parts that must be proved.
- We will use coordinates u and v to work in curved phase space (more on this later).

$$\boxed{\lim_{s\to\infty}(f *_s g) = fg}$$

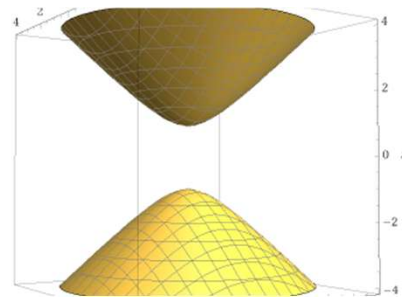
Recovery of the commutative function product

$$\{f,g\} = \frac{(u-v)^2}{2}\left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial v}\right)$$

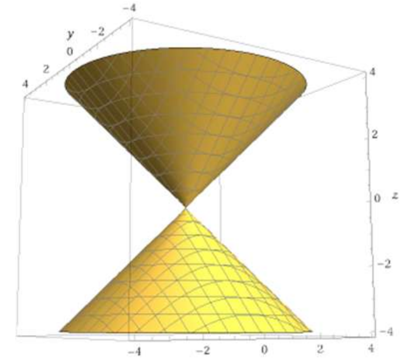
$$\boxed{\lim_{s\to\infty} s(f *_s g - g *_s f) = \{f,g\}}$$

Recovery of the Poisson bracket

Hyperboloids

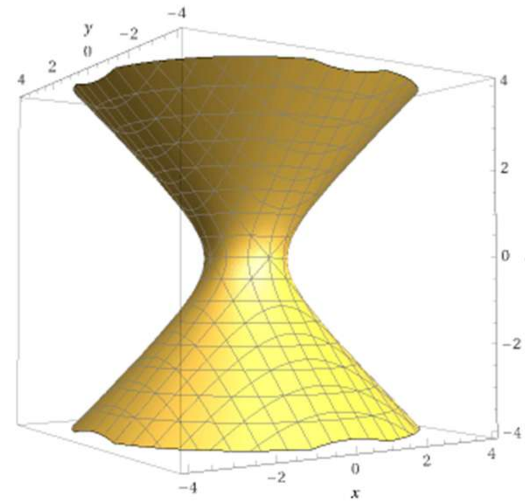


Hyperboloid of
Two Sheets $s^2 < 0$



Infinite Cone $s^2 = 0$

- A surface in $\mathbb{R}^3$ where all points (x, y, z) on the surface satisfy $x^2 + y^2 - z^2 = s^2$ where s is the central radius.
- Since it is a surface, points on the surface can be defined locally with just two coordinates that we call u and v.
- We focus our attention on hyperboloids of one sheet.



Hyperboloid of One Sheet $s^2 > 0$

Goals

- Define symbols and the symbol product constrained to the surface of a hyperboloid of one sheet such that they satisfy the Correspondence Principle.
- Ensure that our symbols and DOs are equivariant
- Draw further corrections between symbols and physical principles.

Coordinate Transforms

- We parameterize the surface of the hyperboloid expressing the coordinates (x, y, z) in $\mathbb{R}^3$ using parameters (u, v).
- This gives us a definition of the Poisson bracket in terms of u and v.

$$\boxed{x = s\frac{u+v}{u-v} \qquad y = s\frac{1-uv}{u-v} \qquad z = s\frac{1+uv}{u-v}}$$

Coordinate transforms

$$\{f, g\} = \frac{(u-v)^2}{2}\left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial v}\right)$$

Hamiltonians

We use the Lie algebra $sl(2,\mathbb{R})$, which has a basis that consists of three generators:

$$E_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad E_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[H, E_+] = 2E_+ \qquad [H, E_-] = -2E_- \qquad [E_+, E_-] = H$$

We can use these to derive the corresponding Hamiltonians (these are symbols):

$$f_{E_+} = \frac{2s}{u-v} \qquad f_{E_-} = -\frac{2suv}{u-v} \qquad f_H = 2s\frac{u+v}{u-v}$$

$$\{f_H, f_{E_+}\}_s = 2f_{E_+} \quad \{f_H, f_{E_-}\}_s = -2f_{E_-} \quad \{f_{E_+}, f_{E_-}\}_s = f_H$$

$$\{f,g\}_s = \frac{(u-v)^2}{2s}\left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial v}\right)$$

Hamiltonians

We find the corresponding DOs that satisfy the needed commutation relations:

$$\hat{f}_{E_+} = -\frac{\partial}{\partial u} \quad \hat{f}_{E_-} = u^2\frac{\partial}{\partial u} + 2su \quad \hat{f}_H = -2u\frac{\partial}{\partial u} - 2s$$

$$[\hat{f}_H, \hat{f}_{E_+}] = 2\hat{f}_{E_+} \quad [\hat{f}_H, \hat{f}_{E_-}] = -2\hat{f}_{E_-} \quad [\hat{f}_{E_+}, \hat{f}_{E_-}] = \hat{f}_H$$

