

高等数学公式汇总

第一章 一元函数的极限与连续

1、常用初等函数公式：

和差角公式：

$$\begin{aligned}\sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\ \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta\end{aligned}$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \cdot \tan \beta}$$

$$\cot(\alpha \pm \beta) = \frac{\cot \alpha \cdot \cot \beta \mp 1}{\cot \beta \pm \cot \alpha}$$

$$sh(\alpha \pm \beta) = sh\alpha ch\beta \pm ch\alpha sh\beta$$

$$ch(\alpha \pm \beta) = ch\alpha ch\beta \pm sh\alpha sh\beta$$

和差化积公式：

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \alpha - \cos \beta = 2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

积化和差公式：

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha + \beta) - \cos(\alpha - \beta)]$$

倍角公式：

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

$$= 1 - 2 \sin^2 \alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

$$\cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha}$$

$$sh 2\alpha = 2 sh \alpha ch \alpha$$

$$ch 2\alpha = 1 + 2 sh^2 \alpha =$$

$$= 2 ch^2 \alpha - 1 = ch^2 \alpha + sh^2 \alpha$$

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$$\sin^2 \alpha + \cos^2 \alpha = 1; \tan^2 x + 1 = \sec^2 x;$$

$$\cot^2 x + 1 = \csc^2 x; ch^2 x - sh^2 x = 1$$

半角公式：

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1-\cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1+\cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1-\cos \alpha}{1+\cos \alpha}} = \frac{1-\cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1+\cos \alpha}$$

$$\cot \frac{\alpha}{2} = \pm \sqrt{\frac{1+\cos \alpha}{1-\cos \alpha}} = \frac{1+\cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1-\cos \alpha}$$

$$\text{双曲正弦: } shx = \frac{e^x - e^{-x}}{2}; \text{ 反双曲正弦: } arshx = \ln(x + \sqrt{x^2 + 1})$$

$$\text{双曲余弦: } chx = \frac{e^x + e^{-x}}{2}; \text{ 反双曲余弦: } archx = \pm \ln(x + \sqrt{x^2 - 1})$$

$$\text{双曲正切: } thx = \frac{shx}{chx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \text{ 反双曲正切: } arthx = \frac{1}{2} \ln \frac{1+x}{1-x}$$

$$(a^3 \pm b^3) = (a \pm b)(a^2 mab + b^2), \quad 1_2 + 2_2 + L + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1_3 + 2_3 + L + n^3 = \frac{n^2(n+1)^2}{4}$$

2、极限

➤ 常用极限: $|q| < 1, \lim_{n \rightarrow \infty} q^n = 0; a > 1, \lim_{n \rightarrow \infty} \sqrt[n]{a} = 1; \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$

➤ 若 $f(x) \rightarrow 0, g(x) \rightarrow \infty$, 则 $\lim [1 \pm f(x)]^{g(x)} = e^{\lim \frac{\ln(1+f(x))}{1/g(x)}} = e^{\lim (1+f(x)) \cdot f(x)} \rightarrow e^{\pm \lim [f(x)g(x)]}$

➤ 两个重要极限

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0; \lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e = \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}}$$

➤ 常用等价无穷小:

$$1 - \cos x \sim \frac{1}{2}x^2; x \sim \sin x \sim \arcsin x \sim \arctan x; \sqrt{1+x} - 1 \sim \frac{1}{2}x;$$

$$a^x - 1 \sim x \ln a; e^x \sim x + 1; (1+x)^a \sim 1 + ax; \ln(1+x) \sim x$$

3、连续：

定义： $\lim_{\Delta x \rightarrow 0} \Delta y = 0; \lim_{x \rightarrow x_0} f(x) = f(x_0)$

极限存在 $\Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0^+} f(x)$ 或 $f(x_-) = f(x_+)$

第二章 导数与微分

基本导数公式：

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \tan \alpha$$

导数存在 $\Leftrightarrow f'_{-}(x_0) = f'_{+}(x_0)$

$C' = 0; (x^a)' = ax^{a-1}; (\sin x)' = \cos x; (\cos x)' = -\sin x; (\tan x)' = \sec^2 x; (\cot x)' = -\csc^2 x;$
 $(\sec x)' = \sec x \cdot \tan x; (\csc x)' = -\csc x \cdot \cot x; (a^x)' = a^x \ln a; (e^x)' = e^x;$

$$(\log_a |x|)' = \frac{1}{x \ln a}; (\ln |x|)' = \frac{1}{x}; (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}; (\arccos x)' = -\frac{1}{\sqrt{1-x^2}};$$

$$(\arctan x)' = \frac{1}{1+x^2}; (\operatorname{arc cot} x)' = -\frac{1}{1+x^2}; (shx)' = hx; (chx)' = shx;$$

$$(thx)' = \frac{1}{ch^2 x}; (arshx)' = \frac{1}{\sqrt{1+x^2}}; (archx)' = \frac{1}{\sqrt{x^2-1}}; (arthx)' = \frac{1}{x^2-1}$$

2、高阶导数：

$$(x^n)^{(k)} = \frac{n!}{(n-k)!} x^{n-k} \Rightarrow (x^n)^{(n)} = n!; (a^x)^{(n)} = a^x \ln^n a \Rightarrow (e^x)^{(n)} = e^x$$

$$\left(\frac{1}{x}\right)^{(n)} = \frac{(-1)^n n!}{x^{n+1}}; \left(\frac{1}{x+a}\right)^{(n)} = \frac{(-1)^n n!}{(x+a)^{n+1}}; \left(\frac{1}{a-x}\right)^{(n)} = \frac{n!}{(a-x)^{n+1}}$$

$$(\sin kx)^{(n)} = k^n \cdot \sin(kx + n \cdot \frac{\pi}{2}); (\cos kx)^{(n)} = k^n \cdot \cos(kx + n \cdot \frac{\pi}{2});$$

$$[\ln(a+x)]^{(n)} = (-1)^{n-1} \frac{(n-1)!}{(a+x)^n} \Rightarrow [\ln(x)]^{(n)} = \left(\frac{1}{x}\right)^{(n-1)} = (-1)^{n-1} \frac{(n-1)!}{x^n}$$

