COLLEGE OF SCIENCE AND ENGINEERING

MA1020

Preparatory Mathematics

LECTURE NOTES
(Including Tutorial Exercises)

Contents

1 Arithmetic

Calculators and computers are a necessary tool for all fields of science and engineering, but we also need to develop the thought processes required to *think* mathematically. In first year university mathematics, the process of developing our logical and problem solving brains is started by mastering the basic techniques of arithmetic and algebra. It is essential to practice until the manipulations become 100% accurate and efficiently routine or second nature. Only then can we move on to apply these techniques to our own area of interest, and also advance to the next level of knowledge. The nature of mathematics is such that we will always build on the previous levels.

1.1 Number Sets

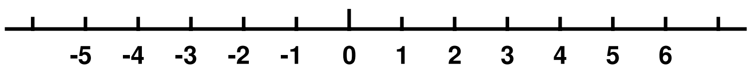
A *set* is a collection of objects with a common property. These objects are called the *elements* of the set.

There are many different sets of numbers, with different properties, that make up the set of real numbers. Some of these are:

- Natural Numbers, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$; positive whole numbers
- Integers, $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$; positive and negative whole numbers including 0.
- Rationals, $\mathbb{Q} = \left\{ \frac{a}{b}, \text{ where } a, b \text{ are integers and } b \neq 0 \right\}$;
- Real Numbers, $\mathbb{R}$; includes all of the above numbers.

1.2 The Number Line

The number line can be used to help us understand some of the arithmetic operations, particularly addition and subtraction. The number line below depicts all the integers, positive and negative.



The number line extends to the right and to the left forever. We say that the number line extends to infinity (∞) on the right, and to negative infinity ($-\infty$) on the left.

All numbers on the number line are ordered. Therefore:

- $6 > 3$ • $3 < 6$ • $14 < 101$ • $-3 > -5$ • $-24 < -10$

1.3 Arithmetic Operations

Arithmetic is based on four operations: Addition (+); Subtraction (−); Multiplication (×); Division (÷). We first demonstrate these operations and their properties using numbers, but keep in mind that the same operations are used for all types of mathematical processes to follow.

1.3.1 Addition

Although we can add any number of numbers, addition is generally a *binary operation*: i.e. we add *two* numbers to get the sum.

The two numbers being added in a sum are called *terms*. In the following examples, the two terms in each sum are positive integers.

Example:

- $4 + 6 = 10$. Think of the number line: start at 4, move 6 to the right $\Rightarrow$ addition.

- $125 + 97 = 222$. For additions with larger terms, use the following setting out:

$$\begin{array}{r} 1\,2\,5 \\ +\quad 9\,7 \\ \hline 2\,2\,2 \end{array}$$

Remember to work from right to left. Add the "units" first, then the "tens", then the "hundreds" etc.

- $$\begin{array}{r} 3\,9\,1\,4 \\ +\quad 6\,4\,9\,7 \\ \hline 1\,0\,4\,1\,1 \end{array}$$

Properties: These properties (or laws) must hold for all additions.

1. The Commutative Law: $4 + 6 = 6 + 4$. i.e. the order of a sum is not important.
2. The Associative Law: $(4 + 6) + 3 = 4 + (6 + 3)$ i.e. the order of grouping is irrelevant.

The associative law allows us to extend addition to more than 2 numbers (or terms).

Example:
$$\begin{aligned} 2 + 3 + 4 + 5 &= ((2 + 3) + 4) + 5 \\ &= (5 + 4) + 5 = 9 + 5 = 14 \end{aligned}$$

Note: *Brackets* – work from *inside to outside,* and always perform the calculation inside the brackets first. This rule is not so important for addition but is very important for when we include other operations.

Example: $12 + ((3 + 4) + 7) + (2 + 6) = 12 + (7 + 7) + 8 = 12 + 14 + 8 = 34$

1.3.2 Subtraction

Even though this is the *inverse* operation of addition (subtraction reverses the effect of addition), we must consider extra examples and properties.

Example:

- $7 + 4 = 11 \iff 11 - 4 = 7$ (and $11 - 7 = 4$). Think of the number line, when subtracting 4 from 11, we move 4 to the *left* of 11 to obtain 7.

- $462 - 136 = 326$ For subtractions with larger terms, use the following setting out:

$$\begin{array}{r} 4\,6\,2 \\ -\quad 1\,3\,6 \\ \hline 3\,2\,6 \end{array}$$

Remember to work from right to left. Subtract the "units" first, then the "tens", then the "hundreds" etc.

Properties:

1. The Commutative Law does not apply: $6 - 4 \neq 4 - 6$. i.e. the order *is* important.

2. The Associative Law does not apply, since

$$\begin{aligned} (12 - 5) - 3 &\neq 12 - (5 - 3) \\ \text{i.e.,}\quad 7 - 3 &\neq 12 - 2 \end{aligned}$$

Note: The use of brackets is very important for subtraction. Remember to perform the calculation inside the brackets first, working from inside to outside brackets.

Example:
$$\begin{aligned} 16 - (12 - (5 - 2)) &= 16 - (12 - 3) \\ &= 16 - 9 = 7 \end{aligned}$$

Note: If brackets are not used, we evaluate additions and subtractions from left to right:

Example:

- $$\begin{aligned} 12 - 5 - 3 &= (12 - 5) - 3 \\ &= 7 - 3 = 4 \end{aligned}$$

- $$\begin{aligned} 24 + 7 - 15 + 6 - 9 &= 31 - 15 + 6 - 9 \\ &= 16 + 6 - 9 \\ &= 22 - 9 \\ &= 13 \end{aligned}$$

Subtraction and Negative Integers

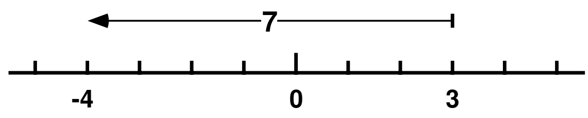
There is a need to consider numbers other than the naturals. Zero and negative integers are required. The set of *all* Integers is denoted by $\mathbb{Z}$.

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

Consider subtracting a larger number from a smaller number:

Example:

- $3 - 7 = -4$. On the number line, start at 3 and move 7 to the *left* to obtain -4.



- Think of a person with \$3 who then loses \$7. The new position is a debt of \$4 (assets of $-\$4$).

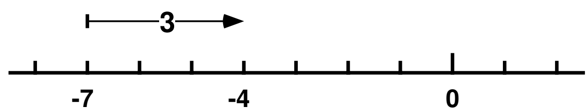
- The summer temperature in Antarctica is 8°C, and this drops overnight by 11°. The temperature is then $(8 - 11)° = -3$°C, or 3° below zero.

Note: Subtracting a positive number gives the same result as adding a negative number.

Note: Another method of subtracting a larger number from a smaller number is to reorder the numbers. We make sure to keep the signs with each number:

Example:

- $3 - 7 = 3 + (-7)$ Note the use of brackets here
  $= -7 + 3$ We can change the order as addition commutes
  $= -4$



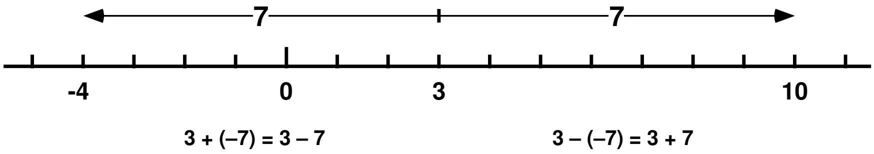
- Losing \$7 is the same as adding a debt of \$7.

- Reordering the numbers when subtracting a larger number from a smaller number is useful when dealing with big numbers. We make sure to keep the signs with each number:

$$
\begin{aligned}
245 - 368 &= -368 + 245 \\
&= -(368 - 245) \qquad \text{Note the use of the distributive law here} \\
&= -123
\end{aligned}
$$

We also consider *subtracting a negative number:*

Example: $3 - (-7) = 3 + 7 = 10$. On the number line, start at 3, subtracting a negative is the same as adding. i.e. subtracting (-7) is equivalent to adding 7, or $-(-7) = +7$.



Summarising addition and subtraction (or sums and differences) for “directed numbers”:

$$
\begin{array}{rcrcr}
3 + (+7) &=& 3 + 7 &=& 10 \\
3 + (-7) &=& 3 - 7 &=& -4 \\
3 - (+7) &=& 3 - 7 &=& -4 \\
3 - (-7) &=& 3 + 7 &=& 10
\end{array}
$$

If in doubt, use brackets. And remember, brackets first, then left to right.

Example:

$$
\begin{aligned}
-15 + (-3 + 4) - 6 - (4 - 9) &= -15 + 1 - 6 - (-5) \\
&= -15 + 1 - 6 + 5 \\
&= -14 - 6 + 5 \\
&= -20 + 5 = -15
\end{aligned}
$$

Note: Addition and subtraction have the same *precedence*, so after we calculate the brackets, work left to right calculating whichever operation is next, addition or subtraction.

EXERCISE 1: Evaluate the following:

1. $5 + 12$
2. $23 + 17$
3. $34 + 29$
4. $125 + 93$
5. $4682 + 1421$
6. $17 - 9$
7. $23 - 16$
8. $131 - 84$
9. $256 - 192$
10. $47256 - 24623$
11. $5 - 9$
12. $13 - 28$
13. $15 - 152$
14. $243 - 835$
15. $6341 - 9517$
16. $3 - 4 + 5 - 6$
17. $12 + 15 - 23 - 9$
18. $-12 + 9 - 48 + 21$
19. $-129 - 213 + 401 - 96$
20. $6351 - 4826 - 1846$
21. $(3 - 5) - (4 + 2)$
22. $22 - (11 - (5 - 12))$
23. $((2 + 4) - 7) - 1$
24. $(((24 - 12) - 6) - 3) + 1$
25. $210 + (-210)$
26. $12 - (-12)$
27. $25 - (6 - 8) + (13 - 19)$
28. $(34 - 45) + (45 - 34)$
29. $-121 - (-121)$
30. $1 - (1 - (1 - (1 - (-1))))$

1.3.3 Multiplication

We start by multiplying integers. The multiplication of two numbers is also called a *product*. The two numbers in the multiplication are often called factors of the product.

Example:

- $7 \times 8 = 56$
- $23 \times 12 = 276$. For products of larger numbers, use the following setting out:

$$\begin{array}{r} 2\,3 \\ \times \quad 1\,2 \\ \hline 4\,6 \\ 2\,3\,0 \\ \hline\hline 2\,7\,6 \end{array}$$

Remember to work from right to left. Multiply the "units" of the second number by the whole of the first number, then the "tens" of the second number by the whole of the first number, then add each line. You can see this is the same as 2×23 plus 10×23.

- A:

$$\begin{array}{rcl} 4\,3\,1\,5 & & \\ \times \quad 3\,1\,2 & & \\ \hline 8\,6\,3\,0 & \leftarrow & 2 \times 4315 \\ 4\,3\,1\,5\,0 & \leftarrow & 10 \times (1 \times 4315) \\ 1\,2\,9\,4\,5\,0\,0 & \leftarrow & 100 \times (3 \times 4315) \\ \hline\hline 1\,3\,4\,6\,2\,8\,0 & & \end{array}$$

B:

4315 x 312
= (4,300 + 15) (300 + 12)
= 1,290,000 + 4,500 + 51,600 + 180
= 1,346, 280

Which method is more optimised? A or B?

The same properties as seen for addition also apply for multiplication of integers.

Properties: These properties (or laws) must hold for all multiplications.

1. The Commutative Law: $4 \times 6 = 6 \times 4$. i.e. the order of a product is not important.
2. The Associative Law: $(4 \times 6) \times 3 = 4 \times (6 \times 3)$ i.e. the order of grouping is irrelevant.

The associative law allows us to extend multiplication to more than 2 numbers (or factors).

Example: $2 \times 5 \times 9 = 10 \times 9 = 90$

or $= 2 \times 45 = 90$

Multiplication involving directed numbers

The "same sign" rule apply here. i.e.

- Multiplying two positives gives a positive
- Multiplying two negatives gives a positive
- Multiplying a positive and a negative gives a negative

Multiplying **Like** signs	$\Rightarrow$ **Positive**
Multiplying **Opposite** signs	$\Rightarrow$ **Negative**

The four possible cases are demonstrated in these examples:

$$\begin{aligned} 4 \times 7 &= (+4) \times (+7) &&= 28 \\ 4 \times (-7) &= (+4) \times (-7) &&= -28 \\ -4 \times 7 &= (-4) \times (+7) &&= -28 \\ -4 \times (-7) &= (-4) \times (-7) &&= 28 \end{aligned}$$

Note: When multiplying more than 2 factors, it is useful to count the number of negative factors. As two negatives multiplied together make a positive, every "pair" of negative factors results in a positive. So the above rule can be expanded to:

- Multiplying factors with an even number of negatives gives a positive
- Multiplying factors with an odd number of negatives gives a negative

Example:

- $2 \times 6 \times 3 = 36$
- $3 \times (-2) \times 5 = -30$
- $-4 \times 5 \times (-7) = 140$
- $-5 \times 2 \times (-4) \times (-2) \times 3 \times (-1) = 240$

Note: We often don't use the "$\times$" sign when writing mathematics, but use brackets in its place. So the last example could be written:

$$(-5)(2)(-4)(-2)(3)(-1) = 240$$

When using algebra with variables, we can even leave out the brackets:

$$2 \times x = (2)(x) = 2x$$

1.3.4 Division

The arithmetic operation of division can be seen as the "inverse of multiplication" or the "opposite process to multiplication"

$$
\begin{array}{ccl}
4 \times 3 = 12 & \iff & 12 \div 3 = 4 \quad (\text{or} \quad \dfrac{12}{3} = 4) \\
\text{and} & & 12 \div 4 = 3
\end{array}
$$

i.e. we can express division in terms of an equivalent product.

Example:

- $24 \div 6 = 4$
- $168 \div 12 = 14$ For division of larger numbers use **long division** with the following setting out:

$$
\begin{array}{r@{\,}r}
 & 1\,4 \\
1\,2\,\big) & \overline{1\,6\,8} \\
- & 1\,2\downarrow \\
 & \overline{4\,8} \\
- & 4\,8 \\
 & \overline{0} \leftarrow \text{remainder} \\
 & =
\end{array}
$$

12 is divided into the first 2 digits, '16', which goes 1 time with a remainder of 4. The remainder is calculated by subtracting 1×12 from 16 which is 4. The next step is to 'bring down' the next digit, '8', and repeat the procedure.

Note: As the final remainder is zero, both 12 and 14 are factors of 168. i.e. the equivalent multiplication is: $12 \times 14 = 168$.

- $3435 \div 11$ Here we will see that 11 is not a factor of 3435. Again, we use long division.

$$
\begin{array}{r@{\,}r}
 & 3\,1\,2 \\
1\,1\,\big) & \overline{3\,4\,3\,5} \\
 & -3\,3 \\
 & \overline{1\,3} \\
 & -1\,1 \\
 & \overline{2\,5} \\
- & 2\,2 \\
 & \overline{3} \leftarrow \text{remainder} \\
 & =
\end{array}
$$

11 is divided into the first 2 digits, '34', which goes 3 times with a remainder of 1. The remainder is calculated by subtracting $3 \times 11 = 33$ from 34 which is 1. The next step is to 'bring down' the next digit, '3', and repeat the procedure. 11 into 13 goes 1 time with a remainder of 2, $1 \times 11 = 11$, $13 - 11 = 2$, bring down the 5, etc.

Note: Here the remainder is 3. As this is non-zero, we know that 11 is not a factor of 3435. i.e. it does not divide evely.

However, we can still express this division as an equation with useful information as:

$$3435 \div 11 = 312 \quad \text{with a remainder of } 3$$

Another way to write this is using fractions, much more of them later:

$$\frac{3435}{11} = 312 + \frac{3}{11}$$

Note that the second fraction has the remainder on the top line, and the divisor on the bottom line. We can also express this in an equivalent form as:

$$3435 = 312 \times 11 + 3$$

Properties

- The Commutative Law does not apply for division. i.e. $12 \div 3 \neq 3 \div 12$
- The Associative Law does not apply. i.e. $(32 \div 8) \div 4 = 1$ but $32 \div (8 \div 4) = 16$

Note: Again, take note of brackets. Calculate inside the brackets first. If no brackets are present, work from left to right.

Division involving directed numbers

Note: As for Multiplication, the "same sign" rule also applies to division:

Dividing **Like** signs	$\Rightarrow$ **Positive**
Dividing **Opposite** signs	$\Rightarrow$ **Negative**

The four possible cases are demonstrated in these examples:

$$\begin{aligned} 20 \div 5 &= \frac{20}{5} = 4 \\ 20 \div (-5) &= \frac{20}{-5} = -4 \\ -20 \div 5 &= \frac{-20}{5} = -4 \\ -20 \div (-5) &= \frac{-20}{-5} = 4 \end{aligned}$$

Note: Multiplication and Division have the same *precedence*, so after we calculate the brackets, work left to right and it doesn't matter which operation is next, Multiplication or Division.

Example:
$$\begin{aligned} 4 \times (6 \div 3) \div 2 \times 3 \div (2 \times 3) &= 4 \times 2 \div 2 \times 3 \div 6 \\ &= 8 \div 2 \times 3 \div 6 \\ &= 4 \times 3 \div 6 \\ &= 12 \div 6 \\ &= 2 \end{aligned}$$

EXERCISE 2: Evaluate the following:

1. 5×6
2. 120×100
3. 15×9
4. 23×19
5. 357×126
6. $3 \times 4 \times 5$
7. $10 \times 12 \times 6$
8. $(1)(2)(3)(4)(5)(6)(7)$
9. $41 \times 15 \times 52$
10. $31279 \times 46185 \times 0$
11. -7×5
12. $-5 \times (-12)$
13. $(-2)(4)(-6)(8)(-10)$
14. $2 \times (-3 \times 4) \times (-1 \times (-2))$
15. -357×126
16. $45 \div 9$
17. $2301 \div 3$
18. $21 \div (-7)$
19. $-144 \div (-12)$
20. $24 \div (-2) \div (3) \div (-2)$
21. $-48 \div (-32 \div -2)$
22. $9 \times (6 \div 3)$
23. $9 \times 6 \div 3$
24. $-(4 \div 2) \times (-4 \div 2)$
25. $\dfrac{(-1)(1)(-1)}{(1)(-1)(1)}$
26. $\dfrac{8 \times 10 \times 12}{2 \times 4 \times 6}$

1.3.5 Distributive Law

The Distributive Law is used to evaluate or simplify expressions that contain a combination of Addition/Subtraction with Multiplication/Division and brackets. We learn the distributive law using numbers (maybe positive and negative) and keep in mind that this law is very useful in algebraic manipulations.

Consider the expression: $3 \times (7 - 4)$. So far we have learnt to evaluate brackets first, then work left to right. So this gives: $3 \times 3 = 9$. The distributive law gives us another option on how to calculate these types of expressions. We can *expand* the brackets which eliminates them. i.e. $3 \times (7-4) = 3 \times 7 - 3 \times 4 = 21 - 12 = 9$. Note that we arrive at the same answer.

Example:

- $6 \times (5 + 4) = 6 \times 9 = 54$ or $6 \times 5 + 6 \times 4 = 30 + 24 = 54$
- $2(3 - 4) = 2(-1) = -2$ or $2(3 - 4) = 2(3) - 2(4) = 6 - 8 = -2$

Note: Using variables x and y, the Distributive Law can be written as:

$$2(x + y) = 2x + 2y$$

The expansion of the LHS (Left Hand Side) to the RHS (Right Hand Side) is the Distributive Law. The reverse process in which the RHS $\rightarrow$ LHS is called *factorisation*. Much more on this later!

We can expand with any number of terms inside the brackets:

Example:

- $5 \times (1 + 3 - 2) = 5 \times 2 = 10$ or $5 \times 1 + 5 \times 3 - 5 \times 2 = 5 + 15 - 10 = 10$
- $3(a - b + 2c) = 3a - 3b + 6c$

EXERCISE 3: Evaluate by (a) calculating brackets first, and (b) expanding out brackets using the Distributive Law. Confirm that your answers are the same:

1. $2(4 + 6)$
2. $3(5 - 2)$
3. $4(3 - 8)$
4. $5(-2 + 5)$
5. $6(-9 + 3)$
6. $7(-6 - 2)$
7. $8(-1 - 5)$
8. $-2(3 + 5)$
9. $-3(6 - 4)$
10. $-4(2 - 7)$
11. $-5(-3 + 9)$
12. $-6(-12 + 6)$
13. $-7(-9 - 4)$
14. $-8(-11 - 4)$
15. $-1(-1 - 1)$
16. $2(3 + 4 + 5)$
17. $3(5 - 6 - 2)$
18. $(-4)(-1 + 3 + 8)$
19. $(5 + 10 + 15)5$
20. $(-10 + 20 - 5)(-12)$

1.3.6 Powers

Powers are used when a single number is multiplied by itself several times. Consider multiplying 2 by itself 5 times. We write:

$$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$$

In this example, we say "2 is raised to the power of 5". 2 is called the *base* and 5 is the *power* or *exponent* or *index*. There are a number of Index Laws we will investigate in the Algebra chapter.

Example:

- $2^2 = 2 \times 2 = 4$ "2 squared is 4"
- $5^3 = 5 \times 5 \times 5 = 125$ "5 cubed is 125"
- $(-3)^4 = (-3)(-3)(-3)(-3) = 9 \times 9 = 81$ "negative 3 raised to the power of 4 is 81"
- $10^6 = 1\,000\,000$ "10 to the power of 6 is one million. A one with 6 zeros after it"

Notes:

- Any base raised to the power of 1 equals the base. e.g. $5^1 = 5$
- One raised to any power equals 1. e.g. $1^3 = 1 \times 1 \times 1 = 1$
- Any base raised to the power of zero equals 1. e.g. $(-2)^0 = 1$
- Zero raised to any power equals zero. e.g. $0^5 = 0$

1.3.7 Logarithms

Logarithms (often shortened to "logs") are used as an alternate method of expressing numbers raised to powers. Logs allow us to work more closely with the actual power or index of an expression.

We can rewrite statements such as: $2^3 = 8$ ("2 to the power of 3 is 8"), using logarithms. In this case we write: $3 = \log_2 8$ ("3 is the log of 8 to base 2"). Note that in both cases, 2 is the *base*.

These two versions of the same statement are called *equivalent expressions.*

i.e., $2^3 = 8 \underset{\substack{\text{equivalent}\\ \text{expression}}}{\Longleftrightarrow} 3 = \log_2 8$

Also
$$3^2 = 9 \iff 2 = \log_3 9$$
$$10^4 = 10\,000 \iff 4 = \log_{10} 10\,000$$
$$16^{\frac{1}{2}} = 4 \iff \tfrac{1}{2} = \log_{16} 4$$

The expression $\log_a n$ is read as "logarithm of n to the base a."

EXERCISE 4:

Evaluate the following expressions involving powers:

1. 2^6
2. 3^5
3. $(-2)^3$
4. $(-3)^3$
5. $(-5)^1$
6. $2^2 + 2^3$
7. $2^2 \times 2^3$
8. $2^2 \times 2^3 - 2^5$
9. $2^5 \div 2^3$
10. $2^5 \div 2^3 - 2^2$
11. $3^2(5^2 - 4^2) + 5^0$
12. $3^3 - 4^2 \times 2^2 - 3^1$
13. $4^2 \div 2^4 + 3^2 \div 2^3$
14. $2 \times 2^2 \times 2^3 \times 2^4$
15. $((1000 - 10^4) \div 3^2) \div 10^3$

Write the equivalent expression using logarithms for the following:

16. $2^4 = 16$
17. $3^1 = 3$
18. $4^3 = 48$
19. $5^3 = 125$
20. $10^4 = 10000$
21. $1^5 = 1$
22. $3^0 = 1$
23. $8^1 = 8$
24. $a^b = c$
25. $e^x = y$

Write the equivalent expression using powers for the following:

26. $\log_2 64 = 6$
27. $\log_3 27 = 3$
28. $\log_5 625 = 4$
29. $\log_1 1 = 4568$
30. $\log_{10} 1000 = 3$
31. $\log_8 1 = 0$
32. $\log_{12} 1 = 0$
33. $\log_p r = q$
34. $\log_e y = x$
35. $\log_n p = \dfrac{1}{2}$

Law	Addition	Subtraction	Multiplication	Division
Commutative ↔	4 + 6 = 6 + 4	4 - 6 ≠ 6 - 4	4 x 6 = 6 x 4	4 / 6 ≠ 6 / 4
Associative ( )	(4+6)+3 = 4+(6+3)	(4-6)-3 ≠ 4-(6-3)	(4x6)x3 = 4x(6x3)	(4/6)/3 ≠ 4/(6/3)

Calculate bracket first, if any, then work from LEFT to RIGHT →

Ex 2 Q20:

24 / (- 2) / (3) / (- 2)
= (- 12) / 3 / (- 2)
= (- 4) / (- 2)
= 2

24 / (- 2) / (3) / (- 2)

$= \cancel{24}_{\cancel{8}\,\cancel{4}\,2} \times \left(-\frac{1}{\cancel{2}_1}\right) \times \frac{1}{\cancel{3}_1} \times \left(-\frac{1}{\cancel{2}_1}\right)$

$= 2$

1.3.8 Precedence

Precedence is the *order* in which mathematical operations are performed. We learn the precedence rules here for numerical arithmetic and remember that the same rules apply for Algebra. This is the same order used by calculators and computers.

The order of precedence is:

- Brackets
- Powers (or Exponents or Orders)
- Multiplication (and/or Division). Multiplication and Division have the same precedence.
- Addition (and/or Subtraction). Addition and Subtraction have the same precedence.

The following mnemonics are often used:

B	**O**	**M**	**D**	**A**	**S**
Brackets	Orders	Multiplication	Division	Addition	Subtraction

B	**E**	**D**	**M**	**A**	**S**
Brackets	Exponents	Division	Multiplication	Addition	Subtraction

Example:

- $4 + 6 \times 5 = 4 + 30 = 34$
- $2 - 9 \div 3 + 2^3 = 2 - 9 \div 3 + 8 = 2 - 3 + 8 = -1 + 8 = 7$
- $5(12 \div (2^2 - 1) + 6) = 5(12 \div (4 - 1) + 6) = 5(12 \div 3 + 6) = 5(4 + 6) = 5(10) = 50$

EXERCISE 5: Evaluate the following:

1. $5 + 4 \times 3$
2. $5 + (4 \times 3)$
3. $(5 + 4) \times 3$
4. $18 - 5 \times 2$
5. $(18 - 5) \times 2$
6. $18 \div 3 + 6$
7. $18 \div (3 + 6)$
8. $3 \times 6 + 2 - 7 \times 2$
9. $(3 \times (6 + 2)) \times 2$
10. $3 \times 2^2 - 16 \div 2^3$
11. $5(3 - 6) - 6(3 - 6)$
12. $(2(5 - 6) \div 2)^2$
13. $(4)^2 - (2)^2$
14. $(4 - 2)(4 + 2)$
15. $(1 + 1) \times (1^2 - (1 + 1)) \div (1 + 1^1)$

1.4 Factors

The numbers that multiply together to give a product (or the "multiple") are called *factors.*

Example: The product of 6 and 4 is 24: i.e. $6 \times 4 = 24$. So we say that 6 and 4 are *factors* of 24.

The process of determining factors is called *factorisation.* We can "factorise" 24 into a number of factor pairs:

$$24 = 6 \times 4 = 8 \times 3 = 12 \times 2 = 24 \times 1.$$

Some factors can be factorised further:

$$6 = 2 \times 3, \qquad 8 = 2 \times 2 \times 2 = 2^3$$

Note: If we factorise all factors of 24 as far as we can:

$$\begin{array}{llllllll}\therefore & 24 & = & 8 \times 3 & & & = & 2^3 \times 3^1 \\ & & = & 4 \times 6 & = & (2 \times 2) \times (2 \times 3) & = & 2^3 \times 3^1 \\ & & = & 2 \times 12 & = & 2 \times (4 \times 3) & = & 2^3 \times 3^1\end{array}$$

Note:

- 2 and 3 have no factors (except for 1 and the number itself).
- 2 and 3 are called **Prime Numbers**.
- Clearly, 2 is the only *even* prime (Why?).
- The set of Prime Numbers: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 \ldots$
- 24 can be factorised several ways, but we always get the same prime factors.

We always choose to factorise down to prime factors because we know this "factorisation" will be unique. i.e. each number has *unique prime factors.* Prime factors can be used for many useful operations which we will see later. The process of finding prime factors is systematic and should be set out as follows:

Example: Determine the Prime Factors of 20, 36 and 450.

$$
\begin{array}{r|l}
2 & 20 \\ \hline
2 & 10 \\ \hline
5 & 5 \\ \hline
 & 1 \\ \hline
\end{array}
\qquad\qquad
\begin{array}{r|l}
2 & 36 \\ \hline
2 & 18 \\ \hline
3 & 9 \\ \hline
3 & 3 \\ \hline
 & 1 \\ \hline
\end{array}
\qquad\qquad
\begin{array}{r|l}
2 & 450 \\ \hline
3 & 225 \\ \hline
3 & 75 \\ \hline
5 & 25 \\ \hline
5 & 5 \\ \hline
 & 1 \\ \hline
\end{array}
$$

$\therefore \quad 20 = 2^2 \times 5 \qquad 36 = 2^2 \times 3^2 \qquad 450 = 2 \times 3^2 \times 5^2$

1.4.1 HCF: Highest Common Factor

The highest common factor (HCF) is the largest number that divides evenly into every number in a given set. To determine the HCF, we use the prime factorisation of each number. The HCF is equal to the product of the lowest power of each prime factor that is "common" to *all* numbers.

From the previous example: $20 = 2^2 \times 5, \;\; 36 = 2^2 \times 3^2, \;\; 450 = 2 \times 3^2 \times 5^2$. We can see that 2 is the only common prime factor, and 2^1 is the lowest power.

$\therefore$ For the set: 20, 36 and 450; HCF = 2.

Example: Find the HCF of 20, 24, 36 and 40.
The prime factors of these numbers are:

$$
\begin{aligned}
20 &= 2^2 \times 5 \\
24 &= 2^3 \times 3 \\
36 &= 2^2 \times 3^2 \\
40 &= 2^3 \times 5
\end{aligned}
$$

2 is the only prime factor that is **common** to all four numbers. And $2^2 = 4$ is the lowest power of 2. $\therefore$ for the set: 20, 24, 36 and 40; HCF = 4.

The process for finding the Highest Common Factor is:

HCF:	• Factorise each number into its prime factors. • Identify **lowest** power for each **Common** factor. • Multiply.

1.4.2 LCM: Lowest Common Multiple

The Lowest Common Multiple is the smallest number that has a given set of numbers as factors.

As an example, consider multiples of 6 and 8.

- (6): 6, 12, 18, **24**, 30, 36, 42, **48** ...
- (8): 8, 16, **24**, 32, 40, **48**, 56 ...

The smallest multiple of 6 and 8 that is a multiple of *each* is 24. i.e. The Lowest Common Multiple (LCM) of 6 and 8 is 24.

Note: 48 is also a multiple of 6 and 8, but not the *Lowest*.

To determine the LCM, we use prime factors again. In this case the LCM equals the product of all prime factors that appear in *any* factorisation including the *highest* power of each factor.

From the previous example: $6 = 2 \times 3, 8 = 2^3$. We can see the factors that appear are 2 and 3 and the highest powers of each are 2^3 and 3^1.

$\therefore$ For the set: 6 and 8; LCM $= 2^3 \times 3 = 24$.

The process for finding the Lowest Common Multiple is:

LCM:	• Factorise each number into prime factors. • Take **all** factors, each to the **highest** power anywhere. • Multiply.

Example: Find the LCM of 20, 24, 36 and 40.

We have seen that the prime factors of these numbers are:

$$20 = 2^2 \times 5, \quad 24 = 2^3 \times 3, \quad 36 = 2^2 \times 3^2, \quad 40 = 2^3 \times 5$$

The list of factors that appear is: 2 (highest power is 3), 3 (highest power is 2), 5 (highest power is 1).

$\therefore$ LCM $= 2^3 \times 3^2 \times 5^1 = 360$.

Note: The product of all numbers is $691,200$. This is a multiple of all the numbers, but is definitely not the *lowest!*

Note: If the LCM of a set is just the product of the numbers in that set, (e.q.$\{3,4,5\}$), then the numbers are *relatively prime.* i.e. they have no prime factors in common. Thus the LCM of the set $\{3,4,5\}$ is $3 \times 4 \times 5 = 60$.

