Theoretical particle physics

Chenhuan Wang

January 21, 2020

Contents

0 Organisational

Tutorials: Thursday: 8-10, 10-12 Friday: 10-12, 13-15

Exam   consists of four problems

- first quickies

- 2nd-4th: similar in style to homework; two will be very close to homework

One needs 50% of points from homework. May hand in pairs.

Content of the lectures

- Standard Model of particle physics

- Electroweak sector

    - gauge principle

    - Higgs mechanism

    - Yukawa interactions

    - CP-violation

Exercises

- go through basics of computing Feynman diagrams

- not to derive the formalism

- Lagrange $\rightarrow$ Feynman rules $\rightarrow$ amplitudes $\rightarrow$ cross section and decay rates (measured quantities)

Literature

- Halzen and Martin, Quarks and Leptons (a lot of basics of QCD)

- Cheng and Li (includes also quantum field theory topics CP-violation in Standard Model)

- Mark Thomson

- QFT basics

    - Peskin and Schroeder

    - M.Schwartz

    - Ryder

- Okun, Leptons and Quarks

1 Introduction

1.1 Standard Model

The Standard Model is the fundamental theory of nature. There are three interactions included

- electromagnetic

- weak

- strong

- Higgs boson exchange

Electromagnetic and weak interactions can be unified into electroweak interactions.
  In the Sun, all these three interactions and gravity are present

- Photons reaching us clearly indicating QED's presence.

- Neutrinos produced in weak interaction. Four protons to two protons and two neutrons (Helium). Only weak interaction can change the colour of quarks.

- Binding of Helium via strong interaction and binding energy released as kinetic energy.

- Gravity brings protons together and at high temperature to give Helium.

Gauge theories   (Lie groups algebras)

- EM: $\mathbf{U}(1)_{\text{EM}}$, $\mathbf{U}(1)_{\text{Y}}$

- weak: $\mathbf{SU}(2)_{\text{L}}$

- strong: $\mathbf{SU}(3)_{\text{C}}$

Forces in quantum theories involve exchange particles spin 1 vector bosons

- Electromagnetism with photon γ

- Weak interaction with $W^{\pm}$, W^0 and Z^0 (mixture of W^0 and hyper charge) (discovered at CERN)

- Strong interaction with $g^a, a = 1, \ldots, 8$ gluons (discovered at DESY)

particles with spin $\frac{1}{2}$
Leptons

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, e_R^-; \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \mu_R^-; \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, \tau_R^-$$

Quarks

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, u_R, d_R; \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, c_R, s_R; \quad \begin{pmatrix} t \\ b \end{pmatrix}_L, t_R, b_R$$

One complete generation

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, e_R^-; \quad \begin{pmatrix} u \\ d \end{pmatrix}_L, u_R, d_R;$$

To remove any one part, then gauge theory is inconsistent. It is known as "anomaly".

Higgs boson h^0, spin 0
In the Standard Model , Higgs bosons are described by complex scalar fields $\begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$. h^0 is the only fundamental scalar in nature, as far as we know.

1.2 Energy scales

- Binding energy of atoms $1 - 10$eV
- Binding energy of nucleons ≈ 1MeV
- No known binding energy in particle physics
- Protons and neutrons $\approx \Lambda_{QCD} \approx O(100\text{MeV})$

Particles have masses

- Electron $m_e = 511\,\text{keV}$
- Muon $m_\mu = 105\,\text{MeV}$
- Tau $m_\tau = 1.7\,\text{GeV}$
- Neutrinos $m_\nu < 1\,\text{eV}$
- Quarks*
    - $m_u \approx 3\,\text{MeV}$
    - $m_d \approx 5\,\text{MeV}$
    - ...
- Photon $m_\gamma = 0$
- Gluon $m_g = 0$
- Higgs $m_{Higgs} \approx 125\,\text{GeV}$

---

*mass of proton mainly comes from dynamical effect "gluon"

Colliders

- Large Electron-Positron Collider (LEP) operating 1989-2000: $91\,\text{GeV} - 200\,\text{GeV}$

- Tevatron($p\bar{p}$-collider) operating 1983-2011: $800\,\text{GeV} - 2\,\text{TeV}$

- LHC operating since 2010: $7\,\text{TeV} - 13\,\text{TeV}$

1.3 Natural units

It is convenient to use the unit system, where we define

$$\hbar = c = 1, \tag{1.3.1}$$
$$k_B = 1. \tag{1.3.2}$$

Everything is expressed in term of powers of energy.

$$1\,\text{fm} = 1 \times 10^{-15}\,\text{m} = 5\,\text{GeV}^{-1}$$

2 Lorentz Transformation

2.1 Introduction

Metric (used for distance measuring) is defined as

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.1.1}$$

In string and general relativity, people tend to use the signature $\mathrm{diag}(-,+,+,+)$.

By defining the four-momentum, the relativistic dispersion relation can be given

$$p = (E, \boldsymbol{p}), \tag{2.1.2}$$

$$p^2 = E^2 - \boldsymbol{p}^2 = m^2. \tag{2.1.3}$$

Light is always light-like

$$k^2 = t^2 - (x^2 + y^2 + z^2) = 0.$$

Greek indices always go from 0 to 3

$$r^2 = g_{\mu\nu} r^\mu r^\nu = t^2 - \boldsymbol{r}^2. \tag{2.1.4}$$

Distance between two spacetime point is defined via

$$|r_A - r_B| = \sqrt{(r_A - r_B) \cdot (r_A - r_B)} = \sqrt{r_A^2 + r_B^2 - 2r_A \cdot r_B}. \tag{2.1.5}$$

2.2 Lorentz Transformation

Lorentz transformation is a transformation between two inertial frames moving with constant velocity $\boldsymbol{v}$ with respect to each other (boosts).

$$x = (x_0, x_1, x_2, x_3),$$
$$x' = (x'_0, x'_1, x'_2, x'_3),$$
$$x_0 = t.$$

We define (in SI unit)

$$\beta = \frac{v}{c} \quad \text{or} \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c}. \tag{2.2.1}$$

Coordinates in these two frames are related like

$$x'_0 = \gamma(x_0 - \beta x_1),$$
$$x'_1 = \gamma(x_1 - \beta x_0),$$
$$x'_2 = x_2,$$
$$x'_3 = x_3.$$

Inverse transformation can be achieved with $\beta \mapsto -\beta$
  γ-factor is defined via

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}. \tag{2.2.2}$$

Since the particles cannot travel faster than light, $|\boldsymbol{\beta}| \leq 1$, then $\gamma \geq 1$.
  Alternative parametrization using rapidity ζ

$$\beta = \tanh(\zeta), \quad \gamma = \cosh(\zeta),$$
$$\gamma\beta = \sinh(\zeta).$$

Insert this into equation (2.2.2)

$$x'_0 = x_0 \cosh(\zeta) - x_1 \sinh(\zeta),$$
$$x'_1 = x_0 \sinh(\zeta) - x_1 \cosh(\zeta).$$

We can turn this into matrices

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix},$$
$$x = \Lambda x'. \tag{2.2.3}$$

2.3 Mathematical Properties of Lorentz Transformation

Distance is invariant under Lorentz transformation

$$s^2 = x_0^2 - x_1^2 + x_2^2 + x_3^2 = x^2. \tag{2.3.1}$$

Lorentz transformation includes

- Rotation and boosts

- Parity $\boldsymbol{x} \mapsto -\boldsymbol{x}$

- Time reversal $t \mapsto -t$

We can also expand it with translation. It then turns to Poincaré group.