牛顿-莱布尼兹公式：

$$(uv)^{(n)} = \sum_{k=0}^n C_k u^{(n-k)} v^{(k)} \\ = u^{(n)}v + n u^{(n-1)}v' + \frac{n(n-1)}{2!} u^{(n-2)}v'' + \dots + \frac{n(n-1)L(n-k+1)}{k!} u^{(n-k)}v^{(k)} + \dots + uv^{(n)}$$

3、微分：

$$\Delta y = f(x + \Delta x) - f(x) = dy + o(\Delta x); dy = f'(x_0)\Delta x = f'(x)dx;$$

连续 \Rightarrow 极限存在 \Leftrightarrow 收敛 \Rightarrow 有界； 可微 \Leftrightarrow 可导 \Leftrightarrow 左导=右导 \Rightarrow 连续；

不连续 \Rightarrow 不可导

第三章 微分中值 定理与微分的应用**基本定理**

拉格朗日中值定理: $f(b) - f(a) = f'(\xi)(b - a), \xi \in (a, b)$

柯西中值定理: $\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(\xi)}{F'(\xi)}, \xi \in (a, b)$

当 $F(x) = x$ 时，柯西中值定理就是拉格朗日中值定理。

2、

泰勒公式: $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$

余项: $R_n(x) = \begin{cases} o((x - x_0)^n) \\ \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}(x - x_0)^{n+1}; (\xi \in (x_0, x), \theta \in (0, 1)) \end{cases}$

麦克劳林公式: $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}; (\theta \in (0, 1))$

常用初等函数的展式:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + R_n(x); R_n(x) = \frac{e^{\theta x}}{(n+1)!}x^{n+1}; (\theta \in (0, 1))$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - L + (-1)^{m-1} \frac{x^{2m-1}}{(2m-1)!} + R_{2m}(x); R_{2m}(x) = \frac{\sin[\theta x + (2m+1)\frac{\pi}{2}]}{(2m+1)!} x^{2m+1}; (\theta \in (0,1))$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - L + (-1)^m \frac{x^{2m}}{(2m)!} + R_{2m+1}(x); R_{2m+1}(x) = \frac{\cos[\theta x + (m+1)\pi]}{(2m+2)!} x^{2m+2}; (\theta \in (0,1))$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^4}{3!} - L + (-1)^{n-1} \frac{x^n}{n} + R_n(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1};$$

$$R_n(x) = \frac{(-1)^n}{(n+1)(1+\theta x)^{n+1}} x^{n+1}; (\theta \in (0,1))$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + L + \frac{\alpha(\alpha-1)L(\alpha-n+1)}{n!} x^n + R_n(x);$$

$$R_n(x) = \frac{\alpha(\alpha-1)L(\alpha-n)}{(n+1)!} (1+\theta x)^{\alpha-n-1} x^{n+1}; (\theta \in (0,1))$$

$$\Rightarrow \frac{1}{1+x} = \ln'(1+x) = 1 - x + x^2 + L + (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

3、

$$\text{弧微分公式: } ds = \sqrt{1+y'^2} dx = \sqrt{x'(t)+y'(t)^2} dt = \sqrt{\rho^2 + \rho'^2} d\theta$$

平均曲率: $\bar{K} = \left| \frac{\Delta \alpha}{\Delta s} \right|$. ($\Delta \alpha$: 从M点到M'点, 切线斜率的倾角变化量; Δs : MM'弧长)

$$\text{M点的曲率: } K = \lim_{\Delta s \rightarrow 0} \left| \frac{\Delta \alpha}{\Delta s} \right| = \left| \frac{d\alpha}{ds} \right| = \frac{|y''|}{\sqrt{(1+y'^2)^3}} = \frac{|\varphi'(t)\psi''(t) - \varphi''(t)\psi'(t)|}{[\varphi'^2(t) + \psi'^2(t)]^{\frac{3}{2}}}.$$

直线的曲率: $K = 0$; 半径为R的圆的曲率: $K = \frac{1}{R}$.

$$\text{曲线在点M处的曲率半径: } \rho = \frac{1}{K} = \frac{\sqrt{(1+y'^2)^3}}{|y''|}$$

第四章 不定积分

常用不定积分公式：

$$\int f(x)dx = F(x) + C; \quad (\int f(x)dx)' = f(x); \quad \int F'(x)dx = F(x) + C$$

$$\int x^{\mu} dx = \frac{x^{\mu+1}}{\mu+1} + C (\mu \neq -1); \quad \int \frac{1}{x} dx = \ln|x| + C;$$

$$\int a^x dx = \frac{a^x}{\ln a} + C; \quad \int e^x dx = e^x + C;$$

$$\int \sin x dx = -\cos x + C; \quad \int \cos x dx = \sin x + C;$$

$$\int \tan x dx = -\ln|\cos x| + C; \quad \int \cot x dx = \ln|\sin x| + C;$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C;$$

$$\int \csc x dx = \ln|\csc x - \cot x| + C = \ln\left|\tan\frac{x}{2}\right| + C = -\ln|\csc x + \cot x| + C;$$

$$\int \sec^2 x dx = \int \frac{dx}{\cos^2 x} = \tan x + C; \quad \int \csc^2 x dx = \int \frac{dx}{\sin^2 x} = -\cot x + C;$$

$$\int \sec x \cdot \tan x dx = \sec x + C; \quad \int \csc x \cdot \cot x dx = -\csc x + C;$$

$$\int shx dx = chx + C; \quad \int chx dx = shx + C;$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C = -\arccos x + C; \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C;$$

$$\int \frac{dx}{1+x^2} = \arctan x + C = -\operatorname{arccot} x + C; \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan \frac{x}{a} + C;$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C; \quad \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \frac{a+x}{a-x} + C;$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2}) + C;$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{x}{2} \sqrt{x^2 \pm a^2} + \frac{a^2}{2} \ln(x + \sqrt{x^2 \pm a^2}) + C;$$

$$\int \sqrt{a^2-x^2} dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C$$

2、常用凑微分公式：

$$\begin{aligned}\frac{dx}{\sqrt{x}} &= 2d\sqrt{x}; \quad \frac{dx}{x^2} = -d\left(\frac{1}{x}\right); \quad \frac{dx}{x} = d(\ln x); \\ \frac{xdx}{\sqrt{1+x^2}} &= d(\sqrt{1+x^2}); \quad \left(1-\frac{1}{x^2}\right)dx = d\left(x+\frac{1}{x}\right) \\ \frac{dx}{\cos x \sin x} &= d(\ln \tan x);\end{aligned}$$

