

高等数学公式汇总

第一章 一元函数的极限与连续

1、常用初等函数公式：

和差角公式：

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \cdot \tan \beta}$$

$$\cot(\alpha \pm \beta) = \frac{\cot \alpha \cdot \cot \beta \mp 1}{\cot \beta \pm \cot \alpha}$$

$$\operatorname{sh}(\alpha \pm \beta) = \operatorname{sh} \alpha \operatorname{ch} \beta \pm \operatorname{ch} \alpha \operatorname{sh} \beta$$

$$\operatorname{ch}(\alpha \pm \beta) = \operatorname{ch} \alpha \operatorname{ch} \beta \pm \operatorname{sh} \alpha \operatorname{sh} \beta$$

和差化积公式：

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \alpha - \cos \beta = 2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

积化和差公式：

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha + \beta) - \cos(\alpha - \beta)]$$

倍角公式：

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

$$= 1 - 2 \sin^2 \alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

$$\cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha}$$

$$\operatorname{sh} 2\alpha = 2 \operatorname{sh} \alpha \operatorname{ch} \alpha$$

$$\operatorname{ch} 2\alpha = 1 + 2 \operatorname{sh}^2 \alpha =$$

$$= 2 \operatorname{ch}^2 \alpha - 1 = \operatorname{ch}^2 \alpha + \operatorname{sh}^2 \alpha$$

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$$\sin^2 \alpha + \cos^2 \alpha = 1; \tan^2 x + 1 = \sec^2 x;$$

$$\cot^2 x + 1 = \csc^2 x; \operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$$

半角公式：

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}$$

$$\cot \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}} = \frac{1 + \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 - \cos \alpha}$$

$$\text{双曲正弦: } \operatorname{sh} x = \frac{e^x - e^{-x}}{2}; \text{ 反双曲正弦: } \operatorname{arsh} x = \ln(x + \sqrt{x^2 + 1})$$

$$\text{双曲余弦: } \operatorname{ch} x = \frac{e^x + e^{-x}}{2}; \text{ 反双曲余弦: } \operatorname{arch} x = \pm \ln(x + \sqrt{x^2 - 1})$$

$$\text{双曲正切: } \operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \text{ 反双曲正切: } \operatorname{arth} x = \frac{1}{2} \ln \frac{1+x}{1-x}$$

$$(a^3 \pm b^3) = (a \pm b)(a^2 \mp ab + b^2), \quad 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

2、极限

$$\text{➤ 常用极限: } |q| < 1, \lim_{n \rightarrow \infty} q^n = 0; a > 1, \lim_{n \rightarrow \infty} \sqrt[n]{a} = 1; \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

$$\text{➤ 若 } f(x) \rightarrow 0, g(x) \rightarrow \infty, \text{ 则 } \lim_{x \rightarrow \infty} [1 \pm f(x)]^{g(x)} = e^{\lim_{x \rightarrow \infty} \frac{\ln(1 \pm f(x))}{1/g(x)}} \xrightarrow{\frac{\ln(1 \pm f(x))}{1/g(x)} \sim \pm f(x)} e^{\pm \lim_{x \rightarrow \infty} [f(x)g(x)]}$$

➤ 两个重要极限

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0; \lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e = \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}}$$

➤ 常用等价无穷小:

$$1 - \cos x \sim \frac{1}{2} x^2; x \sim \sin x \sim \arcsin x \sim \arctan x; \sqrt[n]{1+x} - 1 \sim \frac{1}{n} x;$$

$$a^x - 1 \sim x \ln a; e^x \sim x + 1; (1+x)^a \sim 1 + ax; \ln(1+x) \sim x$$

3、连续：

定义： $\lim_{\Delta x \rightarrow 0} \Delta y = 0; \lim_{x \rightarrow x_0} f(x) = f(x_0)$

极限存在 $\Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0^+} f(x)$ 或 $f(x_0^-) = f(x_0^+)$

第二章 导数与微分

基本导数公式：

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \tan \alpha$$

导数存在 $\Leftrightarrow f'(x_0^-) = f'(x_0^+)$

$C' = 0$; $(x^a)' = ax^{a-1}$; $(\sin x)' = \cos x$; $(\cos x)' = -\sin x$; $(\tan x)' = \sec^2 x$; $(\cot x)' = -\csc^2 x$;

$(\sec x)' = \sec x \cdot \tan x$; $(\csc x)' = -\csc x \cdot \cot x$; $(a^x)' = a^x \ln a$; $(e^x)' = e^x$;

$(\log_a |x|)' = \frac{1}{x \ln a}$; $(\ln |x|)' = \frac{1}{x}$; $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$; $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$;

$(\arctan x)' = \frac{1}{1+x^2}$; $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$; $(\operatorname{sh} x)' = \operatorname{ch} x$; $(\operatorname{ch} x)' = \operatorname{sh} x$;

$(\operatorname{th} x)' = \frac{1}{\operatorname{ch}^2 x}$; $(\operatorname{arsh} x)' = \frac{1}{\sqrt{1+x^2}}$; $(\operatorname{arch} x)' = \frac{1}{\sqrt{x^2-1}}$; $(\operatorname{arth} x)' = \frac{1}{x^2-1}$

2、高阶导数：

$$(x^n)^{(k)} = \frac{n!}{(n-k)!} x^{n-k} \Rightarrow (x^n)^{(n)} = n!; (a^x)^{(n)} = a^x \ln^n a \Rightarrow (e^x)^{(n)} = e^x$$

$$\left(\frac{1}{x}\right)^{(n)} = \frac{(-1)^n n!}{x^{n+1}}; \left(\frac{1}{x+a}\right)^{(n)} = \frac{(-1)^n n!}{(x+a)^{n+1}}; \left(\frac{1}{a-x}\right)^{(n)} = \frac{n!}{(a-x)^{n+1}}$$

$$(\sin kx)^{(n)} = k^n \cdot \sin(kx + n \cdot \frac{\pi}{2}); (\cos kx)^{(n)} = k^n \cdot \cos(kx + n \cdot \frac{\pi}{2});$$

$$[\ln(a+x)]^{(n)} = (-1)^{n-1} \frac{(n-1)!}{(a+x)^n} \Rightarrow [\ln(x)]^{(n)} = \left(\frac{1}{x}\right)^{(n-1)} = (-1)^{n-1} \frac{(n-1)!}{x^n}$$

牛顿-莱布尼兹公式：

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$$\begin{aligned}(uv)^{(n)} &= \sum_{k=0}^n C_n^k u^{(n-k)} v^{(k)} \\ &= u^{(n)} v + n u^{(n-1)} v' + \frac{n(n-1)}{2!} u^{(n-2)} v'' + \dots + \frac{n(n-1)\dots(n-k+1)}{k!} u^{(n-k)} v^{(k)} + \dots + u v^{(n)}\end{aligned}$$

