Inflationary Redistribution vs. Trading Opportunities:

A New-Monetarist Heterogeneous-agent Quantitative Theory[*]

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Abstract

We develop a New-Monetarist heterogenous-agent model applicable towards quantitative questions. Our framework combines tractable competitive search frictions and endogenous Walrasian market participation. Market incompleteness arises from equilibrium frictions in non-Walrasian markets (where money is essential) and endogenous limited participation in Walrasian markets for liquidity risk management. The model accounts for opposing intensive-versus-extensive margins of trade-off of inflation tax. Quantitatively, welfare falls, but liquid-wealth inequality falls and then rises, with inflation. The latter is because the extensive margins dominate for sufficiently high inflation.

JEL Codes: E0; E4; E5; E6; C6

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1 Introduction

In this paper, we propose an heterogeneous-agent New Monetarist model which is applicable towards quantitative questions. Using this basic quantitative theory, we ask what are the distributional consequences of anticipated inflation tax on the economy in the long run. We also use it to quantify the welfare cost of inflation.[1]

We make two contributions in this paper. Our first contribution is in developing a quantitative model with competitive search-and-matching frictions featuring endogenous Walrasian market participation. Apart from these two main frictions, there are no other ad-hoc or parametrized frictions. Using this model, we show that the effect of anticipated inflation tax can be non-motonone on liquid-wealth inequality, and as a result, the welfare cost of inflation can be quantitatively sizeable. In standard heterogeneous-agent monetary models (see, e.g., Imrohoroğlu and Prescott, 1991a; Akyol, 2004; Boel and Camera, 2009; Meh et al., 2010), inflation tends to be a redistributive tax, since there is only an intensive margin through which inflation tax works. That is, with increasing inflation, agents would like to reduce their money holdings. Those with high balances reduce their holdings more relative to those at the bottom end of the distribution. This tends to lower average money balance. Hence, inflation acts as a progressive tax.[2]

However, there is an opposing extensive margin effect in our model. With higher inflation, agents are also spending faster in decentralized trades and entering the Walrasian market to rebalance their liquidity more frequently.[3] Higher Walrasian-market participa-

[1] This long-standing question has been studied through various lenses, beginning from a statistical money demand function approach (Bailey, 1956) to general equilibrium models without agent heterogeneity (see, e.g., Lucas, 2000; Dotsey and Ireland, 1996), and, to models with equilibrium distributions (see, e.g., Imrohoroğlu and Prescott, 1991b,a; Erosa and Ventura, 2002; Molico, 2006; Chiu and Molico, 2010; Camera and Chien, 2014; Wen, 2015).

[2] However this conclusion need not be robust, as it may depend on the nature of idiosyncratic shocks, financial structure and the sensitivity of labor supply to real wage changes: Camera and Chien (2014). For example, in their model with a reduced-form transaction technology that is free from inflationary effects, Erosa and Ventura (2002) showed that for sufficiently high returns to scale of the technology, inflation can become a regressive tax. In a different Bewley economy, where money plays a pure role of buffer-stock insurance, Wen (2015) showed that inflation can increase agents' consumption risks by tightening poorer agents' ad-hoc borrowing limits. We have similar conclusions, but we do not impose ad hoc transactions technology or borrowing limits.

[3] In a random-matching model of money, Molico (2006) showed that a "real balance effect" —i.e., agents choosing to carry less money balances into decentralized trades as inflation increases which results in a higher amount of money paid per goods—tends to work against the progressive-tax effect of inflation. As we will show, there is a similar property in our model, but with a twist. In our setting with competitive search in decentralized trades, there is an additional extensive margin effect: Higher inflation exacts a greater downsize risk of not matching for agents by reducing the equilibrium matching probability for buyers (*contra*. random matching). Although expected money carried in each decentralized trade will be lower per payment for goods, with lower equilibrium probability of matching, agents who match don't have to reduce consumption as much—this trade-off between matching probability and quantity of goods in the competitive search environment (see, e.g., Peters, 1984, 1991; Moen, 1997; Burdett et al., 2001; Julien et al., 2008; Shi, 2008) helps to amplify the speed at which agents expect to get rid of their money in decentralized trades.

In a follow-up paper, Chiu and Molico (2010) introduce an extensive margin of costly participation in centralized markets. In contrast to our setting, theirs maintain a random matching assumption in decentralized-market meetings. Also, with competitive search, as originally pointed out by Menzio et al. (2013), the comtion implies that there are more agents at the bottom end of the equilibrium distribution at the end of each period. These are agents who will enter the Walrasian market in the subsequent period. Also, there are less agents at the top end since they top up with less liquidity in the Walrasian market, and, they spend faster in the decentralized market.

We calibrate our model to U.S. data to quantitatively analyze the resolution of the tension between these two opposing forces. We find that the Gini inequality index rises (eventually) with inflation, although it initially falls with inflation at lower levels of inflation. This is because for low inflation ranges, the redistributive effect dominates the extensive margin, but the tension goes the other way, when inflation is high enough. (That is, there is an inflation-inequality "smile".)

Our second contribution is in terms of efficient enumeration and computation of endogenous fair lotteries when value functions are neither concave nor convex. Such a property arises because of the bilinear interaction between endogenous probabilities of matching and the payoffs arising from matching outcomes (see also Menzio et al., 2013). We propose a novel and fast computational solution method, taking insights from computational geometry, to efficiently compute all possible equilibrium lotteries. We then provide an efficient class of open-source methods for solving and analyzing such equilibria.[4]

A slight adaptation of our model can be shown to inherit from the model of Rocheteau and Wright (2005), in particular, their version with competitive search in decentralized markets. Our extension here deals with an equilibrium non-degenerate distribution of agent types. Also, if we specialized our model in another direction, we return to the theoretical setting of Menzio et al. (2013). In Menzio et al. (2013) they provide a characterization for the existence of a unique stationary monetary equilibrium with tractable non-degenerate distribution of agents. However, their characterization only holds in a zero-inflation setting. Our work here takes the characterization further in dealing with existence of stationary monetary equilibria when inflation can be away from zero.[5] Our work relates to and builds upon these two intellectual foundations, and goes further on a quantitative-theoretic trajectory.

Another class of elegant models with microfounded liquidity demand, that have non-degenerate distributions, can be found in Rocheteau et al. (2018a) and Rocheteau et al.

putation of equilibria is much more tractable—there is block recursivity between agents' dynamic programs and the determination of equilibrium distributions and relative prices. With random-search and bargaining, one needs to keep track of agent distributions as state variables when solving their dynamic programs.

[4]Source Python codes are available from our public GitHub repository.

[5]In fact, this is also what Sun and Zhou (2018) do, but they assume that all agents in a DM must exit it in one period and must enter a CM (with quasilinear preferences) thereafter. In their equilibrium, agent heterogeneity (i.e., non-degeneracy of the equilibrium distribution of agents) needs to be preserved by additionally assuming that there are idiosyncratic exogenous preference shocks to agents in the CM. In contrast, we can have a non-degenerate distribution of agents in our model, since not all agents will find it worthwhile to enter the CM at any period, in an equilibrium.

(2018b). The former is a continuous-time setting and the latter is in discrete time. In their settings, decentralized search and matching processes are done through random matching (or, equivalently, agents face an early consumption demand shock that requires them to trade in anonymous decentralized markets). In contrast, our purpose here is to understand and quantify the effects of extensive-versus-intensive margins that arise from competitive search decentralized markets.

2 Model environment

Our incomplete-markets framework combines the tractable features of competitive search and matching frictions with endogenous participation of agents in a (complete) Walrasian centralized market.[6] There is a decentralized market (DM) with competitive search and matching friction and a Walrasian centralized market (CM). Time is discrete and indexed by $t \in \mathbb{N}$. Hereinafter, we will denote $X := X_t$ and $X_{+1} := X_{t+1}$ for dynamic variables.

2.1 Money supply

Money is taken to be any asset that can be used as a medium of exchange—i.e., a tradable claim on consumption goods that does not require contracting on specific individual trader's characteristics or trading histories. We assume that the total stock of money in the economy M grows according to the process

$$\frac{M_{+1}}{M} = 1 + \tau, \tag{2.1}$$

where $\tau > \beta - 1$.

Labor is the numeraire good. If we denote ωM as the current nominal wage rate, where ω is normalized nominal wage (i.e., nominal wage rate per units of M), then a dollar's worth of money is equivalent to $1/\omega M$ units of labor. The variable ω will be endogenously determined in a monetary equilibrium. If M is the beginning of period aggregate stock of money in circulation, then $1/\omega = M \times 1/\omega M$ is the beginning of period real aggregate (per-capita) stock of money, measured in units of labor. Denote (equilibrium) nominal wage growth as $\gamma(\tau) \equiv \omega_{+1} M_{+1}/(\omega M)$. Later, for a stationary monetary equilibrium, we will require that equilibrium nominal wage grows at the same rate as money supply, i.e., $\gamma(\tau)|_{(\omega_{+1}=\omega)} = M_{+1}/M$.

[6] Market incompleteness will arise from two features of the model: First, equilibrium matching in non-Walrasian markets (where money is essential) implies that agents face ex-ante uncertainty over being able to exchange and consume in those markets. Since agents are "anonymous" in these markets, their individual trading risks cannot be insured away by exchanging private state-contingent securities. As a result, we have ex-post agent heterogeneity. Second, because agents have a convex preference over goods from the Walrasian centralized and the non-Walrasian decentralized markets, there is endogenous limited participation in Walrasian markets. Agents engage in the Walrasian markets for liquidity risk management.

2.2 Markets, agents, commodities and information

There are two types of markets that open simultaneously every period: a centralized market (CM) and a decentralized market (DM). There is a measure one of individuals who will decide at the beginning of each date which market to participate in. Firms act in both CM and DM at the same time. An individual can only be in the CM or DM at a given time period. In the DM, individuals shop for special goods q. In the CM individuals supply labor l, and, consume a general good C. A firm in the CM hires labor to produce the general CM good and the special DM goods. We describe the CM and DM markets in turn.

In the CM, two markets are open: A competitive spot market for labor and a competitive general good market; the latter is equivalent to a competitive market for trading in a complete set of individual-state-contingent consumption claims. Agents trade in these securities to insure their consumption risk, which arises from their heterogeneous trading histories upon exit from the DM. They may still demand money as a precaution against the need for liquidity in anonymous markets in the DM.

In the DM, we have a setting similar to Menzio et al. (2013) where there is an information friction: Buyers of special DM goods, q, are anonymous and cannot trade using private claims or cannot undertake contracts with selling firms. As a result, the only medium of exchange is money. There is a finite set of *types* of individuals and goods, I. There is a continuum of individuals and firms of type $i \in I$, where an individual i consumes good i and produces good $i+1$ (mod-$|I|$). A type i firm hires labor service from type $i-1$ (mod-$|I|$) individuals (from the CM spot labor market) and transforms it (linearly) into the same amount of DM good i. Each i-type firm commits to posted terms of trade in all submarkets it chooses to enter. Buyers of good i direct their search toward these submarkets that sell good i, by choosing the best terms of trade offered. However, as we will see, these buyers will have to balance their decision on terms of trades against the probability of getting matched. Since firms and buyers choose which submarket to participate in, a type i buyer will only participate in the submarkets where type i firms sell.[7]

2.2.1 Preference representation

The per-period utility function of an individual is

$$U(C) - h(l) + u(q). \tag{2.2}$$

[7] Hereinafter, the explicit dependency on the type of good $i \in I$ will become unnecessary. It will turn out that terms of trade in every submarket, indexed by pairs of buyers' willingness to pay and consume $\{(x, q)\}$, identifies an equivalent class of submarkets.

We assume that the functions U and u are continuously differentiable, strictly increasing, strictly concave, $U_1, u_1 > 0$, $U_{11}, u_{11} < 0$, and the following boundary conditions hold: $U(0) = u(0) = U_1(\infty) = u_1(\infty) = 0$, and $U_1(0), u_1(0) < \infty$. Also, we assume that $h(l) = Al$, where $A > 0$ is some constant.[8] Also, the upper bound on money holdings $\bar{m} \in (0, \infty)$ and preferences (through parameter A) are such that:[9]

$$0 < \bar{m} < (U_1)^{-1}(A) \iff A < U_1(\bar{m}) < U_1(0). \tag{2.3}$$

2.2.2 Matching technology in the DM

We follow the assumptions of Menzio et al. (2013) in the setting below. Let $\theta \in \mathbb{R}_+$ denote the ratio of trading posts to buyers in a submarket—i.e., its market tightness. In a submarket with tightness θ, the probability that a buyer is matched with a trading post is $b = \lambda(\theta)$. The probability a trading post is matched with a buyer is $s = \rho(\theta) := \lambda(\theta)/\theta$. We assume that the function $\lambda : \mathbb{R}_+ \to [0,1]$ is strictly increasing, with $\lambda(0) = 0$, and $\lambda(\infty) = 1$. The function $\rho(\theta)$ is strictly decreasing, with $\rho(0) = 1$, and $\rho(\infty) = 0$. We can re-write a trading post's matching probability $s = \rho(\theta) = \rho \circ \lambda^{-1}(b) \equiv \mu(b)$. Observe that the matching function μ is a decreasing function, and that $\mu(0) = 1$ and $\mu(1) = 0$. Assume that $1/\mu(b)$ is strictly convex in b.

2.2.3 Firms

Consider a firm $i \in [0,1]$ that takes the CM good's relative price p (in units of labor) as given. The firm hires labor on the spot market and transforms hired labor services into Y units of CM good linearly. At the same time, in the DM, the firm takes the terms of trade, respectively measured by buyers' payment and demand of the good, (x, q), in a given submarket for some type-i good as given, and chooses the measure of trading posts (viz., shops) $\mathrm{d}N(x, q)$ to open in each submarket.[10] The firm in the DM takes the (equilibrium) probability of being matched with a buyer $s(x, q)$ as given. If x is what a matched buyer is willing to pay for q, then $x \cdot s(x, q)$ is the firm's expected revenue in submarket (x, q). To produce q the firm must hire $c(q)$ units of labor. Hence $s(x, q)c(q)$ is its expected labor wage bill at submarket (x, q). We assume that $q \mapsto c(q)$ is a continuous convex function. The firm also pays a per-period fixed cost k of creating the trading post in submarket

[8] We will use the notational convention, $f_i(x_1, ..., x_n) \equiv \partial f(x_1, ..., x_n)/\partial x_i$, to denote the value of the partial derivative of a function f with respect to its i-th variable. Likewise, f_{ij} will denote its cross-partial derivative function with respect to the j-th variable.

[9] This regularization will ensure that the agent's labor effort is always finite, $l^\star(m, \omega) \in \mathrm{int}(\mathbb{R}_+)$, and that in all dates, money balances are bounded, $m \in [0, \bar{m}]$. Note that this assumption is similar to the one in Sun and Zhou (2018), except that in the latter, the authors' equivalent of A is the upper bound on some random variable (a preference shock).

[10] This is equivalent to stating that the firms commit to posted terms of trade in the particular submarket(s).

(x, q). The firm's value is:

$$\pi(p, x, q; k) = \max_{Y \in \mathbb{R}_+} \{pY - Y\} + \max_{\mathrm{d}N(x,q) \in \mathbb{R}_+} \int \{s(x, q) [x - c(q)] - k\} \, \mathrm{d}N(x, q), \qquad \forall (x, q). \tag{2.4}$$

The first term on the RHS is the firm's value from operating in the CM. The second, is its DM total expected value across all submarkets it chooses to operate in.

2.3 Individuals' decisions

An individual is identified by her current money balance (measured in units of labor), m. Given policy τ, her decisions also depend on knowing the aggregate wage ω. Denote the relevant state vector as $\mathbf{s} := (m, \omega) \in S = [0, \bar{m}] \times [0, +\infty)$.[11] At the beginning of a period (ex ante), an individual decides whether to work and consume in the CM, or, whether to be a buyer in the frictional DM. Ex post, if the agent has positive initial money balance as a DM buyer, he continues searching for a trading post. Also, ex post, another agent is in the CM either because she had previously expended all her money in a DM submarket, or, she finds it optimal to go to CM even with positive money balance.[12] We describe these different ex-post agents' problems in turn, and then, we will describe an agent's ex-ante decision problem.

2.3.1 Ex-post individual in the CM

Suppose now we have an individual $\mathbf{s} := (m, \omega)$ who begins the current period in the CM. The individual takes policy, τ, and the sequence of aggregate prices, $(\omega, \omega_{+1}, ...)$, as given. Her value from optimally consuming C, supplying labor l, and accumulating end-of-period money balance y, is

$$W(\mathbf{s}) = \max_{(C,l,y) \in \mathbb{R}^3_+} \left\{ U(C) - h(l) + \beta \bar{V}(\mathbf{s}_{+1}) : pC + y \leq m + l, \ m_{+1} = \frac{\omega y + \tau}{\omega_{+1}(1 + \tau)} \right\}, \tag{2.5}$$

[11] In a steady state equilibrium, ω is a constant.

[12] Our assumption here is different to that of Sun and Zhou (2018). In Sun and Zhou (2018), all individuals get to go into the CM deterministically in one period, given that they are currently in the DM. Individuals choose whether to go into the DM submarkets, as in our ex ante agent's problem. However, since Sun and Zhou (2018) also have a quasilinear preference representation in the CM, they also require that agents receive idiosyncratic preference shocks (i.e., labor supply shocks) to prevent degeneracy of the equilibrium distribution of agents. In contrast, our assumption here still preserves non-degeneracy without an additional assumption of exogenous preference shocks, since there is always a positive measure of agents who will be stuck trading in the DM submarkets for some time before some of them get to go to the CM.

where $\bar{V} : S \to \mathbb{R}$ is her continuation value function, to be fully described in Section 2.3.3. This continuation value function yields her next-period expected total payoff from state m_{+1}.[13]

2.3.2 Ex-post individual buyer in the DM

Now we focus on an individual who has just decided to be a DM buyer. The buyer can only visit one trading post at a given time by directing search to its terms of trade, (x, q). The individual buyer, $\mathbf{s} := (m, \omega)$, has initial value:

$$B(\mathbf{s}) = \max_{x \in [0,m], q \in \mathbb{R}_+} \left\{ \beta \left[1 - b\left(x, q\right)\right] \left[\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)\right] \right.$$
$$\left. + \, b\left(x, q\right) \left[u(q) + \beta \bar{V}\left(\frac{\omega\left(m - x\right) + \tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)\right] \right\}. \tag{2.6}$$

Consider the first two terms on the RHS of the functional (2.6): With probability $1 - b(x, q)$ the buyer fails to match with the trading post and must thus continue the next period with his initial money balance subject to inflationary transfer. With the complementary probability $b\left(x, q\right)$ he matches with a trading post (x, q), pays the seller x in exchange for a flow payoff $u\left(q\right)$, and then continues into the next period with his net balance, also subject to inflationary tranfers.

2.3.3 Ex ante decision

At the beginning of a period, an agent $\mathbf{s} := (m, \omega)$ plays a fair lottery $(\pi_1, 1 - \pi_1)$ over the prizes $\{z_1, z_2\}$ which yields the value

$$\bar{V}\left(\mathbf{s}\right) = \max_{\pi_1 \in [0,1], z_1, z_2} \left\{ \pi_1 \tilde{V}(z_1, \omega) + \left(1 - \pi_1\right) \tilde{V}\left(z_2, \omega\right) : \pi_1 z_1 + (1 - \pi_1) z_2 = m \right\}; \tag{2.7}$$

and, given a lottery outcome z, the individual decides which markets to participate in, and her value becomes

$$\tilde{V}\left(z, \omega\right) = \begin{cases} \max_{a \in \{0,1\}} \left\{ aW\left(z - \chi, \omega\right) + \left(1 - a\right) B\left(z, \omega\right) \right\}, & z - \chi \geq -y_{\max}\left(\omega; \tau\right) \\ B\left(z, \omega\right), & \text{otherwise} \end{cases},$$

[13] The continuation state for the individual, m_{+1}, is derived as follows: At the end of the CM, the individual would have accumulated balance y (measured in units of labor). In current units of nominal money, this is $\omega M \times y$. At the beginning of next period, each individual gets a nominal transfer of new money τM (population is normalized to size 1). In units of labor next period, the beginning-of-period balance would thus be $m_{+1} = \left(\omega M y + \tau M\right) / \left(\omega_{+1} M_{+1}\right)$. Replacing for M / M_{+1} with the money supply process in (2.1) gives the expression for the individual's continuation state m_{+1} in (2.5).

$$(2.8)$$

where

$$y_{\max}(\omega;\tau) := \begin{cases} \min\left\{\bar{m} - \frac{\tau}{\omega}, \bar{m}\right\}, & \text{if } \tau \geq 0, \\ \max\left\{\bar{m} - \frac{\tau}{\omega}, \bar{m}\right\}, & \text{if } \tau < 0, \end{cases} \tag{2.9}$$

is a natural upper bound on CM saving (in real money balances). We derive this upper bound in the Online Appendix B.

Observe that in equation (2.8), contingent on realizing a lottery payoff z, the outcome of the lottery also induces the pure action of going to the DM or the CM. If the agent decides to go to the CM, he must pay a fixed cost $\chi \geq 0$ (measured in units of labor) to participate in the CM. This fixed cost component is interpretable as a barrier to participation in liquidity-risk managementin the CM for some agents.

Note that the DM-buyer value function B may not be strictly concave due to equilibrium search externalities, which will be inherited by the ex-post value function $\tilde{V}$. It will be profitable for the individual to play a mixed strategy over the set of pure actions of going to the DM or CM.[14]

Remark 1. In the ex-ante market participation problem (2.8), there is a limited short-sale (I.O.U.) constraint $z - \chi \geq -y_{\max}(\omega;\tau)$. It may be possible that an agent, whose state is such that $m < \chi$, when faced with deciding to go to the CM, may still find it optimal to issue an I.O.U. worth $m - \chi$ at the beginning of a CM, and go to work in the CM immediately to repay the shortfall $m - \chi$. Since the fixed cost is levied in the CM, and in the CM promises or contracts are completely sustainable, then a limited amount of short selling (I.O.U.) is possible. The limit on the short sale $m - \chi$, is equivalent to agents exerting the maximal CM labor effort $l_{\max}(\omega;\tau)$ and not saving anything in the CM. In Online Appendix C we derive this limit of $-y_{\max}(\omega;\tau)$.