3、有特殊技巧的积分

$$(1) \int \frac{dx}{a \sin x + b \cos x} = \frac{1}{\sqrt{a^2 + b^2}} \int \frac{1}{\sin(x + \varphi)} dx$$

$$(2) \int \frac{c \sin x + d \cos x}{a \sin x + b \cos x} dx = Ax + B \ln |a \sin x + b \cos x| + C$$

$$(3) \int \frac{x^2 + 1}{x^4 + 1} dx = \int \frac{1}{(x - \frac{1}{x})^2 + (\sqrt{2})^2} d\left(x - \frac{1}{x}\right)$$

第五章 定积分

1、基本概念

$$\int_a^b f(x) dx = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{i}{n}\right) \frac{1}{n} = F(b) - F(a) = F(x) \Big|_a^b, \quad (F'(x) = f(x))$$

连续 \Rightarrow 可积；有界+有限个间断点 \Rightarrow 可积；

可积 \Rightarrow 有界；连续 \Rightarrow 原函数存在

$$\Phi(x) = \int_a^x f(t) dt \Rightarrow \Phi'(x) = f(x)$$

$$\frac{d}{dx} \int_{\psi(x)}^{\varphi(x)} f(t) dt = f[\varphi(x)]\varphi'(x) - f[\psi(x)]\psi'(x)$$

$$\int_b^a f(x) dx = \int_{\beta}^{\alpha} f(\varphi(t))\varphi'(t) dt, \quad \int_b^a u(x) dv(x) = u(x)v(x) - \int_b^a v(x) du(x)$$

2、常用定积分公式：

$$\int_{-a}^a f(x)dx = \int_0^a [f(x) + f(-x)]dx ;$$

$f(x)$ 为偶函数, $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$; $f(x)$ 为奇函数, $\int_{-a}^a f(x)dx = 0$

$$\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx; \quad \int_0^{\frac{\pi}{2}} xf(\sin x)dx = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} f(\sin x)dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x)dx$$

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx; \quad \int_a^{a+nT} f(x)dx = n \int_0^T f(x)dx$$

Wallis 公式 :

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{n-1}{n} I_{n-2} = \begin{cases} \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} L \cdot \frac{n-3}{n-2} \cdot \frac{n-1}{n}, & n \text{ 为正偶数} \\ \frac{2}{3} \cdot \frac{4}{5} L \cdot \frac{n-3}{n-2} \cdot \frac{n-1}{n}, & n \text{ 为正奇数} \end{cases}$$

无穷限积分:

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{b \rightarrow +\infty} \int_b^0 f(x)dx = F(+\infty) - F(a);$$

$$\int_a^b f(x)dx = \lim_{b \rightarrow +\infty} \int_b^a f(x)dx = F(-\infty) - F(a);$$

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{b \rightarrow +\infty} \int_a^b f(x)dx + \lim_{a \rightarrow -\infty} \int_a^b f(x)dx = F(+\infty) - F(-\infty)$$

瑕积分:

$$\int_a^b f(x)dx = \lim_{t \rightarrow a^+} \int_a^t f(x)dx = F(b) - \lim_{t \rightarrow a^+} F(t);$$

$$\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx = \lim_{t \rightarrow b^-} F(t) - F(a);$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^{+\infty} \frac{1}{x^p} dx, p > 1 \text{ 收敛}, p \leq 1 \text{ 发散}; \quad \int_a^1 \frac{1}{x^p} dx, 0 < p < 1 \text{ 收敛}, p \geq 1 \text{ 发散}$$

$$\tau(n) = \int_0^{+\infty} e^{-x} x^{n-1} dx = (n-1)!; \quad \tau(n+1) = n \cdot \tau(n) = n!; \quad \tau(1) = 1;$$

$$\tau\left(\frac{1}{2}\right) = \sqrt{\pi} \Rightarrow \int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

第六章 定积分应用

1、平面图形的面积：

直角坐标情形： $A = \int_a^b |f(x)| dx ; A = \int_a^b |f(x) - g(x)| dx ; A = \int_c^d |\varphi(y) - \psi(y)| dy$

参数方程情形： $A = \int_{\alpha}^{\beta} \psi(t) d\varphi(t) = \int_{\alpha}^{\beta} \psi(t) \varphi'(t) dt; (\varphi(\alpha) = a; \varphi(\beta) = b)$

极坐标情形： $A = \frac{1}{2} \int_{\alpha}^{\beta} \rho^2(\theta) d\theta$

2、空间立体的体积：

由截面面积： $V = \int_a^b A(x) dx$

$V = \int_a^b \pi f^2(x) dx; V = \int_a^b \pi [f^2(x) - g^2(x)] dx$ (x 为积分变量)

旋转体：绕 x 轴旋转：

$V = \int_c^d 2\pi y |\varphi(y)| dy; V = \int_c^d 2\pi y |\varphi(y) - \psi(y)| dy$ (y 为积分变量)

$V = \int_a^b 2\pi x |f(x)| dx = \int_a^b 2\pi x |f(x) - g(x)| dx$ (x 为积分变量)

绕 y 轴旋转：

$V = \int_c^d \pi [\varphi^2(y) - \psi^2(y)] dy$ (y 为积分变量)

3、平面曲线的弧长：

$$s = \int_{\alpha}^{\beta} \sqrt{\varphi'^2(t) + \psi'^2(t)} dt = \int_a^b \sqrt{1 + f'^2(x)} dx = \int_{\alpha}^{\beta} \sqrt{\rho^2(\theta) + \rho'^2(\theta)} d\theta$$

变力做功： $W = \int_a^b F(x) dx$

抽水做功：克服重力做功=质量 $\times g \times$ 高度， $dW = dM \cdot g \cdot h = \rho \cdot dV \cdot g \cdot h$

液体压力做功：压力=压强 \times 面积， $dF = pdA = \rho \cdot g \cdot h \cdot dA$

第七章 向量代数与空间解析几何

两点间距离公式：

$$|M_1 - M_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2},$$

$$\mathbf{a} = (a_x, a_y, a_z) = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}; \quad \mathbf{b} = (b_x, b_y, b_z) = b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}$$