EXERCISE 6:

Find the HCF of the following sets of numbers:

1. $\{8,10,12,16\}$
2. $\{12,24,42\}$
3. $\{12,18,36\}$
4. $\{18,36,54\}$
5. $\{21,30,69\}$
6. $\{21,32,55\}$

Find the LCM for the following sets of numbers:

7. $\{6,8,18\}$
8. $\{12,18,54\}$
9. $\{8,20,45\}$
10. $\{8,9,10\}$
11. $\{4,8,16,32\}$
12. $\{18,21,28,32\}$

1.5 Fractions

For operations such as division, we require a more extended set of numbers than the integers, i.e. we need to evaluate: $3 \div 12$ or $\dfrac{3}{12}$, which does not give us an integer.

These are the *fractions* or *Rational Numbers*, denoted by $\mathbb{Q}$ ('rational' from ratio). Fractions allow us to work with division by numbers that are not factors. i.e. numbers that do not divide evenly into the original number.

Consider $3 \div 12$, we need the number that multiplies 12 to get 3. i.e. if we sub-divide 3 items among 12, how many does each receive? The answer is a quarter.

$$\frac{3}{12} = \frac{\cancel{3} \times 1}{\cancel{3} \times 4} = \frac{1}{4}$$

Here $\dfrac{3}{12}$ is reduced to its *lowest terms* of $\dfrac{1}{4}$, by canceling the HCF of 3 in the top and bottom lines of the fraction (called the *Numerator* and *Denominator*).

We don't normally use the $(\div)$ sign. Usually write $\dfrac{3}{12}$ or $3/12$, rather than $3 \div 12$.

Example: Reduce the following fractions to lowest terms:

$$\frac{18}{24} = \frac{9}{12} = \frac{3}{4}, \qquad \frac{36}{54} = \frac{2 \times 18}{2 \times 27} = \frac{2 \times 2 \times 9}{2 \times 3 \times 9} = \frac{\cancel{2} \times 2 \times \cancel{3} \times \cancel{3}}{\cancel{2} \times 3 \times \cancel{3} \times \cancel{3}} = \frac{2}{3}$$

Note: Either divide out common factors progressively, or simplify by finding the HCF of the numerator and denominator.

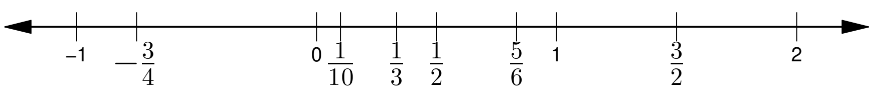
The rationals are all numbers of the form p/q, where p and q are integers, but $q \neq 0$:

$$\mathbb{Q} = \left\{ \quad \ldots \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{1}{5}, \frac{5}{1}, \frac{1}{6}, \frac{2}{5}, \frac{3}{4}, \ldots \quad \right\}$$

Note: Any fraction with a denominator equaling zero, is *undefined.*

Consider: $15 \div 5$. The answer is that number, which when multiplied by 5 gives 15, in this case, 3 $(15 = 3 \times 5 \Leftrightarrow 15 \div 5 = 3)$.
Now consider: $4 \div 0$. We require the number that multiplies by 4 to get 0. There is *no reasonable answer!* i.e. **division by zero is prohibited.**

Rational numbers (fractions) can also be put onto a number line. Foe example:



1.5.1 Proper and Improper Fractions

A *Proper Fraction* has the numerator $<$ denominator. e.g. $\dfrac{2}{5}$

An *Improper Fraction* has the numerator $>$ denominator. e.g. $\dfrac{17}{5}$

A *Mixed Number* is a combination of a whole number plus a fraction. i.e. $\dfrac{17}{5}$ can be written as a mixed numeral:

$$\frac{17}{5} = \frac{15+2}{5} = \frac{15}{5} + \frac{2}{5} = 3 + \frac{2}{5} = 3\frac{2}{5}$$

This mixed numeral is "Three and two fifths".

Note: This is equivalent to: (i) $17 \div 5 = 3$ with remainder 2, or (ii) $17 = 3 \times 5 + 2$

1.5.2 Multiplication of Fractions

The process of multiplying fractions has two steps: • Multiply the numerators, and • Multiply the denominators.

Example: $\dfrac{3}{4} \times \dfrac{2}{5} = \dfrac{3 \times 2}{4 \times 5} = \dfrac{6}{20} = \dfrac{\cancel{2} \times 3}{\cancel{2} \times 10} = \dfrac{3}{10}$

Note: The answer of $\dfrac{6}{20}$ is incomplete. Always reduce fractions to their lowest form.

The process is simplified by canceling common factors from the numerator and denominator before multiplying.

i.e. $\dfrac{3}{4} \times \dfrac{2}{5} = \dfrac{3}{\underset{2}{\cancel{4}}} \times \dfrac{\overset{1}{\cancel{2}}}{5} = \dfrac{3 \times 1}{2 \times 5} = \dfrac{3}{10}$

Example: $\dfrac{15}{42} \times \dfrac{14}{25} = \dfrac{\overset{3}{\cancel{15}}}{\underset{6}{\cancel{42}}} \times \dfrac{\overset{2}{\cancel{14}}}{\underset{5}{\cancel{25}}} = \dfrac{3}{6} \times \dfrac{2}{5} = \dfrac{1}{\cancel{2}} \times \dfrac{\cancel{2}}{5} = \dfrac{1}{5}$

1.5.3 Powers of Fractions

We have previously seen that powers of numbers are equivalent to multiplying a number by itself where the power tells us how many times to multiply. As a fraction represents a number, the same rules for powers must apply.

When evaluating powers of fractions we can either multiply the fraction by itself, or we can take powers of both the numerator and denominator.

Example: $\left(\frac{2}{3}\right)^2 = \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2}{3 \times 3} = \frac{4}{9}$

or equivalently:

$$\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$$

Note: This rule for powers of fractions can be written as:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

1.5.4 Addition/Subtraction of Fractions

The process of adding (or subtracting) fractions is more complicated than multiplication.

We can immediately add fractions of the same type, that is, fractions that have the same denominator:

$$\frac{2}{7} + \frac{3}{7} = \frac{2+3}{7} = \frac{5}{7}$$

To add/subtract fractions with different denominators, we must first transform each fraction so that they have the same denominator. To do this we "find the lowest common denominator". For integers, the lowest common denominator is the Lowest Common Multiple (LCM). This is the most important application of the LCM.

Example: Evaluate $\frac{1}{6}+\frac{5}{8}$.

The LCM of 6 and 8 is 24. i.e. $6 = 2 \times 3$, $8 = 2^3$. LCM $= 2^3 \times 3 = 24$.

To transform each fraction so that the common denominator is 24, we multiply the first fraction (both numerator and denominator) by 4, and the second fraction by 3.

$$\frac{1}{6}+\frac{5}{8}=\frac{1\times 4}{24}+\frac{5\times 3}{24}=\frac{4+15}{24}=\frac{19}{24}$$

Note: Remember the precedence rules and work from left to right where necessary.

Example: Evaluate $\frac{2}{3}+\frac{1}{2}-\frac{3}{5}$ LCM of $\{2,3,5\}$ is 30

$$\begin{aligned}
&= \frac{2}{3}\times\frac{10}{10}+\frac{1}{2}\times\frac{15}{15}-\frac{3}{5}\times\frac{6}{6}\\
&= \frac{20}{30}+\frac{15}{30}-\frac{18}{30}\\
&= \frac{20+15-18}{30}\\
&= \frac{17}{30}
\end{aligned}$$

1.5.5 Reciprocals

Before we divide fractions, we need to define the *reciprocal*.

We can see that $3 \times \frac{1}{3} = 1$. In this case, $\frac{1}{3}$ is the multiplicative inverse of 3.

Equivalently, 1/3 is the *Reciprocal* of 3. That is the number which, when multiplied by 3 gives 1.

Thus, if x represents any number (except zero): $\boxed{\frac{1}{x} \text{ is the RECIPROCAL of } x}$

Consider a fraction: e.g. $\frac{3}{4}$. The reciprocal of this fraction is the number that when multiplied by $\frac{3}{4}$ gives 1. In this case, the reciprocal is $\frac{4}{3}$ as: $\frac{3}{4}\times\frac{4}{3}=1$.

We can extend this to all fractions: $\boxed{\frac{b}{a} \text{ is the RECIPROCAL of } \frac{a}{b}}$

Given any fraction, the reciprocal is the fraction resulting from swapping the numerator and the denominator.

1.5.6 Division of Fractions

This follows immediately from reciprocals. We know that dividing any number by 3 is equivalent to taking one third of that number, i.e., multiplying by $1/3$, which is the *reciprocal* of 3:

i.e. $21 \div 3 = 21 \times \frac{1}{3} = \frac{21}{1} \times \frac{1}{3} = \frac{21}{3} = 7$

This can be extended to all divisions involving fractions:

$$\boxed{\text{Division} \iff \text{Multiplication by Reciprocal}}$$

Example:

$$\frac{6}{1/2} = 6 \div \frac{1}{2} = 6 \times \frac{2}{1} = 12.$$

This example asks how many times does $\frac{1}{2}$ divide into 6? Or, how many halves are there in 6? The answer is 12.

Example:

- $\frac{4}{2/3} = 4 \times \frac{3}{2} = \cancel{4}^{2} \times \frac{3}{\cancel{2}} = 6$
- $1\frac{3}{4} \div 3\frac{1}{2} = \left(\frac{4}{4} + \frac{3}{4}\right) \div \left(\frac{6}{2} + \frac{1}{2}\right) = \frac{7}{4} \div \frac{7}{2} = \frac{7}{4} \times \frac{2}{7} = \frac{\cancel{7}}{\cancel{4}_{2}} \times \frac{\cancel{2}}{\cancel{7}} = \frac{1}{2}$

Note: The precedence rules (BOMDAS) apply to caluclations involving fractions:

Example: Evaluate $\frac{4}{5} + 3\frac{1}{5} \times \left(1\frac{1}{2}\right)^2$

$$= \frac{4}{5} + \frac{16}{5} \times \left(\frac{3}{2}\right)^2$$

$$= \frac{4}{5} + \frac{16}{5} \times \frac{9}{4}$$

$$= \frac{4}{5} + \frac{4}{5} \times \frac{9}{1}$$

$$= \frac{4}{5} + \frac{36}{5}$$

$$= \frac{40}{5}$$

$$= 8$$

EXERCISE 7: Evaluate the following expressions involving fractions:

1. $\frac{1}{2} + \frac{1}{3}$
2. $\frac{3}{4} - \frac{1}{3}$
3. $\frac{5}{6} + \frac{2}{3}$
4. $\frac{4}{5} - \frac{2}{7}$
5. $\frac{11}{21} - \frac{5}{18}$
6. $\frac{1}{3} + \frac{1}{4} + \frac{1}{6}$
7. $\frac{3}{5} - \frac{1}{2} + \frac{3}{4}$
8. $\frac{4}{3} - \frac{7}{6} + \frac{1}{4}$
9. $\frac{7}{24} - \frac{5}{16} + \frac{3}{20} - \frac{1}{15}$
10. $10 \times \frac{1}{2}$
11. $10 \div (-2)$
12. $10 \div \frac{1}{2}$
13. $2 \div 10$
14. $2 \div \frac{1}{10}$
15. $\frac{3}{8} \times \frac{4}{9}$
16. $\left(-\frac{1}{2}\right) \times \left(-\frac{1}{3}\right)$
17. $\left(\frac{2}{3}\right)^3$
18. $\left(-\frac{3}{4}\right)^2$
19. $-\left(\frac{3}{4}\right)^2$
20. $\frac{3}{4} \div \frac{3}{4}$
21. $\frac{4}{3} \div \frac{5}{4}$
22. $\left(\frac{3}{4} \div \frac{2}{3}\right)$
23. $1\frac{1}{2} \div \frac{3}{4}$
24. $\frac{13}{20} \Big/ \frac{39}{16}$
25. $\frac{3}{7}$ of 35
26. $2\frac{1}{2} - 1\frac{1}{4}$
27. $\frac{2}{5} \times \frac{1}{3}$
28. $\frac{3}{5} \Big/ \frac{3}{4}$
29. $\frac{1}{3}\left(\frac{1}{2} - \frac{2}{3}\right)$
30. $\frac{8}{15} \times \frac{25}{32}$
31. $\frac{3}{4} \times 1\frac{1}{2} \times 3\frac{1}{2}$
32. $\left(-\frac{5}{8}\right) \times \frac{8}{11}$
33. $\left(-\frac{2}{3}\right) \times \left(-\frac{15}{7}\right)$
34. $\frac{3}{5}$ of $11\frac{1}{4}$
35. $\frac{1}{5} \times \frac{2}{3} + \frac{2}{5} \div \frac{4}{5}$
36. $\left(\frac{4}{3} - \frac{2}{5} \times \frac{1}{3}\right) \times \frac{1}{4} + \frac{1}{2}$
37. $6\frac{1}{4} \div \left(2\frac{1}{2} + 5\right)$
38. $\left(\frac{2}{3}\right)^2 \Big/ \left(\frac{1}{2}\right)^3$
39. $\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 - \frac{1}{2}$
40. $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7}$

1.6 Decimals and Scientific notation

Our arithmetic is based on the number 10 (base-10 arithmetic). It is the foundation of our number system. Computers use a binary system in which every number can be represented as a combination of 0's and 1's.

We have seen powers of ten: $10 = 10^1$, $100 = 10 \times 10 = 10^2$, $1000 = 10 \times 10 \times 10 = 10^3$ etc.

Every number can be written in terms of powers of 10.

Example:

- $29 = 20 + 9 = 2 \times 10^1 + 9$
- $576 = 500 + 70 + 6 = 5 \times 10^2 + 7 \times 10^1 + 6$
- $2103 = 2000 + 100 + 3 = 2 \times 10^3 + 1 \times 10^2 + 3$

In these examples, all the numbers are integers and greater than 1. We need to represent numbers between zero and one, and all other fractions. For this, we use *decimals*.

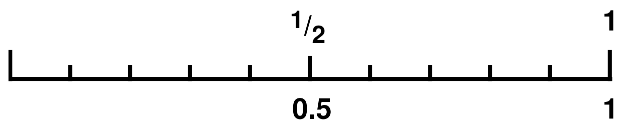
1.6.1 Decimals

Often referred to as *decimal fractions*, decimals are another method of writing some fractions, represented by an integer divided by a specific power of 10.

We first note that any integer can be represented as a decimal. i.e. 3 can be written as 3.0. We usually do not write the decimal point if a single zero follows it.

We can also write any integer as a fraction with a denominator as a power of 10: $3 = \frac{3}{1} = \frac{3}{10^0}$

We need a method for converting other fractions to decimals. Consider the fraction $\frac{1}{2}$: this is exactly half way between zero and one on the number line.



To write $\frac{1}{2}$ as a decimal, we first need the denominator to be a power of 10: $\frac{1}{2} = \frac{5}{10} = 0.5$ and the decimal representation of $\frac{1}{2}$ is written as 0.5.

When we divide by a power of 10, we move the decimal point to the left by the same number of spaces as the power. So in the last example, we start with 5.0, and because we divide by 10^1 we move the decimal point one space to the left giving us 0.5.

Example:

- $\dfrac{3}{10} = \dfrac{3}{10^1} = 0.3 \quad \text{(Move d.p. 1 place left)}$
- $\dfrac{3}{1000} = \dfrac{3}{10^3} = 0.003 \quad \text{(Move d.p. 3 places left)}$
- $\dfrac{3}{4} = \dfrac{75}{100} = 0.75$

Improper fractions (when the numerator > denominator) can also be converted to decimals in the same way.

Example:

- $\dfrac{6}{5} = \dfrac{12}{10} = 1.2$
- $\dfrac{2105}{1000} = \dfrac{2105}{10^3} = 2.105$

We can also write this method of "dividing by a power of ten" with everything on the top line to help us in moving the decimal point. To do this we note that:

$$\frac{1}{10} = 10^{-1}, \qquad \frac{1}{100} = \frac{1}{10^2} = 10^{-2}, \qquad \frac{1}{10^n} = 10^{-n}$$

Here we are using *negative powers*. We will cover this in more detail later, but we can see that any power of ten in the denominator equals a negative power of ten in the numerator.

With this notation, we have a method of calculating decimals that are either multiplied or divided by powers of 10. We follow these rules:

- When multiplying by 10 with a positive power $\Rightarrow$ move the decimal point to the right
- When multiplying by 10 with a negative power $\Rightarrow$ move the decimal point to the left

Example:

- $5.2 \times 10^2 = 520$
- $0.000061 \times 10^4 = 0.61$
- $21.3 \times 10^{-2} = 0.213$
- $0.007 \times 10^{-3} = 0.000007$

1.6.2 Scientific Notation

Scientific notation is a standard method used to represent numbers – usually very large or very small, but can be used for all rational numbers. It removes the need to write a lot of zeros in a number so that we are writing the most important or significant figures.
e.g. for 0.000341, only the digits 341 are significant; the position of the decimal point shows that

$$0.000341 = \frac{341}{1,000,000} \quad \text{or} \quad \frac{3.41}{10,000} = \frac{3.41}{10^4} = 3.41 \times 10^{-4}$$

This last form is the standard for Scientific Notation.

We can check the scientific notation by multiplying out the power of 10.

Notes:

- Express all numbers using scientific notation as: $X \times 10^n$. X is a decimal number between 1 and 10 ($1 \leqslant X < 10$). The decimal point is **always** placed *immediately after the first non-zero digit* (e.g., $49.01 = 4.901 \times 10^1$, $0.543 = 5.43 \times 10^{-1}$).
- If the exponent n is positive ($n > 0$) the number is greater than 1; if $n < 0$ the number lies between 0 and 1.
- For negative numbers, prefix with a minus sign (e.g., $-225 = -2.25 \times 10^2$).
- By emphasising the *significant* digits of a number, scientific notation helps to maintain accuracy in calculations.

Example: Write the following in Scientific Notation

- $3471.2 = 3.4712 \times 10^3$
- $0.00001401 = 1.401 \times 10^{-5}$
- $-8.95 = -8.95 \times 10^0$
- $(1.2 \times 10^3) \times (2.0 \times 10^6) = 1.2 \times 2.0 \times 10^3 \times 10^6 = 2.4 \times 10^9$
- $(1.7 \times 10^2) \div (3.4 \times 10^4) = \dfrac{1.7 \times 10^2}{3.4 \times 10^4} = \dfrac{1.7}{3.4} \times \dfrac{10^2}{10^4} = \dfrac{1}{2} \times 10^{2-4} = 0.5 \times 10^{-2} = 5.0 \times 10^{-3}$

EXERCISE 8:

Write the following numbers in Scientific Notation:

1. 6321
2. 58
3. 4.2
4. 0.5
5. -0.087
6. 500 000
7. 0.00004
8. 9

Write the following as decimal numbers:

9. 6.4×10^{-3}
10. 4.9×10^{1}
11. 5×10^{4}
12. 3.14×10^{0}
13. 7.32×10^{-1}
14. -1.23×10^{2}
15. 7.21×10^{-6}

Evaluate the following and express the results in Scientific Notation:

16. $(3 \times 10^{3})^{3}$
17. $(3 \times 10^{2}) \times (5 \times 10^{3})$
18. $(2.2 \times 10^{3}) \div (1.1 \times 10^{1})$
19. $(4 \times 10^{-3})^{2}$
20. $(2.4 \times 10^{-2}) \div (1.2 \times 10^{-5})$
21. $(1.6 \times 10^{5})^{1/2}$

2 Algebra

Wikipedia: Algebra (from Arabic al-jebr meaning ”reunion of broken parts”) is one of the broad parts of mathematics. In its most general form algebra is the study of symbols and the rules for manipulating symbols and is a unifying thread of almost all mathematics. Algebra is essential for any study of mathematics, science, or engineering, as well as applications in medicine and economics etc.

Algebra differs from arithmetic in the use of letters to stand for numbers that are either "*unknown*" or may take many values ("*variables*"). For example, if $x + 2 = 5$ the letter x is unknown, but we can discover its value: $x = 3$. In $E = mc^2$, the letters E and m are variables, and the letter c is a constant. Algebra gives methods for solving equations and expressing formulas that are much easier than writing everything out in words.

We will be using the standard arithmetic and algebraic operations to manipulate the following types of mathematical statements:

- **Expression:** Letter symbols (unknowns or variables) and numbers combined with arithmetic operation signs, brackets, powers etc.

$$\text{e.g.} \qquad 2x+1, \qquad (p+q)^2, \qquad \frac{3x^2-2}{x}, \qquad ax^2+bx+c$$

- **Equation:** Shows that two expressions have equal value by linking them with an equals sign (=). We are often required to solve equations, i.e. determine the value or values that satisfy the equation, or make the equation true.

$$\text{e.g.} \qquad 3x = 6, \qquad 5x+7 = 2x-4, \qquad \frac{2a+5}{3} = \frac{1}{2}, \qquad ax^2+bx+c = 0$$

- **Identity:** A type of equation that is a "rule". Identities are always true no matter what value is used for the variables (if any are present).

$$\text{e.g.} \qquad 1+1 = 2, \qquad a(b+c) = ab+ac, \qquad \frac{4x^2}{2x} = 2x, \qquad \sin^2\theta + \cos^2\theta = 1$$

- **Formula:** A type of equation that links a number of different variables. When the values of some variables are known, others can be determined.

$$\text{e.g.} \qquad A = \pi r^2, \qquad v = u+at, \qquad E = mc^2, \qquad P = 2(l+w)$$

2.1 Linear Equations

When working with equations, the word "linear" signifies that the highest power the variable (or unknown) is raised to is 1. You may remember the general formula for a straight line: $y = mx + c$. In this formula, x and y are the variables and they are both raised to the power of 1. The sketch of such equations are straight lines or "linear".

In this section we learn the techniques required to "solve" linear equations. These techniques incorporate the arithmetic operations from the last chapter. When we solve an equation, we determine the "solution", or the value of the unknown that satisfies the equation, making it known.

Example: $x + 5 = 9$, $5x = 25$, $3x - 4 = 5$.

Each is true for only certain values of x — the solution (solutions in some cases). The first example states: "A number (x) plus 5 equals 9." This unknown number is 4, so $x = 4$ is the (only) solution.

We need to solve equations *systematically.* The three examples above are *linear equations*: i.e. they contain only terms in x (x^1) and constants (x^0). However, an equation of the form $x^2 + 5x + 4 = 0$ is not linear, it is a *quadratic equation.* We cover linear equations before quadratics.

Most linear equation have *exactly one solution.* Exceptions are pathological examples that have no solution, e.g.

$$x + 3 = x + 4 \qquad \text{and} \qquad (2x + 1) - (x - 7) = x + 8.$$

The first is inconsistent. The second is an *identity.*

Our solution methods are governed by a basic principle. Consider an equation as a <u>balance</u>. *Provided* we do the *same thing to each side*, we maintain that balance.

To keep both sides of an equation equal – what we do to one side, we MUST do to the other

Next we introduce some of the methods and techniques used to solve linear equations. All solution techniques and algebraic manipulations must obey the principle above.

The objective is to find the solution, or the value of the unknown (usually x). Therefore, we need to get x *by itself* on the LHS (left hand side).

2.1.1 Add/Subtract to both sides

If we add any number to both sides of an equation, we do not change the balance: we obtain an *equivalent* equation which has the *same* solution.

Example:

$$x + 5 = 9 \qquad \text{has a solution: } x = 4$$

$$\text{(add 3 to each side)} \quad \Rightarrow \quad x + 8 = 12 \qquad \text{also has a solution: } x = 4$$

So the solution is unchanged from adding/subtracting the same number to both sides. But we need to manipulate the equation to obtain $x =$ **?**. i.e. x by itself on the LHS.

Example

$$\begin{aligned} x + 5 &= 9 \\ x + 5 - 5 &= 9 - 5 \quad \text{(Subtract 5 from each side)} \\ x &= 4 \end{aligned}$$

Thus, we have achieved our aim: to get x by itself on the LHS.

Note: As a *shortcut*, this procedure can be performed by "transposing" the $+5$ on the LHS to a -5 on the RHS. Thus

$$\begin{aligned} x + 5 &= 9 \\ x &= 9 - 5 \qquad \text{(Transpose } +5 \text{ on LHS to } -5 \text{ on RHS)} \\ x &= 4 \end{aligned}$$

2.1.2 Multiply/Divide both sides

Consider the example: $5x = 25$. To get x by itself on the LHS, we need to divide the LHS by 5. Again, to keep the balance, we must divide the RHS by 5 as well:

Example

$$\begin{aligned} 5x &= 25 \\ \frac{\cancel{5}x}{\cancel{5}} &= \frac{25}{5} \qquad \text{(Divide both sides by 5)} \\ x &= 5 \end{aligned}$$

Note: As a *shortcut*, this procedure can also be performed by "transposing" the $\times 5$ on the LHS to a $\div 5$ on the RHS. This also transposes a factor in the numerator on the LHS to a factor in the denominator of the RHS. Thus

$$\begin{aligned} 5x &= 25 \\ x &= \frac{25}{5} \qquad \text{(Transpose } \times 5 \text{ on LHS to } \div 5 \text{ on RHS)} \\ x &= 5 \end{aligned}$$

2.1.3 Combined Add/Subtract and Multiply/Divide

Often we require both of the manipulations above to solve equations, i.e. rearrange the terms and the factors to get x by itself on the LHS. Usually, this will be done by carrying out the add/subtract operations first, then the multiply/divide operations. Whichever operations are performed we must remember to:

- perform the same operation to both sides – keep the balance
- we require x by itself on the LHS

Example Solve $3x - 4 = 5$
To get x by itself, we must isolate the term containing x (or $3x$) first, as follows:

$$\begin{aligned} 3x - 4 &= 5 & \\ 3x - 4 + 4 &= 5 + 4 & \text{[Add 4 both sides]} \\ 3x &= 9 & \\ \frac{\cancel{3}x}{\cancel{3}} &= \frac{9}{3} & \text{[Divide both sides by 3]} \\ x &= 3 & \end{aligned}$$

Or if we use the shortcuts discussed above:

$$\begin{aligned} 3x - 4 &= 5 & \\ 3x &= 5 + 4 & (-4 \text{ on the LHS becomes } +4 \text{ on the RHS }) \\ 3x &= 9 & \\ x &= \frac{9}{3} & \text{(factor 3 in the numerator of the LHS} \\ & & \text{becomes factor in the denominator of the RHS)} \\ x &= 3 & \end{aligned}$$

Check: We can *always* verify a "solution" by checking it. i.e. we substitute the solution for x into the original equation and check that LHS = RHS.

$$\text{LHS} = (3x - 4)_{x=3} = 3(3) - 4 = 9 - 4 = 5.$$

$$\text{RHS} = 5.$$

$\therefore$ LHS = RHS, so $x = 3$ is the solution. In this case, $x = 3$ is the only solution. Recall that linear equations have only one solution.

Example: Solve the following linear equations:

1.
$$\begin{aligned} \frac{p}{3} + 7 &= 10 \\ \frac{p}{3} &= 10 - 7 \\ \frac{p}{3} &= 3 \\ p &= 3 \times 3 \\ p &= 9 \end{aligned}$$

Check: LHS $= \dfrac{p}{3} + 7 = \dfrac{9}{3} + 7 = 3 + 7 = 10 =$ RHS

2.
$$\begin{aligned} 9 - 5a &= 24 \\ -5a &= 24 - 9 \\ -5a &= 15 \\ a &= \frac{15}{-5} \\ a &= -3 \end{aligned}$$

Check: LHS $= 9 - 5a = 9 - 5(-3) = 9 + 15 = 24 =$ RHS

Note: There are usually several different methods that can be used to solve each equation. All methods must follow the same rule: "perform the same operation to both sides".

Example 2 above:
$$\begin{aligned} 9 - 5a &= 24 \\ 9 &= 24 + 5a \\ 5a + 24 &= 9 \\ 5a &= 9 - 24 \\ 5a &= -15 \\ a &= \frac{-15}{5} \\ a &= -3 \end{aligned}$$

EXERCISE 9: Solve the following linear equations:

1. $x + 3 = 9$
2. $x - 6 = -2$
3. $p - 4 = -8$
4. $5 - q = 1$
5. $7 - w = 12$
6. $5 = 11 - y$
7. $-t - 9 = -6$
8. $12 = x + 12$
9. $15 = -p - 15$
10. $-1 = 1 - x$
11. $4x = 12$
12. $7y = 28$
13. $-15 = 3t$
14. $2 = 5p$
15. $-24 = -6q$
16. $\frac{u}{5} = 3$
17. $5 = \frac{v}{5}$
18. $\frac{3w}{4} = 9$
19. $48 = \frac{4x}{3}$
20. $5x = \frac{1}{2}$
21. $2x + 7 = 15$
22. $3a - 6 = 15$
23. $-11 + 7y = 24$
24. $2 - 3x = 11$
25. $14 = -3x - 4$
26. $\frac{p}{5} + 3 = -5$
27. $\frac{3z}{2} - 3 = 9$
28. $11 + \frac{5y}{6} = 6$
29. $5 + \frac{2m}{3} = 2$
30. $\frac{1}{2} = \frac{2x}{5} - \frac{3}{4}$

2.1.4 Linear Equations with Unknowns on Both Sides

We are often required to solve linear equations that contain more than one term that includes the unknown. e.g. $7x - 3 = 2x - 1$

We remember the same principle as before: "to keep the balance, what we do to one side of the equation, we MUST do to the other". And we use the same procedures as before, i.e. add/subtract or multiply/divide the same to both sides of the equation to keep the balance. Our first goal is to get all terms with the unknown onto one side (usually the LHS) and then proceed as we have before.

Example

$$\begin{aligned}
7x - 3 &= 2x - 1 \\
7x - 3 - 2x &= 2x - 1 - 2x && \text{[Subtract } 2x \text{ both sides]} \\
5x - 3 &= -1 \\
5x - 3 + 3 &= -1 + 3 && \text{[Add 3 both sides]} \\
5x &= 2 \\
\frac{\cancel{5}x}{\cancel{5}} &= \frac{2}{5} && \text{[Divide by 5 both sides]} \\
x &= \frac{2}{5}
\end{aligned}$$

Note: Using the methods of transposing terms will give us the same answer:

$$\begin{aligned}
7x - 3 &= 2x - 1 \\
7x - 3 - 2x &= -1 && [2x \text{ on RHS becomes } -2x \text{ on LHS}] \\
5x - 3 &= -1 \\
5x &= -1 + 3 && [-3 \text{ on LHS becomes 3 on RHS}] \\
5x &= 2 \\
x &= \frac{2}{5} && \text{[5 in numerator of LHS becomes 5 in denominator of RHS]}
\end{aligned}$$

Note: In this example, we have performed a very important operation! That is, the process of **"gathering like terms"**. On the LHS we saw that: $7x - 2x = 5x$. In this sum the terms are called **"like terms"** which means they both contain the same unknown (x), raised to the same power (1). The numbers in front are different, but that's OK!, we can still add them. We do more gathering like terms later.

2.1.5 Linear Equations with Brackets

Linear equations will often include brackets, either on one side or possibly both. e.g. $3(5-2x) = 3-5(x-1)$. It's important to note that here there are unknowns inside both sets of brackets. So we can't evaluate the inside of the brackets.

Our general strategy for solving these equations is to expand out the brackets first. We do this with the distributive law. This will eliminate the brackets and we will return to a type of equation we've solved previously.

Note: Make sure to take care with signs.
Example:

$$\begin{aligned}
3(5-2x) &= 3-5(x-1) \\
15-6x &= 3-5x+5 && \text{[Expand brackets. Signs!]} \\
15-6x &= 8-5x && \text{[Collect like terms as you go]} \\
-6x+5x &= 8-15 && \text{[Transpose } x\text{'s to LHS, numbers to RHS]} \\
-x &= -7 \\
x &= 7
\end{aligned}$$

2.1.6 Linear Equations with Fractions

Linear equations will often include fractions, either on one side or possibly both. Our general strategy for solving these equations is to eliminate the fractions first, giving us a type of equation we've solved previously. There are two cases we will consider: (1) A single fraction on each side, and (2) more than one fraction on one side or possibly both.

Case 1: A single fraction on each side of the equation: e.g. $\dfrac{2x}{3} = \dfrac{3}{4}$.

This case applies when everything on the LHS and everything on the RHS is contained in a single fraction on each side. To eliminate the fractions in this case, we transpose the denominator of each side across as a factor in the numerator on the other side. This process is sometimes called **"cross multiplying"**

Example

$$\begin{aligned}
\frac{2x}{3} &= \frac{3}{4} \\
(2x)(4) &= (3)(3) && \text{[LHS 3, and RHS 4 in denominator have transposed]} \\
8x &= 9 \\
x &= \frac{9}{8} && \text{[Divide both sides by 8]}
\end{aligned}$$

Note: Another option in this case is to multiply *every term* by the LCM of the whole equation.

Example: $$\frac{2x}{3} = \frac{3}{4}$$

LCM of 3 and 4 = 12: multiply *all* terms by 12.

$$\overset{4}{\cancel{12}} \times \frac{2x}{\underset{1}{\cancel{3}}} = \overset{3}{\cancel{12}} \times \frac{3}{\underset{1}{\cancel{4}}} \qquad \text{[Multiply both sides by 12]}$$

$$8x = 9 \qquad \text{[Simplify]}$$

$$x = \frac{9}{8} \qquad \text{[Divide both sides by 8]}$$

Case 2: More than one fraction on one side or possibly both: e.g. $\dfrac{3(x-2)}{5} - \dfrac{x}{3} = 2$.