2.3.4 Special cases

An expanded version of our model can be shown to relate to two elegant intellectual origins. In the Online Appendix A, we entertain the additional feature of an exogenous probability α that ex-ante, each agent may go to the CM costlessly. In that extended setting, we can relate to two well-known models in the literature—i.e., the representative-agent random-matching model with competitive search DM of Rocheteau and Wright

[14] The externality problem shows up mathematically as the bilinear and non-concave interaction between $b(x,q)$ and $u(q)$ in the DM-buyer's objective function in (2.6). These two terms, respectively, are interpretable as an aggregate extensive margin (i.e., how likely is a buyer to trade) and an intensive margin (i.e., how much of q to consume).

(2005), and, the block-recursive ex-post heterogeneous agent model of Menzio et al. (2013).[15]

2.4 Monetary equilibrium

Clearly there exists a non-monetary equilibrium whereby no agent will participate in the DM. In this paper, we restrict attention to the case of a monetary equilibrium. Hereinafter, whenever we refer to "monetary equilibrium", or "equilibrium", we mean a recursive monetary equilibrium—one in which agent's decision functions are recursive and time-invariant maps. In what follows, we first characterize the equilibrium strategy of firms (section 2.4.1), the equilibrium value and decision functions of agents in the CM (section 2.4.2) and in the DM (section 2.4.3), and then close the equilibrium notion by describing the market clearing conditions (section 2.4.4). At the end of this section, we restrict attention to and define formally the notion of a stationary monetary equilibrium (SME).

2.4.1 Equilibrium strategy of firms

A firm's problem is static. We can characterize the equilibrium behavior of a firm given p (in the CM) and any operative submarket (x, q) in the DM. Free entry in the CM will render zero profits to firms in equilibrium, and thus, $p = 1$. Likewise, free entry and zero-profit in the DM with competitive search will imply that

$$r(x,q) := s(x,q)\,[x - c(q)] - k \leq 0, \qquad \text{and,} \qquad \mathrm{d}N(x,q) \geq 0, \tag{2.10}$$

where the weak inequalities would hold with complementary slackness: For a submarket (x, q) such that $r(x, q) < 0$, the firm optimally chooses $\mathrm{d}N(x, q) = 0$. If $r(x, q) = 0$, then the firm is indifferent on $\mathrm{d}N(x, q) \in (0, +\infty)$. We can also deduce that $r(x, q) > 0$ cannot be an equilibrium: If expected profit is positive, then the linear program of the firm in the DM yields an optimum of $\mathrm{d}N(x, q) = +\infty$, but this violates the requirement of zero profits in equilibrium.[16] We will restrict attention to an equilibrium where equation (2.10) also holds for submarkets not visited by any buyer.[17]

[15]The main difference in one limit of our model to Rocheteau and Wright (2005) is that in Rocheteau and Wright (2005), some measure of households become sellers in the DM each period. In our setting, non-DM-buyer households are, in a sense, sellers only insofar as supplying labor to firms that create trading posts in the DM. This is a feature inherited from Menzio et al. (2013).

Our difference to Menzio et al. (2013) is that here, agents derive consumption value in the CM and they need not exhaust all their liquid wealth before deciding to enter the CM again. We extend the theoretical analyses of Menzio et al. (2013) in the direction of inflationary monetary equilibria and prove their existence. We also provide an efficient computational method for solving these models, thus taking the new monetarist literature closer to mainstream quantitative macroeconomics.

[16]If we re-label $N(x, q)$ as the equilibrium distribution of trading post across submarkets, condition (2.10) implies that aggregate profit in the DM is zero: $\int \{s(x,q)\,[x - c(q)] - k\}\,\mathrm{d}N(x,q) = 0$.

[17]Justification for this off-equilibrium-path restriction can be rationalized via a "trembling-hand" sort of argument: Suppose there is some exogenous perturbation that induces an infinitesimally small measure of buyers to show up in every submarket. Given a non-zero measure of buyers present in a submarket, if firms' expected profit is still negative in that submarket, i.e., $r(x, q) < 0$, then the market will not be active. This

From (2.10), we can deduce that

$$s(x,q) \equiv \mu \circ b\,(x,q) = \begin{cases} \frac{k}{x-c(q)} & \iff x - c(q) > k \\ 1 & \iff x - c(q) \leq k \end{cases}. \tag{2.11}$$

Observe that the firm's probability of matching with a buyer, $s(x,q)$ depends only on the posted terms of trade (x,q). Likewise, the buyer's probability of matching with a firm, $b(x,q)$, given the matching technology $\mu : [0,1] \to [0,1]$. Thus, in any submarket with positive measure of buyers, the market tightness, $\theta(x,q) \equiv b(x,q)/s(x,q)$, is neccessarily and sufficiently determined by free entry into the submarket. Moreover, the terms of trade of a submarket (x,q) is sufficient to identify the submarket.

It will be convenient for later to note that we have in equilibrium, implicitly, a relation between q and (x,b). That is, in any equilibrium, each active trading post will produce and trade the quantity:

$$q = Q(x,b) \equiv c^{-1}\left[x - \frac{k}{\mu(b)}\right], \tag{2.12}$$

given payment x and its matching probability $s = \mu(b)$. This relation will allow us to perform a change of variables, and re-write the buyers' problems below in terms choices over (x,b), instead of over (x,q).

2.4.2 Equilibrium and the CM individual

Let us denote $\mathcal{C}[0,\bar{m}]$ as the set of continuous and increasing functions with domain $[0,\bar{m}]$. Then $\mathcal{V}[0,\bar{m}] \subset \mathcal{C}[0,\bar{m}]$ denotes the set of continuous, increasing and concave functions on the domain $[0,\bar{m}]$. We have the following observations of any CM individual's value and policy functions, which apply to both a steady-state equilibrium or along a dynamic equilibrium transition. (Since this is a standard dynamic programming result, proofs of these results are relegated to the online appendix.)

Theorem 1. *Assume $\tau/\omega < \bar{m}$. For a given sequence of prices $\{\omega, \omega_{+1}, ...\}$, the value function of the individual beginning in the CM, $W(\cdot,\omega)$, has the following properties:*

1. *$W(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$, i.e., it is continuous, increasing and concave on $[0,\bar{m}]$. Moreover, it is linear on $[0,\bar{m}]$.*

2. *The partial derivative functions $W_1(\cdot,\omega)$ and $\bar{V}_1(\cdot,\omega_{+1})$ exist and satisfy the first-order*

restriction is commonly used in the directed search literature (see, e.g., Menzio et al., 2013; Acemoglu and Shimer, 1999; Moen, 1997).

condition

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega y^{\star}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\begin{cases}\leq A, & y^{\star}(m,\omega)\geq 0\\ \geq A, & y^{\star}(m,\omega)\leq y_{\max}(\omega;\tau)\end{cases}, \tag{2.13}$$

and the envelop condition:

$$W_1(m,\omega)=A, \tag{2.14}$$

where $y^{\star}(m,\omega)=m+l^{\star}(m,\omega)-C^{\star}(m,\omega)$, $l^{\star}(m,\omega)$ *and* $C^{*}(m,\omega)$, *respectively, are the associated optimal choices on labor effort and consumption in the CM.*

3. *The stationary Markovian policy rules* $y^{\star}(\cdot,\omega)$ *and* $l^{\star}(\cdot,\omega)$ *are scalar-valued and continuous functions on* $[0,\bar{m}]$.
   (a) *The function* $y^{\star}(\cdot,\omega)$, *is constant valued on* $[0,\bar{m}]$.
   (b) *The optimizer* $l^{\star}(\cdot,\omega)$ *is an affine and decreasing function on* $[0,\bar{m}]$.
   (c) *Moreover, for every* (m,ω), *the optimal choice* $l^{\star}(m,\omega)$ *is finite valued:* $0<l_{\min}\leq l^{\star}(m)\leq l_{\max}(\omega;\tau)<+\infty$, *where there is a very small* $l_{min}>0$ *and* $l_{\max}(\omega;\tau):=y_{\max}(\omega;\tau)+U^{-1}(A)<2U^{-1}(A)\in(0,\infty)$.

In the proof to Theorem 1, we also derive the equilibrium decisions of the CM agent. We show that in an equilibrium, CM consumption is

$$C^{\star}(m,\omega)\equiv\bar{C}^{\star}=(U_1)^{-1}(pA), \tag{2.15}$$

a finite and non-negative constant. Equilibrium CM asset decision will depend on the aggregate state ω, i.e.,

$$y^{\star}(m,\omega)=\bar{y}^{\star}(\omega) \tag{2.16}$$

and this satisfies the first-order condition (2.13). Finally, from the budget constraint, we can obtain the equilibrium labor supply function as

$$l^{\star}(m,\omega)=\bar{C}^{\star}+\bar{y}^{\star}(\omega)-m. \tag{2.17}$$

Note that $l^{\star}(m,\omega)$ is single-valued, continuous, affine and decreasing in m.

2.4.3 Equilibrium DM buyer

Observe that since $\bar{V}(\cdot,\omega),W(\cdot,\omega)\in\mathcal{V}[0,\bar{m}]$ (i.e., are continuous, increasing and concave), then by (A.1), $\bar{V}(\cdot,\omega)\in\mathcal{V}[0,\bar{m}]$. In an equilibrium, the DM buyer's problem in

(2.6) can be re-written as

$$B(\mathbf{s}) = \max_{x\in[0,m],b\in[0,1]} \left\{ \beta(1-b)\left[\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\omega)}, \omega_{+1}\right)\right] \right.$$
$$\left. + b\left[u \circ Q(x,b) + \beta\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]\right\}. \quad (2.18)$$

It appears as if the buyer is choosing his matching probability b along with payment x. However this is just a change of variables utilizing the equilibrium relation (2.12) between quantity q and terms of trade (x, b). From this we can begin to see that there will be a trade-off to the buyer, in terms of an extensive margin (i.e., trading opportunity b), and, an intensive margin (i.e., how much to pay x).

The operator defined by (2.18) clearly does not preserve concavity: The objective function in (2.18) is not jointly concave in the decisions (x, b) and state m, since it is bilinear in the function b and the value function $\bar{V}$, or the flow payoff function u. However, we can still show that the resulting DM buyers' optimal choice functions for (x, b), denoted by $(x^\star, b^\star)$, are monotone, continuous, and have unique selections, using lattice programming arguments.

The following theorem summarizes the properties of a DM agent's value and policy functions:[18]

Theorem 2 (DM value and policy functions)**.** *For a given sequence of prices $\{\omega, \omega_{+1}, ...\}$, the following properties hold.*

1. *For any $\bar{V}(\cdot, \omega_{+1}) \in \mathcal{V}[0, \bar{m}]$, the DM buyer's value function is increasing and continuous in money balances, $B(\cdot; \omega) \in \mathcal{C}[0, \bar{m}]$.*

2. *For any $m \leq k$, DM buyers' optimal decisions are $b^\star(m, \omega) = x^\star(m, \omega) = q^\star(m, \omega) = 0$, and $B(m, \omega) = \beta\bar{V}[\phi(m, \omega), \omega_{+1}]$, where $\phi(m, \omega) := (\omega m + \tau) / [\omega_{+1}(1 + \tau)]$.*

3. *At any (m, ω), where $m \in [k, \bar{m}]$ and the buyer matching probability is positive $b^\star(m, \omega) > 0$:*

   (a) *The optimal selections $(x^\star, b^\star, q^\star)(m, \omega)$ and $\phi^\star(m, \omega) := \phi[m - x^\star(m, \omega), \omega]$, are unique, continuous, and increasing in m.*

   (b) *The buyer's marginal valuation of money $B_1(m, \omega)$ exists if and only if $\bar{V}_1[\phi(m, \omega), \omega]$ exists.*

   (c) *$B(m, \omega)$ is strictly increasing in m.*

[18] Theorem 2 is a generalization of the observation of Menzio et al. (2013) in two aspects: (i) We have additional endogenous CM participation in our model; and (ii) the theorem extends beyond steady state equilibria to encompass equilibrium along a dynamic transition.

(d) the optimal policy functions $b^\star$and $x^\star$, respectively, satisfy the first-order conditions

$$u \circ Q\left[x^\star(m,\omega), b^\star(m,\omega)\right] + b^\star(m,\omega)\left(u \circ Q\right)_2\left[x^\star(m,\omega), b^\star(m,\omega)\right]$$
$$= \beta\left[\bar{V}\left(\phi\left(m,\omega\right),\omega_{+1}\right) - \bar{V}\left(\phi^\star\left(m,\omega\right),\omega_{+1}\right)\right], \quad (2.19)$$

and,

$$\left(u \circ Q\right)_1\left[x^\star(m,\omega), b^\star(m,\omega)\right] = \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\phi^\star(m,\omega),\omega_{+1}\right]. \quad (2.20)$$

We prove these results in sequence, in Online Appendix D. Here, we summarize briefly the workings behind these results: Part 1 of the Theorem uses standard results from optimization and can be found in Lemma 1 of the appendix. Part 2 is proven as Lemma 2 in the appendix, and its insight here is simple: If buyers do not carry enough money to at least pay for a trading post's fixed cost, no firm will want to set up that post in equilibrium, and so the buyers get nothing. Part 3(a) is proven as Lemma 3 using the fact that a log-transform of the DM buyer's objective function is jointly concave in the choice variables (x, b), and is continuous in m (fixing the aggregate state). It nevertheless satisfies an increasing difference—and therefore, supermodularity—property. Thus, by lattice programming arguments, we can show that the Dm buyer's optimal policies are increasing in m. Lemmata 4 and 5 in the appendix, together establish Parts 3(b) and 3(c): These show that whenever a buyer has a chance of matching, her value function is differentiable. As a result, we can also characterize her best response in terms of a matching probability (extensive margin) and spending level (intensive margin) via Euler equations in Part 3(d), and this is proven in Lemma 6.

2.4.4 Market clearing

Goods in CM. In equilibrium, the total production of CM good equals its demand:

$$Y = C \equiv U^{-1}\left(A\right).$$

Goods in DM. Given equilibrium policy functions, $x^\star$ and $b^\star$, and, equilibrium distribution of money G and wage ω, equation (2.12) pins downs market clearing for each submarket in the set of equilibrium submarkets $\left\{(x(m,\omega), b(m,\omega)) : m \in \text{supp}(G(\cdot,\omega))\right\}$.

By Walras' Law, the requirement that all markets described above clear implies that money demanded must also equal money supplied:

$$\frac{1}{\omega} = \int m \mathrm{d}G(m;\omega) > 0. \tag{2.21}$$

Since M is the beginning of period aggregate stock of money in circulation, then the LHS of (2.21), $1/\omega = M \times 1/\omega M$, is the beginning of period real aggregate stock of money, measured in units of labor. The RHS of (2.21) is beginning of period aggregate demand, or holdings, of real money balances measured in the same unit.

2.5 Existence of a SME with unique distribution

For the rest of the paper, we focus on a stationary monetary equilibrium (SME), which comprises the characterizations from Section 2.4, where the sequence of prices are constant: $\omega = \omega_{+1}$.

Definition 1. A *stationary monetary equilibrium* (SME), given exogenous monetary policy τ, is a

- list of value functions $\mathbf{s} \mapsto (W, B, \bar{V})(\mathbf{s})$, satisfying the Bellman functionals: (2.5), (2.6), and jointly, (A.1)-(2.8);
- a list of corresponding decision rules $\mathbf{s} \mapsto (l^\star, y^\star, b^\star, x^\star, q^\star, z^\star, \pi^\star)(\mathbf{s})$ supporting the value functions;
- a market tightness function $\mathbf{s} \mapsto \mu \circ b^\star(\mathbf{s})$ given a matching technology μ, satisfying firms' profit maximizing strategy (2.11) and (2.12) at all active trading posts;
- an ergodic distribution of real money balances $G(\mathbf{s})$ satisfying an equilibrium law of motion

$$G(E) = T(G)(E) := \int P(\mathbf{s}, E) \mathrm{d}G(\mathbf{s}) \qquad \forall E \in \mathcal{B}(S) \tag{2.22}$$

  where, $\mathcal{B}(S)$ is the Borel σ-algebra generated by open subsets of the product state space S, and, $\mathbf{s} \mapsto P(\mathbf{s}, \cdot)$ is a Markov kernel induced by $(l^\star, x^\star, q^\star, z^\star, \pi^\star)$ and $\mu \circ b^\star$under τ; and,
- a wage rate function $\mathbf{s} \mapsto \omega(\mathbf{s})$ satisfying the money stock adding up condition (2.21).

At this point, we note that it will not be difficult to show that there is a unique distribution of agents in a SME. However, whether a SME is unique remains elusive to us as the frequency function $dG(m;\omega)$ does not admit a closed form expression in terms of known functions, and in general, it will also depend on the equilibrium candidate ω.[19]

The following theorem ensures that in our computations below there exists a steady state, stationary monetary equilibrium, and for each steady state equilibrium ω, there is a unique distribution of agents.

Theorem 3. *There is a SME with a unique nondegenerate distribution G.*

We prove this in Online Appendix E. The idea here is to first show that a composite Bellman functional for each agent satisfies Banach's fixed point theorem. Then, from Theorems 1 and 3, we know that agents' decision functions are monotone and continuous. This implies that for fixed ω, the equilibrium Markov operator on a current distribution of agents G is a monotone map and satisfies measurability conditions. We can easily argue that there monotone mixing as a result of the equilibrium self-map (2.22) on the space of distributions G, and conclude that there is a unique fixed point (in a weak-convergence sense).

3 Quantitative analyses

Finding a SME requires numerical computation. In this section, we calibrate the model to the US economy and then we illustrate the dynamic behavior of an individual agent in the model. We will pause to discuss the underlying forces and trade-offs at work that will help us to understand the SME outcomes. This will also help guide our understanding at the end of this section, where we provide some comparative SME analyses in terms of allocative, distributional and welfare outcomes.

[19]This statement is also true for the original Menzio et al. (2013) setting, if the authors' model had money supply growth. The intractability of their version of the frequency function $dG(m;\omega)$ under money supply growth comes about from the modeller no longer being able to work out analytically how long it will take for DM-unmatched buyers' balances to get eroded by inflation, before they have to go to work again. In contrast, the variation in Sun and Zhou (2018) admits an analytical form for $dG(m;\omega)$ and as a result they can show that there is a unique SME. This special result arises from their assumption that all types of agents in the DM must deterministically enter the CM *after one round* of trade (or no trade) in the DM, so that the aggregate demand for money in their model can be analytically described by a composition of equilibrium decision functions with well-behaved properties and an assumed exogenous distribution of CM preference shocks. In their model, without an exogenous distribution of CM preference shocks, given the market timing assumptions, there would be no distribution of agents since preferences are quasilinear in their CM.

Our setting yields a modelling trade-off in the opposite direction: In contrast to Sun and Zhou (2018), we do not require the latter assumption to preserve distributional non-degeneracy. However, our relaxation here would come at an analytical cost on the form of the frequency function $dG(m;\omega)$. In our opinion, the loss of tractability in this respect is not too severe: Our equilibrium characterization remains computational feasible. In fact, it retains the feature that agents' decision rules depend on the aggregate state only insofar as the scalar aggregate variable, ω. This is because, unlike heterogeneous-agent neoclassical growth models or random matching models, the market clearing conditions in competitive search do not require the conjecture of an entire distribution of assets in order to pin down terms of trade. In that sense, our algorithm for finding a SME will be similar to that used for computing neoclassical heterogeneous-agent models. In fact, with aggregate shocks (e.g., to τ) our setting will also imply an accurate application of the (originally heuristic) Krusell and Smith (1998) algorithm to an exact problem. (A similar point was previously discussed in Menzio et al. 2013, pp.2294-2295.)

3.1 Calibration

The CM and DM preference functions are, respectively,

$$U(C) = \frac{(C + c_{min})^{1-\sigma_{CM}} - (c_{min})^{1-\sigma_{CM}}}{1 - \sigma_{CM}}, \text{ and, } u(q) = \bar{U}_{DM} \frac{(q + q_{min})^{1-\sigma_{DM}} - (q_{min})^{1-\sigma_{DM}}}{1 - \sigma_{DM}}.$$

The matching function is such that $\mu(b) = 1 - b$. All the parameters of the model are listed in Table 1.

Table 1: Benchmark estimates

Parameter	Value	Empirical Targets	Description
$1 + \tau$	$(1 + 0.0089)^{1/4}$	Inflation rate[a]	Inflation rate
$1 + i$	$(1 + 0.0385)^{1/4}$	3-month T-bill rate[a]	Nominal interest rate
β	0.99879	-	Discount factor, $\frac{(1+i)}{(1+\tau)}$
χ	0.00297	Aux reg. $(i, M/PY)$[b]	CM participation cost
k	0.003997	Aux reg. $(i, M/PY)$[c]	Price-posting cost
$\bar{U}_{DM}$	407.77	Mean Hours ($\frac{1}{3}$)	Preference scale
σ_{DM}	1.0	-Normalized	DM risk aversion
σ_{CM}	2	-Normalized	CM risk aversion
A	1	-Normalized	Labor disutility scale
$\bar{m}$	$0.99 \times U^{-1}(A) = 0.987$	-	Sufficient condition (2.3)

[a] Mean nominal interest and inflation rates in the data are annual.
[b] The auxiliary statistics (data) are from a spline function fitted to the data on annual observations of the (3-month T-bill) nominal interest rate (i) and Lucas-Nicolini New-M1-to-GDP ratio (M/PY).
[c] The point elasticity refers to the elasticity of M/PY with respect to i, evaluated at the data mean of i.

There are three sets of parameters in the model. First, some parameters can be pinned down by direct (external) calibration to observable data statistics. These are the benchmark inflation (or money-supply-growth) rate τ and the discount factor β, which is pinned down from using the Fisher relation.

Second, we calibrate the triple, $(\chi, k, \bar{U}_{DM})$, to match some data-implied auxiliary statistics. These targets are summarized in Table 2. In particular, since there is no closed-form aggregate money demand relationship from the model, we use a third-order B-spline to fit to the Lucas and Nicolini (2015) "New" M1-to-GDP ratio ($M1/PY$) and three-month U.S. Treasury Bill rate (i) annual data series. (We used the sample spanning from 1915 to 2007.) This is the auxiliary statistical model whose elasticity of $M1/PY$ with respect to i, evaluated at the data mean of i is used as one of the calibration target. The other two targets are the mean of $M1/PY$ and the mean hours worked.[20]

[20] A Jupyter Notebook available openly on our GitHub site (https://github.com/phantomachine/csim) documents the steps of how we constructed the auxiliary empirical model. Ideally, we would like to conduct more formal inference using indirect estimation, however, because this is a general equilibrium model,

Third, some remaining model parameters that are not identifiable are normalized and these are reported in Table 1. Without loss, we set $c_{min} = q_{min} = 0.001$.

