On Splines and Their Use in Computer Graphics and Generating Curves

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Abstract

Splines are used everywhere around us, from fonts to animations. They are the way that computers deal with curves and because of that have a large range of possible uses. Since they are so useful, there are also a lot of ways to generate these functions and lots of formats they come in. In general, they are fairly computationally cheap which makes them so useful and can be extended in a multitude of ways. In this paper, the goal would be to use splines to smoothly interpolate values either for control or for path generation.

Contents

1 Introduction

Splines are not a specific equation, they are more a class of equations that have similar properties. They are simply piecewise polynomials, and are often parameterized so that they could "loop over themselves" and break the vertical line test. One of the interesting aspects of splines is that they keep this property even at low degrees. The word spline is derived from wooden splines that curve given some constraints and are often used in things like braces for ships. The major benefit of splines is how they are suited for computers. Almost all curves on a computer from fonts to animation paths are described quickly and accurately by Bèzier curves and other such splines. This means that these splines are around us all of the time and are almost invisible if you do not know where to look. There are many different kinds of splines with different methods of formulation. However, this paper will mostly be focused on Hermite and Bèzier curves.

2 Bèzier Curves

2.1 Theory

Bèzier curves can be made with a linear combination of Bernstein polynomials (Casselman, 1984). Bernstein polynomials were first used in the proof of the Stone-Weierstrass theorem which states that every continuous function defined on a closed interval $[a, b]$ can be approximated as closely as desired by a polynomial (Pinkus, 2000). A Bernstein polynomial (2) is defined by a linear combination of Bernstein basis functions (1).

Bernstein Polynomials:

$$b_{\nu,n} = \binom{n}{\nu} x^{\nu}(1-x)^{\nu-n} \tag{1}$$

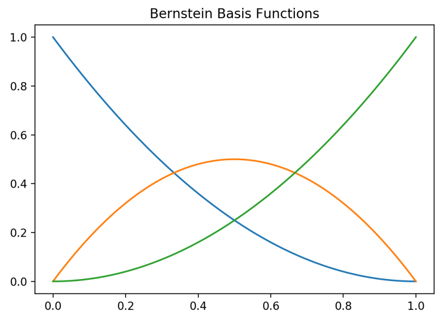
$$B_n(x) = \sum_{\nu=0}^{n} \beta_{\nu} b_{\nu,n}(x) \tag{2}$$

$$\lim_{n\to\infty} B_n(f) = f \tag{3}$$

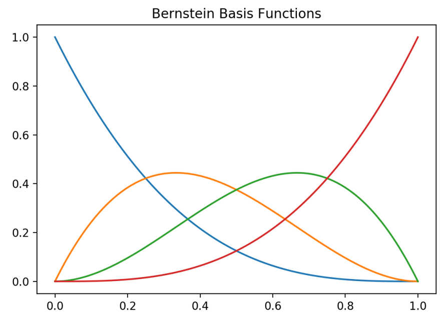
n: is the degree of the Bernstein function

ν: is number of the $n+1$ equations of a function of degree n

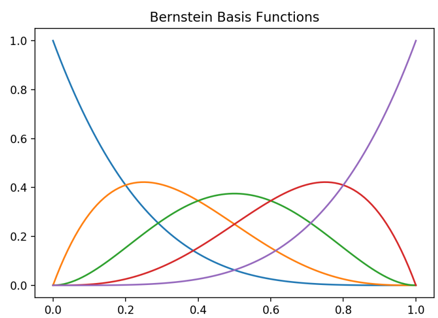
Figure 1: Basis functions for different degrees, Listing 1



