Properties of Symbols on a Hyperboloid

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Abstract.
An area of mathematical physics of particular interest is the development of mathematical tools to describe the transition between the classical and quantum regimes. Classical and quantum mechanics both make use of the Hamiltonian, an important function for modeling physical systems. However, constructing a quantized Hamiltonian that remains valid in the classical limit is still a challenge. We give here an approach for deriving the differential operators and their corresponding symbols needed for producing quantized Hamiltonians that remain valid in the classical limit. We consider Hamiltonians constrained to the surface of a hyperboloid of one sheet since it has a symmetry group, and the symbol-operator correspondence in question is invariant under this symmetry group.

PROOFS

Lemma 1

$$\binom{k}{p} + \binom{k}{p-1} = \binom{k+1}{p} \tag{1}$$

for positive integers p and k where $1 \leq p \leq k$.

Proof

$$\begin{gathered}
\binom{k}{p} + \binom{k}{p-1} = \frac{k!}{p!(k-p)!} + \frac{k!}{(p-1)!(k-p+1)!} \\
= \frac{k!}{p!(k-p)!}\left(\frac{k-p+1}{k-p+1}\right) + \frac{k!}{(p-1)!(k-p+1)!}\left(\frac{p}{p}\right) \\
= \frac{k!(k-p+1)+k!p}{p!(k-p+1)!} = \frac{k!(k-p+1+p)}{p!(k-p+1)!} = \frac{(k+1)!}{p!\big((k+1)-p\big)!} = \binom{k+1}{p}\square
\end{gathered} \tag{2}$$

Lemma 2

$$\lim_{s\to\infty} \frac{(-2s)_k}{(-2s)^k} = 1 \tag{3}$$

for positive integer k where $(-2s)_k = (-2s)(-2s-1)(-2s-2)...(-2s-k+1)$.

Proof

$$\begin{aligned}
\lim_{s\to\infty} \frac{(-2s)_k}{(-2s)^k} &= \lim_{s\to\infty} \frac{(-2s)(-2s-1)(-2s-2)...(-2s-k+1)}{(-2s)^k} \\
&= \lim_{s\to\infty} \left(\frac{-2s}{-2s}\right)\left(\frac{-2s-1}{-2s}\right)\left(\frac{-2s-2}{-2s}\right)...\left(\frac{-2s-k+1}{-2s}\right) \\
&= \lim_{s\to\infty}(1)\left(1+\frac{1}{2s}\right)\left(1+\frac{2}{2s}\right)...\left(1+\frac{k-1}{2s}\right) = 1\square
\end{aligned} \tag{4}$$

Theorem 1

The kth order derivative of the product of two functions $f(u)$ and $g(u)$ is given by the formula

$$\left(\frac{\partial}{\partial u}\right)^{k}[f(u)g(u)] = \sum_{p=0}^{k}\binom{k}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k-p} f(u)\right]\left[\left(\frac{\partial}{\partial u}\right)^{p} g(u)\right] \tag{5}$$

where $\binom{k}{p}$ is the usual binomial coefficient defined by $\binom{k}{p} = \frac{k!}{p!(k-p)!}$.

Proof

We will prove the formula by mathematical induction. To prove the base step, we set $k = 1$ and get

$$\frac{\partial}{\partial u}[f(u)g(u)] = \binom{1}{0}\left[\frac{\partial}{\partial u}f(u)\right]g(u) + \binom{1}{1}f(u)\left[\frac{\partial}{\partial u}g(u)\right]. \tag{6}$$

Simplifying the RHS, we get

$$\frac{\partial}{\partial u}[f(u)g(u)] = \frac{\partial f(u)}{\partial u}g(u) + f(u)\frac{\partial g(u)}{\partial u} \tag{7}$$

which is the well-known product rule. Thus the base step is true.