Given a reasonable fit of our model to the empirical targets, we can then use the calibration above as the benchmark model.

Table 2: Implied Targets

	(Auxiliary) Statistics		
	Model	Data Target	Description
	0.18	0.15	Mean M/PY^a
Parameters $(\chi, k, \bar{U}_{CM})$	-1.46	-1.13	$(i, M/PY)$ point elasticity^b
	0.33	0.33	Mean Hours $(\frac{1}{3})$

[a] The auxiliary statistics (data) are from a spline function fitted to the data on annual observations of the (3-month T-bill) nominal interest rate (i) and Lucas-Nicolini New-M1-to-GDP ratio (M/PY).

[b] The point elasticity refers to the elasticity of M/PY with respect to i, evaluated at the data mean of i.

3.2 A novel computational scheme

We solve for a SME following the pseudocode in our Online Appendix F. Our solution method uses a novel insight that refines the computation of the value of the lottery problem. Recall that the directed search problem makes the value function $\tilde{V}(\cdot, \omega)$ non-concave. Since there may exist lotteries that make agents better off than playing pure strategies over participating in DM (as buyer) or CM (as consumer/worker), we have to devise a means for finding these lotteries that convexify the graph of the function $\tilde{V}(\cdot, \omega)$.[21]

An existing way to do this in the literature is to use a grid $\mathrm{M}_g := \{0 < \cdots < \bar{m}\}$ to approximate the function's original domain of $[0, \bar{m}]$. Then, around each finite element of M_g, one must check if there is a linear segment that *approximately* convexifies graph $[\tilde{V}(\cdot, \omega)]$.[22] This approximation scheme works fine when we only have a lottery where the lower bound in M_g is included, i.e., a lottery on a set like $\{z_1, z_2\}$, where $z_1 = 0$, and, $z_2 < \bar{m}$. It becomes less accurate when lotteries may exist on upper segments of the function, i.e., lotteries on sets like $\{z_1', z_2'\}$, where $0 < z_1' < z_2' < \bar{m}$, but we have no prior knowlege of what z_1' is. This is because in practice (on the computer) it is not feasible to implement this check which is meant to be done at every element on the domain $[0, \bar{m}]$,

solution time is still an impediment to full-scale estimation and statistical inference.

[21] Interestingly, there is parallel similarity between our problem here and those in computational dynamic games. In the latter, non-convexities may sometimes arise in equilibrium payoff sets, and convexification of these payoff correspondence images are rationalized through a public randomization (sunspot) device, instead of lotteries or behavior strategies (see, e.g., Kam and Stauber, 2016).

[22] See part (v) of the proof of Theorem 3.5 in Menzio et al. (2013) for an exact theoretical underpinning of this scheme. We thank Amy Sun for sharing her MATLAB code for Menzio et al. (2013) which confirms this usage.

not its approximant M_g. As a result, its implementation on M_g may be prone to introducing non-negligible approximation errors, especially when the mesh of M_g is coarse. Thus, one would have to make M_g very fine, but, this will come at the cost of the overall SME solution time.

Instead, we propose a novel alternative here. We can exploit the property that $\tilde{V}(\cdot,\omega)$ has a bounded and convex domain, so then there exists a smallest convex set that contains $g\tilde{V} := \text{graph}\left[\tilde{V}(\cdot,\omega)\right]$, i.e., $\text{conv}\left(g\tilde{V}\right)$. This set is indeed $\text{graph}\left[\bar{V}(\cdot,\omega)\right]$. We utilize `SciPy`'s interface to the fast `QHULL` algorithm to back out a finite set of coordinates representing the convex hull, i.e., $\text{graph}\left[\bar{V}(\cdot,\omega)\right]$. Given these points, we approximate the function $\bar{V}(\cdot,\omega)$ by interpolation on a chosen continuous basis function. We use the family of linear B-splines available from `SciPy`'s `interpolate` class for this purpose. As a residual of this exercise, we can very quickly and directly determine the entire set of possible lotteries that exists with minimal loss of precision, for any given non-convex/concave function $\tilde{V}(\cdot,\omega)$.[23]

3.3 Benchmark SME

In Figure 1, we plot the SME value functions (V, B, W) in the benchmark economy. In the benchmark economy, our algorithm finds that two lottery segments exist. The solid blue line is the graph of $W(\cdot,\omega)$. The dashed green line is the graph of $B(\cdot,\omega)$. The upper envelop of these two graphs give us $\tilde{V}(\cdot,\omega)$, the thick solid green line. Denote $\text{conv}\{\cdot\}$ as the convex-hull set operator. The solid magenta graph is the graph of $V(\cdot,\omega)$ obtained through our convex-hull approximation scheme, once we have located all the intersecting coordinates between the set $\text{graph}\left[\tilde{V}(\cdot,\omega)\right]$ and the upper envelope of the set $\text{conv}\left\{\text{graph}\left[\tilde{V}(\cdot,\omega)\right], (0,0), (\bar{m},0)\right\}$.

[23]Detailed comments on how this is done can be found in the method `V` in our `Python` class `cssegmod.py`. We implement our solution in pure `Python` 2.7/3.4 (with OpenMPI parallelization of the agent decision problems on 24 logical cores). We have only tested our solutions on a Dell Precision T7810 workstation (with Intel Xeon E5-2680 v3, 2.50GHz, processors) running on the Centos 7.2 GNU/Linux operating system. In all our experiments, we have monotone convergence towards a unique SME solution, regardless of initial guesses on ω and $\bar{V}(\cdot,\omega)$. The average time taken to find the SME is between 90 to 120 seconds, given our hardware setting.

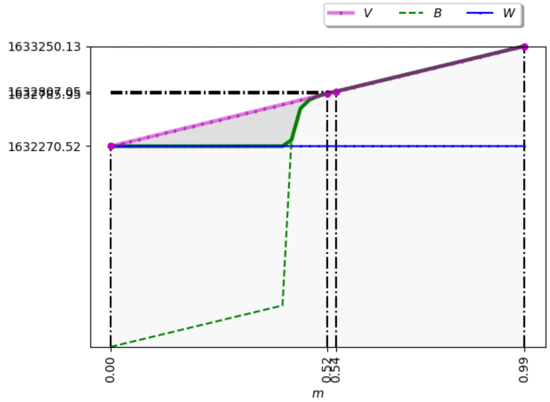

Figure 1: Value functions for benchmark economy.

Sustaining the equilibrium value functions are the policy functions $(l^{\star}, b^{\star}, x^{\star}, q^{\star})$, and the lottery policies $(\pi_1, 1-\pi_1)$ and $(\pi_1', 1-\pi_1')$ over the prize supports (z_1, z_2) and (z_1', z_2'), where $\pi_1(m, \omega) = (z_2 - m) / (z_2 - z_1)$ and $\pi_1'(m, \omega) = (z_2' - m) / (z_2' - z_1')$.

The other policy functions can be seen in Figure 2 on the following page. Consider the panel depicting the graph of the CM labor supply function. As shown earlier in (2.17), labor supply is affine and decreasing in money balance. There are three shaded patches in the Figure's panels. The dark-red patch corresponds to the region where $m \in [0, k)$, i.e., an agent will never match nor trade in the DM. The orange patches (one of which overlaps the dark-red patch) are the regions of the agent's state space in which a lottery may be played—i.e, $[z_1, z_2]$ and $[z_1', z_2']$. What matters for each agent in the SME is then the loci of these policy functions outside of the orange patch, but including the points on its boundary. These will be consistent with the equilibrium's ergodic state space of agents. As proven in Theorem 2, the policy functions $(b^{\star}, x^{\star}, q^{\star})$ are monotone in m in the relevant subspace where an agent can exist at any point in time. The relevant ergodic subspace of $[0, \bar{m}]$ in equilibrium is given by $\{z_1, [z_2, z_1'], [z_2', \bar{m}]\} = \{0, [0.52, 0.54], [0.98.., \bar{m}]\}$ in the benchmark economy in Figure 1 or Figure 2.

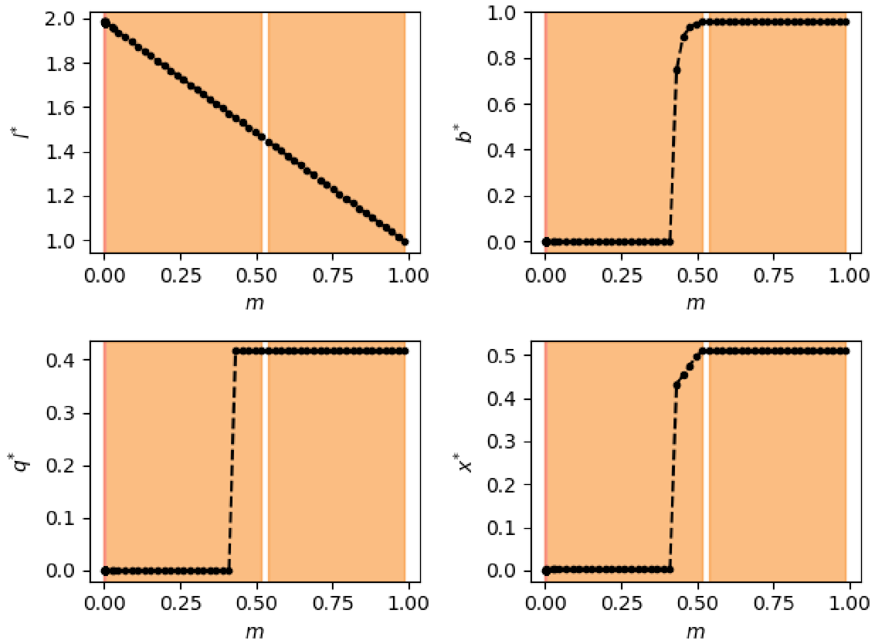

Figure 2: Markov policy functions in the benchmark economy

Given the information about our benchmark SME's active or relevant agent state space, and, the corresponding policy functions, we can simulate an agent's outcomes. To do so, one may begin from any initial agent named (m, ω) and apply the decision rules computed earlier, as in Figure 2. Details of the algorithms for simulating the SME outcomes can be found in our Online Appendix G. Readers interested in the SME behavior of agents may take a detour here to study the benchmark-calibrated SME simulation outcomes in Online Appendix H. Otherwise, we may proceed to discuss the equilibrium trade-offs faced by agents (i.e., the model mechanism) in the next section.

3.4 Trade-offs: inspecting the mechanism

We would like to explain the mechanism behind equilibrium behavior, and the attendant welfare and redistributive consequences of inflation. Since the equilibrium solution is non-analytic, we can at least identify the opposing forces underlying the effect of inflation on a corresponding SME as a function of inflation policy τ. The key insight here is that there will be an intensive-margin effect of inflation, as in all other heterogeneous agent models (i.e., either reduced-form or random-matching based models); but there will also be a countervailing extensive-margin force in this model. For the sake of exposition, we will artificially decompose our discussion of these two opposing forces according to activities in the CM and the DM.

CM-participation intensive vs. extensive margins. Positive inflation, $\tau > 0$, induces the following trade-offs: On one hand, with inflation, individuals would like to visit the CM more frequently to work and consume there (since in the CM money is not needed for exchange). On the other hand, given a real fixed cost $\chi \geq 0$ of entering the CM, higher inflation means that low-balance agents in the DM will face a greater barrier to engage in liquidity management in the CM. This is because of two possibilities: (i) their natural short-sale constraints in (2.8) may be violated if inflation is too high, i.e., $m - \chi < -y_{max}(\omega; \tau) < 0$, and so they choose to stay in the DM and are more likely to keep realizing a bad draw of the zero balance lottery prize; or (ii) their short-sale constraints in (2.8) are not binding, but the value of going to CM, $W(m - \chi, \omega)$ is still dominated by the value of going to the DM, $B(m, \omega)$. However, in equilibrium, in order to continue deriving consumption value in the DM, an agent would also need to ensure that he has sufficient balance to pay to go back to the CM often enough to maintain enough precautionary saving of money.

These trade-offs imply two margins for a precautionary motive for agents with respect to incomplete consumption insurance: Either they work harder each time in the CM and bear the cost of holding excess money balances (*intensive labor-CM margin*), or, they work less in each CM instance, reduce their spending in each DM exchange, and ensure that they are more likely to be able to afford to go to the CM frequently (*extensive labor-CM margin*).

DM-specific intensive vs. extensive margins. There is another trade-off with respect to the DM not present in standard general equilibrium models of money, or, in random matching models. Consider the equilibrium description of firms' optimal behavior in relation to DM production and profit maximization (2.12). Given the firms' best response in a SME, we can deduce the following about a potential DM buyer: $Q_1(x, b) > 0$, $Q_2(x, b) < 0$, $Q(x, b)$ is weakly concave, and $Q_{12}(x, b) = 0$. In words, we have another tension here: On one hand, faced with a given probability of matching, b, with a trading post, the more a buyer is willing to pay, x, the more q she can consume. (This is the *intensive margin* of DM trade—i.e., how much one can purchase.) On the other hand, given a required payment, x, a buyer who seeks to match with higher probability, b, must tolerate eating less q (This is the *extensive margin* of DM trade—i.e., trading opportunities.)

From Theorem 2, we know that if a DM buyer brings in more (less) money balance every period, then x will be higher (lower) and b will be higher (lower). But the tension just outlined above gives an ambiguous resolution on x or q. Thus the intensive margin faces a countervailing force in the extensive margin within the DM as well, as this will interact with the CM-participation intensive and extensive margin trade-off as well.

Remark 2. Superficially, it may appear that we complicate the model's mechanism with an additional CM extensive-margin via the fixed cost parameter χ. We would like to emphasize that the device $\chi \geq 0$ is merely for quantitative fit of the model, given our auxiliary empirical targets for the model. Theoretically, the extensive-margin CM-participation decision is still present even if $\chi = 0$. Why? From equation (2.2) the flow preference function is strictly concave, implying that they would like to consume both CM and DM goods in their infinite lifetimes. We confirm this in a robustness check for the case that $\chi = 0$, in our Online Appendix I.2. There we show that the same qualitative conclusions arise, as in our benchmark-calibrated model results to be discussed next.

3.5 Resolution of forces: Benchmark economy

We now consider the benchmark calibration of the model to resolve the intensive-versus-extensive tensions in the face of higher inflation which we identified in the last section. Let us begin with Figure 3. On the horizontal axes of the panels, we are increasing the (quarterly) steady-state inflation rate, $\tau \in (\beta - 1, 0.025]$. On the vertical axes, we measure the average outcomes for each corresponding economy's equilibrium outcomes under policy τ. We denote each equilibrium as SME(τ). These outcomes are plotted as solid blue circles (with the exceptions of the green diamond and red square points, respectively, indicating the economies SME(τ) at the benchmark τ and at $\tau = 0.025$). The solid gray patches depict the fifth and ninety-fifth percentiles of the distribution finite-sample simulations from each economy.[24]

[24]For each fixed economy SME(τ), we construct a synthetic panel of agents by Monte Carlo. That is, we simulate 1000 independent time-series outcomes for individuals. Each individual series has 10,000 periods. See Online Appendix G for the pseudocode for individual time-series simulations.

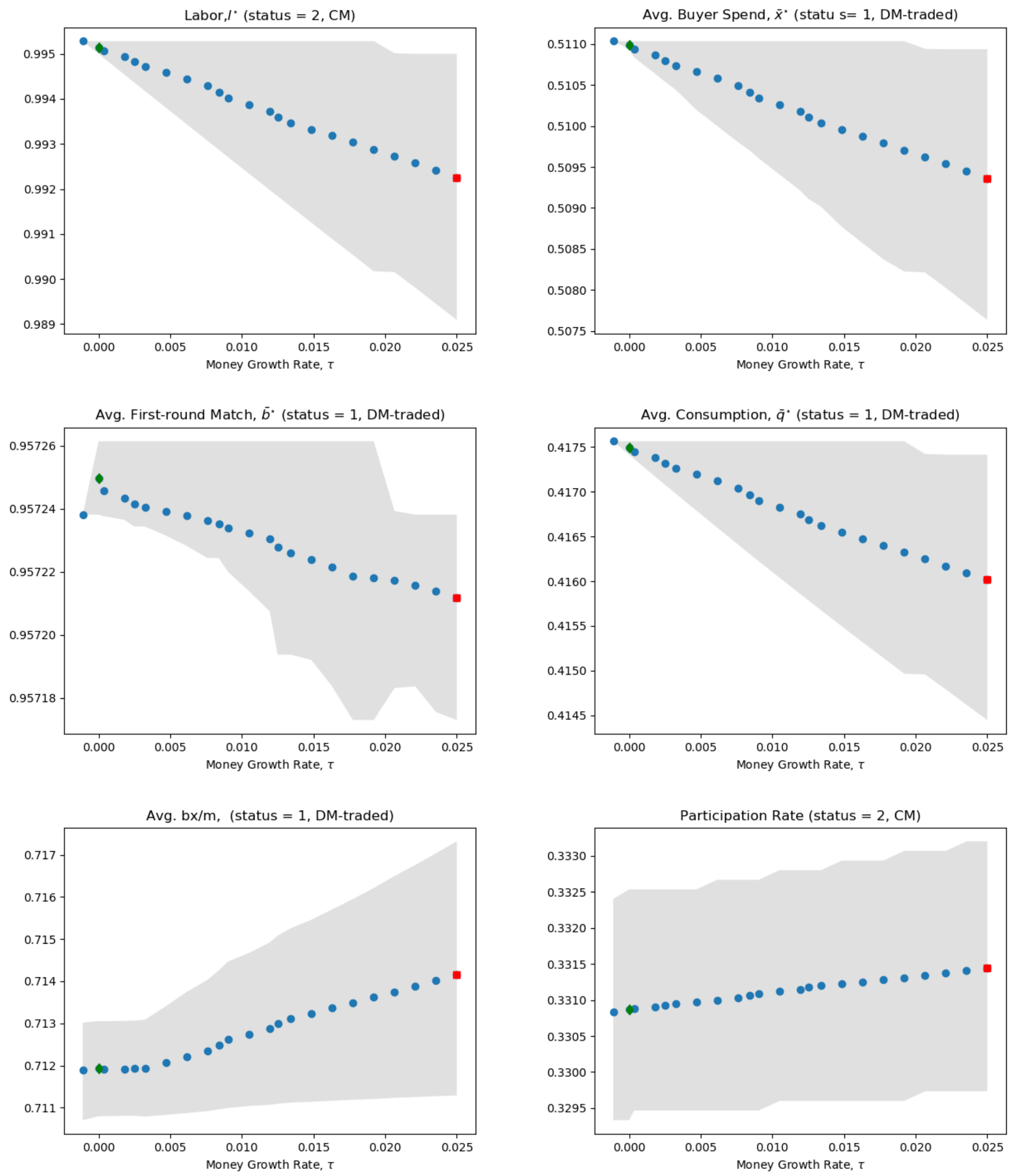

Figure 3: Comparative steady states — allocations

Consider going from left to right and top to bottom through the panels in Figure 3. What we observe is that for each succesive economy, SME(τ), with higher inflation τ, there is an outcome of lower average labor supply (l), lower DM spending (x), lower probability of matching (b) and consumption (q), but higher speed of transactions (bx/m) in the DM and higher rate of participation in the CM.[25]

The mechanism works out as follows: As inflation rises, the cost of holding money increases, so agents carry less money out of the CM by lowering their labor supply. Since

[25]In the first panel of Figure 3, for each economy τ, the average labor supply outcome has been normalized by that economy's upper-bound on labor supply $l_{\max}(\omega;\tau)$, hence the numbers are bounded above by unity.

agents' optimal decision rules $(x^{\star}, q^{\star}, b^{\star})$ are monotone increasing in their real balances whenever they are successfully matched (see Theorem 2) agents will also be paying and consuming less in the DM on average.

However, the downside risk, $1 - b^{\star}(m)$, of not getting matched increases with inflation. This exacerbates the cost of holding money for DM buyers who are unmatched. As a result, even though we observe that aggregate $(x^{\star}, q^{\star}, b^{\star})$ are decreasing with inflation, what matters for DM agents is how quickly they can dispose of a given amount of liquidity they carry into each DM round, in exchange for DM goods. Thus, we see the (average) payment in the DM across buyers, bx, is falling slower than the fall in their average money balances—i.e., the ratio bx/m indicating the speed of transactions in the DM is increasing with inflation. (This echoes the DM extensive margin that we identified above.) Another consequence of this is that agents would also go to the CM more often to manage their liquidity.

We have just rationalized the observation that agents consume less (intensive margins in both markets) in return for being able to trade faster in DM and to visit the CM more often (extensive-margin effects in both markets).

3.6 Distributional and Welfare effect of inflation

From the last section, we can also work out the distributional and aggregate welfare implications of inflation. In Figure 4, we see that average money balance is falling with inflation (as explained in the last section). A measure of inequality in the distribution of money (liquid assets), the Gini index, falls with inflation, for a sequence of low-inflation economies, whereas it increases with inflation, for higher-inflation economies. Average (relative) welfare—measured as how much equivalent variation in stationary consumption (CEV) agents are willing to give up, in order to move back to a zero-inflation economy—falls with inflation.[26]

The interesting observation here is that although welfare is decreasing (i.e., the welfare cost of inflation is rising), there is an "inequality smile" in liquid wealth as inflation increases. Why? This comes back to the two opposing forces that we have highlighted earlier.

[26] To measure average welfare we consider the ex-ante average value:

$$Z_{\tau} := \int \bar{V}(m, \omega; \tau) \mathrm{d}G(m, \omega; \tau).$$

For a chosen reference economy, SME(τ_0) with ex-ante average value as Z_{τ_0}, for any other SME(τ) we define a welfare measure as compensating equivalent variation in CM consumption measure:

$$CEV\left(\tau|\tau_0\right) = \left[\frac{U^{-1}\left(Z_{\tau}\right)}{U^{-1}\left(Z_{\tau_0}\right)} - 1\right] \times 100\%.$$

In the comparisons, we set $\tau_0 = 0$, i.e., the zero-inflation economy.

On the one hand, the intensive margin tends to exact a *redistributive force*: With increasing inflation, agents are economizing on their money holdings. Those with high balances are reducing their holdings more relative to those at the bottom end of the distribution. Moreover, the upper bound on liquidity demand in the CM is falling with inflation. Thus, average money balance is falling with inflation. That is inflation is a redistributive tax, just like in all other heterogeneous-agent monetary models that precede ours.