In this case, we eliminate all fractions at once giving us an equation of a previous type. We use the second option from the last case, i.e. multiply *every term* by the LCM of the whole equation.

Example:

$$\frac{3(x-2)}{5} - \frac{x}{3} = 2$$

$$\frac{3x-6}{5} - \frac{x}{3} = 2 \qquad \text{[Expand brackets]}$$

$$\overset{3}{\cancel{15}} \times \frac{3x-6}{\cancel{5}} - \overset{5}{\cancel{15}} \times \frac{x}{\cancel{3}} = 15 \times 2 \qquad [\textbf{all} \text{ terms} \times \text{LCM} = 15]$$

$$3(3x-6) - 5x = 30$$

$$9x - 18 - 5x = 30 \qquad \text{[Expand brackets]}$$

$$4x = 48$$

$$x = \frac{48}{4}$$

$$x = 12$$

Note: Another option is to add the two fractions on the LHS together giving us a single fraction on each side which is case 1 above. i.e.

$$\begin{aligned}
\frac{3(x-2)}{5} - \frac{x}{3} &= 2 \\
\frac{3 \times 3(x-2)}{15} - \frac{5 \times x}{15} &= 2 && \text{[LCM of } \{5,3\} = 15] \\
\frac{9(x-2) - 5x}{15} &= 2 \\
9x - 18 - 5x &= 30 && \text{[Expand brackets and transpose 15]} \\
4x &= 48 \\
x &= 12
\end{aligned}$$

EXERCISE 10: Solve the following linear equations:

1. $3x + 3 = x + 9$
2. $6p - 1 = 7 - 2p$
3. $2 - a = 18 - 5a$
4. $2t - 9 = 6t + 15$
5. $u - 3u = 12 - u + 5u$
6. $5x - 2x = 3x + x$
7. $2z - 3z = 2z + 3z$
8. $5 - w = 11 + w$
9. $1 - q = q - 1$
10. $6 = -x - x - x - x$
11. $4y = 3(y + 2)$
12. $3(x - 2) + 2x = 4$
13. $-2(2 - 2t) = 2(2t + 2)$
14. $4(x - 2) - 2(x - 5) = 0$
15. $-4 - 5p = 1 - 2(2p + 4)$
16. $3(a + 4) - 8(a - 3) = 1$
17. $4(1 - 2x) - 3(2 - 3x) = 7 - 2x$
18. $0 = 10 - u(3 + 12)$
19. $\frac{1}{2}(b + 4) = \frac{1}{3}(b - 2)$
20. $\frac{1}{6}(3x + 3) - \frac{3}{2}(x - 5) = 1$
21. $\frac{3x}{4} = \frac{1}{2}$
22. $\frac{2x}{3} + \frac{x}{2} = 14$
23. $\frac{3a}{5} - \frac{a}{3} = 4$
24. $\frac{2y}{5} - \frac{1}{3} = \frac{y}{6}$
25. $\frac{x + 2}{3} = \frac{x}{4}$
26. $\frac{b + 1}{2} = \frac{4b - 2}{5}$
27. $\frac{3x + 1}{2} - \frac{2x - 7}{3} = 7$
28. $\frac{2x - 1}{5} - \frac{3x + 1}{2} = \frac{2}{5}$
29. $\frac{1}{8m} = \frac{3}{4}$
30. $\frac{8}{a - 5} = -4$

2.1.7 Applied Linear Equations — Word Problems

In this section we use the solution methods in the previous sections to solve applied problems. These problems are presented in the form of a sentence or two which contain information about the situation and ask a question. We are required to *formulate* the problem, i.e. convert the worded statements into corresponding symbolic expression or equations. If the equations are linear, we can solve them, and then answer the original question.

Example: A girl had a certain amount of money. After getting an additional \$8, she had three times as much money as before. How much did she have originally?

We first define the symbol used for the unknown: Let the original amount (in dollars) be x.

We then use the given information to form an equation. Here, the statement "getting an additional \$8" tells us to include $x + 8$, and "three times as much money as before" tells us to include $3x$. The equation is:

$$x + 8 = 3x$$

We solve the equation to determine the unknown:

$$\begin{aligned} x + 8 &= 3x \\ 8 &= 2x \\ x &= 4 \end{aligned}$$

We state the solution: The original amount is \$4.
Note: Stating just $x = 4$ is *not* sufficient for a solution.

> The basic principles for formulating and solving word problems:
>
> - Identify the unknown (x) from the question.
> - Transform the given information into an equation.
> - Solve the equation.
> - Use the solution to answer the original question.

Example: A man is 3 times as old as his son. In 12 years time, the father will be twice as old as the boy. How old is each now?

Let the present age of the boy be x years. So the father's age is now $3x$. This uses all information in the first sentence, and sets up the variables x and $3x$ as the present ages.

"In 12 years time" the age of each person will be 12 more than at present. So the boy's age will be $x + 12$, and the father's age will be $3x + 12$.

We can now use the remaining information in the second sentence to form the equation.

$$3x + 12 = 2(x + 12)$$
$$3x + 12 = 2x + 24$$
$$\therefore \quad x = 12$$

The boy's current age is 12, and his father's current age is 36 (3×12).

Note: You should always check your answer by going back to the original question and make sure your answers agree with the problem.

EXERCISE 11: Solve the following applied problems:

1. Write down expressions for: (a) the sum of x and 6; (b) 3 times w plus 2; (c) 6 more than the product of p and 5; (d) 3 times the sum of $7d$ and 2; (e) 10 times the difference of $7x$ and 1.
2. Write down expressions for the perimeter P of each of the following rectangles in terms of **either** the length l **or** the width w, depending on which is the more convenient, if: (a) the length is 3 times the width; (b) the length is 5 cm more than the width; (c) the width is 1 cm less than twice the length.
3. In each of the following, write down and solve the equation: (a) adding 6 to the number x gives 22; (b) if 11 is added to m the result is 5; (c) if 11 is subtracted from 3 times a number, the result is 16; (d) twice the number x equals 9 more than half of x.
4. A girl's mother is twice as old as she is. If the sum of their ages is 63, how old is the mother?
5. Apples cost twice as much as pears. If the total cost of buying one apple and one pear is 36 cents, what is the cost of each?
6. The perimeter of a rectangular yard is 32 m. If the length is 3 times the breadth, find its dimensions.
7. One number is 4 times another and their sum is 15. Letting the value of the smaller number be x, write down the equation that corresponds to the first sentence. Solve, to find the two numbers.
8. The length of a room is 2 m more than 3 times the width. If the perimeter is 36 m, find the room's dimensions.
9. $60 is divided among 3 people, labelled A, B and C. If B receives $10 more than C, while A gets twice the amount of B and C combined, how much does each receive?
10. An athlete can jog at twice her walking pace. By walking for a total of 1 h 40 m and jogging for a total of 30 minutes she covers a total distance of 16 km. What is her walking pace in kph (kilometres per *hour*)?
11. A square garden has a 1 m-wide path inside its boundary. If the areas of the two squares differ by 44 m^2, how long is one side of the outer square?
12. Ticket prices at a concert are $5 for children and $12 for adults. If a total of 150 people attend, and total takings are $1492, how many of the audience are children?
13. A boy walked for 2 hours and jogged for 1 hour. If he jogs at an average of 2 times walking pace, how fast does he walk if the total distance covered is 28 km?

2.2 Index Laws

This section is an extension of the work on powers from the last chapter. We often have a number, an unknown, or a variable, multiplied by itself several times. Rather than writing every factor, we use powers as a shorthand notation.

Definition: $a^n = \underbrace{a \times a \times a \times \cdots \times a}_{n \text{ factors}}$

where a is a real number, called the base, and the number n is called the power, index or exponent. The expression a^n is read as the "the n^{th} power of a", or "a raised to the power of n".

Example:

- $(-5)(-5)(-5)(-5) = (-5)^4$
- $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^3$

2.2.1 Positive Indices

We now state the index laws for positive integer exponents.

$$\boxed{\begin{array}{l} \text{Index Laws for positive indices.} \\ \text{Let } m, n \in \mathbb{N} \text{ and } a, b \in \mathbb{R}. \\ \\ \quad \text{IL1.} \quad a^m \times a^n = a^{m+n} \\ \quad \text{IL2.} \quad \dfrac{a^m}{a^n} = a^{m-n} \\ \quad \text{IL3.} \quad \left(a^m\right)^n = \left(a^n\right)^m = a^{nm} \\ \quad \text{IL4.} \quad (ab)^n = a^n b^n \\ \quad \text{IL5.} \quad \left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n} \end{array}}$$

Example: $a^3 \times a^5 = a^{3+5} = a^8$ (applying IL1)

This can be shown by:

$$\text{LHS} = a^3 \times a^5 = \underbrace{(a \times a \times a)}_{3 \text{ factors}} \times \underbrace{(a \times a \times a \times a \times a)}_{+ \quad 5 \text{ factors}} = a^8 = \text{RHS}$$

Example: $\frac{a^5}{a^3} = a^{5-3} = a^2$ (applying IL2)

This can be shown by:

$$\text{LHS} = \frac{a^5}{a^3} = \frac{\cancel{a} \times \cancel{a} \times \cancel{a} \times a \times a}{\cancel{a} \times \cancel{a} \times \cancel{a}} = a^2 = \text{RHS}$$

Example: $\left(a^5\right)^3 = a^{5\times 3} = a^{15}$ (applying IL3)

This can be shown by:

$$\text{LHS} = \left(a^5\right)^3 = (a^5)(a^5)(a^5) = a^{5+5+5} = a^{15} = \text{RHS}$$

2.2.2 Zero and Negative Indices

We now extend the definition of a^n to allow for zero and negative indices. First we derive a result for when the index is zero.

$$\begin{aligned} &\text{Using IL1.} && a^m \times a^n = a^{m+n} \\ &\text{For } m = 0, && a^0 \times a^n = a^n \\ &\text{so} && a^0 = \frac{a^n}{a^n} = 1, \qquad \text{provided } a \neq 0. \end{aligned}$$

$$\boxed{\text{IL6.} \quad a^0 = 1, \qquad (a \neq 0)}$$

Example:

- $5^0 = 1$
- $(2x)^0 = 1$
- $(ax + b)^0 = 1$
- $(a^2b^0c)^2 = a^4b^0c^2 = a^4(1)c^2 = a^4c^2$

We can now derive an index law for negative indices.

$$\begin{aligned} &\text{Using IL2.} && \frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0 \\ &\text{For } m = 0, && \frac{a^0}{a^n} = \frac{1}{a^n} = a^{0-n} = a^{-n} \end{aligned}$$

$$\boxed{\text{IL7.} \quad \frac{1}{a^n} = a^{-n}, \qquad (a \neq 0)}$$

We are often required to express quantities with positive indices only. We use a combination of the index laws above to simplify expressions with negative indices and transform them to include positive indices only.

Example: Express the following with positive indices.

- $4^{-2} = \dfrac{1}{4^2} \qquad \left(= \dfrac{1}{(2^2)^2} = \dfrac{1}{2^4} \right)$
- $\left(\dfrac{1}{2}\right)^{-3} = \left(2^{-1}\right)^{-3} = 2^3$
- $\dfrac{3^{-2} \times 6^3}{9^{-3} \times 2^{-1}} = \dfrac{3^{-2} \times (3 \times 2)^3}{\left(3^2\right)^{-3} \times 2^{-1}} = 3^{-2} \times 3^3 \times 2^3 \times 3^6 \times 2^1 = 2^4 \times 3^7$

EXERCISE 12:

Evaluate the following:

1. $2^2 \times 2^3$
2. $3^{12} \times 3^3$
3. $(-5)^0$
4. $(3^3)^2$
5. $4^5 \times 4^0$
6. 12^{-1}
7. $3^5 \div 3^3$
8. $5^{-2}/5^{-3}$
9. $(2^{-3})^2$
10. $(2^3)^{-2}$
11. $(-4)^{-2}$
12. $(-2)^{-3}$
13. -2^{-3}
14. $\dfrac{2^{-1}4^{-3}6^3}{3^3 2^{-3}}$
15. $\dfrac{(-3)^4(-3)^{-3}}{(-3)^{-2}}$

Simplify and express your answer with positive indices:

16. $2a^0 \times 4a^5$
17. $5a \times 3a^3$
18. $7a^3b \times 2a^2b^5$
19. $(2y^2z^5)^3$
20. $15y^8 \div 3y^2$
21. $\dfrac{6a^0}{18a^3}$
22. $\dfrac{24a^2c^5}{36b^3c^8}$
23. $3x^{-2}$
24. $\dfrac{1}{p^{-1}}$
25. $\dfrac{1}{4c^{-5}}$
26. $(2x^5)(x^{-2}y^3)$
27. $\dfrac{12a^2bc^5}{4a^3b^4c^3}$
28. $\dfrac{2x^{-2}y^4}{10x^{-3}y^{-2}}$
29. $\dfrac{(2xy^3)^2}{4x^3y^5}$
30. $(2x^2y)^3(2x^3y)^{-2}$
31. $\dfrac{(2p^2)^3}{t^4} \bigg/ \dfrac{4p^4}{3t^7}$
32. $\dfrac{3^{-2}x^{-1}}{3^{-3}x^{-2}}$
33. $\dfrac{a^2b^3c^{-4}}{a^4b^{-1}c^{-5}}$
34. $\dfrac{2^n4^{n+1}}{8^{n-2}}$
35. $\left(5^0x^2y^{-1}\right)^{-1}$
36. $\dfrac{\left(2x^{-2}y^3\right)^2}{8x^{-4}y}$
37. $\left(\dfrac{3a^2}{4b^{-1}}\right)^{-3}\left(\dfrac{4}{a}\right)^{-5}$

Prove the following:

38. $\dfrac{2^{n+1}4^n}{8^{n-1}} = 16$
39. $\dfrac{25^{2n}5^{1-n}}{(5^3)^n} = 5$

2.2.3 Using Index Laws to Solve Equations

The index laws form part of the fundamental algebraic processes which are often required to solve equations. Some of the techniques for solving equations using the index laws are demonstrated in the following examples.

Example: Find x, given $5^x = 125$.

$$\begin{aligned} 5^x &= 125 \\ \therefore\ 5^x &= 5^3 \qquad \text{(same base so we equate the indices)} \\ \therefore\ x &= 3. \end{aligned}$$

Example: Given $\dfrac{2^{x+4}}{4^{2x-1}} = 1$, find x.

$$\begin{aligned} \frac{2^{x+4}}{4^{2x-1}} &= 1 \\ \therefore\ 2^{x+4} &= 4^{2x-1} \\ \therefore\ 2^{x+4} &= \left(2^2\right)^{2x-1} = 2^{4x-2} \end{aligned}$$

Equating indices, we get

$$\begin{aligned} x + 4 &= 4x - 2 \\ \therefore\ 6 &= 3x \\ \therefore\ x &= 2. \end{aligned}$$

EXERCISE 13:

Solve for x:

1. $3^{x-1} = 27$
2. $\dfrac{2^{x-3}}{4^{1-x}} = 1$
3. $3^{2x+1} = \dfrac{1}{9}$
4. $(5^x - 25)(3^x - 27) = 0$
5. $(2^x - 1)\left(3^x - \dfrac{1}{9}\right) = 0$

2.2.4 Rational Indices

We can extend the index laws to rational indices, i.e. powers that are fractions. All the index laws will hold provided the base $a > 0$.

Notation: Some popular fractional powers are written as roots:

- $a^{1/2} = \sqrt{a}$: The square root of a.
- $a^{1/3} = \sqrt[3]{a}$: The cube root of a
- $a^{1/5} = \sqrt[5]{a}$: The fifth root of a
- $a^{2/3} = \sqrt[3]{a^2} = \left(\sqrt[3]{a}\right)^2$

  Thus $\quad 8^{2/3} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$

The above index laws will hold for fractional indices, provided $a > 0$.

Examples: Use the index laws to simplify the following:

- $32^{2/5} = (2^5)^{2/5} = 2^{5\times(2/5)} = 2^2 = 4$
- $125^{-2/3} = \dfrac{1}{125^{2/3}} = \dfrac{1}{(5^3)^{2/3}} = \dfrac{1}{5^2} = \dfrac{1}{25}$
- $\left(\dfrac{9}{49}\right)^{-1/2} = \left(\dfrac{49}{9}\right)^{1/2} = \dfrac{7}{3}$

EXERCISE 14:

Evaluate the following:

1. $49^{1/2}$
2. $36^{1/2}$
3. $25^{1/2}$
4. $9^{3/2}$
5. $9^{-3/2}$
6. $-9^{3/2}$
7. $9^{-1/2}$
8. $16^{3/4}$
9. $(-9)^{3/2}$
10. $4^{3/2}$
11. $4^{-3/2}$
12. $8^{-2/3}$
13. $27^{-2/3}$
14. $16^{-3/4}$
15. $8^{1/2} \times 2^{-1/2}$
16. $3^{2/3} \times 9^{1/3}$
17. $3^{2/3} \times 9^{1/6}$
18. $2^{1/2} \div 2^{-3/2}$
19. $3^{2/3} \div 9^{1/6}$
20. $8^{1/3} \times 4^{1/2}$
21. $2^{2/3} \times 8^{1/9}$
22. $(-27)^{-2/3} \times 9^{3/2}$
23. $\sqrt[3]{4}\Big/\sqrt{2}$
24. $2^{\frac{2}{3}}4^{\frac{1}{6}}$
25. $243^{-\frac{2}{5}}$
26. $\dfrac{3^{\frac{1}{3}} \times 3^{\frac{5}{6}} \times 3^{-\frac{1}{6}}}{3^0 \times 3}$
27. $\dfrac{2^{\frac{1}{2}} \times 3^{\frac{1}{2}} \times 4^{\frac{1}{2}} \times 12^{\frac{1}{4}}}{3^{\frac{3}{4}}}$

2.3 Expanding and Factorising

In the last chapter, we saw one form of expanding and factorising through the use of the Distributive Law: $a(b+c) = ab+ac$. We used the distributive law for numbers, but exactly the same rule can be used for expressions involving variables or unknowns.

The process of multiplying all terms inside the brackets by the factor outside the brackets is called **Expansion**. This process eliminates the brackets as they have been expanded.

The opposite process to expansion is **Factorisation**. i.e. we start with the RHS $[ab + ac]$ and create factors by introducing brackets and *"taking out"* common factors to give the LHS $[a(b + c)]$. Note that in the LHS we have two factors: a and $(b + c)$. This process is using the distributive law in reverse.

Example:

- Expansion: $x(2x + 1) = x \times 2x + x \times 1 = 2x^2 + x$
- Factorisation: $3p^3 - 6p = 3p \times p^2 - 3p \times 2 = 3p(p^2 - 2)$

Recall that when simplifying fractions, we are required to cancel common factors from the numerator and denominator: e.g. $\dfrac{8}{20} = \dfrac{4}{10} = \dfrac{2}{5}$.

We follow the same procedure for fractions involving variables or unknowns,
i.e. **cancel common factors from the numerator and denominator.**

Note: We are often required to factorise the numerator and/or denominator of a fraction first to see if there are any common factors that we can cancel.

Example: $\dfrac{4xy + 8y}{2xy^2 + 4y^2} = \dfrac{4y(x + 2)}{2y^2(x + 2)} = \dfrac{{}^{2}\cancel{4}\cancel{y}\cancel{(x + 2)}}{\cancel{2}y^{\cancel{2}}\cancel{(x + 2)}} = \dfrac{2}{y}$

Note: We can only cancel factors – we cannot cancel one term of a sum in the top or bottom of a fraction – only factors.

EXERCISE 15:

Expand the following, and simplify where possible:

1. $3(x+2)$
2. $5(2x-3)$
3. $-2(x+3)$
4. $-3(p+2q)$
5. $-(x-2)$
6. $-2(4y-3)$
7. $a(a+3)$
8. $-x(1-x)$
9. $4(x+1)+2(x-2)$
10. $2(3a+2)-(1-4a)$
11. $3(3x-5)-2(2x+4)$
12. $3(y-1)-(5y-1)$
13. $3(2x+3y)-(x-y)$
14. $5x^2(1-x)-x(x^2+2)$

Factorise the following by "taking out" all common factor:

15. $4x+4y$
16. $3p+pq$
17. $4x-12y$
18. $6a+6$
19. $ab+b$
20. $8x-12xy$
21. x^2-3x
22. $9mn-3n^2$
23. $xy+wx-3x$
24. $6x^2-9x+12$
25. $-2x+6$
26. $-6x-9x^2$
27. $-18a^3-6a^2$
28. $15mn-9m^2-3m$
29. $x(y+z)+3(y+z)$
30. $3(2a-1)-b(2a-1)$
31. $k(k+2)+(k+2)$
32. $(x-y)-z(x-y)$
33. $x(x+2)-2(x+2)$
34. $(x+3)^2-(x+3)$
35. $(1-a)-a(1-a)$
36. $7a-5a^2$
37. $-8xy+4y^2$
38. $20pq-15qr$
39. $-9y^2-15y^3$
40. $(x+3)^2+2(x+3)$
41. $(2x-1)-(2x-1)^2$

Simplify by cancelling out common factors in both the numerator and denominator:

42. $\dfrac{4x}{x}$
43. $\dfrac{3y}{6y}$
44. $\dfrac{12xy}{8xz}$
45. $\dfrac{8(a+4)}{2(a+1)}$
46. $\dfrac{6(x-y)}{2(x-y)}$
47. $\dfrac{4a(a-1)}{2(a-1)}$
48. $\dfrac{4x(x-2)}{12x(x+2)}$
49. $\dfrac{(2x+1)(x-2)}{(x+2)(2x+1)}$
50. $\dfrac{4a+4b}{4}$
51. $\dfrac{p+q}{4(p+q)}$
52. $\dfrac{3(x-4)}{2x-8}$
53. $\dfrac{6m+6n}{12m-12n}$
54. $\dfrac{2s-6}{s-3}$
55. $\dfrac{12x-8y}{15x-10y}$
56. $\dfrac{xy+xz}{xy-xz}$
57. $\dfrac{ab-b}{ab}$
58. $\dfrac{a^2-a}{3a-3}$
59. $\dfrac{(x+2)^2}{3x+6}$
60. $\dfrac{x^2-1}{x-1}$
61. $\dfrac{x^2-9}{x+3}$

2.3.1 Identities for Expanding and Factorising

The processes of expansion and factorisation can be extended by repeated use of the distributive law allowing us to define the following **Binomial Expansion Identities**:

- $(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd$ **F.O.I.L.**
- $\left.\begin{array}{l}(a+b)^2 = a^2 + 2ab + b^2 \\ (a-b)^2 = a^2 - 2ab + b^2\end{array}\right\}$ Perfect Squares
- $(a+b)(a-b) = a^2 - b^2$ Difference of Two Squares

Examples:

- $(x+2)(x-4) = x^2 - 4x + 2x - 8 = x^2 - 2x - 8$
- $(p+5)^2 = (p)^2 + (2)(p)(5) + (5)^2 = p^2 + 10p + 25$
- $(2y-3)^2 = (2y)^2 - (2)(2y)(3) + (3)^2 = 4y^2 - 12y + 9$
- $(3w-1)(3w+1) = (3w)^2 - (1)^2 = 9w^2 - 1$

These expansion identities can be used in reverse to give the equivalent
Binomial Factorisation Identities:

- $a^2 + 2ab + b^2 = (a+b)^2$
- $a^2 - 2ab + b^2 = (a-b)^2$
- $a^2 - b^2 = (a+b)(a-b)$

Note: $a^2 + b^2$ has no REAL FACTORS.

Examples:

- $x^2 + 6x + 9 = (x+3)^2$
- $(5x-2)^2 - (2x-3)^2$ (difference of two squares)

  Let $a = 5x - 2$ and $b = 2x - 3$, so that

$$\begin{aligned}(5x-2)^2 - (2x-3)^2 &= a^2 - b^2 = (a+b)(a-b) \\ &= [(5x-2) - (2x-3)][(5x-2) + (2x-3)] \\ &= (3x+1)(7x-5).\end{aligned}$$

By repeated use of the distributive law, we can expand any number of binomials.

If we have three different factors, we multiply two factors using the standard distributive law first, then expand out the remaining brackets using an extension of the distributive law.

Example:

$$\begin{aligned}(x+2)(x-3)(x-2) &= (x+2)(x^2-5x+6)\\ &= x(x^2-5x+6)+2(x^2-5x+6) \quad \text{(Note: three terms inside brackets)}\\ &= x^3-5x^2+6x+2x^2-10x+12\\ &= x^3-3x^2-4x+12.\end{aligned}$$

EXERCISE 16:

Expand the following using the binomial expansion identities
(FOIL, perfect squares, difference of two squares):

1. $(x+4)(y+2)$
2. $(2x+3)(y+2)$
3. $(5x-1)(z+3)$
4. $(a-2b)(3c-d)$
5. $(x+1)(x+3)$
6. $(a-2)(a-5)$
7. $(p-3)(p+2)$
8. $(a-4)(a+4)$
9. $(5n-2)(5n+2)$
10. $(x+3)(x+3)$
11. $(2y-5)(2y-5)$
12. $(x-y+1)(x-y-1)$
13. $(x+1)^2$
14. $(x-3)^2$
15. $(2y+1)^2$
16. $(3w-4)^2$
17. $(2x+3y)^2$
18. $(-p-q)^2$
19. $(x+2)(x-2)$
20. $(x-5)(x+5)$
21. $(2x+3)(2x-3)$
22. $(ab-cd)^2$
23. $(2x+4)^2$
24. $(3x-5)(3x+5)$
25. $3a^2(2a+7)$
26. $(4x-5y)(2x+3y)$
27. $(x-2)(x+4)$
28. $(7x-2)(7x+2)$
29. $(3x+10y)(3x-10y)$
30. $(3x+10y)(3x+10y)$
31. $(2x-3)^2$
32. $(2x^2-3)(x^2+5)$
33. $(3-2x)(5-x)$
34. $(a+2)(a^2+a+2)$
35. $\left(\frac{3}{x}-2x\right)^2$

Factorise the following using the binomial factorisation identities
(Perfect squares, difference of two squares):

36. x^2-4
37. x^2+2x+1
38. a^2-9
39. y^2+4y+4
40. p^2-25
41. y^2+9
42. p^2-6p+9
43. f^2-49
44. $1-y^2$
45. $x^2-10x+25$
46. x^2-y^2
47. $4x^2+4x+1$
48. $4a^2-9$
49. $1-16x^2$
50. $8q^2-40q+50$
51. $9b^2-16$
52. $36x^2+25$
53. $9x^2-6x+1$
54. $a^2-\frac{1}{9}$
55. $16x^2-9y^2$
56. $36m^2-49n^2$
57. $(x-2)^2-9$
58. $2c^2-50$

2.3.2 Factorising Quadratic Expressions

Recall that a linear expression is of the form: $ax + b$, where a and b are constants. The two terms in this expression contain x raised to the power of 1, and x raised to the power of zero ($x^0 = 1$). A quadratic expression is defined as:

$$\mathbf{ax^2 + bx + c}$$

Notes:

- The highest power of x is 2. A quadratic may also include x to the power of 1, and x to the power of zero. None of the powers of x are negative or a fraction.
- $a \neq 0$. If $a = 0$ the expression is not a quadratic, but b and c may equal 0.
- a, b and c are constants called **coefficients**.

Examples of quadratic expressions include:

$$2x^2 - 3x + 5, \qquad 3y^2 + \frac{y}{2} - \frac{1}{4}, \qquad x^2 - \sqrt{2}\,x + \frac{1}{\sqrt{2}}, \qquad 4p^2 - 9, \qquad m^2 + m, \qquad 10q^2$$

As we know the "form" of the variables (one raised to the power of 2, one raised to the power of 1, and one raised to the power of 0), it is the coefficients (a, b and c) that define all the properties of the quadratic.

Some quadratic expressions will factorise into two linear factors, each involving integers.
e.g. $x^2 - x - 6 = (x + 2)(x - 3)$. Check this by expanding the RHS. However the quadratic: $x^2 - 4x + 1$ cannot be factorised in this way. i.e. the numbers that appear in the linear factors are not integers. But it may have factors involving other numbers, i.e. the factors may include fractions or square roots. We need systematic procedures for factorising different types of quadratics, depending on the values of the coefficients.

2.3.3 The Discriminant

Consider the quadratic expression: $ax^2 + bx + c, \quad a \neq 0$. The **discriminant** is denoted by the Greek letter Delta Δ, and is defined as:

$$\boxed{\Delta = b^2 - 4ac}$$

The discriminant is calculated from the coefficients a, b and c that define the quadratic and is a quantity seen in **The Quadratic Formula** which we'll be using later to solve quadratic equations. We are dealing with expressions here, not equations. The discriminant provides information about the "nature" of the linear factors of the quadratic. In particular:

- $\Delta > 0 \quad \Rightarrow \quad$ quadratic has 2 linear factors
- $\Delta = 0 \quad \Rightarrow \quad$ quadratic has 1 repeated linear factor
- $\Delta < 0 \quad \Rightarrow \quad$ quadratic has no linear factors

Note: If $\sqrt{\Delta} = \sqrt{b^2 - 4ac}$ is an integer, the quadratic has 2 linear factors involving integers or fractions, but not irrational numbers.

Example:

1. $x^2 - 16 \qquad (a = 1, b = 0, c = -16)$
   $\therefore\ \Delta = b^2 - 4ac = (0)^2 - 4(1)(-16) = 64.$ Note: $\sqrt{\Delta} = \sqrt{64} = 8$
   $\therefore\ x^2 - 16$ has 2 linear factors.

2. $x^2 + 9x \qquad (a = 1, b = 9, c = 0)$
   $\therefore\ \Delta = b^2 - 4ac = (9)^2 - 4(1)(0) = 81.$ Note: $\sqrt{\Delta} = \sqrt{81} = 9$
   $\therefore\ x^2 + 9x$ has 2 linear factors.

3. $x^2 + 8x + 16 \qquad (a = 1, b = 8, c = 16)$
   $\therefore\ \Delta = b^2 - 4ac = (8)^2 - 4(1)(16) = 64 - 64 = 0.$ Note: $\sqrt{\Delta} = \sqrt{0} = 0$
   $\therefore\ x^2 + 8x + 16$ has 1 repeated linear factor.

4. $x^2 + x + 1 \qquad (a = 1, b = 1, c = 1)$
   $\therefore\ \Delta = b^2 - 4ac = (1)^2 - 4(1)(1) = 1 - 4 = -3.$ Note: $\sqrt{\Delta} = \sqrt{-3}$ (Not Possible)
   $\therefore\ x^2 + x + 1$ has no linear factors.

5. $x^2 - 2x - 24 \qquad (a = 1, b = -2, c = -24)$
   $\therefore\ \Delta = b^2 - 4ac = (-2)^2 - 4(1)(-24) = 4 + 96 = 100.$ Note: $\sqrt{\Delta} = \sqrt{100} = 10$
   $\therefore\ x^2 - 2x - 24$ has 2 linear factors.

6. $2x^2 + 7x + 3 \qquad (a = 2, b = 7, c = 3)$
   $\therefore\ \Delta = b^2 - 4ac = (7)^2 - 4(2)(3) = 49 - 24 = 25.$ Note: $\sqrt{\Delta} = \sqrt{25} = 5$
   $\therefore\ 2x^2 + 7x + 3$ has 2 linear factors.

7. $x^2 - 4x + 1 \qquad (a = 1, b = -4, c = 1)$
   $\therefore\ \Delta = b^2 - 4ac = (-4)^2 - 4(1)(1) = 16 - 4 = 12.$ Note: $\sqrt{\Delta} = \sqrt{12} = 2\sqrt{3}$
   $\therefore\ x^2 - 4x + 1$ has 2 linear factors but they don't include rational numbers.

Once the value of the discriminant is known, we choose the appropriate factorisation method. We have already seen how to factorise the first three examples above.

1. In example 1 above, $b = 0$, i.e. no x term. As c is negative ($c = -16$) in this case, we can use the difference of two squares:

$$x^2 - 16 = (x - 4)(x + 4)$$

2. In example 2 above, $c = 0$, i.e. no constant term. Both terms have a common factor of x, so we can use the distributive law:

$$x^2 + 9x = x(x + 9)$$

3. In example 3 above, $\Delta = 0$ which signifies a repeated linear factor, i.e. $(x + A)^2 = x^2 + 2Ax + A^2$. We note that $b = 8 = 2A, \therefore\ A = 4$. Also, $c = A^2 = 16, \therefore\ A = 4$ again. Therefore:
$$x^2 + 8x + 16 = (x + 4)^2$$

For the remaining examples we investigate two factorisation methods: Inspection and two-stage factorisation.