To prove the inductive step, we need to show that replacing k with $k+1$ in Eq. 5 does not alter the validity of the equation. That is, assuming Eq. 5 is true, we need to show that

$$\left(\frac{\partial}{\partial u}\right)^{k+1}[f(u)g(u)] = \sum_{p=0}^{k+1}\binom{k+1}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k+1-p} f(u)\right]\left[\left(\frac{\partial}{\partial u}\right)^{p} g(u)\right]. \tag{8}$$

Starting from the LHS, we have

$$\left(\frac{\partial}{\partial u}\right)^{k+1}[f(u)g(u)] = \frac{\partial}{\partial u}\left[\left(\frac{\partial}{\partial u}\right)^{k}[f(u)g(u)]\right]. \tag{9}$$

Substituting the assumption stated by Eq. 5 gives

$$\left(\frac{\partial}{\partial u}\right)^{k+1}[f(u)g(u)] = \frac{\partial}{\partial u}\left(\sum_{p=0}^{k}\binom{k}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k-p} f(u)\right]\left[\left(\frac{\partial}{\partial u}\right)^{p} g(u)\right]\right). \tag{10}$$

Moving constants to the front and applying the product rule gives

$$\left(\frac{\partial}{\partial u}\right)^{k+1}[f(u)g(u)] = \sum_{p=0}^{k}\binom{k}{p}\left(\left[\left(\frac{\partial}{\partial u}\right)^{k-p+1} f(u)\right]\left[\left(\frac{\partial}{\partial u}\right)^{p} g(u)\right] + \left[\left(\frac{\partial}{\partial u}\right)^{k-p} f(u)\right]\left[\left(\frac{\partial}{\partial u}\right)^{p+1} g(u)\right]\right). \tag{11}$$

Distributing the summation, we have

$$\begin{aligned}\left(\frac{\partial}{\partial u}\right)^{k+1}[f(u)g(u)] = \sum_{p=0}^{k}\binom{k}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k-p+1} f(u)\right]\left[\left(\frac{\partial}{\partial u}\right)^{p} g(u)\right] + \\ \sum_{p=0}^{k}\binom{k}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k-p} f(u)\right]\left[\left(\frac{\partial}{\partial u}\right)^{p+1} g(u)\right]\end{aligned} \tag{12}$$

and isolating the $p = 0$ term of the first sum and the $p = k$ term of the second sum gives us

$$\begin{aligned}\left(\frac{\partial}{\partial u}\right)^{k+1}[f(u)g(u)] = \left[\left(\frac{\partial}{\partial u}\right)^{k+1} f(u)\right]g(u) + \sum_{p=1}^{k}\binom{k}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k-p+1} f(u)\right]\left[\left(\frac{\partial}{\partial u}\right)^{p} g(u)\right] \\ + \sum_{p=0}^{k-1}\binom{k}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k-p} f(u)\right]\left[\left(\frac{\partial}{\partial u}\right)^{p+1} g(u)\right] + f(u)\left[\left(\frac{\partial}{\partial u}\right)^{k+1} g(u)\right].\end{aligned} \tag{13}$$

We can now re-index the second sum to run from 1 to k and combine the two sums to get

$$\left(\frac{\partial}{\partial u}\right)^{k+1}[f(u)g(u)] = \left[\left(\frac{\partial}{\partial u}\right)^{k+1} f(u)\right]g(u) + \sum_{p=1}^{k}\left[\binom{k}{p}+\binom{k}{p-1}\right]\left[\left(\frac{\partial}{\partial u}\right)^{k-p+1} f(u)\right]\left[\left(\frac{\partial}{\partial u}\right)^{p} g(u)\right] + f(u)\left[\left(\frac{\partial}{\partial u}\right)^{k+1} g(u)\right]. \tag{14}$$

By Lemma 1, we simplify and get

$$\left(\frac{\partial}{\partial u}\right)^{k+1}[f(u)g(u)] = \sum_{p=0}^{k+1}\binom{k+1}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k+1-p} f(u)\right]\left[\left(\frac{\partial}{\partial u}\right)^{p} g(u)\right]. \tag{15}$$

Thus the inductive step is true.

Since both the base step and the inductive step are true, the theorem is proved.□

Theorem 2

The symbol of the differential operator $\left(\frac{\partial}{\partial u}\right)^k$ is given by $\frac{(-2s)_k}{(u-v)^k}$, that is,

$$(u-v)^{2s}\left(\frac{\partial}{\partial u}\right)^{k}(u-v)^{-2s} = \frac{(-2s)_k}{(u-v)^k}. \tag{16}$$

Proof

We will prove the formula by mathematical induction. To prove the base step, we set $k=1$ and get

$$(u-v)^{2s}\frac{\partial}{\partial u}(u-v)^{-2s} = (u-v)^{2s}(-2s)(u-v)^{-2s-1} = \frac{-2s}{u-v}. \tag{17}$$

which is the RHS of Eq. 16 when $k=1$. Thus the base step is true.

