Inflationary Redistribution vs. Trading Opportunities:

*Cost of Inflation in a Monetary Model with Non-degenerate Distributions**

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Abstract

We propose a monetary model which features endogenous market incompleteness. Our framework combines the tractable features of competitive search with matching frictions of Menzio et al. (2013) with a costly participation model in a centralized market with complete insurance. Equilibrium market incompleteness arises because of: (i) an externality trading off matching opportunities with consumption in non-Walrasian markets where money becomes essential; and (ii) agents have to make costly participation decisions to enter complete consumption insurance markets. We identify two types of opposing (i.e., intensive-versus-extensive) margins of trade-offs in the face of anticipated inflation tax. Numerically, we find that the extensive margins tend to dominate, resulting in average welfare falling and wealth inequality rising with inflation. We also propose a novel computational solution method, taking insights from computational geometry, to efficiently solve for a monetary equilibrium.

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1 Question and Motivation

IN THIS PAPER, we end with the following question: *How big are the welfare cost and redistributive effects of inflation?* This long-standing question has been studied through various lenses, beginning from a statistical money demand function approach (Bailey, 1956) to general equilibrium models without agent heterogeneity (see, e.g., Lucas, 2000; Dotsey and Ireland, 1996), and, to models with equilibrium distributions (see, e.g., Imrohoroğlu and Prescott, 1991b,a; Chiu and Molico, 2010; Molico, 2006). We revisit this question in the context of our new model in which there is endogenous market incompleteness. Equilibrium-determined market incompleteness arises because: (i) agents' equilibrium behavior can affect their trading opportunities (i.e., extensive margin of trade) in non-Walrasian markets in which contracts are not sustainable and money becomes essential; and (ii) agents get to decide when it is beneficial to participate in markets that would otherwise provide complete insurance of individual consumption risks.

There are two main reasons for a reassessment of the welfare cost and redistributive role of inflation in our alternative framework. First, most real-world markets do not possess features of a hypothetical and informationally superior Walrasian-Arrow-Debreu world. However, most monetary general equilibrium models to date have this feature and "money"—i.e., any asset that is liquid or serves as a medium of exchange—is made essential (or rationalized by the modeller) by restricting agent's behavioral outcomes through black-box asset-in-advance constraint assumptions. For example, if money is strictly taken to be a fiat, non-interest bearing and liquid asset, then the constraint is the well-known cash-in-advance (CIA) constraint. In contrast, in our setting, money is only essential insofar as agents need to participate in decentralized markets in which fundamental information and contractual frictions prevent trade supported by contractual promises or private securities.

Second, and in contrast with search models featuring random matching of traders, we consider a competitive search economy—i.e., a large-markets limit of a directed search economy.[1] We suggest why this may be an important consideration: Directed, competitive search and matching friction introduces an additional opposing force to the usual force of inflationary *redistribution* in heterogenous agent monetary models.[2] This opposing force arises from an *extensive margin*—an extra link between inflation and traders' equilibrium probabilities of matching—which is not present in Walrasian models, nor in non-Walrasian random matching models with *ex-post* pricing. The existence of a trade-off

[1] Arguably, many markets, whether physical or online, do have such features: Sellers commit to and post quantities and prices, and, buyers direct their search towards these sellers. We build upon the elegant theory of Menzio et al. (2013) that rationalizes a Baumol-Tobin (non-neutral) liquidity channel of monetary policy.

[2] Both Walrasian-market heterogeneous agent monetary models (see, e.g., Imrohoroğlu and Prescott, 1991a; Meh et al., 2010) and random-matching search models of money (see, e.g., Molico, 2006; Chiu and Molico, 2010) give rise to the role of inflation as a redistributive tax instrument.

between the *intensive* (terms of trade) margin and *extensive* (trading opportunity) margin is well known in the directed search literature; and it is a consequence of models where buyers know *ex-ante* the possible terms of trade offered by sellers, and the buyers choose what terms of trade to search for (see, e.g., Peters, 1984, 1991; Moen, 1997; Burdett et al., 2001; Julien et al., 2008; Shi, 2008). What we show in this paper is that in a monetary setting, this trade-off can be quantitatively relevant in terms of the effects of inflation.

The model we present in this paper is in the spirit of Menzio et al. (2013) but we allow, generally, for two notions of limited participation in consumption insurance markets: First, we entertain the possibility of pure good luck, whereby with probability $\alpha \in [0,1)$ some agents can enter a (complete) centralized market (CM) costlessly. Second, we also allow for the possibility that agents may decide to enter the CM each period, subject to overcoming a real fixed cost $\chi \geq 0$. If we set the probability of CM participation to zero and re-define the flow payoff in the CM as depending only on leisure in a strictly concave fashion, the model is that of Menzio et al. (2013), which we will refer to as a decentralized market (DM). One can think of this setting as a version of a unified monetary framework initiated by Lagos and Wright (2005). In fact, this is also what Sun and Zhou (2016) do, but they assume that all agents in a DM must exit it in one period and must enter a CM (with quasilinear preferences) thereafter. In their equilibrium, agent heterogeneity (i.e., non-degeneracy of the equilibrium distribution of agents) needs to be preserved by additionally assuming that there are idiosyncratic exogenous preference shocks to agents in the CM. In contrast, we can have a non-degenerate distribution of agents in our model, since not all agents will find it worthwhile to enter the CM at any period, in an equilibrium.

We also have a "more standard" feature like a neoclassical CM sector for two reasons: First, this will allow us to have a better mapping of the theory into reality in terms of calibration of the model to observed data. Second, the feature of probabilistic CM participation will allow us to perform counterfactual experiments on the degree of "market incompleteness" in the model.

Using our proposed framework, we study the individual's ability to insure his or her consumption risk over time. We then consider how anticipated (long run) inflation alters the monetary equilibrium in terms of agents' ability to insure themselves using an imperfect vehicle of insurance such as money, and also in terms of its effect on the distribution of money and welfare.

We identify two types of opposing forces at work in this environment. First, there is the intensive-versus-extensive margin in terms of CM participation, which gives rise to a precautionary motive for holding money: In the face of inflation, agents must either work harder each time in the CM and bear the cost of holding excess money balances (*intensive labor-CM margin*), or, they work less in each CM instance, reduce their spending in each DM exchange, and ensure that they are more likely to be able to afford to go to the CM frequently (*extensive labor-CM margin*). Second, there is another intensive-versus-extensive margin in terms of DM activity. With more inflation, if agents know they may end up in the DM sometimes, then on one hand they would like to be able to pay more and consume more in that market. (We call this force the *DM intensive margin*.) However, at each DM sub-market which suffers from equilibrium matching externality, they would also like to have a greater probability of trading too (i.e., *DM extensive margin*).

We show that for plausible parametrizations of the model, the extensive margins tend to dominate, resulting in a reduction of average welfare, and, a rise in inequality of (money) wealth, as inflation rises. We estimate that the cost of increasing inflation from 0 to 10 percent (per annum), involves as much as 1.6 percent per annum loss in welfare-equivalent consumption. We also show that if agents can borrow against future CM incomes, inequality of real money wealth can actually increase with inflation. As a minor contribution, we also propose a novel computational solution method, taking insights from computational geometry, to efficiently solve for a monetary equilibrium.

2 Model environment

There is a decentralized market (DM) in the style of Menzio et al. (2013) and an Arrow-Debreu-Walras centralized market (CM). Time is implicitly indexed by $t \in \mathbb{N}$. Hereinafter, we will denote $X := X_t$ and $X_{+1} := X_{t+1}$ for dynamic variables.

2.1 Money supply

Money is taken to be any asset that can be used as a medium of exchange—i.e., a tradable claim on consumption goods that does not require contracting on specific individual trader's characteristic or trading history. We assume that the total stock of money in the economy M grows according to the process

$$\frac{M_{+1}}{M} = 1 + \tau, \tag{2.1}$$

where $\tau > \beta - 1$.

Labor is the numeraire good. If we denote ωM as the current nominal wage rate, where ω is normalized nominal wage (i.e., nominal wage rate per units of M), then a dollar's worth of money is equivalent to $1/\omega M$ units of labor. The variable ω will be determined in a monetary equilibrium. If M is the beginning of period aggregate stock of money in circulation, then $1/\omega = M \times 1/\omega M$ is the beginning of period real aggregate (per-capita) stock of money, measured in units of labor. Denote (equilibrium) nominal wage growth as $\gamma(\tau) \equiv \omega_{+1}M_{+1}/(\omega M)$. Later, for a steady-state monetary equilibrium, we will require that equilibrium nominal wage grows at the same rate as money supply, i.e., $\gamma(\tau)|_{(\omega_{+1}=\omega)} = M_{+1}/M$.

2.2 Markets, agents, commodities and information

There are two types of markets open every period: a centralized market (CM) and a decentralized market (DM). There is a measure one of individuals who will decide at the beginning of each date which market to participate in. Firms can act in both types of markets in CM and DM simultaneously. An individual can only be in the DM or CM at a given time period. In the DM, individuals shop for special goods q. In the CM individuals supply labor l, and, consume a general good C. A firm in the CM hires labor to produce the general CM good. We describe the CM and DM markets in turn.

In the CM, two markets are open simultaneously: A competitive spot market for labor and a competitive general good market; the latter is equivalent to a competitive market for trading in a complete set of individual-state-contingent consumption claims. Agents trade in these securities to insure their consumption risk, which arises from their heterogeneous trading histories upon exit from the DM. They may still demand money as a precaution against the need for liquidity in anonymous markets in the DM.

In the DM, we have a setting similar to Menzio et al. (2013) where there is an information friction: Buyers of special DM goods, q, are anonymous and cannot trade using private claims or cannot undertake contracts with selling firms. As a result, the only medium of exchange is money. There is a finite set of *types* of individuals and goods, I. There is a measure-one continuum of individuals and firms of type $i \in I$, where an individual i consumes good i and produces good $i+1$ (mod-$|I|$). A type i firm hires labor service from type $i-1$ (mod-$|I|$) individuals (from the CM spot labor market) and transforms it (linearly) into the same amount of DM good i. Each i-type firm commits to posted terms of trade in all submarkets it chooses to enter. Buyers of good i direct their search toward these submarkets that sell good i, by choosing the best terms of trade offered. However, as we will see, these buyers will have to balance their decision on terms of trades against the probability of getting matched. Since firms and buyers choose which submarket to participate in, a type i buyer will only participate in the submarkets where type i firms sell.[3]

[3]Hereinafter, the explicit dependency on the type of good $i \in I$ will become unnecessary. It will turn out that terms of trade in every submarket, indexed by pairs of buyers' willingness to pay and consume $\{(x, q)\}$, identifies an equivalent class of submarkets.

2.2.1 Preference representation

The per-period utility function of an individual is

$$U(C) - h(l) + u(q). \tag{2.2}$$

We assume that the functions U and u are continuously differentiable, strictly increasing, strictly concave, $U_1, u_1 > 0$, $U_{11}, u_{11} < 0$, and the following boundary conditions hold: $U(0) = u(0) = U_1(\infty) = u_1(\infty) = 0$, and $U_1(0), u_1(0) < \infty$. Also, we assume that $h(l) = Al$, where $A > 0$ is some constant.[4] Also, the upper bound on money holdings $\bar{m} \in (0, \infty)$ and preferences (through parameter A) are such that:[5]

$$0 < \bar{m} < (U_1)^{-1}(A) \iff A < U_1(\bar{m}) < U_1(0). \tag{2.3}$$

2.2.2 Matching technology in the DM

We follow the assumptions of Menzio et al. (2013) in the setting below. Let $\theta \in \mathbb{R}_+$ denote the ratio of trading posts to buyers in a submarket—i.e., its market tightness. In a submarket with tightness θ, the probability that a buyer is matched with a trading post is $b = \lambda(\theta)$. The probability a trading post is matched with a buyer is $s = \rho(\theta) := \lambda(\theta)/\theta$. We assume that the function $\lambda : \mathbb{R}_+ \to [0,1]$ is strictly increasing, with $\lambda(0) = 0$, and $\lambda(\infty) = 1$. The function $\rho(\theta)$ is strictly decreasing, with $\rho(0) = 1$, and $\rho(\infty) = 0$. We can re-write a trading post's matching probability $s = \rho(\theta) = \rho \circ \lambda^{-1}(b) \equiv \mu(b)$. Observe that the matching function μ is a decreasing function, and that $\mu(0) = 1$ and $\mu(1) = 0$. Assume that $1/\mu(b)$ is strictly convex in b.

2.2.3 Firms

Consider a representative firm $i \in [0,1]$ that takes the CM good's relative price p (in units of labor) as given. The firm hires labor on the spot market and transforms hired labor services into Y units of CM good linearly. At the same time, in the DM, the firm takes the terms of trade, respectively measured by buyers' payment and demand of the good, (x, q), in a given submarket for some type-i good as given, and chooses the measure of trading posts (viz., shops) $\mathrm{d}N(x,q)$ to open in each submarket.[6] These assumptions are identical to Sun and Zhou (2016). The firm in the DM takes the (equilibrium) probability

[4] We will use the notational convention, $f_i(x_1, ..., x_n) \equiv \partial f(x_1, ..., x_n)/\partial x_i$, to denote the value of the partial derivative of a function f with respect to its i-th variable. Likewise, f_{ij} will denote its cross-partial derivative function with respect to the j-th variable.

[5] This regularization will ensure that the agent's labor effort is always interior, $l^\star(m,\omega) \in \text{int}(\mathbb{R}_+)$, and that in all dates, money balances are bounded, $m \in [0, \bar{m}]$. Note that this assumption is similar to the one in Sun and Zhou (2016), except that in the latter, the authors' equivalent of A is the upper bound on some random variable (a preference shock).

[6] This is equivalent to stating that the firms commit to posted terms of trade in the particular submarket(s).

of being matched with a buyer $s(x,q)$ as given. If x is what a matched buyer is willing to pay for q, then $x \cdot s(x,q)$ is the firm's expected revenue in submarket (x,q). To produce q the firm must hire $c(q)$ units of labor. Hence $s(x,q)c(q)$ is its expected labor wage bill at submarket (x,q). We assume that $q \mapsto c(q)$ is a continuous convex function. The firm also pays a per-period fixed cost k of creating the trading post in submarket (x,q). The firm's value is:

$$\pi(p,x,q;k) = \max_{Y \in \mathbb{R}_+} \{pY - Y\} + \max_{\mathrm{d}N(x,q) \in \mathbb{R}_+} \int \{s(x,q)\,[x - c(q)] - k\}\,\mathrm{d}N(x,q), \qquad \forall\,(x,q)\,. \tag{2.4}$$

The first term on the RHS is the firm's value from operating in the CM. The second, is its DM total expected value across all submarkets it chooses to operate in.

2.3 Individuals' decisions

An individual is identified by her current money balance (measured in units of labor), m. Given policy τ, her decisions also depend on knowing the aggregate wage ω. Denote the relevant state vector as $\mathbf{s} := (m,\omega) \in S = [0,\bar{m}] \times [0,\bar{\omega}]$.[7] At the beginning of a period (ex ante), an individual decides whether to work and consume in the CM, or, whether to be a buyer in the frictional DM. Ex post, if the agent has positive initial money balance as a DM buyer, he continues searching for a trading post. However, with some probability, that agent may get to go back to the CM to simultaneously work, consume and accumulate money. Also, ex post, another agent is in the CM either because she had previously expended all her money in a DM submarket, or, she had received a shock at the end of a trade that moves her to the CM (but she may still have zero or positive money balance).[8] We describe these different ex-post agents' problems in turn, and then, we will describe an agent's ex-ante decision problem.

[7] In a steady state equilibrium, ω is a constant. However, this definition of the state vector will be relevant when we consider dynamic transitions. We define the upper bounds $\bar{m}$ and $\bar{\omega}$ later. (The former will be specified exogenously and the latter can be shown to exist in equilibrium.)

[8] This probabilistic aspect of the agent's status can be interpreted as an exogenous way to capture limited participation by agents in complete financial markets. Note that under our assumption, the only time an agents gets to decide whether to participate in the CM will be when she has no money balance left. We will see that agents with such zero money balance will have to work more than those with existing positive money holdings who entered the CM by luck.

Our assumption here is different to that of Sun and Zhou (2016). In Sun and Zhou (2016), all individuals get to go into the CM deterministically in one period, given that they are currently in the DM. Individuals choose whether to go into the DM submarkets, as in our ex ante agent's problem. However, since Sun and Zhou (2016) also have a quasilinear preference representation in the CM, they also require that agents receive idiosyncratic preference shocks (i.e., labor supply shocks) to prevent degeneracy of the equilibrium distribution of agents. In contrast, our assumption here still preserves non-degeneracy without an additional assumption of exogenous preference shocks, since there is always a positive measure of agents who will be stuck trading in the DM submarkets for some time before some of them get to go to the CM.

2.3.1 Ex-post individual in the CM

Suppose now we have an individual $\mathbf{s} := (m, \omega)$ who begins the current period in the CM. The individual takes policy, τ, and the sequence of aggregate prices, $(\omega, \omega_{+1}, ...)$, as given. Her value from optimally consuming C, supplying labor l, and accumulating end-of-period money balance y, is

$$W(\mathbf{s}) = \max_{(C,l,y)\in\mathbb{R}^3_+} \left\{ U(C) - h(l) + \beta\bar{V}(\mathbf{s}_{+1}) : pC + y \leq m + l, \ m_{+1} = \frac{\omega y + \tau}{\omega_{+1}(1+\tau)} \right\}, \tag{2.5}$$

where $\bar{V} : S \to \mathbb{R}$ is her continuation value function, to be fully described in Section 2.3.3 on the following page. This continuation value yields her next-period expected total payoff, at a state m_{+1} before she makes any decisions next.[9]

2.3.2 Ex-post individual buyer in the DM

Now we focus on an individual who has just decided to be a DM buyer. The buyer can only visit one trading post at a given time by directing search to its terms of trade, (x, q). The individual buyer, $\mathbf{s} := (m, \omega)$, has initial value:

$$\begin{aligned} B(\mathbf{s}) = \max_{x\in[0,m], q\in\mathbb{R}_+} \Bigg\{ & \beta\left[1 - b(x,q)\right]\left[\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right] \\ & + b(x,q)\left[u(q) + \beta\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]\Bigg\}. \end{aligned} \tag{2.6}$$

Consider the first two terms on the RHS of the functional (2.6): With probability $1 - b(x,q)$ the buyer fails to match with the trading post and must thus continue the next period with his initial money balance subject to inflationary transfer. With the complementary probability $b(x,q)$ he matches with a trading post (x,q), pays the seller x in exchange for a flow payoff $u(q)$, and then continues into the next period with his net balance, also subject to inflationary tranfers. [10]

[9] The continuation state for the individual, m_{+1}, is derived as follows: At the end of the CM, the individual would have accumulated balance y (measured in units of labor). In current units of nominal money, this is $\omega M \times y$. At the beginning of next period, each individual gets a nominal transfer of new money τM (population is normalized to size 1). In units of labor next period, the beginning-of-period balance would thus be $m_{+1} = (\omega M y + \tau M) / (\omega_{+1} M_{+1})$. Replacing for M/M_{+1} with the money supply process in (2.1) gives the expression for the individual's continuation state m_{+1} in (2.5).

[10] Again, note that this continuation value corresponds to the expected total payoff of beginning next period, at a point before the buyer chooses to stay in the DM or to leave for the CM (with some fixed cost).

2.3.3 Ex ante decision

At the beginning of a period, before an arbitrary agent $\mathbf{s} := (m, \omega)$ realizes her DM or CM market participation status, her value is

$$\bar{V}(\mathbf{s}) = \alpha W(\mathbf{s}) + (1-\alpha) V(\mathbf{s}), \tag{2.7}$$

where $V(\mathbf{s})$ is the value of playing a fair lottery $(\pi_1, 1-\pi_1)$ over the prizes $\{z_1, z_2\}$, i.e.,

$$V(\mathbf{s}) = \max_{\pi_1 \in [0,1], z_1, z_2} \left\{ \pi_1 \tilde{V}(z_1, \omega) + (1-\pi_1) \tilde{V}(z_2, \omega) : \pi_1 z_1 + (1-\pi_1) z_2 = m \right\}; \tag{2.8}$$

and, given a lottery outcome z, the individual's value becomes

$$\tilde{V}(z, \omega) = \begin{cases} \max_{a \in \{0,1\}} \{ aW(z-\chi, \omega) + (1-a) B(z, \omega) \}, & z - \chi \geq -y_{\max}(\omega; \tau) \\ B(z, \omega), & \text{otherwise} \end{cases}, \tag{2.9}$$

where

$$y_{\max}(\omega; \tau) := \min\left\{ \bar{m} - \frac{\tau}{\omega}, \bar{m} \right\} \tag{2.10}$$

is a natural upper bound on CM saving (in real money balances). We derive this limit in the Online Appendix A.