Method 1: Inspection

Factorising quadratic expressions by inspection is used when $\sqrt{\Delta} = \sqrt{b^2 - 4ac}$ is an integer which means the linear factors will involve rational numbers.

The simplest case is when the coefficient of $x^2, a = 1$. This process is the reverse of expansion using FOIL

Inspection comes from the expansion of two general linear factors:

$$\begin{aligned}(x + A)(x + B) &= x^2 + Bx + Ax + AB \\ &= x^2 + (A + B)x + AB\end{aligned}$$

The resulting quadratic has $A + B$ as the coefficient of x, and AB as the coefficient of x^0. Therefore given a quadratic of this form, factorising it requires us to find the numbers A and B such that:

- the sum $(A + B)$ is the coefficient of x, i.e. we want A and B such that $A + B = b$
- the product (AB) is the coefficient of x^0, i.e. we want A and B such that $AB = c$.

In summary, we want: *"the two numbers that multiply to give c and add to give b"*

Example (5 above): Factorise: $x^2 - 2x - 24$

From above: $\sqrt{\Delta} = \sqrt{100} = 10$, and as $a = 1$, we can assume the two linear factors will be of the form: $(x + A)(x + B)$, and we need to find A and B.

The two numbers that multiply to give -24 and add to give -2 are -6 and 4, i.e. $-6 \times 4 = -24$ and $-6 + 4 = -2$. We may use the following array to assist:

$$\begin{array}{ccccc} \underline{\quad\quad} & \times & \underline{\quad\quad} & = & -24 \\ \underline{\quad\quad} & + & \underline{\quad\quad} & = & -2 \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} \underline{-6} & \times & \underline{4} & = & -24 \\ \underline{-6} & + & \underline{4} & = & -2 \end{array}$$

Therefore: $x^2 - 2x - 24 = (x - 6)(x + 4)$
We can check the factorisation by expanding out the brackets using FOIL.

Method 2: Two-stage Factorisation

This method is useful when a, the coefficient of x^2, is not equal to 1. Inspection is used with an extra factorisation step.

Consider the following:

$$ax^2 + bx + c \quad \Rightarrow \quad \begin{array}{ccccc} \underline{A} & \times & \underline{B} & = & ac \\ \underline{A} & + & \underline{B} & = & b \end{array} \quad \Rightarrow \quad (ax^2 + Ax) + (Bx + c)$$

It can be shown that if A and B are chosen correctly, factorising the first pair and the second pair of terms separately should give a common binomial in the two sets of brackets which can be further factorised.

Example (6 above): Factorise: $2x^2 + 7x + 3$

$$\begin{array}{ccccc} \underline{\quad\quad} & \times & \underline{\quad\quad} & = & 6 \\ \underline{\quad\quad} & + & \underline{\quad\quad} & = & 7 \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} \underline{6} & \times & \underline{1} & = & 6 \\ \underline{6} & + & \underline{1} & = & 7 \end{array}$$

$$\begin{aligned} \therefore \quad & 2x^2 + 7x + 3 \\ &= 2x^2 + 6x + x + 3 \\ &= (2x^2 + 6x) + (x + 3) \\ &= 2x(x + 3) + (x + 3) \\ &= (2x + 1)(x + 3) \end{aligned}$$

Check the factorisation by expanding out the brackets using FOIL.

Note: The standard inspection may be used for quadratics of this form ($a \neq 1$), making sure to include the value of a in the inspection.

EXERCISE 17: Factorise the following quadratics, where possible:

1. $x^2 - 5x + 6$
2. $x^2 + 7x + 12$
3. $x^2 + 9$
4. $x^2 - 14x + 49$
5. $x^2 - 64$
6. $x^2 + 3x + 2$
7. $x^2 - 3x + 2$
8. $x^2 + 6x + 5$
9. $x^2 - 6x + 5$
10. $x^2 + 6x - 7$
11. $x^2 + 7x + 12$
12. $y^2 + 7y + 10$
13. $x^2 + 5x + 6$
14. $y^2 - 7y + 10$
15. $a^2 - 10a + 24$
16. $p^2 - 9p + 14$
17. $x^2 + x - 2$
18. $y^2 + y - 6$
19. $b^2 - 3b - 10$
20. $c^2 + 6c + 8$
21. $-d^2 + 8d - 12$
22. $x^2 + x + 1$
23. $x^2 - 20x + 36$
24. $2x^2 + 5x - 3$
25. $12x^2 + x - 1$
26. $(x+4)^2 + 2(x+4) + 1$
27. $81 + 18y^2 + y^4$
28. $16x^2 - 49y^2$
29. $50x^2 - 40x + 8$
30. $(2x+y)^2 - (3x-2y)^2$
31. $2x^2 + 5x + 3$
32. $6y^2 - 7y + 2$
33. $3a^2 + 10a + 3$
34. $12x^2 + 10x + 2$
35. $m^2 - 6m + 9$
36. $2x^2 - 50$
37. $15 - 4x - 4x^2$
38. $10x - x^2 - 21$
39. $3x^2 - 6x$
40. $16x^2 - 40x + 25$
41. $2y^2 + 3y + 1$
42. $2y^2 - y - 1$
43. $2y^2 + y - 1$
44. $9p^2 + 3p + 20$
45. $9p^2 - 3p - 20$
46. $9p^2 + 3p - 20$
47. $16p^2 + 4p - 20$

2.4 Solving Quadratic Equations

We now extend the work on quadratic expressions to quadratic equations. As with linear equations, to solve a quadratic equation, we want the values of the unknown (usually x) that satisfy the equation, i.e. make the LHS = RHS.

We assume we can rearrange the equation into the form:

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

where the RHS = 0.

Recall the discriminant: $\Delta = b^2 - 4ac$. We used this value to determine how many factors a quadratic expression has. We can also use the discriminant to tell us how many solutions a quadratic equation has:

- $\Delta > 0 \quad \Rightarrow \quad$ quadratic equation has 2 solutions
- $\Delta = 0 \quad \Rightarrow \quad$ quadratic equation has 1 solution
- $\Delta < 0 \quad \Rightarrow \quad$ quadratic equation has no (*real*) solutions

We use two methods for solving quadratic equations: Factorisation and The Quadratic Formula.

2.4.1 Factorisation

We factorise the LHS into linear factors (if possible) using one of the methods from the previous section, and then use the principle:

$$\text{if} \;\; p \times q = 0, \;\; \text{then either} \;\; p = 0 \;\; \text{or} \;\; q = 0$$

This will then separate the two linear factors into two linear equations that can be solved to find the two solutions to the quadratic equation.

Example: Solve for x: $3x^2 + 5x = 0$.

$$3x^2 + 5x = x(3x + 5) = 0.$$

Either $x = 0$ or $3x + 5 = 0$, so that $x = 0$ and $x = -\dfrac{5}{3}$ are solutions of $3x^2 + 5x = 0$.

We can check the two solutions by substituting each into the LHS of the original equation and making sure this evaluates to zero.

Example: Solve: $2x^2 + 7x + 3 = 0$

Factorise the LHS (using past example): $2x^2 + 7x + 3 = (2x + 1)(x + 3) = 0$
Equate each factor to zero: $2x + 1 = 0$ or $x + 3 = 0$
Solve each linear equation: $2x = -1$ or $x = -3$

$$x = -\frac{1}{2}$$

2.4.2 The Quadratic Formula

The quadratic formula calculates the solutions to the general quadratic equation directly by using the values of a, b and c. The quadratic formula can be written as:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Example: (previous) Solve: $2x^2 + 7x + 3 = 0$

In this case, $a = 2, b = 7, c = 3$. Using the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-7 \pm \sqrt{7^2 - 4(2)(3)}}{2(2)} \\ &= \frac{-7 \pm \sqrt{49 - 24}}{4} \\ &= \frac{-7 \pm 5}{4} \end{aligned}$$

$$x = \frac{-2}{4} \quad \text{or} \quad x = \frac{-12}{4}$$

$$x = -\frac{1}{2} \quad \text{or} \quad x = -3 \qquad \text{as before}$$

EXERCISE 18: Solve the following:

1. $x^2 + 3x = 0$
2. $x^2 - 6x + 8 = 0$
3. $a^2 + 7a + 10 = 0$
4. $x^2 + 6x + 9 = 0$
5. $v^2 - 5v - 36 = 0$
6. $x^2 - x = 12$
7. $x^2 = 9$
8. $2x^2 - 10x + 12 = 0$
9. $3z^2 = 15z$
10. $8x - 2x^2 = 0$
11. $3x^2 + 5x + 2 = 0$
12. $2x^2 + 5x + 4 = 0$
13. $x^2 + 25 = 0$
14. $x^2 + x - 1 = 0$
15. $x^2 + x + 1 = 0$
16. $x^2 - 7x + 6 = 0$
17. $2x^2 + 5x - 12 = 0$
18. $(2x - 1)(x - 2) = 5$
19. $x^4 + 4x^3 + 4x^2 = 0$
20. $x(2x - 1) = 3$
21. $x^2 = 20(x - 5)$
22. $\dfrac{x}{4} + \dfrac{x - 6}{x} = \dfrac{1}{2}$

2.5 Exponentials and Logarithms

Exponentials and Logarithms, and the mathematics involved with each, occur naturally in many branches of science. Applications include problems related to growth and decay. Specific examples include population dynamics and money in the bank. It is important to understand the relationship between exponentials and logarithms, and the techniques or procedures used to solve problems involving them.

This section expands on the work from the last chapter on powers and logarithms for numbers. Recall the equivalent statements:

$$2^3 = 8 \quad \Longleftrightarrow \quad \log_2 8 = 3$$

If a base is raised to a power (or index) which is a variable, then we have an *exponential expression* which can be written as an equivalent logarithmic expression.

Also, we have seen the Index Laws for powers involving variables. In this section, we will develop an equivalent set of Log Rules.

2.5.1 Exponentials

An exponential expression is of the form: a^x, where a is the *base*, with $a \geqslant 0$. The variable x is the *power* or *index* and can be any real number.

Exponential expressions can be simplified using the fundamental rules of algebra including the index laws. Therefore if $a, b > 0$ and x and y are any real numbers we recall that:

IL1. $a^x \times a^y = a^{x+y}$

IL2. $\dfrac{a^x}{a^y} = a^{x-y}$

IL3. $\left(a^x\right)^y = \left(a^y\right)^x = a^{xy}$

IL4. $(ab)^x = a^x b^x$

IL5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

IL6, $a^0 = 1$

IL7. $a^{-x} = \dfrac{1}{a^x}$; $\dfrac{1}{a^{-x}} = a^x$

A very common occurring exponential expression involves Euler's number: $e = 2.71828...$, which is a famous irrational number. "THE" Exponential expression has Euler's number as the base and is written: e^x

This expression occurs frequently in biological sciences, physics, chemistry etc. It is often written as: $\exp x$

Note: The index laws above apply to *all* exponential expressions.

Example: Use index laws to simplify the following:

- $e^x e^{3x} = e^{x+3x} = e^{4x}$
- $\dfrac{e^{4t}}{e^{3t}} = e^{4t-3t} = e^t$
- $(e^t)^4 = e^{t\times 4} = e^{4t}$
- $\sqrt{e^{6t}} = (e^{6t})^{1/2} = e^{6t\times 1/2} = e^{3t}$

2.5.2 Logarithms

We have already seen that the logarithm of a number is simply the index when the number is expressed as a power. The expression $\log_a n$ is read as "logarithm of n to the base a."

In general, if $a \in \mathbb{R}^+$ (positive real) and $x \in \mathbb{R}$, then we can write equivalent statements:

$$\boxed{a^x = n \quad \Longleftrightarrow \quad x = \log_a n}$$

We use the equivalent statements to evaluate expressions involving logarithms:

Example: Evaluate the following:

- $$\begin{aligned} & \log_2 16 \\ \text{Let} \quad & n = \log_2 16 \\ \text{Then} \quad & 2^n = 16 = 2^4 \\ \therefore \quad & n = 4. \end{aligned}$$
- $$\begin{aligned} & \log_5 125 \\ \text{Let} \quad & n = \log_5 125 \\ \text{Then} \quad & 5^n = 125 = 5^3 \\ \therefore \quad & n = 3. \end{aligned}$$

2.5.3 Log Rules

From the index laws given earlier, we can now deduce the following Log Rules.

$$a^0 = 1 \quad \Longleftrightarrow \quad \boxed{0 = \log_a 1} \tag{LL1}$$

$$a^1 = a \quad \Longleftrightarrow \quad \boxed{1 = \log_a a} \tag{LL2}$$

To define the remaining log rules, let $m = a^x, n = a^y$. Therefore $\log_a m = x, \log_a n = y$.

Substituting these expressions into both sides of the index laws gives the following:

$$m \times n = a^x \times a^y = a^{x+y} \quad \Longleftrightarrow \quad \log_a(mn) = x + y = \log_a m + \log_a n$$

$$\text{i.e.} \quad \boxed{\log_a m + \log_a n = \log_a(mn)} \tag{LL3}$$

From IL2 we have:

$$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y} \quad \Longleftrightarrow \quad \log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n$$

$$\text{i.e.} \quad \boxed{\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)} \tag{LL4}$$

If $m = 1$, then: $\log_a\left(\dfrac{1}{n}\right) = \log_a 1 - \log_a n = 0 - \log_a n$

$$\text{i.e.} \quad \boxed{-\log_a n = \log_a\left(\frac{1}{n}\right)} \tag{LL5}$$

Using IL3, we have:

$$m^p = \left(a^x\right)^p = \left(a^p\right)^x = a^{px} \quad \Longleftrightarrow \quad \log_a(m^p) = px = p\log_a m$$

$$\text{i.e.} \quad \boxed{\log_a(m^p) = p\log_a m} \tag{LL6}$$

As with the index laws, the log rules form part of the fundamental techniques and procedures which are commonly used to simplify expressions, and to solve equations.

Example: Simplify the following expressions:

- $\log_a 5 + \log_a 3 = \log_a(5 \times 3) = \log_a 15$
- $\log_a 25 - \log_a 5 = \log_a\left(\dfrac{25}{5}\right) = \log_a 5$
- $\log_a 81 = \log_a 3^4 = 4\log_a 3$

Note: We may be required to use more than one log rule in a single simplification:

Example: Simplify: $3\log_{10} 2 + \log_{10} 18 - 2\log_{10}\left(\dfrac{6}{5}\right)$

$$\begin{aligned}
3\log_{10} 2 + \log_{10} 18 - 2\log_{10}\left(\tfrac{6}{5}\right) &= \underset{\text{(LL6)}}{\log_{10} 2^3} + \log_{10} 18 + \underset{\text{(LL5)}}{2\log_{10}\left(\tfrac{5}{6}\right)} \\
&= \log_{10} 2^3 + \log_{10} 18 + \underset{\text{(LL6)}}{\log_{10}\left(\tfrac{25}{36}\right)} \\
&= \log_{10}\left(8 \times 18 \times \tfrac{25}{36}\right) && \text{using (LL3)} \\
&= \log_{10} 100 = \underset{\text{(LL6)}}{\log_{10} 10^2 = 2}\underset{\text{(LL2)}}{\log_{10} 10 = 2.}
\end{aligned}$$

Natural Logarithms

The equivalent expression for the exponential where Euler's number ($e \simeq 2.71828\cdots$) is the base is called the **natural logarithm**. As with the exponential, Euler's number is the base of the log. i.e.

$$y = e^x \quad \Longleftrightarrow \quad \log_e y = x$$

The function $\log_e x$ is called the natural logarithm of x. It is given the particular notation

$$\log_e x = \ln x = \log x$$

Note:

- If the base of a logarithm is not given, it is assumed to be e.
- All the log rules above apply to natural logs.

Example: Simplify the following

- $\ln 3x^2 + \ln 2x = \ln(3x^2.2x) = \ln 6x^3$
- $$\begin{aligned}\ln 5y^2 + \ln 4y - \ln 10y^2 &= \ln(5y^2.4y) - \ln 10y^2 \\ &= \ln\left(\frac{20y^3}{10y^2}\right) \\ &= \ln 2y\end{aligned}$$
- $$\begin{aligned}3\ln t^3 - 4\ln t^2 &= \ln(t^3)^3 - \ln(t^2)^4 \\ &= \ln t^9 - \ln t^8 \\ &= \ln\left(\frac{t^9}{t^8}\right) \\ &= \ln t\end{aligned}$$

EXERCISE 19:

Evaluate the following

1. $\log_2 32$
2. $\log_3 81^{-1}$
3. $\log_2 16 - \log_2 8$
4. $(\log_2 16)(\log_2 4)$
5. $3\log_5 2 - 2\log_5 4$
6. $\log_{10} 4 + 2\log_{10} 5$
7. $\log_2 \left(\frac{1}{2}\right)^{-18}$

Simplify the following:

8. $\ln x^2 - \ln xy + 4\ln y$
9. $e^6 e^{-6}$
10. $12e^7 \div 6e^2$
11. $\ln e^2$
12. $\ln(e^2 \ln e^3)$

2.5.4 Solving Equations involving Exponentials and Logarithms

We have previously solved linear and quadratic equations. Using the same techniques and procedures, combined with the index laws and the log rules, we investigate equations involving logarithms and exponentials.

When solving all equations, our main goal is to make the unknown (or variable) the subject of the equation. i.e. rearrange the equation to get the unknown by itself on the RHS – remembering to keep the balance of the equation.

Some important and useful identities that are often used in solving equations involving exponentials and logs are derived from the equivalent statements: $a^x = y \Longleftrightarrow \log_a y = x$

If we substitute the second of these equations into the first, and the first into the second, we get:

$$\boxed{a^{\log_a y} = y} \quad \text{and} \quad \boxed{\log_a a^x = x}$$

In particular, if $a = e$, we have $e^x = y \iff x = \ln y$ and

$$\boxed{e^{\ln y} = y} \quad \text{and} \quad \boxed{\ln e^x = x}$$

Examples: Solve for x in the following:

1. $\log_2(x+1) - \log_2(x-1) = 3$

$$\begin{aligned} \text{We have} \quad \log_2\left(\frac{x+1}{x-1}\right) &= 3 && \text{using (LL4)} \\ \therefore \quad \frac{x+1}{x-1} &= 2^3 = 8 && \text{by definition} \\ \therefore \quad x+1 &= 8x-8 \\ 9 &= 7x \\ \therefore \quad x &= \frac{9}{7}. \end{aligned}$$

2. $\ln(3x) + \ln(2) = \ln 12$

$$\begin{aligned} \ln(3x) + \ln(2) &= \ln 12 \\ \ln(3x \times 2) &= \ln 12 \\ \ln(6x) &= \ln 12 \end{aligned}$$

For these logs to be equivalent then

$$\begin{aligned} 6x &= 12 \\ x &= 2 \end{aligned}$$

Note: We can check the answer by substituting $x = 2$ into the original equation.

3. $\log 2x^3 - \log 4x = \log 8$

$$\begin{aligned} \log(2x^3) - \log(4x) &= \log 8 \\ \log\left(\frac{2x^3}{4x}\right) &= \log 8 \\ \log\left(\frac{x^2}{2}\right) &= \log 8 \end{aligned}$$

For these logs to be equivalent then

$$\begin{aligned} \frac{x^2}{2} &= 8 \\ x^2 &= 16 \\ x &= 4 \end{aligned}$$

Again we can check our answer by substituting $x = 4$ into the original equation.

EXERCISE 20: Solve the following:

1. $\log_9 x = \frac{1}{2}$
2. $\log_3 x = 4$
3. $\log_m 81 = 4$
4. $\log_x 1000 = 3$
5. $\log_2 \frac{x}{2} = 5$
6. $\log_3 y = 5$
7. $\log_2 4t = 5$
8. $5\log_{32} x = -3$
9. $\log_2 \frac{y}{3} = 4$
10. $\log_{10} 10^x = 5$
11. $\log_2 x + \log_2(x+2) = 3$
12. $\log_2\left(\frac{1}{x}\right) = 2$
13. $3\ln 2x = 2$
14. $\log_{10} x + \log_{10}(x-3) = 1$
15. $2\ln 5 + \frac{1}{2}\ln 9 - \ln 3 = \ln x$
16. $\log_{10} 2 + 5\log_{10} x - \log_{10} 5 - \log_{10}\left(x^3\right) = \log_{10} 40$.
17. Simplify: $\log_e\left(\frac{1}{e}\right)$
18. If $\log_e x = 0.6$ and $\log_e y = 0.2$, evaluate $\log_e\left(\frac{x^2}{\sqrt{y}}\right)$.
19. If $y = ae^{4t}$, express t in terms of a and y.
20. If $\ln A = bt + \ln P$, express P in terms of the other symbols.

2.6 Formulae

In many fields of study we find mathematical relationships between certain quantities. The equation that describes this relationship is a "formula". We use symbols (usually letters) to denote the parameters involved in the relationship.
Some common examples are:

(1)	$y = mx + c$	Mathematics	Straight line equation
(2)	$v = u + at$	Mechanics	Velocity formula
(3)	$\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$	Electricity	Resistors in parallel
(4)	$\dfrac{1}{v} + \dfrac{1}{u} = \dfrac{1}{f}$	Physics	Lens formula
(5)	$Td = \frac{1}{2}mv^2$	Biomechanics	Animal motion
(6)	$T = 2\pi\sqrt{\ell/g}$	Physics	Pendulum law
(7)	$A = P\Big(1 + \dfrac{r}{n}\Big)^{nt}$	Finance	Compound interest

In (7), P = Principal (initial investment); r = annual interest Rate; t = period of investment (Time, years); n = Number of times/year that interest is compounded. If $nt = 1$, we have Simple Interest over the total period. A is the final Amount (Interest plus Principal).

2.6.1 Substitution

Substitution is a common process used with formulae. We substitute known values of some of the parameters into the formula to calculate the value of the remaining parameter. Usually we substitute values into all the parameters on the RHS to calculate the value of the single parameter on the LHS.

Example: Using the formula in (2) above, find v, given $u = 20$, $a = 10$ and $t = 4$.

Starting with the formula: $v = u + at$

Substitute the given value into the RHS:
$$\begin{aligned} v &= 20 + 10(4) \\ &= 20 + 40 \\ &= 60 \end{aligned}$$

The parameter on the LHS is called the *subject* of the formula. We often need to transform a formula so that one of the other parameters becomes the subject.

2.6.2 Transformation

In some of the above formulae, the parameter on the LHS is given *explicitly,* e.g., $v = \cdots$ in (2), and similarly in (7) for A. In others, e.g. (3), the parameter on the LHS (R), is given *implicitly,* i.e. it is not alone on the top line of the LHS.

Often we need to re-arrange a formula to make one of the other parameters the subject.

Example: Assume we use (2) to find acceleration, a, in terms of u, v and t. If these are all known quantities [e.g., $(u, v, t) = (10, 50, 5)$], formula (2) would read:

$$
\begin{aligned}
& 50 = 10 + 5a \\
\text{Solve for } a: \qquad & 40 = 5a \\
\Rightarrow \quad & a = 8.
\end{aligned}
$$

This is a specific numerical example. It is more useful to express a as a general formula in terms of u, v and t. Basically, we are solving the formula (equation) for the required parameter, using our standard methods. Here, we make a the subject (the solution) of the formula:

$$
\begin{aligned}
v &= u + at \\
v - u &= at && \text{[Subtract } u \text{ each side]} \\
\frac{v-u}{t} &= a && \text{[Divide both sides by } t\text{]} \\
\Rightarrow \quad a &= (v-u)/t.
\end{aligned}
$$

Example: Make R the subject of formula (3) above.

We cover the general case, but also parallel it with a numerical example, where R_1 and R_2 take specific values, e.g., $R_1 = 3$, $R_2 = 2$. We thus see that the general problem is not much more difficult than the specific one.

Specific ($R_1 = 3,\ R_2 = 2$)	**General**
$\dfrac{1}{R} = \dfrac{1}{3} + \dfrac{1}{2}$	$\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$
[Multiply by LCM $= 6R$]	[Multiply by LCM $= R \cdot R_1 \cdot R_2$]
$\Rightarrow \dfrac{6R}{R} = \dfrac{6R}{3} + \dfrac{6R}{2}$	$\Rightarrow \dfrac{RR_1R_2}{R} = \dfrac{RR_1R_2}{R_1} + \dfrac{RR_1R_2}{R_2}$
$6 = 2R + 3R$	$R_1R_2 = R_2R + R_1R$
[Swap LHS & RHS]	[Swap LHS & RHS]
$5R = 6$	$RR_1 + RR_2 = R_1R_2$
	[get R by itself: $\therefore$ factorise LHS]
	$R(R_1 + R_2) = R_1R_2$
$\boxed{R = \dfrac{6}{5}}$	$\boxed{R = \dfrac{R_1R_2}{R_1 + R_2}}$

Note: It may not always be possible to make a particular parameter the subject of a formula.

EXERCISE 21:

In the following, make the term in square brackets the subject of the given formula:

1. $S = l + w$ [w]
2. $m = 5n$ [n]
3. $A = lw$ [l]
4. $C = 2\pi r$ [r]
5. $a = 5b - 2$ [b]
6. $y = mx + c$ [x]
7. $P = 2(l + w)$ [w]
8. $V = \frac{1}{3}\pi r^2 h$ [h]
9. If $S = 2a + (n - 1)d$, find separate formulae for d and n.
10. Make r the subject of the formula $A = P\left(1 + \dfrac{rt}{100}\right)$.
11. Show that making u the subject of the formula $s = ut + \frac{1}{2}at^2$, gives

$$u = \frac{2s - at^2}{2t}$$

---

3 Functions and Graphs

Functions are considered as "the central objects of investigation in most fields of modern mathematics." As mathematics is the most fundamental science that underpins all areas of scientific research, it is important for us to study functions and how they are used.

We will see that a function is a relation between a set of inputs and a set of outputs. The value of each output depends on the value of the input and a definition describing how to manipulate the input to get the corresponding output. The description is usually a formula that defines the function, and the values of the inputs and outputs are denoted by variables.

Often, we are required to draw a picture that describes the relation between the inputs and the outputs. This is called the *"graph"* and is a very useful tool in gaining important knowledge about the features of the relation.

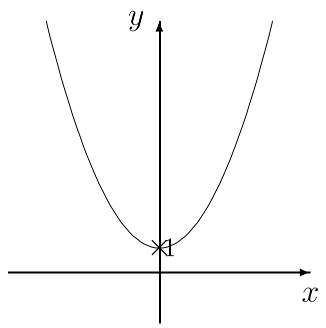
3.1 Definitions and Notation

Definition. A **relation** is a set of ordered pairs (x, y) and is usually defined by a property or rule. In this case, x is an element of the set of possible inputs of the relation, and y is an element of the set of possible outputs.

Definition. The **domain** of the relation is the set of all possible x values, or the inputs. In the ordered pairs (x, y), we always write the value of the domain, x first.

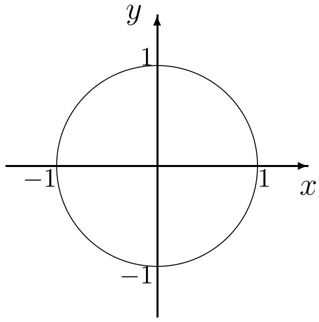
Definition. The **range** of the relation is the set of all possible y values, or the outputs. We always write the value of the range, y second.

Example: $y = x^2 + 1$ is a relation defined by a quadratic expression. In this case, x is a variable that denotes an element of the domain, and y denotes the corresponding element of the range. The graph of this relation can be drawn on a set of axes – one axis represents each variable. We can see the domain and range from the graph.



Domain: $x \in \mathbb{R}$ (all real numbers)
Range: $y \geqslant 1$

Example: $x^2 + y^2 = 1$ is a relation described by an equation involving x and y. The graph of this relation is called the **Unit Circle** and can also be drawn on a set of axes.



Domain: $-1 \leqslant x \leqslant 1$
Range: $-1 \leqslant y \leqslant 1$

Notes:

- The horizontal axis displays values of x from the domain and is called the $x-$axis.
- The vertical axis displays values of y from the range and is called the $y-$axis.
- A single point on the graph of a relation represents one single x value and it's corresponding y value. These single values are called **coordinates**.
- The system used to represent ordered pairs on a graph by coordinates is called the **Cartesian Coordinate System.**
- The two axes are perpendicular (at right angles) and cross at the point $(x, y) = (0, 0)$ which is called the **origin**. The positive $x-$direction is to the right of the origin, and the negative $x-$direction is to the left. The positive $y-$direction is above the origin, and the negative $y-$direction is below.
- The points where the graph cuts the axes are called **intercepts**.

Definition. A **function** is a relation with the property that no two ordered pairs have the same x coordinate. i.e. each input (x value) is related to exactly one output (y value).

Example: Consider the equation of the unit circle: $x^2 + y^2 = 1$. This is <u>not</u> a function as there are points on the curve that have the same x value but two different y values.
i.e. If $x = 0$, then $\quad 0^2 + y^2 = 1 \quad \therefore \quad y = \pm 1$

Example: The equation for the parabola: $y = x^2 + 1$ is a function.

The Vertical Line Test

This test is used on the graph of a relation and determines whether the graph represents a function or not. Consider placing a number of imaginary vertical lines though the graph. Each vertical line corresponds to a single $x-$value. **The Vertical Line Test** states that if a vertical line cuts the curve more than once, the curve does not represent a function. If all vertical lines intersect the curve at most once, then the curve represents a function.

Note: Writing: $f(x) = x^2 + 2x - 3$, denotes the same function as: $y = x^2 + 2x - 3$. i.e. $f(x)$ is the value of y for the given value of x. The value of y changes according to the value of x. Therefore x is called the **independent** variable and y is called the **dependent** variable.

Finding the Domain

Given a function (or a relation), we are often required to determine the domain. This gives us the values of x for which the function is defined. This is important information related to the function and can help in sketching the graph of the function.

The two main rules to remember when finding the domain:

- We cannot take the square root of a negative real number
- We cannot divide a number by zero. i.e. a fraction with a zero denominator is undefined.

Examples:

- The domain of the function: $f(x) = \sqrt{x}$ is $x \geqslant 0$ (as you cannot take the square root of a negative real number). We write this as: $D_f : x \geqslant 0$.
- The domain of the function: $g(x) = \dfrac{1}{x}$ is $x \neq 0$ (all real numbers, except $x = 0$). i.e. $D_g : x \neq 0$.
- The domain of the function $h(x) = \dfrac{1}{\sqrt{x}}$ is $x > 0$. (Note: the point $x = 0$ is excluded because you cannot divide by zero). i.e. $D_h : x > 0$.
- Find the domain of $f(x) = \sqrt{x-3}$. This function is defined if $x - 3 \geqslant 0$, i.e. $x \geqslant 3$. $\therefore \quad D_f\colon\ x \geqslant 3$.
- Find the domain of $g(x) = \dfrac{x+2}{(x-1)(x+1)}$. This function is only defined if $x \neq 1$ and $x \neq -1$. $\therefore \quad D_g\colon\ x \neq -1$ and $x \neq 1$.

  i.e. all real values of x excluding $x = \pm 1$.

Note: Determining the range is a process that follows on after the domain is found. We substitute the values of the domain ($x-$values) into the function and note all the possible resulting $y-$values.

Note: The most convenient method of determining the domain and range is from the graph of the relation or function.

Tutorial Quiz 7: A ball is thrown upwards from a tower, 80m above the ground. The height h in meter (m) of the ball from the ground at time t in seconds is given by: h= -16t^2+64t+80

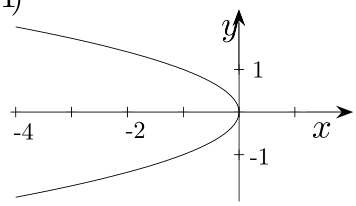
EXERCISE 22:

For each of the following graphs (1)–(4), state

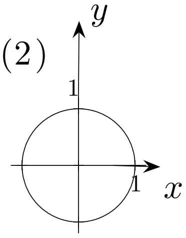
(a) the domain and range, and

(b) whether the relation is a function or not.

(1)



(2)



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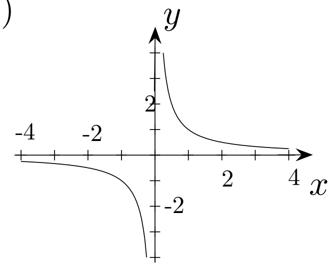

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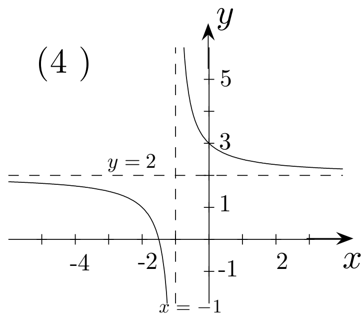


Find the domains of the following functions:

5. $f(x) = 3x + 2$
6. $f(x) = x^2 + 2$
7. $f(x) = \dfrac{10}{x}$
8. $h(x) = \dfrac{x^2}{x-1}$
9. $f(x) = \sqrt{1-x}$
10. $g(x) = \dfrac{4}{\sqrt{4-2x}}$
11. $f(x) = \dfrac{3}{x-1}$
12. $y = 3 - x$
13. $y = (x-1)^2$
14. $y = \sqrt{x+3}$
15. $y = \dfrac{1}{x-2}$
16. $f(x) = \sqrt{x-5}$
17. $f(x) = \dfrac{x}{x^2-16}$
18. $y = \sqrt{x^2}$
19. $y = \dfrac{3}{1-x^2}$
20. $y = \sqrt{1-\sqrt{x}}$
21. $y = \dfrac{1}{\sqrt{x^2-25}}$

3.2 Graphing Functions

The graph of a function (or a relation) is the picture that contains all the ordered pairs (x, y) described by the function (or relation). For the functions we consider in this chapter, the graph will consist of a smooth continuous line or "*curve*".

