MINI-COURSE WARWICK

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This note aims to offer a simple proof of Hrushovski's generalised model theorem for approximate groups when the definable structure consists of all sets. In particular, the proof is model-theory-free and short (4 pages). The proof is essentially that of Krupinski and Pillay [5], but we slightly modified some parts to make it completely elementary.

Nevertheless, I would like to point out that the model-theoretic approach offers deeper insights and stronger conclusions related to the definability of the quasi-model.

1. Models of approximate groups

Recall that

Definition 1.1. *A subset Λ of a group G is an* approximate subgroup *if $e \in \Lambda$, $\Lambda = \Lambda^{-1}$ and there is $F \subset G$ finite such that $\Lambda^2 \subset F\Lambda$.*

If Λ is an approximate subgroup, then $\Lambda^n \subset F^{n-1}\Lambda$ and Λ^n is also an approximate subgroup.

We will talk about a notion of regularity/compactification for approximate groups that has emerged over the last 15 years and has been particularly influential.

1.1. Laminarity.

One of the best-behaved classes of approximate subgroups is the one of neighbourhoods of the identity. The *laminar* approximate subgroups can be modelled out of them.

Definition 1.2. *An approximate subgroup Λ of some group is laminar if there are a locally compact group H and a group homomorphism $f : \langle \Lambda \rangle \to H$ such that:*

- *$f(\Lambda)$ is relatively compact;*
- *$f^{-1}(U)$ for some neighbourhood of the identity U.*

In particular, finite approximate subgroups are always laminar, as are compact approximate subgroups [6].

An influential result of Hrushovski implied that approximate groups of amenable groups are always laminar [2]. Through model-theoretic approaches to combinatorics, this result has had a lasting influence in the study of approximate groups and product growth in groups - it is a key tool in the proof of the Breuillard–Green–Tao structure theorem for approximate groups [1], it has also been used recently to pinpoint minimal product-growth in $SO_3(\mathbb{R})$ [4].

It is an observation from [6] that the cut-and-project construction and laminarity are intertwined. Namely:

Proposition 1.3. *Let Λ be an approximate lattice. Then Λ is laminar if and only if it contains a model set.*

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1.2. **Non-laminarity and generalised models.** As many examples of approximate lattices turn out to be laminar, it is a fair question whether all are. If this were the case, the theory would be much more smooth.

It was, however, noticed relatively early that this cannot be the case. Indeed, using the notion of quasi-morphisms, one can construct approximate groups that are not laminar:

Lemma 1.4. *Let $f : \Gamma \to \mathbb{R}$ be a quasi-morphism not within bounded distance of a group homomorphism. Then $f^{-1}([-R; R])$ is an approximate subgroup that is not laminar.*

Hrushovski then realised in [3] that this could lead to non-laminar approximate lattices.

Lemma 1.5. *Let $\Gamma \subset SL_2(\mathbb{R})$ be a lattice and let $f : \Gamma \to \mathbb{Z}$ be a quasimorphism not within bounded distance of a group homomorphism. Let $X \subset \mathbb{R}$ be any approximate lattice that is not commensurable to a lattice (say, the Penrose tiling (P3)). Then*

$$\Lambda_f := \{(\gamma, f(\gamma) + x) \in SL_2(\mathbb{R}) \times \mathbb{R} : \gamma \in \Gamma, x \in X\}$$

is ***not*** *obtained from the cut-and-project construction.*

While this seems to provide an abundance of counter-examples to extensions of Hrushovski's result or Meyer's result - in particular, showing a stark contrast between amenable and non-amenable groups - in a further breakthrough, Hrushovski showed that these are the "most complicated examples":

Theorem 1.6 (Hrushovski, [3])**.** *Let Λ be an approximate lattice, then there is locally compact group H, a quasi-homomorphism $f : \langle \Lambda \rangle \to H$ with defect K a compact normal subset such that:*

1. *$f(\Lambda)$ is relatively compact;*
2. *there is a neighbourhood of the identity $W \subset H$ such that $f^{-1}(WK) \subset \Lambda^{12}$.*

Such a quasi-homomorphism f is sometimes called a *generalised model.*

2. Hrushovski's generalised model theorem after Krupinski–Pillay

Hrushovski's approach, rooted in model theory, introduces an insightful perspective through his pattern theory. While intriguing connections exist to the notion of patterns explored in the mini-course, a full explanation requires more time. Instead, we will adopt a dynamical approach, examining the topological flow on Stone-Cech spaces via convolution. This method is essentially that of Krupinski and Pillay's proof [5] but simplifies their approach at the expense of relaxing the definability conditions for the generalised model. In particular, no model theory is needed here.