On the other hand there is the opposing extensive margin effect: Agents are also spending faster (conditional on matching) in the DM and entering the CM to rebalance their liquidity more frequently. Higher CM participation implies that we see more agents are the bottom end of the distribution at the end each period (i.e., those who will go to CM next period), and less of agents at the top end since they either top up less in the CM, or they spend faster in the DM.

Therefore, the distribution become more positively skewed (i.e., there is a displacement of mass from the top end to the lower end of the distribution) with inflation. Similarly, we see that the Gini index rises (eventually) with inflation, although it initially falls with inflation at lower levels of inflation. This is because for low inflation ranges, the *redistributive effect* dominates *extensive margin*, but the tension goes the other way, when inflation is high enough.

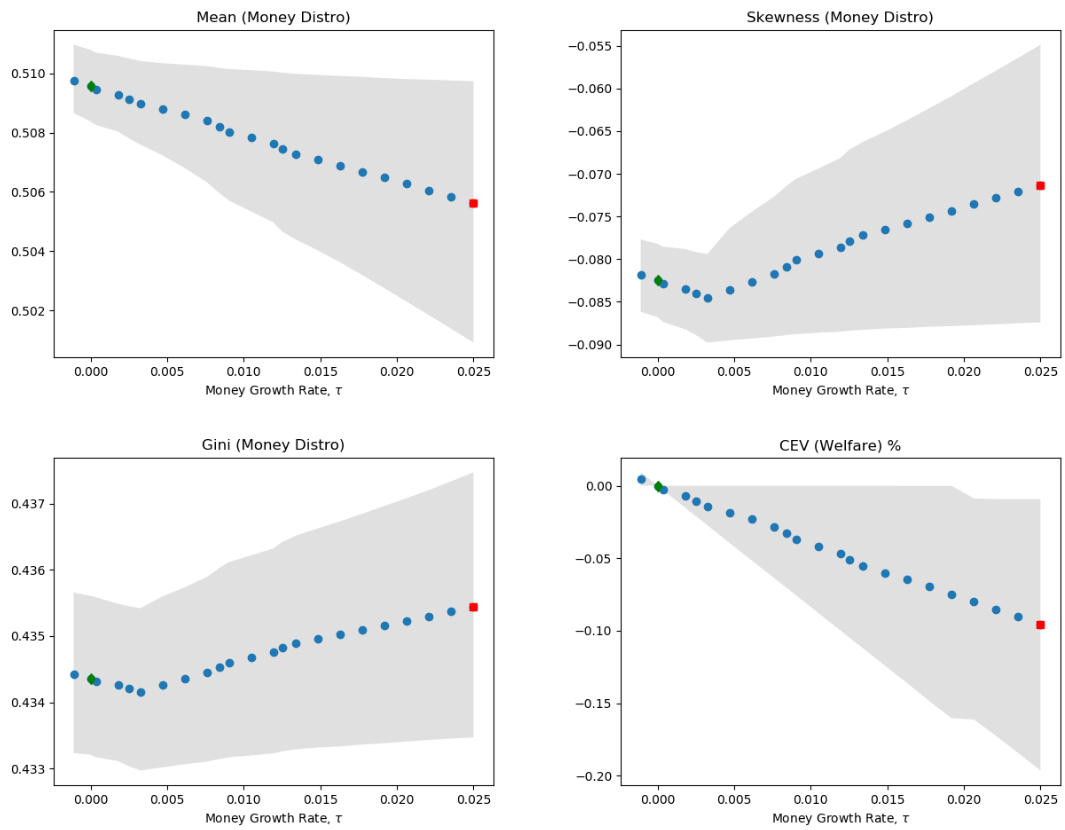

Figure 4: Comparative steady states — distribution

3.7 Welfare cost comparisons

One might be interested to see how this model fares in terms of the welfare cost of inflation. We consider the experiment of moving the benchmark economy from a hypothetical equilibrium at zero inflation (per annum) to ten percent inflation (per annum). The first row gives the welfare cost of this exercise at 0.4% of annual consumption. This is comparable to a version of a limited-participation random-matching model of Chiu and Molico (2010).

In comparison with representative-agent settings such as Lucas (2000) or Lagos and Wright (2005) (which has additional bargaining frictions), the welfare cost in our model is not as high. This is well known, since the redistributive margin of inflation tax is always present in heterogeneous-agent models. In contrast, in heterogeneous-agent models such as Imrohoroğlu and Prescott (1991b), which has more free parameters to govern frictions, one could obtain a welfare cost of inflation as high as 0.9% per annum in CEV terms.

Table 3: Welfare cost (CEV) from 0% to 10% (p.a.) inflation economy.

Economy	**Welfare Cost** (%)[a]	**Remarks**
Benchmark	0.40	
Imrohoroğlu and Prescott (1991b)	0.90	Bewley-CIA[b]-HA[d]
Chiu and Molico (2010)	0.41	RM[c]-HA[d]
Lagos and Wright (2005)	1.32	RM[c]-RA[e]-TIOLI[f]
Lucas (2000)	0.87	CIA[b]-RA[e]

[a] Annualized CEV cost (relative to zero-inflation economy)
[b] CIA: Cash-in-advance model
[c] RM: Random matching model
[d] HA: Heterogeneous agent model
[e] RA: Representative agent model
[f] TIOLI: Take-it-or-leave-it bargaining

3.8 Robustness and variations on the theme

Details for what follows can be found in our Online Appendix I. Specifically, in Online Appendix I.1, we consider two variations or robustness checks on our model assumptions. First, we show that our insights above are robust to alternative parametrization of the fixed-cost parameter χ. (There we show only the case of a doubling of χ, but qualitatively, the baseline results remain across a wide range of χ values. Second, we consider an extreme assumption that agents face a zero-borrowing constraint when overcoming the fixed cost of CM entry, χ: This alternative economy is tantamount to a reparametrization of the borrowing limit (2.9) from the benchmark setting to $y_{\max}(\omega;\tau) = 0$.

Consider the second alternative environment of zero borrowing (when it comes to paying for the CM fixed cost). Given this environment, the results are similar qualitatively across increasing inflation rates. However, when one compares this alternative economy with its benchmark counterpart, at any given level of long-run inflation, we have the following additional insights: In the zero-borrowing-limit economy, average money balance is higher, and, equilibrium extensive margin effects in the DM (i.e., on average how fast agents expend their given DM money holdings) are lower than its corresponding benchmark economy. However, the participation rate in CM is higher, but the Gini index is smaller.

The reason is as follows: In the zero-borrowing economy, agents have a stronger precautionary liquidity-risk insurance motive. Since they cannot borrow to overcome the fixed cost of entering the CM to manage their liquidity needs, then whenever they are in the CM, agents will tend to demand more real balances. Likewise, conditional on being in the DM, agents expect to trade at a lower volume relative to their DM money holdings, as they need to economize on the balance in order to possibly overcome the fixed cost of re-entering the CM. This explains the on-average higher money balance (in comparison to the benchmark economy) and the lower rate of trading in the DM. In return, agents would like to go to the CM more often to demand additional precautionary liquidity. That explains a relatively higher top end of the money distribution relative to the bottom (i.e., a more left-skewed distribution), and hence a lower Gini index, in comparison to the benchmark economy's outcome.

Finally, as alluded to earlier in Section 3.4, we also show in Online Appendix I.2 that χ *per se* is not needed to materialize the model's endogenous extensive-margin forces. That is, if we set $\chi = 0$, the same qualitative pattern arises as in the benchmark economy discussion in Section 3.5. It is nevertheless a useful parameter for quantitative reasons.

4 Conclusion

We proposed a theoretical and quantitative heterogeneous-agent monetary model. In this paper, we have shown that details matter. That is, details matter, both from a logical-theoretic and from a quantitative perspective, when thinking about market frictions and understanding the effects of monetary policy on heterogeneous individuals. We focused on competitive search and matching in markets when money has value in equilibrium exchange—i.e., frictions are not parameterically assumed, but arise from more fundamental features of information and trading environment.

We highlight a new mechanism—an endogenous trade-off between intensive and extensive margins—through which monetary policy has impact on the aggregate economy and welfare. In contrast to well-received wisdom that inflation acts as a redistributive tax, we showed that when there exists endogenous extensive-margin forces that make agents trade-off between their desire to consume more and their desire to be able to trade more frequently. Quantitatively, the effect of inflation tax on liquid-wealth inequality is non-monotone. Thus, the welfare cost of inflation in our model is still sizable, despite the redistributive effect of inflation that tends to induce heterogenous-agent monetary models to produce lower costs of inflation, relative to their representative-agent counterparts.

This is not our final word on quantitative analyses in this class of models. In this paper, we deliberately focused on a single-asset, pure-currency economy in order to have a simple and clear understanding of our new equilibrium relation between extensive- and intensive-margins of trade-off, and, the effect of inflation tax on the trade-off. We think that if we allowed agents to hold additional illiquid assets (say, in the centralized markets), this may further exarcerbate the inequality result in our model. In companion projects (Kam and Lee, 2016; Kam et al., 2016), we explore this conjecture in an expanded setting with liquid and illiquid assets, and, further with aggregate dynamics.[27]

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[27]Since the framework renders agents' Markov decision processes independent from the aggregate distribution, but for an aggregate scalar statistic, we will have an exact Krusell and Smith (1998) sort of algorithm for computing equilibria. This is especially pertinent for the extended setting with aggregate dynamics, as the competitive search refinement means that the solution method can be made more efficient and more precise, than models where one has to brute-force approximate distributions when solving agent problems.

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ONLINE APPENDIX

Inflationary Redistribution vs. Trading Opportunities

Omitted Proofs and Other Results

This document is also available from

https://github.com/phantomachine/csim.

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A Extension and special cases

Consider a variation on the benchmark setting in the paper. In particular, suppose that each agent $\mathbf{s} := (m, \omega)$ has an initial value each period as

$$\bar{V}(\mathbf{s}) = \alpha W(\mathbf{s}) + (1-\alpha) V(\mathbf{s}), \tag{A.1}$$

where $V(\mathbf{s})$ is the value of playing a fair lottery $(\pi_1, 1-\pi_1)$ over the prizes $\{z_1, z_2\}$, i.e.,

$$V(\mathbf{s}) = \max_{\pi_1 \in [0,1], z_1, z_2} \left\{ \pi_1 \tilde{V}(z_1, \omega) + (1-\pi_1) \tilde{V}(z_2, \omega) : \pi_1 z_1 + (1-\pi_1) z_2 = m \right\}; \tag{A.2}$$

is a natural upper bound on CM saving (in real money balances).

The difference between (A.1) and (A.2) and their respective counterparts in (2.7) and (2.8) in Section 2.3.3 on page 8 of the paper, is that there is a measure α of agents who will participate in the CM for sure each period. When $\alpha = 0$, we recover the simpler model used in the main paper.

Also, when $\alpha = 0$, there is no fixed cost of entering the CM ($\chi = 0$), $U(C) = 0$ for all C, and the labor utility function $h(l)$ is strictly convex, we recover the original Menzio et al. (2013) model as a special case.

Note that when $\alpha = 1$ (i.e., agents get to enter the CM deterministically), $\chi = 0$ (there is no fixed cost of entering the CM), and the continuation value from CM is a convexification of $B(\cdot, \omega)$, our model becomes a version of Rocheteau and Wright (2005) with competitive search markets.

All proofs (to results in the paper) below are written with the more general case of $\alpha \in [0, 1)$ in mind.

B CM individual's problem

The following gives the proof of Theorem 1 on page 11 in the paper.

Theorem 1. Assume $\tau/\omega < \bar{m}$. For a given sequence of prices $\{\omega, \omega_{+1}, ...\}$, the value function of the individual beginning in the CM, $W(\cdot, \omega)$, has the following properties:

1. $W(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$, i.e., it is continuous, increasing and concave on $[0, \bar{m}]$. Moreover, it is linear on $[0, \bar{m}]$.

2. The partial derivative functions $W_1(\cdot, \omega)$ and $\bar{V}_1(\cdot, \omega_{+1})$ exist and satisfy the first-order condition

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega y^\star(m,\omega)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\begin{cases}\leq A, & y^\star(m,\omega) \geq 0 \\ \geq A, & y^\star(m,\omega) \leq \min\left\{\bar{m} - \frac{\tau}{\omega}, \bar{m}\right\}\end{cases}, \tag{B.1}$$

and the envelop condition:

$$W_1(m, \omega) = A, \tag{B.2}$$

where $y^\star(m,\omega) = m + l^\star(m,\omega) - C^\star(m,\omega)$, $l^\star(m,\omega)$ and $C^*(m,\omega)$, respectively, are the associated optimal choices on labor effort and consumption in the CM.

3. The stationary Markovian policy rules $y^\star(\cdot, \omega)$ and $l^\star(\cdot, \omega)$ are scalar-valued and continuous functions on $[0, \bar{m}]$.

   (a) The function $y^\star(\cdot, \omega)$, is constant valued on $[0, \bar{m}]$.

   (b) The optimizer $l^\star(\cdot, \omega)$ is an affine and decreasing function on $[0, \bar{m}]$.

   (c) Moreover, for every (m, ω), the optimal choice $l^\star(m,\omega)$ is interior: $0 < l_{\min} \leq l^\star(m) \leq l_{\max}(\omega;\tau) < +\infty$, where there is a very small $l_{\min} > 0$ and $l_{max}(\omega) := \min\left\{\bar{m} - \frac{\tau}{\omega}, \bar{m}\right\} + U^{-1}(A) < 2U^{-1}(A) \in (0, \infty)$.

Proof. (*Part 1*). The individual's problem beginning in the CM (2.5) is:

$$W(\mathbf{s}) = \max_{(C,l,y)\in\mathbb{R}^3_+}\left\{U(C) - Al + \beta\bar{V}(\mathbf{s}_{+1}) : pC + y \leq m + l,\ m_{+1} = \frac{\omega y + \tau}{\omega_{+1}(1+\tau)}\right\}.$$

Since $U_1(C) > 0$ for all C, the budget constraint always binds. Thus we can re-write (2.5) as

$$W(\mathbf{s}) = \max_{(C,y)\in\mathbb{R}_+\times[0,\bar{m}]}\left\{U(C) - A[pC + y - m] + \beta\bar{V}\left(\frac{\omega y + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}. \tag{B.3}$$

Let

$$(C^{\star}, y_c^{\star})(m,\omega) \in \underset{(C,y)\in\mathbb{R}_+\times[0,\bar{m}]}{\arg\max} \left\{ U(C) - A\left[pC + y - m\right] + \beta \bar{V}\left(\frac{\omega y + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}. \tag{B.4}$$

From (B.3), it is clear that $W_1(\cdot,\omega)$ exists on $[0,\bar{m}]$, and moreover, we have the envelope condition $W_1(\cdot,\omega) = A > 0$. This implies that the value function $W(\cdot,\omega)$ is continuous, increasing and concave in m. Moreover it is affine in m.

(*Part 2*). First, we make the following observations: Since U is strictly concave in C, the objective function is strictly concave in C. Moreover, the objective function on the RHS of (B.3) is continuously differentiable with respect to C. The optimal decision, $C^{\star}(m,\omega)$ satisfies the following Karush-Kuhn-Tucker (KKT) conditions:

$$U_1(C) \begin{cases} = pA, & C > 0 \\ < pA, & C = 0 \end{cases}. \tag{B.5}$$

In an equilibrium, $p > 0$ will be finite—in fact, $p = 1$—and, since $A \in (0,\infty)$, then $C^{\star}(m,\omega) \equiv \bar{C}^{\star} = (U_1)^{-1}(pA)$ is a finite and non-negative constant. Thus, we only have to verify that the optimal decision correspondence, given by $l_c^{\star}(m,\omega) \equiv p\bar{C}^{\star} + y_c^{\star}(m,\omega) - m$ at each (m,ω), exists and is at least a convex-valued and upper-semicontinuous (*usc*) correspondence: Fixing $C = \bar{C}^{\star}$, the objective function on the RHS of (B.3) is continuous and concave on the compact choice set $[0,\bar{m}] \ni y$. By Berge's Maximum Theorem, the maximizer $y_c^{\star}(m,\omega)$, or $l_c^{\star}(m,\omega)$, is convex-valued and *usc* on $[0,\bar{m}]$. (In fact, after we further establish that the derivative $V_1(\cdot,\omega_{+1})$ exists, we show below that it will be constant and single-valued with respect to m.)

Second, we take a detour and show that the derivative $\bar{V}_1(\cdot,\omega_{+1})$ exists, in order to characterize a first-order condition with the respect to y. The results below will rely on the observation that since $V(\cdot,\omega_{+1})$ is a concave, real-valued function on $[0,\bar{m}]$, it has right- and left-hand derivatives (see, e.g., Rockafellar, 1970, Theorem 24.1, pp.227-228). Fix $C^{\star}(m,\omega) \equiv \bar{C}^{\star}$. Since $y_c^{\star}(m,\omega)$ is *usc* on $[0,\bar{m}]$, then for all $\varepsilon \in [0,\delta]$, and taking $\delta \searrow 0$, there exists a selection $y^{\star}(m-\varepsilon,\omega) \in y_c^{\star}(m-\varepsilon,\omega)$ feasible to a CM agent m. Similarly, there is a $y^{\star}(m,\omega) \in y_c^{\star}(m,\omega)$ that is feasible to a CM agent $m-\varepsilon$. Moreover, if $l^{\star}(m,\omega) \in l_c^{\star}(m,\omega)$ is an optimal selection associated with $y^{\star}(m,\omega)$, then for an agent at m,

$$W(m,\omega) = \underbrace{U(\bar{C}^{\star}) - Al^{\star}(m,\omega) + \beta\bar{V}\left[\frac{\omega\left[m + l^{\star}(m,\omega) - \bar{C}^{\star}\right] + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}_{\equiv Z[m, y^{\star}(m,\omega)]}$$
$$\geq \underbrace{U(\bar{C}^{\star}) - Al^{\star}(m-\varepsilon,\omega) + \beta\bar{V}\left[\frac{\omega\left[m + l^{\star}(m-\varepsilon,\omega) - \bar{C}^{\star}\right] + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}_{\equiv Z[m, y^{\star}(m-\varepsilon,\omega)]};$$

and, for an agent at $m - \varepsilon$,

$$W(m-\varepsilon,\omega)$$
$$= \underbrace{U(\bar{C}^{\star}) - Al^{\star}(m-\varepsilon,\omega) + \beta\bar{V}\left[\frac{\omega\left[(m-\varepsilon) + l^{\star}(m-\varepsilon,\omega) - \bar{C}^{\star}\right] + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}_{\equiv Z[m-\varepsilon, y^{\star}(m-\varepsilon,\omega)]}$$
$$\geq \underbrace{U(\bar{C}^{\star}) - Al^{\star}(m,\omega) + \beta\bar{V}\left[\frac{\omega\left[(m-\varepsilon) + l^{\star}(m,\omega) - \bar{C}^{\star}\right] + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}_{\equiv Z[m-\varepsilon, y^{\star}(m,\omega)]}.$$

Rearranging these inequalities, we have the following fact:

$$\frac{Z[m, y^{\star}(m-\varepsilon,\omega)] - Z[m-\varepsilon, y^{\star}(m-\varepsilon,\omega)]}{m-(m-\varepsilon)}$$
$$\leq \frac{W(m,\omega) - W(m-\varepsilon,\omega)}{m-(m-\varepsilon)} \leq \frac{Z[m, y^{\star}(m,\omega)] - Z[m-\varepsilon, y^{\star}(m,\omega)]}{m-(m-\varepsilon)},$$

which, after simplifying the denominator and taking limits, yields:

$$\lim_{\varepsilon\searrow 0}\left\{\frac{Z[m, y^{\star}(m-\varepsilon,\omega)] - Z[m-\varepsilon, y^{\star}(m-\varepsilon,\omega)]}{\varepsilon}\right\}$$
$$\leq \lim_{\varepsilon\searrow 0}\left\{\frac{W(m,\omega) - W(m-\varepsilon,\omega)}{\varepsilon}\right\} \leq \lim_{\varepsilon\searrow 0}\left\{\frac{Z[m, y^{\star}(m,\omega)] - Z[m-\varepsilon, y^{\star}(m,\omega)]}{\varepsilon}\right\}$$
$$\Longleftrightarrow$$
$$\beta\lim_{\varepsilon\searrow 0}\left\{\frac{\bar{V}\left[\frac{\omega(m+l^{\star}(m-\varepsilon,\omega)-\bar{C}^{\star})+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right] - \bar{V}\left[\frac{\omega(m-\varepsilon+l^{\star}(m-\varepsilon,\omega)-\bar{C}^{\star})+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}{\varepsilon}\right\} \leq W_1(m,\omega)$$
$$\leq \beta\lim_{\varepsilon\searrow 0}\left\{\frac{\bar{V}\left[\frac{\omega(m+l^{\star}(m,\omega)-\bar{C}^{\star})+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right] - \bar{V}\left[\frac{\omega(m-\varepsilon+l^{\star}(m,\omega)-\bar{C}^{\star})+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}{\varepsilon}\right\}.$$

Since, from (B.3), $W(\cdot,\omega)$ is clearly differentiable with respect to m, the second term in the inequalities above is equal to the partial derivative $W_1(m,\omega)$, which is constant. As $\varepsilon \searrow 0$, there is a selection $l^{\star}(m-\varepsilon,\omega) \to l^{\star}(m,\omega)$, and, by Rockafellar (1970, Theorem

24.1) the first is the left derivative of $V(\cdot, \omega_{+1})$. Moreover, the last term is identical to the first, i.e.,

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega\left(m^{-}+l^{\star}(m,\omega)-\bar{C}^{\star}\right)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right] \le W_1(m,\omega) \le \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega\left(m^{-}+l^{\star}(m,\omega)-\bar{C}^{\star}\right)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right].$$

Therefore, if the optimal selection is interior, these weak inequalities must hold with equality, so we have the left derivative of $\bar{V}$ with respect to the agent's decision variable y as:

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega y^{\star-}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right] = W_1(m,\omega).$$

where $y^{\star-}(m,\omega) \equiv m^{-}+l^{\star}(m,\omega)-\bar{C}^{\star}$.