The graph immediately gives us information about the important features of the function (or relation). For example, the domain, range, regions where the function is increasing or decreasing, and regions where the function is positive or negative.

To *sketch* the graph of a function (or relation), we calculate specific features such as the intercepts with the axes and any turning points. These features are usually sufficient to get a good idea of what the graph of a function looks like. We will demonstrate this for specific types of functions (linear and quadratic) in the next sections.
Notes:

- The $x-$intercepts are the values of the domain that correspond to $y = 0$.
- The $y-$intercepts are the values in the range that correspond to $x = 0$.
- A function may have more than one $x-$intercept, but can only have one $y-$intercept.

3.3 Linear Functions

The graph of a linear function is a straight line. To define a straight line, we need to know the angle of the line, or how steep the line is, and at least one point on the line. Or, if we know two points that both lie on the line, we can simply connect the points. We need to formalise these ideas in order to connect the equation of a linear function to its graph.

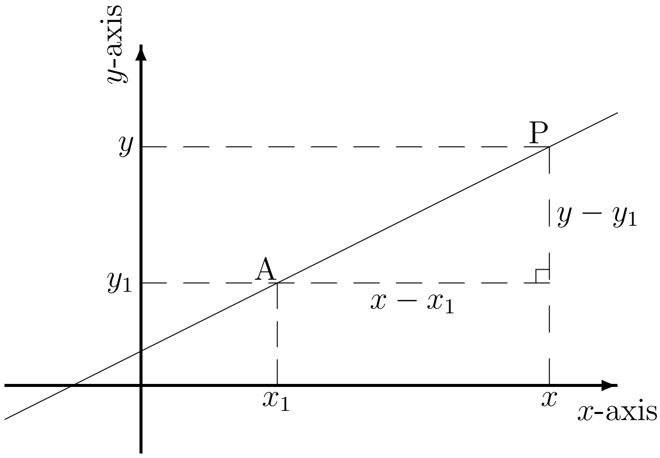
The **standard form** for the equation of a linear function, $y = f(x)$ is:

$$y = mx + c$$

where x is the independent variable, y is the dependent variable, m is the **gradient** (or slope), and c is the $y-$intercept (where $x = 0$). Note that the RHS of the equation is a "*linear*" expression.

If the values of m and c are known, the linear function is fully defined and the graph can be sketched. However, we are often required to determine the equation of the linear function before we can sketch the graph.

Let $A(x_1, y_1)$ be a fixed point on the line and $P(x, y)$ be any other point on the line.



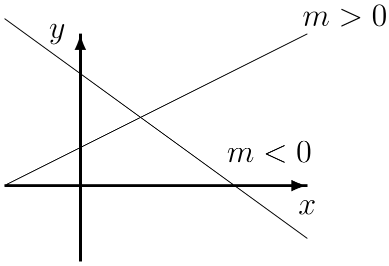
As the gradient of the line = gradient of AP, then: $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y - y_1}{x - x_1}$, and

$$y - y_1 = m(x - x_1)$$

This is another form for the equation to a straight line through the point (x_1, y_1) with a gradient m. This equation can be rearranged into the standard form above.

Note:

- A positive gradient ($m > 0$) represents an increasing line as we move from left to right.
- A negative gradient ($m < 0$) represents a decreasing line as we move from left to right.



EXERCISE 23:

Write down the equations of the straight lines with the following gradients and y-intercepts.

1. $2, \ -7$
2. $-1, \ -3$
3. $\frac{2}{5}, \ -\frac{3}{5}$
4. $0, \ -2$

Draw sketches of the following functions. In each case, write down the gradient and y–intercept, find any $x-$intercepts and mark them on each graph.

5. $y = 2x + 3$
6. $y = 2x - 2$
7. $y = x$
8. $y = -x - 3$
9. $y = \frac{1}{2}x + 1$
10. $y = 2$
11. $x + y = 3$
12. $-2x + y - 2 = 0$
13. $2x - 3y = 6$

There is a missing page between page 72 and 73, where the first 2 cases of finding the equation of a straight line are explained. The 1st case is, given the gradient m and intercept c. The 2nd case is, given the gradient m and one point A. The 3rd case is, given 2 points A & B as explained below. Students can view the first 2 cases from the 5th Lecture Recordings of "Functions & Graphs 1". A short written explanation can be found in the PPT slides "MA1020 Summary &..."

Finding the Equation to a Straight Line

Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on a straight line.

The gradient of the line: $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$

After finding m, c remains unknown. This can be calculated by substituting the coordinates of one of the points into the equation and solving for c.

Example: Find the equation of the straight line that passes through $(1, 3)$ and $(3, -1)$.

$$\text{Gradient: } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-3}{3-1} = \frac{-4}{2} = -2$$

Thus, the equation must be of the form: $y = -2x + c$.

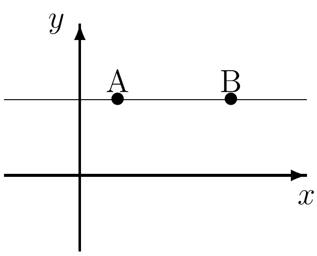
Find c by substituting *either* $(1, 3)$ *or* $(3, -1)$ into the equation.
Substituting $(x, y) = (1, 3)$ into $y = -2x + c$ gives:

$$\begin{aligned} 3 &= -2 \times 1 + c \\ 3 &= -2 + c \\ \Rightarrow \quad c &= 5 \end{aligned}$$

Therefore the equation of the line is: $y = -2x + 5$.

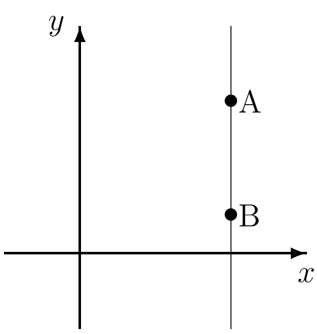
Note: Some special types of straight lines:

- Horizontal Lines:



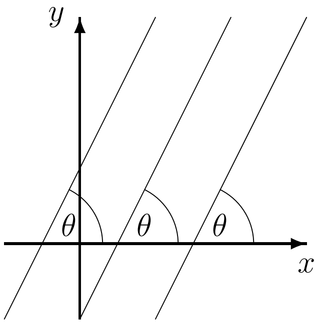
If $y_1 = y_2$, the gradient is zero.

- Vertical Lines:

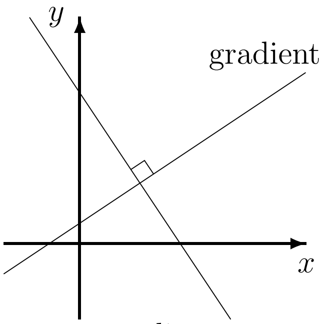


If $x_1 = x_2$, the gradient is not defined.

- Parallel Lines: Lines that have the same gradient.



- Perpendicular Lines: Lines that are at right angles (90 degrees) to each other.



The product of gradients of perpendicular lines is -1.

$$\therefore \quad m_1 m_2 = -1$$

Or, the gradients are negative reciprocals:

$$m_1 = -\frac{1}{m_2}$$

Example: Find the equation of the line passing through the point $(2, -3)$ that is (1) parallel to; (2) perpendicular to the line: $3x + 4y - 5 = 0$.

(1) First, rearrange the equation of the line into standard form:

$$\begin{aligned} 3x + 4y - 5 &= 0 \\ \therefore \quad 4y &= -3x + 5 \\ \therefore \quad y &= -\frac{3}{4}x + \frac{5}{4} \end{aligned}$$

The required line is parallel to this line. Therefore it has the same gradient, i.e. $m = -3/4$. Our line also passes through the point $(2, -3)$

$$\begin{aligned} \therefore \quad y - y_1 &= m(x - x_1) \\ \text{becomes} \quad y - (-3) &= -\frac{3}{4}(x - 2) \\ \therefore \quad y + 3 &= -\frac{3}{4}x + \frac{3}{2} \\ \therefore \quad y &= -\frac{3}{4}x - \frac{3}{2} \end{aligned}$$

(2) The required line is perpendicular to the line: $y = -\dfrac{3x}{4} + \dfrac{5}{4}$

$$\begin{aligned} \therefore \qquad & m = \text{negative reciprocal of } -3/4 \\ i.e. \qquad & m = 4/3 \end{aligned}$$

The line also passes through the point $(2, -3)$,

$$\begin{aligned} \therefore \qquad y - y_1 &= m(x - x_1) \\ \text{becomes } y + 3 &= \frac{4}{3}(x - 2) \\ &= \frac{4x}{3} - \frac{8}{3} \\ \therefore \qquad y &= \frac{4}{3}x - \frac{17}{3}. \end{aligned}$$

EXERCISE 24:

Determine the equations of the straight lines which have the following gradient, and passes through the given point.

1. $3, \ (2, 2)$
2. $-2, \ (3, -6)$
3. $-\frac{3}{4}, \ (8, -3)$
4. $-1, \ (0, 5)$

Find the equations and sketch the straight lines which pass through the pairs of points.

5. $(1, 1), (4, 10)$
6. $(-1, 9), (3, -7)$
7. $(2, 3), (6, 5)$
8. $(-6, 5), (-1, 0)$
9. $(0, 2), (6, 0)$

Find the equation of the following straight lines and sketch their graphs:

10. gradient $\frac{3}{4}$, passing through $(-6, 5)$;
11. passing through $(2, -8)$ and $(7, 2)$
12. parallel to the x–axis and passing through the point $(5, 2)$
13. parallel to the y–axis and passing through the point $(-2, -4)$
14. x–intercept -3, y–intercept -2
15. containing the point $(2, -3)$ and parallel to the line $3x + 2y - 6 = 0$
16. containing the point $(2, -3)$ and perpendicular to the line $3x + 2y - 6 = 0$

Given two points: $A(0, -2)$ and $B(3, 0)$, and the x-coordinate of a point C on AB is 6, find

17. the equation of AB
18. the equation of the line containing the point C and perpendicular to AB.
19. Write down the equation of the straight line that is perpendicular to $y = 2x - 3$ and has a y–intercept of 5.
20. Find the equation of the straight line that is perpendicular to $x + 3y = 4$ and which passes through the point $(2, 4)$.

$y = mx^1 + c$: Linear
$y = ax^2 + bx + c$: quadratic
$y = ax^3 + bx^2 + cx + d$: cubic

3.4 Quadratic Functions

The standard form for a quadratic function is:

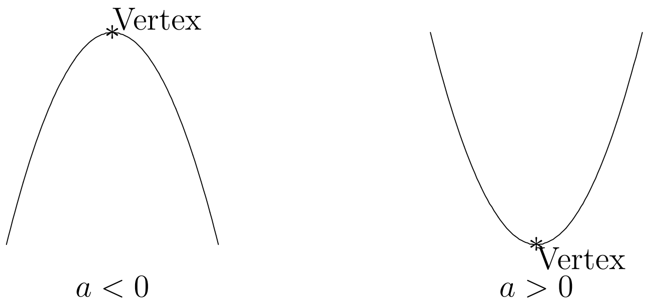
$$y = f(x) = ax^2 + bx + c, \quad a \neq 0$$

where x is the independent variable, y is the dependent variable, and a, b, c are the coefficients that will define the function and its graph. We will investigate how the values of the coefficients translate to the important features of the graph.

The graph of a standard quadratic function is a parabola. A parabola has one turning point called the **vertex**. We will use the values of the coefficients to determine the location of the vertex (i.e. the x and y coordinates).

Note: The sign of the coefficient a, defines the **orientation** of the parabola.

- If $a < 0$ the graph opens downwards. i.e. the vertex is the maximum of the function.
- If $a > 0$, the graph opens upwards. i.e. the vertex is the minimum of the function.



It can be shown that the coordinates of the vertex of a parabola with equation:
$y = ax^2 + bx + c$, are:

$$(x, y) = \left[-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right]$$

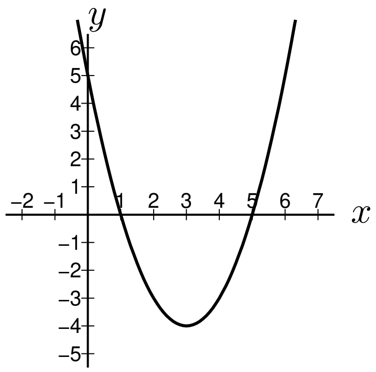
Calculate the $x-$coordinate, then substitute this into the equation to find the $y-$coordinate.

The other important features of the graph of a quadratic function are the $x-$intercepts ($y = 0$) and the $y-$intercepts ($x = 0$). If the graph is in standard form, we can see that the $y = c$ is the $y-$intercept. To calculate the $x-$ intercepts ($y = 0$), we must solve the equation: $ax^2 + bx + c = 0$, which we have seen in the previous chapter.

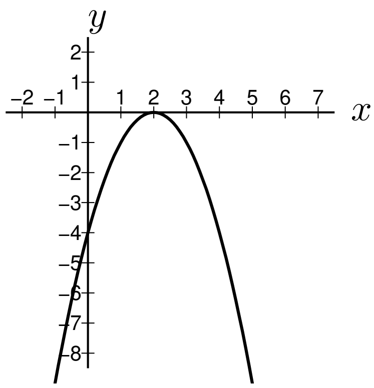
Recall that we have three possibilities when finding solutions to $ax^2 + bx + c = 0$. Two solutions, one solution or no solutions. Therefore, we have three possibilities for the number of $x-$intercepts: Two $x-$intercepts, one $x-$intercept or no $x-$intercepts. The number of $x-$intercepts is decided by the values of the coefficients.

Recall the discriminant: $\Delta = b^2 - 4ac$. We used this value to determine how many factors a quadratic expression has and how many solutions a quadratic equation has. We can also use the discriminant to tell us how many $x-$intercepts the graph of a quadratic function will have. These three sources of information from the discriminant are all related:

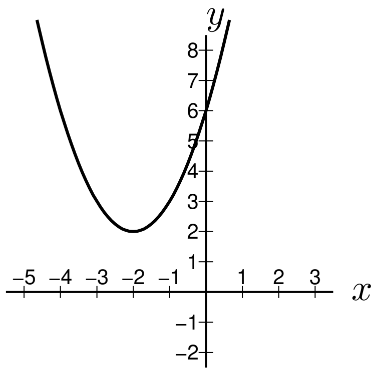
- $\Delta > 0 \quad \Rightarrow \quad$ parabola has 2 $x-$intervepts
- $\Delta = 0 \quad \Rightarrow \quad$ parabola has 1 $x-$intercept
- $\Delta < 0 \quad \Rightarrow \quad$ parabola has no $x-$intercepts



$y = x^2 - 6x + 5$
$y-$int $(x = 0) : y = 5$
$x-$int $(y = 0) : x = 1, 5$



$y = -x^2 + 4x - 4$
$y-$int $(x = 0) : y = -4$
$x-$int $(y = 0) : x = 2$



$y = x^2 + 4x + 6$
$y-$int $(x = 0) : y = 6$
$x-$int $(y = 0)$: No $x-$int

Example: Sketch the graph of: $y = f(x) = 2x^2 - 4x + 1$.

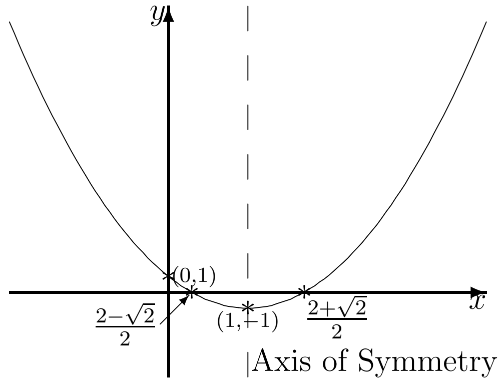
Orientation: $a = 2 \quad > 0 \therefore$ parabola opens up
y-intercept: $x = 0 \quad \Rightarrow \quad y = 0 - 0 + 1 = 1$
x-intercept: $y = 0 \quad \Rightarrow \quad 0 = 2x^2 - 4x + 1$

$$\begin{aligned} \therefore \quad x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{16 - 8}}{4} \\ &= \frac{4 \pm \sqrt{8}}{4} \\ &= \frac{4 \pm 2\sqrt{2}}{4} \\ &= \frac{2 \pm \sqrt{2}}{2} \end{aligned}$$

Vertex: $x = -\dfrac{b}{2a} = \dfrac{4}{4} = 1 \qquad \therefore y = f\left(-\dfrac{b}{2a}\right) = f(1) = 2 \times 1^2 - 4 \times 1 + 1 = -1$

$\therefore$ the vertex lies at $(x, y) = (1, -1)$.

The graph of $y = f(x) = 2x^2 - 4x + 1$ is



EXERCISE 25:

Sketch the graphs of the following quadratics. In doing so, state the value of a and the orientation of the parabola, find the y then any x intercepts, and find the coordinates of the vertex.

1. $y = x^2 + 2x - 3$
2. $y = x^2 + 4x - 5$
3. $y = x^2 - 6x + 5$
4. $y = x^2 - 4x$
5. $y = x^2 - 9$
6. $y = x^2$
7. $y = x^2 - 1$
8. $y = x^2 + 4x + 4$
9. $y = x^2 - 3x$
10. $y = 2x - x^2$
11. $y = 4x^2 + 4x - 3$
12. $y = x^2 - 4x + 2$
13. $y = x^2 - 4x + 6$
14. $y = x^2 + 5x + 4$
15. A stone is projected vertically upwards from the ground. The height $h(t)$ (in metres) above the ground is a function of time t ($t \geq 0$), with rule

$$h(t) = 12.5t - 5t^2$$

Find the greatest height reached.

3.5 Simultaneous Equations

Here we consider the problem of solving more than one equation, e.g., the following pair of equations:

$$x + y = 12, \quad (1)$$
$$x - y = 2. \quad (2)$$

There are **two** unknowns, x and y. In general we need **two** equations to find a solution — an x–value and a y–value that satisfy (1) and (2), *simultaneously.*

Notes:

- Each equation is true for certain values of x and y. In fact, Eqn. (1) can be written as
  $$y = -x + 12,$$
  the equation of a straight line, satisfied by *all* (x, y) lying on this line, e.g., $(2, 10)$, $(6, 6)$, $(15, -3)$, etc.
- We want the particular (x, y) pair for which both equations are *simultaneously* true.
- This should therefore correspond to the point of intersection of two straight lines.

Layout.
1. *Number* your equations.
2. *Align* variables vertically (in columns).
3. *Explain* operations by referring to eqn nos.

3.5.1 Method 1: Elimination

Basic Principle:	Any multiple of one equation, that is added to (or subtracted from) any multiple of another equation gives an equation that remains true.

For the above equations, we see that addition *eliminates* y. We thus get one equation in one unknown (x). This can be solved to give x, and subsequently y.

Example:

$$x + y = 12 \quad (1)$$
$$x - y = 2 \quad (2)$$

$$\begin{aligned} (1) + (2) \rightarrow \qquad 2x &= 14 \\ \therefore \qquad x &= 7 \\ \text{In } (1) \quad 7 + y &= 12 \qquad \text{(Back-substitute)} \\ \therefore \qquad y &= 5 \end{aligned}$$

The solution is $\underline{x = 7,\ y = 5}$, or $(x, y) = (7, 5)$.

<u>Check</u> the solution: LHS $= 3(-2) - 5(-3) = -6 + 15 = 9 =$ RHS.

Example:

$$3x + 2y = 13 \quad (1)$$
$$x - 4y = -5 \quad (2)$$

Simply adding or subtracting the equations cannot eliminate x or y. However, we can eliminate y by making its coefficients 4 and -4.

$$\begin{aligned} (1) \times 2: \qquad 6x + 4y &= 26 \quad \text{(1a)} \\ x - 4y &= -5 \quad (2) \\ (1a) + (2): \qquad 7x &= 21 \\ \therefore \qquad x &= 3 \\ \text{In } (1): \qquad 3(3) + 2y &= 13 \\ 2y &= 4 \\ \therefore \qquad y &= 2 \end{aligned}$$

The solution is $(x, y) = (3, 2)$. Check by substitution the solution into *both* equations.

Alternatively, we could have eliminated x by multiplying Eqn. (2) by 3. Then *subtract* the resulting equations.

3.5.2 Method 2: Substitution

Basic Principle:	Express one variable in terms of the other. Substitute in the *other* equation and solve. Back-substitute to solve for the first variable.

Example:

$$5x + 2y = 4 \tag{1}$$

$$-4x + \ \ y = 15 \tag{2}$$

Easier here to express y in terms of x by using (2).

$$\text{From (2):} \qquad y = 4x + 15 \tag{2a}$$

Substitute for y in the other equation, (1).

$$\begin{aligned} \text{(2a) in (1):} \quad 5x + 2(4x + 15) &= 4 \\ 5x + 8x + 30 &= 4 \\ 13x + 30 &= 4 \\ 13x &= -26 \\ \therefore \qquad x &= -2 \\ \text{From (2a):} \qquad y &= 4(-2) + 15 \\ &= 7. \end{aligned}$$

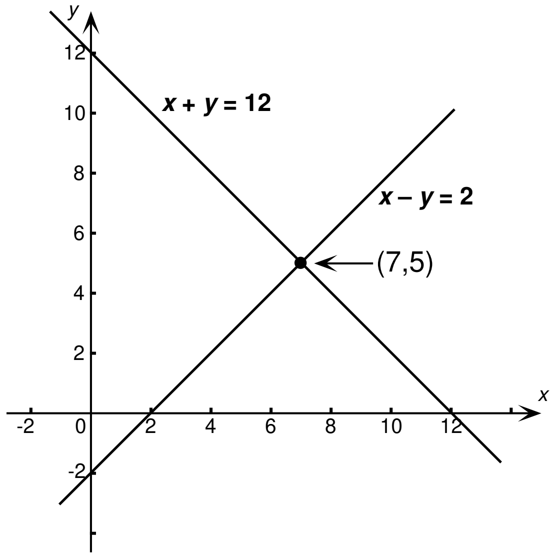
The solution is $(x, y) = (-2, 7)$. Check by substitution.

3.5.3 Geometric/Graphical

We can use graphs to help find the points of intersection of two functions.

Example:

$$x + y = 12 \quad \to \quad y = -x + 12$$
$$x - y = 2 \quad \to \quad y = x - 2$$



The two straight lines intersect at a point $(x, y) = (7, 5)$ which is the solution.

3.5.4 Simultaneous Linear and Nonlinear Equations

When one of the equations is not linear (i.e. does not represent a straight line), we can still try and solve them simultaneously. In this case, there is often more than one solution, i.e. the straight line and the curve may cut at two points (see figure below).

Generally use the method of substitution. Again, we can interpret the problem as the point of intersection of two curves, but one of these will *not* be a straight line.

Example: $x + y = 3$ (1) Straight line
$y = x^2 + 2x - 1$ (2) Parabola

Use the method of substitution:

$$\text{From (1):} \qquad y = -x + 3 \qquad \text{(1a)}$$

Substitute for y in the other equation, (2).

$$\text{(1a) in (2):} \qquad -x + 3 = x^2 + 2x - 1$$
$$0 = x^2 + 3x - 4$$
$$\Rightarrow \qquad x^2 + 3x - 4 = 0$$
$$(x + 4)(x - 1) = 0$$
$$\therefore \qquad x = -4, 1$$

Now back-substitute in Eqn. (1a) to find the y–values.

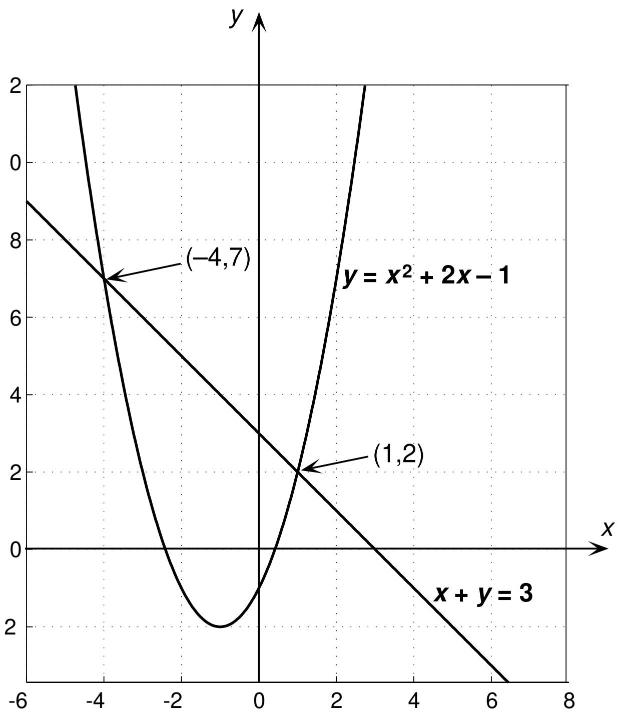
$$
\begin{array}{llll}
x = -4 & \rightarrow & y = -(-4) + 3 & = 4 + 3 = 7; \\
x = 1 & \rightarrow & y = -1 + 3 & = 2.
\end{array}
$$

$\therefore$ Solutions are $(x, y) = (-4, 7) \quad \text{and} \quad (1, 2)$.

Graphically:

- $x + y = 3 \;\Rightarrow\; y = -x + 3$ (Straight line: gradient $= -1$, y–intercept $= +3$.)
- $y = x^2 + 2x - 1 = (x + 1)^2 - 2$ (Parabola opens *up:* vertex $(-1, -2)$, y–intercept $= -1$; x–intercepts from solution of $x^2 + 2x - 1 = 0 \rightarrow x = -1 \pm \sqrt{2}$.)

Now sketch these two curves on the *same* set of axes.



The solutions are the same as those found above.

EXERCISE 26:

Solve the following sets of equations. For those that are asterisked, also find the solution by using a graphical method.

1. $x - y = 1$
   $x + y = 5$
2. $p + 2q = 0$
   $p - 3q = 10$
3. $2x + y = 11$
   $-x + y = -1$
4. $3x + 2y = -5$
   $x + 2y = 1$
5. $^*3x + y = 7$
   $x + 2y = -1$
6. $^*3a + b = 21$
   $2a - 3b = 3$
7. $5x - 12y = -1$
   $11x - 16y = 29$
8. $4x + 5y = -9$
   $3x - 4y = 32$

If equations involve fractions, it is usually simpler to *first* remove denominators by multiplying each equation (i.e., multiply <u>*each*</u> term in the equation concerned) by the Lowest Common Denominator (LCM of all its denominators).

9. $\dfrac{x}{2} + y = 5$
   $2x - \dfrac{y}{3} = 7$
10. $\dfrac{3a}{2} + \dfrac{2b}{3} = 4$
    $a + b = 1$
11. $\dfrac{x-1}{2} + y = 5$
    $\dfrac{1-y}{2} + x = 4$
12. Find two numbers that have a sum of 16 and a product of 48.
13. A bill of \$105 was paid using \$5 and \$10 notes, 12 in all. What was the breakdown of the notes used?
14. At a football match, 2 adults and 3 children paid \$39; 4 adults and 1 child paid \$53. Find the cost of each type of ticket.
15. Find all points of intersection of the parabola: $y = x^2 - 2$ and the straight line $y = x$. Draw the graphs of both functions on the same set of axes.
16. Find any points of intersection of the circle $x^2 + y^2 = 25$ and the straight line $4x - 3y = 25$. Interpret your result sketching graphs of both functions on the same set of axes.

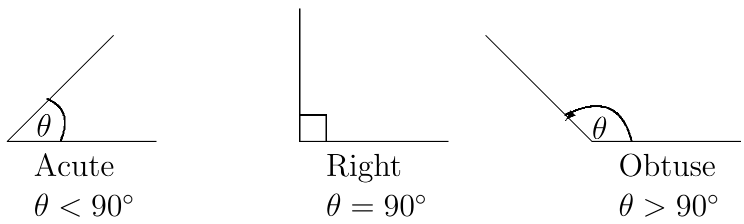
4 Trigonometry

Wikipedia: “Trigonometry (from Greek trigōnon, ”triangle” and metron, ”measure”) is a branch of mathematics that studies relationships involving lengths and angles of triangles. The field emerged during the 3rd century BC from applications of geometry to astronomical studies.”

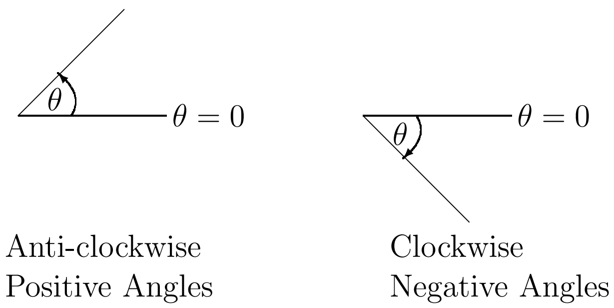
There is many applications of trigonometry and trigonometric functions. For instance, the technique of *triangulation* is used in astronomy to measure the distance to nearby stars, in geography to measure distances between landmarks, and in satellite navigation systems. Trigonometric functions are fundamental to the theory of periodic functions such as those that describe sound and light waves.

4.1 Angles: Degrees and Radians

Angles measure *rotation*. Equivalently rotation can be thought of as “angular displacement”. Often the Greek letter *“theta”* (θ), is used to represent the angle between two lines.



Angles can be measured in a positive or a negative sense. Rotation in an anti-clockwise direction corresponds to a positive angle and rotation in a clockwise direction corresponds to a negative angle:



The size or magnitude of an angle can be measured or expressed with units of degrees or radians. We will be required to transform angles from degrees to radians and vice versa. And we will be required to recognise several common angles that will be very useful in the mathematics of trigonometry.

The transformation between degrees and radians is related to the angle of a point moving around a circle, and the arc length, or the distance traveled around the circle. The length of a curve that makes a complete circle is called the circumference and is calculated by the formula:

$$C = 2\pi r$$

where r is the radius of the circle, and π is the irrational constant:

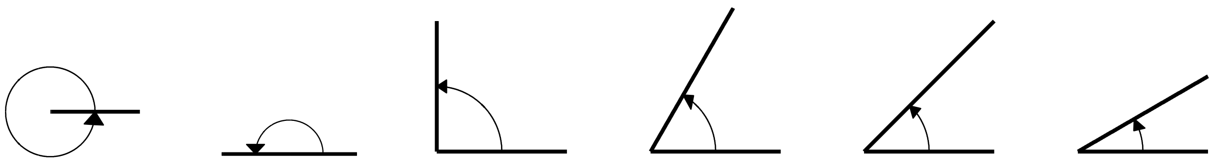
$$\pi = 3.1415926535897932...$$

The angle of a full circle, or one revolution, is 360 degrees ($360°$) and is related to 2π radians ($2\pi^{\mathrm{c}}$) from the formula of the circumference. Thus

$$360° = 2\pi^{\mathrm{c}}$$

We evaluate other common angles from this identity. One half of a full circle is $180°$ or π^{c}.
Note: π and 180 are in no way the same. Only when using the units: $180° = \pi^{\mathrm{c}}$.

The most common angles used are:



$\theta = 360° = 2\pi^{\mathrm{c}}$ $\theta = 180° = \pi^{\mathrm{c}}$ $\theta = 90° = (\pi/2)^{\mathrm{c}}$ $\theta = 60° = (\pi/3)^{\mathrm{c}}$ $\theta = 45° = (\pi/4)^{\mathrm{c}}$ $\theta = 30° = (\pi/6)^{\mathrm{c}}$

Note: When converting angles from degrees into radians, we multiply the angle by: $\dfrac{\pi^{\mathrm{c}}}{180}$.

Example: Convert the following angles from degrees into radians:

$$30° = 30\left(\frac{\pi^{\mathrm{c}}}{180}\right) = \frac{\pi^{\mathrm{c}}}{6}; \qquad 45° = 45\left(\frac{\pi^{\mathrm{c}}}{180}\right) = \frac{\pi^{\mathrm{c}}}{4}; \qquad 90° = 90\left(\frac{\pi^{\mathrm{c}}}{180}\right) = \frac{\pi^{\mathrm{c}}}{2}$$

Note: When converting angles from radians into degrees, we multiply the angle by: $\dfrac{180°}{\pi}$.