3、微分：

$$\Delta y = f(x + \Delta x) - f(x) = dy + o(\Delta x); \quad dy = f'(x_0) \Delta x = f'(x) dx;$$

连续 \Rightarrow 极限存在 \Leftrightarrow 收敛 \Rightarrow 有界； 可微 \Leftrightarrow 可导 \Leftrightarrow 左导=右导 \Rightarrow 连续；

不连续 \Rightarrow 不可导

第三章 微分中值 定理与微分的应用

基本定理

拉格朗日中值定理： $f(b) - f(a) = f'(\xi)(b - a), \xi \in (a, b)$

柯西中值定理： $\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(\xi)}{F'(\xi)}, \xi \in (a, b)$

当 $F(x) = x$ 时，柯西中值定理就是拉格朗日中值定理。

2、

泰勒公式： $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$

余项： $R_n(x) = \begin{cases} o((x - x_0)^n) \\ \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}(x - x_0)^{n+1}; (\xi \in (x_0, x), \theta \in (0, 1)) \end{cases}$

麦克劳林公式： $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}; (\theta \in (0, 1))$

常用初等函数的展式：

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + R_n(x); \quad R_n(x) = \frac{e^{\theta x}}{(n+1)!} x^{n+1}; (\theta \in (0, 1))$$

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$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - L + (-1)^{m-1} \frac{x^{2m-1}}{(2m-1)!} + R_{2m}(x); R_{2m}(x) = \frac{\sin[\theta x + (2m+1)\frac{\pi}{2}]}{(2m+1)!} x^{2m+1}; (\theta \in (0,1))$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - L + (-1)^m \frac{x^{2m}}{(2m)!} + R_{2m+1}(x); R_{2m+1}(x) = \frac{\cos[\theta x + (m+1)\pi]}{(2m+2)!} x^{2m+2}; (\theta \in (0,1))$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^4}{3!} - L + (-1)^{n-1} \frac{x^n}{n} + R_n(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1};$$

$$R_n(x) = \frac{(-1)^n}{(n+1)(1+\theta x)^{n+1}} x^{n+1}; (\theta \in (0,1))$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + L + \frac{\alpha(\alpha-1)L}{n!} (\alpha-n+1) x^n + R_n(x);$$

$$R_n(x) = \frac{\alpha(\alpha-1)L}{(n+1)!} (\alpha-n) (1+\theta x)^{\alpha-n-1} x^{n+1}; (\theta \in (0,1))$$

$$\Rightarrow \frac{1}{1+x} = \ln'(1+x) = 1 - x + x^2 - L + (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

3、

$$\text{弧微分公式: } ds = \sqrt{1+y'^2} dx = \sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{\rho^2 + \rho'^2} d\theta$$

$$\text{平均曲率: } \bar{K} = \left| \frac{\Delta\alpha}{\Delta s} \right|. (\Delta\alpha: \text{从M点到M'点, 切线斜率的倾角变化量; } \Delta s: MM' \text{弧长})$$

$$\text{M点的曲率: } K = \lim_{\Delta s \rightarrow 0} \left| \frac{\Delta\alpha}{\Delta s} \right| = \left| \frac{d\alpha}{ds} \right| = \frac{|y''|}{\sqrt{(1+y'^2)^3}} = \frac{|\varphi'(t)\psi''(t) - \varphi''(t)\psi'(t)|}{[\varphi'^2(t) + \psi'^2(t)]^{\frac{3}{2}}}.$$

$$\text{直线的曲率: } K = 0; \text{ 半径为} R \text{的圆的曲率: } K = \frac{1}{R}.$$

$$\text{曲线在点} M \text{处的曲率半径: } \rho = \frac{1}{K} = \frac{\sqrt{(1+y'^2)^3}}{|y''|}$$

第四章 不定积分

常用不定积分公式：

$$\int f(x)dx = F(x) + C; \quad (\int f(x)dx)' = f(x); \quad \int F'(x)dx = F(x) + C$$

$$\int x^{\mu}dx = \frac{x^{\mu+1}}{\mu+1} + C (\mu \neq -1); \quad \int \frac{1}{x}dx = \ln|x| + C;$$

$$\int a^x dx = \frac{a^x}{\ln a} + C; \quad \int e^x dx = e^x + C;$$

$$\int \sin x dx = -\cos x + C; \quad \int \cos x dx = \sin x + C;$$

$$\int \tan x dx = -\ln|\cos x| + C; \quad \int \cot x dx = \ln|\sin x| + C;$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C;$$

$$\int \csc x dx = \ln|\csc x - \cot x| + C = \ln\left|\tan \frac{x}{2}\right| + C = -\ln|\csc x + \cot x| + C;$$

$$\int \sec^2 x dx = \int \frac{dx}{\cos^2 x} = \tan x + C; \quad \int \csc^2 x dx = \int \frac{dx}{\sin^2 x} = -\cot x + C;$$

$$\int \sec x \cdot \tan x dx = \sec x + C; \quad \int \csc x \cdot \cot x dx = -\csc x + C;$$

$$\int shx dx = chx + C; \quad \int chx dx = shx + C;$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C = -\arccos x + C; \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C;$$

$$\int \frac{dx}{1+x^2} = \arctan x + C = -\operatorname{arccot} x + C; \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan \frac{x}{a} + C;$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C; \quad \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \frac{a+x}{a-x} + C;$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2}) + C;$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{x}{2} \sqrt{x^2 \pm a^2} + \frac{a^2}{2} \ln(x + \sqrt{x^2 \pm a^2}) + C;$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C$$

2、常用凑微分公式：

$$\frac{dx}{\sqrt{x}} = 2d\sqrt{x}; \quad \frac{dx}{x^2} = -d\left(\frac{1}{x}\right); \quad \frac{dx}{x} = d(\ln x);$$

$$\frac{xdx}{\sqrt{1+x^2}} = d(\sqrt{1+x^2}); \quad \left(1 - \frac{1}{x^2}\right)dx = d\left(x + \frac{1}{x}\right)$$

$$\frac{dx}{\cos x \sin x} = d(\ln \tan x);$$

3、有特殊技巧的积分

$$(1) \int \frac{dx}{a \sin x + b \cos x} = \frac{1}{\sqrt{a^2 + b^2}} \int \frac{1}{\sin(x + \varphi)} dx$$

$$(2) \int \frac{c \sin x + d \cos x}{a \sin x + b \cos x} dx = Ax + B \ln |a \sin x + b \cos x| + C$$

$$(3) \int \frac{x^2 + 1}{x^4 + 1} dx = \int \frac{1}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{2})^2} d\left(x - \frac{1}{x}\right)$$