Equation (2.7) says the following: With some exogenous probability α, an agent gets to enter the frictionless CM costlessly, which will allow him to work and save. The value of such an event to the individual is $W(\mathbf{s})$. (Note that if the economy were to all exist as the CM, then there is complete markets insurance of individual consumption risks.) With probability $1-\alpha$ the buyer has to decide whether to enter the CM or DM. Since the DM-buyer value function may not be strictly concave due to equilibrium externality from the matching friction, it will be profitable for the individual to play a mixed strategy over the set of pure actions $a \in \{0, 1\}$, where $a = 0$ ($a = 1$) corresponds to going to the DM (CM).[11] The value to the individual in such an event is $V(\mathbf{s})$, which is defined by (2.8) and (2.9). Observe that in (2.9), contingent on realizing a lottery payoff z, the outcome of the lottery also induces the pure action of going to the DM or the CM. Also, in such $(1-\alpha)$-measure of contingent events, if the agent decides to go to the CM, he

[11]The externality problem shows up mathematically as the bilinear and non-concave interaction between $b(x, q)$ and $u(q)$ in the DM-buyer's objective function in (2.6). These two terms, respectively, are interpretable as an aggregate extensive margin (i.e., how likely is a buyer to trade) and an intensive margin (i.e., how much of q to consume).

must pay a fixed cost $\chi \geq 0$ (measured in units of labor) to participate in the CM. This fixed cost component is interpretable as a barrier to some unlucky and poor individuals to participate in financial markets (which would have allowed further insurance of their consumption risks).

Remark 1. Observe that in the ex-ante market participation problem (2.9), there is a limited short-sale (I.O.U.) constraint $z - \chi \geq -y_{\max}(\omega;\tau)$. It may be possible that an agent, whose state is such that $m < \chi$, when faced with deciding to go to the CM, may still find it optimal to issue an I.O.U. worth $m - \chi$ at the beginning of a CM, and go to work in the CM immediately to repay the shortfall $m - \chi$. Since the fixed cost is levied in the CM, and in the CM promises or contracts are completely sustainable, then a limited amount of short selling (I.O.U.) is possible. The limit on the short sale $m - \chi$, is equivalent to agents exerting the maximal CM labor effort $l_{\max}(\omega;\tau)$ and not saving anything in the CM. In Online Appendix B we derive this limit of $-y_{\max}(\omega;\tau) \equiv -\min\{\bar{m}, \bar{m} - \tau/\omega\}$.

From this we have the following insights:

1. The higher the fixed cost χ, the closer the agent will be to violating this bound, which means that he will then choose not to participate in the CM.

2. Observe also that, the short-selling limit tightens with higher money supply growth rate, τ, but loosens with higher ω (but ω may depend positively on τ in equilibrium).

Thus, inflation (through money supply growth τ) can create an equilibrium tension acting on the *extensive margin of CM participation*. (We discuss this again in Section 3.4 on page 27.)

2.3.4 Special cases

Note that when $\alpha = 1$ (i.e., agents get to enter the CM deterministically), $\chi = 0$ (there is no fixed cost of entering the CM), and the continuation value from CM is a convexification of $B(\cdot, \omega)$, our model becomes a version of Lagos and Wright (2005) with competitive search markets, instead of search with Nash bargaining.

If we retain, the same DM buyer's problem as we have here, then the economy trivially collapses to a frictionless CM every period, and we would have a first-best represtative-agent competitive equilibrium in which money is inessential.

Also, when $\alpha = 0$, there is no fixed cost of entering the CM $(\chi = 0)$, $U(C) = 0$ for all C, and the labor utility function $h(l)$ is strictly convex, we recover the original Menzio et al. (2013) model as a special case.

2.4 Monetary equilibrium (ME)

Clearly there exists a non-monetary equilibrium whereby no agent will participate in the DM. In this paper, we restrict attention to the case of a monetary equilibrium. Hereinafter, whenever we refer to "monetary equilibrium", or "equilibrium", we mean a recursive monetary equilibrium—one in which agent's decision functions are recursive and time-invariant maps. In what follows, we first characterize the equilibrium strategy of firms (section 2.4.1), the equilibrium value and decision functions of agents in the CM (section 2.4.2) and in the DM (section 2.4.3), and then close the equilibrium notion by describing the market clearing conditions (section 2.4.4). At the end of this section, we restrict attention to and define formally the notion of a steady-state or stationary monetary equilibrium (SME).

2.4.1 Equilibrium strategy of firms.

A firm's problem is static. We can characterize the equilibrium behavior of a firm given p (in the CM) and any operative submarket (x, q) in the DM. Free entry in the CM will render zero profits to firms in equilibrium, and thus, $p = 1$. Likewise, free entry and zero-profit in the DM with competitive search will imply that

$$r(x,q) := s(x,q)\,[x - c(q)] - k \leq 0, \qquad \text{and}, \qquad dN(x,q) \geq 0, \tag{2.11}$$

where the weak inequalities would hold with complementary slackness: For a submarket (x, q) such that $r(x, q) < 0$, the firm optimally chooses $\mathrm{d}N(x, q) = 0$. If $r(x, q) = 0$, then the firm is indifferent on $\mathrm{d}N(x, q) \in (0, +\infty)$. We can also deduce that $r(x, q) > 0$ cannot be an equilibrium: If expected profit is positive, then the linear program of the firm in the DM yields an optimum of $\mathrm{d}N(x, q) = +\infty$, but this violates the requirement of zero profits in equilibrium.[12] We will restrict attention to an equilibrium where (2.11) also holds for submarkets not visited by any buyer.[13]

From (2.11), we can deduce that

$$s(x,q) \equiv \mu \circ b\,(x,q) = \begin{cases} \frac{k}{x-c(q)} & \iff x - c(q) > k \\ 1 & \iff x - c(q) \leq k \end{cases}. \tag{2.12}$$

[12]If we re-label $N(x, q)$ as the equilibrium distribution of trading post across submarkets, condition (2.11) implies that aggregate profit in the DM is zero: $\int \{s(x,q)\,[x - c(q)] - k\}\,\mathrm{d}N(x,q) = 0$.

[13]Justification for this off-equilibrium-path restriction can be rationalized via a "trembling-hand" sort of argument: Suppose there is some exogenous perturbation that induces an infinitesimally small measure of buyers to show up in every submarket. Given a non-zero measure of buyers present in a submarket, if firms' expected profit is still negative in that submarket, i.e., $r(x, q) < 0$, then the market will not be active. This restriction is commonly used in the directed search literature (see, e.g., Menzio et al., 2013; Acemoglu and Shimer, 1999; Moen, 1997).

Observe that the firm's probability of matching with a buyer, $s(x,q)$ depends only on the posted terms of trade (x,q). Likewise, the buyer's probability of matching with a firm, $b(x,q)$, given the matching technology $\mu : [0,1] \rightarrow [0,1]$. Thus, in any submarket with positive measure of buyers, the market tightness, $\theta(x,q) \equiv b(x,q)/s(x,q)$, is neccessarily and sufficiently determined by free entry into the submarket. Moreover, the terms of trade of a submarket (x,q) is sufficient to identify the submarket.

It will be convenient for later to note that we have in equilibrium, implicitly, a relation between q and (x,b). That is, in any equilibrium, each active trading post will produce and trade the quantity:

$$q = Q(x,b) \equiv c^{-1}\left[x - \frac{k}{\mu(b)}\right], \tag{2.13}$$

given payment x and its matching probability $s = \mu(b)$. This relation will allow us to perform a change of variable, and re-write the buyers' problems below in terms choices over (x,b), instead of over (x,q).

2.4.2 Equilibrium and the CM individual

Let us denote $\mathcal{C}[0,\bar{m}]$ as the set of continuous and increasing functions with domain $[0,\bar{m}]$. Then $\mathcal{V}[0,\bar{m}] \subset \mathcal{C}[0,\bar{m}]$ denotes the set of continuous, increasing and concave functions on the domain $[0,\bar{m}]$. We have the following observations of any CM individual's value and policy functions, which apply to both a steady-state equilibrium or along a dynamic equilibrium transition. (Proofs of these results are relegated to the online appendix.)

Theorem 1. *Assume $\tau/\omega < \bar{m}$. For a given sequence of prices $\{\omega, \omega_{+1}, ...\}$, the value function of the individual beginning in the CM, $W(\cdot,\omega)$, has the following properties:*

1. *$W(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$, i.e., it is continuous, increasing and concave on $[0,\bar{m}]$. Moreover, it is linear on $[0,\bar{m}]$.*

2. *The partial derivative functions $W_1(\cdot,\omega)$ and $\bar{V}_1(\cdot,\omega_{+1})$ exist and satisfy the first-order condition*

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega y^{\star}(m,\omega)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\begin{cases} \leq A, & y^{\star}(m,\omega) \geq 0 \\ \geq A, & y^{\star}(m,\omega) \leq \min\left\{\bar{m} - \frac{\tau}{\omega}, \bar{m}\right\} \end{cases}, \tag{2.14}$$

and the envelop condition:

$$W_1(m,\omega) = A, \tag{2.15}$$

where $y^{\star}(m,\omega) = m + l^{\star}(m,\omega) - C^{\star}(m,\omega)$, $l^{\star}(m,\omega)$ and $C^{*}(m,\omega)$, respectively, are the associated optimal choices on labor effort and consumption in the CM.

3. *The stationary Markovian policy rules $y^{\star}(\cdot,\omega)$ and $l^{\star}(\cdot,\omega)$ are scalar-valued and continuous functions on $[0,\bar{m}]$.*
   (a) *The function $y^{\star}(\cdot,\omega)$, is constant valued on $[0,\bar{m}]$.*
   (b) *The optimizer $l^{\star}(\cdot,\omega)$ is an affine and decreasing function on $[0,\bar{m}]$.*
   (c) *Moreover, for every (m,ω), the optimal choice $l^{\star}(m,\omega)$ is interior: $0 < l_{\min} \leq l^{\star}(m) \leq l_{\max}(\omega;\tau) < +\infty$, where there is a very small $l_{min} > 0$ and $l_{max}(\omega;\tau) := \min\left\{\bar{m} - \frac{\tau}{\omega}, \bar{m}\right\} + U^{-1}(A) < 2U^{-1}(A) \in (0,\infty)$.*

In the proof to Theorem 1, we also derive the equilibrium decisions of the CM agent. We show that in an equilibrium, CM consumption is

$$C^{\star}(m,\omega) \equiv \bar{C}^{\star} = (U_1)^{-1}(pA), \tag{2.16}$$

a finite and non-negative constant. Equilibrium CM asset decision will depend on the aggregate state ω, i.e.,

$$y^{\star}(m,\omega) = \bar{y}^{\star}(\omega) \tag{2.17}$$

and this satisfies the first-order condition (2.14). Finally, from the budget constraint, we can obtain the equilibrium labor supply function as

$$l^{\star}(m,\omega) = \bar{C}^{\star} + \bar{y}^{\star}(\omega) - m. \tag{2.18}$$

Note that $l^{\star}(m,\omega)$ is single-valued, continuous, affine and decreasing in m.

2.4.3 Equilibrium DM buyer

Observe that since $V(\cdot,\omega), W(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$ (i.e., are continuous, increasing and concave), then by (2.7), $\bar{V}(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$. In an equilibrium, the DM buyer's problem in (2.6) can be re-written as

$$\begin{aligned} B(\mathbf{s}) = \max_{x\in[0,m],b\in[0,1]} \Bigg\{ & \beta(1-b)\left[\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\omega)},\omega_{+1}\right)\right] \\ & + b\left[u \circ Q(x,b) + \beta\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\Bigg\}. \end{aligned} \tag{2.19}$$

It appears as if the buyer is choosing his matching probability b along with payment x. However this is just a change of variables utilizing the equilibrium relation (2.13) between quantity q and terms of trade (x, b). From this we can begin to see that there will be a trade-off to the buyer, in terms of an extensive margin (i.e., trading opportunity b), and, an intensive margin (i.e., how much to pay x).

The operator defined by (2.19) clearly does not preserve concavity: The objective function in (2.19) is not jointly concave in the decisions (x, b) and state m, since it is bilinear in the function b and the value function $\bar{V}$, or the flow payoff function u. However, we can still show that the resulting DM buyers' optimal choice functions for (x, b), denoted by $(x^\star, b^\star)$, are monotone, continuous, and have unique selections, using lattice programming arguments.

The following theorem summarizes the properties of a DM agent's value and policy functions:[14]

Theorem 2 (DM value and policy functions)**.** *For a given sequence of prices $\{\omega, \omega_{+1}, ...\}$, the following properties hold.*

1. *For any $\bar{V}(\cdot, \omega_{+1}) \in \mathcal{V}[0, \bar{m}]$, the DM buyer's value function is increasing and continuous in money balances, $B(\cdot; \omega) \in \mathcal{C}[0, \bar{m}]$.*

2. *For any $m \leq k$, DM buyers' optimal decisions are $b^\star(m, \omega) = x^\star(m, \omega) = q^\star(m, \omega) = 0$, and $B(m, \omega) = \beta \bar{V}[\phi(m, \omega), \omega_{+1}]$, where $\phi(m, \omega) := (\omega m + \tau) / [\omega_{+1}(1 + \tau)]$.*

3. *At any (m, ω), where $m \in [k, \bar{m}]$ and the buyer matching probability is positive $b^\star(m, \omega) > 0$:*

   (a) *The optimal selections $(x^\star, b^\star, q^\star)(m, \omega)$ and $\phi^\star(m, \omega) := \phi[m - x^\star(m, \omega), \omega]$, are unique, continuous, and increasing in m.*

   (b) *The buyer's marginal valuation of money $B_1(m, \omega)$ exists if and only if $\bar{V}_1[\phi(m, \omega), \omega]$ exists.*

   (c) *$B(m, \omega)$ is strictly increasing in m.*

   (d) *the optimal policy functions $b^\star$and $x^\star$, respectively, satisfy the first-order conditions*

$$\begin{aligned} u \circ Q[x^\star(m, \omega), b^\star(m, \omega)] + b^\star(m, \omega)\,(u \circ Q)_2[x^\star(m, \omega), b^\star(m, \omega)] \\ = \beta [\bar{V}(\phi(m, \omega), \omega_{+1}) - \bar{V}(\phi^\star(m, \omega), \omega_{+1})], \end{aligned} \quad (2.20)$$

and,

$$(u \circ Q)_1[x^\star(m, \omega), b^\star(m, \omega)] = \frac{\beta}{1 + \tau} \left(\frac{\omega}{\omega_{+1}}\right) \bar{V}_1[\phi^\star(m, \omega), \omega_{+1}]. \quad (2.21)$$

[14] Theorem 2 is a generalization of the observation of Menzio et al. (2013) in two aspects: (i) We have additional endogenous CM participation in our model; and (ii) the theorem extends beyond steady state equilibria to encompass equilibrium along a dynamic transition.

We prove these results in parts in the online appendix. Part 1 of the Theorem is obtained in Lemma 1, Part 2 is proven as Lemma 2. Part 3(a) is proven as Lemma 3. Lemmata 4 and 5 together establish Parts 3(b) and 3(c). Finally, Lemma 6 establishes Part 3(d).

2.4.4 Market clearing

Goods in CM. In equilibrium, the total production of CM good equals its demand:

$$Y = C \equiv U^{-1}(A).$$

Goods in DM. Given equilibrium policy functions, $x^\star$ and $b^\star$, and, equilibrium distribution of money G and wage ω, equation (2.13) pins downs market clearing for each submarket in the set of equilibrium submarkets $\{(x(m,\omega), b(m,\omega)) : m \in \text{supp}(G(\cdot,\omega))\}$.

Labor market clearing. The demand for labor from CM is Y since the firm has a linear production using labor. All other labor available in the economy must be absorbed by the firms operating in the DM submarkets. Let $z_i^j \equiv z_i^j(m,\omega)$ denote the possible lottery payout, where $\{1,2\} \ni i$ denotes the set of the low and the high prize, from playing a lottery $j \in J$, $\left(\pi_1^j, \pi_2^j\right)$, that spans an arbitrary given individual m at the beginning of the DM. Thus the measure of buyers at each $i = 1,2$ and each $j \in J$ is:

$$\mathscr{M}_b(i,j;\omega) = (1-\alpha)\,\pi_i^j\left(z_i^j;\omega\right).$$

Given the constant returns to scale matching function μ, the *outcome* of market tightness θ in a given submarket satisfies the restriction

$$\frac{b}{\mu(b)} = \frac{\mathscr{M}_s}{\mathscr{M}_b} \equiv \theta.$$

for any respective buyer and seller matching probability *outcomes*, b and $s = \mu(b)$. From this we can work out the measure of trading posts created in equilibrium to meet buyers with holdings z_i^j as

$$\begin{aligned}\mathscr{M}_s(i,j;\omega) &= \frac{b^\star\left(z_i^j;\omega\right)}{\mu\left[b^\star\left(z_i^j;\omega\right)\right]}\mathscr{M}_b(i,j;\omega) \\ &= \frac{b^\star\left(z_i^j;\omega\right)}{\mu\left[b^\star\left(z_i^j;\omega\right)\right]}\left(1-\alpha\right)\pi_i^j\left(z_i^j;\omega\right).\end{aligned}$$

Since a firm hires workers to create a trading post and to produce outcome q at each given submarket (x,q), its expected demand for labor is $k+c(q)\mu\left[b(x,q)\right]$. Applying the zero-profit condition (2.11), we also have that

$$k+c(\left[q^\star\left(z_i^j;\omega\right)\right]\mu\left[b^\star\left(z_i^j;\omega\right)\right]=x^\star\left(z_i^j;\omega\right)\mu\left[b^\star\left(z_i^j;\omega\right)\right],$$

where again, $(q^\star,b^\star,x^\star)$ are equilibrium Markovian policy functions.

Thus, total demand for labor, across all possible submarkets created in equilibrium is:

$$\sum_{j\in J}\sum_{i\in\{1,2\}}\mathscr{M}_s(i,j;\omega)x^\star\left(z_i^j;\omega\right).$$

Therefore, we can write the total demand for labor in the economy as:

$$\begin{aligned}L_D\left(\omega\right) &= Y+\sum_{j\in J}\sum_{i\in\{1,2\}}\mathscr{M}_s(i,j;\omega)x^\star\left(z_i^j;\omega\right) \\ &= Y+\left(1-\alpha\right)\sum_{j\in J}\sum_{i\in\{1,2\}}b^\star\left(z_i^j;\omega\right)\pi_i^j\left(z_i^j;\omega\right)x^\star\left(z_i^j;\omega\right).\end{aligned}\tag{2.22}$$

Define the indicator function for any agent $m\in\text{supp}(G(\cdot,\omega))$:

$$\mathbb{I}_{\{m,CM\}}:=\begin{cases}1 & \text{if } m \text{ goes to CM now} \\ 0 & \text{otherwise}\end{cases}.$$

On the labor supply side, we have that

$$L_S\left(\omega\right)=\int\mathbb{I}_{\{m,CM\}}l^\star(m,\omega)\mathrm{d}G(m;\omega)$$

Given the optimal labor supply of agents in the CM, $l^*(m;\omega)$ derived in (A.6), aggregate labor supplied can be re-written as

$$L_S\left(\omega\right)=\int\mathbb{I}_{\{m,CM\}}\left[\bar{C}^\star+\frac{\bar{y}^\star\left(\omega\right)+\tau}{1+\tau\omega}-m\right]\mathrm{d}G(m;\omega)\tag{2.23}$$

The equilibrium wage rate ω is determined by equating (2.22) and (2.23):

$$\frac{\tau}{1+\tau\omega} = \frac{1}{\int \mathbb{I}_{\{m,CM\}} \mathrm{d}G(m;\omega)} \left[Y + (1-\alpha) \sum_{j\in J} \sum_{i\in\{1,2\}} b^{\star}\left(z_i^j;\omega\right) \pi_i^j \left(z_i^j;\omega\right) x^{\star}\left(z_i^j;\omega\right) \right] - \left[\bar{C}^{\star} + \frac{\bar{y}^{\star}(\omega)}{1+\tau\omega} \right] + \frac{\int \mathbb{I}_{\{m,CM\}} m \mathrm{d}G(m;\omega)}{\int \mathbb{I}_{\{m,CM\}} \mathrm{d}G(m;\omega)}. \tag{2.24}$$

By Walras' Law, the requirement that all markets described above clear implies that money demanded must also equal money supplied:

$$\frac{1}{\omega} = \int m \mathrm{d}G(m;\omega) > 0. \tag{2.25}$$

Since M is the beginning of period aggregate stock of money in circulation, then the LHS of (2.25), $1/\omega = M \times 1/\omega M$, is the beginning of period real aggregate stock of money, measured in units of labor. The RHS of (2.25) is beginning of period aggregate demand, or holdings, of real money balances measured in the same unit.