2.3.1 Tensors

Define a function of original coordinates ($\alpha = 0, 1, 2, 3$)

$$x'^{\alpha} = x'^{\alpha}(x^0, x^1, x^2, x^3). \tag{2.3.2}$$

If x'^{α} transforms like

$$x'^{\alpha} = \frac{x'^{\alpha}}{x^{\beta}} x^{\beta}, \tag{2.3.3}$$

it is called <u>contravariant</u>

Consider derivative $\frac{\partial}{\partial x'^{\alpha}}$

$$\frac{\partial f(x)}{\partial x'^{\alpha}} = \frac{\partial f(x)}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\alpha}}. \tag{2.3.4}$$

We can see the x' is now in the denominator. The objects transformed like this are called <u>covariant</u>.

Consider the following generic objects: A'^{α} contravariant vector

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta}. \tag{2.3.5}$$

B'_{α} is covariant

$$B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta}. \tag{2.3.6}$$

Note (x^0, x^1, x^2, x^3) is contravariant.

The field strength tensor

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta},$$

is contravariant rank 2. Mixed is also allowed

$$H'^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{x^{\delta}}{\partial x'^{\beta}} H^{\delta}_{\gamma}.$$

2.3.2 Inner or scalar product

Lorentz transformation is an isometry with Minkowski scalar product

$$B' \cdot A' = B'_{\alpha} A'^{\alpha},$$

$$= \left(\frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} \right) \left(\frac{\partial x'^{\alpha}}{\partial x^{\gamma}} A^{\gamma} \right),$$

$$= \frac{\partial x^{\beta}}{\partial x^{\gamma}} B_{\beta} A^{\gamma},$$

$$= \delta^{\beta}_{\gamma} B_{\beta} A^{\gamma},$$

$$= B \cdot A.$$

Infinitesimal distance is

$$ds^2 = \left(dx^0 \right)^2 - (d\boldsymbol{x})^2 ,$$

$$= (g_{\alpha\beta} \, dx^{\alpha}) \, dx^{\beta} = dx_{\beta} \, dx^{\beta} . \tag{2.3.7}$$

Thus we can use metric tensor to lower index $\mathrm{d}x_\beta = g_{\alpha\beta}\,\mathrm{d}x^\alpha$

$$A^\alpha = \left(A^0, \boldsymbol{A}\right),$$
$$A_\alpha = \left(A^0, -\boldsymbol{A}\right).$$

2.4 Matrix Representation of Lorentz Transformation

2.4.1 General Properties

We have

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \quad gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}.$$

Then $a \cdot b = (a, gb) = g_{\mu\nu}a^\mu b^\nu = (ga, b) = a^T gb = (ga)^T b$

Lorentz transforming the coordinate

$$x'^\mu = \Lambda^\mu_{\ \nu}x^\nu \Leftrightarrow x' = \Lambda x. \tag{2.4.1}$$

$$
\begin{aligned}
g_{\mu\nu}x^\mu x^\nu &= g_{\sigma\tau}x'^\sigma x'^\tau, \\
&= g_{\sigma\tau}\Lambda^\sigma_{\ \mu}x^\mu \Lambda^\tau_{\ \nu}x^\nu, \\
&= g_{\sigma\tau}\Lambda^\sigma_{\ \mu}\Lambda^\tau_{\ \nu}x^\mu x^\nu.
\end{aligned}
$$

Then we have the *defining* rule of Lorentz group

$$
\begin{aligned}
g_{\mu\nu} &= g_{\sigma\tau}\Lambda^\sigma_{\ \mu}\Lambda^\tau_{\ \nu}, \\
g &= \Lambda^T g\Lambda.
\end{aligned}
\tag{2.4.2}
$$

Properties   of Lorentz transformation

- $|\det(\Lambda)| = 1$

- $\left|\Lambda^0_{\ 0}\right| \geq 1$

The orthochronous Lorentz transformations ($ $)$\Lambda$ forms a group.

Parity does not form a group

$$\Lambda_P = \mathrm{diag}(1, -1, -1, -1) \tag{2.4.3}$$

Time reversal

$$\Lambda_T = \mathrm{diag}(-1, +1, +1, +1) \tag{2.4.4}$$

There are four classes of Lorentz transformations depending on $\left(\mathrm{sgn}(\det(\Lambda)), \mathrm{sgn}(\Lambda^0_{\ 0})\right)$

- $(+,+)\ \Lambda$

- $(-,-)\ \Lambda_T \Lambda$

- $(-,+)\ \Lambda_P \Lambda$

- $(+,-)\ \Lambda_T \Lambda_P \Lambda$

Orthochronous Λ has 6 parameters, 3 for boosts and 3 for rotations. $\Lambda^T g \Lambda = g$ is actually 16 equations. All matrices here are symmetric. Thus 6 of 16 are redundant. There are 10 independent equations. Λ has 16 entries and it has $16 - 10 = 6$ free parameters.

2.4.2 Explicit Construction

We will restrict ourselves in *orthochronous* Lorentz transformations. The exponential function is defines via Taylor expansion. With $L \in \mathbb{R}^{4 \times 4}$

$$\Lambda = e^L = \exp(L).$$

From linear algebra we know (because of the properties of determinant and trace)

$$\det(\Lambda) = \det\!\left(e^L\right) = e^{\mathrm{tr}(L)}. \tag{2.4.5}$$

Since $\det(\Lambda) = 1$, $\mathrm{tr}(L) = 0$, gL is anti-symmetric

$$\Lambda^T g \Lambda = g,$$
$$g \Lambda^T g \Lambda = \mathbb{1}_4,$$
$$g \Lambda^T g = \Lambda^{-1},$$
$$\exp\!\left(g L^T g\right) = \Lambda^{-1} = \exp(-L),$$
$$\Leftrightarrow g L^T g = -L,$$
$$\Leftrightarrow (gL)^T = -gL.$$

Thus

$$L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & L_{12} & 0 & L_{23} \\ L_{03} & L_{13} & L_{23} & 0 \end{pmatrix}.$$

Define 6 basis matrices $S_{1,2,3}$ and $K_{1,2,3}$

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

S_i is the generator of 3-dimensional rotations and K_i is the generator of 3-dimensional boosts. WHAT IS THIS?

$$\hat{\boldsymbol{n}} \in \mathbb{R}^3, \quad |\hat{\boldsymbol{n}}| = 1,$$
$$\hat{\boldsymbol{n}} \cdot \boldsymbol{S} = n_1 S_1 + n_2 S_2 + n_3 S_3,$$
$$(\hat{\boldsymbol{n}} \cdot \boldsymbol{S})^3 = -\hat{\boldsymbol{n}} \cdot \boldsymbol{S},$$
$$(\hat{\boldsymbol{n}} \cdot \boldsymbol{K})^3 = +\hat{\boldsymbol{n}} \cdot \boldsymbol{S}.$$

In the end, with $\boldsymbol{\omega}, \boldsymbol{\zeta} \in \mathbb{R}^3$

$$L = -\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\zeta} \cdot \boldsymbol{K},$$
$$\Lambda = \exp(-\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\zeta} \cdot \boldsymbol{K}).$$

$\boldsymbol{\omega}$ is the axis of rotation, $|\boldsymbol{\omega}|$ is then the angle of rotation. $\tanh |\boldsymbol{\zeta}| = \beta$ and $\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|}$ is the direction of boost. We now will look at concrete examples

- $\boldsymbol{\zeta} = 0, \boldsymbol{\omega} = w\hat{e}_z$

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \omega & \sin \omega & 0 \\ 0 & -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Rotational angle is ω.