To prove the inductive step, we need to show that replacing k with $k+1$ in Eq. 16 does not alter the validity of the equation. That is, assuming Eq. 16 is true, we need to show that

$$(u-v)^{2s}\left(\frac{\partial}{\partial u}\right)^{k+1}(u-v)^{-2s} = \frac{(-2s)_{k+1}}{(u-v)^{k+1}}. \tag{18}$$

Starting from the LHS, we have

$$\begin{aligned}(u-v)^{2s}\left(\frac{\partial}{\partial u}\right)^{k+1}(u-v)^{-2s} &= (u-v)^{2s}\frac{\partial}{\partial u}\left[\left(\frac{\partial}{\partial u}\right)^{k}(u-v)^{-2s}\right] \\ &= (u-v)^{2s}\frac{\partial}{\partial u}\left[(u-v)^{-2s}(u-v)^{2s}\left(\frac{\partial}{\partial u}\right)^{k}(u-v)^{-2s}\right].\end{aligned} \tag{19}$$

Substituting the assumption given by Eq. 16 gives

$$(u-v)^{2s}\left(\frac{\partial}{\partial u}\right)^{k+1}(u-v)^{-2s} = (u-v)^{2s}\frac{\partial}{\partial u}\left[(u-v)^{-2s}\frac{(-2s)_k}{(u-v)^k}\right]. \tag{20}$$

Using the product rule and simplifying gives

$$\begin{aligned}(u-v)^{2s}\left(\frac{\partial}{\partial u}\right)^{k+1}(u-v)^{-2s} &= \frac{(-2s)_k}{(u-v)^{k+1}}(-2s-k) \\ &= \frac{(-2s)_k}{(u-v)^{k+1}}\big(-2s-(k+1)+1\big) = \frac{(-2s)_{k+1}}{(u-v)^{k+1}}.\end{aligned} \tag{21}$$

Thus the inductive step is true.

Since both the base step and the inductive step are true, the theorem is proved.□

Definition

Let S be the space of symbols with elements of the form $\sum_k f_k(u)\frac{1}{(u-v)^k}$ and O be the set of differential operators of the form $\sum_k g_k(u)\left(\frac{\partial}{\partial u}\right)^k$. We define the mappings $\sigma_s : O \longrightarrow S$ and $\tau_s : S \longrightarrow O$ as

$$\sigma_s\left(\sum_k g_k(u)\left(\frac{\partial}{\partial u}\right)^k\right) = \sum_k g_k(u)\frac{(-2s)_k}{(u-v)^k} \tag{22}$$

$$\tau_s\left(\sum_k f_k(u)\frac{1}{(u-v)^k}\right) = \sum_k f_k(u)\frac{1}{(-2s)_k}\left(\frac{\partial}{\partial u}\right)^k \tag{23}$$

using the result of Theorem 2. Then we can define the product of symbols as

$$f *_s g = \sigma_s\big(\tau_s(f)\circ\tau_s(g)\big) \tag{24}$$

where $f, g \in S$ and $\tau_s(f)\circ\tau_s(g)$ is the product of the differential operators $\tau_s(f)$ and $\tau_s(g)$.

Theorem 3

$$\lim_{s\to\infty}(f *_s g) = fg \tag{25}$$

Proof

We first consider $f *_s g$:

$$\begin{aligned} f *_s g &= f(u)\frac{1}{(u-v)^k} *_s g(u)\frac{1}{(u-v)^l} = \sigma_s\left(\tau_s\left(f(u)\frac{1}{(u-v)^k}\right)\circ\tau_s\left(g(u)\frac{1}{(u-v)^l}\right)\right) \\ &= \sigma_s\left(f(u)\frac{1}{(-2s)_k}\left(\frac{\partial}{\partial u}\right)^k\left[g(u)\frac{1}{(-2s)_l}\left(\frac{\partial}{\partial u}\right)^l\right]\right) = \sigma_s\left(f(u)\frac{1}{(-2s)_k(-2s)_l}\left(\frac{\partial}{\partial u}\right)^k\left[g(u)\left(\frac{\partial}{\partial u}\right)^l\right]\right) \end{aligned} \tag{26}$$