第五章 定积分

1、基本概念

$$\int_a^b f(x) dx = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{i}{n}\right) \frac{1}{n} = F(b) - F(a) = F(x) \Big|_a^b, \quad (F'(x) = f(x))$$

连续 \Rightarrow 可积; 有界+有限个间断点 \Rightarrow 可积;

可积 \Rightarrow 有界; 连续 \Rightarrow 原函数存在

$$\Phi(x) = \int_a^x f(t) dt \Rightarrow \Phi'(x) = f(x)$$

$$\frac{d}{dx} \int_{\psi(x)}^{\varphi(x)} f(t) dt = f[\varphi(x)]\varphi'(x) - f[\psi(x)]\psi'(x)$$

$$\int_b^a f(x) dx = \int_{\beta}^{\alpha} f(\varphi(t))\varphi'(t) dt, \quad \int_b^a u(x) dv(x) = u(x)v(x) - \int_b^a v(x) du(x)$$

2、常用定积分公式：

$$\int_{-a}^a f(x)dx = \int_0^a [f(x) + f(-x)]dx ;$$

$$f(x) \text{ 为偶函数, } \int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx ; \quad f(x) \text{ 为奇函数, } \int_{-a}^a f(x)dx = 0$$

$$\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx ; \quad \int_0^{\frac{\pi}{2}} xf(\sin x)dx = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} f(\sin x)dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x)dx$$

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx ; \quad \int_a^{a+nT} f(x)dx = n \int_0^T f(x)dx$$

Wallis 公 式 :

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{n-1}{n} I_{n-2} = \begin{cases} \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{n-3}{n-2} \cdot \frac{n-1}{n}, n \text{ 为正偶数} \\ \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{n-3}{n-2} \cdot \frac{n-1}{n}, n \text{ 为正奇数} \end{cases}$$

无有限积分：

$$\int_a^{+\infty} f(x)dx = \lim_{b \rightarrow +\infty} \int_a^b f(x)dx = F(+\infty) - F(a);$$

$$\int_b^a f(x)dx = \lim_{b \rightarrow -\infty} \int_b^a f(x)dx = F(-\infty) - F(a);$$

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{b \rightarrow +\infty} \int_a^b f(x)dx + \lim_{a \rightarrow -\infty} \int_a^b f(x)dx = F(+\infty) - F(-\infty)$$

瑕积分：

$$\int_a^b f(x)dx = \lim_{t \rightarrow a^+} \int_t^b f(x)dx = F(b) - \lim_{t \rightarrow a^+} F(t);$$

$$\int_b^a f(x)dx = \lim_{t \rightarrow a^+} \int_b^t f(x)dx = \lim_{t \rightarrow a^+} F(t) - F(a);$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^{+\infty} \frac{1}{x^p} dx, p > 1 \text{ 收敛}, p \leq 1 \text{ 发散}; \quad \int_a^1 \frac{1}{x^p} dx, 0 < p < 1 \text{ 收敛}, p \geq 1 \text{ 发散}$$

$$\tau(n) = \int_0^{+\infty} e^{-x} x^{n-1} dx = (n-1)!, \tau(n+1) = n \cdot \tau(n) = n!; \tau(1) = 1;$$

$$\tau\left(\frac{1}{2}\right) = \sqrt{\pi} \Rightarrow \int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

第六章 定积分应用

1、平面图形的面积：

直角坐标情形： $A = \int_a^b |f(x)| dx$; $A = \int_a^b |f(x) - g(x)| dx$; $A = \int_c^d |\varphi(y) - \psi(y)| dy$

参数方程情形： $A = \int_{\alpha}^{\beta} \psi(t) d\varphi(t) = \int_{\alpha}^{\beta} \psi(t) \varphi'(t) dt$; ($\varphi(\alpha) = a$; $\varphi(\beta) = b$)

极坐标情形： $A = \frac{1}{2} \int_{\alpha}^{\beta} \rho^2(\theta) d\theta$

2、空间立体的体积：

由截面面积： $V = \int_a^b A(x) dx$

旋转体：绕 x 轴旋转： $V = \int_a^b \pi f^2(x) dx$; $V = \int_a^b \pi [f^2(x) - g^2(x)] dx$ (x 为积分变量)
 $V = \int_c^d 2\pi y |\varphi(y)| dy$; $V = \int_c^d 2\pi y |\varphi(y) - \psi(y)| dy$ (y 为积分变量)

绕 y 轴旋转： $V = \int_a^b 2\pi x |f(x)| dx = \int_a^b 2\pi x |f(x) - g(x)| dx$; (x 为积分变量)
 $V = \int_c^d \pi [\varphi^2(y) - \psi^2(y)] dy$ (y 为积分变量)

3、平面曲线的弧长：

$s = \int_{\alpha}^{\beta} \sqrt{\varphi'^2(t) + \psi'^2(t)} dt = \int_a^b \sqrt{1 + f'^2(x)} dx = \int_{\alpha}^{\beta} \sqrt{\rho^2(\theta) + \rho'^2(\theta)} d\theta$

变力做功： $W = \int_a^b F(x) dx$

抽水做功： 克服重力做功=质量 $\times g \times$ 高度, $dW = dM \cdot g \cdot h = \rho \cdot dV \cdot g \cdot h$

液体压力做功： 压力=压强 \times 面积, $dF = p dA = \rho \cdot g \cdot h \cdot dA$

第七章 向量代数与空间解析几何

两点间距离公式：

$$|M_1 - M_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2},$$

$$\vec{a} = (a_x, a_y, a_z) = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}; \quad \vec{b} = (b_x, b_y, b_z) = b_x \vec{i} + b_y \vec{j} + b_z \vec{k}$$