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$$\overset{\text{r}}{a \pm b} = (\overset{x}{a} \pm \overset{x}{b}, \overset{y}{a} \pm \overset{y}{b}, \overset{z}{a} \pm \overset{z}{b}); \quad \overset{\text{r}}{\lambda a} = (\overset{x}{\lambda a}, \overset{y}{\lambda a}, \overset{z}{\lambda a})$$

$$\cos \alpha = \frac{a}{|\overset{\text{r}}{a}|} = \frac{a}{\sqrt{\overset{x}{a^2} + \overset{y}{a^2} + \overset{z}{a^2}}},$$

$$\text{方向余弦: } \cos \beta = \frac{a}{|\overset{\text{r}}{a}|} = \frac{a}{\sqrt{\overset{x}{a^2} + \overset{y}{a^2} + \overset{z}{a^2}}}, \quad \text{单位向量: } \overset{\text{r}}{e_a} = \frac{a}{|\overset{\text{r}}{a}|} = (\cos \alpha, \cos \beta, \cos \gamma)$$

$$\cos \gamma = \frac{a}{|\overset{\text{r}}{a}|} = \frac{a}{\sqrt{\overset{x}{a^2} + \overset{y}{a^2} + \overset{z}{a^2}}}$$

$$\text{数量积: } \overset{\text{r}}{a} \cdot \overset{\text{r}}{b} = |\overset{\text{r}}{a}| |\overset{\text{r}}{b}| \cos(a, b) = \overset{x}{a} \overset{x}{b} + \overset{y}{a} \overset{y}{b} + \overset{z}{a} \overset{z}{b}$$

$$\overset{\text{r}}{a} \cdot \overset{\text{r}}{a} = a^2 = |\overset{\text{r}}{a}|^2 \Rightarrow \overset{\text{r}}{i} \cdot \overset{\text{r}}{j} = \overset{\text{r}}{j} \cdot \overset{\text{r}}{k} = \overset{\text{r}}{k} \cdot \overset{\text{r}}{i} = 0, \quad \overset{\text{r}}{i} \cdot \overset{\text{r}}{i} = \overset{\text{r}}{j} \cdot \overset{\text{r}}{j} = \overset{\text{r}}{k} \cdot \overset{\text{r}}{k} = 1$$

$$\text{夹角余弦: } \cos(a, b) = \frac{\overset{\text{r}}{a} \cdot \overset{\text{r}}{b}}{|\overset{\text{r}}{a}| |\overset{\text{r}}{b}|} = \frac{\overset{x}{a} \overset{x}{b} + \overset{y}{a} \overset{y}{b} + \overset{z}{a} \overset{z}{b}}{\sqrt{\overset{x}{a^2} + \overset{y}{a^2} + \overset{z}{a^2}} \sqrt{\overset{x}{b^2} + \overset{y}{b^2} + \overset{z}{b^2}}}$$

$$\text{向量积: } \overset{\text{r}}{a} \times \overset{\text{r}}{b} = (\overset{y}{a} \overset{z}{b} - \overset{z}{a} \overset{y}{b}) \overset{\text{r}}{i} + (\overset{z}{a} \overset{x}{b} - \overset{x}{a} \overset{z}{b}) \overset{\text{r}}{j} + (\overset{x}{a} \overset{y}{b} - \overset{y}{a} \overset{x}{b}) \overset{\text{r}}{k} = \begin{vmatrix} \overset{\text{r}}{i} & \overset{\text{r}}{j} & \overset{\text{r}}{k} \\ \overset{x}{a} & \overset{y}{a} & \overset{z}{a} \\ \overset{x}{b} & \overset{y}{b} & \overset{z}{b} \end{vmatrix}$$

$$\overset{\text{r}}{a} \times \overset{\text{r}}{a} = 0, \quad |\overset{\text{r}}{a} \times \overset{\text{r}}{b}| = |\overset{\text{r}}{a}| |\overset{\text{r}}{b}| \sin(a, b) = S_{\text{平行四边形}},$$

$$\text{空间位置关系: } \overset{\text{r}}{a} \parallel \overset{\text{r}}{b} \Leftrightarrow \overset{\text{r}}{a} \times \overset{\text{r}}{b} = 0 \Leftrightarrow (\exists \alpha, \beta) \alpha \overset{\text{r}}{a} + \beta \overset{\text{r}}{b} = 0 \Leftrightarrow \frac{\overset{x}{b}}{\overset{x}{a}} = \frac{\overset{y}{b}}{\overset{y}{a}} = \frac{\overset{z}{b}}{\overset{z}{a}}$$

$$\overset{\text{r}}{a} \perp \overset{\text{r}}{b} \Leftrightarrow \overset{\text{r}}{a} \cdot \overset{\text{r}}{b} = 0 \Leftrightarrow \overset{x}{a} \overset{x}{b} + \overset{y}{a} \overset{y}{b} + \overset{z}{a} \overset{z}{b} = 0 \Leftrightarrow |\overset{\text{r}}{a} + \overset{\text{r}}{b}| = |\overset{\text{r}}{a} - \overset{\text{r}}{b}|$$

$$\text{平面的方程: 点法式: } A(x - x_0) + B(y - y_0) + C(z - z_0) = 0; \quad \text{一般式: }$$

$$Ax + By + Cz + D = 0$$

$$\text{截距式: } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\text{两平面的夹角: } \cos \theta = \frac{|\overset{\text{r}}{n}_1 \cdot \overset{\text{r}}{n}_2|}{|\overset{\text{r}}{n}_1| |\overset{\text{r}}{n}_2|} = \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}}$$

$$\text{点到平面的距离: } d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$\text{两平行平面的距离: } d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$$

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$$\text{直线与平面的夹角: } \sin \varphi = \frac{\left| \begin{matrix} \mathbf{r} & \mathbf{r} \\ \mathbf{n} \cdot \mathbf{s} \end{matrix} \right|}{\left| \begin{matrix} \mathbf{r} & \mathbf{r} \\ \mathbf{n} & \mathbf{s} \end{matrix} \right|} = \frac{Am + Bn + Cp}{\sqrt{A^2 + B^2 + C^2} \sqrt{m^2 + n^2 + p^2}}$$

空间曲线 C ，曲线的投影 C_{xoy} ，空间立体 Ω ，曲面 Σ ，曲面的投影 D_{xy}

球面: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$

椭圆柱面: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; 双曲柱面: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; 抛物柱面: $x^2 = 2py$