2.5 Existence of a SME with unique distribution

For the rest of the paper, we focus on a stationary monetary equilibrium (SME), which comprises the characterizations from Section 2.4, where the sequence of prices are constant: $\omega = \omega_{+1}$.

Definition 1. A *stationary monetary equilibrium* (SME), given exogenous monetary policy τ, is a

- list of value functions $\mathbf{s} \mapsto (W, B, \bar{V})(\mathbf{s})$, satisfying the Bellman functionals: (2.5), (2.6), and jointly, (2.7)-(2.9);
- a list of corresponding decision rules $\mathbf{s} \mapsto (l^{\star}, y^{\star}, b^{\star}, x^{\star}, q^{\star}, z^{\star}, \pi^{\star})(\mathbf{s})$ supporting the value functions;
- a market tightness function $\mathbf{s} \mapsto \mu \circ b^{\star}(\mathbf{s})$ given a matching technology μ, satisfying firms' profit maximizing strategy (2.12) and (2.13) at all active trading posts;
- an ergodic distribution of real money balances $G(\mathbf{s})$ satisfying an equilibrium law of motion

$$T(G)(E) = \int P(\mathbf{s}, \mathcal{E}) \mathrm{d}G(\mathbf{s}) \qquad \forall E \in \mathcal{E} \tag{2.26}$$

  where, $\mathcal{E}$ is Borel σ-algebra generated by open subsets of the product state space S, and, $\mathbf{s} \mapsto P(\mathbf{s}, \cdot)$ is a Markov kernel induced by $(l^{\star}, x^{\star}, q^{\star}, z^{\star}, \pi^{\star})$ and $\mu \circ b^{\star}$under τ; and,

- a wage rate function $\mathbf{s} \mapsto \omega(\mathbf{s})$ satisfying labor market clearing (2.24), or equivalently, the money stock adding up condition (2.25).

Computationally, it will be easier to work with condition (2.25) instead of (2.24). At this point, we note that it will not be difficult to show that there is a unique distribution of agents in a SME. However, whether a SME is unique remains elusive to us as the frequency function $\mathrm{d}G(m;\omega)$ does not admit a closed form expression in terms of known functions, and in general, it will also depend on the equilibrium candidate ω. [15]

The following theorem ensures that in our computations below there exists a steady state, stationary monetary equilibrium, and for each steady state equilibrium ω, there is a unique distribution of agents.

Theorem 3. *There is a steady-state SME with a unique nondegenerate distribution G.*

Proof. First, we show that the value functions listed in the definition of a SME are unique given ω. For given ω, The CM agent's problem in (2.5) clearly defines a self-map $T_{\omega}^{CM} : \mathcal{V}[0,\bar{m}] \to \mathcal{V}[0,\bar{m}]$, which preserves monotonicity, continuity and concavity (see Theorem 1). However, for fixed ω, the DM buyer's problem in 2.19 defines an operator $T_{\omega}^{DM} : \mathcal{V}[0,\bar{m}] \to \mathcal{C}[0,\bar{m}]$, where $\mathcal{C}[0,\bar{m}] \supset \mathcal{V}[0,\bar{m}]$ is the set of continuous and increasing functions on the domain $[0,\bar{m}]$. This operator does not preserve concavity. Note that $\bar{V}(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$ as previously defined. Now we show that the ex-ante problem in (2.9) and (2.8) defines an operator that maps the CM agent's and the DM buyer's value functions, respectively, $W(\cdot,\omega) = T_{\omega}^{DM}\bar{V}(\cdot,\omega)$ and $B(\cdot,\omega) = T_{\omega}^{DM}\bar{V}(\cdot,\omega)$, back into the set of continuous, increasing and concave functions: $T_{\omega} : \mathcal{V}[0,\bar{m}] \to \mathcal{V}[0,\bar{m}]$. Since

[15]This statement is also true for the original Menzio et al. (2013) setting, if the authors' model had money supply growth. The intractability of their version of the frequency function $\mathrm{d}G(m;\omega)$ under money supply growth comes about from the modeller no longer being able to work out analytically how long it will take for DM-unmatched buyers' balances to get eroded by inflation, before they have to go to work again. In contrast, the variation in Sun and Zhou (2016) admits an analytical form for $\mathrm{d}G(m;\omega)$ and as a result they can show that there is a unique SME. This special result arises from their assumption that all types of agents in the DM must deterministically enter the CM *after one round* of trade (or no trade) in the DM, so that the total labor demand in their model—i.e., their counterpart of equation (2.24)—can be analytically described by a composition of equilibrium decision functions with well-behaved properties and an assumed exogenous distribution of CM preference shocks. In their model, without an exogenous distribution of CM preference shocks, given the market timing assumptions, there would be no distribution of agents since preferences are quasilinear in their CM.

Our setting yields a modelling trade-off in the opposite direction: In contrast to Sun and Zhou (2016), we do not require the latter assumption to preserve distributional non-degeneracy. However, our relaxation here would come at an analytical cost on the form of the frequency function $\mathrm{d}G(m;\omega)$. In our opinion, the loss of tractability in this respect is not too severe: Our equilibrium characterization remains computational feasible. In fact, it retains the feature that agents' decision rules depend on the aggregate state only insofar as the scalar aggregate variable, ω. This is because, unlike heterogeneous-agent neoclassical growth models or random matching models, the market clearing conditions in competitive search do not require the conjecture of an entire distribution of assets in order to pin down terms of trade. In that sense, our algorithm for finding a SME will be similar to that used for computing neoclassical heterogeneous-agent models. In fact, with aggregate shocks (e.g., to τ) our setting will also imply an accurate application of the (originally heuristic) Krusell and Smith (1998) algorithm to an exact problem. (A similar point was previously discussed in Menzio et al. 2013, pp.2294-2295.)

T_ω^{CM} and T_ω^{DM} are monotone functional operators that satisfy discounting with factor $0 < \beta < 1$, then the ex-ante problem in (2.9) and (2.8), which defines the composite operator $T_\omega : \mathcal{V}[0, \bar{m}] \to \mathcal{V}[0, \bar{m}]$, clearly preserves these two properties. (The convexification of the graph of T_ω via lotteries in (2.8) preserves concavity of the image of the operator, thus making it a self-map on $\mathcal{V}[0, \bar{m}]$.) It can be shown that $\mathcal{V}[0, \bar{m}]$ is a complete metric space. Thus $T_\omega : \mathcal{V}[0, \bar{m}] \to \mathcal{V}[0, \bar{m}]$ satisfies Blackwell's conditions, and has a unique fixed point, $\bar{V}(\cdot, \omega) = T_\omega \bar{V}(\cdot, \omega)$, by Banach's fixed point theorem.

Second, we verify the following three properties: (1) By Theorem 1 and Theorem 2, the agent's optimal policies are continuous, single-valued and monotone functions. This implies, for fixed ω, that the Markov kernel $P(\mathbf{s}, \cdot)$ in the distributional operator (2.26) is a probability measure, and, $P(\cdot, \mathcal{E})$ for all Borel subsets $\mathcal{E} \in \mathcal{B}([0, \bar{m}])$ is a measurable function. (2) Since agent's policies are monotone, then $P(\mathbf{s}, \cdot)$ is increasing on $[0, \bar{m}]$. Thus the Markov kernel is a transition probability function. (3) The equilibrium policies clearly dictate that the monotone mixing conditions of Hopenhayn and Prescott (1992) are satisfied: Consider a DM buyer who has zero money balance. With non-zero probability either by pure luck (α) or by choosing a lottery that induces such an outcome, he will enter the CM to work and to accumulate some positive money balance. Likewise, consider an agent, either in the DM or CM with the highest possible initial balance of $\bar{m}$. Again, with non-zero probability, she will decumulate that balance, either by matching and spending that balance down in the DM, or, by working less and consuming more in the CM. These conditions, are sufficient, by Theorem 2 of Hopenhayn and Prescott (1992), for the Markov operator (2.26) to have a unique fixed point—i.e., regardless of an initial distribution of agents, the recursive operation on the initial distribution converges (in the weak* topology) to the same long run distribution G.[16]

Third, the market clearing condition (2.25) is continuous on the RHS: (1) The integrand is clearly continuous in m; and, (2) the distribution $G(\cdot; \omega)$ is continuous in ω in the sense of convergence in the weak* topology (Stokey and Lucas, 1989, Theorem 12.13)—i.e., if $\omega_n \to \omega^\star$, then for each $\omega_n \in \{\omega_n\}_{n \in \mathbb{N}}$, the Markov operator (2.26) defines a (weakly) convergent sequence of distributions: $G(\cdot; \omega_n) \to G(\cdot; \omega^\star)$. The LHS of (2.25) is clearly continuous in ω. Thus a SME exists. □

3 Computational equilibrium analyses

Finding a SME requires numerical computation. In this section, we parametrize the model and then we illustrate the dynamic behavior of an individual agent in a economy in the case where we have positive inflation. We will pause to discuss the underlying

[16] Alternatively, one could check the more relaxed set of necessary and sufficient conditions of Kamihigashi and Stachurski (2014, Theorem 2) to guarantee that there is a unique distribution for a given ω, in a steady state SME.

forces and trade-offs at work that help will help us to understand the SME outcomes. This will also help guide our understanding at the end of this section, where we provide some comparative SME analyses in terms of allocative, distributional and welfare outcomes.

3.1 Parametrization

The CM and DM preference functions are, respectively,

$$U(C) = \bar{U}_{CM} \frac{(C + c_{min})^{1-\sigma_{CM}} - (c_{min})^{1-\sigma_{CM}}}{1 - \sigma_{CM}}, \text{ and, } u(q) = \frac{(q + q_{min})^{1-\sigma_{DM}} - (q_{min})^{1-\sigma_{DM}}}{1 - \sigma_{DM}}.$$

Without loss, we set $c_{min} = q_{min} = 0.001$ and normalize $\sigma_{DM} = 1.01$. The matching function is such that $\mu(b) = 1 - b$. All the parameters of the model are listed in Table 1.

Table 1: Baseline parameterization

Parameter	Value	Unit	Description
β	0.99	-	Quarterly interest rate of $1/\beta - 1 \approx 1\%$
σ_{CM}	2	-	CM risk aversion
σ_{DM}	1.01	-	Normalized DM risk aversion
A	1	-	Scale on labor disutility function
$\bar{m}$	$0.95 \times U^{-1}(A)$	labor	See assumption (2.3)
χ	$0.1 \times \bar{m} = 0.095$	labor	Fixed cost of CM participation
α	0	-	Probability of costless CM entry
k	0.05	labor	Fixed cost of creating a trading post
$\bar{U}_{CM}$	0.002	-	CM preference scale parameter

Caveat emptor: We plan to estimate some key parameters via the method of simulated moments (SMM) later. For now, Table 1 contains an ad-hoc parametrization of the model.

3.2 A novel computational scheme

We solve for a steady state SME following the pseudocode in our Online Appendix D. Our solution method uses a novel insight that refines the computation of the value of the lottery problem. Recall that the directed search problem makes the ex-ante value function $\tilde{V}(\cdot, \omega)$ non-concave. Since there may exist lotteries that make agents better off than playing pure strategies over participating in DM (as buyer) or CM (as consumer/worker), we have to devise a means for finding these lotteries that convexify the graph of the function $\tilde{V}(\cdot, \omega)$.[17]

[17] Interestingly, there is parallel similarity between our problem here and those in computational dynamic games. In the latter, non-convexities may sometimes arise in equilibrium payoff sets, and convexification of these payoff correspondence images are required as a consequence of public randomization, instead of lotteries or behavior strategies (see, e.g., Kam and Stauber, 2016).

An existing way to do this in the literature is to use a grid $\mathrm{M}_g := \{0 < \cdots < \bar{m}\}$ to approximate the function's original domain of $[0, \bar{m}]$. Then, around each finite element of M_g, one must check if there is a linear segment that *approximately* convexifies graph $[\tilde{V}(\cdot, \omega)]$.[18] This approximation scheme works fine when we only have a lottery where the lower bound in M_g is included, i.e., a lottery on a set like $\{z_1, z_2\}$, where $z_1 = 0$, and, $z_2 < \bar{m}$. It becomes less accurate when lotteries may exist on upper segments of the function, i.e., lotteries on sets like $\{z_1', z_2'\}$, where $0 < z_1' < z_2' < \bar{m}$, but we have no prior knowlege of what z_1' is. This is because in practice (on the computer) it is not feasible to implement this check which is meant to be done at every element on the domain $[0, \bar{m}]$, not its approximant M_g. As a result, its implementation on M_g may be prone to introducing non-negligible approximation errors, especially when the mesh of M_g is coarse. Thus, one would have to make M_g very fine, but, this will come at the cost of the overall SME solution time.

Instead, we propose a novel alternative here. We can exploit the property that $\tilde{V}(\cdot, \omega)$ has a bounded and convex domain, so then there exists a smallest convex set that contains $g\tilde{V} :=$ graph $[\tilde{V}(\cdot, \omega)]$, i.e., conv $(g\tilde{V})$,. This set is indeed graph $[V(\cdot, \omega)]$. We utilize `SciPy`'s interface to the fast `QHULL` algorithm to back out a finite set of coordinates representing the convex hull, i.e., graph $[V(\cdot, \omega)]$. Given these points, we approximate the function $V(\cdot, \omega)$ by interpolation on a chosen continuous basis function. We use the family of linear B-splines available from `SciPy`'s `interpolate` class for this purpose. As a residual of this exercise, we can very quickly and directly determine the entire set of possible lotteries that exists with minimal loss of precision, for any given non-convex/concave function $\tilde{V}(\cdot, \omega)$. Detailed comments on how this is done can be found in the method `V` in our `Python` class `cssegmod.py`.[19]

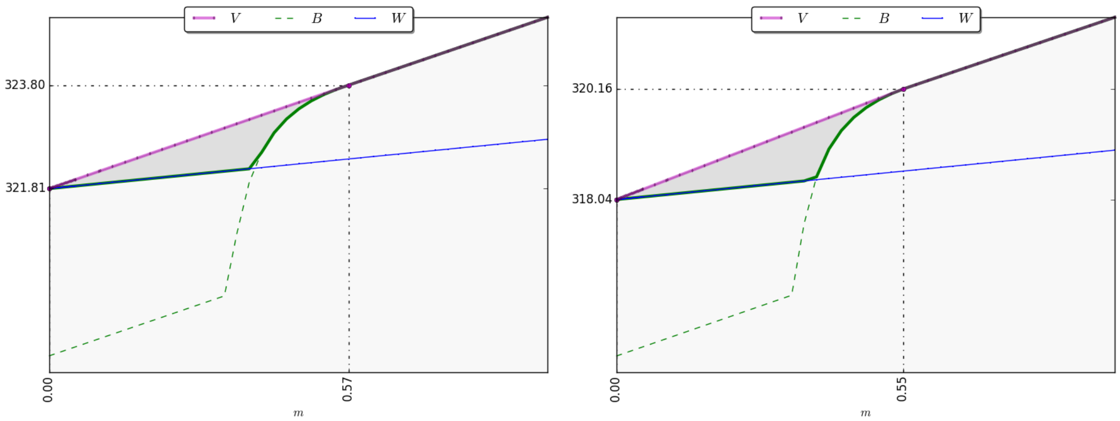
3.3 Example SME by simulation

We provide two examples of the previous discussion in Figure 1 on the next page. In Figure 1(a), we plot the SME value functions (V, B, W) when $\tau = 0$ (i.e., zero steady state inflation rate). In Figure 1(b), we have the SME value functions (V, B, W) when $\tau = 0.1$ (i.e., 10% steady state inflation rate). In both instances of SME, the algorithm finds that only one lottery segment exists. The solid blue line is the graph of $W(\cdot, \omega)$. The dashed green line is the graph of $B(\cdot, \omega)$. The upper envelop of these two graphs give us $\tilde{V}(\cdot, \omega)$, the

[18] See part (v) of the proof of Theorem 3.5 in Menzio et al. (2013) for an exact theoretical underpinning of this scheme. We thank Amy Sun for sharing her `MATLAB` code for Menzio et al. (2013) which confirms this usage.

[19] We implement our solution in pure `Python` 2.7/3.4 (with OpenMPI parallelization of the agent decision problems on 20 logical cores). We have only tested our solutions on a Dell Precision T7810 workstation (with Intel Xeon E5-2680 v3, 2.50GHz, processors) running on the Centos 7.2 GNU/Linux operating system. In all our experiments, we have monotone convergence towards a unique SME solution, regardless of initial guesses on ω and $\tilde{V}(\cdot, \omega)$. The average time taken to find the SME is between 90 to 120 seconds, given our hardware setting.

thick solid green line. Denote conv $\{\cdot\}$ as the convex-hull set operator. The solid magenta graph is the graph of $V(\cdot, \omega)$ obtained through our convex-hull approximation scheme, once we have located all the intersecting coordinates between the set graph $\left[\tilde{V}(\cdot, \omega)\right]$ and the upper envelope of the set conv $\left\{\text{graph}\left[\tilde{V}(\cdot, \omega)\right], (0,0), (\bar{m}, 0)\right\}$.



(a) $\tau = 0$ (0% inflation)

(b) $\tau = 0.1$ (10% inflation)

Figure 1: Value functions from two economies.

Sustaining these equilibrium value functions are the policy functions $(l^{\star}, b^{\star}, x^{\star}, q^{\star})$ and the lottery policy with prizes (z_1, z_2) and lottery $(\pi_1, 1 - \pi_1)$ over the prize support, where $\pi_1(m, \omega) = (z_2 - m) / (z_2 - z_1)$. For example, comparing the cases in Figure 1, when $\tau = 0.1$, the high prize of $z_1 = 0.55$ is smaller than its counterpart of 0.57 when $\tau = 0$. Higher anticipated inflation reduces the lottery payoffs since the real value of money balances get eroded by higher inflation. Also, as expected, higher anticipated inflation will shift the value functions down uniformly.

The other policy functions can be seen in Figure 2 on the following page. Consider the panel depicting the graph of the CM labor supply function. As shown earlier in (2.18), labor supply is affine and decreasing in money balance. There are two shaded patches in the Figure's panels. The earth-red patch corresponds to the region where $m \in [0, k)$, i.e., an agent will never match nor trade in the DM. The orange patch (which overlaps the earth-red patch) is the region of the agent's state space in which a lottery may be played—i.e, $[z_1, z_2] \ni m$, where in this instance, $z_1 = 0$. What matters for each agent in the SME is then the loci of these policy functions outside of the orange patch, but including the points on its boundary. These will be consistent with the equilibrium's ergodic state space of agents. As proven in Theorem 2, the policy functions $(b^{\star}, x^{\star}, q^{\star})$ are monotone in m in the relevant subspace where an agent can exist at any point in time. The relevant ergodic subspace of $[0, \bar{m}]$ in equilibrium is given by $\{z_1, [z_2, \bar{m}]\} = \{0, [0.55, \bar{m}]\}$ in the example of $\tau = 0.1$ in Figure 1(b) or Figure 2.