By similar arguments, we can also prove that the right directional derivative of $\bar{V}(\cdot,\omega_{+1})$ exists, and show that the right derivative of $\bar{V}$ with respect to the agents decision y as:

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega y^{\star+}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right] = W_1(m,\omega),$$

where $y^{\star+}(m,\omega) \equiv m^{+}+l^{\star}(m,\omega)-\bar{C}^{\star}$. From the last two equations, we can conclude that the right and left directional derivatives must agree, and thus, we have the first-order KKT condition (2.13) as, repeated here as

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega y^{\star}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\begin{cases}\le A, & y^{\star}(m,\omega)\ge 0\\ \ge A, & y^{\star}(m,\omega)\le y_{\max}(\omega;\tau)\end{cases},$$

where the weak inequalities apply with complementary slackness.[28] Also, note that in

[28] Consider the feasible choice set for CM saving y: If $\bar{m}$ is an upper bound on end-of-period balance plus transfer (measured in units of labor), then this gives the bounds on end-of-period money balance plus transfer, in current money value, as:

$$\tau M \le \omega M y + \tau M \le \omega M \bar{m},$$

where ωM is current nominal wage. Since there is inflation in nominal wage, then next-period initial balance is current end-of-period nominal money balance normalized by the next period nominal wage $M_{+1}\omega_{+1}$, i.e.,

$$\frac{\tau M}{\omega_{+1}M_{+1}} \le m_{+1} \equiv \frac{\omega M y+\tau M}{\omega_{+1}M_{+1}} \le \begin{cases}\min\left\{\frac{\omega M\bar{m}}{\omega_{+1}M_{+1}},\bar{m}\right\}, & \text{if } \tau\ge 0\\ \max\left\{\frac{\omega M\bar{m}}{\omega_{+1}M_{+1}},\bar{m}\right\}, & \text{if } \tau<0\end{cases}.$$

Using (2.1), we can re-write the above bounds as

$$0 \le y \le y_{\max}(\omega;\tau),$$

which applies in the pair of KKT complementary slackness conditions (2.13). Note that the min operator is introduced to account for the possibility that $\tau < 0$.

the previous proof of Part 1), we have established the envelop condition (2.14):

$$W_1(m,\omega) = A.$$

(*Part 3.*) Observe that given the assumption in (2.3), we have (B.5) always binding: $U'(C) = pA$. Also, observe from(B.5) and (2.13) that an individual's current money holding m and the aggregate state ω have no influence on his optimal decision on consumption, $C^\star(m,\omega) = \bar{C}^\star$, but that $y^\star(m,\omega) = \bar{y}^\star(\omega)$. However, from the budget constraint, m clearly does affect the optimal labor decision,

$$\begin{aligned} l^\star(m,\omega) &= pC^\star(m,\omega) + y^\star(m,\omega) - m \\ &\underset{(p=1)}{\equiv} \bar{C}^\star + \bar{y}^\star(\omega) - m. \end{aligned} \tag{B.6}$$

Clearly, $l^\star(m,\omega)$ is single-valued, continuous, affine and decreasing in m.

Finally, we show that the optimal choice of l will always be interior. Evaluating the budget constraint in terms of optimal choices at the current individual state m,

$$l^\star(m,\omega) = \bar{y}^\star(\omega) - m + \bar{C}^\star.$$

Since $m \in [0,\bar{m}]$, then, the minimal l attains when m is maximal at $\bar{m}$, and, $\bar{y}^\star(\bar{m},\omega) = 0$:

$$l_{\min} := \check{l}^\star(\bar{m},\omega) \equiv 0 - \bar{m} + \bar{C}^\star > 0.$$

The last inequality obtains from (2.3) which requires $\bar{m} < U^{-1}(A)$, and from optimal CM consumption (2.15) which yields $\bar{C}^\star = U^{-1}(A)$, where $p = 1$ in an equilibrium. The maximal l attains when $m = 0$ and $\bar{y}^\star(0,\omega) = y_{\max}(\omega;\tau)$:

$$l_{\max}(\omega,\tau) := y_{\max}(\omega;\tau) - 0 + \bar{C}^\star = y_{\max}(\omega;\tau) + U^{-1}(A) < 2U^{-1}(A). \tag{B.7}$$

Clearly, $l_{\max}(\omega,\tau) < +\infty$. If we do not have hyperinflation, or, if transfers are not excessively large—i.e., if $\tau/\omega < \bar{m}$—then, $l_{\max}(\omega,\tau) > 0$ will be well-defined. So if $\tau/\omega < \bar{m}$, then we will have an interior optimizer for all m: $0 < l_{\min} \leq l^\star(m) \leq l_{\max}(\omega;\tau) < +\infty$. □

C Limited short-sale constraint and CM participation

Here we derive the short-sale constraint that may bind in the ex-ante market participation problem (2.8) in the paper. Suppose an agent were to participate in the CM with initial asset $a = z - \chi$, where z is his ex-ante money balance, and, χ is the fixed cost (in units of labor) of CM participation. Thus if $a < 0$, the agent is said to be short selling, or issuing an I.O.U.

Recall the CM budget constraint is

$$y + C = l + a.$$

The most negative an asset position the agent can attain in an equilibrium is some $\underline{a}$ such that he must work at the maximal amount $l_{\max}(\omega; \tau)$ and cannot afford to save, $y = 0$. From the budget constraint in such an equilibrium, we have:

$$0 + \bar{C}^{\star} = l_{\max}(\omega; \tau) + \underline{a},$$

which then implies that $\underline{a} = \bar{C}^{\star} - l_{\max}(\omega; \tau)$. From (B.7), we can further obtain the simplified expression $\underline{a} = -y_{\max}(\omega; \tau) \equiv -\min\{\bar{m}, \bar{m} - \tau/\omega\}$. Thus the limited short-sale constraint in (2.8) in the paper.

D DM agent's problem

In this section, we provide the omitted proofs leading up to Theorem 2 on page 13 in the paper. Part 1 of the Theorem is obtained in Lemma 1, Part 2 is proven as Lemma 2. Part 3(a) is proven as Lemma 3. Lemmata 4 and 5 together establish Parts 3(b) and 3(c) of the Theorem. Finally, Lemma 6 establishes Part 3(d) of the Theorem.

D.1 DM buyer optimal policies

Recall the DM buyer's problem from (2.18):

$$B(\mathbf{s}) = \max_{x \in [0,m], b \in [0,1]} \{f(x, b; m, \omega)\},$$

where

$$f(x, b; m, \omega) := \beta(1-b)\left[\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right] + b\left[u^{Q}(x, b) + \beta\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right],$$

and, we have re-defined the composite function $u \circ Q$ as u^{Q}. Note that we have not explicitly written $f(x, b; m, \omega)$ as depending on ω_{+1} which is taken as parametric. In an equilibrium, ω_{+1} will be recursively dependent on ω, thus our small sleight of hand here in writing $f(x, b; m, \omega)$.

The following Lemmata 1, 2, 3, 4, 5, and 6 make up Theorem 2. Also, these results will rely on the following statements and notations:

1. Assume $\{\omega, \omega_{+1}, \omega_{+2}...\}$ is a given sequence of prices.

2. Let

$$\phi(m,\omega) := \frac{\omega m + \tau}{\omega_{+1}(1+\tau)},$$

and,

$$\phi^{\star}(m,\omega) = \phi\left[m - x^{\star}(m,\omega), \omega\right].$$

3. Equivalently define the objective function $f(\cdot,\cdot;m,\omega)$ in the DM buyer's problem (2.18) as follows:

$$\begin{aligned} f(x,b;m,\omega) &= \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \\ &\quad + b\left[u^{Q}(x,b) + \beta\bar{V}\left(\phi^{\star}(m,\omega),\omega_{+1}\right) - \beta\bar{V}\left(\phi(m,\omega),\omega_{+1}\right)\right]. \\ &\equiv \beta\bar{V}\left(\phi(m,\omega),\omega_{+1}\right) + bR(x,b;m,\omega). \end{aligned} \tag{D.1}$$

Remark. Observe that maximizing the value of the objective function $f(x,b;m,\omega)$ in the DM buyer's problem (2.18) is equivalent to maximizing the second term, $bR(x,b;m,\omega)$. Note that the function $R(x,b;m,\omega)$ has the interpretation of the DM buyer's surplus from trading with a particular trading post named (x,b), by offering to pay x in exchange for quantity $Q(x,b)$.

Lemma 1. *For any $\bar{V}(\cdot,\omega_{+1}) \in \mathcal{V}[0,\bar{m}]$, the DM buyer's value function is increasing and continuous in money balances, $B(\cdot;\omega) \in \mathcal{C}[0,\bar{m}]$.*

Proof. Since the functions $W(\cdot,\omega_{+1}), V(\cdot,\omega_{+1}) \in \mathcal{C}[0,\bar{m}]$, i.e., are continuous and increasing on $[0,\bar{m}]$, and $\bar{V} := \alpha W + (1-\alpha V)$, then $\bar{V}(\cdot,\omega_{+1}) \in \mathcal{C}[0,\bar{m}]$. The feasible choice set $\Phi(m) := [0,m] \times [0,1]$ is compact, and it expands with m at each $m \in [0,\bar{m}]$. By Berge's Maximum Theorem, the maximizing selections $(x^{\star},b^{\star})(m,\omega) \in \Phi(m)$ exist for every fixed $m \in [0,\bar{m}]$, since the objective function is continuous on a compact choice set (Berge, 1963). Evaluating the Bellman operator (2.18), we have that the value function $B(\cdot,\omega) \in \mathcal{C}[0,\bar{m}]$. □

Lemma 2. *For any $m \leq k$, DM buyers' optimal decisions are such that $b^{\star}(m,\omega) = 0$ and $B(m,\omega) = \beta\bar{V}[\phi(m,\omega),\omega_{+1}]$, where $\phi(m,\omega) := \frac{\omega m+\tau}{\omega_{+1}(1+\tau)}$.*

Proof. Since a buyer's payment x is always constrained above by her initial money balance m in the DM, it will never be optimal for any firm to trade with such a buyer whose $m \leq k$, as the firm will be making an economic loss. In equilibrium it is thus optimal for a buyer $m \leq k$ to optimally not trade and exit the DM with end-of-period balance as m (i.e., with beginning-of-next-period balance $\phi(m, \omega)$ when inflationary transfers are accounted for). As a result, the continuation value is $\bar{V}[\phi(m,\omega),\omega_{+1}]$, and thus, $B(m,\omega) = \beta\bar{V}[\phi(m,\omega),\omega_{+1}]$, if $m \leq k$. □

Lemma 3. *For any (m, ω), where $m \in [k, \bar{m}]$ and the buyer matching probability is positive $b^\star(m,\omega) > 0$, the optimal selections $(x^\star, b^\star, q^\star)(m,\omega)$ and $\phi^\star(m,\omega) := \phi[m - x^\star(m,\omega), \omega]$ are unique, continuous, and increasing in m.*

Observe that the DM buyer's problem has a general structure similar to that of Menzio et al. (2013). The main difference is in the details underlying the buyer's continuation value function, which in our setting is denoted by $\bar{V}(\cdot,\omega)$. Nevertheless, we still have that $\bar{V}(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$. As a consequence the proof of Lemma 3.3 in Menzio et al. (2013) can be adapted to our setting. For the reader's convenience, we repeat the proof strategy of Menzio et al. (2013) below for our model setting in a few steps:

Proof. The DM buyer's problem (2.18) can be re-written as

$$B(\mathbf{s}) = \beta\bar{V}(\phi(m,\omega),\omega_{+1}) + \exp\left\{\max_{x\in[0,m],b\in[0,1]}\{\ln(b) + \ln[R(x,b;m,\omega)]\}\right\}.$$

The optimizers thus must satisfy

$$(x^\star, b^\star)(m,\omega) \in \left\{\arg\max_{x\in[0,m],b\in[0,1]}\{\ln(b) + \ln[R(x,b;m,\omega,\omega_+)]\}\right\}. \tag{D.2}$$

(*Uniqueness and continuity of policies.*) First we establish that the policy functions are continuous, and, at every state, there is a unique optimal selection: Since $u^Q(x,b)$ is continuous, jointly and strictly concave in (x,b), and by assumption, $\bar{V}(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$, then

$$R(x,b;m,\omega) \equiv u^Q(x,b) + \beta\bar{V}(\phi^\star(m,\omega),\omega_{+1}) - \beta\bar{V}(\phi(m,\omega),\omega_{+1})$$

is continuous, jointly and strictly concave in the choice variables (x,b). Also, $\ln(b)$ is strictly increasing and strictly concave in b. Thus the maximand is jointly and strictly concave in (x,b). By Berge's Maximum Theorem, the optimal selections $(x^\star, b^\star)(m,\omega)$ are continuous and unique at any m. Since $c \mapsto c(q)$ is bijective, then

$$q^\star(m,\omega) = c^{-1}[x^\star(m,\omega) - k/\mu(b^\star(m,\omega))]$$

is continuous in m; and so is $\phi^{\star}(m, \omega)$.

(*Monotonicity of policies.*) The remainder of this proof establishes that the policy functions are increasing. The key idea of the proof is in showing that the choice set is a lattice equipped with a partial order, that the choice set is increasing in m, and, has increasing differences on the choice set, and the slices of the buyer's objective is supermodular in each given direction of his choice set. By Theorem 2.6.2 of Topkis (1998), these properties are sufficient to ensure that the buyer's objective function is supermodular. Together, these properties suffice, by Theorem 2.8.1 of Topkis (1998), for showing that the buyer's optimal policies are increasing functions in m.

1. The function $R(\cdot, \cdot, \cdot, \omega)$ in (D.2) has *increasing difference* in (x, b, m) and is therefore supermodular:

   Fix an $m \in [k, \bar{m}]$ and $b \in (0, 1]$. (The case of $b = 0$ is trivially uninteresting.) It suffices to optimize over the function $\ln[R(\cdot, b, m, \omega)]$ in (D.2). Then the optimizer

$$\tilde{x}(b, m, \omega) \in \left\{ \arg \max_{x \in [k, \bar{m}]} \{\ln [R(x, b, m, \omega)]\} \right\}$$

   is unique for each (m, b, ω), since the objective functions is strictly concave.

   Next we show how the value of the objective function has increasing differences in (x, b, m), throughout taking the sequence $\{\omega, \omega_{+1}, ...\}$ as fixed. Thus we will now write $R(x, b, m) \equiv R(x, b, m, \omega)$ to temporarily ease the notation. First, the feasible choice set

$$\mathcal{F}_m := \{(x, b, m) : x \in [0, m], b \in [0, 1], m \in [k, \bar{m}]\},$$

   is a partially ordered set with relation $\leq$, and it has least-upper and greatest-lower bounds. It is therefore a sublattice in $\mathbb{R}^3_+$. Observe that $\mathcal{F}_m$ is increasing in m. Second, pick any $m' > m$, $b' > b$, and $x' > x$ in $\mathcal{F}_m$:

   (a) For fixed x, consider $m' > m$ and $b' > b$. Then, we can write

$$\begin{aligned} R(x, b', m') &- R(x, b, m) \\ &= \left[u^Q(x, b') - u^Q(x, b)\right] \\ &\quad + \beta \left[\bar{V}\left(\frac{\omega(m' - x) + \tau}{\omega_{+1}(1 + \tau)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega(m - x) + \tau}{\omega_{+1}(1 + \tau)}, \omega_{+1}\right)\right] \\ &\quad - \beta \left[\bar{V}\left(\frac{\omega m' + \tau}{\omega_{+1}(1 + \tau)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1 + \tau)}, \omega_{+1}\right)\right]. \end{aligned}$$

   Observe that the RHS is separable in b and m: The first term on the right, $u^Q(x, b') - u^Q(x, b) < 0$, shows increasing difference in b. Likewise the remainder two difference terms on the RHS show increasing differences in m. Overall $R(x,b,m)$ has increasing differences on the lattice $[0,1] \times [0,\bar{m}] \ni (b,m)$.

(b) For fixed m, consider $x' > x$ and $b' > b$. Observe that

$$\begin{aligned} R(x,b,m) - R(x',b,m) &= \left[u^Q(x,b) - u^Q(x',b)\right] \\ &+ \beta\left[\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]. \end{aligned} \tag{D.3}$$

Now, using the expression (D.3) twice below, we have that

$$\begin{aligned} &\left[R(x',b',m) - R(x,b',m)\right] - \left[R(x',b,m) - R(x,b,m)\right] \\ &= \left[u^Q(x',b') - u^Q(x,b')\right] \\ &\qquad + \beta\left[\bar{V}\left(\frac{\omega(m-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\ &\qquad - \left[u^Q(x',b) - u^Q(x,b)\right] \\ &\qquad - \beta\left[\bar{V}\left(\frac{\omega(m-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\ &\qquad\qquad = \left[u^Q(x',b') - u^Q(x,b')\right] - \left[u^Q(x',b) - u^Q(x,b)\right] > 0, \end{aligned}$$

where the last inequality is implied by the fact that $\left(u^Q\right)_{12}(x,b) > 0$. Therefore $R(x,b,m)$ has increasing differences on the lattice $[0,m] \times [0,1] \ni (x,b)$.

(c) For fixed b, consider $x' > x$ and $m' > m$. Observe that

$$\begin{aligned} &\left[R(x',b,m') - R(x,b,m')\right] - \left[R(x',b,m) - R(x,b,m)\right] \\ &= \left[u^Q(x',b) - u^Q(x,b)\right] \\ &\qquad + \beta\left[\bar{V}\left(\frac{\omega(m'-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m'-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\ &\qquad - \left[u^Q(x',b) - u^Q(x,b)\right] \\ &\qquad - \beta\left[\bar{V}\left(\frac{\omega(m-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\ &= \beta\left[\bar{V}\left(\frac{\omega(m'-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m'-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\ &\qquad - \beta\left[\bar{V}\left(\frac{\omega(m-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{m-x+\tau}{1+\tau\omega},\omega_{+1}\right)\right] \geq 0, \end{aligned}$$

where the last weak inequality obtains from the property that $\bar{V}(\cdot,\omega_{+1}) \in \mathcal{V}[0,\bar{m}]$, and $\bar{V}(\cdot,\omega_{+1})$ is therefore weakly concave. Therefore $R(x,b,m)$ has increasing differences on the lattice $[0,m] \times [0,\bar{m}] \ni (x,m)$.

From parts (1a), (1b), and (1c), we can conclude that the objective function $R(\cdot,\cdot,\cdot,\omega)$ has increasing differences on $\mathcal{F}_m$. This suffices to prove that the objective function $R(\cdot,\cdot,\cdot,\omega)$ is supermodular (see Topkis, 1998, Corollary 2.6.1), since the domain of the function is a direct product of a finite set of chains (partially ordered sets with no unordered pair of elements), and, the objective function is real valued (see Topkis, 1978).

2. Since $R(\cdot,b,m)$ is supermodular, for fixed choice b, the optimizer $\tilde{x}(b,m,\omega)$ is increasing in (b,m), for given ω:

   Let $\tilde{x}(b,m,\omega) = \arg\max_{x\in[0,m]} R(x,b,m)$. From part (1a) above, we can deduce that for fixed $\tilde{x}(b,m)$, $\tilde{R}(b,m) \equiv R[\tilde{x}(b,m,\omega),b,m]$ is supermodular on the lattice $[0,1]\times[0,\bar{m}] \ni (b,m)$. Since $R(x,b,m)$ is strictly decreasing in b, then

$$\tilde{R}(b,m) \equiv R[\tilde{x}(b,m,\omega),b,m]$$

   is strictly decreasing in b. Observe that for any $m' \geq m$, where $m',m \in [k,\bar{m}]$, we have

$$\begin{aligned}
&R(x,b,m') - R(x,b,m) \\
&= \beta\left[\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\
&\qquad - \beta\left[\bar{V}\left(\frac{\omega m'+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m'-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \geq 0,
\end{aligned}$$

   since $\bar{V}(\cdot,\omega)$ is concave. Since this inequality holds at each fixed pair(x,b), then,

$$\begin{aligned}
\tilde{R}(b,m) &\equiv R[\tilde{x}(b,m,\omega),b,m] \\
&\leq R[\tilde{x}(b,m,\omega),b,m'] \leq R[\tilde{x}(b,m',\omega),b,m'] \equiv \tilde{R}(b,m').
\end{aligned}$$

   The last weak inequality obtains because the choice set is increasing in m, and so $\tilde{x}(b,m,\omega)$ is a feasible selection for the more relaxed problem whose value is

$$R[\tilde{x}(b,m',\omega),b,m'] = \max_{x\in[0,m']} R(x,b,m').$$

   From these weak inequalities, we can conclude that $\tilde{R}(b,m)$ is increasing in m.

   Now we are ready to apply Theorem 2.8.1 of Topkis (1998) to show that $b^\star(m,\omega)$ increases with m: Let

$$b^\star(m,\omega) = \arg\max_{b\in[0,1]} r(b,m)$$

where $r(b,m) = b \cdot \tilde{R}(b,m)$ and $\tilde{R}(b,m) \equiv R(\tilde{x}(b,m,\omega), b, m, \omega)$. Observe the following identity:

$$\begin{aligned}\left[r(b',m') - r(b,m')\right] - \left[r(b',m) - r(b,m)\right] = \\ b'\left\{\tilde{R}(b',m') - \tilde{R}(b,m') - \left[\tilde{R}(b',m) - \tilde{R}(b,m)\right]\right\} \\ + (b'-b)\left[\tilde{R}(b,m') - \tilde{R}(b,m)\right],\end{aligned}$$

for any $b, b' \in (0,1], m, m' \in [k, \bar{m}]$ where $b' > b$ and $m' > m$. The first term on the RHS is positive, since $b' > 0$ and since $\tilde{R}(b,m)$ is supermodular in (b,m), then Topkis (1998, Theorem 2.6.1) applies, so that $\tilde{R}(b,m)$ has increasing differences on $[0,1] \times [0,\bar{m}]$ (i.e., the terms in the curly braces are positive). Since we have previously established that $\tilde{R}(b,m)$ is increasing in m, and $b' - b > 0$, then the second term on the RHS is also positive. Thus the objective $r(b,m)$ is supermodular on $[0,1] \times [k,\bar{m}] \ni (b,m)$. (Note that the choice set of b does not depend on m.)

Therefore, by Theorem 2.8.1 of Topkis (1998), the optimal selection $b^\star(m,\omega)$ is increasing in m. Since $\tilde{x}(b,m,\omega)$ is increasing in (b,m), for given ω, then we can conclude that the optimal payment choice $x^\star(m,\omega) = \tilde{x}(b^\star(m,\omega), m, \omega)$ is also increasing in m.