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$$\overset{\textbf{r}}{a}\pm\overset{\textbf{r}}{b}=(\overset{x}{a}\pm\overset{x}{b},\overset{y}{a}\pm\overset{y}{b},\overset{z}{a}\pm\overset{z}{b});\ \overset{\textbf{r}}{\lambda a}=(\overset{x}{\lambda a},\overset{y}{\lambda a},\overset{z}{\lambda a})$$

$$\cos\alpha=\frac{\overset{\textbf{r}}{a}}{\left|\overset{\textbf{r}}{a}\right|}=\frac{\overset{x}{a}}{\sqrt{\overset{x}{a}^2+\overset{y}{a}^2+\overset{z}{a}^2}},$$

$$\text{方向余弦：}\cos\beta=\frac{\overset{\textbf{r}}{a}}{\left|\overset{\textbf{r}}{a}\right|}=\frac{\overset{y}{a}}{\sqrt{\overset{x}{a}^2+\overset{y}{a}^2+\overset{z}{a}^2}},\quad\text{单位向量：}\overset{\textbf{r}}{e}_a=\frac{\overset{\textbf{r}}{a}}{\left|\overset{\textbf{r}}{a}\right|}=(\cos\alpha,\cos\beta,\cos\gamma)$$

$$\cos\gamma=\frac{\overset{\textbf{r}}{a}}{\left|\overset{\textbf{r}}{a}\right|}=\frac{\overset{z}{a}}{\sqrt{\overset{x}{a}^2+\overset{y}{a}^2+\overset{z}{a}^2}}$$

$$\text{数量积：}\overset{\textbf{r}}{a}\cdot\overset{\textbf{r}}{b}=\left|\overset{\textbf{r}}{a}\right|\left|\overset{\textbf{r}}{b}\right|\cos(\overset{\textbf{r}}{a},\overset{\textbf{r}}{b})=\overset{x}{a}\overset{x}{b}+\overset{y}{a}\overset{y}{b}+\overset{z}{a}\overset{z}{b}$$

$$\overset{\textbf{r}}{a}\cdot\overset{\textbf{r}}{a}=\overset{\textbf{r}}{a}^2=\left|\overset{\textbf{r}}{a}\right|^2\Rightarrow\overset{\textbf{r}}{i}\cdot\overset{\textbf{r}}{j}=\overset{\textbf{r}}{j}\cdot\overset{\textbf{r}}{k}=\overset{\textbf{r}}{k}\cdot\overset{\textbf{r}}{i}=0,\quad\overset{\textbf{r}}{i}\cdot\overset{\textbf{r}}{i}=\overset{\textbf{r}}{j}\cdot\overset{\textbf{r}}{j}=\overset{\textbf{r}}{k}\cdot\overset{\textbf{r}}{k}=1$$

$$\text{夹角余弦：}\cos(\overset{\textbf{r}}{a},\overset{\textbf{r}}{b})=\frac{\overset{\textbf{r}}{a}\cdot\overset{\textbf{r}}{b}}{\left|\overset{\textbf{r}}{a}\right|\left|\overset{\textbf{r}}{b}\right|}=\frac{\overset{x}{a}\overset{x}{b}+\overset{y}{a}\overset{y}{b}+\overset{z}{a}\overset{z}{b}}{\sqrt{\overset{x}{a}^2+\overset{y}{a}^2+\overset{z}{a}^2}\sqrt{\overset{x}{b}^2+\overset{y}{b}^2+\overset{z}{b}^2}}$$

$$\text{向量积：}\overset{\textbf{r}}{a}\times\overset{\textbf{r}}{b}=(\overset{y}{a}\overset{z}{b}-\overset{z}{a}\overset{y}{b})\overset{\textbf{r}}{i}+(\overset{z}{a}\overset{x}{b}-\overset{x}{a}\overset{z}{b})\overset{\textbf{r}}{j}+(\overset{x}{a}\overset{y}{b}-\overset{y}{a}\overset{x}{b})\overset{\textbf{r}}{k}=\begin{vmatrix}\overset{\textbf{r}}{i}&\overset{\textbf{r}}{j}&\overset{\textbf{r}}{k}\\\overset{x}{a}&\overset{y}{a}&\overset{z}{a}\\\overset{x}{b}&\overset{y}{b}&\overset{z}{b}\end{vmatrix}$$

$$\overset{\textbf{r}}{a}\times\overset{\textbf{r}}{a}=\overset{\textbf{r}}{0},\quad\left|\overset{\textbf{r}}{a}\times\overset{\textbf{r}}{b}\right|=\left|\overset{\textbf{r}}{a}\right|\left|\overset{\textbf{r}}{b}\right|\sin(\overset{\textbf{r}}{a},\overset{\textbf{r}}{b})=S_{\text{平行四边形}},$$

$$\text{空间位置关系：}\overset{\textbf{r}}{a}\parallel\overset{\textbf{r}}{b}\Leftrightarrow\overset{\textbf{r}}{a}\times\overset{\textbf{r}}{b}=\overset{\textbf{r}}{0}\Leftrightarrow(\exists\alpha,\beta)\alpha\overset{\textbf{r}}{a}+\beta\overset{\textbf{r}}{b}=\overset{\textbf{r}}{0}\Leftrightarrow\frac{\overset{x}{b}}{\overset{x}{a}}=\frac{\overset{y}{b}}{\overset{y}{a}}=\frac{\overset{z}{b}}{\overset{z}{a}}$$

$$\overset{\textbf{r}}{a}\perp\overset{\textbf{r}}{b}\Leftrightarrow\overset{\textbf{r}}{a}\cdot\overset{\textbf{r}}{b}=\overset{\textbf{r}}{0}\Leftrightarrow\overset{x}{a}\overset{x}{b}+\overset{y}{a}\overset{y}{b}+\overset{z}{a}\overset{z}{b}=0\Leftrightarrow\left|\overset{\textbf{r}}{a}+\overset{\textbf{r}}{b}\right|=\left|\overset{\textbf{r}}{a}-\overset{\textbf{r}}{b}\right|$$

$$\text{平面的方程：点法式：}\quad A(x-x_0)+B(y-y_0)+C(z-z_0)=0\quad；\quad\text{一般式}\quad：$$

$$Ax+By+Cz+D=0$$

$$\text{截距式：}\quad\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$\text{两平面的夹角：}\cos\theta=\frac{\left|\overset{\textbf{r}}{n}_1\cdot\overset{\textbf{r}}{n}_2\right|}{\left|\overset{\textbf{r}}{n}_1\right|\left|\overset{\textbf{r}}{n}_2\right|}=\frac{\overset{1}{A}\overset{1}{A}+\overset{1}{B}\overset{1}{B}+\overset{1}{C}\overset{1}{C}}{\sqrt{\overset{1}{A}^2+\overset{1}{B}^2+\overset{1}{C}^2}\sqrt{\overset{2}{A}^2+\overset{2}{B}^2+\overset{2}{C}^2}}$$

$$\text{点到平面的距离：}\quad d=\frac{\left|Ax_0+By_0+Cz_0+D\right|}{\sqrt{A^2+B^2+C^2}}$$

$$\text{两平行平面的距离：}\quad d=\frac{\left|D_1-D_2\right|}{\sqrt{A^2+B^2+C^2}}$$

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直线与平面的夹角：
$$\sin \varphi = \frac{\left| \begin{vmatrix} \mathbf{r} & \mathbf{n} \\ \mathbf{s} & \mathbf{s} \end{vmatrix} \right|}{\left| \begin{vmatrix} \mathbf{r} & \mathbf{n} \\ \mathbf{r} & \mathbf{s} \end{vmatrix} \right|} = \frac{Am + Bn + Cp}{\sqrt{A^2 + B^2 + C^2} \sqrt{m^2 + n^2 + p^2}}$$