2.1. **Semigroup structure on the Stone-Cech compactification.** From now on, fix an approximate subgroup Λ generating a group Γ. For some $X \subset \Gamma$, define the Stone–Cech space $S(X)$ as the space of all these ultrafilters of X. Recall that a ultrafilter a is a maximal subset of $\mathcal{P}(X)$ such that:

- $\forall A, B \in a$, $A \cap B \in a$.
- $\forall A \subset B$, if $A \in a$ then $B \in a$.

Equivalently, it is a finitely additive probability law on X taking values in $\{0; 1\}$. Moreover, one can define a topology from the basic open sets $\{a \in S(X) : A \in a\}$ for all $A \subset X$. With a slight abuse of notation, we will denote these subsets $S(A)$. This topology makes $S(X)$ into a totally disconnected compact Hausdorff space, and every clopen set is a basic open set.

Write $S_{\Lambda,\Gamma} := \bigcup_{m\geq 0} S(\Lambda^m)$. Since $e \in \Lambda$, $\Lambda^m \subset \Lambda^{m+1}$. And so we can equip $S_{\lambda,\Gamma}$ with the inductive limit topology. As such, it is a locally compact space. There is, moreover, a natural continuous map $S_{\Lambda,\Gamma} \to S(\Gamma)$ that sends an ultrafiltre of Λ^m to one of Γ.

It is well known that $S(\Gamma)$ is equipped with continuous actions on both sides by Γ. Since for every element $\gamma \in \Gamma$, there is $m \geq 0$ such that $\gamma \in \Lambda^m$, we have that Γ also acts continuously on both sides on $S_{\Lambda,\Gamma}$.

It is well-known that $S(\Gamma)$ admits a natural semi-group operation:

Definition 2.1. *The* convolution *of two ultrafilters $a, b \in S(\Lambda)$ is*

$$A \in a * b \Leftrightarrow \{\gamma \in \Gamma | \gamma^{-1}A \in a\} \in b.$$

This semigroup operation is left-continuous (but never right-continuous). The name convolution comes from another interpretation of the Stone-Cech space. In fact, an ultrafilter can be seen as a finitely additive probability measure with values in $\{0; 1\}$. The convolution of ultrafilters is then simply the convolution of measures. In other words, if X and Y are two independent random variables with law a and b respectively, the law of XY is $a * b$.

But from this interpretation, it is clear that:

Lemma 2.2. (1) *So $a \in S(\Lambda^m), b \in S(\Lambda^n)$ yields that $a * b \in S(\Lambda^{n+m})$;*
(2) *Similarly, if $a * b \in S(\Lambda^m)$ and $b \in S(\Lambda^n)$, then $a \in S(\Lambda^{n+m})$.*

Indeed, for (2) if X takes values in Λ^n almost surely and XY takes values in Λ^m almost surely, then Y takes values in Λ^{n+m} almost surely. And a similar argument works for (1).

While (1) implies that the convolution operation is also well-defined (and continuous) on $S_{\Lambda,\Gamma}$, combined with (2) it implies:

Corollary 2.3. *For all $b \in S_{\Lambda,\Gamma}$, the map $a \mapsto a * b$ is continuous and proper.*

The two actions of Γ can also be reinterpreted in terms of convolution. Indeed, there is an injection $\iota : \Gamma \to S_{\Lambda,\Gamma}$ sending an element to the prinDirac ultrafilter associated (equivalently, the dirac mass at this element). We denote the image of γ by γ. Then $\gamma * a$ (resp. $a * \gamma$) corresponds to the left action (resp. right-action).

So, we are one step closer to the result. We have found a semigroup $S_{\Lambda,\Gamma}$ with a locally compact topology and a semigroup homomorphism ι. The issue is that semigroups with locally compact topology are perhaps not as nice as locally compact groups. In addition, we have *yet* to use the assumption that Λ is an approximate subgroup.