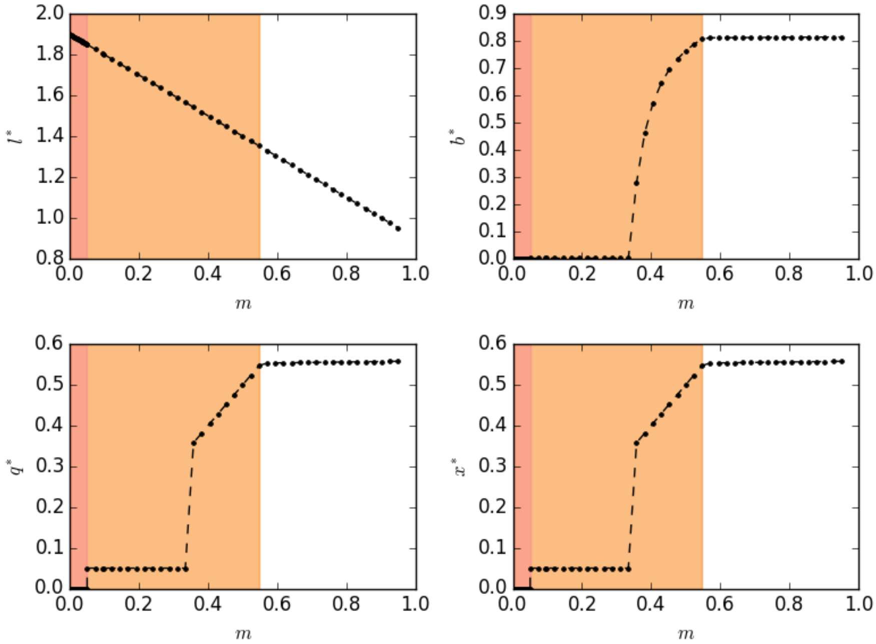


Figure 2: Markov policy functions with $\tau = 0.1$

Given the information about a particular SME's active or relevant agent state space, and, the corresponding policy functions, we can use Monte Carlo methods to simulate an agent's outcomes. To do so, one may begin from any initial agent named (m, ω) and apply the decision rules computed earlier, as in Figure 2. Details of how this is done can be found in our Online Appendix E. This is also relevant for constructing an approximation of the SME's probability distribution over agent states. In our solution method, we simulate an agent over $T = 10^4$ periods to approximate the SME's ergodic distribution.

Figure 3 on the next page shows a subsample of an agent's existence, for the baseline economy with monetary policy $\tau = 0.1$. Corresponding to the DM/CM patterns of spending, we can also observe the subsample's evolution of money balances, in the panel with its vertical axis labelled m, in Figure 3. Here, we can see that at $t = 0$, the agent has his initial real balance as some $m \approx 0.87$. He decides be in the DM, succeeds in matching with a trading post, and spends a fraction of the balance to consume some positive q. In the following period $t = 1$, he begins with some positive balance—because of transfer $\tau/(\omega(1+\tau)) > 0$ combined with his residual balance—but this amount land in the lottery region; and so the agents plays the lottery. He realizes the high prize of $z_2 = 0.57$ in $t = 1$, and so his money balance is z_2. He matches and gets to consume $q > 0$. (Hence, the record $q_1, x_1, b_1 > 0$.) A similar event realizes again in $t = 2$, so the agent again gets to consume in the DM. In $t = 3$, having spent his balance on consuming in the DM the previous period, the agent realizes a low, i.e., $z_1 = 0$, lottery payoff and his initial balance is thus zero. However, the agent is able to borrow against his CM income, and thus decides to take a temporary short asset position of $-\chi$ (although his recorded money balance is $m = z_1 = 0$) and enters the CM to work, repay the entry cost, consume in the CM, and save some money balance.[20] That is why we see a record of -1 for the figure panel labelled “match status” for $t = 3$. Subsequently in $t = 4$, he begins again with positive balance from the last CM trade. At this point, he decides to go shopping in the DM and again, spends it all in one round. he wins the high prize in the lottery, and finds it profitable to pay the fixed cost χ, enters the CM and works.

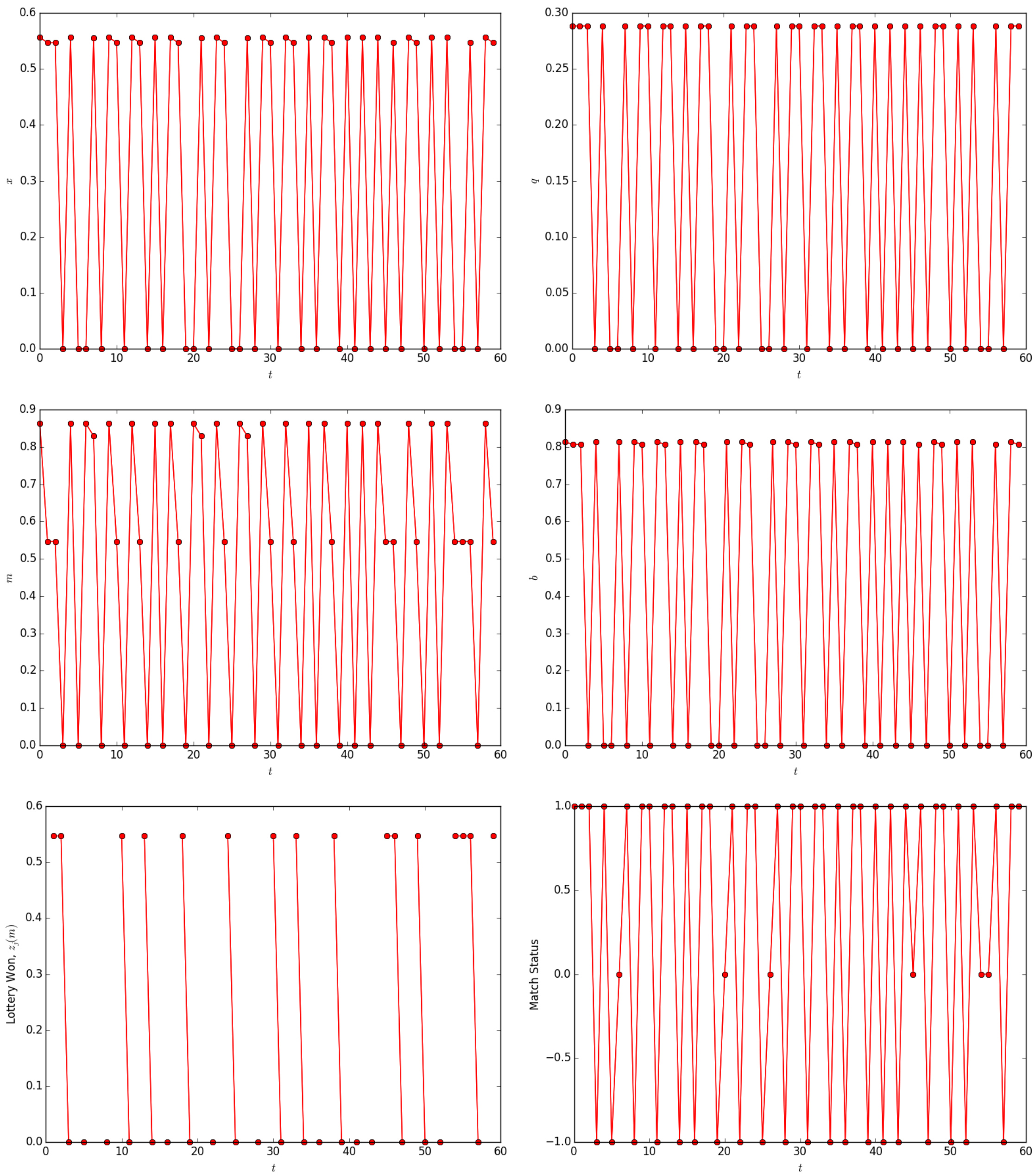
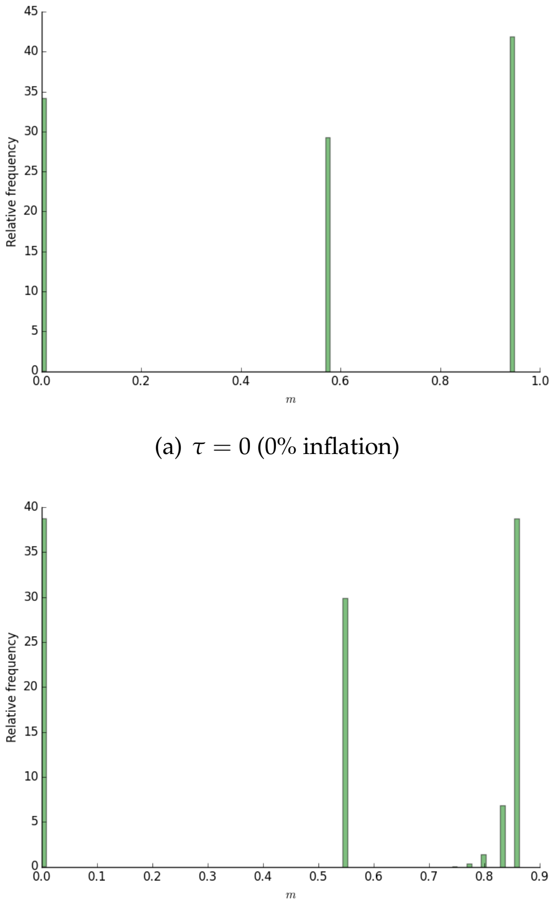


Figure 3: Agent sample path ($\tau = 0.1$). Match Status: 0 (No Match in DM), 1 (Match in DM), -1 (in CM).

[20] See our earlier Remark 1 on page 11.

In summary, we can observe the following about this example economy: Agents can trade more than once in the DM sometimes. This depends on their luck of the draw in their lottery decisions. Agents must also ocassionally pay a fixed cost to enter the CM to load up on money balances. Depending on their money balance, they may sometimes find it worthwhile to borrow against their CM income to pay the fixed cost of CM entry. Thus, we have an equilibrium Baumol-Tobin style of money demand in the model. Because agents endogenously may not have complete consumption insurance, the pattern of consumption, for example, DM q in Figure 3 on the previous page, is not completely smooth. (While not shown, the same would occur in terms of CM consumption.)

The long run distribution of the sample path of m is shown in Figure 4, for the two cases of $\tau = 0$ and $\tau = 0.1$. Observe that in each case, the agents, in terms of money balance m can only spend time in the equilibrium's ergodic subspace of the set of money holding. For example this equilibrium subspace turns out to be $\{z_1, [z_2, \bar{m}]\} = \{0, [0.55, \bar{m}]\}$, when $\tau = 0$. Comparing between the stationary distribution of SME$(\tau = 0)$ in Figure 4(a) and SME$(\tau = 0.1)$ in Figure 4(b), we see that when inflation is higher, more individuals get stuck with zero balances but also, more agents tend to stay at the top end of the distribution. Comparing this with the sample path of m from Figure 3 on the preceding page, we can see that with higher inflation, an individual tends to stay for more periods in the CM, which explains the additional measures of people near the natural upper bound on money balances in the money distribution in Figure 4(b), in contrast with Figure 4(a).



(a) $\tau = 0$ (0% inflation)

(b) $\tau = 0.1$ (10% inflation)

Figure 4: Money distribution under different equilibrium inflation rates

3.4 Mechanism tear-down

We would like to explain the mechanism behind the previous simulation observations, and, behind the welfare and redistributive consequences of inflation to be shown later. Since the equilibrium solution is non-analytic, we can at least identify the opposing forces underlying the effect of inflation on a corresponding SME.

CM-participation intensive vs. extensive margins. Positive inflation has the following trade-offs: On on hand, with inflation individuals would like to visit the CM more frequently to work and consume there (since in the CM money is not needed for exchange).

On the other hand, given a real fixed cost $\chi > 0$ of entering the CM, higher inflation means that low-balance agents in the DM will be more unlikely to pay χ to enter the CM. This is because of two possibilities: (i) their natural short-sale constraints in (2.9) may be violated if inflation is too high, i.e., $m - \chi < -y_{max}(\omega; \tau) < 0$, and so they choose to stay in the DM and are more likely to keep realizing a bad draw of the zero balance lottery prize; or (ii) their short-sale constraints in (2.9) are not binding, but the value of going to CM, $W(m-\chi,\omega)$ is still dominated by the value of going to the DM, $B(m,\omega)$. Nevertheless, in order to continue deriving consumption value in the DM, an agent would also need to ensure that he has sufficient balance to pay to go back to the CM often enough to maintain enough precautionary saving of money.

So these trade-offs imply two margins for a precautionary motive for agents with respect to incomplete consumption insurance: Either they work harder each time in the CM and bear the cost of holding excess money balances (*intensive labor-CM margin*), or, they work less in each CM instance, reduce their spending in each DM exchange, and ensure that they are more likely to be able to afford to go to the CM frequently (*extensive labor-CM margin*).

DM-specific intensive vs. extensive margins. There is another trade-off with respect to the DM not present in standard general equilibrium models of money, or, in random matching models. Consider the equilibrium description of firms' optimal behavior in relation to DM production and profit maximization (2.13). Given the firms' best response in a SME, we can deduce the following about a potential DM buyer: $Q_1(x,b)>0$, $Q_2(x,b)<0$, $Q(x,b)$ is weakly concave, and $Q_{12}(x,b)=0$. In words, we have another tension here: On one hand, faced with a given probability of matching with a trading post, b, the more a buyer is willing to pay, x, the more q she can consume. (This is the *intensive margin* of DM trade—i.e., how much one can purchase.) On the other hand, given a required payment, x, a buyer who seeks to match with higher probability, b, must tolerate eating less q (This is the *extensive margin* of DM trade—i.e., trading opportunities.)

From Theorem 2, we know that if a DM buyer brings in more (less) money balance every period, then x will be higher (lower) and b will be higher (lower). But the tension just outlined above gives an ambiguous resolution on q. Thus the intensive (x or q) margin faces a countervailing force in the intensive margin (b).

All together now. We now consider some numerical results to resolve the *CM-particpation* and the *DM-specific* intensive-versus-extensive tensions, in the face of higher inflation.

Let us begin with the red-cross graphs in the panels in Figure 5, from left to right, and, top to bottom. These correspond to economies with the natural borrowing limit $y_{\max}(\omega;\tau)$ as defined in (2.10). (The purple-square graphs correspond to alternative economies with an ad-hoc zero borrowing constraint assumption, where we set $y_{\max}(\omega;\tau)=0$ as in Aiyagari-Huggett type economies.) On the horizontal axes of the panels, we are increasing the steady-state inflation rate, $\tau\in(\beta-1,0.1]$. On the vertical axes, we measure the average outcomes for each corresponding economy's equilibrium allocation under policy τ; we denote each equilibrium as SME(τ). For agents who participate in the CM to accumulate money wealth, higher anticipated inflation τ will tend to reduce their labor effort whilst increasing their CM participation rate (i.e., the *CM-participation extensive margin* dominates), and they tend to spend (x) and consume (q) less, conditional on finding a match with a trading post, in favor of increasing the likelihood of trading in the DM (b)—i.e., the *DM extensive margin* dominates. Because less labor is supplied to firms in each market, the equilibrium nominal wage per total stock of money (i.e., normalized nominal wage), ω, increases with τ.

In terms of aggregate and distribution outcomes, higher τ results in lower aggregate money holdings. In Figure 6, both mean and median money holdings fall with inflation. Inflation tends to increase wealth inequality (Gini index rises) despite reducing the left-skewness of the money wealth distribution. Note that higher anticipated inflation affects agents unequally since they are risk aversion: Higher inflation means relatively lower (higher) marginal valuation of money to high (low)-balance agents. On one hand, higher inflation (through both the the *CM-* and the *DM-intensive margin*) provides a redistribution of wealth from high-money-balance agents to low-money-balance agents. Hence, the less (negative) skewness of the money distribution as inflation rises. Other the hand (through both the *CM-* and the *DM-extensive margin*), inflation raises the downside risk of agents not getting matched in subsequent trade rounds, and, also raises the cost of not being able to participate in CM as much, and these two effects imply higher-balance agents tend to trade "faster" in DM sub-markets by spending and consuming less in each, and exiting the DM faster. The results shown in Figure 6 suggest that the *extensive margins dominate*. The less negative skewness of the distribution under higher inflation, for example, suggests that the *extensive* margin is offsetting the rich-to-poor redistributive aspect of inflation through the *intensive* channels.

Now we can contrast the economies with the natural borrowing limit with their counterparts with a zero-borrowing limit assumption. Respectively, these are the red-crossed versus the purple-crossed graphs in Figures 5 and 6. Interestingly, in the economies where we have imposed the ad-hoc borrowing limit, agents tend to spend less time in the DM (and more time in the CM), and work less in the CM. In equilibrium, because agents know they can borrow against their future CM income to pay for the cost of entering the CM, they choose to work less—or equivalently, hold less money as precaution for its use in the DM—but as a result, they get much lower consumption in equilibrium. As a result average welfare is lower in economies with the no-borrowing assumption, but, money-balance inequality is also lower.

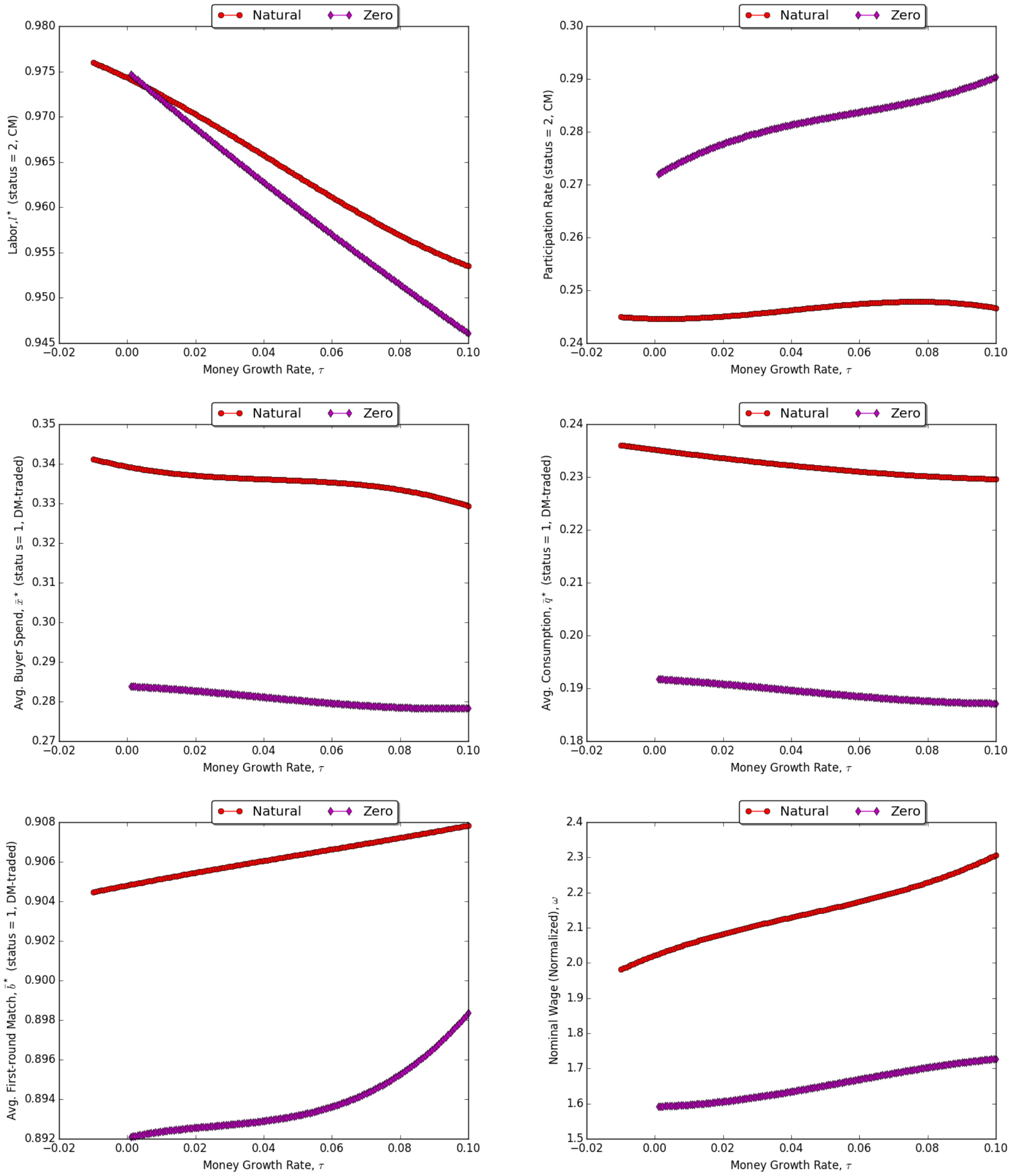


Figure 5: Comparative steady states — allocations

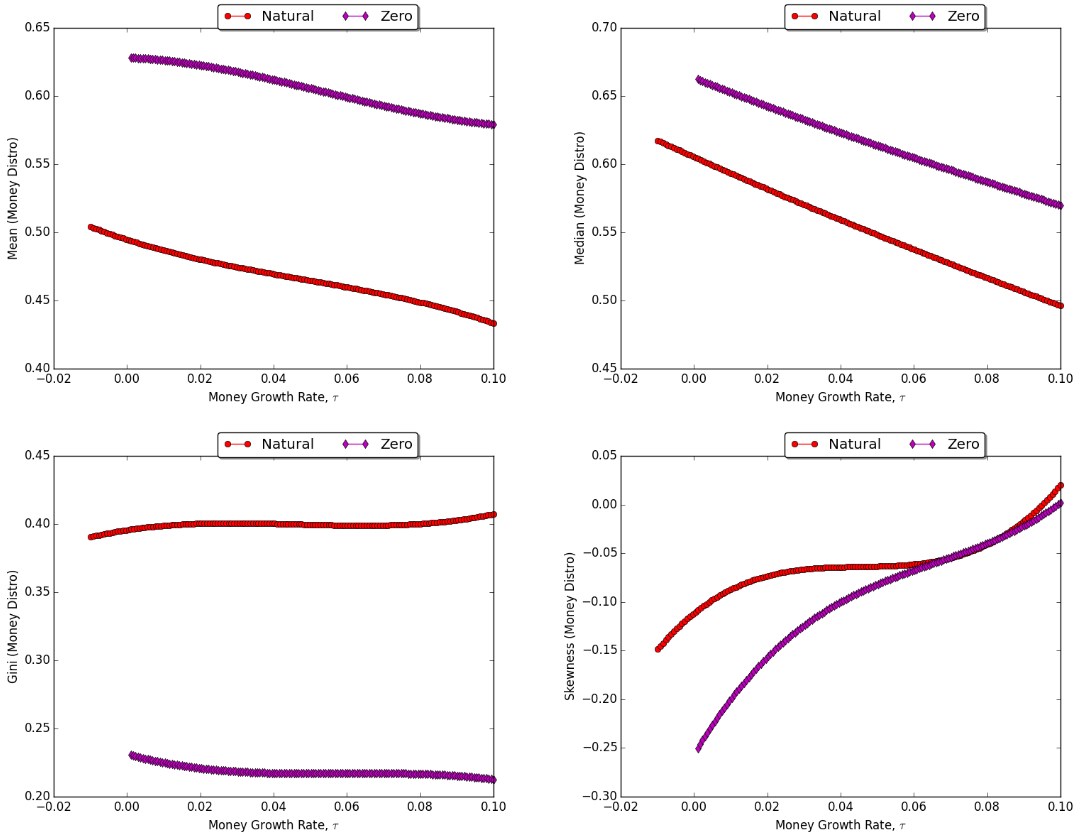


Figure 6: Comparative steady states — distribution

3.5 Welfare cost of inflation

From the last comparative steady-state exercise, we can also work out the aggregate welfare cost of inflation. For example, to measure mean welfare we consider the ex-ante mean value

$$Z_\tau := \int \bar{V}(m, \omega; \tau) \mathrm{d}G(m, \omega; \tau).$$

For a chosen reference economy, SME(τ_0) with ex-ante average value as Z_{τ_0}, for any other SME(τ) we define a welfare measure as compensating equivalent variation in CM consumption measure:

$$CEV\left(\tau | \tau_0\right) = \left[\frac{U^{-1}\left(Z_\tau\right)}{U^{-1}\left(Z_{\tau_0}\right)} - 1\right] \times 100\%.$$