旋转曲面: 圆柱面: $x^2 + y^2 = a^2$; 圆锥面: $z^2 = b^2(x^2 + y^2)$; 双叶双曲面:

$$\frac{x^2}{a^2} - \frac{y^2 + z^2}{c^2} = 1$$

单叶双曲面: $\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1$; 旋转椭球面: $\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$; 旋转抛物面:

$$x^2 + y^2 = 2pz$$

二次曲面:

椭球面: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ($a > 0, b > 0, c > 0$)

抛物面: 椭圆抛物面: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$; 双曲抛物面: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$

单叶双曲面: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; 双叶双曲面: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

椭圆锥面: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

总结

求极限方法:

- 1、极限定义；2、函数的连续性；3、极限存在的充要条件；4、两个准则；
- 5、两个重要极限；6、等价无穷小；7、导数定义；8利用微分中值定理；
- 9、洛必达法则；10、麦克劳林公式展开；

求导法:

- 1、导数的定义（求极限）；2、导数存在的充要条件；3、基本求导公式；
- 4、导数四则运算及反函数求导；5、复合函数求导；6、参数方程确定的函数求

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导：

7、隐函数求导法；8、高阶导数求导法（莱布尼茨公式/常用的高阶导数）；

等式与不等式的证明：

- 1、利用微分中值定理；2、利用泰勒公式展开；3、函数的单调性；
- 4、最大最小值；5、曲线的凸凹性

第八章 多元函数微分法及其应用

一、定义：

$$\frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{d}{dx} f(x, y) \Big|_{x=x_0} = f_x(x_0, y_0) = f(x, y) \Big|_{(x_0, y_0)}$$

二、微分：

$$\lim_{\rho \rightarrow 0} \frac{\Delta z - f_x(x, y)\Delta x - f_y(x, y)\Delta y}{\rho} = 0 \Leftrightarrow \text{可微, 偏导连续} \Rightarrow \text{可微} \Rightarrow \text{连续+偏导存在},$$

$$\text{全微分: } dz = f_x(x, y)dx + f_y(x, y)dy$$

三、隐函数求导：

$$1 \circ F(x, y) = 0 \Rightarrow y = f(x) \text{ 且 } \frac{dy}{dx} = -\frac{F_x}{F_y}.$$

$$2 \circ F(x, y, z) = 0 \Rightarrow z = f(x, y) \text{ 且 }$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_y}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_x}$$

四、曲线的切线和法平面

$$1、\text{ 曲线方程 } L: \begin{cases} x = \varphi(t) \\ y = \psi(t) \\ z = \omega(t) \end{cases}, \text{ 切线: } \frac{(x-x_0)}{\varphi'(t_0)} = \frac{(y-y_0)}{\psi'(t_0)} = \frac{(z-z_0)}{\omega'(t_0)}, \text{ 法平面: }$$

$$\varphi'(t_0)(x-x_0) + \psi'(t_0)(y-y_0) + \omega'(t_0)(z-z_0) = 0$$

$$2、\text{ 曲线方程 } L: \begin{cases} y = y(x) \\ z = z(x) \end{cases}, \text{ 切线: } \frac{x-x_0}{1} = \frac{y-y_0}{y'(x_0)} = \frac{z-z_0}{z'(x_0)}, \text{ 法平面: }$$

$$(x-x_0) + y'(x_0)(y-y_0) + z'(x_0)(z-z_0) = 0$$

$$3、\text{ 曲线方程 } L: \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}, \text{ 切向量 } T = \pm \left\{ \begin{matrix} F_x & F_y & F_z \end{matrix} \right\}_{M_0} \times \left\{ \begin{matrix} G_x & G_y & G_z \end{matrix} \right\}_{M_0}, \text{ 切线: }$$

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$$\frac{x-x_0}{\begin{vmatrix} F & F \\ G & G \end{vmatrix}_{M_0}} = \frac{y-y_0}{\begin{vmatrix} F & F \\ G & G \end{vmatrix}_{M_0}} = \frac{z-z_0}{\begin{vmatrix} F & F \\ G & G \end{vmatrix}_{M_0}}$$

四、曲面的切平面和法线

1、曲面方程: $F(x, y, z) = 0$, 法向量: $\vec{n} = \pm \left\{ \begin{matrix} F_x, F_y, F_z \end{matrix} \right\}_{M_0}$, 切平面:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0, \text{ 法线:}$$

$$\frac{(x - x_0)}{F_x(x_0, y_0, z_0)} = \frac{(y - y_0)}{F_y(x_0, y_0, z_0)} = \frac{(z - z_0)}{F_z(x_0, y_0, z_0)}$$

2 、曲面方程: $z = f(x, y)$, 切平面

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) - (z - z_0) = 0,$$

$$\text{法线: } \frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - z_0}{-1}$$

$$\text{五、方向导数: } \frac{\partial f}{\partial l} \Big|_{M_0} = f_x \Big|_{M_0} \cos \alpha + f_y \Big|_{M_0} \cos \beta + f_z \Big|_{M_0} \cos \gamma$$

$$\text{梯度: } \text{grad } u \Big|_{M_0} = \left\{ f_x, f_y, f_z \right\}_{M_0}$$

第九章: 重积分

一、二重积分:

$$\iint_D f(x, y) d\sigma = \iint_D f(x, y) dx dy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx$$

$$\iint_D f(\rho \cos \theta, \rho \sin \theta) \rho d\rho d\theta = \int_{\alpha}^{\beta} d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(\rho \cos \theta, \rho \sin \theta) \rho d\rho$$

二、三重积分:

$$1、\text{直角坐标系: } \iiint_{\Omega} f(x, y, z) dV = \iint_{D_{xy}} dx dy \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz$$

$$\iiint_{\Omega} f(x, y, z) dV = \int_{c_1}^{c_2} dz \iint_{D(z)} f(x, y, z) dx dy.$$

$$2、\text{柱面坐标系: } \begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \\ z = z. \end{cases} dv = r dr d\theta dz,$$

$$\iiint_{\Omega} f(x, y, z) dv = \int_{\alpha}^{\beta} d\theta \int_{\rho_1(\theta)}^{\rho_2(\theta)} dr \int_{z_1(\rho, \theta)}^{z_2(\rho, \theta)} f(\rho \cos \theta, \rho \sin \theta, z) \rho dz .$$