- $\boldsymbol{\omega} = 0, \boldsymbol{\zeta} = \zeta \hat{e}_x$

$$\Lambda = \begin{pmatrix} \cosh \zeta & -\sinh \zeta & 0 & 0 \\ -\sinh \zeta & \cosh \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Pure general boost $\boldsymbol{\zeta}$

$$\Lambda = \exp(-\boldsymbol{\zeta} \cdot \boldsymbol{K}),$$
$$\boldsymbol{\zeta} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|} \tanh^{-1} |\boldsymbol{\beta}|, \quad \hat{\boldsymbol{\beta}} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|},$$
$$\Lambda = \exp\left(-\hat{\boldsymbol{\beta}} \cdot \boldsymbol{K} \tanh^{-1}(\beta)\right).$$

2.4.3 Algebra of generators

Consider the commutation algebra of $S_{i=1,2,3}$ and $K_{i=1,2,3}$

$$\left[S_i, S_j\right] = \epsilon_{ijk} S_k, \tag{2.4.6}$$
$$\left[S_i, K_j\right] = \epsilon_{ijk} K_k, \tag{2.4.7}$$
$$\left[K_i, K_j\right] = -\epsilon_{ijk} S_k. \tag{2.4.8}$$

The last equation causes Thomas precession in atomic physics.

Choose a different basis

$$S_+ = \frac{1}{2}(S + iK),$$ (2.4.9)

$$S_- = \frac{1}{2}(S - iK).$$ (2.4.10)

Then we can calculate the algebra

$$\left[S_{+,i}, S_{+,j}\right] = i\epsilon_{ijk}S_{+,k},$$ (2.4.11)

$$\left[S_{-,i}, S_{-,j}\right] = i\epsilon_{ijk}S_{-,k},$$ (2.4.12)

$$\left[S_{+,i}, S_{-,j}\right] = 0.$$ (2.4.13)

In other word, the algebras are decoupled. This familiar algebra is angular momentum algebra **SU**(2).

Classification by two numbers (j_+, j_-)

$$j_+ = 0, \frac{1}{2}, 1, \ldots,$$

$$j_- = 0, \frac{1}{2}, 1, \ldots.$$

Dimension $= (2j_+ + 1)(2j_- + 1)$.

A field is scalar field if $j_+ = j_- = 0$. One fundamental example of scalar field is Higgs boson. Other scalar particles are just bound states.

There are two possible states with spin $\frac{1}{2}$: $(j_+, j_-) = (\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. For $(\frac{1}{2}, 0)$ it is

$$S_+ = \frac{1}{2}\sigma,$$

$$S_- = 0,$$

$$S = \frac{1}{2}\sigma,$$

$$iK = \frac{1}{2}\sigma.$$

For $(0, \frac{1}{2})$ there is "-".

So two types of spin $\frac{1}{2}$ fermions

$$\left(\frac{1}{2}, 0\right) \to e_L^-,$$

$$\left(0, \frac{1}{2}\right) \to e_R^-.$$

They have different transformation law. W boson only to e_L^- but photon couples to both.

Under parity transformation $e_L^- \leftrightarrow e_R^-$. Both particles are needed for theory to be invariant under parity transformation, like EM and strong interactions.

3 Relativistic Quantum Field Theory

In non-relativistic quantum mechanics, the dispersion relation is

$$E = \frac{\boldsymbol{p}^2}{2m}.$$

After promoting the momentum and energy into operators,

$$E \mapsto i\hbar \frac{\partial}{\partial t},$$

$$\boldsymbol{p} \rightarrow -i\hbar \boldsymbol{\nabla},$$

we have the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(x,t) + \frac{1}{2m}\boldsymbol{\nabla}^2 \psi(x,t) = 0. \qquad (3.0.1)$$

Density of probability is defined via

$$\rho = |\psi|^2 = \psi\psi^*. \qquad (3.0.2)$$

It obeys the continuity equation

$$-\frac{\partial}{\partial t}\int_V \rho \, \mathrm{d}V = \int \boldsymbol{j} \cdot \boldsymbol{n} \, \mathrm{d}S \,,$$

$$= \int_V \boldsymbol{\nabla} \cdot \boldsymbol{j} \, \mathrm{d}V \,,$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0. \qquad (3.0.3)$$

Writing this explicitly

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}\left(\psi\psi^*\right),$$

$$= \psi\frac{\partial \psi^*}{\partial t} + \psi^*\frac{\partial \psi}{\partial t},$$

$$= \frac{i}{2m}\left(\psi^*\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\psi\right),$$

$$\Rightarrow \boldsymbol{j} = -\frac{i}{2m}\left(\psi^*\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\psi\right). \qquad (3.0.4)$$

If we have a plane wave state, as an example

$$\psi = N\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x} - iEt},$$

$$\boldsymbol{j} = \frac{\boldsymbol{p}}{m}|N|^2.$$

3.1 Relativistic wave equation

Now we enter the relativistic regime and the dispersion relation is

$$E^2 = \boldsymbol{p}^2 + m^2.$$

Four-momentum is defined as before

$$p^\mu = (E, \boldsymbol{p}) \quad p_\mu = (E, -\boldsymbol{p}),$$
$$p^2 = m^2.$$

Promoting energy and momentum into operators,

$$p^\mu \mapsto i\partial^\mu,$$
$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial^2 t} - \nabla^2.$$

we have then Klein-Gordon (KG) equation

$$(\partial_\mu \partial^\mu + m^2)\phi(\boldsymbol{x}, t) = 0. \tag{3.1.1}$$

The current in KG-theory is conserved as well

$$j^\mu = (\rho, \boldsymbol{j}) = i\left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right) \tag{3.1.2}$$
$$\partial_\mu j^\mu = 0 \tag{3.1.3}$$

An example solution is

$$\phi = N e^{-ip \cdot x},$$
$$j^\mu = 2p^\mu |N|^2.$$

In terms of Lorentz transformation

$$\rho \sim E.$$

Energies of particles

$$E^2 = \boldsymbol{p}^2 + m^2,$$
$$E = \pm \sqrt{\boldsymbol{p}^2 + m^2}.$$

It also implies negative probability

$$E > 0 \mapsto \rho > 0,$$
$$E < 0 \mapsto \rho < 0.$$

3.2 Feynman-Stückelberg Interpretation of negative energy states

Current of "electron" with $E, \boldsymbol{p}$ and charge $-e$ is
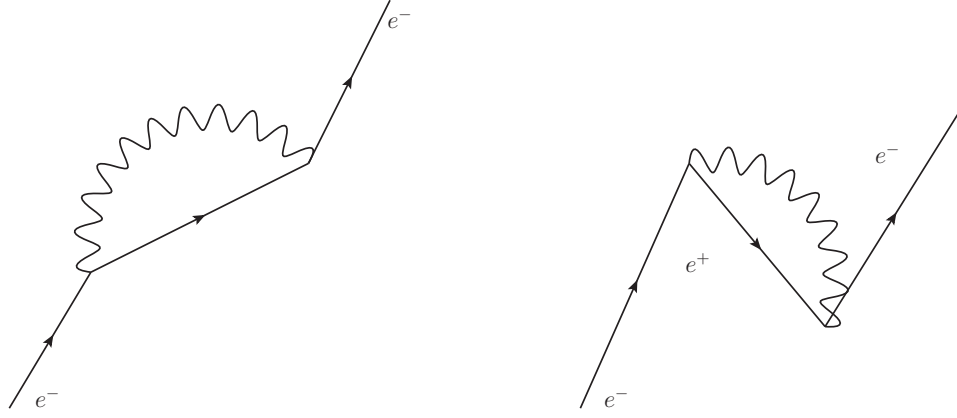
$$j^\mu_{e^-} = 2e|N|^2 (E, \boldsymbol{p})$$

Figure 3.1: scattering process; horizontal time-axis; in the second diagram a electron positron pair is produced

Current of "positron" with $E, \boldsymbol{p}$ and charge $+e$ is

$$j^{\mu}_{e^+} = 2e|N|^2(E, \boldsymbol{p}) = -2e|N|^2(-E, -\boldsymbol{p})$$

We can think of $E < 0$ solution as particle flying backwards in time or $E > 0$ anti-particle forwards in time.