We plot this welfare measure across each SME with increasing long run inflation rates. From Figure 7, we see that the mean welfare measure attain an optimum for some small, negative inflation rate. This is away from the usual Friedman rule that says optimal inflation is at $\beta - 1$, because of the additional market incompleteness arising for agents not being able to trade in the CM completely and also not being able to trade all the time in the DM.

Consider the left-hand panel of Figure 7. In this thought experiment, we show the mean welfare cost of inflation for two settings: natural-borrowing (red-crosses) and zero-borrowing (purple-squares) limit economies. In the zero-borrowing economies, inflation reduces welfare more than the natural borrowing-limit economies. The intuition is obvious: In the former, agents are worse off since they face an additional no-borrowing constraint.

Now consider the right-hand panel of Figure 7. In this thought experiment, we contrast an economy where the fixed cost of CM participation, χ, is zero, with a setting where $\chi = 0.3$. The outcomes are depicted as the blue-circle graph for the former, and as green-diamond graphs for the latter. A higher fixed cost of CM participation would amplify the extensive margins outlined earlier. As a consequence, the graphs depicting CM market participation and average matching probability, for each SME(τ), shift down with higher χ. Also average consumption, q, and spending, x, in the DM are also lower with χ, since agents find it more costly to enter the CM, and so must economize on their money holdings. As a consequence, aggregate money balances shift down with χ. The effect on wealth inequality is not so clear.

Overall, the welfare statistic tends fall with positive inflation. For the baseline parametrization of the model (red-squares), relative to the reference economy at $\tau_0 \searrow \beta - 1$, would cost welfare in terms of a reduction in equivalent consumption by about 0.2 percent a quarter, or 0.8 percent per annum. This cost rises to about 1.4 percent per annum if the fixed cost were $\chi = 0.3$. Or, this welfare cost rises to about 1.6 percent per annum if agents cannot borrow against their CM incomes to participate in the CM.

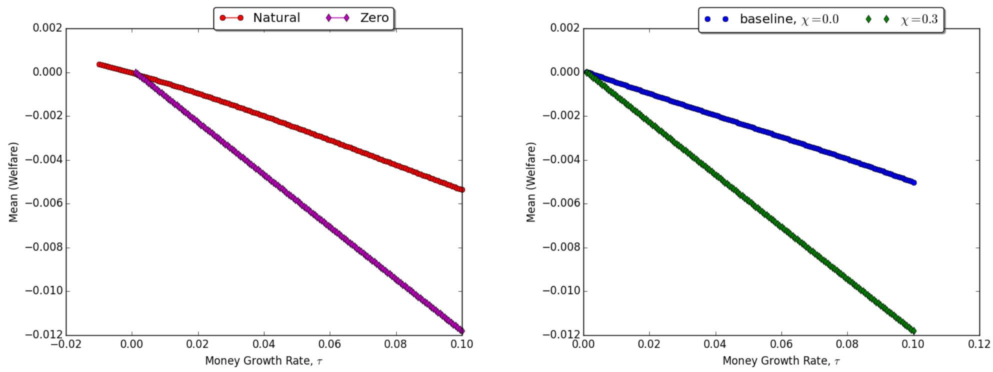


Figure 7: Comparative steady states — welfare

4 Conclusion

In companion projects (Kam and Lee, 2016; Kam et al., 2016), we explore the dynamics of the model and also study unanticipated monetary policy shocks.

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Online Appendix

Inflationary Redistribution vs. Trading Opportunities

Omitted Proofs and Other Bits and Bobs

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A CM individual's problem

The following gives the proof of Theorem 1 on page 13 in the paper.

Theorem 1. Assume $\tau/\omega < \bar{m}$. For a given sequence of prices $\{\omega, \omega_{+1}, ...\}$, the value function of the individual beginning in the CM, $W(\cdot, \omega)$, has the following properties:

1. $W(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$, i.e., it is continuous, increasing and concave on $[0, \bar{m}]$. Moreover, it is linear on $[0, \bar{m}]$.

2. The partial derivative functions $W_1(\cdot, \omega)$ and $\bar{V}_1(\cdot, \omega_{+1})$ exist and satisfy the first-order condition

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega y^{\star}(m,\omega)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\begin{cases}\leq A, & y^{\star}(m,\omega) \geq 0\\ \geq A, & y^{\star}(m,\omega) \leq \min\left\{\bar{m} - \frac{\tau}{\omega}, \bar{m}\right\}\end{cases}, \tag{A.1}$$

   and the envelop condition:

$$W_1(m,\omega) = A, \tag{A.2}$$

   where $y^{\star}(m,\omega) = m + l^{\star}(m,\omega) - C^{\star}(m,\omega)$, $l^{\star}(m,\omega)$ and $C^{*}(m,\omega)$, respectively, are the associated optimal choices on labor effort and consumption in the CM.

3. The stationary Markovian policy rules $y^{\star}(\cdot, \omega)$ and $l^{\star}(\cdot, \omega)$ are scalar-valued and continuous functions on $[0, \bar{m}]$.

   (a) The function $y^{\star}(\cdot, \omega)$, is constant valued on $[0, \bar{m}]$.

   (b) The optimizer $l^{\star}(\cdot, \omega)$ is an affine and decreasing function on $[0, \bar{m}]$.

   (c) Moreover, for every (m, ω), the optimal choice $l^{\star}(m,\omega)$ is interior: $0 < l_{\min} \leq l^{\star}(m) \leq l_{\max}(\omega;\tau) < +\infty$, where there is a very small $l_{\min} > 0$ and $l_{max}(\omega) := \min\left\{\bar{m} - \frac{\tau}{\omega}, \bar{m}\right\} + U^{-1}(A) < 2U^{-1}(A) \in (0, \infty)$.

Proof. (*Part 1*). The individual's problem beginning in the CM (2.5) is:

$$W(\mathbf{s}) = \max_{(C,l,y)\in\mathbb{R}^3_+}\left\{U(C) - Al + \beta\bar{V}(\mathbf{s}_{+1}) : pC + y \leq m + l,\ m_{+1} = \frac{\omega y + \tau}{\omega_{+1}(1+\tau)}\right\}.$$

Since $U_1(C) > 0$ for all C, the budget constraint always binds. Thus we can re-write (2.5) as

$$W(\mathbf{s}) = \max_{(C,y)\in\mathbb{R}_+\times[0,\bar{m}]} \left\{ U(C) - A\left[pC + y - m\right] + \beta\bar{V}\left(\frac{\omega y + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \right\}. \quad \text{(A.3)}$$

Let

$$(C^\star, y_c^\star)(m,\omega) \in \underset{(C,y)\in\mathbb{R}_+\times[0,\bar{m}]}{\arg\max} \left\{ U(C) - A\left[pC + y - m\right] + \beta\bar{V}\left(\frac{\omega y + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \right\}. \quad \text{(A.4)}$$

From (A.3), it is clear that $W_1(\cdot,\omega)$ exists on $[0,\bar{m}]$, and moreover, we have the envelope condition $W_1(\cdot,\omega) = A > 0$. This implies that the value function $W(\cdot,\omega)$ is continuous, increasing and concave in m. Moreover it is affine in m.

(*Part 2*). First, we make the following observations: Since U is strictly concave in C, the objective function is strictly concave in C. Moreover, the objective function on the RHS of (A.3) is continuously differentiable with respect to C. The optimal decision, $C^\star(m,\omega)$ satisfies the following Karush-Kuhn-Tucker (KKT) conditions:

$$U_1(C) \begin{cases} = pA, & C > 0 \\ < pA, & C = 0 \end{cases}. \quad \text{(A.5)}$$

In an equilibrium, $p > 0$ will be finite—in fact, $p = 1$—and, since $A \in (0,\infty)$, then $C^\star(m,\omega) \equiv \bar{C}^\star = (U_1)^{-1}(pA)$ is a finite and non-negative constant. Thus, we only have to verify that the optimal decision correspondence, given by $l_c^\star(m,\omega) \equiv p\bar{C}^\star + y_c^\star(m,\omega) - m$ at each (m,ω), exists and is at least a convex-valued and upper-semicontinuous (*usc*) correspondence: Fixing $C = \bar{C}^\star$, the objective function on the RHS of (A.3) is continuous and concave on the compact choice set $[0,\bar{m}] \ni y$. By Berge's Maximum Theorem, the maximizer $y_c^\star(m,\omega)$, or $l_c^\star(m,\omega)$, is convex-valued and *usc* on $[0,\bar{m}]$. (In fact, after we further establish that the derivative $V_1(\cdot,\omega_{+1})$ exists, we show below that it will be constant and single-valued with respect to m.)

Second, we take a detour and show that the derivative $\bar{V}_1(\cdot,\omega_{+1})$ exists, in order to characterize a first-order condition with the respect to y. The results below will rely on the observation that since $V(\cdot,\omega_{+1})$ is a concave, real-valued function on $[0,\bar{m}]$, it has right- and left-hand derivatives (see, e.g., Rockafellar, 1970, Theorem 24.1, pp.227-228). Fix $C^\star(m,\omega) \equiv \bar{C}^\star$. Since $y_c^\star(m,\omega)$ is *usc* on $[0,\bar{m}]$, then for all $\varepsilon \in [0,\delta]$, and taking $\delta \searrow 0$, there exists a selection $y^\star(m-\varepsilon,\omega) \in y_c^\star(m-\varepsilon,\omega)$ feasible to a CM agent m. Similarly, there is a $y^\star(m,\omega) \in y_c^\star(m,\omega)$ that is feasible to a CM agent $m - \varepsilon$. Moreover, if $l^\star(m,\omega) \in l_c^\star(m,\omega)$ is an optimal selection associated with $y^\star(m,\omega)$, then for an agent at m,

$$W(m,\omega) = \underbrace{U(\bar{C}^\star) - Al^\star(m,\omega) + \beta\bar{V}\left[\frac{\omega\left[m + l^\star(m,\omega) - \bar{C}^\star\right] + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}_{\equiv Z[m, y^\star(m,\omega)]}$$
$$\geq \underbrace{U(\bar{C}^\star) - Al^\star(m-\varepsilon,\omega) + \beta\bar{V}\left[\frac{\omega\left[m + l^\star(m-\varepsilon,\omega) - \bar{C}^\star\right] + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}_{\equiv Z[m, y^\star(m-\varepsilon,\omega)]};$$

and, for an agent at $m - \varepsilon$,

$$W(m-\varepsilon,\omega)$$
$$= \underbrace{U(\bar{C}^\star) - Al^\star(m-\varepsilon,\omega) + \beta\bar{V}\left[\frac{\omega\left[(m-\varepsilon) + l^\star(m-\varepsilon,\omega) - \bar{C}^\star\right] + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}_{\equiv Z[m-\varepsilon, y^\star(m-\varepsilon,\omega)]}$$
$$\geq \underbrace{U(\bar{C}^\star) - Al^\star(m,\omega) + \beta\bar{V}\left[\frac{\omega\left[(m-\varepsilon) + l^\star(m,\omega) - \bar{C}^\star\right] + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}_{\equiv Z[m-\varepsilon, y^\star(m,\omega)]}.$$

Rearranging these inequalities, we have the following fact:

$$\frac{Z[m, y^\star(m-\varepsilon,\omega)] - Z[m-\varepsilon, y^\star(m-\varepsilon,\omega)]}{m-(m-\varepsilon)}$$
$$\leq \frac{W(m,\omega) - W(m-\varepsilon,\omega)}{m-(m-\varepsilon)} \leq \frac{Z[m, y^\star(m,\omega)] - Z[m-\varepsilon, y^\star(m,\omega)]}{m-(m-\varepsilon)},$$

which, after simplifying the denominator and taking limits, yields:

$$\lim_{\varepsilon\searrow 0}\left\{\frac{Z[m, y^\star(m-\varepsilon,\omega)] - Z[m-\varepsilon, y^\star(m-\varepsilon,\omega)]}{\varepsilon}\right\}$$
$$\leq \lim_{\varepsilon\searrow 0}\left\{\frac{W(m,\omega) - W(m-\varepsilon,\omega)}{\varepsilon}\right\} \leq \lim_{\varepsilon\searrow 0}\left\{\frac{Z[m, y^\star(m,\omega)] - Z[m-\varepsilon, y^\star(m,\omega)]}{\varepsilon}\right\}$$
$$\Longleftrightarrow$$
$$\beta\lim_{\varepsilon\searrow 0}\left\{\frac{\bar{V}\left[\frac{\omega(m+l^\star(m-\varepsilon,\omega)-\bar{C}^\star)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right] - \bar{V}\left[\frac{\omega(m-\varepsilon+l^\star(m-\varepsilon,\omega)-\bar{C}^\star)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}{\varepsilon}\right\} \leq W_1(m,\omega)$$
$$\leq \beta\lim_{\varepsilon\searrow 0}\left\{\frac{\bar{V}\left[\frac{\omega(m+l^\star(m,\omega)-\bar{C}^\star)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right] - \bar{V}\left[\frac{\omega(m-\varepsilon+l^\star(m,\omega)-\bar{C}^\star)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}{\varepsilon}\right\}.$$

Since, from (A.3), $W(\cdot,\omega)$ is clearly differentiable with respect to m, the second term in the inequalities above is equal to the partial derivative $W_1(m,\omega)$, which is constant. As $\varepsilon \searrow 0$, there is a selection $l^\star(m-\varepsilon,\omega) \to l^\star(m,\omega)$, and, by Rockafellar (1970, Theorem

24.1) the first is the left derivative of $V(\cdot,\omega_{+1})$. Moreover, the last term is identical to the first, i.e.,

$$\frac{\beta}{1+\tau\omega}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega\left(m^-+l^\star(m,\omega)-\bar{C}^\star\right)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]$$
$$\leq W_1(m,\omega)\leq\frac{\beta}{1+\tau\omega}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega\left(m^-+l^\star(m,\omega)-\bar{C}^\star\right)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right].$$

Therefore, if the optimal selection is interior, these weak inequalities must hold with equality, so we have the left derivative of $\bar{V}$ with respect to the agent's decision variable y as:

$$\frac{\beta}{1+\tau\omega}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega y^{\star-}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]=W_1(m,\omega).$$

where $y^{\star-}(m,\omega)\equiv m^-+l^\star(m,\omega)-\bar{C}^\star$.

By similar arguments, we can also prove that the right directional derivative of $\bar{V}(\cdot,\omega_{+1})$ exists, and show that the right derivative of $\bar{V}$ with respect to the agents decision y as:

$$\frac{\beta}{1+\tau\omega}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega y^{\star+}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]=W_1(m,\omega),$$

where $y^{\star+}(m,\omega)\equiv m^++l^\star(m,\omega)-\bar{C}^\star$. From the last two equations, we can conclude that the right and left directional derivatives must agree, and thus, we have the first-order KKT condition (2.14) as, repeated here as

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega y^\star(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\begin{cases}\leq A, & y^\star(m,\omega)\geq 0\\ \geq A, & y^\star(m,\omega)\leq\min\left\{\bar{m}-\frac{\tau}{\omega},\bar{m}\right\}\end{cases},$$

where the weak inequalities apply with complementary slackness.[21] Also, note that in

[21]Consider the feasible choice set for CM saving y: If $\bar{m}$ is an upper bound on end-of-period balance plus transfer (measured in units of labor), then this gives the bounds on end-of-period money balance plus transfer, in current money value, as:

$$\tau M\leq\omega My+\tau M\leq\omega M\bar{m},$$

where ωM is current nominal wage. Since there is inflation in nominal wage, then next-period initial balance is current end-of-period nominal money balance normalized by the next period nominal wage $M_{+1}\omega_{+1}$, i.e.,

$$\frac{\tau M}{\omega_{+1}M_{+1}}\leq m_{+1}\equiv\frac{\omega My+\tau M}{\omega_{+1}M_{+1}}\leq\min\left\{\frac{\omega M\bar{m}}{\omega_{+1}M_{+1}},\bar{m}\right\}.$$

Using (2.1), we can re-write the above bounds as

$$0\leq y\leq\min\left\{\bar{m}-\frac{\tau}{\omega},\bar{m}\right\}\equiv y_{\max}(\omega;\tau),$$

which applies in the pair of KKT complementary slackness conditions (2.14). Note that the min operator is introduced to account for the possibility that $\tau<0$.

the previous proof of Part 1), we have established the envelop condition (2.15):

$$W_1(m, \omega) = A.$$

(*Part 3*.) Observe that given the assumption in (2.3), we have (A.5) always binding: $U'(C) = pA$. Also, observe from(A.5) and (2.14) that an individual's current money holding m and the aggregate state ω have no influence on his optimal decision on consumption, $C^{\star}(m, \omega) = \bar{C}^{\star}$, but that $y^{\star}(m, \omega) = \bar{y}^{\star}(\omega)$. However, from the budget constraint, m clearly does affect the optimal labor decision,

$$\begin{aligned} l^{\star}(m, \omega) &= pC^{\star}(m, \omega) + y^{\star}(m, \omega) - m \\ &\underset{(p=1)}{\equiv} \bar{C}^{\star} + \bar{y}^{\star}(\omega) - m. \end{aligned} \tag{A.6}$$

Clearly, $l^{\star}(m, \omega)$ is single-valued, continuous, affine and decreasing in m.