Applying Eq. 5 gives

$$\begin{aligned} f *_s g &= \sigma_s\left(f(u)\frac{1}{(-2s)_k(-2s)_l}\sum_{p=0}^{k}\binom{k}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k-p}g(u)\right]\left(\frac{\partial}{\partial u}\right)^{p+l}\right) \\ &= f(u)\frac{1}{(-2s)_k(-2s)_l}\sum_{p=0}^{k}\binom{k}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k-p}g(u)\right]\frac{(-2s)_{p+l}}{(u-v)^{p+l}}. \end{aligned} \tag{27}$$

We now have a useful formula:

$$f *_s g = f(u)\sum_{p=0}^{k}\frac{(-2s)_{p+l}}{(-2s)_k(-2s)_l}\binom{k}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k-p}g(u)\right]\frac{1}{(u-v)^{p+l}} \tag{28}$$

We now consider the limit $\lim_{s\to\infty}(f *_s g)$ by taking the limit of the terms in Eq. 28 that depend on s:

$$\begin{aligned} \lim_{s\to\infty}\frac{(-2s)_{p+l}}{(-2s)_k(-2s)_l} &= \lim_{s\to\infty}\frac{(-2s)_{p+l}}{(-2s)_k(-2s)_l}\frac{(-2s)^k}{(-2s)^k}\frac{(-2s)^l}{(-2s)^l}\frac{(-2s)^{p+l}}{(-2s)^{p+l}} \\ &= \lim_{s\to\infty}\frac{(-2s)^k}{(-2s)_k}\frac{(-2s)^l}{(-2s)_l}\frac{(-2s)_{p+l}}{(-2s)^{p+l}}(-2s)^{p-k} \end{aligned} \tag{29}$$

Applying Eq. 33 gives

$$\lim_{s\to\infty}\frac{(-2s)_{p+l}}{(-2s)_k(-2s)_l}=\lim_{s\to\infty}(-2s)^{p-k}=\begin{cases}1 & \text{if } p=k\\ 0 & \text{if } p<k\end{cases}. \tag{30}$$

Plugging this result into Eq. 28 gives

$$\begin{aligned}\lim_{s\to\infty}(f *_s g)&=f(u)g(u)\frac{1}{(u-v)^{k+l}}\\ &=\left(f(u)\frac{1}{(u-v)^k}\right)\left(g(u)\frac{1}{(u-v)^l}\right)=fg.\end{aligned} \tag{31}$$

Thus the theorem is proved.□

Definition The Poisson bracket in (u,v) coordinates of two continuous functions $f(u,v)$ and $g(u,v)$ is

$$\{f,g\}=\frac{(u-v)^2}{2}\left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial v}-\frac{\partial g}{\partial u}\frac{\partial f}{\partial v}\right). \tag{32}$$

Theorem 4

$$\lim_{s\to\infty}s(f *_s g-g *_s f)=\{f,g\} \tag{33}$$

Proof

We begin by considering the LHS of Eq. 33. Applying the formula given by Eq. 28, we get

$$\begin{aligned}\lim_{s\to\infty}s(f *_s g-g *_s f)=\lim_{s\to\infty}s\Bigg(f(u)\sum_{p=0}^{k}\frac{(-2s)_{p+l}}{(-2s)_k(-2s)_l}\binom{k}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k-p}g(u)\right]\frac{1}{(u-v)^{p+l}}\\ -g(u)\sum_{q=0}^{l}\frac{(-2s)_{q+k}}{(-2s)_k(-2s)_l}\binom{l}{q}\left[\left(\frac{\partial}{\partial u}\right)^{l-q}f(u)\right]\frac{1}{(u-v)^{q+k}}\Bigg).\end{aligned} \tag{34}$$

We isolate the last term of each sum and get

$$\begin{aligned}&\lim_{s\to\infty}s(f *_s g-g *_s f)\\ &=\left(\lim_{s\to\infty}f(u)\sum_{p=0}^{k-1}\frac{s(-2s)_{p+l}}{(-2s)_k(-2s)_l}\binom{k}{p}\left[\left(\frac{\partial}{\partial u}\right)^{k-p}g(u)\right]\frac{1}{(u-v)^{p+l}}+\lim_{s\to\infty}\frac{s(-2s)_{k+l}}{(-2s)_k(-2s)_l}f(u)g(u)\frac{1}{(u-v)^{k+l}}\right)\\ &-\left(g(u)\sum_{q=0}^{l-1}\frac{s(-2s)_{q+k}}{(-2s)_k(-2s)_l}\binom{l}{q}\left[\left(\frac{\partial}{\partial u}\right)^{l-q}f(u)\right]\frac{1}{(u-v)^{q+k}}+\lim_{s\to\infty}\frac{s(-2s)_{k+l}}{(-2s)_k(-2s)_l}f(u)g(u)\frac{1}{(u-v)^{k+l}}\right).\end{aligned} \tag{35}$$