In a relativistic systems we need to remember following points

- anti-particles

- particle numbers are not conserved

3.3 Electrodynamics (spin 1)

Maxwell equations are

$$\boldsymbol{E} = -\vec{\nabla}\phi - \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}, \tag{3.3.1}$$

$$\boldsymbol{B} = \vec{\nabla} \times \boldsymbol{A}, \tag{3.3.2}$$

$$\vec{\nabla} \times \boldsymbol{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B}, \tag{3.3.3}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{3.3.4}$$

Field strength tensor and four-potential are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3.3.5}$$

$$A^\mu(x) = (\phi, \boldsymbol{A}).$$

The fields can be calculated from it

$$E^i = F^{0i} = \partial^i A^0 - \partial^0 A^i, \tag{3.3.6}$$

$$B_i = -\epsilon_{ijk}\partial^i A^k = -\epsilon_{0ijk}F^{jk}. \tag{3.3.7}$$

Often it is useful to use the dual tensor

$$\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\sigma\tau} F^{\sigma\tau}. \tag{3.3.8}$$

The second set of maxwell equations is

$$\partial^{\mu} \tilde{F}_{\mu\nu} = 0. \tag{3.3.9}$$

The other set of two equations is

$$\partial_{\nu} F^{\mu\nu} = 4\pi j^{\mu}. \tag{3.3.10}$$

E, B are observable, A is not. A^{μ} is not uniquely fixed by E and B. It has the following gauge symmetry

$$\tilde{A}_{\mu} = A_{\mu} + \partial_{\mu}\Lambda(x, t). \tag{3.3.11}$$

Use this transformation to get the gauge conditions

$$\partial_{\mu} A^{\mu} = 0. \tag{3.3.12}$$

Plugging it back then we have the relativistic wave equation

$$\partial_{\mu} \partial^{\mu} A^{\nu} = 0. \tag{3.3.13}$$

it essentially is Klein-Gordon equation with mass $m = 0$.

A^{μ} is a vector with spin 1

$$(j_+, j_-) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

It implied it has two transverse degrees of freedom. It has spin 1 properties: $+1, 0, -1$, in which 0 mode does not exist.

3.4 Description of Fermions

Original motivation for Dirac is that he wants a linear equation in E or $\frac{\partial}{\partial t}$

$$p^{\mu} \mapsto i\partial^{\mu}.$$

Take the ansatz

$$ih\frac{\partial}{\partial t}\psi = H\psi,$$
$$= (\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m)\psi,$$

but $\boldsymbol{\alpha}$ and β unknown. It still has to obey the relativistic energy relation

$$H^2 = (\alpha_i p_i + \beta m)(\alpha_i p_i + \beta m),$$
$$\overset{!}{=} \boldsymbol{p}^2 + m^2,$$
$$= \alpha_i \alpha_j p_i p_j + \beta^2 m^2 + \alpha_i \beta p_i m + \beta \alpha_j p_j m.$$

From this we demand

$$\beta^2 = 1, \tag{3.4.1}$$

$$\alpha_i^2 = 1, \tag{3.4.2}$$

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0, \tag{3.4.3}$$

$$\alpha_i\beta + \beta\alpha_i = 0. \tag{3.4.4}$$

So α and β cannot be just $\mathbb{C}$ or $\mathbb{R}$ numbers (cannot be Grassmann valued either), but (can be proven to be) hermitian traceless matrices with eigenvalue ± 1. In addition, it only exits in even dimensions. Since α_i and β are 4×4 matrices. ψ has to be a 4-component spinor.

For parity conservation need $(\frac{1}{2}, 0) \bigoplus (0, \frac{1}{2})$. Thus

$$\begin{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix}_{2\times2} & \\ & \begin{pmatrix} & \\ & \end{pmatrix}_{2\times2} \end{pmatrix}.$$

There are different sets of α_i, β which satisfy the conditions. They are called representations. Dirac-Pauli representation

$$\alpha_i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \tag{3.4.5}$$

$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \tag{3.4.6}$$

with σ^i the Pauli matrices.

Weyl (chiral) representation

$$\alpha^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \tag{3.4.7}$$

$$\beta = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{3.4.8}$$

They are mainly used in high energy physics ($E \gg m$).

3.4.1 Gamma Matrices

We now define 4 gamma matrices γ^μ, $\mu = 0, 1, 2, 3$

$$\gamma^\mu = (\beta, \beta\boldsymbol{\alpha}). \tag{3.4.9}$$

Note that having an index does not make it Lorentz vector.

The *Clifford algebra* is defined as following

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{3.4.10}$$

In Dirac-Pauli representation, they are

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \tag{3.4.11}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{3.4.12}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.4.13}$$

Gamma matrices γ^0 and γ^5 are interchanged in Weyl representation.

Rewriting the Dirac equation using gamma matrices

$$i\partial_t \psi = (\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m)\,\psi$$

$$i\partial_t \psi = -i\boldsymbol{\alpha} \cdot \vec{\nabla}\psi + m\beta\psi$$

$$i\beta\partial_t \psi = -i\beta\boldsymbol{\alpha} \cdot \vec{\nabla}\psi + m\psi$$

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0 \tag{3.4.14}$$

Conventionally we use ϕ for spin 0 particle and A_μ for spin 1.

It is convenient to also have an equation for $\psi^\dagger$. First one can show $\gamma^{\dagger\mu} = \gamma^0 \gamma^\mu \gamma^0$.

- $\mu = 0$: $\gamma^0 = \beta$ and $\gamma^{\dagger 0} = \gamma^0 \gamma^0 \gamma^0 \Rightarrow \beta^2 = \mathbb{1}_4$.

- $\gamma^{\dagger\mu} = (\beta\alpha^k)^\dagger = (\alpha^k)^\dagger \beta^\dagger = \alpha^k \beta = \beta^2 \alpha^k \beta = \beta\gamma^k \beta = \gamma^0 \gamma^k \gamma^0$.

$$i\gamma^0 \partial_0 \psi + i\gamma^k \partial_k \partial - m\psi = 0,$$

$$-i\partial_0 \psi^\dagger (\gamma^0)^\dagger - i(\partial_k \psi^\dagger)\gamma^{\dagger k} - m\psi^\dagger = 0,$$

$$-i\partial_0 \psi^\dagger \gamma^0 - i(\partial_k \psi^\dagger)\gamma^0 \gamma^k \gamma^0 - m\psi^\dagger = 0,$$

define $\bar{\psi} = \psi^\dagger \gamma^0$

$$, -i\partial_0 \bar{\psi}\gamma^0 - i\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0,$$

$$i(\partial_\mu \bar{\psi})\gamma^\mu + m\bar{\psi} = 0. \tag{3.4.15}$$

3.4.2 Free Particle Solution to Dirac Equation

The Dirac equation read

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0,$$

multiplying $\gamma^\nu \partial_\nu$ from left

$$i\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \psi - m\gamma^\nu \partial_\nu \psi = 0,$$

$$i\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \psi + im^2 \psi = 0.$$

The product of two gamma matrices is

$$\gamma^\mu \gamma^\nu = \frac{1}{2}\left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right),$$

$$= \frac{1}{2}\left(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu + 2g^{\mu\nu}\right),$$

$$= \frac{1}{2}[\gamma^\mu, \gamma^\nu] + g^{\mu\nu}.$$

The commutator is anti-symmetric and multiplying to symmetric tensor (derivatives in the equation) the term must vanish.