3. The decision $q^\star(m,\omega)$ is monotone in m:

We perform a change of decision variables. Denote $a \equiv \varphi + c(q)$, where, $\varphi \equiv m - x$. Then we have a change of the DM buyer's decision variables from (x,q) to (a,q). From (2.11), we can re-write $m - x = a - c(q)$ and $b = \mu^{-1}[k/(m-a)]$. Since $b \in [0,1]$, the domain of a is $[0, m-k]$, and the domain for q is $[0,a]$. The buyer's problem from (2.18) is thus equivalent to writing

$$\begin{aligned}B(m,\omega) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) = \max_{a\in[0,m-k], q\in[0,a]} \left\{\mu^{-1}\left(\frac{k}{m-a}\right)\left[u(q)\right.\right. \\ \left.\left. + \beta\bar{V}\left(\frac{\omega(a-q)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]\right\}. \quad \text{(D.4)}\end{aligned}$$

Recall we take the sequence $(\omega, \omega_{+1}, \ldots)$ as parametric here. This problem can be broken down into two steps: Fix (a,ω). Find the optimal q for any a, to be denoted by $\tilde{q}(a,\omega)$, and then, find the optimal a given (a,ω), to be denoted by $a^\star(m,\omega)$. Then we can deduce the optimal $q^\star(m,\omega) \equiv \tilde{q}[a^\star(m,\omega), m, \omega]$. We details these steps below:

(a) For any fixed a and (m,ω), $\tilde{q}(a,\omega)$ induces the value

$$J(a,\omega) = \max_{q\in[0,a]} \left\{u(q) + \beta\bar{V}\left(\frac{\omega(a-q)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}. \quad \text{(D.5)}$$

Observe that q and J do not depend on m, given a fixed a. The objective function on the RHS is clearly supermodular on the lattice $[0, m-k] \times [0, a] \ni (a, q)$. Since the objective function is strictly concave, the selection $\tilde{q}(a, \omega)$ in unique for every a, given ω. Also, the choice set $[0, a]$ increases with a, and, the objective function is increasing. Therefore, respectively by Theorems 2.8.1 (increasing optimal solutions) and 2.7.6 (preservation of supermodularity) of Topkis (1998), we have that $\tilde{q}(a, \omega)$ and $J(a, \omega)$ are increasing in a.

(b) Given best response $\tilde{q}(a, \omega)$, the optimal $a^{\star}(m, \omega)$ choice satisfies

$$a^{\star}(m, \omega) = \arg \max_{a \in [0, m-k]} g(a, m, \omega),$$

where

$$g(a, m, \omega) = \mu^{-1}\left(\frac{k}{m-a}\right)\left[J(a, \omega) - \beta \bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right].$$

(Again, note that we have suppressed dependencies on ω_{+1} since this is taken as parametric by the agent, and, in equilibrium ω_{+1} recursively depends on ω.)

Consider the case $J(a, \omega) - \beta \bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \geq 0$. Since $\mu(b)$ is strictly decreasing in b, and $1/\mu(b)$ is strictly convex in b, then $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly increasing in m, strictly decreasing in a, and is strictly supermodular in (a, m). Pick any $a', a \in [0, m-k]$, and any $m', m \in [k, \bar{m}]$, such that $a' > a$ and $m' > m$. We have the identity:

$$\begin{aligned}
&[g(a', m', \omega) - g(a, m', \omega)] - [g(a', m, \omega) - g(a, m, \omega)] = \\
&\left[\mu^{-1}\left(\frac{k}{m'-a'}\right) - \mu^{-1}\left(\frac{k}{m-a'}\right)\right][J(a', \omega) - J(a, \omega)] \\
&+ \left[\mu^{-1}\left(\frac{k}{m'-a}\right) - \mu^{-1}\left(\frac{k}{m'-a'}\right)\right] \\
&\quad \times \left[\beta \bar{V}\left(\frac{\omega m' + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \beta \bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right] \\
&+ \left[\mu^{-1}\left(\frac{k}{m'-a'}\right) - \mu^{-1}\left(\frac{k}{m'-a}\right) - \mu^{-1}\left(\frac{k}{m-a'}\right) - \mu^{-1}\left(\frac{k}{m-a}\right)\right] \\
&\qquad \times \left[J(a, \omega) - \beta \bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]
\end{aligned}$$

The first term on the RHS is positive since $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly increasing in m, and we have previously shown that $J(a, \omega)$ is increasing in a. The second term on the RHS is positive since $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly decreasing in a, and, $\tilde{V}(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$. The last term on the RHS is positive since $\mu^{-1}\left(\frac{k}{m-a}\right)$ is supermodular, and therefore its first term in the product shows increasing differences Topkis (1998, Theorem 2.6.1). Its last term in the product is positive under the case we are considering. Therefore the LHS is positive, and this suffices to establish that $g(a, m, \omega)$ is strictly supermodular (Topkis, 1998, Theorem 2.8.1).

Finally, since the choice set $[0, m-k]$ is increasing in m, the solution $a^{\star}(m, \omega)$ is also increasing in m Topkis (1998, Theorem 2.6.1). Since we have established in part (3a) that $\tilde{q}(m, \omega)$ is increasing in a, then, $q^{\star}(m, \omega) \equiv \tilde{q}[a^{\star}(m, \omega), \omega]$ is also increasing in m.

4. The decision $\phi^{\star}(m, \omega)$ is monotone in m:

Similar to the procedure in the last part, we perform a change of decision variables via $a \equiv \varphi + c(q)$, where, $\varphi \equiv m - x$. The domain for φ is $[0, \min\{m, a\}]$. However, an optimal choice under $b > 0$ means that we will have $\varphi < m$ (the end of period residual balance is less than the beginning of period money balance). This is because, if $\varphi = m$ then it must be that $x = 0$, i.e., the buyer pays nothing; but this is not optimal for the buyer if the buyer faces a positive probability of matching $b > 0$. Moreover, $\varphi < a$, if $u'(0)$ is sufficiently large—i.e., the buyer can always increase utility by raising x (thus lowering φ such that $\varphi < a$ attains). Thus the upper bound on φ will never be binding. As such, the buyer's problem from (2.18) can be re-written as

$$\begin{aligned} B(m,\omega) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) = \max_{a\in[0,m-k],\varphi\geq 0} \Bigg\{\mu^{-1}\left(\frac{k}{m-a}\right)\Bigg[u^{C}(a-\varphi) \\ +\beta\bar{V}\left(\frac{\omega\varphi + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\Bigg]\Bigg\}, \end{aligned} \tag{D.6}$$

where $u^{C}(q) := u \circ c^{-1}(q)$, which is continuously differentiable with respect to $q \geq 0$. For fixed $a \in [0, m-k]$, denote the value

$$J(a, \omega) = \max_{\varphi \geq 0}\left\{u^{C}(a-\varphi) + \beta\bar{V}\left(\frac{\omega\varphi + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}, \tag{D.7}$$

and the optimizer,

$$\tilde{\varphi}(a, \omega) = \arg\max_{\varphi \geq 0}\left\{u^{C}(a-\varphi) + \beta\bar{V}\left(\frac{\omega\varphi + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}. \tag{D.8}$$

Denote also $\tilde{q}(a, \omega) = c^{-1}[a - \tilde{\varphi}(a, \omega)]$.

Given $\tilde{\varphi}(a,\omega)$, the optimal choice over a, i.e., $a^{\star}(m,\omega)$, solves

$$B(m,\omega) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) = \max_{a\in[0,m-k]}\left\{\mu^{-1}\left(\frac{k}{m-a}\right)\left[J(a,\omega) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\right\}.$$

Applying the similar logic in the proof in part 3 on page OA-§.D. 14, we can show that $\tilde{\varphi}(a,\omega)$ is increasing in a; that $a^{\star}(m,\omega)$ is increasing in m, and therefore, $\varphi^{\star}(m,\omega) \equiv \tilde{\varphi}[a^{\star}(m,\omega),\omega]$ is increasing in m. Finally, since

$$\phi^{\star}(m,\omega) := [\omega\varphi^{\star}(m,\omega) + \tau] / [\omega_{+1}(1+\tau)],$$

which is a linear transform of $\varphi^{\star}(m,\omega)$, then $\phi^{\star}(m,\omega)$ is increasing with m, since $\omega/[\omega_{+1}(1+\tau)] > 0$.

□

D.2 DM buyer value function and first-order conditions

Let us return to the DM buyer's problem re-written as (D.6) in part (4) of the proof of Lemma 3 on page OA-§.D. 10. The buyer's decision problem over $\varphi \equiv m - x$, for any fixed decision $a \equiv \varphi + c(q)$, yields the value $J(a,\omega)$ as defined in equation (D.7) of that proof. The following intermediate results says that the value function $J(\cdot,\omega)$ is differentiable with respect to a and its marginal value can be related to primitives, i.e.:

Lemma 4. *The marginal value of $J(\cdot,\omega)$ agrees with the flow DM marginal utility with respect to the buyer's payment x,*

$$J_1(a,\omega) = u'[\tilde{q}(a,\omega)] \equiv \left(u^{Q}\right)_1[x^{\star}(m,\omega), b^{\star}(m,\omega)] > 0. \tag{D.9}$$

Proof. Consider the problem described in equations (D.6) and (D.7). Observe that since $\varphi \equiv a - c(q)$, then

$$\tilde{\varphi}(a,\omega) = \arg\max_{\varphi\geq 0}\left\{u^{C}(a-\varphi) + \beta\bar{V}\left(\frac{\omega\varphi + \tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right\} \tag{D.10}$$

is continuous with respect to a: There is some $\delta' > 0$, such that for all $\varepsilon \in [0,\delta']$, the choices $\tilde{\varphi}(a+\varepsilon,\omega)$ and $\tilde{\varphi}(a-\varepsilon,\omega)$ exist. Moreover the optimal selection $\tilde{\varphi}(a,\omega)$ is unique since the objective function in (D.10) is strictly concave by virtue of u^{C} being strictly concave and $\bar{V}$ being concave. Denote also $\tilde{q}(a,\omega) = c^{-1}[a - \tilde{\varphi}(a,\omega)]$, where the choices $\tilde{q}(a+\varepsilon,\omega)$ and $\tilde{q}(a-\varepsilon,\omega)$ also exist, by continuity of c^{-1} in its argument.

To verify (D.9), we can use the perturbed choices, $\tilde{\varphi}(a+\varepsilon,\omega)$ and $\tilde{\varphi}(a-\varepsilon,\omega)$, for evaluating right- and left-derivatives of the functions u^C, $\bar{V}$ and J, in order to "sandwich" the derivative function $J_1(\cdot,\omega)$ and arrive at the claimed result. For notational convenience below, we define the following function

$$K_\omega\left[a,\tilde{\varphi}(a,\omega)\right] \equiv u^C\left(a-\tilde{\varphi}(a,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right).$$

Consider first the right derivatives: Take $\delta' \searrow 0$ such that for all $\varepsilon \in [0,\delta']$, the choice $\tilde{\varphi}(a+\varepsilon,\omega)$ is affordable for a buyer a. Since $\tilde{\varphi}(\cdot,\omega)$ is an optimal policy satisfying (D.8), then under action $\tilde{\varphi}(a,\omega)$ we must have that

$$\begin{aligned}
J(a,\omega) &= u^C\left(a-\tilde{\varphi}(a,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\\
&\geq u^C\left(a-\tilde{\varphi}(a+\varepsilon,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a+\varepsilon,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\\
\Leftrightarrow J(a,\omega) &= K_\omega\left[a,\tilde{\varphi}(a,\omega)\right] \geq K_\omega\left[a,\tilde{\varphi}(a+\varepsilon,\omega)\right].
\end{aligned}$$

Again, take $\delta' \searrow 0$ such that $\forall\varepsilon \in [0,\delta']$, the choice $\tilde{\varphi}(a,\omega)$ is affordable for buyer $a+\varepsilon$. Since $\tilde{\varphi}(\cdot,\omega)$ is an optimal policy satisfying (D.8), then under $\tilde{\varphi}(a+\varepsilon,\omega)$ we must have that

$$\begin{aligned}
J(a+\varepsilon,\omega) &= u^C\left(a+\varepsilon-\tilde{\varphi}(a+\varepsilon,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a+\varepsilon,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\\
&\geq u^C\left(a+\varepsilon-\tilde{\varphi}(a,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\\
\Leftrightarrow J(a+\varepsilon,\omega) &= K_\omega\left[a+\varepsilon,\tilde{\varphi}(a+\varepsilon,\omega)\right] \geq K_\omega\left[a+\varepsilon,\tilde{\varphi}(a,\omega)\right].
\end{aligned}$$

Re-write the two inequalities above as

$$\begin{aligned}
\frac{K_\omega\left[a+\varepsilon,\tilde{\varphi}(a,\omega)\right]-K_\omega\left[a,\tilde{\varphi}(a,\omega)\right]}{\varepsilon} &\leq \frac{J(a+\varepsilon,\omega)-J(a,\omega)}{\varepsilon}\\
&\leq \frac{K_\omega\left[a+\varepsilon,\tilde{\varphi}(a+\varepsilon,\omega)\right]-K_\omega\left[a,\tilde{\varphi}(a+\varepsilon,\omega)\right]}{\varepsilon}.
\end{aligned}$$

Since the composite function u^C—and therefore the objective function in (D.7)—is differentiable with respect to a, $J_1(\cdot,\omega)$ clearly exists. Therefore, the right derivative of this value function must agree with its partial derivative: $\lim_{\varepsilon\searrow 0} J(a+\varepsilon,\omega) = J_1(a,\omega)$. Using this fact, the inequalities above imply

$$\lim_{\varepsilon \searrow 0} \frac{K_\omega \left[a+\varepsilon, \tilde{\varphi}(a,\omega)\right] - K_\omega \left[a, \tilde{\varphi}(a,\omega)\right]}{\varepsilon} \leq J_1\left(a,\omega\right)$$
$$\leq \lim_{\varepsilon \searrow 0} \frac{K_\omega \left[a+\varepsilon, \tilde{\varphi}(a+\varepsilon,\omega)\right] - K_\omega \left[a, \tilde{\varphi}(a+\varepsilon,\omega)\right]}{\varepsilon}.$$

Moreover, by continuity of $\tilde{\varphi}(\cdot,\omega)$, we have that $\lim_{\varepsilon \searrow 0} \tilde{\varphi}(a+\varepsilon,\omega) = \tilde{\varphi}(a,\omega)$, so the inequalities above collapse to

$$u'\left[\tilde{q}(a^+,\omega)\right] := \lim_{\varepsilon \searrow 0} \frac{u^C\left(a+\varepsilon-\tilde{\varphi}(a,\omega)\right) - u^C\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon} \leq J_1\left(a,\omega\right)$$
$$\leq \lim_{\varepsilon \searrow 0} \frac{u^C\left(a+\varepsilon-\tilde{\varphi}(a,\omega)\right) - u^C\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon} =: u'\left[\tilde{q}(a^+,\omega)\right].$$

However, the first and the last term in the inequalities above are identical, and they are the same as the right derivative of u with respect to $q := \tilde{q}(a,\omega)$, i.e., $u'\left[\tilde{q}(a^+,\omega)\right]$. Thus, it must be that $u'\left[\tilde{q}(a^+,\omega)\right] = J_1\left(a,\omega\right)$.

Using similar arguments as above, we can also consider the left-hand-side perturbation about a, to evaluate $\tilde{\varphi}(a-\varepsilon,\omega)$. It can be shown that

$$u'\left[\tilde{q}(a^-,\omega)\right] := \lim_{\varepsilon \searrow 0} \frac{u^C\left(a-\varepsilon-\tilde{\varphi}(a,\omega)\right) - u^C\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon} \leq J_1\left(a,\omega\right)$$
$$\leq \lim_{\varepsilon \searrow 0} \frac{u^C\left(a-\varepsilon-\tilde{\varphi}(a,\omega)\right) - u^C\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon} =: u'\left[\tilde{q}(a^-,\omega)\right],$$

so that $u'\left[\tilde{q}(a^-,\omega)\right] = J_1\left(a,\omega\right)$.

Combining the two arguments above, we have that

$$u'\left[\tilde{q}(a,\omega)\right] = u'\left[\tilde{q}(a^+,\omega)\right] = u'\left[\tilde{q}(a^-,\omega)\right] = J_1\left(a,\omega\right) > 0.$$

Finally, the equivalence $u'\left[\tilde{q}(a,\omega)\right] = \left(u^Q\right)_1\left[x^\star(m,\omega), b^\star(m,\omega)\right]$ can be derived using standard calculus, since the composite function $u^Q \equiv u \circ Q$ is a known continuously differentiable function in its arguments (x,b). The assumption on u that marginal utility is everywhere positive renders $u'\left[\tilde{q}(a,\omega)\right] > 0$. This completes the proof of the claim. $\square$

Lemma 5. *At any (m,ω), where $m \in [k, \bar{m}]$ and the buyer matching probability is positive $b^{\star}(m,\omega) > 0$,*

1. the buyer's marginal valuation of money $B_1(m,\omega)$ exists if and only if $\bar{V}_1\left[\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega\right]$ exists; and
2. $B(m,\omega)$ is strictly increasing in m.

Proof. Lemma 3 implies that $\tilde{q}(a,\omega)$ is increasing in a. Since we have shown that $u'[\tilde{q}(a,\omega)] = J_1(a,\omega) > 0$, then $J_1(a,\omega)$ is also decreasing in a. Since $J(a,\omega)$ is clearly increasing in a, then we conclude that it is also concave in a. The term $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly decreasing and strictly concave in a. Therefore the objective function in (D.6) is strictly concave in a. Thus maximizing (D.6) over a yields a unique optimal selection $a^{\star}(m,\omega)$. Moreover, the objective function in (D.6) is continuously differentiable with respect to a; and using (D.9) we can show that $a^{\star}(m,\omega)$ satisfies the first-order condition:[29]

$$J\left(a^{\star}(m,\omega),\omega\right) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) + u'\left[\tilde{q}\left(a^{\star}(m,\omega),\omega\right)\right]\cdot\frac{k\cdot\mu'\left[b^{\star}(m,\omega)\right]b^{\star}(m,\omega)}{\mu\left[b^{\star}(m,\omega)\right]^2}\begin{cases}=0, & \text{if } a^{\star}(m,\omega) < m-k \\ <0, & \text{if } a^{\star}(m,\omega) = m-k\end{cases}. \tag{D.11}$$

Observe that $b^{\star}(m,\omega) > 0$ implies the buyer has more than enough initial balance for purchasing $q^{\star}(m,\omega)$, i.e.,

$$m - \varphi^{\star}(m,\omega) > c\left[q^{\star}(m,\omega)\right] + k \implies a(m,\omega) \equiv \varphi^{\star}(m,\omega) + c\left[q^{\star}(m,\omega)\right] < m-k.$$

Since $a^{\star}(m,\omega) < m-k$, and $a^{\star}(m,\omega)$ is continuous in m, then there is an $\epsilon > 0$ such that the following selections are also feasible: $a^{\star}(m+\epsilon,\omega) < m-k$, and, $a^{\star}(m,\omega) < (m-\epsilon)-k$. Define the open ball $\mathbf{N}_{\epsilon}(m) := (m-\epsilon, m+\epsilon)$. Note that for any $m' \in$

[29] Note that $b = \mu^{-1}\left(\frac{k}{m-a}\right)$. The term $\mathrm{d}b/\mathrm{d}a = k/(m-a)^2 \times 1/\mu'[b]$ can be derived using the implicit function theorem: Define $H(a,b) = k/(m-a) - \mu[b] = 0$. Then $\mathrm{d}b/\mathrm{d}a = -H_a(a,b)/H_b(a,b)$, which yields the result. The first-order condition is thus derived as

$$\left[J\left(a^{\star}(m,\omega),\omega\right) - \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]\frac{k}{(m-a^{\star}(m,\omega))^2}\frac{1}{\mu'\left[b^{\star}(m,\omega)\right]} + J_1\left(a^{\star}(m,\omega),\omega\right)\mu^{-1}\left(\frac{k}{m-a^{\star}(m,\omega)}\right)\begin{cases}=0, & \text{if } a^{\star}(m,\omega) < m-k \\ <0, & \text{if } a^{\star}(m,\omega) = m-k\end{cases}.$$

Moreover, since $k/(m-a) = \mu(b)$, we can write $\mathrm{d}b/\mathrm{d}a = k/(m-a)^2 \times 1/\mu'[b] \equiv [\mu(b)]^2/k \times 1/\mu'[b]$, and using the relation (D.9), the first-order condition can be further simplified to (D.11).

$\mathbf{N}_\epsilon(m)$, the selection $a^\star(m',\omega)$ is feasible for an agent m; and $a^\star(m,\omega)$ is feasible for agent m'.