空间曲线 C ，曲线的投影 C_{xoy} ，空间立体 Ω ，曲面 Σ ，曲面的投影 D_{xy}

球面：
$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

椭圆柱面： $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ；双曲柱面： $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ；抛物柱面： $x^2 = 2py$

旋转曲面：圆柱面： $x^2 + y^2 = a^2$ ；圆锥面： $z^2 = b^2(x^2 + y^2)$ ；双叶双曲面：

$$\frac{x^2}{a^2} - \frac{y^2 + z^2}{c^2} = 1$$

单叶双曲面： $\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1$ ；旋转椭球面： $\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$ ；旋转抛物面：

$$x^2 + y^2 = 2pz$$

二次曲面：

椭球面： $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (a > 0, b > 0, c > 0)$

抛物面：椭圆抛物面： $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$ ；双曲抛物面： $\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$

单叶双曲面： $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ ；双叶双曲面： $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

椭圆锥面： $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

总结

求极限方法：

- 1、极限定义；
- 2、函数的连续性；
- 3、极限存在的充要条件；
- 4、两个准则；
- 5、两个重要极限；
- 6、等价无穷小；
- 7、导数定义；
- 8 利用微分中值定理；
- 9、洛必达法则；
- 10、麦克劳林公式展开；

求导法：

- 1、导数的定义（求极限）；
- 2、导数存在的充要条件；
- 3、基本求导公式；
- 4、导数四则运算及反函数求导；
- 5、复合函数求导；
- 6、参数方程确定的函数求

导；

7、隐函数求导法；8、高阶导数求导法（莱布尼茨公式/常用的高阶导数）；

等式与不等式的证明：

1、利用微分中值定理；2、利用泰勒公式展开；3、函数的单调性；

4、最大最小值；5、曲线的凹凸性

第八章 多元函数微分法及其应用

一、定义：

$$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \left. \frac{d}{dx} f(x, y_0) \right|_{x=x_0} = f'_x(x_0, y_0) = f'_x(x, y) \Big|_{(x_0, y_0)}$$

二、微分：

$$\lim_{\rho \rightarrow 0} \frac{\Delta z - f'_x(x, y)\Delta x - f'_y(x, y)\Delta y}{\rho} = 0 \Leftrightarrow \text{可微}, \text{偏导连续} \Rightarrow \text{可微} \Rightarrow \text{连续} + \text{偏导存在},$$

$$\text{全微分: } dz = f'_x(x, y)dx + f'_y(x, y)dy$$

三、隐函数求导：

$$1^\circ \quad F(x, y) = 0 \Rightarrow y = f(x) \text{ 且 } \frac{dy}{dx} = -\frac{F'_x}{F'_y}.$$

$$2^\circ \quad F(x, y, z) = 0 \Rightarrow z = f(x, y) \text{ 且}$$

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z}, \quad \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}$$

四、曲线的切线和法平面

$$1、\text{曲线方程 } L: \begin{cases} x = \varphi(t) \\ y = \psi(t) \\ z = \omega(t) \end{cases}, \text{切线: } \frac{(x-x_0)}{\varphi'(t_0)} = \frac{(y-y_0)}{\psi'(t_0)} = \frac{(z-z_0)}{\omega'(t_0)}, \text{法平面:}$$

$$\varphi'(t_0)(x-x_0) + \psi'(t_0)(y-y_0) + \omega'(t_0)(z-z_0) = 0$$

$$2、\text{曲线方程 } L: \begin{cases} y = y(x) \\ z = z(x) \end{cases}, \text{切线: } \frac{x-x_0}{1} = \frac{y-y_0}{y'(x_0)} = \frac{z-z_0}{z'(x_0)}, \text{法平面:}$$

$$(x-x_0) + y'(x_0)(y-y_0) + z'(x_0)(z-z_0) = 0$$

$$3、\text{曲线方程 } L: \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}, \text{切向量 } \vec{T} = \pm \{F'_x, F'_y, F'_z\}_{M_0} \times \{G'_x, G'_y, G'_z\}_{M_0}, \text{切线:}$$

$$\frac{x-x_0}{\left[\begin{smallmatrix} F_x \\ G_y \end{smallmatrix}\right]_{M_0}}=\frac{y-y_0}{\left[\begin{smallmatrix} F_y \\ G_z \end{smallmatrix}\right]_{M_0}}=\frac{z-z_0}{\left[\begin{smallmatrix} F_z \\ G_x \end{smallmatrix}\right]_{M_0}}$$

四、曲面的切平面和法线

1、曲面方程： $F(x,y,z)=0$ ，法向量： $\vec{n}=\pm\left\{\begin{smallmatrix} F_x \\ F_y \\ F_z \end{smallmatrix}\right\}_{M_0}$ ，切平面：

$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$ ，法线：

$$\frac{(x-x_0)}{F_x(x_0,y_0,z_0)}=\frac{(y-y_0)}{F_y(x_0,y_0,z_0)}=\frac{(z-z_0)}{F_z(x_0,y_0,z_0)}$$

2、曲面方程： $z=f(x,y)$ ，切平面

$$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)-(z-z_0)=0,$$

法线：
$$\frac{x-x_0}{f_x(x_0,y_0)}=\frac{y-y_0}{f_y(x_0,y_0)}=\frac{z-z_0}{-1}$$

五、方向导数：
$$\left.\frac{\partial f}{\partial l}\right|_{M_0}=f_x\Big|_{M_0}\cos\alpha+f_y\Big|_{M_0}\cos\beta+f_z\Big|_{M_0}\cos\gamma$$

梯度：
$$\operatorname{gradu}\Big|_{M_0}=\left\{f_x,f_y,f_z\right\}_{M_0}$$

第九章：重积分

一、二重积分：

$$\iint_D f(x,y)d\sigma=\iint_D f(x,y)dxdy=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy=\int_c^d dy\int_{\psi_1(x)}^{\psi_2(x)} f(x,y)dx$$

$$\iint_D f(\rho\cos\theta,\rho\sin\theta)\rho d\rho d\theta=\int_\alpha^\beta d\theta\int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(\rho\cos\theta,\rho\sin\theta)\rho d\rho$$