3、球面坐标系：

$$\begin{cases} x = r \sin \varphi \cos \theta, \\ y = r \sin \varphi \sin \theta, dv = r^2 \sin \varphi dr d\varphi d\theta, \\ z = r \cos \varphi. \end{cases}$$

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \int_{\alpha}^{\beta} d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} d\varphi \int_{r_1(\theta, \varphi)}^{r_2(\theta, \varphi)} f(r \sin \varphi \cos \theta, r \sin \varphi \sin \theta, r \cos \varphi) r^2 \sin \varphi dr.$$

二、重积分的应用：

$$1、\text{体积: } V = \iiint_{\Omega} dx dy dz = \iint_{D_{xy}} [z_2(x, y) - z_1(x, y)] dx dy$$

$$2、\text{曲面 } \Sigma: z = f(x, y) \text{ 面积: } S = \iint_{D_{xy}} \sqrt{1 + f'_x(x, y)^2 + f'_y(x, y)^2} dx dy$$

$$3、\text{质量: } M = \iint_D \rho(x, y) d\sigma \text{ 或 } M = \iiint_{\Omega} \mu(x, y, z) dv$$

4、质心 (\bar{x}, \bar{y}) :

$$\bar{x} = \frac{\iint_D x \rho(x, y) d\sigma}{M}, \bar{y} = \frac{\iint_D y \rho(x, y) d\sigma}{M} \text{ 或}$$

$$\bar{x} = \frac{\iiint_{\Omega} x \mu(x, y, z) dv}{\iiint_{\Omega} \mu(x, y, z) dv}, \bar{y} = \frac{\iiint_{\Omega} y \mu(x, y, z) dv}{\iiint_{\Omega} \mu(x, y, z) dv}, \bar{z} = \frac{\iiint_{\Omega} z \mu(x, y, z) dv}{\iiint_{\Omega} \mu(x, y, z) dv}$$

$$5、\text{转动惯量: } I_x = \iint_D y^2 \rho(x, y) d\sigma, I_y = \iint_D x^2 \rho(x, y) d\sigma, I_o = \iint_D (x^2 + y^2) \rho(x, y) d\sigma$$

$$\text{或 } I_x = \iiint_{\Omega} (y^2 + z^2) \mu(x, y, z) dv, I_y = \iiint_{\Omega} (z^2 + x^2) \mu(x, y, z) dv$$

$$I_z = \iiint_{\Omega} (x^2 + y^2) \mu(x, y, z) dv, I_o = \iiint_{\Omega} (x^2 + y^2 + z^2) \mu(x, y, z) dv$$

第十章：曲线积分和曲面积分

一、第一类曲线积分：(对弧长的曲线积分)：

$$\begin{aligned} \int_L f(x, y) ds &= \int_{\alpha}^{\beta} f(\varphi(t), \psi(t)) \sqrt{\varphi'^2(t) + \psi'^2(t)} dt = \int_a^b f(x, y(t)) \sqrt{1 + y'^2(t)} dx \\ &= \int_{\alpha}^{\beta} f(\rho(\theta) \cos \theta, \rho(\theta) \sin \theta) \sqrt{\rho^2(\theta) + \rho'^2(\theta)} d\theta \end{aligned}$$

$$\int_L f(x, y, z) ds = \int_{\alpha}^{\beta} f(\varphi(t), \psi(t), \omega(t)) \sqrt{\varphi'^2(t) + \psi'^2(t) + \omega'^2(t)} dt$$

二、第二类曲线积分（对坐标的曲线积分）：

1、计算公式：

$$\int P(x, y)dx + Q(x, y)dy = \int [P(x, y)\cos \alpha + Q(x, y)\cos \beta]ds$$

$$= \int_a^b [P(\varphi(t), \psi(t))\varphi'(t) + Q(\varphi(t), \psi(t))\psi'(t)]dt$$

2、格林公式：

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x} \right) dx dy = \oint_{\partial D^+} P dx + Q dy = \int_{\partial D} (P \cos \alpha + Q \cos \beta) ds$$

3、Stokes 公式：

$$\text{Stokes 公式: } \oint_{\Gamma} P dx + Q dy + R dz =$$

$$\iint_{\Sigma} \begin{vmatrix} dy dz & dz dx & dx dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \iint_{\Sigma} \begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS = \pm \iint_{D_{xy}} f(x, y, z) dx dy$$

$$4、\text{封闭曲线围城的面积: } A = \frac{1}{2} \oint_{\partial D^+} x dy - y dx$$

三、第一类曲面积分：

$$\Sigma: z = z(x, y): \iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \sqrt{1 + Z_x^2 + Z_y^2} dx dy$$

四、第二类曲面积分：

1、计算公式：

$$\iint_{\Sigma} \mathbf{F}(x, y, z) \cdot \mathbf{u} dS = \iint_{\Sigma} P(x, y, z) dy dz + Q(x, y, z) dz dx + R(x, y, z) dx dy$$

$$= \iint_{\Sigma} \mathbf{F}(x, y, z) \cdot \mathbf{e}_n dS = \iint_{\Sigma} (P \cos \alpha + Q \cos \beta + R \cos \gamma) dS$$

$$\iint_{\Sigma} R(x, y, z) dx dy = \iint_{\Sigma} R[x, y, z(x, y)] dx dy; \iint_{\Sigma} R(x, y, z) dx dy = - \iint_{\Sigma} R[x, y, z(x, y)] dx dy$$

$$\iint_{\Sigma} P(x, y, z) dy dz = \pm \iint_{D_{yz}} p(x(y, z), y, z) dy dz; \iint_{\Sigma} Q(x, y, z) dz dx = \pm \iint_{D_{zx}} p(x, y(z, x), z) dz dx$$

2、投影转化法：

$$\Sigma: z = z(x, y), dy dz = \frac{\cos \alpha}{\cos \gamma} dx dy = -z_x dx dy, dz dx = \frac{\cos \beta}{\cos \gamma} dx dy = -z_y dx dy$$

$$\Sigma: F(x, y, z) = 0, dy dz = \frac{F_x}{F_z} dx dy, dz dx = \frac{F_y}{F_z} dx dy$$