Example: Convert the following angles from radians into degrees.

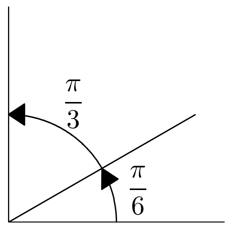
$$\frac{\pi^{\mathrm{c}}}{3} = \frac{\pi}{3} \times \frac{180°}{\pi} = 60° \qquad \frac{3\pi^{\mathrm{c}}}{2} = \frac{3\pi}{2} \times \frac{180°}{\pi} = 270° \qquad \frac{4\pi^{\mathrm{c}}}{3} = \frac{4\pi}{3} \times \frac{180°}{\pi} = 240°$$

Note: When angles are measured in degrees it is important to include the units (the degree sign). If the angle is in radians, the units can be omitted. If the units are not specified, we assume the angle is expressed in radians.

Notes: Angles can be added/subtracted in the same way as adding/subtracting real numbers:

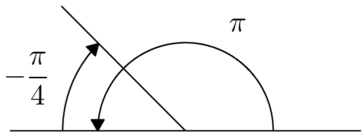
Example: $\dfrac{\pi}{6} + \dfrac{\pi}{3} = \dfrac{\pi + 2\pi}{6} = \dfrac{3\pi}{6} = \dfrac{\pi}{2}$

This sum can be represented graphically as:



Example: $\pi - \dfrac{\pi}{4} = \dfrac{4\pi - \pi}{4} = \dfrac{3\pi}{4}$

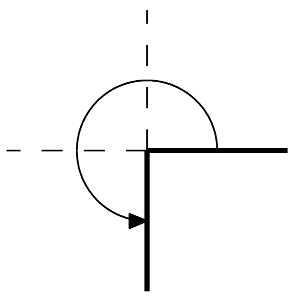
Which can be represented graphically as:



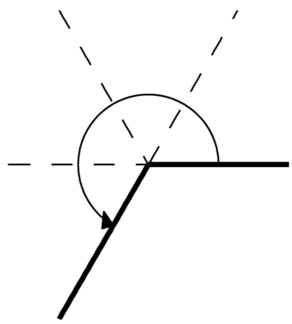
Note: One advantage of using radians is that it provides an easy way to evaluate and sketch angles other than the common angles above. In the previous examples, we saw that the angles 270° and 240° are equivalent to $\dfrac{3\pi}{2}$ and $\dfrac{4\pi}{3}$ respectively. To sketch these we can usually write angles as multiples of the common angles $\left[\dfrac{\pi}{6}, \dfrac{\pi}{4}, \dfrac{\pi}{3}, \dfrac{\pi}{2}\right]$. i.e.:

$$\frac{3\pi}{2} = 3 \times \frac{\pi}{2} \qquad \text{and} \qquad \frac{4\pi}{3} = 4 \times \frac{\pi}{3}$$

Which can be sketched as:

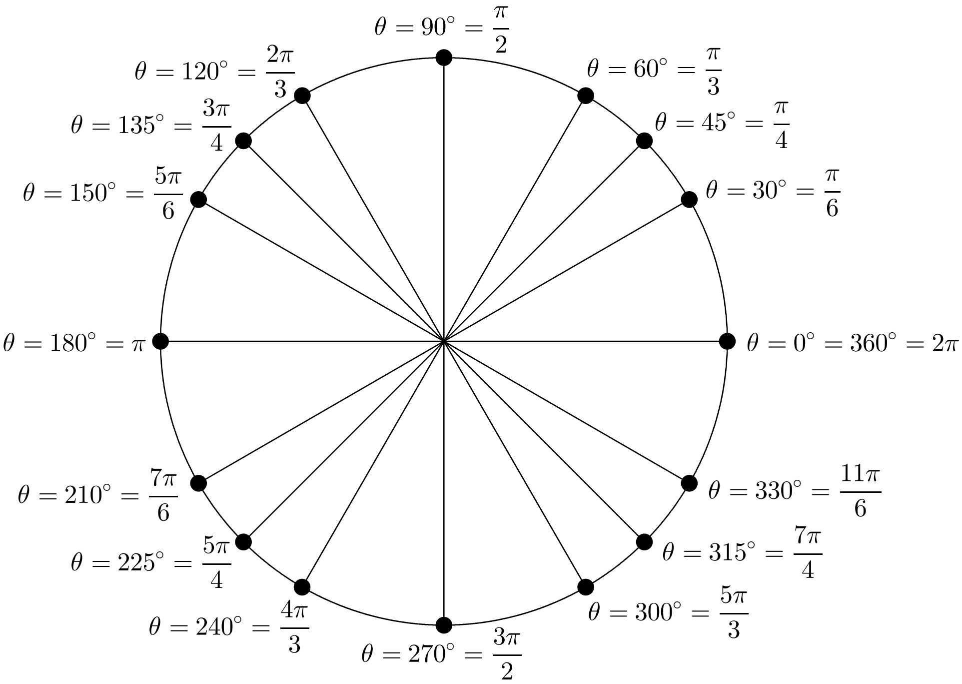


$$\theta = 270° = \frac{3\pi}{2}$$



$$\theta = 240° = \frac{4\pi}{3}$$

We will be required to work with all the multiples of the common angles $\left[\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right]$, that are within one revolution. These angles can all be sketched as follows:



EXERCISE 27:

Convert the following angles from degrees into radians. Check your answers with the figures above:

1. 90°	5. 45°	9. 30°	13. 60°
2. 180°	6. 135°	10. 120°	14. 150°
3. 270°	7. 225°	11. 210°	15. 240°
4. 360°	8. 315°	12. 300°	16. 330°

Convert the following angles from radians into degrees. Check your answers with the figures above:

17. $\pi/2$	21. π	25. $3\pi/2$	29. 2π
18. $\pi/3$	22. $2\pi/3$	26. $4\pi/3$	30. $5\pi/3$
19. $\pi/4$	23. $3\pi/4$	27. $5\pi/4$	31. $7\pi/4$
20. $\pi/6$	24. $5\pi/6$	28. $7\pi/6$	32. $11\pi/6$

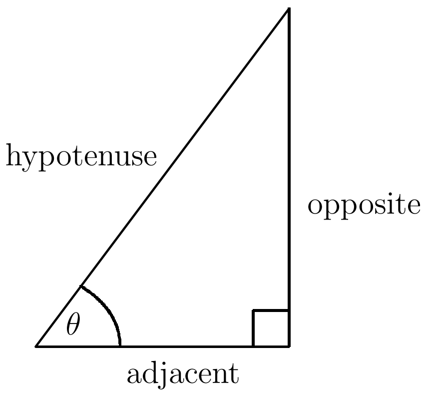
Evaluate the following and represent each sum graphically:

33. $\frac{\pi}{2} + \pi$	35. $\frac{3\pi}{2} - \frac{\pi}{4}$	37. $\frac{\pi}{4} - \frac{\pi}{3} + \frac{\pi}{6}$
34. $\frac{3\pi}{4} + \frac{\pi}{2}$	36. $2\pi - \frac{5\pi}{4}$	38. $-\frac{\pi}{2} + \frac{11\pi}{4}$

4.2 Trigonometric Functions

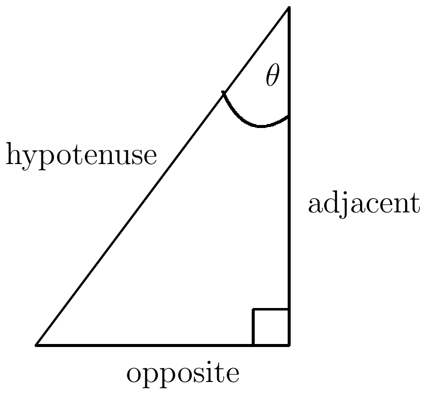
Trigonometry is associated with right-angle triangles, which is a triangle with one angle equal to 90° or $\pi/2$.

Trigonometric functions are defined in terms of ratios (or fractions) of lengths of the sides of a right-angled triangle. Consider the triangle below:



The side opposite the right-angle is called the *hypotenuse*.

If the angle of interest is θ, the *adjacent* side and hypotenuse form the angle while the *opposite* side is furthest from the angle.



If the angle of interest θ, changes to the other possible angle, the *adjacent* side and the *opposite* side change position.

Definition: We define the standard trigonometric functions as:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

One method used to recall these definitions is to remember *"soh-cah-toa"*, which can be written:

$$\mathrm{S}\,\frac{\mathrm{O}}{\mathrm{H}}\,\mathrm{C}\,\frac{\mathrm{A}}{\mathrm{H}}\,\mathrm{T}\,\frac{\mathrm{O}}{\mathrm{A}}$$

Some of the common mnemonics that are used to remember the definitions are:

- "Some Old Hags Can't Always Hide Their Old Age"
- "Some Old Hippy Caught Another Hippy Tripping On Acid"
- "Silly Old Henry Caught Albert Hugging Two Old Aunts"

Note: From the definitions above, we can see that the trigonometric functions are related to each other. In particular:

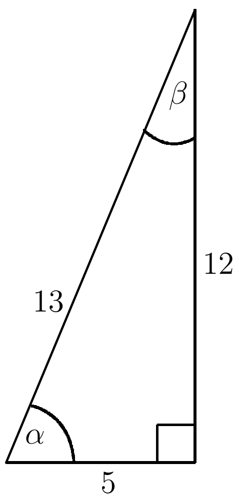
$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\text{opposite}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\text{opposite}}{\text{adjacent}} = \tan\theta$$

This is an important **trigonometric identity** and is usually expressed as:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \text{for all } \theta$$

We will investigate other trigonometric identities later.

Example: Given the triangle below, calculate: $\sin\alpha, \cos\alpha, \tan\alpha, \sin\beta, \cos\beta, \tan\beta$.
Check your answers by proving that: $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$, and $\tan\beta = \dfrac{\sin\beta}{\cos\beta}$



$$\sin\alpha = \frac{12}{13} \qquad \sin\beta = \frac{5}{13}$$

$$\cos\alpha = \frac{5}{13} \qquad \cos\beta = \frac{12}{13}$$

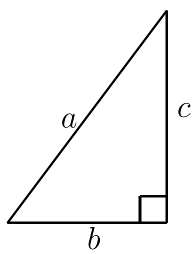
$$\tan\alpha = \frac{12}{5} \qquad \tan\beta = \frac{5}{12}$$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{12/13}{5/13} = \frac{12}{13} \times \frac{13}{5} = \frac{12}{5} = \tan\alpha$$

$$\frac{\sin\beta}{\cos\beta} = \frac{5/13}{12/13} = \frac{5}{13} \times \frac{13}{12} = \frac{5}{12} = \tan\beta$$

Note: In this example, we have calculated all the trigonometric function evaluations without knowing the angles α or β. This is possible as we knew the lengths of all sides.

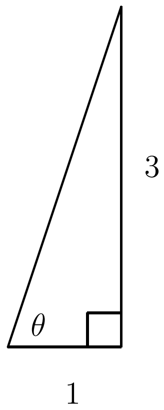
Definition: **Pythagoras' Theorem** is a relation among the three sides of a right triangle and states that: "the square of the hypotenuse is equal to the sum of the square of the other two sides." This can be expressed by the equation:



$$a^2 = b^2 + c^2$$

Note: We use Pythagoras' Theorem to calculate the lengths of the sides of a triangle if they are required for a trigonometric function evaluation.

Example: Calculate $\sin\theta$:



Let the length of the hypotenuse be: x

$\therefore\ x^2 = 1^2 + 3^2 = 1 + 9 = 10$

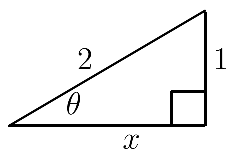
$\therefore\ x = \sqrt{10}$

$\therefore\ \sin\theta = \dfrac{3}{\sqrt{10}}$

Note: Given one trigonometric function evaluation, we can use Pythagoras' theorem and the function definitions to find all other trigonometric function evaluations. This is done without knowing the value of the angle.

Example: If $\sin\theta = \dfrac{1}{2}$, find $\cos\theta$ and $\tan\theta$.

First draw a triangle and add on any given information:



Letting x be the length of the other side, use Pythagoras to find x:

$$\begin{aligned} 1^2 + x^2 &= 2^2 \\ x^2 &= 4 - 1 = 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

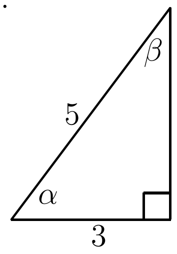
As x is the length of a side, we take the positive answer: $x = \sqrt{3}$.

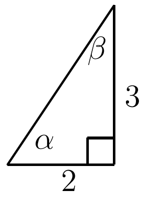
$$\therefore\ \cos\theta = \frac{\sqrt{3}}{2} \qquad \text{and} \qquad \tan\theta = \frac{1}{\sqrt{3}}$$

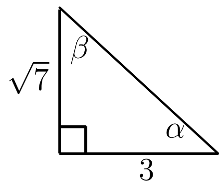
EXERCISE 28:

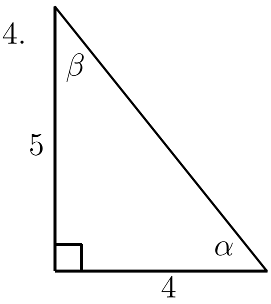
Given the following right triangles, use Pythagoras' Theorem to calculate the missing length, then write out values for: $\sin\alpha, \cos\alpha, \tan\alpha, \sin\beta, \cos\beta, \tan\beta$.

Check your answers by proving that: $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$, and $\tan\beta = \dfrac{\sin\beta}{\cos\beta}$

1.


2.


3.


4.


Given the following trigonometric function evaluation, in each case calculate the remaining two trigonometric function evaluations:

5. $\sin\theta = \dfrac{4}{5}$

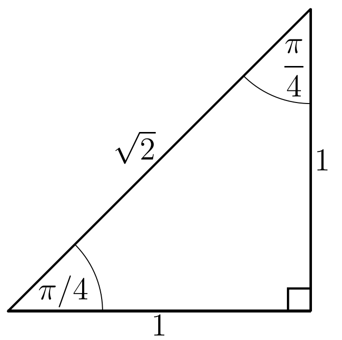
6. $\cos\theta = \dfrac{12}{13}$

7. $\tan\theta = \dfrac{2}{5}$

8. A ladder is $6m$ long and leans against the wall of a building with the base of the ladder $2m$ from the wall. How far up the wall is the top of the ladder?

4.3 Common Triangles

In the previous section, we saw the common angles that are in a full revolution. Most of the common angles we use are multiples of $[\pi/6, \pi/4, \pi/3]$. It is important for us to know the trigonometric function evaluations of these angles. We do this by the following common triangles.

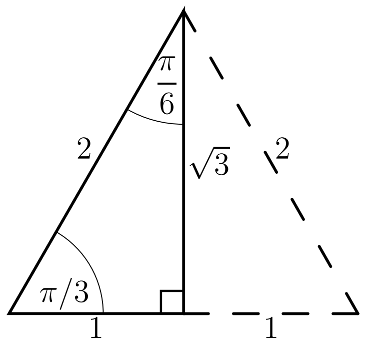


$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\tan\frac{\pi}{4} = 1$$

	30°	45°	60°
sin	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$



$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cos\frac{\pi}{3} = \frac{1}{2} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan\frac{\pi}{3} = \sqrt{3} \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

Note: The second triangle is half of an equilateral triangle which has all sides the same length and all angles the same ($60° = \pi/3$).

Note: We remember the triangles, along with: S $\frac{\text{O}}{\text{H}}$ C $\frac{\text{A}}{\text{H}}$ T $\frac{\text{O}}{\text{A}}$, and use them to generate all the information on the right. i.e. don't remember each function evaluation, just the triangles.

Note: From the second common triangle, we can see that:

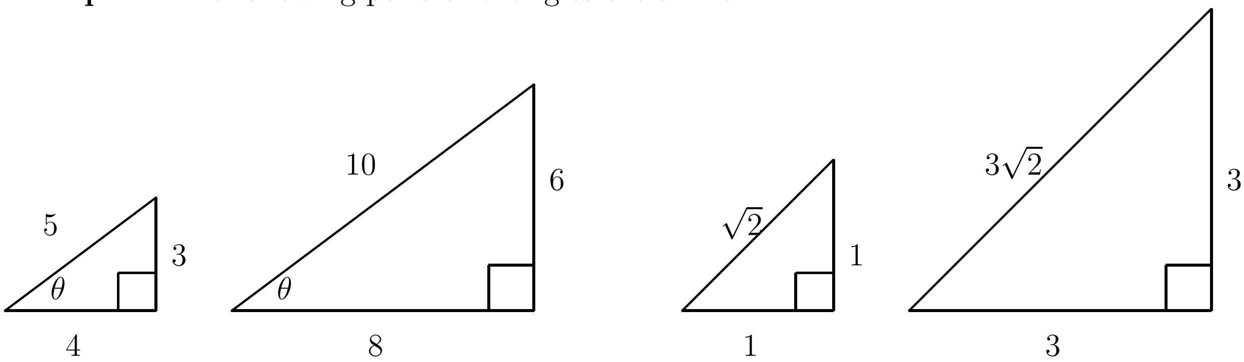
$$\sin\frac{\pi}{3} = \sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$
$$\cos\frac{\pi}{3} = \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$$

Similarly, from the first common triangle, we can see that:

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4}$$

Definition: Two triangles that have the same three angles but different lengths are called **Similar Triangles**.

Example: The following pairs of triangles are similar.



Note: The trigonometric ratios for similar triangles are equal:

• In the first pair of triangles:

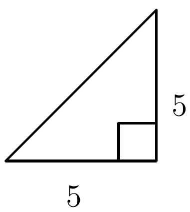
$$\sin\theta = \frac{3}{5}, \qquad \text{and} \qquad \sin\theta = \frac{6}{10} = \frac{3}{5}$$

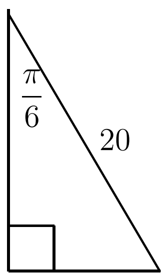
• In the second pair of triangles, both angles other than the right angle must be $\pi/4$, therefore

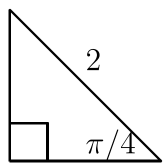
$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \qquad \text{and} \qquad \sin\frac{\pi}{4} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

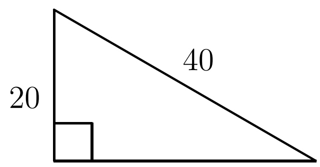
EXERCISE 29:

For the following triangles, calculate all missing lengths of sides and angles:

1. 

2. 

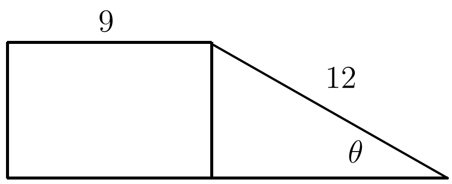
3. 

4. 

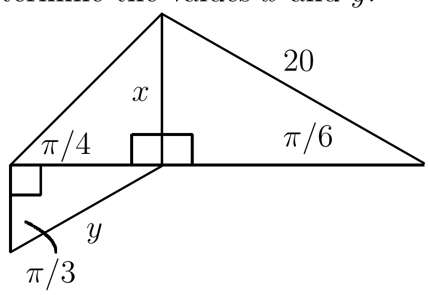
5. A square has sides of length $8cm$. Calculate the length of the diagonal.

6. A man standing on a house $25m$ tall, looks down at $45°$ and sees a cat on the ground. How far does the cat need to walk to reach the house?

7. The rectangle in the figure below has a perimeter of $30cm$. Find the angle θ.



8. Determine the values x and y:

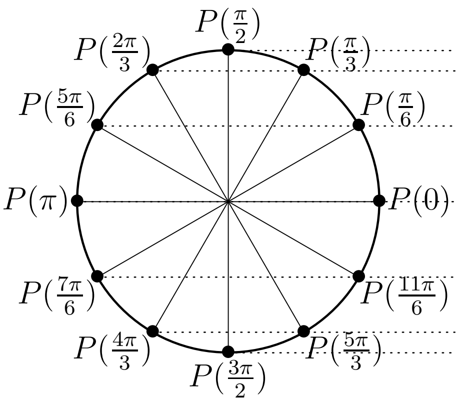


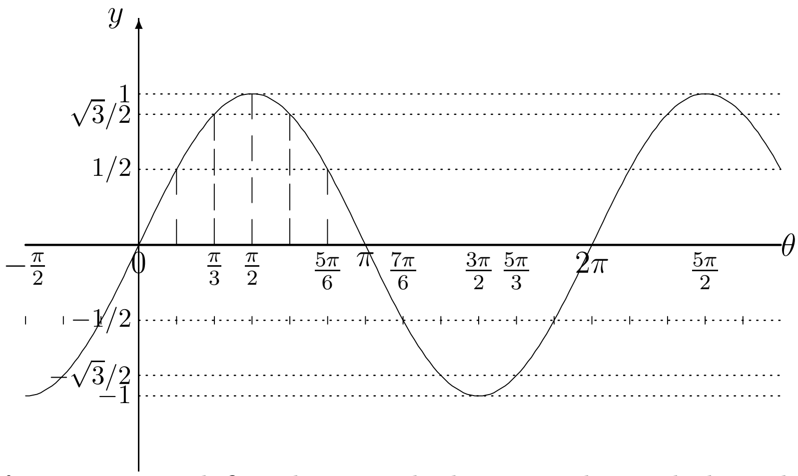
4.4 Graphs of Trigonometric Functions

4.4.1 Graphs of: $f(\theta) = \sin\theta$, $f(\theta) = \cos\theta$, and $f(\theta) = \tan\theta$

We display a table of some of the common angles (θ) and the values of $y = f(\theta) = \sin\theta$. The table forms a set of coordinates which can be placed on a set of (θ, y) axes.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0





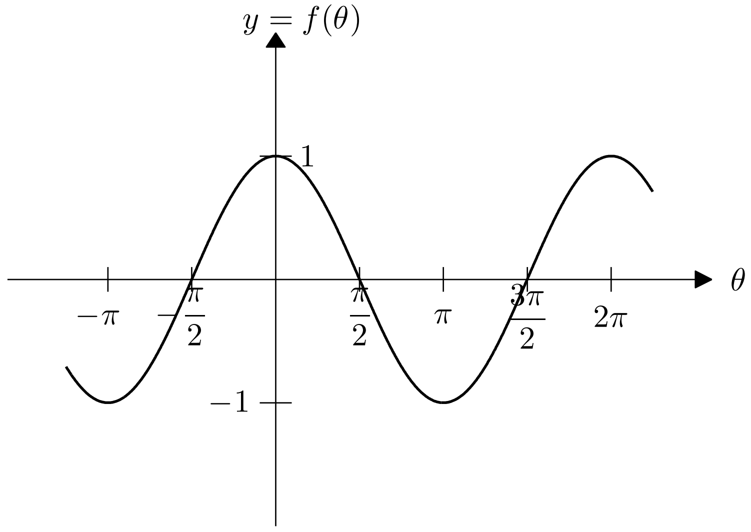
The graph of $y = f(\theta) = \sin\theta$ continues indefinitely in each direction, beyond the values shown in the table. The basic shape,  being repeated over and over again. This curve is called '**the sine curve**' or '**the sine wave**'. This function is said to be '**periodic**' with '**period** 2π' because the length of one complete wave is 2π.

Note: The graph of: $y = f(\theta) = \sin\theta$

Students keen to know more, about calculation of sin, cos & tan for common angles larger than 90 degree (by relating them to the first quadrant), may refer to Supplementary Lecture Notes found on LearnJCU. This skill is needed to solve Q22 of Ex 30.

- "periodic" with period: $T = 2\pi$.
- $y-$intercept $(\theta = 0)$: $y = 0$, which is the origin $(0, 0)$
- $\theta-$intercepts $(y = 0)$: $\theta = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots,$
  i.e. $\theta-$int: $\theta = n\pi,\ n = 0, \pm1, \pm2, \ldots$
- Turning points: $\left(\pm\frac{\pi}{2}, 1\right), \left(\pm\frac{3\pi}{2}, -1\right), \left(\pm\frac{5\pi}{2}, 1\right), \ldots$
- Domain: $D_f : \theta \in \mathbb{R}$, Range: $R_f = -1 \leqslant y \leqslant 1$
- $y = \sin\theta$ is an **odd** function

Similarly, we can sketch the graph of the function: $y = f(\theta) = \cos\theta$.



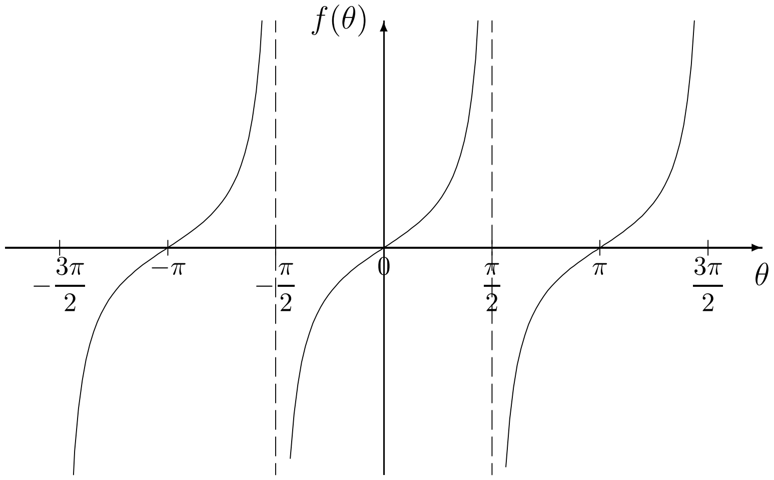
Note: The graph of: $y = f(\theta) = \cos\theta$

- "periodic" with period: $T = 2\pi$.
- $y-$intercept $(\theta = 0)$: $y = 1$
- $\theta-$intercepts $(y = 0)$: $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$
- Turning points: $(0, 1), (\pm\pi, -1), (\pm 2\pi, 1), (\pm 3\pi, -1), \ldots$
- Domain: $D_f : \theta \in \mathbb{R}$, Range: $R_f = -1 \leqslant y \leqslant 1$
- $y = \cos\theta$ is an **even** function

Similarly, we can sketch the graph of the function: $y = f(\theta) = \tan\theta$.

Note: Recall the identity: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. Therefore $\tan\theta = 0$ when $\sin\theta = 0$ and $\tan\theta$ is undefined when $\cos\theta = 0$.

i.e. the $\theta-$intercepts of $\sin\theta$ will be the $\theta-$intercepts of $\tan\theta$, and the $\theta-$intercepts of $\cos\theta$ will be vertical asymptotes of $\tan\theta$.



Note: The graph of: $y = f(\theta) = \tan\theta$

- "periodic" with period: $T = \pi$.
- $y-$intercept ($\theta = 0$): $y = 0$, which is the origin $(0, 0)$
- $\theta-$intercepts ($y = 0$): $\theta = n\pi,\ n = 0, \pm 1, \pm 2, \ldots$
- Vertical Asymptotes (y is undefined): $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$
- Domain: $D_f : \theta \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$, Range: $R_f = y \in \mathbb{R}$
- $y = \tan\theta$ is an **odd** function

4.4.2 Graphs of: $f(\theta) = A\sin\theta + k$ and $f(\theta) = A\cos\theta + k$

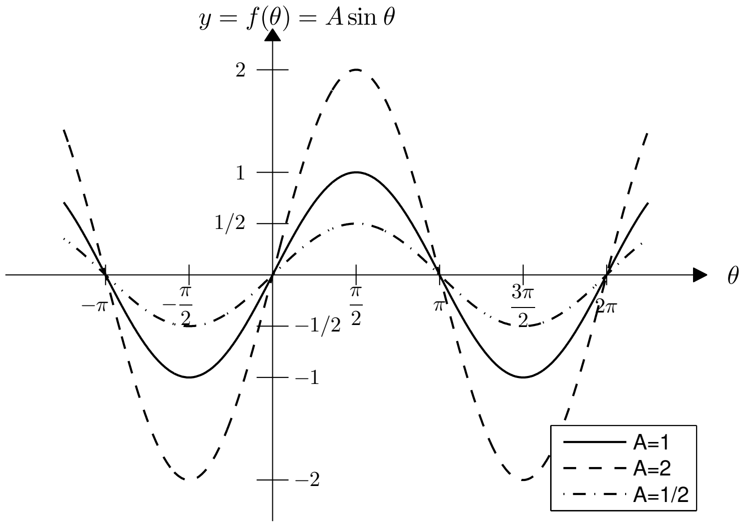
The constants A and k in these formulae cause transformations of the basic sine and cosine graphs.

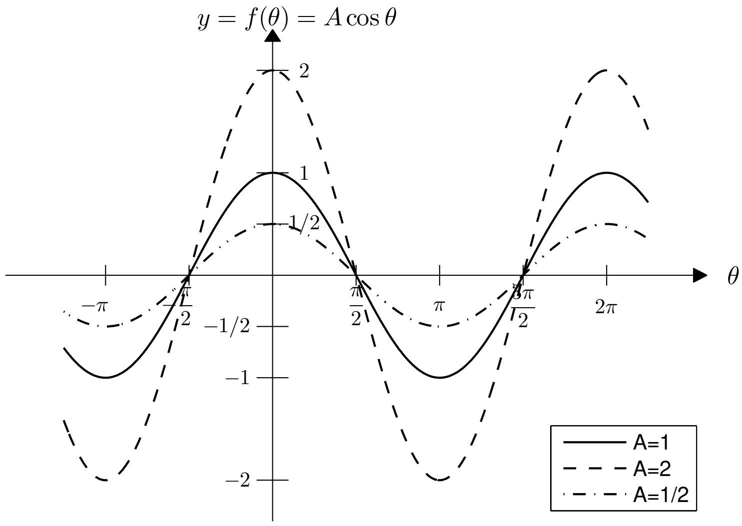
$|A|$ is called the **Amplitude** of the graph, and is defined as "half the distance between the maximum and minimum values of the function (the y-values).

The amplitude acts as a multiplying factor in the $y-$axis. The graphs of $y = \sin\theta$ and $y = \cos\theta$ have a maximum value of 1 and a minimum value of -1. So it follows that $y = A\sin\theta$ and $y = A\cos\theta$ have a maximum value of A and a minimum value of $-A$. The effect of the A is to dilate the graph parallel to the y-axis (i.e. either stretch or compress).

i.e. $\begin{array}{l} -1 \leqslant \sin\theta \leqslant 1 \quad \Rightarrow \quad -A \leqslant A\sin\theta \leqslant A \\ -1 \leqslant \cos\theta \leqslant 1 \quad \Rightarrow \quad -A \leqslant A\cos\theta \leqslant A \end{array}$

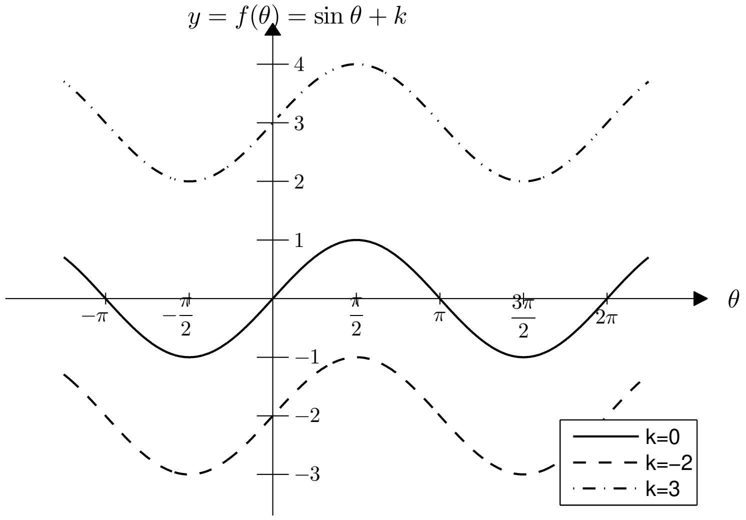
Compare the graphs below for $y = A\sin\theta$ and $y = A\cos\theta$, for $A = 1/2, 1, 2$.





The constant k is called the **vertical translation**. It has the effect of moving the curve k units up or down the y-axis according to whether k is positive (up) or negative (down). It does not alter the amplitude of the graph.

Compare the graphs below for $y = \sin\theta + k$ for $k = -2, 0, 3$.

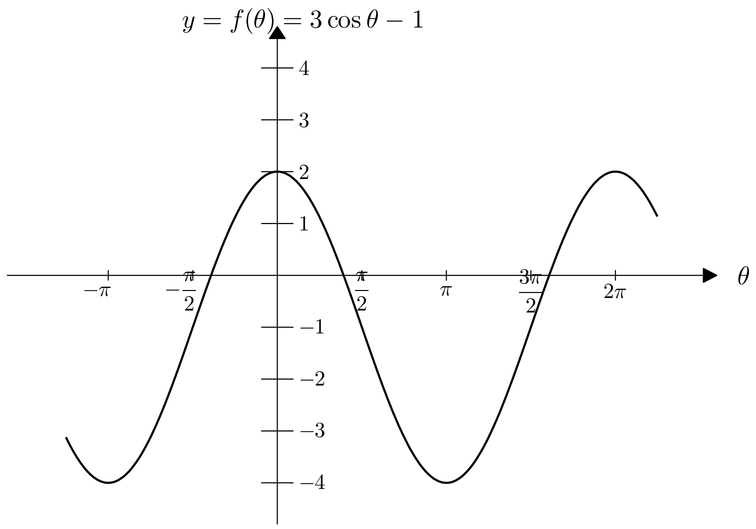


Note: The amplitude of all the graphs in the above figure is: $A = 1$.