二、三重积分：

1、直角坐标系：
$$\iiint_\Omega f(x,y,z)dV=\iint_{D_{xy}} dxdy\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz$$

$$\iiint_\Omega f(x,y,z)dv=\int_{c_1}^{c_2} dz\iint_{D(z)} f(x,y,z)dxdy.$$

2、柱面坐标系：
$$\begin{cases} x=r\cos\theta, \\ y=r\sin\theta, \, dv=rdrd\theta \, dz, \\ z=z. \end{cases}$$

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$$\iiint_{\Omega} f(x, y, z) dv = \int_{\alpha}^{\beta} d\theta \int_{\rho_1(\theta)}^{\rho_2(\theta)} dr \int_{z_1(\rho, \theta)}^{z_2(\rho, \theta)} f(\rho \cos \theta, \rho \sin \theta, z) \rho dz .$$

3、球面坐标系：

$$\begin{cases} x = r \sin \varphi \cos \theta, \\ y = r \sin \varphi \sin \theta, \\ z = r \cos \varphi. \end{cases} dv = r^2 \sin \varphi dr d\varphi d\theta,$$

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \int_{\alpha}^{\beta} d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} d\varphi \int_{r_1(\theta, \varphi)}^{r_2(\theta, \varphi)} f(r \sin \varphi \cos \theta, r \sin \varphi \sin \theta, r \cos \varphi) r^2 \sin \varphi dr.$$

二、重积分的应用：

1、体积： $V = \iiint_{\Omega} dx dy dz = \iint_{D_{xy}} [z_2(x, y) - z_1(x, y)] dx dy$

2、曲面 Σ ： $z = f(x, y)$ 面积： $S = \iint_{D_{xy}} \sqrt{1 + f_x'^2(x, y) + f_y'^2(x, y)} dx dy$

3、质量： $M = \iint_D \rho(x, y) d\sigma$ 或 $M = \iiint_{\Omega} \mu(x, y, z) dv$

4、质心 (\bar{x}, \bar{y}) ：

$$\bar{x} = \frac{\iint_D x \rho(x, y) d\sigma}{M}, \bar{y} = \frac{\iint_D y \rho(x, y) d\sigma}{M} \text{ 或}$$

$$\bar{x} = \frac{\iiint_{\Omega} x \mu(x, y, z) dv}{\iiint_{\Omega} \mu(x, y, z) dv}, \bar{y} = \frac{\iiint_{\Omega} y \mu(x, y, z) dv}{\iiint_{\Omega} \mu(x, y, z) dv}, \bar{z} = \frac{\iiint_{\Omega} z \mu(x, y, z) dv}{\iiint_{\Omega} \mu(x, y, z) dv}$$

5、转动惯量： $I_x = \iint_D y^2 \rho(x, y) d\sigma, I_y = \iint_D x^2 \rho(x, y) d\sigma, I_o = \iint_D (x^2 + y^2) \rho(x, y) d\sigma$

或 $I_x = \iiint_{\Omega} (y^2 + z^2) \mu(x, y, z) dv, I_y = \iiint_{\Omega} (z^2 + x^2) \mu(x, y, z) dv$
 $I_z = \iiint_{\Omega} (x^2 + y^2) \mu(x, y, z) dv, I_o = \iiint_{\Omega} (x^2 + y^2 + z^2) \mu(x, y, z) dv$

第十章：曲线积分和曲面积分

一、第一类曲线积分：（对弧长的曲线积分）：

$$\int_L f(x, y) ds = \int_{\alpha}^{\beta} f(\varphi(t), \psi(t)) \sqrt{\varphi'(t)^2 + \psi'(t)^2} dt = \int_a^b f(x, y(t)) \sqrt{1 + y'^2(t)} dx$$

$$= \int_{\alpha}^{\beta} f(\rho(\theta) \cos \theta, \rho(\theta) \sin \theta) \sqrt{\rho^2(\theta) + \rho'^2(\theta)} d\theta$$

$$\int_L f(x, y, z) ds = \int_{\alpha}^{\beta} f(\varphi(t), \psi(t), \omega(t)) \sqrt{\varphi'(t)^2 + \psi'(t)^2 + \omega'(t)^2} dt$$

二、第二类曲线积分（对坐标的曲线积分）：

1、计算公式：

$$\int P(x,y)dx+Q(x,y)dy=\int [P(x,y)\cos \alpha +Q(x,y)\cos \beta]ds$$

$$=\int_a^b [P(\varphi (t),\psi (t))\varphi '(t)+Q(\varphi (t),\psi (t))\psi '(t)]dt$$

2、格林公式：

$$\iint_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy=\oint_{\partial D^+}Pdx+Qdy=\int_{\partial D}(P\cos \alpha +Q\cos \beta)ds$$

3、Stokes 公式：

$$\text{Stokes 公式：}\oint_{\Gamma}Pdx+Qdy+Rdz=$$

$$\iint_{\Sigma}\begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}=\iint_{\Sigma}\begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}dS=\pm\iint_{D_{xy}}f(x,y,z)dxdy$$

$$4、\text{封闭曲线围城的面积：}A=\frac{1}{2}\oint_{\partial D^+}xdy-ydx$$

三、第一类曲面积分：

$$\Sigma:z=z(x,y):\iint_{\Sigma}f(x,y,z)dS=\iint_{D_{xy}}f(x,y,z(x,y))\sqrt{1+Z_x^2+Z_y^2}dxdy$$

四、第二类曲面积分：

1、计算公式：

$$\iint_{\Sigma}\mathbf{F}(x,y,z)d\mathbf{S}=\iint_{\Sigma}P(x,y,z)dydz+Q(x,y,z)dzdx+R(x,y,z)dxdy$$

$$=\iint_{\Sigma}\mathbf{F}(x,y,z)\cdot\mathbf{e}_n dS=\iint_{\Sigma}(P\cos \alpha +Q\cos \beta +R\cos \gamma)dS$$

$$\iint_{\Sigma}R(x,y,z)dxdy=\iint_{\Sigma}R[x,y,z(x,y)]dxdy;\iint_{\Sigma}R(x,y,z)dxdy=-\iint_{\Sigma}R[x,y,z(x,y)]dxdy$$

$$\iint_{\Sigma_{\text{上侧}}}P(x,y,z)dydz=\pm\iint_{D_{yz}}p(x(y,z),y,z)dydz;\iint_{\Sigma_{\text{下侧}}}Q(x,y,z)dzdx=\pm\iint_{D_{zx}}p(x,y(z,x),z)dzdx$$