We then find that each component of spinor satisfies the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\psi_i = 0. \tag{3.4.16}$$

Thus we can write the solution as plane-wave

$$\psi = u(p)\mathrm{e}^{-ipx} \tag{3.4.17}$$

$u(p)$ is also a 4-component object but as function p not x.

Insert this back into Dirac equation, then we have Dirac equation in momentum space

$$\left(\gamma^\mu p_\mu - m\right)u(p) = 0. \tag{3.4.18}$$

Solution by considering Dirac-Pauli representation

$$\left(\not{p} - m\right)u(p) = \begin{pmatrix} (E - m)\mathbb{1} & -\boldsymbol{p}\cdot\boldsymbol{\sigma} \\ \boldsymbol{p}\cdot\boldsymbol{\sigma} & -(E + m)\mathbb{1} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix}.$$

If the reference frame is so chosen (rest frame) that $\boldsymbol{p} = 0$ then $E = \pm m$.

- $E = +m$

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- $E = -m$

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In the case when $\boldsymbol{p} \neq 0$, there are a set of coupled equations

$$\begin{aligned} \boldsymbol{\sigma}\cdot\boldsymbol{p}\, u_B &= (E - m)u_A, \\ \boldsymbol{\sigma}\cdot\boldsymbol{p}\, u_A &= (E + m)u_B. \end{aligned} \tag{3.4.19}$$

Again there are two cases

- $E > 0$
  Define two-component spinors

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

  Make the ansatz

$$u_A^{(s)} = \chi^{(s)}, \qquad u_B^{(s)} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E + m}\chi^{(s)},$$

$$u(\boldsymbol{p}) = N\begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \end{pmatrix}. \tag{3.4.20}$$

- $E < 0$

$$u(\boldsymbol{p}) = N\begin{pmatrix} -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \tag{3.4.21}$$

One can show $u^{\dagger(r)}u^{(s)} = N^2 \delta^{rs}$.

Two-fold degeneracy (in energy) in each case. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $E > 0$ and $E < 0$. There must be another observable which commutes with H and $\boldsymbol{p}$.

$$H = \gamma^i p_i + \gamma^0 m, \boldsymbol{S} \cdot \hat{P} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} \cdot \boldsymbol{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{p} \end{pmatrix}. \tag{3.4.22}$$

Helicity operator is defined by

$$\frac{1}{2}\boldsymbol{\sigma} \cdot \hat{p} = \frac{1}{2} \begin{pmatrix} \hat{p}_z & \hat{p}_x + i\hat{p}_y \\ \hat{p}_x - i\hat{p}_y & -\hat{p}_z \end{pmatrix}, \tag{3.4.23}$$

$$\det(\boldsymbol{\sigma} \cdot \hat{p}) = -\hat{p}^2 = -1. \tag{3.4.24}$$

Determinant is the product of two eigenvalues, then

$$\lambda_1 + \lambda_2 = 0$$
$$\lambda_1 \cdot \lambda_2 = 1$$
$$\lambda_{1,2} = \pm 1$$

Antiparticle solution $u^{(3,4)}(-\boldsymbol{p})e^{-i(-p)x} = v^{(2,1)}$

$$\left(\not{p} + m\right)v(\boldsymbol{p}) = 0. \tag{3.4.25}$$

Normalization is

$$\int \rho \, dV = 2E, \tag{3.4.26}$$

$$N = \sqrt{E + m}. \tag{3.4.27}$$

Completeness relation (spin sums)

$$\sum u^{(s)}(p)\bar{u}^{(s)}(p) = \left(\not{p} + m\right), \tag{3.4.28}$$

$$\sum v^{(s)}(p)\bar{v}^{(s)}(p) = \left(\not{p} - m\right). \tag{3.4.29}$$

Define a projector projecting out positive and negative energy states

$$\Lambda_\pm = \frac{\pm\not{p} + m}{2m}. \tag{3.4.30}$$

In Chiral (Weyl) representation, the Dirac equation is

$$\left(\not{p} - m\right)u(\boldsymbol{p}) = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix}\begin{pmatrix} u_L \\ u_R \end{pmatrix}, \tag{3.4.31}$$

with $\bar{\sigma} = (\sigma^0, -\boldsymbol{\sigma})$ and $\sigma^0 = \mathbb{1}_2$.

Weyl equations are

$$-mu_L + p \cdot \sigma u_R = 0,$$
$$p \cdot \bar{\sigma} u_L - m u_R = 0. \tag{3.4.32}$$

If $m = 0$, the equations decouple from each other.

3.4.3 Lorentz Transformation of Spinor

We want to construct Lagrangians involving the fields ψ, A_μ and ψ. The reason we choose Lagrangians as opposed to Hamiltonians, is that Hamiltonian H is associated with energy of system, which is not Lorentz (or relativistic) invariant. But Lagrangian density is $S = \int dt = \int d^4x \mathcal{L}$. In natural unit, the action is dimensionless. One can in addition prove d^4x is Lorentz invariant. The fact that $\mathcal{L}$ is Lorentz invariant, means that $\mathcal{L}$ need to have at least 2 spin-$\frac{1}{2}$ fields ψ or $\bar{\psi}$, to make a spin-0.

We are interested in the objects like $\bar{\psi}(\gamma^\mu)\psi$. Representation of the Lorentz group can be split into $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$. In Weyl or chiral representation, ψ_L has $\left(\frac{1}{2}, 0\right)$ and ψ_R $\left(0, \frac{1}{2}\right)$.

Here we are taking a different approach from the paper [1]. First define

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

(3.4.33)

We want to find out the transformation of $\psi = u(p)e^{-ip\cdot x}$. Note that the second factor is written in covariant form, i.e. Lorentz invairant. Consider the Dirac equation in two different frames $x' = \Lambda x$

$$i\gamma^\mu \frac{\partial\psi(x)}{\partial x^\mu} - m\psi(x) = 0,$$

(3.4.34)

$$i\gamma^\mu \frac{\partial\psi'(x')}{\partial x'^\mu} - m\psi'(x') = 0.$$

(3.4.35)

We make the ansatz that the spinor transforms with S

$$\psi'(x') = S\psi(x).$$

(3.4.36)

S must be independent of x. Plug 3.4.36 into 3.4.35

$$i\gamma^\mu \frac{\partial}{\partial x'^\mu}[S\psi(x)] - mS\psi(x) = 0,$$

$$iS^{-1}\gamma^\mu S \frac{\partial\psi(x)}{\partial x^\nu} \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{=\Lambda^{-1}} - m\psi(x) = 0.$$

This equation must be the same as 3.4.34, then we get

$$S^{-1}\gamma^\mu S \left[\Lambda^{-1}\right]^\nu_\mu = \gamma^\nu,$$

$$S^{-1}\gamma^\mu S = \Lambda^\mu_\nu \gamma^\nu.$$

(3.4.37)