Finally, we show that the optimal choice of l will always be interior. Evaluating the budget constraint in terms of optimal choices at the current individual state m,

$$l^{\star}(m, \omega) = \bar{y}^{\star}(\omega) - m + \bar{C}^{\star}.$$

Since $m \in [0, \bar{m}]$, then, the minimal l attains when m is maximal at $\bar{m}$, and, $\bar{y}^{\star}(\bar{m}, \omega) = 0$:

$$l_{\min} := \check{l}^{\star}(\bar{m}, \omega) \equiv 0 - \bar{m} + \bar{C}^{\star} > 0.$$

The last inequality obtains from (2.3) which requires $\bar{m} < U^{-1}(A)$, and from optimal CM consumption (2.16) which yields $\bar{C}^{\star} = U^{-1}(A)$, where $p = 1$ in an equilibrium. The maximal l attains when $m = 0$ and $\bar{y}^{\star}(0, \omega) = y_{\max}(\omega; \tau)$:

$$l_{\max}(\omega, \tau) := y_{\max}(\omega; \tau) - 0 + \bar{C}^{\star} = \min\left\{\bar{m} - \frac{\tau}{\omega}, \bar{m}\right\} + U^{-1}(A) < 2U^{-1}(A). \tag{A.7}$$

Clearly, $l_{\max}(\omega, \tau) < +\infty$. If we do not have hyperinflation, or, if transfers are not excessively large—i.e., if $\tau/\omega < \bar{m}$—then, $l_{\max}(\omega, \tau) > 0$ will be well-defined. So if $\tau/\omega < \bar{m}$, then we will have an interior optimizer for all m: $0 < l_{\min} \leq l^{\star}(m) \leq l_{\max}(\omega; \tau) < +\infty$. □

B Limited short-sale constraint and CM participation

Here we derive the short-sale constraint that may bind in the ex-ante market participation problem (2.9) in the paper. Suppose an agent were to participate in the CM with initial asset $a = z - \chi$, where z is his ex-ante money balance, and, χ is the fixed cost (in units of labor) of CM participation. Thus if $a < 0$, the agent is said to be short selling, or issuing an I.O.U.

Recall the CM budget constraint is

$$y + C = l + a.$$

The most negative an asset position the agent can attain in an equilibrium is some $\underline{a}$ such that he must work at the maximal amount $l_{\max}(\omega;\tau)$ and cannot afford to save, $y = 0$. From the budget constraint in such an equilibrium, we have:

$$0 + \bar{C}^{\star} = l_{\max}(\omega;\tau) + \underline{a},$$

which then implies that $\underline{a} = \bar{C}^{\star} - l_{\max}(\omega;\tau)$. From (A.7), we can further obtain the simplified expression $\underline{a} = -y_{\max}(\omega;\tau) \equiv -\min\{\bar{m}, \bar{m} - \tau/\omega\}$. Thus the limited short-sale constraint in (2.9) in the paper.

C DM agent's problem

In this section, we provide the omitted proofs leading up to Theorem 2 on page 15 in the paper. Part 1 of the Theorem is obtained in Lemma 1, Part 2 is proven as Lemma 2. Part 3(a) is proven as Lemma 3. Lemmata 4 and 5 together establish Parts 3(b) and 3(c) of the Theorem. Finally, Lemma 6 establishes Part 3(d) of the Theorem.

C.1 DM buyer optimal policies

Recall the DM buyer's problem from (2.19):

$$B(\mathbf{s}) = \max_{x\in[0,m],b\in[0,1]} \{f(x,b;m,\omega)\},$$

where

$$f(x,b;m,\omega) := \beta(1-b)\left[\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] + b\left[u^{Q}(x,b) + \beta\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right],$$

and, we have re-defined the composite function $u \circ Q$ as u^{Q}. Note that we have not explicitly written $f(x,b;m,\omega)$ as depending on ω_{+1} which is taken as parametric. In an equilibrium, ω_{+1} will be recursively dependent on ω, thus our small sleight of hand here in writing $f(x,b;m,\omega)$.

The following Lemmata 1, 2, 3, 4, 5, and 6 make up Theorem 2. Also, these results will rely on the following statements and notations:

1. Assume $\{\omega, \omega_{+1}, \omega_{+2}...\}$ is a given sequence of prices.

2. Let

$$\phi(m,\omega) : = \frac{\omega m + \tau}{\omega_{+1}(1+\tau)},$$

and,

$$\phi^{\star}(m,\omega) = \phi\left[m - x^{\star}(m,\omega), \omega\right].$$

3. Equivalently define the objective function $f(\cdot,\cdot;m,\omega)$ in the DM buyer's problem (2.19) as follows:

$$\begin{aligned} f(x,b;m,\omega) = \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \\ + b\left[u^{Q}(x,b) + \beta\bar{V}\left(\phi^{\star}(m,\omega),\omega_{+1}\right) - \beta\bar{V}\left(\phi(m,\omega),\omega_{+1}\right)\right]. \\ \equiv \beta\bar{V}\left(\phi(m,\omega),\omega_{+1}\right) + bR\left(x,b;m,\omega\right). \end{aligned} \tag{C.1}$$

Remark. Observe that maximizing the value of the objective function $f(x,b;m,\omega)$ in the DM buyer's problem (2.19) is equivalent to maximizing the second term, $bR(x,b;m,\omega)$. Note that the function $R(x,b;m,\omega)$ has the interpretation of the DM buyer's surplus from trading with a particular trading post named (x,b), by offering to pay x in exchange for quantity $Q(x,b)$.

Lemma 1. *For any $\bar{V}(\cdot,\omega_{+1}) \in \mathcal{V}[0,\bar{m}]$, the DM buyer's value function is increasing and continuous in money balances, $B(\cdot;\omega) \in \mathcal{C}[0,\bar{m}]$.*

Proof. Since the functions $W(\cdot,\omega_{+1}), V(\cdot,\omega_{+1}) \in \mathcal{C}[0,\bar{m}]$, i.e., are continuous and increasing on $[0,\bar{m}]$, and $\bar{V} := \alpha W + (1-\alpha V)$, then $\bar{V}(\cdot,\omega_{+1}) \in \mathcal{C}[0,\bar{m}]$. The feasible choice set $\Phi(m) := [0,m] \times [0,1]$ is compact, and it expands with m at each $m \in [0,\bar{m}]$. By Berge's Maximum Theorem, the maximizing selections $(x^{\star},b^{\star})(m,\omega) \in \Phi(m)$ exist for every fixed $m \in [0,\bar{m}]$, since the objective function is continuous on a compact choice set (Berge, 1963). Evaluating the Bellman operator (2.19), we have that the value function $B(\cdot,\omega) \in \mathcal{C}[0,\bar{m}]$. □

Lemma 2. *For any $m \leq k$, DM buyers' optimal decisions are such that $b^{\star}(m,\omega) = 0$ and $B(m,\omega) = \beta\bar{V}[\phi(m,\omega),\omega_{+1}]$, where $\phi(m,\omega) := \frac{\omega m+\tau}{\omega_{+1}(1+\tau)}$.*

Proof. Since a buyer's payment x is always constrained above by her initial money balance m in the DM, it will never be optimal for any firm to trade with such a buyer whose $m \leq k$, as the firm will be making an economic loss. In equilibrium it is thus optimal for a buyer $m \leq k$ to optimally not trade and exit the DM with end-of-period balance as m (i.e., with beginning-of-next-period balance $\phi(m, \omega)$ when inflationary transfers are accounted for). As a result, the continuation value is $\bar{V}[\phi(m, \omega), \omega_{+1}]$, and thus, $B(m, \omega) = \beta\bar{V}[\phi(m, \omega), \omega_{+1}]$, if $m \leq k$. □

Lemma 3. *For any (m, ω), where $m \in [k, \bar{m}]$ and the buyer matching probability is positive $b^{\star}(m, \omega) > 0$, the optimal selections $(x^{\star}, b^{\star}, q^{\star})(m, \omega)$ and $\phi^{\star}(m, \omega) := \phi[m - x^{\star}(m, \omega), \omega]$ are unique, continuous, and increasing in m.*

Observe that the DM buyer's problem has a general structure similar to that of Menzio et al. (2013). The main difference is in the details underlying the buyer's continuation value function, which in our setting is denoted by $\bar{V}(\cdot, \omega)$. Nevertheless, we still have that $\bar{V}(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$. As a consequence the proof of Lemma 3.3 in Menzio et al. (2013) can be adapted to our setting. For the reader's convenience, we repeat the proof strategy of Menzio et al. (2013) below for our model setting in a few steps:

Proof. The DM buyer's problem (2.19) can be re-written as

$$B(\mathbf{s}) = \beta\bar{V}(\phi(m, \omega), \omega_{+1}) + \exp\left\{\max_{x\in[0,m], b\in[0,1]} \{\ln(b) + \ln[R(x, b; m, \omega)]\}\right\}.$$

The optimizers thus must satisfy

$$(x^{\star}, b^{\star})(m, \omega) \in \left\{\arg\max_{x\in[0,m], b\in[0,1]} \{\ln(b) + \ln[R(x, b; m, \omega, \omega_{+})]\}\right\}. \tag{C.2}$$

(*Uniqueness and continuity of policies.*) First we establish that the policy functions are continuous, and, at every state, there is a unique optimal selection: Since $u^{Q}(x, b)$ is continuous, jointly and strictly concave in (x, b), and by assumption, $\bar{V}(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$, then

$$R(x, b; m, \omega) \equiv u^{Q}(x, b) + \beta\bar{V}(\phi^{\star}(m, \omega), \omega_{+1}) - \beta\bar{V}(\phi(m, \omega), \omega_{+1})$$

is continuous, jointly and strictly concave in the choice variables (x, b). Also, $\ln(b)$ is strictly increasing and strictly concave in b. Thus the maximand is jointly and strictly concave in (x, b). By Berge's Maximum Theorem, the optimal selections $(x^{\star}, b^{\star})(m, \omega)$ are continuous and unique at any m. Since $c \mapsto c(q)$ is bijective, then

$$q^{\star}(m, \omega) = c^{-1}[x^{\star}(m, \omega) - k/\mu(b^{\star}(m, \omega))]$$

is continuous in m; and so is $\phi^{\star}(m,\omega)$.

(*Monotonicity of policies.*) The remainder of this proof establishes that the policy functions are increasing. The key idea of the proof is in showing that the choice set is a lattice equipped with a partial order, that the choice set is increasing in m, and, has increasing differences on the choice set, and the slices of the buyer's objective is supermodular in each given direction of his choice set. By Theorem 2.6.2 of Topkis (1998), these properties are sufficient to ensure that the buyer's objective function is supermodular. Together, these properties suffice, by Theorem 2.8.1 of Topkis (1998), for showing that the buyer's optimal policies are increasing functions in m.

1. The function $R(\cdot,\cdot,\cdot,\omega)$ in (C.2) has *increasing difference* in (x,b,m) and is therefore supermodular:

   Fix an $m \in [k,\bar{m}]$ and $b \in (0,1]$. (The case of $b = 0$ is trivially uninteresting.) It suffices to optimize over the function $\ln[R(\cdot,b,m,\omega)]$ in (C.2). Then the optimizer

   $$\tilde{x}(b,m,\omega) \in \left\{ \arg\max_{x\in[k,\bar{m}]} \{\ln[R(x,b,m,\omega)]\} \right\}$$

   is unique for each (m,b,ω), since the objective functions is strictly concave.

   Next we show how the value of the objective function has increasing differences in (x,b,m), throughout taking the sequence $\{\omega,\omega_{+1},...\}$ as fixed. Thus we will now write $R(x,b,m) \equiv R(x,b,m,\omega)$ to temporarily ease the notation. First, the feasible choice set

   $$\mathcal{F}_m := \{(x,b,m) : x \in [0,m], b \in [0,1], m \in [k,\bar{m}]\},$$

   is a partially ordered set with relation $\leq$, and it has least-upper and greatest-lower bounds. It is therefore a sublattice in $\mathbb{R}^3_+$. Observe that $\mathcal{F}_m$ is increasing in m. Second, pick any $m' > m$, $b' > b$, and $x' > x$ in $\mathcal{F}_m$:

   (a) For fixed x, consider $m' > m$ and $b' > b$. Then, we can write

   $$\begin{aligned} R(x,b',m') &- R(x,b,m) \\ &= \left[u^Q(x,b') - u^Q(x,b)\right] \\ &\quad + \beta\left[\bar{V}\left(\frac{\omega(m'-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\ &\quad - \beta\left[\bar{V}\left(\frac{\omega m'+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]. \end{aligned}$$

   Observe that the RHS is separable in b and m: The first term on the right, $u^Q(x,b') - u^Q(x,b) < 0$, shows increasing difference in b. Likewise the remainder two difference terms on the RHS show increasing differences in m. Overall $R(x,b,m)$ has increasing differences on the lattice $[0,1]\times[0,\bar{m}]\ni(b,m)$.

(b) For fixed m, consider $x'>x$ and $b'>b$. Observe that

$$\begin{aligned}R(x,b,m)-R(x',b,m) &= \left[u^Q(x,b)-u^Q(x',b)\right]\\ &+\beta\left[\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega(m-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right].\end{aligned}\tag{C.3}$$

Now, using the expression (C.3) twice below, we have that

$$\begin{aligned}&\left[R(x',b',m)-R(x,b',m)\right]-\left[R(x',b,m)-R(x,b,m)\right]\\ &=\left[u^Q(x',b')-u^Q(x,b')\right]\\ &\qquad+\beta\left[\bar{V}\left(\frac{\omega(m-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\\ &\qquad-\left[u^Q(x',b)-u^Q(x,b)\right]\\ &\qquad-\beta\left[\bar{V}\left(\frac{\omega(m-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\\ &\qquad\qquad=\left[u^Q(x',b')-u^Q(x,b')\right]-\left[u^Q(x',b)-u^Q(x,b)\right]>0,\end{aligned}$$

where the last inequality is implied by the fact that $\left(u^Q\right)_{12}(x,b)>0$. Therefore $R(x,b,m)$ has increasing differences on the lattice $[0,m]\times[0,1]\ni(x,b)$.

(c) For fixed b, consider $x'>x$ and $m'>m$. Observe that

$$\begin{aligned}&\left[R(x',b,m')-R(x,b,m')\right]-\left[R(x',b,m)-R(x,b,m)\right]\\ &=\left[u^Q(x',b)-u^Q(x,b)\right]\\ &\qquad+\beta\left[\bar{V}\left(\frac{\omega(m'-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega(m'-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\\ &\qquad-\left[u^Q(x',b)-u^Q(x,b)\right]\\ &\qquad-\beta\left[\bar{V}\left(\frac{\omega(m-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\\ &=\beta\left[\bar{V}\left(\frac{\omega(m'-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega(m'-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\\ &\qquad-\beta\left[\bar{V}\left(\frac{\omega(m-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)-\bar{V}\left(\frac{m-x+\tau}{1+\tau\omega},\omega_{+1}\right)\right]\geq 0,\end{aligned}$$

where the last weak inequality obtains from the property that $\bar{V}(\cdot,\omega_{+1})\in\mathcal{V}[0,\bar{m}]$, and $\bar{V}(\cdot,\omega_{+1})$ is therefore weakly concave. Therefore $R(x,b,m)$ has increasing differences on the lattice $[0,m]\times[0,\bar{m}]\ni(x,m)$.

From parts (1a), (1b), and (1c), we can conclude that the objective function $R(\cdot,\cdot,\cdot,\omega)$ has increasing differences on $\mathcal{F}_m$. This suffices to prove that the objective function $R(\cdot,\cdot,\cdot,\omega)$ is supermodular (see Topkis, 1998, Corollary 2.6.1), since the domain of the function is a direct product of a finite set of chains (partially ordered sets with no unordered pair of elements), and, the objective function is real valued (see Topkis, 1978).

2. Since $R(\cdot,b,m)$ is supermodular, for fixed choice b, the optimizer $\tilde{x}(b,m,\omega)$ is increasing in (b,m), for given ω:

   Let $\tilde{x}(b,m,\omega) = \arg\max_{x\in[0,m]} R(x,b,m)$. From part (1a) above, we can deduce that for fixed $\tilde{x}(b,m)$, $\tilde{R}(b,m) \equiv R[\tilde{x}(b,m,\omega),b,m]$ is supermodular on the lattice $[0,1]\times[0,\bar{m}] \ni (b,m)$. Since $R(x,b,m)$ is strictly decreasing in b, then

   $$\tilde{R}(b,m) \equiv R[\tilde{x}(b,m,\omega),b,m]$$

   is strictly decreasing in b. Observe that for any $m' \geq m$, where $m',m \in [k,\bar{m}]$, we have

   $$\begin{aligned} &R(x,b,m') - R(x,b,m) \\ &= \beta\left[\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\ &\qquad - \beta\left[\bar{V}\left(\frac{\omega m'+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m'-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \geq 0, \end{aligned}$$

   since $\bar{V}(\cdot,\omega)$ is concave. Since this inequality holds at each fixed pair(x,b), then,

   $$\begin{aligned} \tilde{R}(b,m) &\equiv R[\tilde{x}(b,m,\omega),b,m] \\ &\leq R[\tilde{x}(b,m,\omega),b,m'] \leq R[\tilde{x}(b,m',\omega),b,m'] \equiv \tilde{R}(b,m'). \end{aligned}$$

   The last weak inequality obtains because the choice set is increasing in m, and so $\tilde{x}(b,m,\omega)$ is a feasible selection for the more relaxed problem whose value is

   $$R[\tilde{x}(b,m',\omega),b,m'] = \max_{x\in[0,m']} R(x,b,m').$$

   From these weak inequalities, we can conclude that $\tilde{R}(b,m)$ is increasing in m.

   Now we are ready to apply Theorem 2.8.1 of Topkis (1998) to show that $b^\star(m,\omega)$ increases with m: Let

   $$b^\star(m,\omega) = \arg\max_{b\in[0,1]} r(b,m)$$

where $r(b,m) = b \cdot \tilde{R}(b,m)$ and $\tilde{R}(b,m) \equiv R(\tilde{x}(b,m,\omega), b, m, \omega)$. Observe the following identity:

$$\begin{aligned}
\left[r(b',m') - r(b,m')\right] - \left[r(b',m) - r(b,m)\right] = \\
b'\left\{\tilde{R}(b',m') - \tilde{R}(b,m') - \left[\tilde{R}(b',m) - \tilde{R}(b,m)\right]\right\} \\
+ (b'-b)\left[\tilde{R}(b,m') - \tilde{R}(b,m)\right],
\end{aligned}$$

for any $b, b' \in (0,1], m, m' \in [k, \bar{m}]$ where $b' > b$ and $m' > m$. The first term on the RHS is positive, since $b' > 0$ and since $\tilde{R}(b,m)$ is supermodular in (b,m), then Topkis (1998, Theorem 2.6.1) applies, so that $\tilde{R}(b,m)$ has increasing differences on $[0,1] \times [0,\bar{m}]$ (i.e., the terms in the curly braces are positive). Since we have previously established that $\tilde{R}(b,m)$ is increasing in m, and $b' - b > 0$, then the second term on the RHS is also positive. Thus the objective $r(b,m)$ is supermodular on $[0,1] \times [k,\bar{m}] \ni (b,m)$. (Note that the choice set of b does not depend on m.)

Therefore, by Theorem 2.8.1 of Topkis (1998), the optimal selection $b^\star(m,\omega)$ is increasing in m. Since $\tilde{x}(b,m,\omega)$ is increasing in (b,m), for given ω, then we can conclude that the optimal payment choice $x^\star(m,\omega) = \tilde{x}(b^\star(m,\omega), m, \omega)$ is also increasing in m.

3. The decision $q^\star(m,\omega)$ is monotone in m:

   We perform a change of decision variables. Denote $a \equiv \varphi + c(q)$, where, $\varphi \equiv m - x$. Then we have a change of the DM buyer's decision variables from (x,q) to (a,q). From (2.12), we can re-write $m - x = a - c(q)$ and $b = \mu^{-1}[k/(m-a)]$. Since $b \in [0,1]$, the domain of a is $[0, m-k]$, and the domain for q is $[0,a]$. The buyer's problem from (2.19) is thus equivalent to writing

$$\begin{aligned}
B(m,\omega) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) = \max_{a\in[0,m-k], q\in[0,a]} \left\{\mu^{-1}\left(\frac{k}{m-a}\right)\left[u(q)\right.\right. \\
\left.\left. + \beta\bar{V}\left(\frac{\omega(a-q)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]\right\}. \quad \text{(C.4)}
\end{aligned}$$

   Recall we take the sequence $(\omega, \omega_{+1}, ...)$ as parametric here. This problem can be broken down into two steps: Fix (a,ω). Find the optimal q for any a, to be denoted by $\tilde{q}(a,\omega)$, and then, find the optimal a given (a,ω), to be denoted by $a^\star(m,\omega)$. Then we can deduce the optimal $q^\star(m,\omega) \equiv \tilde{q}[a^\star(m,\omega), m, \omega]$. We details these steps below:

   (a) For any fixed a and (m,ω), $\tilde{q}(a,\omega)$ induces the value

$$J(a,\omega) = \max_{q\in[0,a]} \left\{u(q) + \beta\bar{V}\left(\frac{\omega(a-q)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}. \quad \text{(C.5)}$$

Observe that q and J do not depend on m, given a fixed a. The objective function on the RHS is clearly supermodular on the lattice $[0, m-k] \times [0, a] \ni (a, q)$. Since the objective function is strictly concave, the selection $\tilde{q}(a, \omega)$ in unique for every a, given ω. Also, the choice set $[0, a]$ increases with a, and, the objective function is increasing. Therefore, respectively by Theorems 2.8.1 (increasing optimal solutions) and 2.7.6 (preservation of supermodularity) of Topkis (1998), we have that $\tilde{q}(a, \omega)$ and $J(a, \omega)$ are increasing in a.