2、投影转化法：

$$\Sigma:z=z(x,y),dydz=\frac{\cos \alpha}{\cos \gamma}dxdy=-z_xdxdy,dzdx=\frac{\cos \beta}{\cos \gamma}dxdy=-z_ydxdy$$

$$\Sigma:F(x,y,z)=0,dydz=\frac{F_x}{F_z}dxdy,dzdx=\frac{F_y}{F_z}dxdy$$

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3、高斯公式：

$$\oiint_{\Sigma} Pdydz + Qdzdx + Rdxdy = \iiint_{\Omega} (P \cos \alpha + Q \cos \beta + R \cos \gamma) dS$$

$$= \pm \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV. \quad (\Sigma \text{ 为 } \partial\Omega \text{ 外侧时取+; } \Sigma \text{ 为 } \partial\Omega \text{ 内侧时取-})$$

$$\vec{A}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}, \quad u = u(x, y, z) \Rightarrow$$

$$\text{散度: } \operatorname{div} \vec{A} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}; \text{ 梯度: } \operatorname{gradu} = (u_x, u_y, u_z)$$

$$\operatorname{div}(\operatorname{gradu}) = u_{xx} + u_{yy} + u_{zz}; \text{ 旋度: } \operatorname{rot} \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

第十一章 无穷级数

一、常数项级数 $\sum_{n=1}^{\infty} u_n$

$$1、\text{常用级数: 等比级数/几何级数: } \sum_{n=0}^{\infty} q^n \begin{cases} \text{收} = \frac{1}{1-q} & |q| < 1 \\ \text{发} & |q| \geq 1 \end{cases}$$

$$P\text{级数: } \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{收} & p > 1 \\ \text{发} & 0 < p \leq 1 \end{cases}; \text{ 交错}P\text{级数: } \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p} \text{ 收敛} \begin{cases} \text{绝对收敛} & p > 1 \\ \text{条件收敛} & 0 < p \leq 1 \end{cases}$$

2、正项级数: $u_n \geq 0$

基本定理: 收敛 \Leftrightarrow 部分和有上界 $S_n < \sigma$

比较审敛法: 大收小收, 小发大发

比较审敛法的极限形式: 同阶: 同收同发; 低阶: 同收; 高阶: 同发

$$\text{比值/根值审敛法: } \rho = \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} \quad (\rho = \lim_{n \rightarrow \infty} \sqrt[n]{u_n}) \Rightarrow \begin{cases} < 1, \text{ 收敛} \\ > 1, \text{ 发散} \\ = 1, \text{ 失效} \end{cases}$$

$$3、\text{交错级数: } \sum_{n=1}^{\infty} (-1)^{n-1} u_n \quad (u_n \geq 0)$$

$$\text{莱布尼茨审敛法: } \begin{cases} u_{n+1} \leq u_n \\ \lim_{n \rightarrow \infty} u_n = 0 \end{cases} \Rightarrow \text{级数收敛, } S \leq u_1, |r_n| \leq u_{n+1}$$

$$\text{绝对收敛: } \sum_{n=1}^{\infty} |u_n| \text{ 收敛} \Rightarrow \sum_{n=1}^{\infty} u_n \text{ 收敛, 条件收敛: } \sum_{n=1}^{\infty} u_n \text{ 收敛而 } \sum_{n=1}^{\infty} |u_n| \text{ 发散, 发散}$$

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4、任意项级数：

- 利用定义：部分和有极限 $\lim_{n \rightarrow \infty} S_n = \begin{cases} S, & \text{收敛} \\ \infty, & \text{发散} \end{cases}$;
- 利用收敛的必要条件： $\lim_{n \rightarrow \infty} u_n \neq 0 \Rightarrow$ 发散；
- 利用正项级数（比值/根植）审敛法：

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| \left(\rho = \lim_{n \rightarrow \infty} \sqrt[n]{|u_n|} \right) \Rightarrow \begin{cases} < 1, & \text{绝对收敛} \Rightarrow \text{收敛} \\ > 1, & \text{绝对值发散} \Rightarrow \text{发散} \\ = 1, & \text{失效} \end{cases}$$

二、幂级数： $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

1、收敛半径： $\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \left(\rho = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \right) \Rightarrow R = \begin{cases} 1/\rho, & 0 < \rho < \infty \\ 0, & \rho = \infty \\ \infty, & \rho = 0 \end{cases}$

2、常用等式：

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (|x| < 1), \quad \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \quad (|x| < 1), \quad \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x} \quad (|x| < 1)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \quad (-1 \leq x < 1), \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1+x) \quad (-1 < x \leq 1)$$

$$\sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2} \quad (|x| < 1),$$

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1} = \sum_{n=1}^{\infty} \frac{1}{2n-1} x^{2n-1} = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| \quad (|x| < 1)$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad (|x| < 1)$$

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$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots; \quad x \in (-\infty, +\infty)$$

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} + \cdots; \quad x \in (-\infty, +\infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots; \quad x \in (-\infty, +\infty)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots; \quad x \in (-1, 1]$$

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} x^n$$

$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n + \cdots; \quad x \in (-1, 1)$$

3、泰勒展开：

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n, \quad R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1}, \quad (\xi \in (x_0, x))$$

$$\Leftrightarrow \lim_{n \rightarrow \infty} R_n(x) = 0$$

三、傅里叶级数： $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$1、T = 2\pi: f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = S(x),$$

($x \in (-\infty, +\infty)$, 且 $x \neq$ 间断点)

$$\text{其中 } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, (n=0, 1, 2, \dots); \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, (n=1, 2, \dots).$$

$$(\text{间断点处}, S(x) = \frac{f(x^-) + f(x^+)}{2})$$

$$\text{若 } f(x) \text{ 为奇函数} \Rightarrow \text{正弦级数} \quad (a_n = 0, b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx);$$

$$\text{若 } f(x) \text{ 为偶函数} \Rightarrow \text{余弦级数} \quad (b_n = 0, a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx);$$

$$2、T = 2l: f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}), \quad (x \in (-\infty, +\infty), \text{ 且 } x \neq \text{间断点})$$

$$\text{其中 } a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx, (n=0, 1, 2, \dots); \quad b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx, (n=1, 2, \dots).$$

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3、非周期函数 $f(x)$,

(1) $x \in [-l, l]$: $f(x)$ — 周期延拓 $\rightarrow F(x)$ 展开 \rightarrow 限制

$$f(x) = S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}), (x \in (-l, l))$$

$$(x = \pm l \text{ 时}, S(x) = \frac{f(-l+) + f(l-)}{2})$$

(2) $x \in [0, l]$: $f(x)$ — 奇延拓/偶延拓 \rightarrow — 周期延拓 $\rightarrow F(x)$ 展开 \rightarrow 限制

$$\text{奇延拓: } f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, (x \in (0, l));$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad (n = 1, 2, \dots); (x = 0 \text{ 或 } l \text{ 时}, S(x) = 0);$$

$$\text{偶延拓: } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad (x \in [0, l])$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx \quad (n = 0, 1, 2, \dots), \text{ 端点处不间断。}$$