The transformation can be parametrized using $\omega^{\mu\nu}$ or $\boldsymbol{\omega}$ and $\boldsymbol{\zeta}$

$$\psi'_\alpha(x') = \left[\exp\left(-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\right)\right]^\beta_\alpha \psi_\beta$$

(3.4.38)

$$= [\exp(-i\boldsymbol{\omega}\cdot\boldsymbol{S} - i\boldsymbol{\zeta}\cdot\boldsymbol{K})]^\beta_\alpha \psi_\beta$$

(3.4.39)

$M_{\mu\nu}$ is anti-symmetric in μ and ν. The generators are

$$S_i = \frac{1}{2}\epsilon_{ijk}M^{jk},$$

$$K_i = -M_{0i}.$$

Convertion of parameters is

$$\omega_i = \frac{1}{2}\epsilon_{ijk}\omega^{jk},$$
$$\zeta_i = \omega^{0i}.$$

Lorentz transformation containing rotations and boosts, but infinitesimal version (infinitesimally different from $\mathbb{1}$)

$$\Lambda^\nu_{\ \mu} = \delta^\nu_{\ \mu} + \epsilon^\nu_{\ \mu}. \tag{3.4.40}$$

We will show in the exercise the expression of transformation under boosts and rotations (proper orthochronous Lorentz transformation)

$$S_L = \mathbb{1}_4 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu},$$
$$S_L^{-1} = \mathbb{1}_4 + \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}.$$

It satisfies the defining relation of S 3.4.37

$$S^{-1}\gamma^\mu S,$$
$$= \left(\mathbb{1}_4 + \frac{i}{4}\sigma_{\alpha\beta}\epsilon^{\alpha\beta}\right)\gamma^\mu\left(\mathbb{1}_4 - \frac{i}{4}\sigma_{\sigma\lambda}\epsilon^{\sigma\lambda}\right),$$
$$= \gamma^\mu + \frac{i}{4}\left[\sigma_{\alpha\beta}, \gamma^\mu\right]\epsilon^{\alpha\beta} + O\left(\epsilon^2\right),$$
$$= \gamma^\mu + \epsilon^\mu_{\ \nu}\gamma^\nu + O\left(\epsilon^2\right),$$
$$= \Lambda^\mu_{\ \nu}\gamma^\nu + O\left(\epsilon^2\right).$$

Here the commutator and anti-commutator are both used.

Parity transformation cannot be written in infinitesimal form. Thus it is often called (one of) discrete transformation

$$\Lambda_P = \text{diag}(+1, -1, -1, -1). \tag{3.4.41}$$

Parity symmetry is naturally violated. Especially in weak interaction, since neutrinos are left-handed ν_L

$$S_P^{-1}\gamma^\mu S_P = \Lambda^\mu_{P\ \nu}\gamma^\nu. \tag{3.4.42}$$

- For $\mu = 0$

$$S_P^{-1}\gamma^0 S_P = \gamma^0.$$

- For $\mu = i$

$$S_P^{-1}\gamma^i S_P = -\gamma^i.$$

It is easy to show $S_P = \gamma^0$ satisfies the equations. Then the spinor transform under parity like

$$\psi'(t, -\boldsymbol{x}) = \gamma^0\psi(t, \boldsymbol{x}). \tag{3.4.43}$$

In Dirac-Pauli representation $\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \psi' = \begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_3 \\ -\psi_4 \end{pmatrix}.$$

In chiral representation $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$ and

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \psi' = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}.$$

Recall $\bar\psi = \psi^\dagger \gamma^0$, so

$$\begin{aligned} \bar\psi' &= \psi'^\dagger \gamma^0 = \psi^\dagger S^\dagger \gamma^0, \\ &= \psi^\dagger \gamma^0 S^{-1}, \\ &= \bar\psi S^{-1}. \end{aligned}$$

It can easily be shown using the explicit expression of S_L and S_L. Parity transformation is γ^0.
There are 16 different bilinear $\bar\psi A \psi$.

- $A = \mathbb{1}_4$

$$\bar\psi'\psi' = (\bar\psi S^{-1})(S\psi) = \bar\psi\psi.$$

- $A = \gamma^\mu$

$$\bar\psi'\gamma^\mu\psi' = \bar\psi S^{-1}\gamma^\mu S\psi = \Lambda^\mu_{\ \nu}(\bar\psi\gamma^\nu\psi)$$

It transforms under (proper orthochronous Lorentz transformation Λ) exactly like a Lorentz vector. Under parity

$$\bar\psi'\gamma^\mu\psi' = \begin{cases} \bar\psi'\gamma^0\psi', & \mu = 0 \\ -\bar\psi'\gamma^i\psi'. & \mu = 1,2,3 \end{cases}$$

- $A = \gamma^5$

$$\begin{aligned} \bar\psi'\gamma^5\psi' &= \bar\psi S_L^{-1}\gamma^5 S_L\psi, \\ &= \bar\psi\gamma^5\psi. \end{aligned}$$

$\{\gamma^\mu, \gamma^5\} = 0$ and $\sigma_{\mu\nu}$ contains two γs It is a scalar under boosts and rotations. Under parity it is pseudoscalar, like pions

$$\gamma^5 S_P = \gamma^5\gamma^0 = -S_P\gamma^5.$$

Thus

$$\bar\psi'\gamma^5\psi' = -\bar\psi'\gamma^5\psi'.$$

For experiments, look it up in Introduction to HEP, Perkins

Chiral spinors (in chiral representation)

$$P_R = \frac{1}{2}\left(\mathbb{1}_4 + \gamma^5\right) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}, \tag{3.4.44}$$

$$P_L = \frac{1}{2}\left(\mathbb{1}_4 - \gamma^5\right) = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.4.45}$$

If ψ written like $\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$, it is in chiral representation

$$P_L\psi = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} = \psi_L,$$

$$P_R\psi = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix} = \psi_R.$$

These operators are projectors

$$P_R^2 = \frac{1}{4}(\mathbb{1}_4 + \gamma^5)(\mathbb{1}_4 + \gamma^5) = \frac{1}{4}(\mathbb{1}_4 + 2\gamma^5 + (\gamma^5)^2) = P_R,$$

$$P_L^2 = P_L.$$

They also project onto complete space meaning $P_L + P_R = \mathbb{1}_4$.

Left-handed particles have spin and momentum in the opposite direction and right-handed in the same direction. In Dirac-Pauli representation in high energy limit

$$\gamma^5 u^{(5)} \approx \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{p} \end{pmatrix}.$$

To show this take the free particle solution ($E > 0$)

$$u^{(s)} = N\begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E+m}\chi^{(s)} \end{pmatrix}$$

$$\gamma^5 u^{(s)} = N\begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \approx N\begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{p}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

$$= N\boldsymbol{\sigma} \cdot \hat{p}\begin{pmatrix} \chi^{(s)} \\ \boldsymbol{\sigma} \cdot \hat{p}\chi^{(s)} \end{pmatrix} = \boldsymbol{\sigma} \cdot \hat{p}u^{(s)}$$

In general $(\boldsymbol{\sigma} \cdot \boldsymbol{a})^2 = \boldsymbol{a}^2\mathbb{1}$ holds. At high energy limit ($E \gg m$), γ^5 helicity operator, but not at low energy. Chirality $\psi_{L,R}$ is always a good quantum number.