(b) Given best response $\tilde{q}(a, \omega)$, the optimal $a^{\star}(m, \omega)$ choice satisfies

$$a^{\star}(m, \omega) = \arg \max_{a \in [0, m-k]} g(a, m, \omega),$$

where

$$g(a, m, \omega) = \mu^{-1}\left(\frac{k}{m-a}\right)\left[J(m, \omega) - \beta \bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right].$$

(Again, note that we have suppressed dependencies on ω_{+1} since this is taken as parametric by the agent, and, in equilibrium ω_{+1} recursively depends on ω.)

Consider the case $J(a, \omega) - \beta \bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \geq 0$. Since $\mu(b)$ is strictly decreasing in b, and $1/\mu(b)$ is strictly convex in b, then $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly increasing in m, strictly decreasing in a, and is strictly supermodular in (a, m). Pick any $a', a \in [0, m-k]$, and any $m', m \in [k, \bar{m}]$, such that $a' > a$ and $m' > m$. We have the identity:

$$\begin{aligned}
&[g(a', m', \omega) - g(a, m', \omega)] - [g(a', m, \omega) - g(a, m, \omega)] = \\
&\left[\mu^{-1}\left(\frac{k}{m'-a'}\right) - \mu^{-1}\left(\frac{k}{m-a'}\right)\right]\left[J(a', \omega) - J(a, \omega)\right] \\
&+ \left[\mu^{-1}\left(\frac{k}{m'-a}\right) - \mu^{-1}\left(\frac{k}{m'-a'}\right)\right] \\
&\qquad \times \left[\beta \bar{V}\left(\frac{\omega m' + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \beta \bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right] \\
&+ \left[\mu^{-1}\left(\frac{k}{m'-a'}\right) - \mu^{-1}\left(\frac{k}{m'-a}\right) - \mu^{-1}\left(\frac{k}{m-a'}\right) - \mu^{-1}\left(\frac{k}{m-a}\right)\right] \\
&\qquad\qquad\qquad \times \left[J(a, \omega) - \beta \bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]
\end{aligned}$$

The first term on the RHS is positive since $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly increasing in m, and we have previously shown that $J(a, \omega)$ is increasing in a. The second term on the RHS is positive since $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly decreasing in a, and, $\tilde{V}(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$. The last term on the RHS is positive since $\mu^{-1}\left(\frac{k}{m-a}\right)$ is supermodular, and therefore its first term in the product shows increasing differences Topkis (1998, Theorem 2.6.1). Its last term in the product is positive under the case we are considering. Therefore the LHS is positive, and this suffices to establish that $g(a, m, \omega)$ is strictly supermodular (Topkis, 1998, Theorem 2.8.1).

Finally, since the choice set $[0, m-k]$ is increasing in m, the solution $a^{\star}(m, \omega)$ is also increasing in m Topkis (1998, Theorem 2.6.1). Since we have established in part (3a) that $\tilde{q}(m, \omega)$ is increasing in a, then, $q^{\star}(m, \omega) \equiv \tilde{q}[a^{\star}(m, \omega), \omega]$ is also increasing in m.

4. The decision $\phi^{\star}(m, \omega)$ is monotone in m:

Similar to the procedure in the last part, we perform a change of decision variables via $a \equiv \varphi + c(q)$, where, $\varphi \equiv m - x$. The domain for φ is $[0, \min\{m, a\}]$. However, an optimal choice under $b > 0$ means that we will have $\varphi < m$ (the end of period residual balance is less than the beginning of period money balance). This is because, if $\varphi = m$ then it must be that $x = 0$, i.e., the buyer pays nothing; but this is not optimal for the buyer if the buyer faces a positive probability of matching $b > 0$. Moreover, $\varphi < a$, if $u'(0)$ is sufficiently large—i.e., the buyer can always increase utility by raising x (thus lowering φ such that $\varphi < a$ attains). Thus the upper bound on φ will never be binding. As such, the buyer's problem from (2.19) can be re-written as

$$B(m,\omega) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) = \max_{a \in [0, m-k], \varphi \geq 0} \left\{\mu^{-1}\left(\frac{k}{m-a}\right)\left[u^C(a-\varphi) + \beta\bar{V}\left(\frac{\omega\varphi + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]\right\}, \quad \text{(C.6)}$$

where $u^C(q) := u \circ c^{-1}(q)$, which is continuously differentiable with respect to $q \geq 0$. For fixed $a \in [0, m-k]$, denote the value

$$J(a, \omega) = \max_{\varphi \geq 0}\left\{u^C(a-\varphi) + \beta\bar{V}\left(\frac{\omega\varphi + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}, \quad \text{(C.7)}$$

and the optimizer,

$$\tilde{\varphi}(a, \omega) = \arg\max_{\varphi \geq 0}\left\{u^C(a-\varphi) + \beta\bar{V}\left(\frac{\omega\varphi + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}. \quad \text{(C.8)}$$

Denote also $\tilde{q}(a, \omega) = c^{-1}[a - \tilde{\varphi}(a, \omega)]$.

Given $\tilde{\varphi}(a,\omega)$, the optimal choice over a, i.e., $a^\star(m,\omega)$, solves

$$B(m,\omega)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)= \max_{a\in[0,m-k]}\left\{\mu^{-1}\left(\frac{k}{m-a}\right)\left[J(a,\omega)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\right\}.$$

Applying the similar logic in the proof in part 3 on page OA-§.C. 13, we can show that $\tilde{\varphi}(a,\omega)$ is increasing in a; that $a^\star(m,\omega)$ is increasing in m, and therefore, $\varphi^\star(m,\omega)\equiv\tilde{\varphi}[a^\star(m,\omega),\omega]$ is increasing in m. Finally, since

$$\phi^\star(m,\omega):=[\omega\varphi^\star(m,\omega)+\tau]/[\omega_{+1}(1+\tau)],$$

which is a linear transform of $\varphi^\star(m,\omega)$, then $\phi^\star(m,\omega)$ is increasing with m, since $\omega/[\omega_{+1}(1+\tau)]>0$.

□

C.2 DM buyer value function and first-order conditions

Let us return to the DM buyer's problem re-written as (C.6) in part (4) of the proof of Lemma 3 on page OA-§.C. 9. The buyer's decision problem over $\varphi\equiv m-x$, for any fixed decision $a\equiv\varphi+c(q)$, yields the value $J(a,\omega)$ as defined in equation (C.7) of that proof. The following intermediate results says that the value function $J(\cdot,\omega)$ is differentiable with respect to a and its marginal value can be related to primitives, i.e.:

Lemma 4. *The marginal value of $J(\cdot,\omega)$ agrees with the flow DM marginal utility with respect to the buyer's payment x,*

$$J_1(a,\omega)=u'[\tilde{q}(a,\omega)]\equiv\left(u^Q\right)_1[x^\star(m,\omega),b^\star(m,\omega)]>0. \tag{C.9}$$

Proof. Consider the problem described in equations (C.6) and (C.7). Observe that since $\varphi\equiv a-c(q)$, then

$$\tilde{\varphi}(a,\omega)=\arg\max_{\varphi\geq0}\left\{u^C(a-\varphi)+\beta\bar{V}\left(\frac{\omega\varphi+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right\} \tag{C.10}$$

is continuous with respect to a: There is some $\delta'>0$, such that for all $\varepsilon\in[0,\delta']$, the choices $\tilde{\varphi}(a+\varepsilon,\omega)$ and $\tilde{\varphi}(a-\varepsilon,\omega)$ exist. Moreover the optimal selection $\tilde{\varphi}(a,\omega)$ is unique since the objective function in (C.10) is strictly concave by virtue of u^C being strictly concave and $\bar{V}$ being concave. Denote also $\tilde{q}(a,\omega)=c^{-1}[a-\tilde{\varphi}(a,\omega)]$, where the choices $\tilde{q}(a+\varepsilon,\omega)$ and $\tilde{q}(a-\varepsilon,\omega)$ also exist, by continuity of c^{-1} in its argument.

To verify (C.9), we can use the perturbed choices, $\tilde{\varphi}(a+\varepsilon,\omega)$ and $\tilde{\varphi}(a-\varepsilon,\omega)$, for evaluating right- and left-derivatives of the functions u^C, $\bar{V}$ and J, in order to "sandwich" the derivative function $J_1(\cdot,\omega)$ and arrive at the claimed result. For notational convenience below, we define the following function

$$K_\omega\left[a,\tilde{\varphi}(a,\omega)\right] \equiv u^C\left(a-\tilde{\varphi}(a,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right).$$

Consider first the right derivatives: Take $\delta' \searrow 0$ such that for all $\varepsilon \in [0,\delta']$, the choice $\tilde{\varphi}(a+\varepsilon,\omega)$ is affordable for a buyer a. Since $\tilde{\varphi}(\cdot,\omega)$ is an optimal policy satisfying (C.8), then under action $\tilde{\varphi}(a,\omega)$ we must have that

$$\begin{aligned}
J(a,\omega) &= u^C\left(a-\tilde{\varphi}(a,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\\
&\geq u^C\left(a-\tilde{\varphi}(a+\varepsilon,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a+\varepsilon,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\\
\Leftrightarrow J(a,\omega) &= K_\omega\left[a,\tilde{\varphi}(a,\omega)\right] \geq K_\omega\left[a,\tilde{\varphi}(a+\varepsilon,\omega)\right].
\end{aligned}$$

Again, take $\delta' \searrow 0$ such that $\forall\varepsilon \in [0,\delta']$, the choice $\tilde{\varphi}(a,\omega)$ is affordable for buyer $a+\varepsilon$. Since $\tilde{\varphi}(\cdot,\omega)$ is an optimal policy satisfying (C.8), then under $\tilde{\varphi}(a+\varepsilon,\omega)$ we must have that

$$\begin{aligned}
J(a+\varepsilon,\omega) &= u^C\left(a+\varepsilon-\tilde{\varphi}(a+\varepsilon,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a+\varepsilon,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\\
&\geq u^C\left(a+\varepsilon-\tilde{\varphi}(a,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\\
\Leftrightarrow J(a+\varepsilon,\omega) &= K_\omega\left[a+\varepsilon,\tilde{\varphi}(a+\varepsilon,\omega)\right] \geq K_\omega\left[a+\varepsilon,\tilde{\varphi}(a,\omega)\right].
\end{aligned}$$

Re-write the two inequalities above as

$$\begin{aligned}
\frac{K_\omega\left[a+\varepsilon,\tilde{\varphi}(a,\omega)\right]-K_\omega\left[a,\tilde{\varphi}(a,\omega)\right]}{\varepsilon} &\leq \frac{J(a+\varepsilon,\omega)-J(a,\omega)}{\varepsilon}\\
&\leq \frac{K_\omega\left[a+\varepsilon,\tilde{\varphi}(a+\varepsilon,\omega)\right]-K_\omega\left[a,\tilde{\varphi}(a+\varepsilon,\omega)\right]}{\varepsilon}.
\end{aligned}$$

Since the composite function u^C—and therefore the objective function in (C.7)—is differentiable with respect to a, $J_1(\cdot,\omega)$ clearly exists. Therefore, the right derivative of this value function must agree with its partial derivative: $\lim_{\varepsilon\searrow 0} J(a+\varepsilon,\omega) = J_1(a,\omega)$. Using this fact, the inequalities above imply

$$\lim_{\varepsilon\searrow 0}\frac{K_{\omega}\left[a+\varepsilon,\tilde{\varphi}(a,\omega)\right]-K_{\omega}\left[a,\tilde{\varphi}(a,\omega)\right]}{\varepsilon}\leq J_{1}\left(a,\omega\right)$$
$$\leq\lim_{\varepsilon\searrow 0}\frac{K_{\omega}\left[a+\varepsilon,\tilde{\varphi}(a+\varepsilon,\omega)\right]-K_{\omega}\left[a,\tilde{\varphi}(a+\varepsilon,\omega)\right]}{\varepsilon}.$$

Moreover, by continuity of $\tilde{\varphi}(\cdot,\omega)$, we have that $\lim_{\varepsilon\searrow 0}\tilde{\varphi}(a+\varepsilon,\omega)=\tilde{\varphi}(a,\omega)$, so the inequalities above collapse to

$$u'\left[\tilde{q}(a^{+},\omega)\right]:=\lim_{\varepsilon\searrow 0}\frac{u^{C}\left(a+\varepsilon-\tilde{\varphi}(a,\omega)\right)-u^{C}\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon}\leq J_{1}\left(a,\omega\right)$$
$$\leq\lim_{\varepsilon\searrow 0}\frac{u^{C}\left(a+\varepsilon-\tilde{\varphi}(a,\omega)\right)-u^{C}\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon}=:u'\left[\tilde{q}(a^{+},\omega)\right].$$

However, the first and the last term in the inequalities above are identical, and they are the same as the right derivative of u with respect to $q:=\tilde{q}(a,\omega)$, i.e., $u'\left[\tilde{q}(a^{+},\omega)\right]$. Thus, it must be that $u'\left[\tilde{q}(a^{+},\omega)\right]=J_{1}\left(a,\omega\right)$.

Using similar arguments as above, we can also consider the left-hand-side perturbation about a, to evaluate $\tilde{\varphi}(a-\varepsilon,\omega)$. It can be shown that

$$u'\left[\tilde{q}(a^{-},\omega)\right]:=\lim_{\varepsilon\searrow 0}\frac{u^{C}\left(a-\varepsilon-\tilde{\varphi}(a,\omega)\right)-u^{C}\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon}\leq J_{1}\left(a,\omega\right)$$
$$\leq\lim_{\varepsilon\searrow 0}\frac{u^{C}\left(a-\varepsilon-\tilde{\varphi}(a,\omega)\right)-u^{C}\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon}=:u'\left[\tilde{q}(a^{-},\omega)\right],$$

so that $u'\left[\tilde{q}(a^{-},\omega)\right]=J_{1}\left(a,\omega\right)$.

Combining the two arguments above, we have that

$$u'\left[\tilde{q}(a,\omega)\right]=u'\left[\tilde{q}(a^{+},\omega)\right]=u'\left[\tilde{q}(a^{-},\omega)\right]=J_{1}\left(a,\omega\right)>0.$$

Finally, the equivalence $u'\left[\tilde{q}(a,\omega)\right]=\left(u^{Q}\right)_{1}\left[x^{\star}(m,\omega),b^{\star}(m,\omega)\right]$ can be derived using standard calculus, since the composite function $u^{Q}\equiv u\circ Q$ is a known continuously differentiable function in its arguments (x,b). The assumption on u that marginal utility is everywhere positive renders $u'\left[\tilde{q}(a,\omega)\right]>0$. This completes the proof of the claim. □

Lemma 5. *At any (m,ω), where $m \in [k,\bar{m}]$ and the buyer matching probability is positive $b^\star(m,\omega) > 0$,*

1. the buyer's marginal valuation of money $B_1(m,\omega)$ exists if and only if $\bar{V}_1\left[\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega\right]$ exists; and
2. $B(m,\omega)$ is strictly increasing in m.

Proof. Lemma 3 implies that $\tilde{q}(a,\omega)$ is increasing in a. Since we have shown that $u'[\tilde{q}(a,\omega)] = J_1(a,\omega) > 0$, then $J_1(a,\omega)$ is also decreasing in a. Since $J(a,\omega)$ is clearly increasing in a, then we conclude that it is also concave in a. The term $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly decreasing and strictly concave in a. Therefore the objective function in (C.6) is strictly concave in a. Thus maximizing (C.6) over a yields a unique optimal selection $a^\star(m,\omega)$. Moreover, the objective function in (C.6) is continuously differentiable with respect to a; and using (C.9) we can show that $a^\star(m,\omega)$ satisfies the first-order condition:[22]

$$\begin{aligned} J\left(a^\star(m,\omega),\omega\right) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) \\ + u'\left[\tilde{q}\left(a^\star(m,\omega),\omega\right)\right]\cdot\frac{k\cdot\mu'\left[b^\star(m,\omega)\right]b^\star(m,\omega)}{\mu\left[b^\star(m,\omega)\right]^2} \begin{cases} = 0, & \text{if } a^\star(m,\omega) < m-k \\ < 0, & \text{if } a^\star(m,\omega) = m-k \end{cases}. \end{aligned} \tag{C.11}$$

Observe that $b^\star(m,\omega) > 0$ implies the buyer has more than enough initial balance for purchasing $q^\star(m,\omega)$, i.e.,

$$m - \varphi^\star(m,\omega) > c\left[q^\star(m,\omega)\right] + k \implies a(m,\omega) \equiv \varphi^\star(m,\omega) + c\left[q^\star(m,\omega)\right] < m-k.$$

Since $a^\star(m,\omega) < m-k$, and $a^\star(m,\omega)$ is continuous in m, then there is an $\epsilon > 0$ such that the following selections are also feasible: $a^\star(m+\epsilon,\omega) < m-k$, and, $a^\star(m,\omega) < (m-\epsilon)-k$. Define the open ball $\mathbf{N}_\epsilon(m) := (m-\epsilon, m+\epsilon)$. Note that for any $m' \in$

[22] Note that $b = \mu^{-1}\left(\frac{k}{m-a}\right)$. The term $\mathrm{d}b/\mathrm{d}a = k/(m-a)^2 \times 1/\mu'[b]$ can be derived using the implicit function theorem: Define $H(a,b) = k/(m-a) - \mu[b] = 0$. Then $\mathrm{d}b/\mathrm{d}a = -H_a(a,b)/H_b(a,b)$, which yields the result. The first-order condition is thus derived as

$$\begin{aligned} \left[J\left(a^\star(m,\omega),\omega\right) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\frac{k}{(m-a^\star(m,\omega))^2}\frac{1}{\mu'\left[b^\star(m,\omega)\right]} \\ + J_1\left(a^\star(m,\omega),\omega\right)\mu^{-1}\left(\frac{k}{m-a^\star(m,\omega)}\right) \begin{cases} = 0, & \text{if } a^\star(m,\omega) < m-k \\ < 0, & \text{if } a^\star(m,\omega) = m-k \end{cases}. \end{aligned}$$

Moreover, since $k/(m-a) = \mu(b)$, we can write $\mathrm{d}b/\mathrm{d}a = k/(m-a)^2 \times 1/\mu'[b] \equiv [\mu(b)]^2/k \times 1/\mu'[b]$, and using the relation (C.9), the first-order condition can be further simplified to (C.11).

$\mathbf{N}_{\epsilon}(m)$, the selection $a^{\star}(m', \omega)$ is feasible for an agent m; and $a^{\star}(m,\omega)$ is feasible for agent m'.