第十二章 微分方程

一、基本类型的一阶微分方程：

1、可分离变量方程： $\frac{dy}{dx} = f(x)g(y)$, 分离变量，两边积分 $\int \frac{dy}{g(y)} = \int f(x) dx$

2、一阶线性微分方程： $\frac{dy}{dx} + P(x)y = Q(x)$

$$\begin{cases} Q(x) = 0 & \text{齐次: 通解: } y = e^{-\int P(x) dx}, \\ Q(x) \neq 0 & \text{非齐次: 通解: } y = e^{-\int P(x) dx} (\int Q(x) e^{\int P(x) dx} dx + C) \end{cases}$$

3、全微分方程： $P(x, y)dx + Q(x, y)dy = 0$ (其中 $P_y = Q_x$)

通解： $u(x, y) = C$. (1)、分项组合法；

(2)、特殊路径法： $u(x, y) = \int_{x_0}^x P(x, y_0) dx + \int_{y_0}^y Q(x_0, y) dy = C$.

(3)、偏积分法；

$$\left. \begin{aligned} P(x, y) = \frac{\partial u}{\partial x} &\Rightarrow u(x, y) = \int P(x, y) dx + c(y) \\ Q(x, y) = \frac{\partial u}{\partial y} &\Rightarrow c'(y) = Q - \frac{\partial}{\partial y} \int P(x, y) dx = \varphi(y) \end{aligned} \right\} \Rightarrow u(x, y) = \int P(x, y) dx + \int \varphi(y) dy$$

二、可化为基本类型的一阶微分方程：

(1) 齐次方程： $\frac{dy}{dx} = f(\frac{y}{x})$ 或 $\frac{dy}{dx} = f(\frac{ax+by}{ax^2+by^2})$, 令 $u = \frac{y}{x}$

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(2) 准齐次方程: $\frac{dy}{dx} = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$

$$\left\{ \begin{array}{l} \text{若 } \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \text{ 令 } \begin{cases} x = X + h \\ y = Y + k \end{cases}, \text{ (} h, k \text{ 由 } \begin{cases} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{cases} \text{ 解得)} \\ \Rightarrow \frac{dY}{dX} = f\left(\frac{a_1X+b_1Y}{a_2X+b_2Y}\right), \text{ 再令 } u = \frac{Y}{X}. \\ \text{若 } \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0, \frac{dy}{dx} = \frac{k(a_1x+b_1y)+c}{a_1x+b_1y+c_2} = f(a_1x+b_1y). \text{ 令 } u = a_1x+b_1y. \end{array} \right.$$

(3) $\frac{dy}{dx} = f(ax+by+c)$ 令 $u = ax+by+c$.

(4) 伯努利方程: $\frac{dy}{dx} + P(x)y = Q(x)y^\alpha (\alpha \neq 0, 1)$, 令 $z = y^{1-\alpha} \Rightarrow \frac{dz}{dx} + (1-\alpha)P(x)z = (1-\alpha)Q(x)$

(5) $P(x, y)dx + Q(x, y)dy = 0$ (其中 $P_y \neq Q_x$) $\Rightarrow \frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)}$

(6) 关于x的线性方程 / 伯努利方程:

$$\frac{dx}{dy} + P(y)x = Q(y); \quad \frac{dx}{dy} + P(y)x = Q(y)x^\alpha, \text{ 令 } z = x^{1-\alpha}$$

(7) $P(x, y)dx + Q(x, y)dy = 0$ (其中 $P_y \neq Q_x$)

求积分因子方法:

1、分项组合法: 常用全微分公式;

2、公式法:

(1) 方程有形如 $u(x)$ 的积分因子 $\Leftrightarrow \frac{1}{Q} (P_y - Q_x) = \varphi(x) \Rightarrow u(x) = ce^{\int \varphi(x) dx}$

(2) 方程有形如 $u(y)$ 的积分因子 $\Leftrightarrow \frac{1}{P} (P_y - Q_x) = \psi(y) \Rightarrow u(y) = ce^{-\int \psi(y) dy}$

(3) 齐次方程的积分因子 $u(x, y) = \frac{1}{xP + yQ}$

三、可降阶的高阶微分方程:

(1) $\frac{d^ny}{dx^n} = f(x)$ 连续积分n次;

(2) $y'' = f(x, y')$, 令 $y' = p$, 则 $y'' = p' \Rightarrow p' = f(x, p)$

(3) $y'' = f(y, y')$, 令 $y' = p$, 则 $y'' = p \frac{dp}{dy} \Rightarrow p \frac{dp}{dy} = f(y, p)$

四、二阶常系数齐次线性微分方程

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$$y'' + py' + qy = 0 \Leftrightarrow \text{特征方程: } r^2 + pr + q = 0$$

$$\Delta = p^2 - 4q > 0, r_1 \neq r_2 \Rightarrow \text{通解: } y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$$\Delta = p^2 - 4q = 0, r_1 = r_2 \Rightarrow \text{通解: } y = (C_1 + C_2 x) e^{r_1 x}$$

$$\Delta = p^2 - 4q < 0, r_{1,2} = \alpha \pm i\beta \Rightarrow \text{通解: } y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

四、二阶常系数非齐次线性微分方程

$$y'' + p y' + q y = f(x) \text{ 通解 } y(x) = \text{齐次通解 } Y(x) + \text{非齐次特解 } y^*(x)$$

$$(1) f(x) = e^{\lambda x} P_m(x) \Rightarrow \text{特解形式 } y^* = x^k Q_m(x) e^{\lambda x} \begin{pmatrix} \lambda \text{不是特征根} & k=0 \\ \lambda \text{是特征单根} & k=1 \\ \lambda \text{是特征重根} & k=2 \end{pmatrix}$$

$$(2) f(x) = f(x) = e^{\lambda x} [P_l(x) \cos \omega x + P_n(x) \sin \omega x]$$

$$\Rightarrow \text{特解形式 } y^* = x e^{\lambda x} [R_m^{(1)}(x) \cos \omega x + R_m^{(2)}(x) \sin \omega x] \begin{pmatrix} \lambda + i\omega \text{不是特征根} & k=0 \\ \lambda + i\omega \text{是特征根} & k=1 \end{pmatrix}$$