2.2. **Finding a subgroup via idempotents.** We will now apply an extension of the Ellis–Numakura lemma to find suitable subgroup structures in $S_{\Lambda,\Gamma}$. The Ellis–Numakura lemma provides idempotents (which will be the identity element of the group) and minimal ideals (which will be the base space of the group). But what does this lemma say?

Lemma 2.4 (Ellis–Numakura)**.** *Let S be a semigroup that is a locally compact space and such that the group operation is left-continuous and proper. Suppose there is a compact subset $K \subset S$ such that for every $s \in S$ there is $r \in S$ with $rs \in K$. Then S has a minimal ideal M and an idempotent element $u \in M$.*

The proof of this lemma is the proof of the original Ellis–Numakura verbatim.

Proof. Because of the assumption, any non-empty ideal intersects K. By Zorn's lemma, there is therefore a minimal one for inclusion, call it M, and it intersects K. Now, for any $m \in M$, Mm is closed (by properness) and an ideal. So $Sm = M$ by minimality.

Now, for any $m \in M$ the set $L = \{n \in M | nm = m\}$ is compact (by properness) and non-empty. But it is also a subsemigroup. There is K a minimal compact subsemigroup in L by Zorn's lemma. Take $u \in K$, and $K' := \{k \in K : ku = u\}$. Then K' is non-empty because $Ku = K$ by minimality and a compact subsemigroup. So $K = K'$. In particular, $u^2 = u$. □

We are now ready to construct the quasi-homomorphism f. Let us first pinpoint the target of the quasi-homomorphism. We can apply the Ellis-Numakura lemma to $S_{\Lambda,\Gamma}$. This provides an ideal M and an idempotent u. We claim that uM is a subgroup with identity element u.

Proof of the claim. That u is a left-inverse is clear. Indeed, for all $m \in u * M$, $M * m = M$. So any m has $n \in M$ such that $n * m = u$. Then $u * n * m = u^{*2} = u$. So m has a left-inverse. Thus it is a group. □

2.3. **The quasi-homomorphism.** Now that we have identified the target group, let us identify the quasi-homomorphism. Since we already have a homomorphism $\iota : \Gamma \to S_{\Lambda,\Gamma}$ it is particularly natural to look for a type of projection $S_{\Lambda,\Gamma} \to u * M$. Perhaps the most naive choice is $a \mapsto u * a * m$ for any $m \in M$. But since $u \in M$, one might as well choose $m = u$. So our projection is

$$\begin{aligned} f : \Gamma &\longrightarrow u * M \\ \gamma &\longmapsto u * \gamma * u. \end{aligned}$$

Why would it behave nicely with respect to the product? Well,

$$\begin{aligned} f(\gamma_1) * f(\gamma_2) &= (u * \gamma_1 * u) * (u * \gamma_2 * u) \\ &= u * \gamma_1 * u * \gamma_2 * u \\ &= u * u^{\gamma_1^{-1}} * u * \gamma_1\gamma_2 * u = u * u^{\gamma_1^{-1}} * u * f(\gamma_1\gamma_2) \end{aligned}$$

where $u^{\gamma_1^{-1}} = \delta_{\gamma_1^{-1}} * u * \delta_{\gamma_1}$.

So the defect D is contained in the set $\{u * u^{\gamma} * u | \gamma \in \Gamma\} \subset uM$. We are done if we can show that it is contained in a compact normal subset. But each u^{γ} is also an idempotent. So if the set of idempotents is relatively compact, then D is relatively compact.

Proof that the defect is relatively compact. Let u be an idempotent and X, Y to independent random variables with law u. Then $Z := XY$ has law $u * u = u$. Since $u \in S_{\Lambda,\Gamma}$ there is $m \geq 0$ such that $u \in S(\Lambda^m)$. As $\Lambda^m \subset F\Lambda$ for some finite subset F, there is $f \in F$ such that both Z and X take values in $f\Lambda$ with probability > 0. But u is a finitely additive probability law taking $\{0; 1\}$ values, so $Z, X \in f\Lambda$ almost surely. So, almost surely $Y = X^{-1}Z \in (f\Lambda)^{-1}(f\Lambda) = \Lambda^2$. In other words, all idempotents are contained in $S(\Lambda^2)$. So $D \subset S(\Lambda^6)$. □

Corollary 2.5. *Because Γ acts co-compactly on $S_{\Lambda,\Gamma}$ and the defect is compact, $f(\Gamma)$ is relatively dense in $u * M$.*

It remains to be proven that the defect is normal. Since conjugation by Dirac masses sends idempotent to idempotents, it is already clear that the defect is normal for this action.