3.4.4 Angular Momentum

There must be another observable which commutes with H and p ($u = u(\boldsymbol{p})$)

$$\boldsymbol{\Sigma} \cdot \hat{p} = \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{p} \end{pmatrix}, \tag{3.4.46}$$

$$H = \boldsymbol{\alpha} \cdot P + \beta m, \tag{3.4.47}$$

$$[H, \boldsymbol{\Sigma} \cdot \hat{p}] = 0. \tag{3.4.48}$$

Angular momentum operator is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{P}. \tag{3.4.49}$$

Use the relation $\left[\hat{x}_i, \hat{P}_j\right] = i\delta_{ij}$ and check the commutation

$$
\begin{aligned}
[H, L_1] &= [\boldsymbol{\alpha} \cdot \boldsymbol{P}, x_2 P_3 - x_3 P_2], \\
&= [\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3, x_2 P_3 - x_3 P_2], \\
&= \alpha_2 [P_2, x_2] P_3 - \alpha_3 [P_3, x_3] P_2, \\
&= -i (\alpha_2 P_3 - \alpha_3 P_2), \\
&= -i(\alpha \times P)_1.
\end{aligned}
$$

Thus $[H, \boldsymbol{L}] = -i(\boldsymbol{\alpha} \times P) \neq 0$. In other word, L not conserved.

But we observe

$$[H, \boldsymbol{\Sigma}] = 2i(\boldsymbol{\alpha} \times P). \tag{3.4.50}$$

Thus define total angular momentum, second term describe the intrinsic angular momentum of particles

$$\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\boldsymbol{\Sigma}, \tag{3.4.51}$$

and J is conserved.

3.4.5 Charge conjugation

Equation describing fermions couple to electromagnetic current

$$\left[\gamma^\mu \left(i\partial_\mu + eA_\mu\right) - m\right]\psi = 0, \tag{3.4.52}$$

with e the electric charge of electron and $A_\mu(x)$ vector potential in electromagnetism.

For positron the charge is the opposite

$$\left[\gamma^\mu \left(i\partial_\mu - eA_\mu\right) - m\right]\psi^C = 0. \tag{3.4.53}$$

ψ^C is the charge conjugate of ψ. We want to know the relation between $\psi \leftrightarrow \psi^C$. Take complex conjugate of equation 3.4.52

$$\left[-(\gamma^\mu)^* \left(i\partial_\mu - eA_\mu\right) - m\right]\psi^* = 0,$$

We postulate a matrix C so that $\psi^C = C\gamma^0\psi^*$ and multiply equation it by $C\gamma^0$ from left

$$C\gamma^0 \left[-(\gamma^\mu)^* \left(i\partial_\mu - eA_\mu\right) - m\right]\psi^* = 0. \tag{3.4.54}$$

It must be the same as equation 3.4.53. Thus

$$
\begin{aligned}
-(C\gamma^0)(\gamma^\mu)^* &= \gamma^\mu C\gamma^0, \\
-C(\gamma^\mu)^T &= \gamma^\mu C, \\
C\gamma^{\mu T} C^{-1} &= -\gamma^\mu, \\
C &= i\gamma^0\gamma^2.
\end{aligned}
\tag{3.4.55}
$$

Then we find

$$\psi^C = C\gamma^0\psi^* = C\bar{\psi}^T. \tag{3.4.56}$$

It would be interesting to compare $P_L(\psi^C)$ and $(P_L\psi)^C$

3.4.6 Continuity equation for ψ

Particle density is

$$\rho = \psi^\dagger \psi = \bar{\psi}\gamma^0 \psi. \tag{3.4.57}$$

Multiply $\bar{\psi}$ from left to dirac equation and ψ from right to conjugated Dirac equation

$$\bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0, \qquad i(\partial_\mu \bar{\psi})\gamma^\mu \psi + m\bar{\psi}\psi = 0,$$
$$\bar{\psi}\gamma^\mu(\partial_\mu \psi) + (\partial_\mu \bar{\psi})\gamma^\mu \psi = 0,$$
$$\partial_\mu(\bar{\psi}\gamma^\mu \psi) = 0,$$
$$\partial_\mu j^\mu = 0.$$

3.5 Classical Field Theory

Reminder Lagrangian and action are

$$L = L(q_i, \dot{q}_i, t), \qquad S = \int dt\, L.$$

Variation principle

$$\delta S = 0,$$

is equivalent to Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}.$$

Going to field theory, we transform $q(t) \rightarrow \phi(x)$ and $L \rightarrow \mathcal{L}$

$$\mathcal{L} = \mathcal{L}\left(\phi, \frac{\partial \phi}{\partial x^\mu}, x_\mu\right),$$
$$S = \int d^4 x\, \mathcal{L}(\phi, \frac{\partial \phi}{\partial x^\mu}, x_\mu).$$

Using the variation principle

$$\delta S = \sum_i \frac{dS}{d\alpha_i}\delta\alpha_i,$$
$$= \sum_i \int d^4 x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial \alpha_i}\delta\alpha_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial(\partial_\mu \phi)}{\partial \alpha_i}\delta\alpha_i \right\},$$
$$= \sum_i \int d^4 x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial \alpha_i} - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial \phi}{\partial \alpha_i}\right) \right\}\delta\alpha_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial \phi}{\partial \alpha_i}\delta\alpha_i \Big|_{x_0}^{x_1}.$$

Then we have our new Euler-Lagrange equations for continuous fields

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0. \tag{3.5.1}$$

Conservation Laws and Noether Theorem   Reminder: Under transformation $t \mapsto t' = t + \delta b$, system stays invariant if $\delta_T L = \frac{\mathrm{d}}{\mathrm{d}t}(\delta\Omega)$. $\delta\Omega = 0$ if L invariant.

The change in L is

$$\delta_T L = L(q, \dot{q}, t + \delta b) - L(q, \dot{q}, t) = \frac{\partial L}{\partial t}\delta b.$$

The time dependence of Lagrangian on time is

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}, \\
&= \frac{\partial L}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}, \\
&= \frac{\partial L}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}}\dot{q}\right].
\end{aligned}$$

System invariant if

$$\frac{\partial L}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}(\delta\Omega). \tag{3.5.2}$$

So we find that if L not dependent explicitly on t, $\mathrm{d}L / \mathrm{d}t = 0$

If $\partial L/\partial t = 0$, the conserved quantity

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[L - \frac{\partial L}{\partial \dot{q}}\dot{q}\right] &= 0, \\
L - \frac{\partial L}{\partial \dot{q}}\dot{q} &= -H = L - \dot{q}p,
\end{aligned}$$

using Legendre transformation and $\partial L/\partial \dot{q}$ the general momentum.

Analogously for fields

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\mu} &= \left(\frac{\partial \mathcal{L}}{\partial \phi}\right)\frac{\partial \phi}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\rho\phi)}\partial_\rho(\partial_\mu\phi), \\
&= \left(\partial_\rho\frac{\partial \mathcal{L}}{\partial(\partial_\rho\phi)}\right)\partial_\mu\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho\phi)}\partial_\rho\partial_\mu\phi, \\
\partial_\mu\mathcal{L} &= \partial_\rho\left[\frac{\partial \mathcal{L}}{\partial(\partial_\rho\phi)}\partial_\mu\phi\right], \\
\Rightarrow \partial_\rho&\left[\mathcal{L}\delta_\mu^\rho - \frac{\partial \mathcal{L}}{\partial(\partial_\rho\phi)}\partial_\mu\phi\right] = 0.
\end{aligned}$$

This conserved quantity is energy momentum tensor T_μ^ρ.