Given that $a^{\star}(m,\omega)$ is optimal for agent m, and since $\varphi^{\star}(m,\omega) = \tilde{\varphi}[a^{\star}(m,\omega)]$, then we have the buyer's optimal value as

$$\begin{aligned}
B(m,\omega) = \beta \bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) + \max_{a\in[0,m-k],\varphi\geq 0} \Bigg\{ \mu^{-1}\left(\frac{k}{m-a}\right) \\
\times \left[u \circ c^{-1}(a-\varphi) + \beta \bar{V}\left(\frac{\omega\tilde{\varphi}+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \beta \bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\} \\
= F(a^{\star}(m,\omega), m) \geq F(a^{\star}(m+\epsilon,\omega), m).
\end{aligned}$$

where $F(a,m) := \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) + \mu^{-1}\left(\frac{k}{m-a}\right)\left[J(a,\omega) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]$. Similarly, for agent $m+\epsilon$, it must be that

$$B(m+\epsilon,\omega) = F(a^{\star}(m+\epsilon,\omega), m+\epsilon) \geq F(a^{\star}(m,\omega), m+\epsilon).$$

Clearly,

$$\begin{aligned}
\frac{F(a^{\star}(m,\omega), m+\epsilon) - F(a^{\star}(m,\omega), m)}{\epsilon} \leq \frac{B(m+\epsilon,\omega) - B(m,\omega)}{\epsilon} \\
\leq \frac{F(a^{\star}(m+\epsilon,\omega), m+\epsilon) - F(a^{\star}(m+\epsilon,\omega), m)}{\epsilon}.
\end{aligned}$$

Since $F(a,m)$ is continuous and concave in a, and, $a^{\star}(m,\omega)$ is continuous in m, the following limits exist (Rockafellar, 1970, Theorem 24.1, pp.227-228), and the inequality ordering is preserved in the limit:

$$\begin{aligned}
\lim_{\epsilon\searrow 0} \frac{F(a^{\star}(m,\omega), m+\epsilon) - F(a^{\star}(m,\omega), m)}{\epsilon} \leq \lim_{\epsilon\searrow 0} \frac{B(m+\epsilon,\omega) - B(m,\omega)}{\epsilon} \\
\leq \lim_{\epsilon\searrow 0} \frac{F(a^{\star}(m+\epsilon,\omega), m+\epsilon) - F(a^{\star}(m+\epsilon,\omega), m)}{\epsilon}.
\end{aligned}$$

Since $\lim_{\epsilon\searrow 0} a^{\star}(m+\epsilon,\omega) = a^{\star}(m,\omega)$, the inequalities above are equivalent to

$$\begin{aligned}
& b^{\star}(m,\omega)\left[J_1(a^{\star}(m,\omega),\omega) - \frac{\beta}{1+\tau\omega}\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right] \\
& + \frac{\beta}{1+\tau\omega}\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \\
& \quad \leq B_1(m^{+},\omega) \\
& \quad\quad \leq b^{\star}(m,\omega)\left[J_1(a^{\star}(m,\omega),\omega) - \frac{\beta}{1+\tau\omega}\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right] \\
& \quad\quad\quad + \frac{\beta}{1+\tau\omega}\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right),
\end{aligned}$$

where

$$\bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)$$
$$:= \lim_{\epsilon \searrow 0} (1+\tau)\left(\frac{\omega_{+1}}{\omega}\right)\left[\bar{V}\left(\frac{\omega(m+\epsilon)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]/\epsilon.$$

However, observe that the first and the last terms in the inequalities are identical. Thus we must have that the right derivative of $B(\cdot, \omega)$ satisfies

$$B_1(m^+, \omega) = b^\star(m, \omega)\left[J_1(a^\star(m, \omega), \omega) - \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]$$
$$+ \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right).$$

By a similar process to arrive at the left derivative of $B(\cdot, \omega)$, we have

$$B_1(m^-, \omega) = b^\star(m, \omega)\left[J_1(a^\star(m, \omega), \omega) - \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]$$
$$+ \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right),$$

where

$$\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)$$
$$:= (1+\tau)\left(\frac{\omega_{+1}}{\omega}\right)\lim_{\epsilon \searrow 0}\left\{\frac{1}{\epsilon}\left[\bar{V}\left(\frac{\omega(m-\epsilon)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]\right\}.$$

Using the result from (C.9) in Lemma 4 on page OA-§.C. 16, we can re-write these right- and left-derivative functions, respectively, as

$$B_1(m^+, \omega) = b^\star(m, \omega)\left(u^Q\right)_1[x^\star(m, \omega), x^\star(m, \omega)]$$
$$+ \frac{\beta[1-b^\star(m, \omega)]}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right), \quad \text{(C.12)}$$

and,

$$B_1(m^-, \omega) = b^\star(m, \omega)\left(u^Q\right)_1[x^\star(m, \omega), x^\star(m, \omega)]$$
$$+ \frac{\beta[1-b^\star(m, \omega)]}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right). \quad \text{(C.13)}$$

From (C.12) and (C.13), it is apparent that $B_1(m, \omega)$ exists if and only if the left- and right-derivatives of $\bar{V}(\cdot, \omega_{+1})$ exist and they agree at the continuation state from m, i.e., if

$$\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) = \bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) = \bar{V}_1\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right).$$

This proves the first part of the statement in the Lemma.

Since $\bar{V}(\cdot, \omega_{+1}) \in \mathcal{V}[0, \bar{m}]$, it is concave and increasing in m, and therefore,

$$\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \geq \bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \geq 0.$$

Since we assumed $b^\star(m, \omega) \in (0, 1]$, and by Lemma 4, we have

$$J_1(a^\star(m, \omega), \omega) \equiv \left(u^Q\right)_1 [x^\star(m, \omega), x^\star(m, \omega)] > 0,$$

then (C.12) and (C.13) imply that the first-order left and right derivatives of $B(\cdot, \omega_{+1})$ satisfy:

$$B_1(m^-, \omega) \geq B_1(m^+, \omega) \geq b^\star(m, \omega)\left(u^Q\right)_1 [x^\star(m, \omega), x^\star(m, \omega)] > 0.$$

From this ordering, we can conclude that if $b^\star(m, \omega) > 0$, the buyer's valuation $B(m, \omega_{+1})$ is *strictly* increasing with his money balance, m. This proves the last part of the statement in the Lemma. □

Lemma 6. *For any (m, ω), where $m \in [k, \bar{m}]$ and the buyer matching probability is positive $b^\star(m, \omega) > 0$, the optimal policy functions $b^\star$and $x^\star$, respectively, satisfy the first-order conditions* (2.20) *and* (2.21).

Proof. We want to show that the first order conditions characterizing the optimal policy functions $b^\star$and $x^\star$, are indeed (2.20) and (2.21). It is immediate that the objective function (2.19) is continuously differentiable with respect to the choice $b \in [0, 1]$. Holding fixed x, if the optimal choice for b is interior, $b^\star(m, \omega) \in (0, 1)$, then it must satisfies the first order condition (2.20) with respect to b:

$$\begin{aligned} u^Q[x^\star(m,\omega), b^\star(m,\omega)] + b^\star(m,\omega)\left(u^Q\right)_2 [x^\star(m,\omega), b^\star(m,\omega)] \\ = \beta\left[\bar{V}(\phi(m,\omega), \omega_{+1}) - \bar{V}(\phi^\star(m,\omega), \omega_{+1})\right]. \end{aligned}$$

The first order condition with respect to x is more subtle. We can derive it by first defining one-sided derivatives of $B(\cdot, \omega)$. Assuming beginning-of-next-period residual balance after current DM trade is positive—i.e.,

$$\phi^{\star}(m,\omega)=\frac{\omega\left[m-x^{\star}(m,\omega)\right]+\tau}{\omega_{+1}(1+\tau)}>0. \tag{C.14}$$

Since (C.14) holds, and since we have shown in Lemma 3 that $x^{\star}(m,\omega)$ and $\phi^{\star}(m,\omega)$ are continuous in $m\in[k,\bar{m}]$, then

$$(\phi^{\star})^{+}(m,\omega):=\frac{\omega\left[m+\varepsilon-x^{\star}(m,\omega)\right]+\tau}{\omega_{+1}(1+\tau)},$$

and,

$$(\phi^{\star})^{-}(m,\omega):=\frac{\omega\left[m-\varepsilon-x^{\star}(m,\omega)\right]+\tau}{\omega_{+1}(1+\tau)},$$

exist and are feasible (or affordable). From (2.19), the DM buyer's one-sided derivatives of $B(\cdot,\omega)$—i.e., its left- or right-marginal valuation of initial money balance—are, respectively,

$$\begin{aligned}B_1\left(m^{+},\omega\right)=&\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\\&\times\left\{\left[1-b^{\star}(m,\omega)\right]\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}(1+\tau)},\omega\right)+b^{\star}(m,\omega)\,\bar{V}_1\left[(\phi^{\star})^{+}(m,\omega),\omega_{+1}\right]\right\},\end{aligned} \tag{C.15}$$

and,

$$\begin{aligned}B_1\left(m^{-},\omega\right)=&\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\\&\times\left\{\left[1-b^{\star}(m,\omega)\right]\bar{V}_1\left(\frac{\omega m^{-}+\tau}{\omega_{+1}(1+\tau)},\omega\right)+b^{\star}(m,\omega)\,\bar{V}_1\left[(\phi^{\star})^{-}(m,\omega),\omega_{+1}\right]\right\},\end{aligned} \tag{C.16}$$

where

$$\begin{aligned}&\bar{V}_1\left(\frac{\omega m^{\pm}+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\\&:=(1+\tau)\left(\frac{\omega_{+1}}{\omega}\right)\lim_{\varepsilon\searrow 0}\left\{\frac{1}{\varepsilon}\left[\bar{V}\left(\frac{\omega(m\pm\epsilon)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega\right)\right]\right\}.\end{aligned}$$

From Lemma 5, we have shown by change of variable, that the one-sided derivatives of $B(\cdot,\omega)$ also satisfy (C.15) and (C.16). These are repeated here for convenience as the following equations:

$$B_1\left(m^+,\omega\right) = \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\left[1-b^\star\left(m,\omega\right)\right]\bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}\left(1+\tau\right)},\omega\right) + b^\star\left(m,\omega\right)\left(u^Q\right)_1\left[x^\star\left(m,\omega\right),b^\star\left(m,\omega\right)\right], \quad \text{(C.17)}$$

and,

$$B_1\left(m^-,\omega\right) = \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\left[1-b^\star\left(m,\omega\right)\right]\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}\left(1+\tau\right)},\omega\right) + b^\star\left(m,\omega\right)\left(u^Q\right)_1\left[x^\star\left(m,\omega\right),b^\star\left(m,\omega\right)\right]. \quad \text{(C.18)}$$

From the last term on the RHS of each of equations (C.15), (C.16), (C.17), and, (C.18), we have the observation that

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\left(\phi^\star\right)^+\left(m,\omega\right),\omega_{+1}\right] = \frac{\beta}{1+\tau\omega}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\left(\phi^\star\right)^-\left(m,\omega\right),\omega_{+1}\right] = \left(u^Q\right)_1\left[x^\star\left(m,\omega\right),b^\star\left(m,\omega\right)\right].$$

Since these marginal valuation functions are evaluated at the DM buyer's optimal choice, it must be that $\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\left(\phi^\star\right)^+\left(m,\omega\right),\omega_{+1}\right] = \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\left(\phi^\star\right)^-\left(m,\omega\right),\omega_{+1}\right] = \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\phi^\star(m,\omega),\omega_{+1}\right]$, and, that this satisfies the first order condition (2.21), which is

$$\left(u^Q\right)_1\left[x^\star(m,\omega),b^\star(m,\omega)\right] = \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\phi^\star(m,\omega),\omega_{+1}\right].$$

□

D Algorithm for finding a SME

We compute a SME as follows.

1. Fix a guess ω and guess $\bar{V}\left(\cdot,\omega\right) \in \mathcal{V}\left[0,\bar{m}\right]$.
2. Solve for CM policy and value functions:
   - We know $C^\star\left(m,\omega\right) = \bar{C}^\star$ already using equation (2.16).
   - For fixed $\bar{C}^\star$, and, given guess of $\bar{V}\left(\cdot,\omega\right)$, iterate on Bellman equation (A.3) solving a one-dimensional (1D) optimization problem over choice $y^\star\left(\cdot,\omega\right)$.
     - Note: By equation (2.17), the solution $y^\star\left(\cdot,\omega\right) = \bar{y}^*\left(\omega\right)$ should be a constant with respect to m.

- Back out $l^{\star}(m, \omega)$ using the binding budget constraint (2.18).
- Store value function $W^{\star}(\cdot, \omega)$.

3. Solve for DM policy and value functions:

   - For each $m \leq k$, set
     - $b^{\star}(m, \omega) = x^{\star}(m, \omega) = q^{\star}(m, \omega) = 0$
     - $B(m, \omega) = \beta \bar{V}[\phi(m, \omega), \omega_{+1}]$,

     where $\phi(m, \omega) := (m + \tau)/(1 + \tau\omega)$.
   - For each $m \in [k, \bar{m}]$,
     - Invert first-order condition (2.21) to obtain implicit $b[m, x(m, \omega), \omega]$.
     - Plug the implicit expression for $b[m, x(x, \omega), \omega]$ into Bellman equation (2.19), and do a 1D optimization over choices $x(m, \omega)$.
     - Get optimizer $x^{\star}(m, \omega)$ and corresponding value $B^{\star}(m, \omega)$.
     - Use previous step to now back out $b^{\star}(m, \omega)$.

4. Solve ex-ante decision problem:

   - Given approximants $W^{\star}(m, \omega)$ and $B^{\star}(m, \omega)$, solve the lottery problem (2.9) and (2.8).
   - Get policies $\left\{\pi_1^{j,\star}(m, \omega)\right\}_{j \in J}$ and $\left\{z_1^{j,\star}(m, \omega), z_2^{j,\star}(m, \omega)\right\}_{j \in J}$, where J is endogenous to the solution of (2.9) and (2.8).
   - Get value of the problem (2.9) and (2.8) as $V^{\star}(\cdot, \omega)$.

5. Construct the approximant of the ex-ante value function, $\bar{V}^{\star}(\cdot, \omega) = (1 - \alpha) V^{\star}(\cdot, \omega) + \alpha W^{\star}(\cdot, \omega)$.

6. Given policy functions from Steps 2-4, construct limiting distribution $G(\cdot, \omega)$ solving the implicit Markov map (2.26). (See details in Section E on page OA-§.E. 26.)

   - Check if market clearing condition (2.25) holds.
   - If not,
     - generate new guess and set $\omega \leftarrow \omega_{new}$;
     - set $\bar{V}(\cdot, \omega) \leftarrow \bar{V}^{\star}(\cdot, \omega)$; and
     - repeat Steps 2-6 again until (2.25) holds.

Algorithm 1 on page OA-§.E. 26 summarizes the steps above with reference to function names in our actual `Python` implementation. Algorithm 1 is called `SolveSteadyState` in our `Python` class file `cssegmod.py`.

Algorithm 1 Solving for an SME

Require: $\alpha \in [0,1)$, $\omega > 0$, $\bar{V}(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$, $N_{\max} > 0$

for $n \leq N_{\max}$ **do**
  $(W^\star, \bar{C}^\star, l^\star, y^\star) \leftarrow \texttt{WorkerProblem}(\bar{V}, \omega)$
  $(B^\star, b^\star, x^\star, q^\star) \leftarrow \texttt{BuyerProblem}(\bar{V}, \omega)$
  $\tilde{V} \leftarrow \max\{B^\star(\cdot,\omega), W^\star(\cdot - \chi, \omega)\}$
  $\left(V^\star, \{z^{\star,j}, \pi^{\star,j}\}_{j\in J}\right) \leftarrow \texttt{ConvexHull}\left[\text{graph}\left(\tilde{V}\right)\right]$
  $\bar{V}^\star \leftarrow \alpha W^\star + (1-\alpha)V^\star$
  $\mathbf{v} \leftarrow (\bar{V}, B, W)$
  $\mathbf{p} \leftarrow \langle \{\pi_1^{j,\star}, z_1^{j,\star}, z_2^{j,\star}\}_{j\in J}, (b^\star, x^\star, y^\star, \bar{C}^\star)\rangle$
  $G \leftarrow \texttt{Distribution}(\mathbf{p}, \mathbf{v})$
  $\omega^\star \leftarrow \texttt{MarketClearing}(G)$
  $e \leftarrow \max\{\|\bar{V}^\star - \bar{V}\|, \|\omega^\star - \omega\|\}$
  if $e < \varepsilon$ **then**
    STOP
  else
    $(\bar{V}, \omega) \leftarrow (\bar{V}^\star, \omega^\star)$
    CONTINUE
  end if
end for
  return $\mathbf{p}, \mathbf{v}, G, \omega^\star$

E Monte Carlo simulation of stationary distribution

We use a Monte Carlo method to approximate the steady-state distribution of agents at each fixed value of the aggregate state ω, in the `Distribution` step in Algorithm 1.

For any current outcome of an agent named (m, ω) we can evaluate her ex-post optimal choices in either the CM (2.5), or the DM (2.6). The outcomes of the decision at each current state for an agent is summarized in Algorithm 2. In words, these go as follows: First, we must identify where the agent is currently in (DM or CM). Second, we evaluate the corresponding decisions and record the agent's end-of-period money balance as m'. Associated with each realized identity m we would also have a record of the agent's actions in that period, e.g., $y^\star(m,\omega)$ and $l^\star(m,\omega)$ if the agent was in the CM, or, $x^\star(m,\omega)$ and $b^\star(m,\omega)$ if she was in the DM.

Algorithm 2 is then embedded in Algorithm 3 below, the Monte Carlo approximation of the steady state distribution at ω. We begin, without loss, from an agent who had just accumulated money balances at the end of a CM, and track the evolution of the agent's money balances over the horizon $T \to +\infty$. Theorem 3 implies that if ω is any candidate equilibrium price, and $G(\cdot,\omega)$ is the unique limiting distribution of agents associated with the candidate equilibrium, then the agent will visit each of all possible states $(m,\omega) \in \text{supp}G(\cdot,\omega)$ with frequency $\mathrm{d}G(m,\omega)$, as $T \to +\infty$.

Algorithm 3 does the following:

Algorithm 2 `ExPostDecisions( )`

Require: ω, $(B, W) \leftarrow \mathbf{v}$, $(b^\star, x^\star, y^\star) \leftarrow \mathbf{p}$

if $W(m - \chi, \omega) \geq B(m, \omega)$ **then**
    $m' \leftarrow y^\star(m - \chi, \omega)$
else
    Get $u \sim \mathbf{U}[0, 1]$
    if $u \in [0, b^\star(m, \omega)]$ **then**
        Get $x^\star(m, \omega) > 0$
        Get $b^\star(m, \omega) > 0$
        $m' \leftarrow m - x^\star(m, \omega)$
    else
        $x^\star(m, \omega) \leftarrow 0$
        $b^\star(m, \omega) \leftarrow 0$
        $m' \leftarrow m$
    end if
end if
    return m'

1. Begin with an arbitrary agent m.
2. At the start of each date $t \leq T$:
   (a) The agent realizes the shock $z \sim (\alpha, 1 - \alpha)$.
   (b) Conditional on the shock z, the agent goes to the CM for sure (and costlessly), or, makes the ex-ante lottery decision.
   (c) If the agent has to solve the ex-ante decision problem, then we evaluate the corresponding ex-post decisions of the agent.

The main output of Algorithm 3 is the list m^T, which stores the stochastic realization of an agent's money balances each period. The long run distribution of the sample m^T is used to approximate $G(\cdot, \omega)$. Algorithms 2 and 3 can be found in the `Python` class `cssegmod.py`, respectively, as methods `ExPostDecisions` and `Distribution`.

Note that the function `Distribution` will be called each time we have an updated guess of ω. Because the Monte-Carlo problem is serially dependent, the only way to speed up the evaluations at this point is to compile it to machine code and execute it on the fly. The user will have the option to exploit `Numba` (a `Python` interface to the `LLVM` just-in-time compiler).

Algorithm 3 `Distribution( )`

Require: $\mathbf{v} \leftarrow (\bar{V}, B, W)$, $\mathbf{p} \leftarrow \langle \{\pi_1^{j,\star}, z_1^{j,\star}, z_2^{j,\star}\}_{j \in J}, (b^\star, x^\star, y^\star, \bar{C}^\star) \rangle$, T, ω

Get $\phi(m, \omega) \leftarrow \frac{m+\tau}{(1+\tau\omega)(1-\delta)}$
Set $m^T \leftarrow \emptyset$
$m \leftarrow y^\star(0, \omega)$
for $t \leq T$ **do**
    $m \leftarrow \phi(m, \omega)$
    Get $u \sim \mathbf{U}[0, 1]$
    if $u \in [0, \alpha]$ **then**
        $m' \leftarrow y^\star(m, \omega)$
    else
        if $\exists j \in J$ and $m \in \left[z_1^{j,\star}(m, \omega), z_2^{j,\star}(m, \omega)\right]$ **then**
            Get $u \sim \mathbf{U}[0, 1]$
            if $u \in \left[0, \pi_1^{j,\star}(m, \omega)\right]$ **then**
                $m \leftarrow z_1^{j,\star}(m, \omega)$
            else
                $m \leftarrow z_2^{j,\star}(m, \omega)$
            end if
        end if
        $m' \leftarrow$ `ExPostDecisions`$(m, \omega, \mathbf{p}, \mathbf{v})$
    end if
    Set $m^T \leftarrow m^T \cup \{m\}$
    Set $m \leftarrow m'$
end for
    return m^T