3、高斯公式：

$$\begin{aligned} & \iint_{\Sigma} P dy dz + Q dz dx + R dx dy = \iint_{\Omega} (P \cos \alpha + Q \cos \beta + R \cos \gamma) dS \\ &= \pm \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV. \quad (\Sigma \text{为} \partial \Omega_+ \text{外侧时取+; } \Sigma \text{为} \partial \Omega_- \text{内侧时取-}) \end{aligned}$$

$$4 \mathbf{A}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}, \quad u = u(x, y, z) \Rightarrow$$

散度： $\operatorname{div} \mathbf{A} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$; 梯度： $\operatorname{grad} u = (u_x, u_y, u_z)$

$$\operatorname{div}(\operatorname{grad} u) = u_{xx} + u_{yy} + u_{zz}; \text{ 旋度: } \operatorname{rot} \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

第十一章 无穷级数

一、常数项级数 $\sum_{n=1}^{\infty} u_n$

1、常用级数：等比级数 / 几何级数： $\sum_{n=0}^{\infty} q^n$ $\begin{cases} \text{收} = \frac{1}{1-q} & |q| < 1 \\ \text{发} & |q| \geq 1 \end{cases}$

P 级数： $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{收} & P > 1 \\ \text{发} & 0 < P \leq 1 \end{cases}$; 交错 P 级数： $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$ 收敛 $\begin{cases} \text{绝对收敛} & P > 1 \\ \text{条件收敛} & 0 < P \leq 1 \end{cases}$

2、正项级数： $u_n \geq 0$

基本定理：收敛 \Leftrightarrow 部分和有上届 $S_n < \sigma$

比较审敛法：大收小收，小发大发

比较审敛法的极限形式：同阶：同收同发；低阶：同收；高阶：同发

比值/根值审敛法： $\rho = \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n}$ ($\rho = \lim_{n \rightarrow \infty} \sqrt[n]{u_n}$) $\Rightarrow \begin{cases} < 1, \text{ 收敛} \\ > 1, \text{ 发散} \\ = 1, \text{ 失效} \end{cases}$

3、交错级数： $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ ($u_n \geq 0$)

莱布尼茨审敛法： $\begin{cases} u_{n+1} \leq u_n \\ \lim_{n \rightarrow \infty} u_n = 0 \end{cases} \Rightarrow$ 级数收敛， $S \leq u_1, |r_n| \leq u_{n+1}$

绝对收敛： $\sum_{n=1}^{\infty} |u_n|$ 收敛 $\Rightarrow \sum_{n=1}^{\infty} u_n$ 收敛，条件收敛： $\sum_{n=1}^{\infty} u_n$ 收敛而 $\sum_{n=1}^{\infty} |u_n|$ 发散，发散

4、任意项级数：

- 利用定义：部分和有极限 $\lim_{n \rightarrow \infty} S_n = \begin{cases} S, & \text{收敛} \\ \infty, & \text{发散} \end{cases}$ ；

利用收敛的必要条件： $\lim u_n \neq 0 \Rightarrow$ 发散；

- 利用正项级数（比值/根植）审敛法：

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| (\rho = \lim_{n \rightarrow \infty} \sqrt[n]{|u_n|}) \Rightarrow \begin{cases} < 1, & \text{绝对收敛} \Rightarrow \text{收敛} \\ > 1, & \text{绝对值发散} \Rightarrow \text{发散} \\ = 1, & \text{失效} \end{cases}$$

二、幂级数： $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

$$1、\text{收敛半径：} \rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| (\rho = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}) \Rightarrow R = \begin{cases} 1/\rho, & 0 < \rho < \infty \\ 0, & \rho = \infty \\ \infty, & \rho = 0 \end{cases}$$

2、常用等式：

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} (|x| < 1), \quad \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} (|x| < 1), \quad \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x} (|x| < 1)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \quad (-1 \leq x < 1), \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1+x) \quad (-1 < x \leq 1)$$

$$\sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \quad (|x| < 1),$$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1} = \sum_{n=1}^{\infty} \frac{1}{2n-1} x^{2n-1} = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| \quad (|x| < 1)$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad (|x| < 1)$$

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$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots; \quad x \in (-\infty, +\infty) \\ \sin x &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} + \cdots; \quad x \in (-\infty, +\infty) \\ \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots; \quad x \in (-\infty, +\infty) \\ \ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots; \quad x \in (-1, 1] \\ (1+x)^\alpha &= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} x^n \\ &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n + \cdots; \quad x \in (-1, 1) \end{aligned}$$

3、泰勒展开：

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-x_0)^n, a_n = \frac{1}{n!} f^{(n)}(x_0), R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1}, (\xi \in (x_0, x)) \\ \Leftrightarrow \lim_{n \rightarrow \infty} R_n(x) &= 0 \end{aligned}$$

三、傅里叶级数： $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$1、T=2\pi: f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = S(x), \quad (x \in (-\infty, +\infty), \text{且 } x \neq \text{间断点})$$

$$\text{其中 } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, (n=0,1,2,\dots); \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, (n=1,2,\dots).$$

$$(\text{间断点处, } S(x) = \frac{f(x^-) + f(x^+)}{2})$$

$$\text{若 } f(x) \text{ 为奇函数} \Rightarrow \text{正弦级数 } (a_n = 0, b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx);$$

$$\text{若 } f(x) \text{ 为偶函数} \Rightarrow \text{余弦级数 } (b_n = 0, a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx);$$

$$2、T=2l: f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}), \quad (x \in (-\infty, +\infty), \text{且 } x \neq \text{间断点})$$

$$\text{其中 } a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx, (n=0,1,2,\dots); \quad b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx, (n=1,2,\dots).$$

3、非周期函数 $f(x)$,

(1) $x \in [-l, l]$: $f(x)$ — 周期延拓 $\rightarrow F(x)$ 展开 \rightarrow 限制

$$f(x) = S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}), (x \in (-l, l))$$

$$(x = \pm l \text{ 时}, S(x) = \frac{f(-l) + f(l)}{2})$$

(2) $x \in [0, l]$: $f(x)$ — 奇延拓/偶延拓 \rightarrow 周期延拓 $\rightarrow F(x)$ 展开 \rightarrow 限制

$$\text{奇延拓: } f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, (x \in (0, l));$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad (n = 1, 2, \dots); (x = 0 \text{ 或 } l \text{ 时}, S(x) = 0);$$

$$\text{偶延拓: } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad (x \in [0, l])$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx \quad (n = 0, 1, 2, \dots), \text{ 端点处不间断。}$$