Given that $a^\star(m,\omega)$ is optimal for agent m, and since $\varphi^\star(m,\omega) = \tilde{\varphi}[a^\star(m,\omega)]$, then we have the buyer's optimal value as

$$\begin{aligned} B(m,\omega) &= \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) + \max_{a\in[0,m-k],\varphi\geq 0}\left\{\mu^{-1}\left(\frac{k}{m-a}\right)\right. \\ &\quad \left.\times\left[u\circ c^{-1}(a-\varphi)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right\}\right. \\ &= F(a^\star(m,\omega),m) \geq F(a^\star(m+\epsilon,\omega),m). \end{aligned}$$

where $F(a,m) := \beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)+\mu^{-1}\left(\frac{k}{m-a}\right)\left[J(a,\omega)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]$. Similarly, for agent $m+\epsilon$, it must be that

$$B(m+\epsilon,\omega) = F(a^\star(m+\epsilon,\omega),m+\epsilon) \geq F(a^\star(m,\omega),m+\epsilon).$$

Clearly,

$$\begin{aligned} \frac{F(a^\star(m,\omega),m+\epsilon)-F(a^\star(m,\omega),m)}{\epsilon} &\leq \frac{B(m+\epsilon,\omega)-B(m,\omega)}{\epsilon} \\ &\leq \frac{F(a^\star(m+\epsilon,\omega),m+\epsilon)-F(a^\star(m+\epsilon,\omega),m)}{\epsilon}. \end{aligned}$$

Since $F(a,m)$ is continuous and concave in a, and, $a^\star(m,\omega)$ is continuous in m, the following limits exist (Rockafellar, 1970, Theorem 24.1, pp.227-228), and the inequality ordering is preserved in the limit:

$$\begin{aligned} \lim_{\epsilon\searrow 0}\frac{F(a^\star(m,\omega),m+\epsilon)-F(a^\star(m,\omega),m)}{\epsilon} &\leq \lim_{\epsilon\searrow 0}\frac{B(m+\epsilon,\omega)-B(m,\omega)}{\epsilon} \\ &\leq \lim_{\epsilon\searrow 0}\frac{F(a^\star(m+\epsilon,\omega),m+\epsilon)-F(a^\star(m+\epsilon,\omega),m)}{\epsilon}. \end{aligned}$$

Since $\lim_{\epsilon\searrow 0} a^\star(m+\epsilon,\omega) = a^\star(m,\omega)$, the inequalities above are equivalent to

$$\begin{aligned} & b^\star(m,\omega)\left[J_1(a^\star(m,\omega),\omega)-\frac{\beta}{1+\tau}\bar{V}_1\left(\frac{\omega m^+ +\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\ & + \frac{\beta}{1+\tau}\bar{V}_1\left(\frac{\omega m^+ +\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) \\ & \quad \leq B_1(m^+,\omega) \\ & \qquad \leq b^\star(m,\omega)\left[J_1(a^\star(m,\omega),\omega)-\frac{\beta}{1+\tau}\bar{V}_1\left(\frac{\omega m^+ +\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\ & \qquad\qquad + \frac{\beta}{1+\tau}\bar{V}_1\left(\frac{\omega m^+ +\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right), \end{aligned}$$

where

$$\bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)$$
$$:= \lim_{\epsilon \searrow 0}(1+\tau)\left(\frac{\omega_{+1}}{\omega}\right)\left[\bar{V}\left(\frac{\omega(m+\epsilon)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]/\epsilon.$$

However, observe that the first and the last terms in the inequalities are identical. Thus we must have that the right derivative of $B(\cdot, \omega)$ satisfies

$$B_1(m^+, \omega) = b^\star(m,\omega)\left[J_1(a^\star(m,\omega),\omega) - \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]$$
$$+ \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right).$$

By a similar process to arrive at the left derivative of $B(\cdot, \omega)$, we have

$$B_1(m^-, \omega) = b^\star(m,\omega)\left[J_1(a^\star(m,\omega),\omega) - \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]$$
$$+ \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right),$$

where

$$\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)$$
$$:= (1+\tau)\left(\frac{\omega_{+1}}{\omega}\right)\lim_{\epsilon \searrow 0}\left\{\frac{1}{\epsilon}\left[\bar{V}\left(\frac{\omega(m-\epsilon)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]\right\}.$$

Using the result from (D.9) in Lemma 4 on page OA-§.D. 17, we can re-write these right- and left-derivative functions, respectively, as

$$B_1(m^+, \omega) = b^\star(m,\omega)\left(u^Q\right)_1[x^\star(m,\omega), b^\star(m,\omega)]$$
$$+ \frac{\beta[1-b^\star(m,\omega)]}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right), \quad \text{(D.12)}$$

and,

$$B_1(m^-, \omega) = b^\star(m,\omega)\left(u^Q\right)_1[x^\star(m,\omega), b^\star(m,\omega)]$$
$$+ \frac{\beta[1-b^\star(m,\omega)]}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right). \quad \text{(D.13)}$$

From (D.12) and (D.13), it is apparent that $B_1(m, \omega)$ exists if and only if the left- and right-derivatives of $\bar{V}(\cdot, \omega_{+1})$ exist and they agree at the continuation state from m, i.e., if

$$\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) = \bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) = \bar{V}_1\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right).$$

This proves the first part of the statement in the Lemma.

Since $\bar{V}(\cdot, \omega_{+1}) \in \mathcal{V}[0, \bar{m}]$, it is concave and increasing in m, and therefore,

$$\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \geq \bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \geq 0.$$

Since we assumed $b^\star(m, \omega) \in (0, 1]$, and by Lemma 4, we have

$$J_1(a^\star(m, \omega), \omega) \equiv \left(u^Q\right)_1 [x^\star(m, \omega), b^\star(m, \omega)] > 0,$$

then (D.12) and (D.13) imply that the first-order left and right derivatives of $B(\cdot, \omega_{+1})$ satisfy:

$$B_1(m^-, \omega) \geq B_1(m^+, \omega) \geq b^\star(m, \omega)\left(u^Q\right)_1 [x^\star(m, \omega), b^\star(m, \omega)] > 0.$$

From this ordering, we can conclude that if $b^\star(m, \omega) > 0$, the buyer's valuation $B(m, \omega_{+1})$ is *strictly* increasing with his money balance, m. This proves the last part of the statement in the Lemma. □

Lemma 6. *For any (m, ω), where $m \in [k, \bar{m}]$ and the buyer matching probability is positive $b^\star(m, \omega) > 0$, the optimal policy functions $b^\star$and $x^\star$, respectively, satisfy the first-order conditions* (2.19) *and* (2.20)*.*

Proof. We want to show that the first order conditions characterizing the optimal policy functions $b^\star$and $x^\star$, are indeed (2.19) and (2.20). It is immediate that the objective function (2.18) is continuously differentiable with respect to the choice $b \in [0, 1]$. Holding fixed x, if the optimal choice for b is interior, $b^\star(m, \omega) \in (0, 1)$, then it must satisfies the first order condition (2.19) with respect to b:

$$\begin{aligned} u^Q[x^\star(m,\omega), b^\star(m,\omega)] + b^\star(m,\omega)\left(u^Q\right)_2 [x^\star(m,\omega), b^\star(m,\omega)] \\ = \beta[\bar{V}(\phi(m,\omega), \omega_{+1}) - \bar{V}(\phi^\star(m,\omega), \omega_{+1})]. \end{aligned}$$

The first order condition with respect to x is more subtle. We can derive it by first defining one-sided derivatives of $B(\cdot, \omega)$. Assuming beginning-of-next-period residual balance after current DM trade is positive—i.e.,

$$\phi^{\star}(m,\omega) = \frac{\omega\left[m - x^{\star}(m,\omega)\right] + \tau}{\omega_{+1}(1+\tau)} > 0. \tag{D.14}$$

Since (D.14) holds, and since we have shown in Lemma 3 that $x^{\star}(m,\omega)$ and $\phi^{\star}(m,\omega)$ are continuous in $m \in [k, \bar{m}]$, then

$$(\phi^{\star})^{+}(m,\omega) := \frac{\omega\left[m + \varepsilon - x^{\star}(m,\omega)\right] + \tau}{\omega_{+1}(1+\tau)},$$

and,

$$(\phi^{\star})^{-}(m,\omega) := \frac{\omega\left[m - \varepsilon - x^{\star}(m,\omega)\right] + \tau}{\omega_{+1}(1+\tau)},$$

exist and are feasible (or affordable). From (2.18), the DM buyer's one-sided derivatives of $B(\cdot,\omega)$—i.e., its left- or right-marginal valuation of initial money balance—are, respectively,

$$\begin{aligned} B_1\left(m^{+},\omega\right) = & \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right) \\ & \times\left\{\left[1 - b^{\star}(m,\omega)\right]\bar{V}_1\left(\frac{\omega m^{+} + \tau}{\omega_{+1}(1+\tau)},\omega\right) + b^{\star}(m,\omega)\,\bar{V}_1\left[(\phi^{\star})^{+}(m,\omega),\omega_{+1}\right]\right\}, \end{aligned} \tag{D.15}$$

and,

$$\begin{aligned} B_1\left(m^{-},\omega\right) = & \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right) \\ & \times\left\{\left[1 - b^{\star}(m,\omega)\right]\bar{V}_1\left(\frac{\omega m^{-} + \tau}{\omega_{+1}(1+\tau)},\omega\right) + b^{\star}(m,\omega)\,\bar{V}_1\left[(\phi^{\star})^{-}(m,\omega),\omega_{+1}\right]\right\}, \end{aligned} \tag{D.16}$$

where

$$\begin{aligned} & \bar{V}_1\left(\frac{\omega m^{\pm} + \tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) \\ & := (1+\tau)\left(\frac{\omega_{+1}}{\omega}\right)\lim_{\varepsilon\searrow 0}\left\{\frac{1}{\varepsilon}\left[\bar{V}\left(\frac{\omega(m\pm\epsilon)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)},\omega\right)\right]\right\}. \end{aligned}$$

From Lemma 5, we have shown by change of variable, that the one-sided derivatives of $B(\cdot,\omega)$ also satisfy (D.15) and (D.16). These are repeated here for convenience as the following equations:

$$B_1\left(m^+,\omega\right)=\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\left[1-b^{\star}\left(m,\omega\right)\right]\bar{V}_1\left(\frac{\omega m^+ +\tau}{\omega_{+1}\left(1+\tau\right)},\omega\right)$$
$$+\,b^{\star}\left(m,\omega\right)\left(u^Q\right)_1\left[x^{\star}\left(m,\omega\right),b^{\star}\left(m,\omega\right)\right], \quad \text{(D.17)}$$

and,

$$B_1\left(m^-,\omega\right)=\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\left[1-b^{\star}\left(m,\omega\right)\right]\bar{V}_1\left(\frac{\omega m^- +\tau}{\omega_{+1}\left(1+\tau\right)},\omega\right)$$
$$+\,b^{\star}\left(m,\omega\right)\left(u^Q\right)_1\left[x^{\star}\left(m,\omega\right),b^{\star}\left(m,\omega\right)\right]. \quad \text{(D.18)}$$

From the last term on the RHS of each of equations (D.15), (D.16), (D.17), and, (D.18), we have the observation that

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\left(\phi^{\star}\right)^+\left(m,\omega\right),\omega_{+1}\right]=\frac{\beta}{1+\tau\omega}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\left(\phi^{\star}\right)^-\left(m,\omega\right),\omega_{+1}\right]$$
$$=\left(u^Q\right)_1\left[x^{\star}\left(m,\omega\right),b^{\star}\left(m,\omega\right)\right].$$

Since these marginal valuation functions are evaluated at the DM buyer's optimal choice, it must be that $\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\left(\phi^{\star}\right)^+\left(m,\omega\right),\omega_{+1}\right]=\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\left(\phi^{\star}\right)^-\left(m,\omega\right),\omega_{+1}\right]=\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\phi^{\star}(m,\omega),\omega_{+1}\right]$, and, that this satisfies the first order condition (2.20), which is

$$\left(u^Q\right)_1\left[x^{\star}(m,\omega),b^{\star}(m,\omega)\right]=\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\phi^{\star}(m,\omega),\omega_{+1}\right].$$

□

E Proof of Theorem 3

Proof. First, we show that the value functions listed in the definition of a SME are unique given ω. For given ω, The CM agent's problem in (2.5) clearly defines a self-map $T_{\omega}^{CM}:\mathcal{V}\left[0,\bar{m}\right]\rightarrow\mathcal{V}\left[0,\bar{m}\right]$, which preserves monotonicity, continuity and concavity (see Theorem 1). However, for fixed ω, the DM buyer's problem in 2.18 defines an operator $T_{\omega}^{DM}:\mathcal{V}\left[0,\bar{m}\right]\rightarrow\mathcal{C}\left[0,\bar{m}\right]$, where $\mathcal{C}\left[0,\bar{m}\right]\supset\mathcal{V}\left[0,\bar{m}\right]$ is the set of continuous and increasing functions on the domain $\left[0,\bar{m}\right]$. This operator does not preserve concavity. Note that $\bar{V}(\cdot,\omega)\in\mathcal{V}\left[0,\bar{m}\right]$ as previously defined. Now we show that the ex-ante problem in (2.8) and (2.7) defines an operator that maps the CM agent's and the DM buyer's value functions, respectively, $W(\cdot,\omega)=T_{\omega}^{DM}\bar{V}(\cdot,\omega)$ and $B(\cdot,\omega)=T_{\omega}^{DM}\bar{V}(\cdot,\omega)$, back into the set of continuous, increasing and concave functions: $T_{\omega}:\mathcal{V}\left[0,\bar{m}\right]\rightarrow\mathcal{V}\left[0,\bar{m}\right]$. Since

T_ω^{CM} and T_ω^{DM} are monotone functional operators that satisfy discounting with factor $0 < \beta < 1$, then the ex-ante problem in (2.8) and (2.7), which defines the composite operator $T_\omega : \mathcal{V}[0, \bar{m}] \to \mathcal{V}[0, \bar{m}]$, clearly preserves these two properties. (The convexification of the graph of T_ω via lotteries in (2.7) preserves concavity of the image of the operator, thus making it a self-map on $\mathcal{V}[0, \bar{m}]$.) It can be shown that $\mathcal{V}[0, \bar{m}]$ is a complete metric space. Thus $T_\omega : \mathcal{V}[0, \bar{m}] \to \mathcal{V}[0, \bar{m}]$ satisfies Blackwell's conditions, and has a unique fixed point, $\bar{V}(\cdot, \omega) = T_\omega \bar{V}(\cdot, \omega)$, by Banach's fixed point theorem.

Second, we verify the following three properties: (1) By Theorem 1 and Theorem 2, the agent's optimal policies are continuous, single-valued and monotone functions. This implies, for fixed ω, that the Markov kernel $P(\mathbf{s}, \cdot)$ in the distributional operator (2.22) is a probability measure, and, $P(\cdot, E)$ for all Borel subsets $E \in \mathcal{B}([0, \bar{m}])$ is a measurable function. (2) Since agent's policies are monotone, then $P(\mathbf{s}, \cdot)$ is increasing on $[0, \bar{m}]$. Thus the Markov kernel is a transition probability function. (3) The equilibrium policies clearly dictate that the monotone mixing conditions of Hopenhayn and Prescott (1992) are satisfied: Consider a DM buyer who has zero money balance. With non-zero probability either by pure luck (α) or by choosing a lottery that induces such an outcome, he will enter the CM to work and to accumulate some positive money balance. Likewise, consider an agent, either in the DM or CM with the highest possible initial balance of $\bar{m}$. Again, with non-zero probability, she will decumulate that balance, either by matching and spending that balance down in the DM, or, by working less and consuming more in the CM. These conditions, are sufficient, by Theorem 2 of Hopenhayn and Prescott (1992), for the Markov operator (2.22) to have a unique fixed point—i.e., regardless of an initial distribution of agents, the recursive operation on the initial distribution converges (in the weak* topology) to the same long run distribution G.[30]

Third, the market clearing condition (2.21) is continuous on the RHS: (1) The integrand is clearly continuous in m; and, (2) the distribution $G(\cdot; \omega)$ is continuous in ω in the sense of convergence in the weak* topology (Stokey and Lucas, 1989, Theorem 12.13)—i.e., if $\omega_n \to \omega^\star$, then for each $\omega_n \in \{\omega_n\}_{n \in \mathbb{N}}$, the Markov operator (2.22) defines a (weakly) convergent sequence of distributions: $G(\cdot; \omega_n) \to G(\cdot; \omega^\star)$. The LHS of (2.21) is clearly continuous in ω. Thus a SME exists. □

F Algorithm for finding a SME

The following algorithm presumes the more general setting from Section A, which allowed for a new parameter $\alpha \in [0, 1]$. We compute a SME as follows.

[30] Alternatively, one could check the more relaxed set of necessary and sufficient conditions of Kamihigashi and Stachurski (2014, Theorem 2) to guarantee that there is a unique distribution for a given ω, in a steady state SME.

1. Fix a guess ω and guess $\bar{V}(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$.

2. Solve for CM policy and value functions:

   - We know $C^\star(m, \omega) = \bar{C}^\star$ already using equation (2.15).
   - For fixed $\bar{C}^\star$, and, given guess of $\bar{V}(\cdot, \omega)$, iterate on Bellman equation (B.3) solving a one-dimensional (1D) optimization problem over choice $y^\star(\cdot, \omega)$.
     - Note: By equation (2.16), the solution $y^\star(\cdot, \omega) = \bar{y}^*(\omega)$ should be a constant with respect to m.
   - Back out $l^\star(m, \omega)$ using the binding budget constraint (2.17).
   - Store value function $W^\star(\cdot, \omega)$.

3. Solve for DM policy and value functions:

   - For each $m \leq k$, set
     - $b^\star(m, \omega) = x^\star(m, \omega) = q^\star(m, \omega) = 0$
     - $B(m, \omega) = \beta \bar{V}[\phi(m, \omega), \omega_{+1}]$,

     where $\phi(m, \omega) := (m + \tau)/(1 + \tau\omega)$.
   - For each $m \in [k, \bar{m}]$,
     - Invert first-order condition (2.20) to obtain implicit $b[m, x(m, \omega), \omega]$.
     - Plug the implicit expression for $b[m, x(x, \omega), \omega]$ into Bellman equation (2.18), and do a 1D optimization over choices $x(m, \omega)$.
     - Get optimizer $x^\star(m, \omega)$ and corresponding value $B^\star(m, \omega)$.
     - Use previous step to now back out $b^\star(m, \omega)$.

4. Solve ex-ante decision problem:

   - Given approximants $W^\star(m, \omega)$ and $B^\star(m, \omega)$, solve the lottery problem (2.8) and (2.7).
   - Get policies $\left\{\pi_1^{j,\star}(m, \omega)\right\}_{j \in J}$ and $\left\{z_1^{j,\star}(m, \omega), z_2^{j,\star}(m, \omega)\right\}_{j \in J}$, where J is endogenous to the solution of (2.8) and (2.7).
   - Get value of the problem (2.8) and (2.7) as $V^\star(\cdot, \omega)$.

5. Construct the approximant of the ex-ante value function, $\bar{V}^\star(\cdot, \omega) = (1 - \alpha) V^\star(\cdot, \omega) + \alpha W^\star(\cdot, \omega)$.

6. Given policy functions from Steps 2-4, construct limiting distribution $G(\cdot, \omega)$ solving the implicit Markov map (2.22). (See details in Section G on page OA-§.G. 28.)

   - Check if market clearing condition (2.21) holds.

- If not,
  - generate new guess and set $\omega \leftarrow \omega_{new}$;
  - set $\bar{V}(\cdot,\omega) \leftarrow \bar{V}^{\star}(\cdot,\omega)$; and
  - repeat Steps 2-6 again until (2.21) holds.

Algorithm 1 summarizes the steps above with reference to function names in our actual `Python` implementation. Algorithm 1 is called `SolveSteadyState` in our `Python` class file `cssegmod.py`.

Algorithm 1 Solving for an SME

Require: $\alpha \in [0,1)$, $\omega > 0$, $\bar{V}(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$, $N_{\max} > 0$

for $n \leq N_{\max}$ **do**

  $(W^{\star}, \bar{C}^{\star}, l^{\star}, y^{\star}) \leftarrow$ `WorkerProblem`$(\bar{V},\omega)$

  $(B^{\star}, b^{\star}, x^{\star}, q^{\star}) \leftarrow$ `BuyerProblem`$(\bar{V},\omega)$

  $\tilde{V} \leftarrow \max\{B^{\star}(\cdot,\omega), W^{\star}(\cdot - \chi,\omega)\}$

  $\left(V^{\star}, \{z^{\star,j}, \pi^{\star,j}\}_{j\in J}\right) \leftarrow$ `ConvexHull` $\left[\text{graph}\left(\tilde{V}\right)\right]$

  $\bar{V}^{\star} \leftarrow \alpha W^{\star} + (1-\alpha)V^{\star}$

  $\mathbf{v} \leftarrow (\bar{V}, B, W)$

  $\mathbf{p} \leftarrow \langle \{\pi_1^{j,\star}, z_1^{j,\star}, z_2^{j,\star}\}_{j\in J}, (b^{\star}, x^{\star}, y^{\star}, \bar{C}^{\star})\rangle$

  $G \leftarrow$ `Distribution`$(\mathbf{p},\mathbf{v})$

  $\omega^{\star} \leftarrow$ `MarketClearing`(G)

  $e \leftarrow \max\{\|\bar{V}^{\star} - \bar{V}\|, \|\omega^{\star} - \omega\|\}$

  if $e < \varepsilon$ **then**

    STOP

  else

    $(\bar{V},\omega) \leftarrow (\bar{V}^{\star},\omega^{\star})$

    CONTINUE

  end if

end for

  return $\mathbf{p}, \mathbf{v}, G, \omega^{\star}$

G Monte Carlo simulation of stationary distribution

We use a Monte Carlo method to approximate the steady-state distribution of agents at each fixed value of the aggregate state ω, in the `Distribution` step in Algorithm 1. Again, the following algorithm presumes the more general setting from Section A, which allowed for a new parameter $\alpha \in [0,1]$.

For any current outcome of an agent named (m,ω) we can evaluate her ex-post optimal choices in either the CM (2.5), or the DM (2.6). The outcomes of the decision at each current state for an agent is summarized in Algorithm 2. In words, these go as follows: First, we must identify where the agent is currently in (DM or CM). Second, we evaluate the corresponding decisions and record the agent's end-of-period money balance as m'.

Associated with each realized identity m we would also have a record of the agent's actions in that period, e.g., $y^\star(m,\omega)$ and $l^\star(m,\omega)$ if the agent was in the CM, or, $x^\star(m,\omega)$ and $b^\star(m,\omega)$ if she was in the DM.

Algorithm 2 `ExPostDecisions()`

Require: ω, $(B, W) \leftarrow \mathbf{v}$, $(b^\star, x^\star, y^\star) \leftarrow \mathbf{p}$

if $W(m-\chi,\omega) \geq B(m,\omega)$ **then**
  $m' \leftarrow y^\star(m-\chi,\omega)$
else
  Get $u \sim \mathbf{U}[0,1]$
  if $u \in [0, b^\star(m,\omega)]$ **then**
    Get $x^\star(m,\omega) > 0$
    Get $b^\star(m,\omega) > 0$
    $m' \leftarrow m - x^\star(m,\omega)$
  else
    $x^\star(m,\omega) \leftarrow 0$
    $b^\star(m,\omega) \leftarrow 0$
    $m' \leftarrow m$
  end if
end if
  return m'

Algorithm 2 is then embedded in Algorithm 3 below, the Monte Carlo approximation of the steady state distribution at ω. We begin, without loss, from an agent who had just accumulated money balances at the end of a CM, and track the evolution of the agent's money balances over the horizon $T \to +\infty$. Theorem 3 implies that if ω is any candidate equilibrium price, and $G(\cdot,\omega)$ is the unique limiting distribution of agents associated with the candidate equilibrium, then the agent will visit each of all possible states $(m,\omega) \in \text{supp}G(\cdot,\omega)$ with frequency $\mathrm{d}G(m,\omega)$, as $T \to +\infty$.