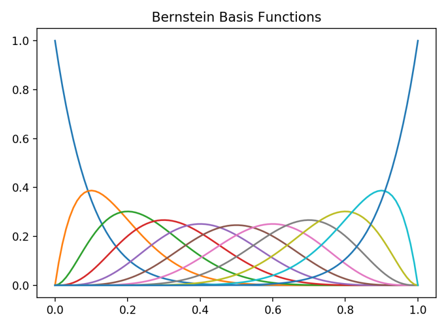
(a) $n = 2$



(b) $n = 3$



(a) $n = 4$



(b) $n = 10$

Bernstein functions are only approximations and to perfectly match the original function you would need an infinite degree Bernstein polynomials similar to Taylor series (3). The algorithm you use to generate a Bézier curve given your control points is called De Castlejau's algorithm. It is a recursive algorithm (4) that takes your control points and uses them for interpolation. This method is a little more complicated than the explicit form (6) which is a summation of the control points times the basis functions of a given degree n. Then to generate any possible curve, parameterize the X and Y axes and fun De Castlejau's algorithm on the scalar x and y quantities.

$$\beta_n^{(0)} = \beta_\nu, i = 0, \ldots, n \tag{4}$$
$$\beta_i^{(j)} = \beta_\nu^{(j-1)}(1 - t_0) + \beta_\nu^{(j-1)} t_0 \tag{5}$$

$j = 1, \ldots, n$

$\nu = 0, \ldots, n - j$

$$B(t) = \sum_{\nu=0}^{n} \beta_\nu b_{\nu,n}(t) \tag{6}$$

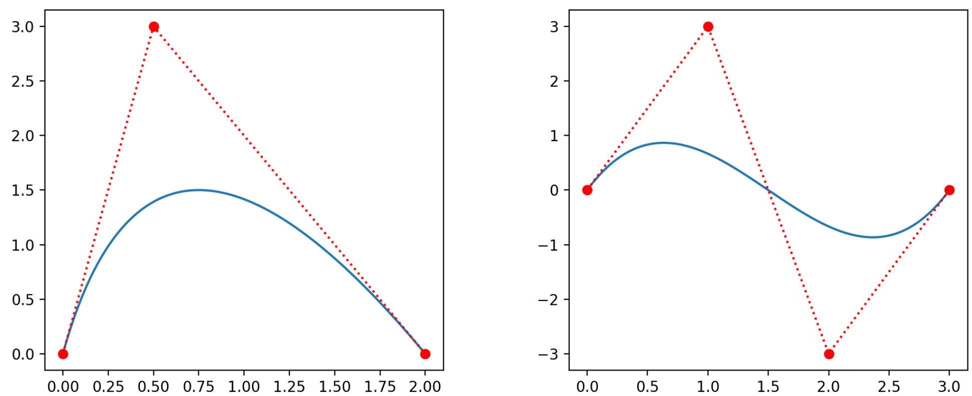
β_ν: is the ith control point

$b_{\nu,n}(t)$: is the Bernstein basis function for the number and degree

2.2 Application

There are practically infinite number of uses for Bèzier curves. They and other similar methods are used to display all curves in computers which means that anything make industrially with a curve has a Bèzier curve or other spline involved. However, the hidden ubiquitous use of Bèzier curves in particular are fonts. All PostScript fonts, the basis of PDFs, use cubic Bèzier curves (a) and TrueType fonts, abbreviated to ttf, use quadratic curves (b).

Figure 3: Cubic and Quadratic Curves, Listing 2



(a) Cubic curve, 3 control points (b) Quadratic curve, 4 control points

In fact, most of the computer generated graphics like advertisements contain Bèzier curves. The popular software used for such things is Adobe Illustrator®which has tools for generating curves with specific parameters being 2 anchor and 2 control points which is exactly the quadratic Bèzier curve.