Example: Sketch the graph of the function: $y = 3\cos\theta - 1$

We note that this is a cos function with an amplitude of 3 and a vertical translation of -1.

So we consider the standard cos graph ($-1 \leqslant \cos\theta \leqslant 1$), that has been stretched in the vertical direction with an amplitude of 3 ($-3 \leqslant 3\cos\theta \leqslant 3$) and then shifted down the $y-$axis by 1 unit ($-4 \leqslant 3\cos\theta - 1 \leqslant 2$):



EXERCISE 30:

State (a) the amplitude, (b) the vertical translation and (c) the range for

1. $y = \cos\theta - 3$
2. $y = 2\sin x + 1$
3. $y = \dfrac{\cos t}{2} + \dfrac{1}{2}$
4. $x = 10\sin t - 10$
5. $p = -5\cos\alpha$
6. $r = 3(\sin q - 2)$
7. $X = 2\pi\cos\beta - \pi$
8. $d = 100\sin\theta + e$
9. $f = \dfrac{\cos x + 1}{10}$

Write down the trigonometric function described by:

10. the cosine function with amplitude 1/2, vertical translation 2 units downwards

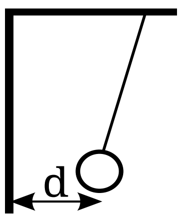
11. the sine function with amplitude 10, shifted 5 units up

Graph the following functions clearly labelling the axes, intercepts and turning points:

12. $y = 2\sin\theta - 1$
13. $y = 3\sin x + 2$
14. $x = 10\sin t - 10$
15. $y = 3\cos\theta + 1$
16. $y = 4\cos x - 2$
17. $x = 100\cos t + 50$
18. $y = \tan\theta + 1$
19. $y = \tan x - 2$
20. $x = 2\tan t + 2$

21. Sketch the curve $y = \pi\cos x + \pi$ for $-\pi \le x \le \pi$.

22. A pendulum hangs from a ceiling as shown. As the pendulum swings, the distance d in centimetres from one wall of the room depends on the time t in seconds since it was set in motion. The equation for the distance d as a function of t is given as $d = 30\cos\left(\dfrac{\pi}{3}t\right) + 80, t \ge 0$. Find:



(a) The distance from the pendulum to the wall after 2 seconds.

(b). The time when the pendulum at a maximum distance from the wall.

(~~b~~ c) The maximum distance of the pendulum from the wall.

5 Calculus

Calculus was developed in the 17th century by Isaac Newton and Gottfried Leibniz and has many applications in science, engineering and economics and can solve many problems that algebra alone cannot.

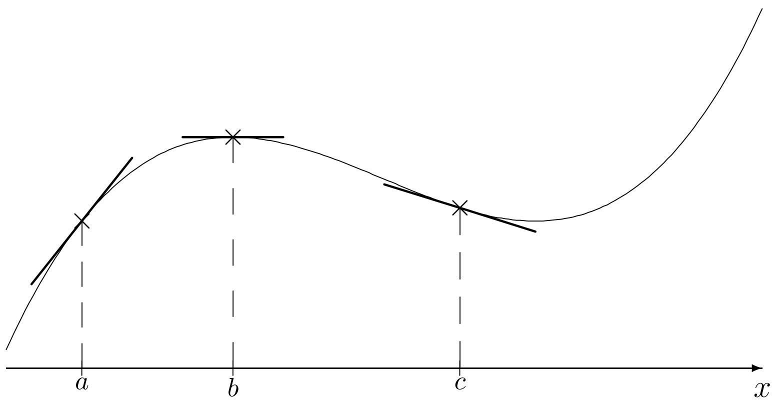
Calculus is the mathematical study of change, in the same way that geometry is the study of shape and algebra is the study of operations and their application to solving equations. It has two major branches, differential calculus (concerning rates of change and slopes of curves), and integral calculus (concerning accumulation of quantities and the areas under and between curves).

5.1 The Gradient of a Curve

Recall that the gradient of a straight line is constant ($y = mx + c$). The gradient of a curve is constantly changing, i.e. a different gradient for a different $x-$value.

Definition: A **tangent** to a curve is a straight line that touches a curve at a single point without crossing or intersecting the curve.

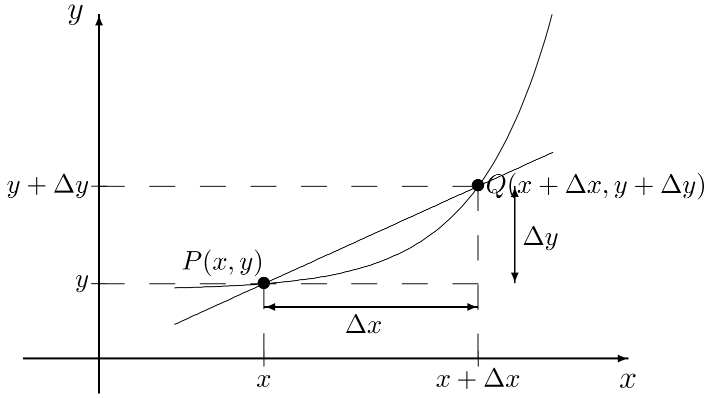
The figure below displays the graph of a cubic and three different tangents. The tangent at $x = a$ has a positive gradient, the tangent at $x = b$ is horizontal and therefore has a zero gradient, and the tangent at $x = c$ has a negative gradient:



Note that every point on the curve (different $x-$value) will have an associated tangent with a different gradient.

Definition: The **Gradient of a Curve** or the gradient at any point on a curve is defined as the gradient of the tangent to the curve at that point.

Consider $y = f(x)$, with $P(x, y)$ being any point on the curve and Q a neighbouring point, also lying on the curve. The coordinates of Q will be $(x + \Delta x, y + \Delta y)$ where $y + \Delta y = f(x + \Delta x)$. Thus $\Delta y = f(x + \Delta x) - f(x)$.



The chord (straight line) through PQ has gradient

$$m_{PQ} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

As we take Q closer to P then eventually the slope of PQ will approach the actual slope of the tangent at P. The gradient of the tangent at P is defined as the limit of the gradient of the chord PQ as $\Delta x \to 0$.

Notation: $\lim\limits_{x \to a} f(x)$ is read as "the limit of $f(x)$ as x approaches a".

i.e. Gradient of tangent at P is: $\lim\limits_{\Delta x \to 0} m_{PQ} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \text{Derivative of } f(x) \text{ at } P.$$

$$\text{i.e. } \boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

Note: $f'(x)$ is a new function giving the gradient of the curve, $y = f(x)$, as a function of x.

The *"derivative of y with respect to x"* is represented using the notation: $\frac{dy}{dx}$, or the dash notation: $f'(x)$.

If $y = f(x)$, $\frac{dy}{dx} = f'(x)$ gives the gradient of the curve at the point x.

The derivative can also be written as $\frac{d}{dx}(y)$, meaning to take the derivative of the function, y, with respect to x.

Note: We often write "*with respect to x*" as: w.r.t. x

5.2 Rules for Differentiation

Even though the definition above can be used to find the derivative of any function, it is convenient for us to remember a set of rules to apply to different types of functions.

The definition above has been used to prove the following rules for differentiation.

- If c is a constant then $\dfrac{dc}{dx} = 0$
- $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for all $x \in \mathbb{R}$
- $\dfrac{d}{dx}(cx^n) = cnx^{n-1}$
- $\dfrac{d}{dx}(f+g) = \dfrac{df}{dx} + \dfrac{dg}{dx}$, and $\dfrac{d}{dx}(f-g) = \dfrac{df}{dx} - \dfrac{dg}{dx}$

Example:

$$\begin{aligned} y &= 5x^3 + 6\sqrt{x} - \frac{1}{x^3} + \frac{2}{x^{3/2}} + 4 \\ &= 5x^3 + 6x^{1/2} - x^{-3} + 2x^{-3/2} + 4 \\ \therefore \quad \frac{dy}{dx} &= 15x^2 + 6 \times \frac{1}{2}x^{-1/2} - (-3)x^{-4} + 2 \times \frac{-3}{2}x^{-5/2} + 0 \\ &= 15x^2 + \frac{3}{\sqrt{x}} + \frac{3}{x^4} - \frac{3}{x^{5/2}}. \end{aligned}$$

Note: If we are given a function, the derivative gives us the gradient of a tangent at any point. If we are also given the coordinates of a point, we have enough information to find the equation of the tangent to the curve at the given point.

Example: Find the equation of the tangent to $f(x) = x^3 - 5x - 1$ at the point $(-2, 1)$.

$$\begin{aligned} \text{Since} \qquad f(x) &= x^3 - 5x - 1 \\ \text{then} \qquad f'(x) &= 3x^2 - 5 \\ \therefore \quad f'(-2) &= 3(-2)^2 - 5 = 7 \end{aligned}$$

The tangent at $(-2, 1)$ has $m = 7$, therefore the equation to the tangent is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ \text{i.e.} \quad y - 1 &= 7(x + 2) \\ \text{i.e.} \quad y &= 7x + 15. \end{aligned}$$

Note: Often we are required to determine points on the graph of a function that have a specific gradient. Points that have a zero gradient (a derivative equal to zer0), have a horizontal tangent which occurs at a turning point.

Example: Find the coordinates of any points on the following curve where the tangent is horizontal.

$$f(x) = x^3 + 3x^2 - 9x + 5.$$
$$\therefore \quad f'(x) = 3x^2 + 6x - 9.$$

The tangent is horizontal if it has zero slope. i.e., $m = f'(x) = 0$.

$$\therefore \quad 3x^2 + 6x - 9 = 0.$$
$$\therefore \quad x^2 + 2x - 3 = 0.$$
$$\therefore \quad (x+3)(x-1) = 0.$$
$$\therefore \quad x = 1, -3.$$

When $x = 1$, $f(1) = 1{+}3{-}9{+}5 = 0$. When $x = -3$, $f(-3) = (-3)^3{+}3(-3)^2{-}9(-3){+}5 = 32$. Therefore, at the points $(1, 0)$ and $(-3, 32)$ the tangents have zero gradient.

EXERCISE 31:

Differentiate each of the following:

1. $y = 2$
2. $y = \sqrt{3}$
3. $y = 3x + 4$
4. $y = x - 100$
5. $y = -x + 1$
6. $y = 5 - 5x$
7. $y = \pi + (\pi + 1)x$
8. $y = x \cos \dfrac{\pi}{4}$
9. $y = x^2 + \pi^2$
10. $y = x + x^2 + x^3 + x^4$
11. $y = 4x^5 + 3x^4 + 2x^3$
12. $y = 2x - 3x^3 + \dfrac{1}{2}x^2 - x^5$
13. $y = \dfrac{1}{2}x^3 - \dfrac{3}{4}x^4 - \dfrac{2}{5}x^3$
14. $y = \dfrac{3}{x^{-1}} - \dfrac{5}{2x^{-2}}$
15. $y = x^{-1} + x^{-2} + x^{-3}$
16. $y = \dfrac{1}{x} + \dfrac{2}{x^2} + \dfrac{3}{x^3}$
17. $y = 5x^3 - \dfrac{1}{5x^3}$
18. $y = 7\sqrt{x} + x\sqrt{7}$
19. $y = 4x^{-3/4} - \dfrac{1}{2}x^{-2}$
20. $y = \dfrac{1/2}{x^{1/2}} + \dfrac{5/6}{x^{5/6}}$
21. If $f(x) = x^3 + 2x^2 - 3x + 1$, find $f'(x)$ and evaluate $f'(0)$, $f'(1)$, $f'(2)$, $f'(-1)$.
22. Find the equation of the tangent to the curve $f(x) = x^3 - 2x + 1$ at $(2, 5)$.
23. Find the coordinates of the points on $f(x) = x^3 + 5x^2 + 3x - 2$ that have a horizontal tangent.
24. Find the coordinates of the point on $f(x) = \sqrt{x}$ where the tangent has slope $\dfrac{1}{2}$.

5.2.1 The Product Rule

We are often required to differentiate functions that are the product of two other functions.

If $u(x)$ and $v(x)$ are differentiable functions, then the derivative of their product, $y = u(x)v(x)$, is given by **The Product Rule**:

$$\boxed{\frac{dy}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}} \quad \text{or} \quad \boxed{(uv)' = uv' + u'v}$$

Example: Differentiate $y = (3x+2)(2x^2 - 3x + 4)$.

Let $u = 3x + 2$ and $v = 2x^2 - 3x + 4$, then

$\frac{du}{dx} = 3$ and $\frac{dv}{dx} = 4x - 3$, and

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= (3x+2)(4x-3) + (2x^2 - 3x + 4)3 \\ &= 12x^2 - 9x + 8x - 6 + 6x^2 - 9x + 12 \\ &= 18x^2 - 10x + 6\end{aligned}$$

Note: The product rule transforms a more complicated derivative into the sum of two simpler derivatives.

5.2.2 The Chain Rule

Often we are required to differentiate the function of a function, or a composite function.

If $y = f(u)$ and $u = g(x)$, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

Note: There is no dash notation for the chain rule as the dash represents a derivative with respect to x. In the chain rule we differentiate with respect to x and u.

Example: Differentiate $y = (2x+1)^3$

Let $u = 2x + 1$, so that $y = u^3$. We have $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = 2$.

$$\text{Chain Rule:} \qquad \begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 3u^2 \times 2 \\ &= 6(2x+1)^2.\end{aligned}$$

Example: Differentiate $y = \dfrac{1}{(x^2-1)^2}$

Let $u = x^2 - 1$, so that $y = u^{-2}$. We have $\dfrac{dy}{du} = -2u^{-3}$ and $\dfrac{du}{dx} = 2x$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= -\frac{2}{u^3} \times 2x \\ &= -\frac{4x}{(x^2-1)^3}.\end{aligned}$$

Example: Differentiate $y = (2x+1)^2$

(A) Direct: Expand

$$\begin{aligned} y &= 4x^2 + 4x + 1 \\ \frac{dy}{dx} &= 8x + 4 \\ &= 4(2x+1).\end{aligned}$$

(B) Chain Rule:

Let $y = u^2$ where $u = 2x + 1$

$$\begin{aligned}\frac{dy}{du} &= 2u = 2(2x+1) \\ \frac{du}{dx} &= 2\end{aligned}$$

$$\begin{aligned}\therefore\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 2(2x+1)(2) \\ &= 4(2x+1).\end{aligned}$$

Exercise: Differentiate $y = (3x^2 + x + 2)^{10}$.

The Chain Rule provides the only practical method here.
Let $y = u^{10}$ where $u = 3x^2 + x + 2$. We have $\dfrac{dy}{du} = 10u^9$, and $\dfrac{du}{dx} = 6x + 1$

$$\begin{aligned}\therefore\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 10u^9\ (6x+1) \\ &= 10(6x+1)(3x^2+x+2)^9\end{aligned}$$

5.2.3 The Quotient Rule

We are often required to differentiate functions that are the quotient (or ratio) of two other functions.

If $u(x)$ and $v(x)$ are differentiable functions, then the derivative of their quotient, $y = \dfrac{u(x)}{v(x)}$, is given by **The Quotient Rule**:

$$\boxed{\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}} \qquad \text{or} \qquad \boxed{\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}}$$

Example: Differentiate $\quad y = \dfrac{2x+1}{4x-3}$.

Let $u = 2x+1$ and $v = 4x-3$. Then $\dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = 4$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\
&= \frac{(4x-3)2 - (2x+1)4}{(4x-3)^2} \\
&= \frac{8x - 6 - 8x - 4}{(4x-3)^2} \\
&= -\frac{10}{(4x-3)^2}
\end{aligned}$$

Example: Differentiate $\quad y = \dfrac{x}{1+x^2}$.

Let $u = x$ and $v = 1 + x^2$. Then $\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 2x$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\
&= \frac{(1+x^2)\times 1 - x \times 2x}{(1+x^2)^2} \\
&= \frac{1 + x^2 - 2x^2}{(1+x^2)^2} \\
&= \frac{1-x^2}{(1+x^2)^2}.
\end{aligned}$$

Note: If we write $y = \dfrac{u}{v} = uv^{-1}$, then the product rule can be used to differentiate functions of this type.

EXERCISE 32:

Use either the product rule, chain rule or quotient rule to differentiate the following:

1. $y = (2x - 5)(x + 4)$
2. $f(x) = (x^2 - 1)(2x + 3)$
3. $y = (2 - 3x^2)(2x^3 - 3)$
4. $f(x) = (x^4 + x^3)(x^2 + x)$
5. $y = (2x^2 - x + 1)(3x^2 - 2x - 2)$
6. $f(x) = x^3(4x^5 + x)$
7. $y = x^{-2}(4 + 3x^{-3})$
8. $f(x) = (\sqrt{x} + 1)(\sqrt{x} - 1)$
9. $y = (3x^2 - 1)^3$
10. $f(x) = (2 - 3x^5)^{20}$
11. $y = \sqrt{x^2 + 2}$
12. $f(x) = \dfrac{1}{(x^2 - 5)^3}$
13. $y = (3x^2 - x^{-2})^{-1/2}$
14. $f(x) = \left(\dfrac{1}{x} - \dfrac{1}{\sqrt{x}}\right)^{-1}$
15. $f(x) = \dfrac{x^2 + 3}{3x^3}$
16. $y = \dfrac{x^2 + x}{x}$
17. $y = \dfrac{1}{1 + x^2}$
18. $y = \dfrac{x + 1}{\sqrt{x}}$
19. $y = \dfrac{x}{1 + 2x^2}$
20. $f(x) = \dfrac{x}{1 + \sqrt{x}}$

5.3 Higher Order Derivatives

Given a function, $y = f(x)$, the derivative, $y' = f'(x) = \dfrac{dy}{dx}$, is the gradient of the curve.

Note: $y' = \dfrac{d}{dx}(y)$ is a measure of the rate at which y changes with respect to a change in x.

The derivative may have a derivative of its own, i.e. we differentiate the derivative.
This is called the **Second Derivative**:

$$y'' = f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Note: $y'' = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is a measure of the rate at which y' changes with respect to a change in x.

Similarly, the third derivative is: $y''' = f'''(x) = \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{d^3y}{dx^3}$, etc.

Example:
$$\begin{aligned} f(x) &= x^3 - 6x^2 + 4x - 1 \\ f'(x) &= 3x^2 - 12x + 4 \\ f''(x) &= 6x - 12 \\ f'''(x) &= 6. \end{aligned}$$

Geometrically: On the graph of a given function, $y = f(x)$, the gradient of the curve (the gradient of the tangent at a point), is the change in y with respect to x and is given by the first derivative, $y' = f'(x) = \dfrac{dy}{dx}$. The change in gradient with respect to a change in x is given by the second derivative, $y'' = f''(x) = \dfrac{d^2y}{dx^2}$.

Note: We use both the first and second derivatives in sketching graphs in the next section.

Application: If the displacement, s, of a body moving in a straight line is described by the function: $s(t)$, where t is time, then the instantaneous velocity (at a specific time) is the rate of change of displacement with respect to time and is given by the first derivative: $v(t) = \dfrac{ds}{dt}$. The acceleration at a specific time is the rate of change of velocity with respect to time and is given by the first derivative of the velocity which is the second derivative of the displacement: $a(t) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$

EXERCISE 33:

Find the first and second derivative for the following:

1. $f(x) = 2x + 1$
2. $y = 4x^2$
3. $f(x) = 2x^3 - 3x^2 + 4x - 1$
4. $x = 5t - 7t^3$
5. $g = \dfrac{1}{t}$
6. $y = \dfrac{1}{\sqrt{x}}$
7. $x = (2t - 1)(3t + 2)$
8. $f(x) = \dfrac{x}{1 + x}$

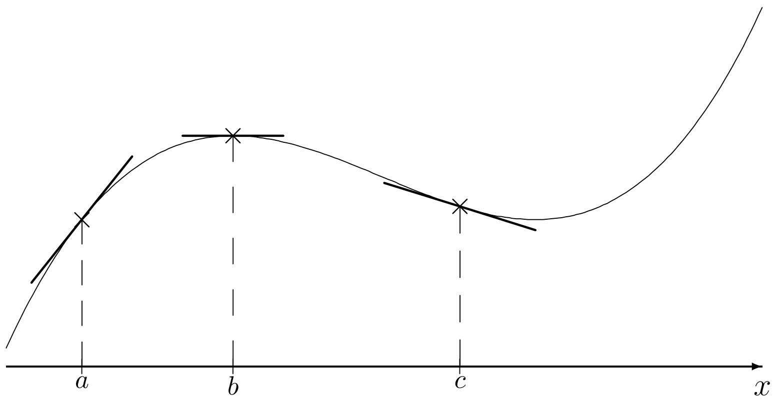
Find the first and second derivative for the following, and then sketch the original function as well as the first and second derivatives on the same axes:

9. $y = x^2$
10. $x = t^2 + t$
11. $y = 2x - x^2$
12. $y = (2x + 1)(x - 2)$

5.4 Application of Derivatives - Sketching

We use the first and second derivative to find information about the important features of functions. This assists us in sketching the graphs of functions and will be added to the process of sketching graphs developed in chapter 3.

Recall the previous graph of a cubic function:



Some of the information we gain by investigating the derivative is:

- At $x = a$: The gradient of the tangent is positive. Therefore $f'(x) > 0$ and $f(x)$ is an increasing function here.
- At $x = c$: The gradient of the tangent is negative. Therefore $f'(x) < 0$ and $f(x)$ is a decreasing function here.
- At $x = b$: The tangent is horizontal which has a zero gradient. Therefore $f'(x) = 0$.

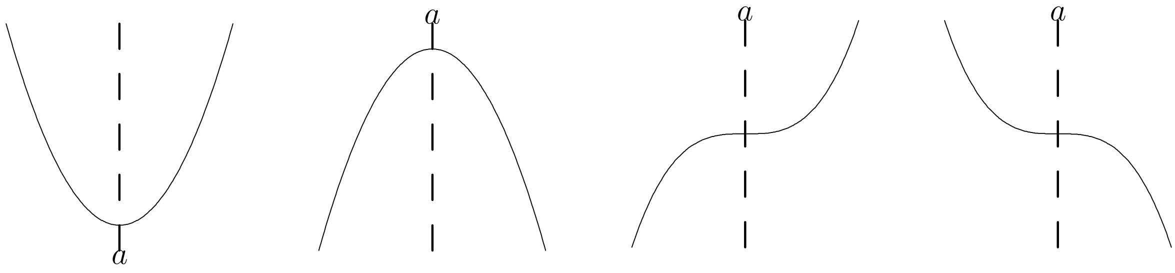
5.4.1 Stationary Points – Turning Points and Points of Inflection

Definition: A **stationary point** is a point on a graph where $f'(x) = 0$.

We will consider two types of stationary points:

- **Turning points** – local maxima or local minima
- **Points of inflection**

The different types of stationary points ($f'(x) = 0$) are:



$x < a,\ f'(x) < 0$	$x < a,\ f'(x) > 0$	$x < a,\ f'(x) > 0$	$x < a,\ f'(x) < 0$
$x = a,\ f'(x) = 0$	$x = a,\ f'(x) = 0$	$x = a,\ f'(x) = 0$	$x = a,\ f'(x) = 0$
$x > a,\ f'(x) > 0$	$x > a,\ f'(x) < 0$	$x > a,\ f'(x) > 0$	$x > a,\ f'(x) < 0$
local minimum	local maximum	point of inflection	point of inflection

First Derivative – Location of the stationary point

To find the location, or the $x-$coordinate, of the stationary points, we find the first derivative of the function ($f'(x)$), equate the derivative to zero (let $f'(x) = 0$), and solve for x.

We then substitute the $x-$values into the original function to find the corresponding $y-$values.

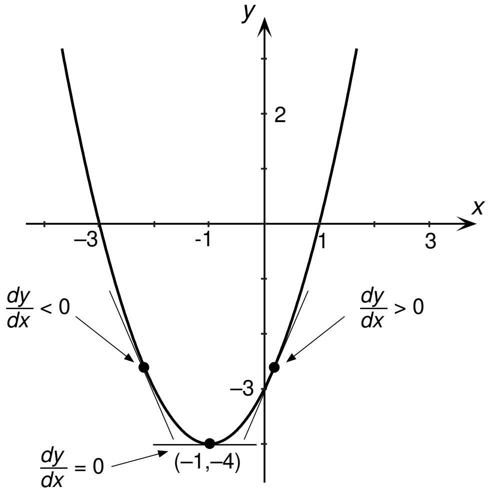
Example: Find the stationary point on the graph of the function: $y = x^2 + 2x - 3$

Note: This is a quadratic equation therefore the graph will be a parabola.
The stationary point where $\dfrac{dy}{dx} = 0$, is the vertex or the turning point.

We have already seen that as, $a = 1 > 0$, the parabola will open upwards, and the vertex will be a local minimum.

$$\frac{dy}{dx} = 2x + 2 = 0$$
$$\therefore\ 2x = -2$$
$$\therefore\ x = -1$$
$$\therefore\ y = f(-1) = (-1)^2 + 2(-1) - 3 = -4$$

The coordinates of the Vertex of the parabola are: $(x, y) = -1, -4$



Note: For any quadratic of the form: $f(x) = ax^2+bx+c$, the derivative is: $f'(x) = 2ax+b$. To find the coordinates of the vertex, we let $f'(x) = 0$:

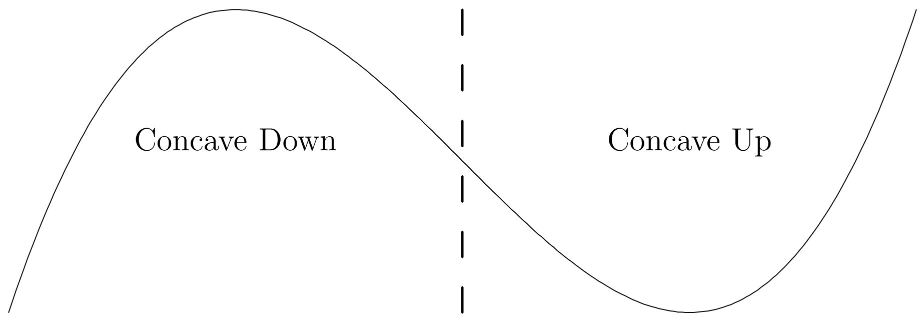
$$\begin{aligned} \therefore \quad 2ax + b &= 0 \\ 2ax &= -b \\ x &= -\frac{b}{2a} \end{aligned}$$

This is the equation for the $x-$coordinate of the vertex from chapter 3.

Second Derivative – Type of stationary point

Once we know the location ($x-$values) of any stationary points, we use the second derivative of the function to determine what type of stationary point each is.

We consider the **Concavity** of a function. The figure shows two regions of a function with different concavity.



$f'(x)$ is decreasing from left to right. Therefore $f''(x)$, the change in gradient w.r.t. x, is negative. i.e. $f''(x) < 0$.

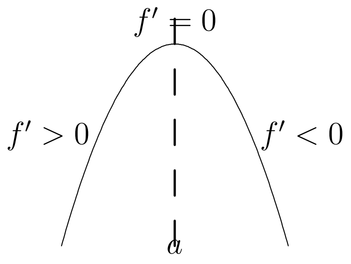
$f'(x)$ is increasing from left to right. Therefore $f''(x) > 0$.

If $f''(x) < 0$ the curve is concave down.
If $f''(x) > 0$ the curve is concave up.

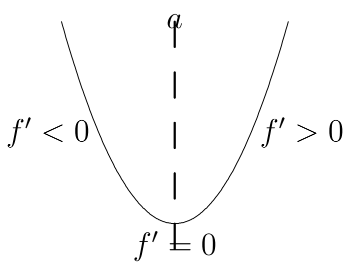
We use the information about concavity to determine the nature of stationary points.

If $\dfrac{dy}{dx} = 0$, we have a stationary point. We use the sign of $\dfrac{d^2y}{dx^2}$ at that point to determine if we have a maximum or a minimum.

If $f''(x) < 0$ the stationary point is a maximum.
If $f''(x) > 0$ the stationary point is a minimum.



If $f'(a) = 0$ and $f''(a) < 0$ then we have a local maximum at $(a, f(a))$.



If $f'(a) = 0$ and $f''(a) > 0$ then we have a local minimum at $(a, f(a))$.

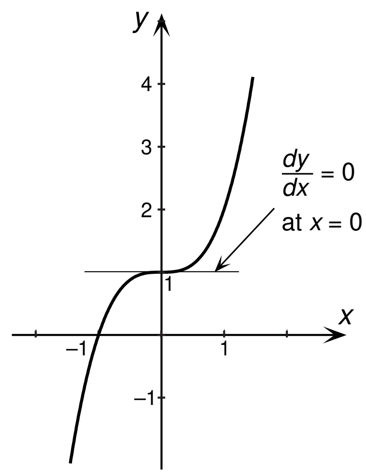
Note: If $\dfrac{d^2y}{dx^2} = 0$ at the stationary point, this indicates a point of inflection.

Previous Example:
For $y = x^2 + 2x - 3$, the coordinates of the stationary point are $(x, y) = (-1, 4)$.

$$\frac{dy}{dx} = 2x + 2, \qquad \therefore \ \frac{d^2y}{dx^2} = 2 > 0, \therefore \text{ stationary point is a minimum.}$$

Example: Find the location and the type of stationary point on the graph of: $y = x^3 + 1$

$$\begin{aligned} && y &= x^3 + 1 \\ \text{Let } \frac{dy}{dx} = 0: && \frac{dy}{dx} &= 3x^2 = 0 \\ & \therefore & x &= 0 \\ && y &= 3(0)^2 + 1 = 1 \end{aligned}$$



So there is a stationary point at $(x, y) = (0, 1)$

$$\frac{d^2y}{dx^2} = 6x \therefore \text{ at } x = 0, \frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad \text{Point of inflection}$$

5.5 Cubic Functions

The general equation for a cubic function is:

$$y = f(x) = ax^3 + bx^2 + cx + d, \qquad a \neq 0$$

Again, the values of the coefficients (a, b, c, d) define the important features of the graph of this function. These are the orientation, $x-$ and $y-$intercepts and the turning points.

The sign of the coefficient a, defines the orientation of the graph.

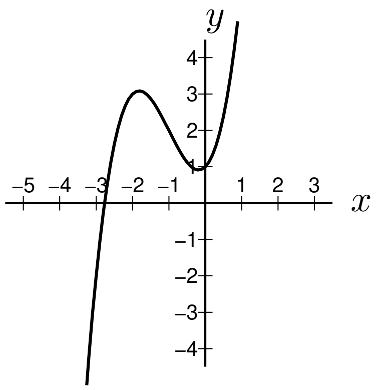
Orientation: If $a > 0$, the curve is similar to 

If $a < 0$, the curve is similar to 

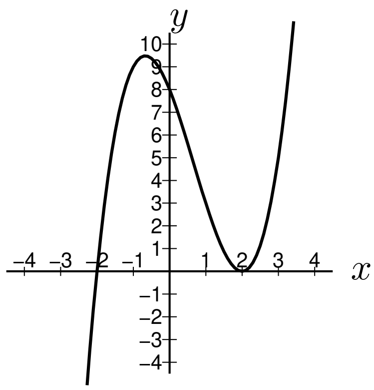
From the standard equation, we can see that the $y-$intercept $(x = 0)$ is: $y = d$

To calculate the $x-$intercepts $(y = 0)$ we solve the factorised form of the equation.

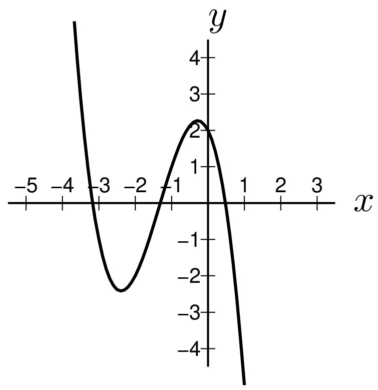
The graph of a cubic function may have 1, 2 or 3 $x-$intercepts, depending on the values of the coefficients.



$y = x^3 + 3x^2 + x + 1$
One $x-$intercepts



$y = x^3 - 2x^2 - 4x + 8$
Two $x-$intercepts



$y = -x^3 - 4x^2 - 2x + 2$
Three $x-$intercepts

The graph of a cubic function may have two turning points. We will use differentiation to find the coordinates of the turning points: i.e. we solve: $\dfrac{dy}{dx} = 0$. Once the values of the $x-$coordinates of the turning points are known, we substitute them into the equation to determine the $y-$coordinates.