We now have homomorphisms relating DOs and symbols:

$$\frac{2s}{u-v} \mapsto -\frac{\partial}{\partial u} \qquad -\frac{2suv}{u-v} \mapsto u^2\frac{\partial}{\partial u} + 2su$$

$$2s\frac{u+v}{u-v} \mapsto -2u\frac{\partial}{\partial u} - 2s$$

A Physical Application

DO and Symbol Mappings

We generalize our findings so far to construct the space of symbols and the space of DOs:

$$S = \left\{ \sum_k f_k(u) \frac{1}{(u-v)^k} \right\}$$

$$O = \left\{ \sum_k f_k(u) \left(\frac{\partial}{\partial u} \right)^k \right\}$$

We can then use the mappings $\sigma_s : O \to S$ and $\tau_s : S \to O$ defined as:

$$\sigma_s \left(\sum_k g_k(u) \left(\frac{\partial}{\partial u} \right)^k \right) = \sum_k g_k(u) \frac{(-2s)_k}{(u-v)^k}$$

$$\tau_s \left(\sum_k f_k(u) \frac{1}{(u-v)^k} \right) = \sum_k f_k(u) \frac{1}{(-2s)_k} \left(\frac{\partial}{\partial u} \right)^k$$

Result - The Symbol Product

- We can now the define the symbol product that we have been looking for!
- The result of this product is another symbol.
- It follows from the composition of DOs.
- It can be shown that it satisfies both parts of the Correspondence Principle.
- It can also be shown to be have equivariance under the symmetry group SL(2,R).

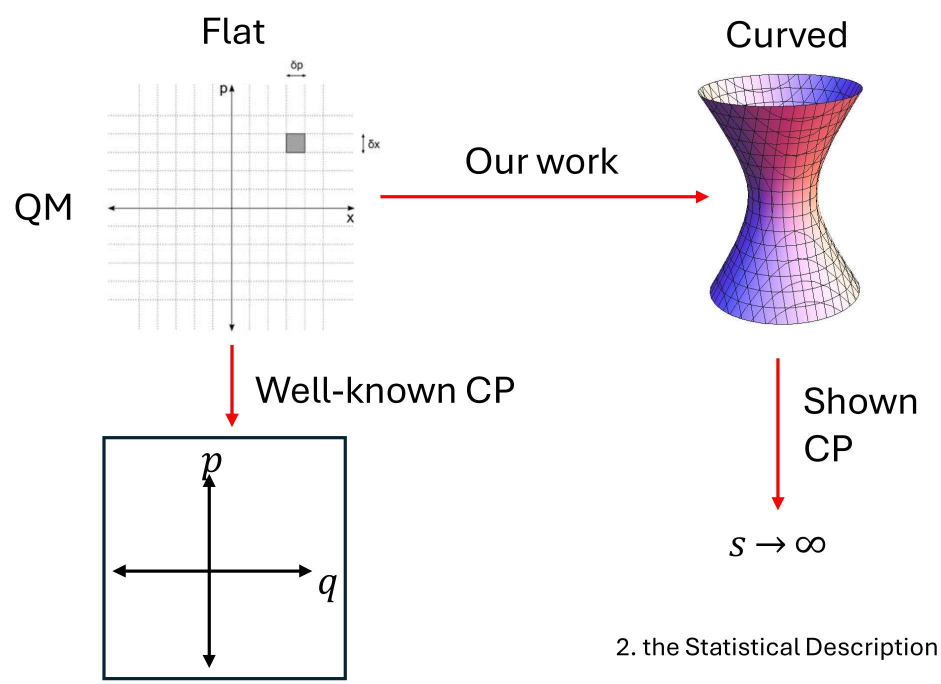
$$f *_s g = \sigma_s\Big(\underbrace{\tau_s(f)}_{\text{DO}} \circ \underbrace{\tau_s(g)}_{\text{DO}}\Big)$$

Discussion

- We have demonstrated a procedure for developing and using the symbol product on a hyperboloid.
- This extends our ability to perform QM calculations to curved phase spaces.
- The Correspondence Principle means that our symbol product remains valid in the classical limit.

Conclusion

- Commonly-used DOs in QM often lead to messy calculations, especially in curved phase spaces.
- We have shown symbols constrained to a hyperboloid's surface and shown that they satisfy the Correspondence Principle.
- Future work: Learn more about the functions on the hyperboloid, e.g. various subalgebras.



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Acknowledgements

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$$f *_s g = \sigma_s\left(\tau_s(f) \circ \tau_s(g)\right)$$