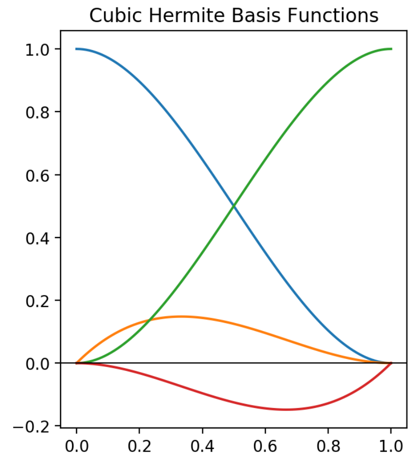
The other use for Bèzier curves are in optimization problems. Since they are used to approximate a curve and the control point can be related to the derivative of a function, they are very useful for optimization problems. They can be used to minimize path lengths (Jusko, 2016) and can even have restrictions such as a maximum acceleration or velocity (G. Jolly et al., 2009). The acceleration and velocity limits are caused by the control points which can be controlled since they are tangential to their anchor points meaning the angle is related to the derivative of the function and the magnitude of the control point is related to the acceleration or multiple control points. These values can even be used to model resistance in a parameterized manner so yet another use of Bèzier curves would be the generation of an optimal airfoil (Rogalsky et al., 2000).

3 Hermite

3.1 Theory

A cubic Hermite spline takes two control points and the derivatives. There are also higher order Hermite splines such as quintic splines that take 2 points their derivatives as well as their second derivatives. For this section, however, we will be focusing on cubic splines. These cubic splines have 4 basis functions much like the Bernstein basis functions for Bèzier curves. In fact, you can express Hermite basis functions in terms of their Bernstein counterparts (equation 1); however, a linear combination of Bernstein Basis functions approximates any polynomial. Two of those functions are for the points and then there are 2 for the derivatives.

Figure 4: Hermite Bases



$$
\begin{aligned}
h_{0,0}(t) &= (1+2t)(1-t)^2 = \beta_0(t) + \beta_1(t) \\
h_{1,0}(t) &= t(1-t)^2 = \frac{\beta_1(t)}{3} \\
h_{0,1}(t) &= t^2(3-2t) = \beta_2(t) + \beta_3(t) \\
h_{1,1}(t) &= t^2(1-t) = \frac{-\beta_2(t)}{3}
\end{aligned} \tag{7}
$$

$h_{0,0}$: is function for the first point

$h_{0,1}$: is function for the first derivative

$h_{1,0}$: is function for the second point

$h_{1,1}$: is function for the second derivative

β_n: Bernstein basis function with degree 3 for index n

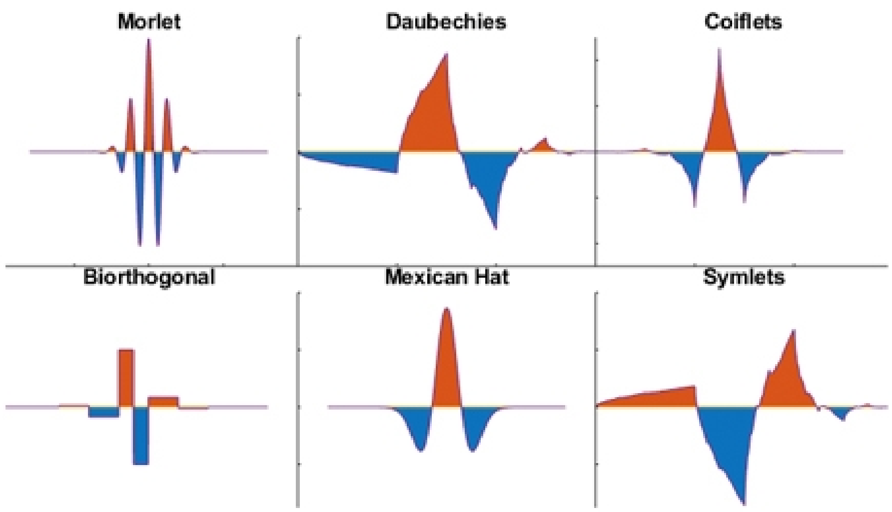
Another way to describe Hermite splines as well as Bèzier curves is through matrices (8). These matrices are equivalent the basis functions. The benefit of using matrices is that it is easier to change between different types of splines for different purposes. For example if one were making a application they could pass in the basis matrix to generate Hermite, Bèzier, Catmull-Rom, or other forms.