Example: Sketch the graph of the cubic function $f(x) = x^3 - 3x$, including the orientation, $x-$ and $y-$intercepts and any stationary points.

Orientation: $a = 1 > 0 \quad \therefore$ 

y intercepts: $x = 0 \quad \therefore \quad y = 0$

x intercepts: $y = 0 \quad \therefore$

$$\begin{aligned} x^3 - 3x &= 0 \\ x(x^2 - 3) &= 0 \\ x(x - \sqrt{3})(x + \sqrt{3}) &= 0 \\ \therefore \quad x &= 0, \pm\sqrt{3} \end{aligned}$$

Turning Points: $f'(x) = 0$

$$\begin{aligned} f'(x) = 3x^2 - 3 &= 3(x^2 - 1) = 0 \\ \therefore \quad x^2 - 1 &= 0 \\ \therefore \quad x &= \pm 1 \end{aligned}$$

When $x = 1$, $y = 1 - 3 = -2$.
When $x = -1$, $y = -1 + 3 = 2$.
Turning points at $(1, -2)$ and $(-1, 2)$.

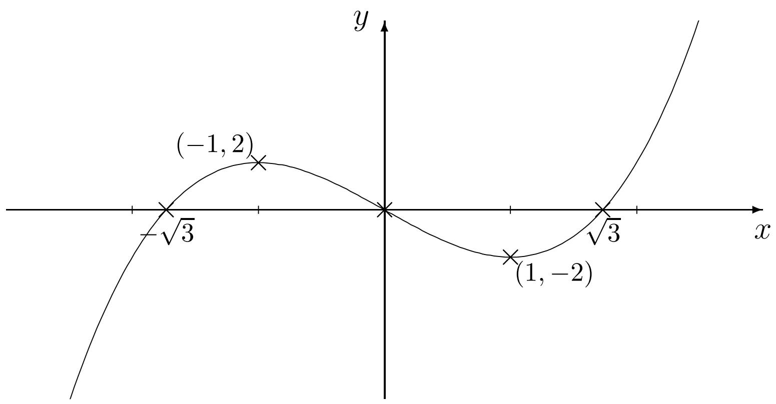
Concavity :

$$\begin{aligned} f'(x) &= 3x^2 - 3 \\ \therefore \quad f''(x) &= 6x \end{aligned}$$

At $x = 1$: $\quad f''(1) = 6 > 0 \quad \therefore$ minimum at $(1, -2)$

At $x = -1$: $\quad f''(x) = 6x$

$\therefore \quad f''(-1) = -6 < 0 \quad \therefore$ maximum at $(-1, 2)$



EXERCISE 34:

Sketch graphs of the following showing all intercepts and turning points including their concavity.

1. $f(x) = x^3 - 6x$ 2. $y = x^3 - x^2 - 8x + 12 = (x+3)(x-2)^2$

Any assessment question is more like that of Q2 & Q3, than Q4 (rev) or Q7(rev).

3. $y = -x^3 - 3x^2 + 20 = (x-2)(x^2 + 5x + 10)$

~~4.~~ $y = x^3 - 7x^2 - 10x - 8 = (x-1)(x-2)(x-4)$

This numbering corresponds to the sample solution given. There is no sample solution to the original Q4, Q5 & Q6 (which are similar the replacement Q4, Q6 & Q7.)

~~5.~~ $y = -x^3 + 2x^2 + 3x$

~~6.~~ $y = x^3 - 4x^2 + 4x$

Find the turning points of the following, but do not sketch the graph:

Q4(rev) x-intercept is not easy to solve; use approximation

4 ~~7.~~ $y = x^3 - 6x^2 + 9x + 1$ 6 ~~9.~~ $y = x^4 - 2x^3$

(Q6(rev) this function has a point of inflexion as
f'(x) = 4x^3 - 6x^2 = x^2(4x-6) = 0 (turn pt)
f"(x) = 12x^2 - 12x = 0 (inflexion pt) @ x = 0)

5 ~~8.~~ $y = 3x^4 - 4x^3 - 12x^2$ 7 ~~10.~~ $y = x^3 - x^2 - x + 4$

Q7(rev) x-intercept is not easy to solve; use approximation

A Answers to Selected Exercises

EXERCISE 1:
1. 17 2. 40 3. 63 4. 218 5. 6103 6. 8 7. 7 8. 47 9. 64
10. 22633 11. -4 12. -15 13. -137 14. -592 15. -3176 16. -2
17. -5 18. -30 19. -37 20. -321 21. -8 22. 4 23. -2 24. 4
25. 0 26. 24 27. 21 28. 0 29. 0 30. -1

EXERCISE 2:
1. 30 2. 12000 3. 135 4. 437 5. 44982 6. 60 7. 720 8. 5040
9. 31980 10. 0 11. -35 12. 60 13. -3840 14. -48 15. -44982 16. 5
17. 767 18. -3 19. 12 20. 2 21. -3 22. 18 23. 18 24. 4 25. -1
26. 20

EXERCISE 3:
1. 20 2. 9 3. -20 4. 15 5. -36 6. -56 7. -48 8. -16 9. -6
10. 20 11. -30 12. 36 13. 91 14. 120 15. 2 16. 24 17. -9 18. -40
19. 150 20. -60

EXERCISE 4:
1. 64 2. 243 3. -8 4. -27 5. -5 6. 12 7. 32 8. 0 9. 4 10. 0
11. 82 12. -40 13. $2+1/8$ 14. 1024 15. -1 16. $4=\log_2 16$ 17. $1=\log_3 3$
18. $3=\log_4 48$ 19. $3=\log_5 125$ 20. $4=\log_{10} 10000$ 21. $5=\log_1 1$ 22. $0=log_3 1$
23. $1=\log_8 8$ 24. $b=\log_a c$ 25. $x=\log_e y$ 26. $2^6=64$ 27. $3^3=27$ 28. $5^4=625$
29. $1^{4568}=1$ 30. $10^3=1000$ 31. $8^0=1$ 32. $12^0=1$ 33. $p^q=r$ 34. $e^x=y$
35. $n^{1/2}=p$

EXERCISE 5:
1. 17 2. 17 3. 27 4. 8 5. 26 6. 12 7. 2 8. 6 9. 48 10. 10
11. 3 12. 1 13. 12 14. 12 15. -1

EXERCISE 6:
1. 2 2. 6 3. 6 4. 18 5. 3 6. 1 7. 72 8. 108 9. 360 10. 360
11. 32 12. 2016

EXERCISE 7:
1. 5/6 2. 5/12 3. 3/2 4. 18/35 5. 31/126 6. 3/4 7. 17/20 8. 5/12
9. 1/16 10. 5 11. -5 12. 20 13. 1/5 14. 20 15. 1/6 16. 1/6
17. 8/27 18. 9/16 19. $-9/16$ 20. 1 21. 16/15 22. 9/8 23. 2 24. 4/15
25. 15 26. 5/4 27. 2/15 28. 4/5 29. $-1/18$ 30. 5/12 31. 63/16
32. $-5/11$ 33. 10/7 34. 27/4 35. 19/30 36. 4/5 37. 5/6 38. 32/9
39. $-5/8$ 40. 1/7

EXERCISE 8:
1. 6.321×10^3 2. 5.8×10^1 3. 4.2×10^0 4. 5×10^{-1} 5. -8.7×10^{-2} 6. 5×10^5
7. 4×10^{-5} 8. 9×10^0 9. 0.0064 10. 49 11. $50\,000$ 12. 3.14 13. 0.732
14. -123 15. 0.00000721 16. 2.7×10^{10} 17. 1.5×10^6 18. 2×10^2 19. 1.6×10^{-5}
20. 2×10^3 21. 4×10^2

EXERCISE 9:
1. $x = 6$ 2. $x = 4$ 3. $p = -4$ 4. $q = 4$ 5. $w = -5$ 6. $y = 6$ 7. $t = -3$
8. $x = 0$ 9. $p = -30$ 10. $x = 2$ 11. $x = 3$ 12. $y = 4$ 13. $t = -5$ 14. $p = 2/5$
15. $q = 4$ 16. $u = 15$ 17. $v = 25$ 18. $w = 12$ 19. $x = 36$ 20. $x = 1/10$
21. $x = 4$ 22. $a = 7$ 23. $y = 5$ 24. $x = -3$ 25. $x = -6$ 26. $p = -40$
27. $z = 8$ 28. $y = -6$ 29. $m = -9/2$ 30. $x = 25/8$

EXERCISE 10:
1. $x = 3$ 2. $p = 1$ 3. $a = 4$ 4. $t = -6$ 5. $u = -2$ 6. $x = 0$ 7. $z = 0$
8. $w = -3$ 9. $q = 1$ 10. $x = -3/2$ 11. $y = 6$ 12. $x = 2$ 13. No Solution
14. $x = -1$ 15. $p = 3$ 16. $a = 7$ 17. $x = 3$ 18. $u = 2/3$ 19. $b = -16$
20. $x = 7$ 21. $x = 2/3$ 22. $x = 12$ 23. $a = 15$ 24. $y = 10/7$ 25. $x = -8$
26. $b = 3$ 27. $x = 5$ 28. $x = -1$ 29. $m = 1/6$ 30. $a = 3$

EXERCISE 11:
1. (a) $x + 6$ (b) $3w + 2$ (c) $5p + 6$ (d) $3(7d + 2)$ (e) $10(7x - 1)$ 2. $P = 2(l + w)$
(a) $P = 8w$ (b) $P = 4w + 10$ (c) $P = 6l - 2$ 3. (a) $x = 16$ (b) $m = -6$ (c) $x = 9$
(d) $x = 6$ 4. 42 5. Pear: $12c$, Apple: $24c$ 6. Length: $12m$, Breadth: $4m$ 7. $3, 12$
8. $14m \times 4m$ 9. A: $40, B: $15, C: $5 10. Walk: $6km/h$ 11. $12m$ 12. 44
13. Walk: $7km/h$

EXERCISE 12:
1. 2^5 2. 3^{15} 3. 1 4. 3^6 5. 4^5 6. $1/12$ 7. 3^2 8. 5 9. $1/2^6$
10. $1/2^6$ 11. $1/4^2$ 12. $-1/2^3$ 13. $-1/2^3$ 14. $1/2$ 15. -3^3 16. $8a^5$
17. $15a^4$ 18. $14a^5b^6$ 19. $8y^6z^{15}$ 20. $5y^6$ 21. $1/3a^3$ 22. $2a^2/3b^3c^3$ 23. $3/x^2$
24. p 25. $c^5/4$ 26. $2x^3y^3$ 27. $3c^2/ab^3$ 28. $xy^6/5$ 29. y/x 30. $2y$
31. $6p^2t^3$ 32. $3x$ 33. b^4c/a^2 34. 2^8 35. y/x^2 36. $y^5/2$ 37. $1/3^3 4^2 ab^3$
38. Proof 39. Proof

EXERCISE 13:
1. 4 2. -5 3. $-3/2$ 4. 2or3 5. -2 ~~6. 1~~

EXERCISE 14:
1. 7 2. 6 3. 5 4. 27 5. $1/27$ 6. -27 7. $1/3$ 8. 8 9. $N.P.$ 10. 8
11. $1/8$ 12. $1/4$ 13. $1/9$ 14. $1/8$ 15. 2 16. $3^{4/3}$ 17. 3 18. 4 19. $3^{1/3}$
20. 4 21. 2 22. 3 23. $2^{1/6}$ 24. 2 25. $1/9$ 26. 1 27. 4

EXERCISE 15:

1. $3x+6$ 2. $10x-15$ 3. $-2x-6$ 4. $-3p-6q$ 5. $-x+2$ 6. $-8y+6$
7. a^2+3a 8. $-x+x^2$ 9. $6x$ 10. $10a+3$ 11. $5x-23$ 12. $-2y-2$
13. $5x+10y$ 14. $-6x^3+5x^2-2x$ 15. $4(x+y)$ 16. $p(3+q)$ 17. $4(x-3y)$
18. $6(a+1)$ 19. $b(a+1)$ 20. $4x(2-3y)$ 21. $x(x-3)$ 22. $3n(3m-n)$
23. $x(y+w-3)$ 24. $3(2x^2-3x+4)$ 25. $-2(x-3)$ 26. $-3x(2+3x)$ 27. $-6a^2(3a+1)$
28. $3m(5n-3m-1)$ 29. $(y+z)(x+3)$ 30. $(2a-1)(3-b)$ 31. $(k+2)(k+1)$
32. $(x-y)(1-z)$ 33. $(x+2)(x-2)$ 34. $(x+3)(x+2)$ 35. $(1-a)^2$ 36. $a(7-5a)$
37. $-4y(2x-y)$ 38. $5q(4p-3r)$ 39. $-3y^2(3+5y)$ 40. $(x+3)(x+5)$ 41. $2(2x-1)(1-x)$
42. 4 43. 1/2 44. $3y/2z$ 45. $4(a+4)/(a+1)$ 46. 3 47. $2a$ 48. $(x-2)/3(x+2)$
49. $(x-2)/(x+2)$ 50. $a+b$ 51. 1/4 52. 3/2 53. $(m+n)/2(m-n)$ 54. 2
55. 4/5 56. $(y+z)/(y-z)$ 57. $(a-1)/a$ 58. $a/3$ 59. $(x+2)/3$ 60. $x+1$
61. $x-3$

EXERCISE 16:

1. $xy+2x+4y+8$ 2. $2xy+4x+3y+6$ 3. $5xz+15x-z-3$ 4. $3ac-ad-6bc+2bd$
5. x^2+4x+3 6. $a^2-7a+10$ 7. p^2-p-6 8. a^2-16 9. $25n^2-4$ 10. x^2+6x+9
11. $4y^2-20y+25$ 12. $x^2-2xy+y^1-1$ 13. x^2+2x+1 14. x^2-6x+9
15. $4^y2+4y+1$ 16. $9w^2-24w+16$ 17. $4x^2+12xy+9y^2$ 18. $p^2+2pq+q^2$ 19. x^2-4
20. x^2-25 21. $4x^2-9$ 22. $a^2b^2-2abcd+c^2d^2$ 23. $4x^2+16x+16$ 24. $9x^2-25$
25. $6a^3+21a^2$ 26. $8x^2+2xy-15y^2$ 27. x^2+2x-8 28. $49x^2-4$ 29. $9x^2-100y^2$
30. $9x^2+60xy+100y^2$ 31. $4x^2-12x+9$ 32. $2x^4+7x^2-15$ 33. $2x^2-13x+15$
34. a^3+3a^2+4a+4 35. $4x^2-12+9/x^2$ 36. $(x-2)(x+2)$ 37. $(x+1)^2$ 38. $(a+3)(a-3)$
39. $(y+2)^2$ 40. $(p+5)(p-5)$ 41. No Factors 42. $(p-3)^2$ 43. $(f+7)(f-7)$
44. $(1+y)(1-y)$ 45. $(x-5)^2$ 46. $(x+y)(x-y)$ 47. $(2x+1)^2$ 48. $(2a-3)(2a+3)$
49. $(1-4x)(1+4x)$ 50. $2(2q-5)^2$ 51. $(3b+4)(3b-4)$ 52. No Factors 53. $(3x-1)^2$
54. $(a-1/3)(a+1/3)$ 55. $(4x-3y)(4x+3y)$ 56. $(6m-7n)(6m+7n)$ 57. $(x-5)(x+1)$
58. $2(c-5)(c+5)$

EXERCISE 17:

1. $(x-3)(x-2)$ 2. $(x+3)(x+4)$ 3. No Factors 4. $(x-7)^2$ 5. $(x-8)(x+8)$
6. $(x+1)(x+2)$ 7. $(x-2)(x-1)$ 8. $(x+1)(x+5)$ 9. $(x-1)(x-5)$
10. $(x-1)(x+7)$ 11. $(x+3)(x+4)$ 12. $(y+2)(y+5)$ 13. $(x+2)(x+3)$
14. $(y-2)(y-5)$ 15. $(a-6)(a-4)$ 16. $(p-2)(p-7)$ 17. $(x-1)(x+2)$
18. $(y+3)(y-2)$ 19. $(b-5)(b+2)$ 20. $(c+2)(c+4)$ 21. $-(d-2)(d-6)$
22. No Factors 23. $(x-2)(x-18)$ 24. $(2x-1)(x+3)$ 25. $(3x+1)(4x-1)$ 26. $(x+5)^2$
27. $(9+y)^2$ 29. $(4x+7y)(4x-7y)$ 29. $2(5x-2)^2$ 30. $(5x-y)(-x+3y)$
31. $(x+1)(2x+3)$ 32. $(2y-1)(3y-2)$ 33. $(3a+1)(a+3)$ 34. $2(2x+1)(3x+1)$
35. $(m-3)^2$ 36. $2(x+5)(x-5)$ 37. $(3-2x)(5+2x)$ 38. $-(x-3)(x-7)$
39. $3x(x-2)$ 40. $(4x-5)^2$ 41. $(2y+1)(y+1)$ 42. $(y-1)(2y+1)$ 43. $(y+1)(2y-1)$
44. No Factors 45. $(3p+4)(3p-5)$ 46. $(3p-4)(3p+5)$ 47. $4(p-1)(4p+5)$

EXERCISE 18:

1. $0,-3$ 2. $2,4$ 3. $-5,-2$ 4. -3 5. $9,-4$ 6. $4,-3$ 7. $3,-3$
8. $2,3$ 9. $0,5$ 10. $0,4$ 11. $-1,-2/3$ 12. No Solutions 13. No Solutions
14. $(-1+\sqrt{5})/2,(-1-\sqrt{5})/2$ 15. No Solutions 16. $1,6$ 17. $-4,3/2$ 18. $-1/2,3$
19. $0,-2$ 20. $-1,3/2$ 21. 10 22. $-6,4$

EXERCISE 19:
1. 5 2. -4 3. 1 4. 8 5. $\log_5(1/2)$ 6. 2 7. 18 8. $\ln(xy^3)$
9. 1 10. $2e^5$ 11. 2 12. $2+\ln 3$

EXERCISE 20:
1. 3 2. 81 3. 3 4. 10 5. 64 6. 243 7. 8 8. $1/8$ 9. 48
10. 5 11. 2 12. $1/4$ 13. $(1/2)e^{2/3}$ 14. 5 15. 25 16. 10 17. -1
18. 1.1 19. $t=(1/4)\ln(y/a)$ 20. $P=Ae^{-bt}$

EXERCISE 21:
1. $w=S-l$ 2. $n=\frac{m}{5}$ 3. $l=\frac{A}{w}$ 4. $r=\frac{C}{2\pi}$ 5. $b=\frac{a+2}{5}$ 6. $x=\frac{y-c}{m}$
7. $w=\frac{P}{2}-l$ or $w=\frac{P-2l}{2}$ 8. $h=\frac{3V}{\pi r^2}$ 9. $d=\frac{S-2a}{n-1}$, $n=1+\frac{S-2a}{d}=\frac{d+S-2a}{d}$
10. $r=\frac{100(A-P)}{Pt}$ or $r=\frac{100}{t}\left(\frac{A}{P}-1\right)$

EXERCISE 22:
1. (a) $D: x\leqslant 0, R: y\in\mathbb{R}$ (b) Function 2. (a) $D: -1\leqslant x\leqslant 1, R: -1\leqslant y\leqslant 1$
(b) Not a Function 3. (a) $D: x\neq 0, R: y\neq 0$ (b) Function 4. (a) $D: x\neq -1, R: y\neq 2$
(b) Function 5. $D_f: x\in\mathbb{R}$ 6. $D_f: x\in\mathbb{R}$ 7. $D_f: x\neq 0$ 8. $D_h: x\neq 1$
9. $D_f: x\leqslant 1$ 10. $D_g: x<2$ 11. $D_f: x\neq 1$ 12. $D_y: x\in\mathbb{R}$ 13. $D_y: x\in\mathbb{R}$
14. $D_y: x\geqslant -3$ 15. $D_y: x\neq 2$ 16. $D_f: x\geqslant 5$ 17. $D_f: x\neq -4, x\neq 4$
18. $D_f: x\in\mathbb{R}$ 19. $D_y: x\neq -1, x\neq 1$ 20. $D_y: 0\leqslant x\leqslant 1$ 21. $D_y: x<-5$ or $x>5$

EXERCISE 23:
1. $y=2x-7$ 2. $y=-x-3$ 3. $5y=2x-3$ 4. $y=-2$
5. $m=2$, $c=3$, $x-$int: $x=-3/2$ 6. $m=2$, $c=-3$, $x-$int: $x=1$
7. $m=1$, $c=0$, $x-$int: $x=0$ 8. $m=-1$, $c=-1$, $x-$int: $x=-3$
9. $m=1/2$, $c=1$, $x-$int: $x=-2$ 10. $m=0$, $c=2$, $x-$int: None
11. $m=-1$, $c=3$, $x-$int: $x=3$ 12. $m=2$, $c=2$, $x-$int: $x=-1$
13. $m=2/3$, $c=-2$, $x-$int: $x=3$

EXERCISE 24:
1. $y=3x-4$ 2. $y=-2x$ 3. $3x+4y=3$ 4. $x+y=5$ 5. $y=3x-2$
6. $y=-4x+5$ 7. $2y=x+4$ 8. $y=-x-1$ 9. $x+3y=6$ 10. $4y=3x+38$
11. $y=2x-12$ 12. $y=2$ 13. $x=-2$ 14. $2x+3y+6=0$ 15. $2y+3x=0$
16. $3y=2x-13$ 17. $y=(2/3)x-2$ 18. $y=-(3/2)x+11$ 19. $y=-(1/2)x+5$
20. $y=3x-2$

EXERCISE 25:
1. $a=1\Rightarrow$ MIN, $y-$int: $y=-3$, $x-$int: $x=-3,1$, Vertex $(x,y)=(-1,-4)$
2. $a=1\Rightarrow$ MIN, $y-$int: $y=-5$, $x-$int: $x=-5,1$, Vertex $(x,y)=(-2,-9)$
3. $a=1\Rightarrow$ MIN, $y-$int: $y=5$, $x-$int: $x=5,1$, Vertex $(x,y)=(3,-4)$
4. $a=1\Rightarrow$ MIN, $y-$int: $y=0$, $x-$int: $x=0,4$, Vertex $(x,y)=(2,-4)$
5. $a=1\Rightarrow$ MIN, $y-$int: $y=-9$, $x-$int: $x=-3,3$, Vertex $(x,y)=(0,-9)$
6. $a=1\Rightarrow$ MIN, $y-$int: $y=0$, $x-$int: $x=0$, Vertex $(x,y)=(0,0)$
7. $a=1\Rightarrow$ MIN, $y-$int: $y=-1$, $x-$int: $x=\pm 1$, Vertex $(x,y)=(0,-1)$
8. $a=1\Rightarrow$ MIN, $y-$int: $y=4$, $x-$int: $x=-2$, Vertex $(x,y)=(-2,0)$
9. $a=1\Rightarrow$ MIN, $y-$int: $y=0$, $x-$int: $x=0,3$, Vertex $(x,y)=(3/2,-9/4)$
10. $a=-1\Rightarrow$ MAX, $y-$int: $y=0$, $x-$int: $x=0,2$, Vertex $(x,y)=(1,1)$

11. $a = 4 \Rightarrow$ MIN, $y-$int: $y = -3$, $x-$int: $x = 1/2, -3/2$, Vertex $(x, y) = (-1/2, -4)$
15 ~~12~~. $7.8125m$ 12 ~~13~~. $a = 1 \Rightarrow$ MIN, $y-$int: $y = 2$, $x-$int: $x = 2 \pm \sqrt{2}$, Vertex $(x, y) = (2, -2)$
13 ~~14~~. $a = 1 \Rightarrow$ MIN, $y-$int: $y = 6$, $x-$int: None, Vertex $(x, y) = (2, 2)$
14 ~~15~~. $a = 1 \Rightarrow$ MIN, $y-$int: $y = 4$, $x-$int: $x = -1, -4$, Vertex $(x, y) = (-5/2, -9/4)$
15. Max height reached is 125/16 = 7.8125m when t = 5/4 = 1.25 sec

EXERCISE 26:
1. $(3, 2)$ 2. $(4, -2)$ 3. $(4, 3)$ 4. $(-3, 2)$ 5. $(3, -2)$ 6. $(6, 3)$ 7. $(7, 3)$
8. $(4, -5)$ 9. $(x, y) = (4, 3)$ 10. $(a, b) = (4, -3)$ 11. $(x, y) = (5, 3)$ 12. 12 and 4
13. 3 @ \$5; 9 @ \$10 14. \$12 (adult) and \$5 (child) 15. $(-1,-1)$ & $(2,2)$
16 ~~15~~. $(x, y) = (4, -3)$. The line is tangent to the circle at this point.

EXERCISE 27:
1. $\pi/2$ 2. π 3. $3\pi/2$ 4. 2π 5. $\pi/4$ 6. $3\pi/4$ 7. $5\pi/4$ 8. $7\pi/4$ 9. $\pi/6$
10. $2\pi/3$ 11. $7\pi/6$ 12. $5\pi/3$ 13. $\pi/3$ 14. $5\pi/6$ 15. $4\pi/3$ 16. $11\pi/6$
17. $90°$ 18. $60°$ 19. $45°$ 20. $30°$ 21. $180°$ 22. $120°$ 23. $135°$ 24. $150°$
25. $270°$ 26. $240°$ 27. $225°$ 28. $210°$ 29. $360°$ 30. $300°$ 31. $315°$ 32. $330°$
33. $3\pi/2$ 34. $5\pi/4$ 35. $5\pi/4$ 36. $3\pi/4$ 37. $\pi/12$ 38. $9\pi/4$

EXERCISE 28:
1. $l = 4, \sin\alpha = \frac{4}{5}, \cos\alpha = \frac{3}{5}, \tan\alpha = \frac{4}{3}, \sin\beta = \frac{3}{5}, \cos\beta = \frac{4}{5}, \tan\beta = \frac{3}{4}$
2. $l = \sqrt{13}, \sin\alpha = \frac{3}{\sqrt{13}}, \cos\alpha = \frac{2}{\sqrt{13}}, \tan\alpha = \frac{3}{2}, \sin\beta = \frac{2}{\sqrt{13}}, \cos\beta = \frac{3}{\sqrt{13}}, \tan\beta = \frac{2}{3}$
3. $l = 4, \sin\alpha = \frac{\sqrt{7}}{4}, \cos\alpha = \frac{3}{4}, \tan\alpha = \frac{\sqrt{7}}{3}, \sin\beta = \frac{3}{4}, \cos\beta = \frac{\sqrt{7}}{4}, \tan\beta = \frac{3}{\sqrt{7}}$
4. $l = \sqrt{41}, \sin\alpha = \frac{5}{\sqrt{41}}, \cos\alpha = \frac{4}{\sqrt{41}}, \tan\alpha = \frac{5}{4}, \sin\beta = \frac{4}{\sqrt{41}}, \cos\beta = \frac{5}{\sqrt{41}}, \tan\beta = \frac{4}{5}$
5. $\cos\theta = \frac{3}{5}, \tan\theta = \frac{4}{3}$ 6. $\sin\theta = \frac{5}{13}, \tan\theta = \frac{5}{12}$ 7. $\sin\theta = \frac{2}{\sqrt{29}}, \cos\theta = \frac{5}{\sqrt{29}}$ 8. $\sqrt{32} = 5.66m$

EXERCISE 29:
1. $x = 5\sqrt{2}, \alpha = \beta = \pi/4$ 2. $\alpha = \pi/3, x = 10, y = 10\sqrt{3}$ 3. $x = y = \sqrt{2}, \alpha = \beta = \pi/4$
4. $x = 20\sqrt{3}, \alpha = \pi/6, \beta = \pi/3$ 5. $8\sqrt{2}cm$ 6. $25m$ 7. $\theta = \pi/6$ 8. $x = 10, y = 20/\sqrt{3}$

EXERCISE 30:
See Solutions

EXERCISE 31:
1. $\frac{dy}{dx} = 0$ 2. $\frac{dy}{dx} = 0$ 3. $\frac{dy}{dx} = 3$ 4. $\frac{dy}{dx} = 1$ 5. $\frac{dy}{dx} = -1$ 6. $\frac{dy}{dx} = -5$ 7. $\frac{dy}{dx} = \pi + 1$
8. $\frac{dy}{dx} = \cos\frac{pi}{4}$ 9. $\frac{dy}{dx} = 2x$ 10. $\frac{dy}{dx} = 1 + 2x + 3x^2 + 4x^3$ 11. $\frac{dy}{dx} = 20x^4 + 12x^3 + 6x^2$
12. $\frac{dy}{dx} = 2 - 9x^2 + x - 5x^4$ 13. $\frac{dy}{dx} = \frac{3}{2}x^2 - 3x^3 - \frac{6}{5}x^2$ 14. $\frac{dy}{dx} = 3 - 5x$
15. $\frac{dy}{dx} = -x^{-2} - 2x^{-3} - 3x^{-4}$ 16. $\frac{dy}{dx} = -\frac{1}{x^2} - \frac{4}{x^3} - \frac{9}{x^4}$ 17. $\frac{dy}{dx} = 15x^2 + \frac{3}{5x^4}$ 18. $\frac{dy}{dx} = \frac{\sqrt{7}}{2\sqrt{x}} + \sqrt{7}$
19. $\frac{dy}{dx} = -3x^{-7/4} + x^{-3}$ 20. $\frac{dy}{dx} = -\frac{1}{4x^{3/2}} - \frac{25}{36x^{11/6}}$
21. $f'(x) = 3x^2 + 4x - 3, f'(0) = -3, f'(1) = 4, f'(2) = 17, f'(-1) = -4$ 22. $y = 10x - 15$
23. $(x, y) = (-3, 7)$ and $(-\frac{1}{3}, -\frac{67}{27})$ 24. $(x, y) = (1, 1)$

EXERCISE 32:
1. $\frac{dy}{dx} = 4x + 3$ 2. $f'(x) = 6x^2 + 6x - 2$ 3. $\frac{dy}{dx} = -30x^4 + 12x^2 + 18x$
4. $f'(x) = 2x^3(x + 2)(3x + 2)$ 5. $\frac{dy}{dx} = 24x^3 - 21x^2 + 2x$ 6. $f'(x) = 4x^3(8x^4 + 1)$
7. $\frac{dy}{dx} = x^{-3}(-8 - 15x^{-3})$ 8. $f'(x) = 1$ 9. $\frac{dy}{dx} = 18x(3x^2 - 1)^2$ 10. $f'(x) = 300x^4(2 - 3x^5)^{19}$
11. $\frac{dy}{dx} = x/\sqrt{x^2 + 2}$ 12. $f'(x) = -62/(x^2 - 5)^4$ 13. $\frac{dy}{dx} = \frac{3x^4+1}{(3x^4-1)^{3/2}}$ 14. $f'(x) = \frac{2-x}{2(1-\sqrt{x})^2}$

12. $f'(x) = \dfrac{-6x}{(x^2-5)^4}$

15. $f'(x) = -\frac{x^2+9}{3x^4}$ 16. $\frac{dy}{dx} = 1$ 17. $\frac{dy}{dx} = -\frac{2x}{(1+x^2)^2}$ 18. $\frac{dy}{dx} = \frac{x-1}{2x\sqrt{x}}$

19. $\frac{dy}{dx} = \frac{1-2x^2}{(1+2x^2)^2}$ 20. $f'(x) = \frac{2+\sqrt{x}}{2(1+\sqrt{x})^2}$

EXERCISE 33:

1. $f'(x) = 2, f''(x) = 0$ 2. $\frac{dy}{dx} = 8x, \frac{d^2y}{dx^2} = 8$ 3. $f'(x) = 6x^2 - 6x + 4, f''(x) = 12x - 6$

4. $\frac{dx}{dt} = 5 - 21t^2, \frac{d^2x}{dt^2} = -42t$ 5. $\frac{dg}{dt} = -\frac{1}{t^2}, \frac{d^2g}{dt^2} = \frac{2}{t^3}$ 6. $\frac{dy}{dx} = -\frac{1}{2x^{3/2}}, \frac{d^2y}{dx^2} = \frac{3}{4x^{5/2}}$

7 ~~8~~. $\frac{dy}{dx} = 12t + 1, \frac{d^2y}{dx^2} = 12$

8 ~~11~~. $f'(x) = \frac{1}{(1+x)^2}, f''(x) = -\frac{2}{(1+x)^3}$ 9 - 12 ~~12 – 17~~. See Solutions

EXERCISE 34:

1– 7 ~~6~~. See Solutions 4 ~~7~~. $(1, 5)$ MAX, $(3, 1)$ MIN 5 ~~8~~. $(-1, -5), (0, 0), (2, -32)$

6 ~~9~~. $(0, 0), (3/2, -27/16)$ 7 ~~10~~. $(1, 3)$ MIN, $(-\frac{1}{3}, \frac{113}{27})$ MAX