In general, formalising proofs using the notion of random variables is delicate. This is because the probability measures we consider are *finitely* additive, and product measures are not well defined. Model theory offers a beautiful approach to this, but this is beyond the scope of this note. But all we need - and have used so far - is a weak notion of inversion. We do not need random variables for this.

Definition 2.6. *For $a, b \in S_{\Lambda,\Gamma}$ define $a \bullet b = \bigcap_{A \in a, B \in b} S(AB)$ and $a^{\bullet -1} := \{A^{-1} | A \in a\}$.*

Note that $a \bullet b$ is a singleton if and only if a and b are Dirac masses. While $a * b$ is the product of independent variables, $a \bullet b$ can be understood as the union of all products of random variables with law a and b respectively, and all possible joint laws. More generally, $a \bullet b$ is much more naturally a product on closed subsets of $S_{\Lambda,\Gamma}$, but we will not need that.

The only feature we need is the weak inversion formula $\bullet$ provides:

Lemma 2.7. *For $a, b \in S_{\Lambda,\Gamma}$, $a \in (a * b) \bullet b^{\bullet -1}$.*

Proof. This is clear from the random variable interpretation. From the filter point of view, we prove it as follows. If $A \in a * b$ and $B \in b$, then $\{\gamma \in \Gamma | A\gamma^{-1} \in a\} \in b$. But $A\{\gamma \in \Gamma | A\gamma^{-1} \in a\} \in a$ and $B \cap \{\gamma \in \Gamma | A\gamma^{-1} \in a\} \in b$. So $AB \supset A\left(B \cap \{\gamma \in \Gamma | A\gamma^{-1} \in a\}\right)$ i.e. $AB \in a$. So $a \in \bigcap_{A \in a*b, B \in b} S(AB) = (a * b) \bullet b^{\bullet -1}$. □

Moreover:

Fact 2.8. *For all $a \in S_{\Lambda,\Gamma}$, $a \bullet a^{\bullet -1} \subset S(\Lambda^2)$. Hence, $u \subset S(\Lambda^2)$*

Proof. We have already seen that $\Lambda f \in a$ for some $f \in \Gamma$ (equivalently, Λf has probability 1). So $a \bullet a^{\bullet -1} \subset S(\Lambda f (f\Lambda)) = S(\Lambda^2)$.

We have $u \subset (u * u) \bullet u^{\bullet -1} = u \bullet u^{\bullet -1} \subset S(\Lambda^2)$. □

Remark 2.9. *I encourage the reader to rephrase the following proof in an informal way using the intuition of random variables.*

Proof that the defect is contained in a compact normal subset. Take $\gamma, \lambda \in \Gamma$ and let $u * u^\gamma * u \in D$. Then

$$(u * u^\gamma * u)(u * \lambda * u) = u * \lambda * \left(u^{\lambda^{-1}} * u^{\gamma\lambda^{-1}} * u^{\lambda^{-1}} * u\right). \tag{1}$$

Since $u^{\lambda^{-1}}$ and $u^{\gamma\lambda^{-1}}$ are also idempotents, $u^{\lambda^{-1}} * u^{\gamma\lambda^{-1}} * u^{\lambda^{-1}} * u \in S(\Lambda^8)$. If we could write,

$$u * \lambda * \left(u^{\lambda^{-1}} * u^{\gamma\lambda^{-1}} * u^{\lambda^{-1}} * u\right) = u * \lambda * u * \left(u^{-1} * u^{\lambda^{-1}} * u^{\gamma\lambda^{-1}} * u^{\lambda^{-1}} * u\right)$$

for a suitable notion of u^{-1} we would be done as $u^{\lambda^{-1}} * u^{\gamma\lambda^{-1}} * u^{\lambda^{-1}} * u \in S(\Lambda^8)$. Thanks to the weak inversion formula:

Fact 2.10. $u * \lambda \in u * \lambda * u * S(\Lambda^2)$

Proof of the fact. This is essentially the same proof as Fact 2.10. Indeed,

$$(u * \lambda * u)^{-1} * u * \lambda \in \left((u * \lambda * u)^{-1} * u * \lambda * u\right) \bullet u^{\bullet -1} = u \bullet u^{\bullet -1} \subset S(\Lambda^2).$$

□

Finally, from (1) and Fact 2.10,

$$(u * u^\gamma * u)(u * \lambda * u) \in (u * \lambda * u) * S(\Lambda^{10}).$$

In other words, D is contained in a compact normal subset of $S(\Lambda^{10})$.