$$p = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{n-1} \\ p \\ p_{n+1} \\ p_{n+2} \end{bmatrix} \tag{8}$$

3.2 Application

Interpolating splines are useful for a host of different applications. Unlike Bèzier curves, Hermite splines are not as ubiquitous, but they have more specific uses. They are often used for statistics since they can generate intermediate values between data points as well as being used for regression and filtering (Giles, 2010). Hermite splines are also used for path generation for simple mobile or differential robots. In the case of (Boeing, 2008) they use genetic algorithms to optimize Hermite paths to simulate bipedal, tripedal, and "snake-like" motion. One of the biggest uses of Hermite splines are wavelets (Averbuch et al., 2007).

Figure 5: The different Types of Wavelets



These are wavelets because of how they are not infinite like sin waves. So in signal processing there are Fourier Transforms which isolates the frequency and amplitude of a signal which is the global view. Wavelets allow one to find the local transformations such as edges or blobs. Most of the purely academic uses of Hermite splines are in modeling different forms of wavelets or other signal analysis. For example, in (Uhlmann et al., 2014) they are using wavelets for edge detection as well as for modeling the blobs, or objects that make said edges, to automatically analyze medical photos. As previously alluded to, these interpolating splines can be used for smoothing or filtering data. Hermite splines can be used to model the output of a wave (Park et al., 2008) which is an inversion of the wavelets which take a spline and output data, these process can be reversed to map data to splines in signal processing.

4 Conclusion

Bèzier and Hermite splines are just two examples in a sea of possible splines. These are two simple types that are useful for outlining the two archetypes, interpolating and weighted, while still being computationally simple. There are often "better" or more specialized splines for any given use case; however, that misses the point of their power and generality. The naively simple task of making an approximation of curve between points has much more depth than one can possibly cover. The simplicity and generality also explains how splines are everywhere.

References

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5 Appendix

Listing 1: Bernstein Basis Functions

```python
from numpy import linspace as lin  # linespace
from scipy.special import binom  # binomial coefficient
import matplotlib.pyplot as plt  # plotting
from math import pow # power function

x = lin(0, 1, 100)  # Input as 100 floats between 0 and 1
num: int = 10  # The degree

# returns the value from the Bernstein basis function at the point x
def bbf(i: int, n: int, x: float) -> float:
    return binom(n, i) * pow(x, i) * pow((1 - x), (n-i))

# Iterates through all of the functions for the degree
for c in range(num + 1):
    # Plots all of the functions
    plt.plot(x, [bbf(c, num, i) for i in x])

plt.title("Bernstein Basis Functions")  # Title
plt.show()  # displays graphs
```

Listing 2: Bèzier Curves

```python
from numpy import linspace as lin  # linespace
from scipy.special import binom  # binomial coefficient
import matplotlib.pyplot as plt  # plotting
from math import pow # power function

cpx: [float] = [0.0, 0.5, 2.0]
cpy: [float] = [0.0, 3.0, 0.0]

t = lin(0, 1, 100)  # Input as 100 floats between 0 and 1
num: int = 2  # The degree 0 indexed

# returns the value from the Bernstein basis function at the point x
def bbf(i: int, n: int, x: float) -> float:
    return binom(n, i) * pow(x, i) * pow((1 - x), (n-i))

# Generalized De Casteljau's Explicit formula
def f(a: [float], t: float) -> float:
    ret = 0
    for i in range(num + 1):
        ret += a[i] * bbf(i, num, t)
    return ret

def x(t: float) -> float: return f(cpx, t) # implemented on X's control points
def y(t: float) -> float: return f(cpy, t) # implemented on Y's control points
```

```

plt.plot([x(i) for i in t], [y(i) for i in t]) # plots X and Y
plt.plot(cpx, cpy, 'ro') # plots the control points
plt.plot(cpx, cpy, 'r:') # plots lines between control points
plt.show()
```