Algorithm 3 does the following:

1. Begin with an arbitrary agent m.
2. At the start of each date $t \leq T$:
   (a) The agent realizes the shock $z \sim (\alpha, 1-\alpha)$.
   (b) Conditional on the shock z, the agent goes to the CM for sure (and costlessly), or, makes the ex-ante lottery decision.
   (c) If the agent has to solve the ex-ante decision problem, then we evaluate the corresponding ex-post decisions of the agent.

The main output of Algorithm 3 is the list m^T, which stores the stochastic realization of an agent's money balances each period. The long run distribution of the sample m^T is used to approximate $G(\cdot,\omega)$. Algorithms 2 and 3 can be found in the `Python` class `cssegmod.py`, respectively, as methods `ExPostDecisions` and `Distribution`.

Note that the function `Distribution` will be called each time we have an updated guess of ω. Because the Monte-Carlo problem is serially dependent, the only way to speed up the evaluations at this point is to compile it to machine code and execute it on the fly. The user will have the option to exploit `Numba` (a `Python` interface to the `LLVM` just-in-time compiler).

Algorithm 3 `Distribution( )`

Require: $\mathbf{v} \leftarrow (\bar{V}, B, W)$, $\mathbf{p} \leftarrow \langle \{\pi_1^{j,\star}, z_1^{j,\star}, z_2^{j,\star}\}_{j\in J}, (b^\star, x^\star, y^\star, \bar{C}^\star)\rangle$, T, ω

Get $\phi(m,\omega) \leftarrow \frac{m+\tau}{(1+\tau\omega)(1-\delta)}$
Set $m^T \leftarrow \emptyset$
$m \leftarrow y^\star(0,\omega)$
for $t \leq T$ **do**
    $m \leftarrow \phi(m,\omega)$
    Get $u \sim \mathbf{U}[0,1]$
    if $u \in [0,\alpha]$ **then**
        $m' \leftarrow y^\star(m,\omega)$
    else
        if $\exists j \in J$ and $m \in \left[z_1^{j,\star}(m,\omega), z_2^{j,\star}(m,\omega)\right]$ **then**
            Get $u \sim \mathbf{U}[0,1]$
            if $u \in \left[0, \pi_1^{j,\star}(m,\omega)\right]$ **then**
                $m \leftarrow z_1^{j,\star}(m,\omega)$
            else
                $m \leftarrow z_2^{j,\star}(m,\omega)$
            end if
        end if
        $m' \leftarrow$ `ExPostDecisions`$(m,\omega,\mathbf{p},\mathbf{v})$
    end if
    Set $m^T \leftarrow m^T \cup \{m\}$
    Set $m \leftarrow m'$
end for
    return m^T

H Sample SME outcome for an agent

Figure 5 on page OA-§.H. 32 shows a subsample of an agent's existence, for the baseline economy. Corresponding to the DM/CM patterns of spending, we can also observe the subsample's evolution of money balances, in the panel with its vertical axis labelled m, in Figure 5. Here, we can see that at $t = 0$, the agent has his initial real balance as some m. He decides to be in the DM, succeeds in matching with a trading post, and spends a fraction of the balance to consume some positive q. In the following period $t = 1$, he begins with some positive balance—because of transfer $\tau/(\omega(1+\tau)) > 0$ combined with his residual balance—but this amount land in the lottery region; and so the agents plays the lottery. He realizes the high prize of $z_2 = 0.52$ in $t = 1$, and so his money balance is z_2. He matches and gets to consume $q > 0$. (Hence, the record $q_1, x_1, b_1 > 0$.) A similar event realizes again in $t = 2$, so the agent again gets to consume in the DM. In $t = 3$, having spent his balance on consuming in the DM the previous period, the agent realizes a low, i.e., $z_1 = 0$, lottery payoff and his initial balance is thus zero. However, the agent is able to borrow against his CM income, and thus decides to take a temporary short asset position of $-\chi$ (although his recorded money balance is $m = z_1 = 0$) and enters the CM to work, repay the entry cost, consume in the CM, and save some money balance.[31] That is why we see a record of -1 for the figure panel labelled "match status" for $t = 3$. Subsequently in $t = 4$, he begins again with positive balance from the last CM trade. At this point, he decides to go shopping in the DM and again, spends it all in one round. he wins the high prize in the lottery, and finds it profitable to pay the fixed cost χ, enters the CM and works.

[31]See our earlier Remark 1 on page 9.

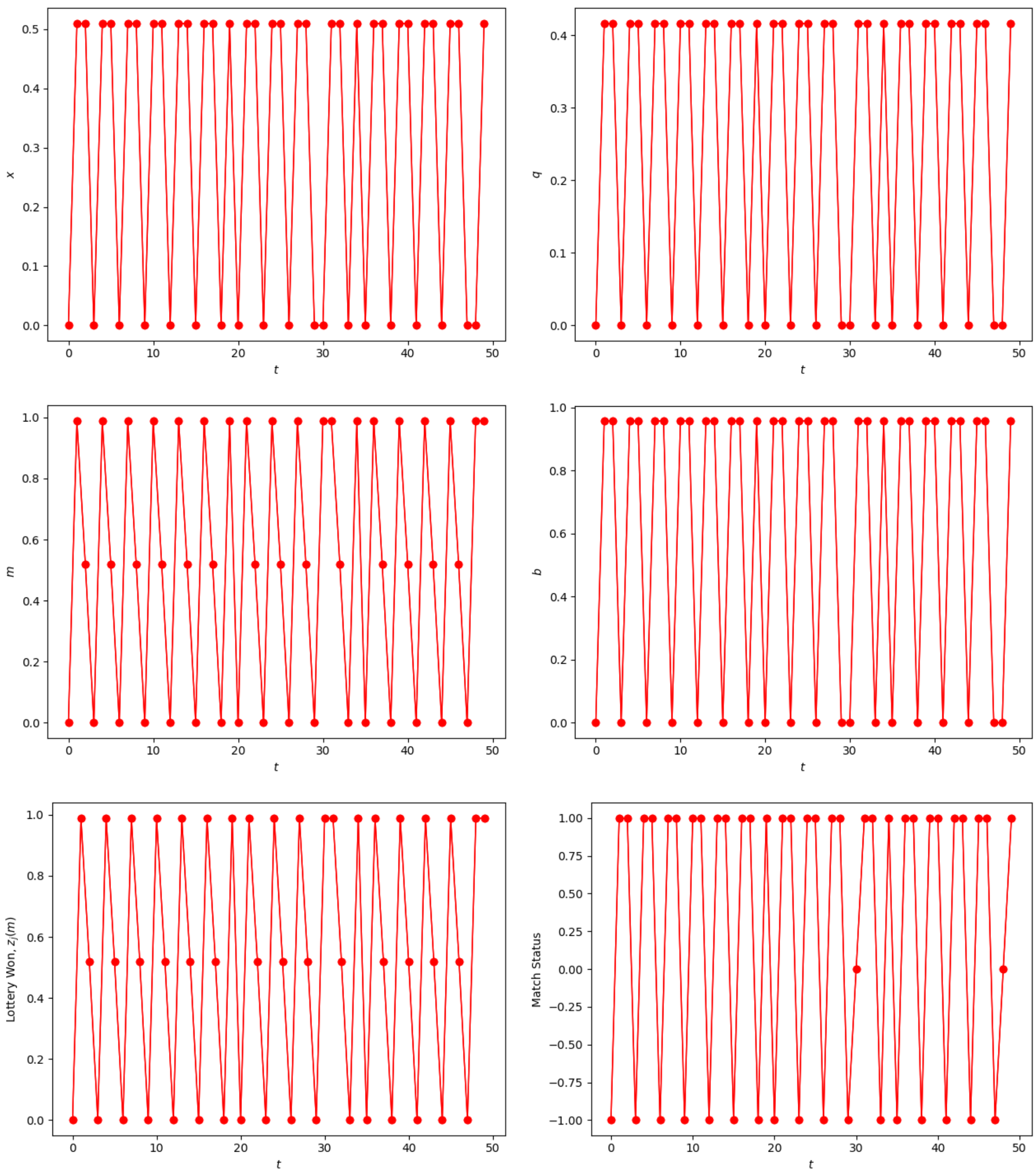

Figure 5: Agent sample path (Benchmark economy). Match Status: 0 (No Match in DM), 1 (Match in DM), −1 (in CM).

In summary, we can observe the following from our simulation: Agents can trade more than once in the DM sometimes. This depends on their luck of the draw in their lottery outcomes. Agents must also pay a fixed cost to enter the CM to load up on money balances. Depending on their money balance, they may sometimes find it worthwhile to borrow against their CM income to pay the fixed cost of CM entry. Thus, we have an equilibrium Baumol-Tobin type of money sepnding cycle in the model. Because agents endogenously may not have complete consumption insurance, the pattern of consumption, for example, DM q in Figure 5, is not completely smooth. (While not shown, the same would occur in terms of CM consumption.)

The long run distribution of the sample path of m is shown in Figure 6, for the benchmark economy. Observe that in each case, the agents, in terms of money balance m can only spend time in the equilibrium's ergodic subspace of the set of money holding. For example this equilibrium subspace turns out to be $\{z_1, [z_2, z_1'], [z_2', \bar{m}]\} = \{0, [0.52, 0.54], [0.99, \bar{m}]\}$ in our baseline economy.

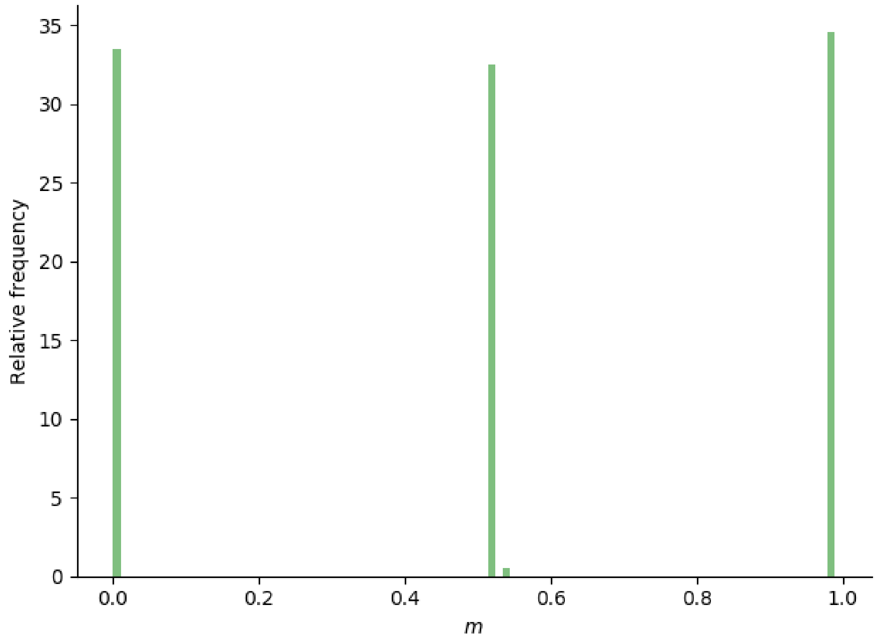

Figure 6: Money distribution

I Robustness and variations on the benchmark economy

This section elaborates in more detail the conclusions made in the paper. Here, we consider two variations or robustness checks on our model assumptions.

First, we show that our insights above are robust to alternative parametrization of the fixed-cost parameter χ. (There we show only the case of a doubling of χ, but qualitatively, the baseline results remain across a wide range of χvalues.

Second, we consider an extreme assumption that agents face a zero-borrowing constraint when overcoming the fixed cost of CM entry, χ: This alternative economy is tantamount to a reparametrization of the borrowing limit (2.9) in the benchmark setting.

We consider the extreme case of $\chi = 0$ in the last part of this section.

I.1 Two separate variations: Higher fixed cost and zero-borrowing

Consider first the second environment. The results are qualitatively similar, across increasing inflation rates, to that of the benchmark economy (i.e., the economy with a natural short-sale constraint on overcoming the CM fixed cost). However, for any fixed inflation rate, when one compares this alternative economy with its benchmark counterpart, we have the following additional insights.

τ	Benchmark	Benchmark, $2 \times \chi$	Zero-borrowing Limit
0.000000	0.509568	0.509569	0.525787
0.008394	0.508201	0.508201	0.524484
0.025000	0.505621	0.505621	0.518749

Table 4: Robustness and variations — Mean money holdings

τ	Benchmark	Benchmark, $2 \times \chi$	Zero-borrowing Limit
0.000000	0.711928	0.711885	0.678093
0.008394	0.712490	0.712474	0.678659
0.025000	0.714158	0.714148	0.687622

Table 5: Robustness and variations — DM expected spending relative to holdings

From Table 4 and 5 we can see the following: In the zero-borrowing-limit economy, average money balance is higher, and, equilibrium extensive margin effects in the DM (i.e., on average how fast agents expend their given DM money holdings) are lower than its corresponding benchmark economy.

However, from Tables 6, 7 and 8, we see that the participation rate in CM is higher, but money distribution is less left-skewed or the Gini index is smaller.

The reason is as follows: In the zero-borrowing economy, agents have a stronger precautionary liquidity-risk insurance motive. Since they cannot borrow to overcome the

τ	Benchmark	Benchmark, $2 \times \chi$	Zero-borrowing Limit
0.000000	0.330868	0.330855	0.333409
0.008394	0.331062	0.331057	0.333645
0.025000	0.331440	0.331435	0.333817

Table 6: Robustness and variations — CM participation rate

τ	Benchmark	Benchmark, $2 \times \chi$	Zero-borrowing Limit
0.000000	-0.082472	-0.082479	-0.101590
0.008394	-0.080854	-0.080855	-0.099131
0.025000	-0.071332	-0.071332	-0.063506

Table 7: Robustness and variations — Skewness

fixed cost of entering the CM to manage their liquidity needs, then whenever they are in the CM, agents will tend to demand more real balances. Likewise, conditional on being in the DM, agents expect to trade at a lower volume relative to their DM money holdings, as they need to economize on the balance in order to possibly overcome the fixed cost of re-entering the CM. This explains the on-average higher money balance (in comparison to the benchmark economy) and the lower rate of trading in the DM. In return, agents would like to go to the CM more often to demand additional precautionary liquidity. That explains a relatively higher top end of the money distribution relative to the bottom (i.e., a more left-skew distribution), and hence a lower Gini index, in comparison to the benchmark economy's outcome.

A similar reasoning also applies in the first alternative case where we doubled the fixed cost in the benchmark economy.

I.2 Results from limit economy when $x = 0$

The figures below were experiments conducted with the benchmark economy, except for $\chi = 0$. It is clear that in the setting the same qualitative results arises as in the benchmark economy discussion in Section 3.5 in the main paper.

τ	Benchmark	Benchmark, $2 \times \chi$	Zero-borrowing Limit
0.000000	0.434360	0.434351	0.409190
0.008394	0.434528	0.434525	0.409407
0.025000	0.435441	0.435439	0.412800

Table 8: Robustness and variations — Gini index

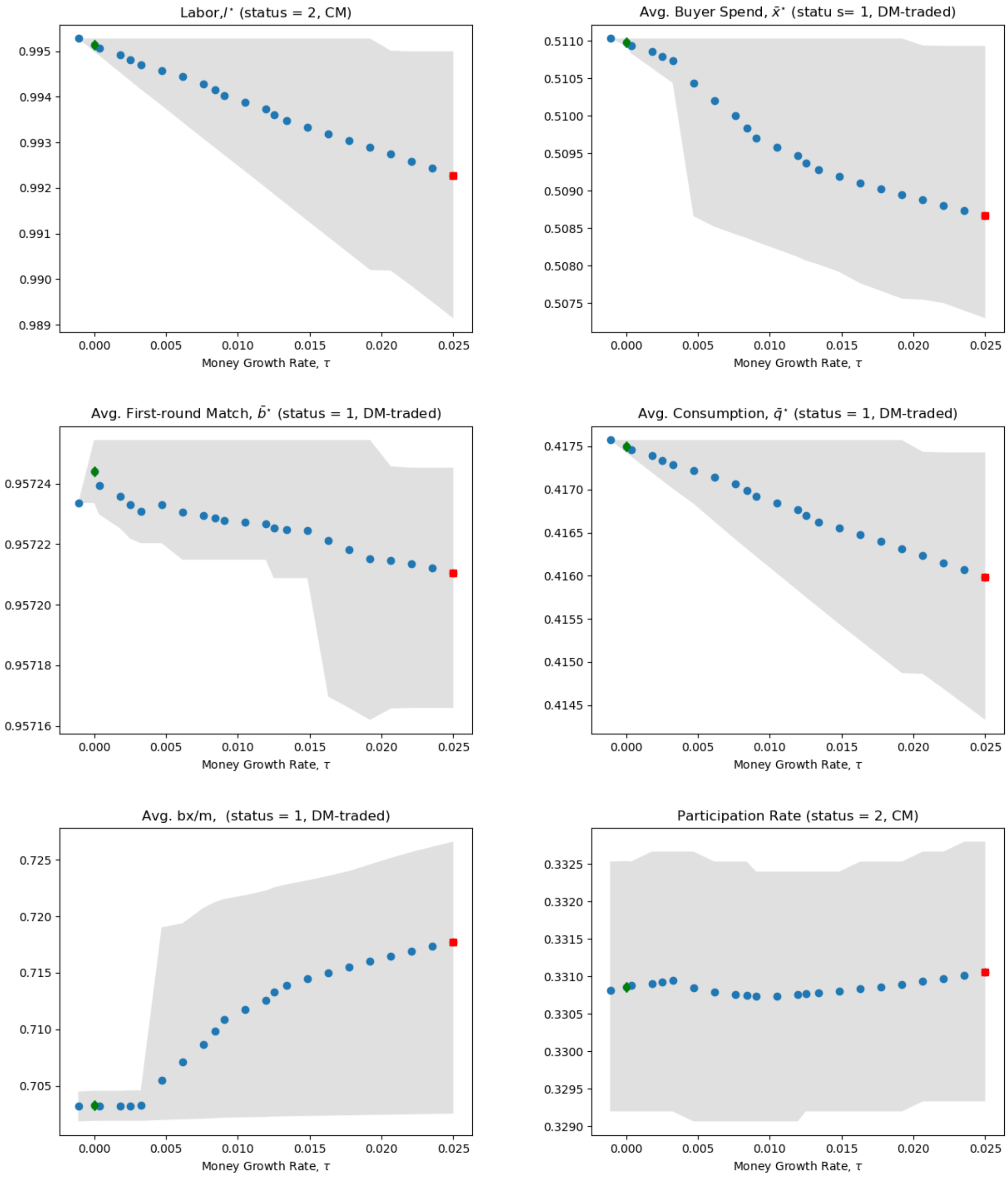

Figure 7: Comparative steady states — allocations (Benchmark but for $\chi = 0$).

Thus, it may have appeared that we complicated the model's mechanism with an additional CM extensive-margin via the fixed cost parameter χ in the benchmark setting. However, the robustness check shows that the same forces are at work even when $\chi = 0$.

Theoretically, the extensive-margin CM-participation decision is still present even if $\chi = 0$. Why? As we discussed in the main paper, because the flow preference function of agents is strictly concave, they would like to consume both CM and DM goods in their infinite lifetimes. As a result agents will still transit from CM to DM recurrently, even when it is completely costless to participate in the CM where markets are complete, and even when in the DM, there is a risk that agents may not get to match and consume, and thus face a liquidity-holdings and insurance risk.

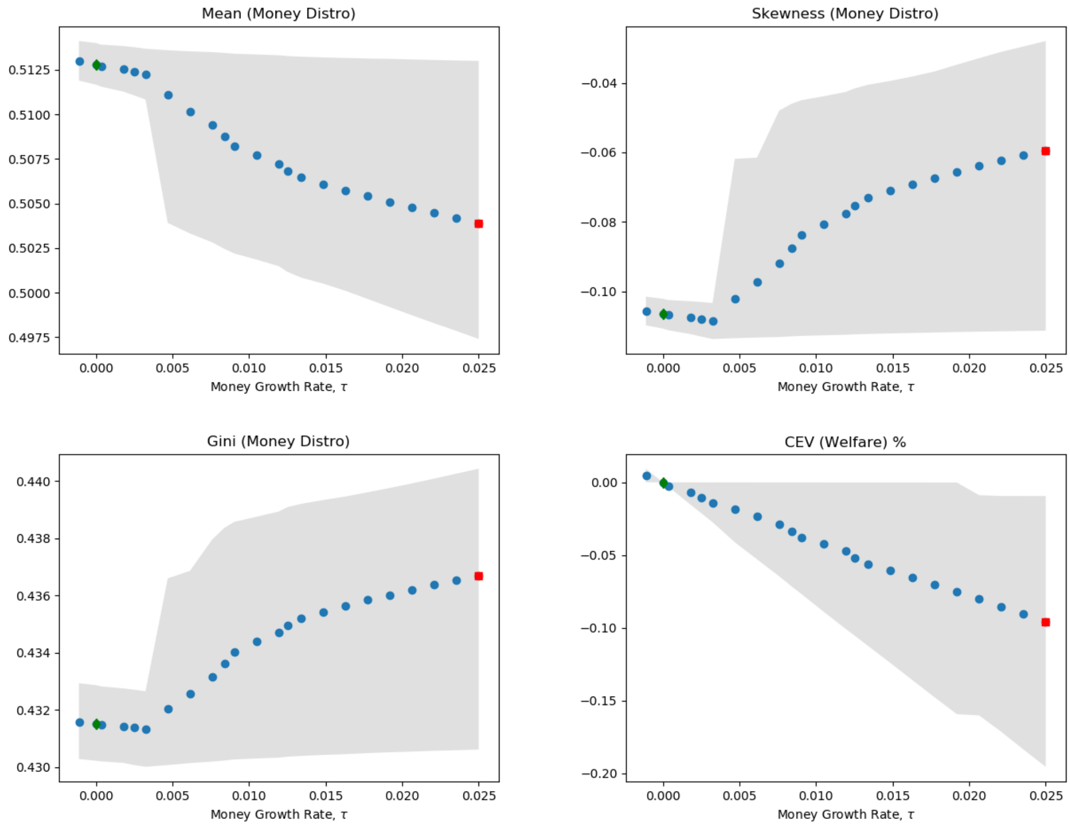

Figure 8: Comparative steady states — distribution (Benchmark but for $\chi = 0$)