第十二章 微分方程

一、基本类型的一阶微分方程:

1、可分离变量方程: $\frac{dy}{dx} = f(x)g(y)$, 分离变量, 两边积分 $\int \frac{dy}{g(y)} = \int f(x) dx$

2、一阶线性微分方程: $\frac{dy}{dx} + P(x)y = Q(x)$

$$\begin{cases} Q(x) = 0 & \text{齐次: 通解: } y = e^{-\int P(x) dx}, \\ Q(x) \neq 0 & \text{非齐次: 通解: } y = e^{-\int P(x) dx} (\int Q(x) e^{\int P(x) dx} dx + C) \end{cases}$$

3、全微分方程: $P(x, y)dx + Q(x, y)dy = 0$ (其中 $P_x = Q_y$)

通解: $u(x, y) = C$. (1)、分项组合法;

(2)、特殊路径法: $u(x, y) = \int_{x_0}^x P(x, y_0) dx + \int_{y_0}^y Q(x, y) dy = C$.

(3)、偏积分法;

$$\left. \begin{array}{l} P(x, y) = \frac{\partial u}{\partial x} \Rightarrow u(x, y) = \int P(x, y) dx + c(y) \\ Q(x, y) = \frac{\partial u}{\partial y} \Rightarrow c'(y) = Q - \frac{\partial}{\partial y} \int P(x, y) dx = \varphi(y) \end{array} \right\} \Rightarrow u(x, y) = \int P(x, y) dx + \int \varphi(y) dy$$

二、可化为基本类型的一阶微分方程:

(1) 齐次方程: $\frac{dy}{dx} = f(\frac{y}{x})$ 或 $\frac{dy}{dx} = f(\frac{ax+by}{a^2x+b^2y})$, 令 $u = \frac{y}{x}$

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(2) 准齐次方程: $\frac{dy}{dx} = f\left(\frac{ax+by+c}{a_1x+b_1y+c_1}\right)$

$$\begin{cases} \text{若 } \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \text{ 令 } \begin{cases} x = X + h, \\ y = Y + k \end{cases} \text{ (} h, k \text{ 由 } \begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases} \text{ 解得)} \\ \Rightarrow \frac{dY}{dX} = f\left(\frac{a_1X + b_1Y}{a_2X + b_2Y}\right), \text{ 再令 } u = \frac{Y}{X}. \\ \text{若 } \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0, \frac{dy}{dx} = \frac{k(a_1x + b_1y) + c_1}{a_1x + b_1y + c_2} = f(a_1x + b_1y). \text{ 令 } u = a_1x + b_1y. \end{cases}$$

(3) $\frac{dy}{dx} = f(ax + by + c)$ 令 $u = ax + by + c$.

(4) 伯努利方程: $\frac{dy}{dx} + P(x)y = Q(x)y^\alpha$ ($\alpha \neq 0, 1$), 令 $z = y^{1-\alpha} \Rightarrow \frac{dz}{dy} + (1-\alpha)P(x)z = (1-\alpha)Q(x)$

(5) $P(x, y)dx + Q(x, y)dy = 0$ (其中 $P_y \neq Q_x$) $\Rightarrow \frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)}$

(6) 关于 x 的线性方程 / 伯努利方程:

$\frac{dx}{dy} + P(y)x = Q(y); \quad \frac{dx}{dy} + P(y)x = Q(y)x^\alpha$, 令 $z = x^{1-\alpha}$

(7) $P(x, y)dx + Q(x, y)dy = 0$ (其中 $P_y \neq Q_x$)

求积分因子方法:

1、分项组合法: 常用全微分公式;

2、公式法:

(1) 方程有形如 $u(x)$ 的积分因子 $\Leftrightarrow \frac{1}{Q} (P_y - Q_x) = \varphi(x) \Rightarrow u(x) = ce^{\int \varphi(x) dx}$

(2) 方程有形如 $u(y)$ 的积分因子 $\Leftrightarrow \frac{1}{P} (P_y - Q_x) = \psi(y) \Rightarrow u(y) = ce^{-\int \psi(y) dy}$

(3) 齐次方程的积分因子 $u(x, y) = \frac{1}{xP + yQ}$

三、可降阶的高阶微分方程:

(1) $\frac{d^n y}{dx^n} = f(x)$ 连续积分 n 次;

(2) $y'' = f(x, y')$, 令 $y' = p$, 则 $y'' = p' \Rightarrow p' = f(x, p)$

(3) $y'' = f(y, y')$, 令 $y' = p$, 则 $y'' = p \frac{dp}{dy} \Rightarrow p \frac{dp}{dy} = f(y, p)$

四、二阶常系数齐次线性微分方程

$y'' + py' + qy = 0 \Leftrightarrow$ 特征方程: $r^2 + pr + q = 0$

$\Delta = p^2 - 4q > 0, r_1 \neq r_2 \Rightarrow$ 通解: $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

$\Delta = p^2 - 4q = 0, r_1 = r_2 \Rightarrow$ 通解: $y = (C_1 + C_2 x) e^{r_1 x}$

$\Delta = p^2 - 4q < 0, r_{1,2} = \alpha \pm i\beta \Rightarrow$ 通解: $y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$

四、二阶常系数非齐次线性微分方程

$y'' + p y' + q y = f(x)$ 通解 $y(x) =$ 齐次通解 $Y(x) +$ 非齐次特解 $y^*(x)$

$$(1) f(x) = e^{\lambda x} P_m(x) \Rightarrow \text{特解形式} y^* = x^k Q_m(x) e^{\lambda x} \begin{cases} \lambda \text{不是特征根} & k=0 \\ \lambda \text{是特征单根} & k=1 \\ \lambda \text{是特征重根} & k=2 \end{cases}$$

$$(2) f(x) = f(x) = e^{\lambda x} [P_l(x) \cos \omega x + P_n(x) \sin \omega x] \\ \Rightarrow \text{特解形式} y^* = x e^{\lambda x} [R_m^{(1)}(x) \cos \omega x + R_m^{(2)}(x) \sin \omega x] \begin{cases} \lambda + iw \text{不是特征根} & k=0 \\ \lambda + iw \text{是特征根} & k=1 \end{cases}$$