The isolated terms cancel, and we consider the limit of the terms of the first sum that depend on s and simplify as shown in the proof of theorem 3:

$$\lim_{s\to\infty}\frac{s(-2s)_{p+l}}{(-2s)_k(-2s)_l}=\lim_{s\to\infty}s(-2s)^{p-k}=(-2)^{p-k}\lim_{s\to\infty}s^{p-k+1}=\begin{cases}-\frac{1}{2} & \text{if } p=k-1\\ 0 & \text{if } p<k-1\end{cases} \tag{36}$$

We can use a similar approach for the terms of the second sum to show that

$$\lim_{s\to\infty} \frac{s(-2s)_{q+k}}{(-2s)_k(-2s)_l} = \begin{cases} -\frac{1}{2} & \text{if } q = l-1 \\ 0 & \text{if } q < l-1 \end{cases}. \tag{37}$$

Plugging these results into Eq. 35 gives

$$\begin{aligned} \lim_{s\to\infty} s(f *_s g - g *_s f) = f(u)\Big(-\frac{1}{2}\Big)\binom{k}{k-1}\frac{\partial g(u)}{\partial u}\frac{1}{(u-v)^{k+l-1}} - g(u)\Big(-\frac{1}{2}\Big)\binom{l}{l-1}\frac{\partial f(u)}{\partial u}\frac{1}{(u-v)^{k+l-1}} \\ = -\frac{1}{2(u-v)^{k+l-1}}\left(kf(u)\frac{\partial g(u)}{\partial u} - lg(u)\frac{\partial f(u)}{\partial u}\right). \end{aligned} \tag{38}$$

We now have

$$\lim_{s\to\infty} s(f *_s g - g *_s f) = \frac{1}{2(u-v)^{k+l-1}}\left(lg(u)\frac{\partial f(u)}{\partial u} - kf(u)\frac{\partial g(u)}{\partial u}\right). \tag{39}$$

Now we consider the RHS of Eq. 33:

$$\begin{aligned} \{f,g\} = \frac{(u-v)^2}{2}\left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial v}\right) \\ = \frac{(u-v)^2}{2}\left(\frac{\partial}{\partial u}\left[f(u)\frac{1}{(u-v)^k}\right]\frac{\partial}{\partial v}\left[g(u)\frac{1}{(u-v)^l}\right] - \frac{\partial}{\partial u}\left[g(u)\frac{1}{(u-v)^l}\right]\frac{\partial}{\partial v}\left[f(u)\frac{1}{(u-v)^k}\right]\right) \\ = \frac{(u-v)^2}{2}\left(\left[\frac{\partial f(u)}{\partial u}\frac{1}{(u-v)^k} - kf(u)\frac{1}{(u-v)^{k+1}}\right]\left[lg(u)\frac{1}{(u-v)^{l+1}}\right]\right. \\ \left. - \left[\frac{\partial g(u)}{\partial u}\frac{1}{(u-v)^l} - lg(u)\frac{1}{(u-v)^{l+1}}\right]\left[kf(u)\frac{1}{(u-v)^{k+1}}\right]\right) \\ = \frac{(u-v)^2}{2}\left(lg(u)\frac{\partial f(u)}{\partial u}\frac{1}{(u-v)^{k+l+1}} - klf(u)g(u)\frac{1}{(u-v)^{k+l+2}}\right. \\ \left. -kf(u)\frac{\partial g(u)}{\partial u}\frac{1}{(u-v)^{k+l+1}} + klf(u)g(u)\frac{1}{(u-v)^{k+l+2}}\right) \end{aligned} \tag{40}$$

The non-derivative terms cancel, leaving us with

$$\{f,g\} = \frac{1}{2(u-v)^{k+l-1}}\left(lg(u)\frac{\partial f(u)}{\partial u} - kf(u)\frac{\partial g(u)}{\partial u}\right) \tag{41}$$

Since the RHS of Eq. 39 is the same as the RHS of Eq. 41, the theorem is proved.□