□

2.4. **Correcting the topology.** In the previous section we constructed a quasi-homomorphism to a group that happens to be a locally compact space. However, because $*$ is not right-semi-continuous, the group is not locally compact. We will make the topology coarser to resolve this issue. We will leave some details to the reader, see [7, App. A] for more.

Definition 2.11. *Let $a \in S_{\Lambda,\Gamma}$ and $A \subset S_{\Lambda,\Gamma}$. Define $a \circ A$ as the set of all limits $\lim g_i * p_i$ where $(g_i)_{i\in I}$ is a net of elements of Γ converging to a and $(p_i)_{i\in I}$ is a net of elements of A.*
Let $A \subset u * M$, then $cl_\tau(A) := u \circ A \cap u * M$.

From the definition, it is clear that $cl_\tau(A)$ is closed in the topology inherited from $S_{\Lambda,\Gamma}$. In addition, the definition of convolution implies the striking formula $a \circ (b \circ A) = (ab) \circ A$ and from left-continuity $aA \subset a * A$.

Fact 2.12. (1) *if $A \subset u * M \cap S(\Lambda^m)$, then $cl_\tau(A) \subset S(\Lambda^{m+2})$;*
(2) *if $A \subset u * M \setminus S(\Lambda^m)$, then $cl_\tau(A) \subset u * M \setminus S(\Lambda^{m-2})$.*

Indeed, if $(g_i)_{i\in I}$ is a net converging to $u \in S(\Lambda^2)$, then $g_i \in \Lambda^2$ from a certain point on. This observation, combined with $S(\Lambda^m)$ being clopen for all $m \geq 0$, concludes.
As a consequence:

Corollary 2.13. *Define the τ-topology as the topology generated by the family of closed sets $cl_\tau(A)$ on $u * M$.*
The group $u*M$ equipped with the τ-topology is also a locally compact space (but potentially not Hausdorff).

As it turns out, this topology makes the group operation on $u * M$ separately continuous.

Lemma 2.14. *The group operation on $u * M$ is separately continuous for the τ-topology.*

Proof. From left-continuity and properness of $*$, we have for all $A \subset uM$ and $b \in uM$,

$$cl_\tau(A) * b = cl_\tau(Ab).$$

This also proves left-continuity of $*$ in the τ-topology.
The τ-topology is designed to make right-continuity of $a * \bullet$ for $a \in uM$. If B is any closed subset for the τ-topology, $a \circ B \cap uM$ then $a^{-1} * (a \circ B \cap uM) \subset u \circ B \cap uM = B$. So $a \circ B \cap uM \subset a * B$. Then:

$$a * B \subset cl_\tau(a * B) \subset u \circ (a * B) \cap uM \subset ua \circ B \cap uM \subset a \circ B \cap uM \subset a * B$$

which can be established easily from the definition of the τ-topology. So $a * B = cl_\tau(a * B)$.

2.5. **Conclusion.** The map

$$\begin{aligned} f : \Gamma &\longrightarrow u * M \\ \gamma &\longmapsto u * \gamma * u. \end{aligned}$$

has defect contained in a normal subset of $S(\Lambda^{10})$, hence compact in the τ-topology. Moreover, uM is locally compact with $S(\Lambda^5)$ a relatively compact neighbourhood of u (Fact 2.12). But

$$f^{-1}(S(\Lambda^5) * D) \subset f^{-1}(S(\Lambda^{15})) \subset \Lambda^{20}$$

by Lemma 2.2. Finally, uM is not Hausdorff, but $cl_\tau(\{u\}) \subset S(\Lambda^4)$ so we can quotient it out if we want a Hausdorff target.

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