Examples   Real scalar field, $\phi(x) \in \mathbb{R}$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \tag{3.5.3}$$

Since $L = T - V$, we interpret first term as kinetic and second as potential, which has minimum at $\phi = 0$. Since action is dimensionless, $[\phi] = 1$. Take Euler-Lagrange

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi,$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \partial^\mu \phi,$$

put together

$$\left(\partial_\mu \partial^\mu + m^2 \right) \phi = 0.$$

This is just Klein-Gordon equation. In Feynman rules we have the propagator and mass term as interaction. One can also have complex scalar field $\phi(x) \in \mathbb{C}$ and ϕ, ϕ^* are independent fields.

Example   Fermionic field has the following Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi. \tag{3.5.4}$$

$[\psi] = \frac{3}{2}$. $\bar{\psi}$ and ψ are independent fields. Solve Euler-Lagrange we get the familiar Dirac equation

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} i\gamma_\mu \delta^\mu \psi - m\psi = 0. \tag{3.5.5}$$

Example   Spin 1 field $A_\mu(x)$ is just Lagrangian for electromagnetism with source

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu, \tag{3.5.6}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.5.7}$$

Equation of motion can be computed

$$\frac{\partial \mathcal{L}}{\partial (\partial_\rho A_\sigma)} = \frac{1}{4} 4 \delta_{\rho\mu} \delta_{\nu\sigma} F^{\mu\nu} = -F^{\rho\sigma},$$

$$\frac{\partial \mathcal{L}}{\partial A_\sigma} = -j^\sigma,$$

then we have the maxwell equations

$$-\partial_\rho F^{\rho\sigma} + j^\sigma = 0.$$

Using Lorenz gauge we can get

$$\partial_\mu \partial^\mu A^\sigma = j^\sigma.$$

In principle we can add a term $A_\mu A^\mu = A^2$, which is Lorentz invariant by construction

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2} m^2 A_\mu A^\mu. \tag{3.5.8}$$

so each component satisfies Klein-Gordon equation for $j^\mu = 0$. Later we will see j^μ corresponds to an interaction, e.g. $e\bar{\psi}\gamma^\mu \psi$.

3.6 Gauge Theories

Hermann Weyl first considered the gauge theory (*Eichtheorie*). We start with Dirac Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi.$$

If we add a phase $\psi(x) \mapsto \psi'(x) = e^{i\alpha}\psi(x)$ with $\alpha \in \mathbb{R}$ and constant, we find that the Lagrangian stays invariant

$$\begin{aligned}
\mathcal{L}' &= i\bar{\psi}'\gamma_\mu\partial^\mu\psi' - m\bar{\psi}'\psi', \\
&= i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi, \\
&= \mathcal{L}.
\end{aligned}$$

Since α is just a number, it commutes with derivative and gamma matrices. It is a continuous symmetry and one can apply Noether theorem and compute the Noether current. The set of transformation $\left\{e^{i\alpha}\right\}_{\alpha\in\mathbb{R}}$ forms a $\mathbf{U}(1)$ (gauge) group. If the parameter does not depend on coordinates, i.e. constant, it is called global transformation.

Now consider the case $\alpha = \alpha(x)$, not constant but a continuous function. In other word, we allow a local change. The transformation of mass term is

$$\begin{aligned}
\bar{\psi}'\psi' &= e^{-i\alpha(x)}\bar{\psi}e^{i\alpha(x)}\psi, \\
&= \bar{\psi}\psi.
\end{aligned}$$

Kinetic term transforms like

$$\begin{aligned}
\bar{\psi}'\gamma_\mu\partial^\mu\psi' &= e^{-i\alpha(x)}\bar{\psi}\gamma_\mu\partial^\mu\left(e^{i\alpha(x)}\psi\right), \\
&= \bar{\psi}\gamma_\mu\left(\partial^\mu + i\partial^\mu\alpha(x)\right)\psi.
\end{aligned}$$

But we can still demand local invariance and we need to modify $\mathcal{L}$ by introducing a term $\bar{\psi}\gamma_\mu A^\mu(x)\psi$ in Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + \bar{\psi}\gamma_\mu A^\mu\psi. \tag{3.6.1}$$

We also need a transformation property for $A_\mu(x)$ to get Lagrangian invariant

$$A^\mu \mapsto A'^\mu = A^\mu + \partial^\mu\alpha(x). \tag{3.6.2}$$

Then the transformation of this extra term is

$$\begin{aligned}
\bar{\psi}'\gamma_\mu A'^\mu\psi' &= \bar{\psi}e^{-i\alpha(x)}\gamma_\mu\left(A^\mu + \partial^\mu\alpha(x)\right)\cdot e^{i\alpha(x)}\psi, \\
&= \bar{\psi}\gamma_\mu\left(A^\mu + \partial^\mu\alpha(x)\right)\psi.
\end{aligned}$$

The second term cancels the extra terms with $\partial^\mu\alpha(x)$ because of local transformation.

Introduce a charge $e\bar{\psi}\gamma_\mu A^\mu\psi = j_\mu A^\mu$

$$\begin{aligned}
\mathcal{L} &= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + e\bar{\psi}\gamma_\mu A^\mu\psi, \\
&= i\bar{\psi}\gamma^\mu\mathrm{D}_\mu\psi - m\bar{\psi}\psi, \tag{3.6.3}
\end{aligned}$$

with $\mathrm{D}_\mu = \partial_\mu - ieA_\mu$ covariant derivative and $(\mathrm{D}_\mu\psi)' = e^{i\alpha(x)}\mathrm{D}_\mu\psi$.

Gauge field A_μ is introduced into Lagrangian and a kinetic term is needed to obtain a wave equation is needed

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + e\bar{\psi}\gamma_\mu A^\mu\psi.$$

(3.6.4)

Thus we have an additional Feynman rule

 $= -ie\gamma^\mu.$ (3.6.5)

By demanding invariance under local phase transformation, we introduce $A_\mu(x)$ and an interaction term in Lagrangian. We call A_μ a gauge field, meaning that it is not physical observable. The gauge group is an abelian group (commutative).

One can also check a mass term for gauge field $m^2 A_\mu A^\mu$ is not invariant under $U(1)$ gauge. So gauge or local principle excludes photon mass term. Experimentally we are able to measure the value of ξ by looking at the electromagnetic interaction. If photon has mass, the Coulomb force has the following form

$$F \sim \frac{1}{4\pi\epsilon}\frac{q_1 q_2}{r^{2+\xi}}.$$

Calculation of loop diagrams like 3.2 leads to most precisely computed object, anomalous magnetic
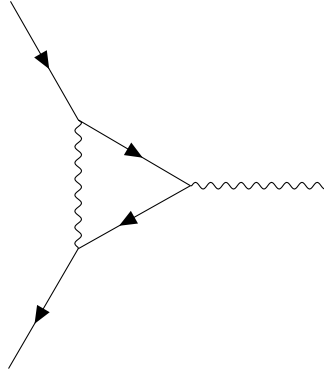

Figure 3.2: Anomalous magnetic dipole moment

moment of electron. In fact this calculation is so accurate, people use this to define the electron charge e.

3.7 Non-Abelian Gauge Theories

We will restrict ourselves in $\mathbf{SU}(N)$ groups, special unitary group of dimension N. It has $N^2 - 1$ generators. Weak interaction is described by $\mathbf{SU}(2)_\mathrm{L}$ and strong interaction $\mathbf{SU}(3)_\mathrm{C}$. The Lagrangian has the following form

$$\mathcal{L}_0 = \bar{q}_j(x)(i\gamma^\mu\partial_\mu - m)q_j,$$

with $j = 1, \ldots, N$. The transformation of the spinor field is

$$q(x) \mapsto q'(x) = e^{i\alpha_a(x)T_a} q(x),$$

with T_a a $N \times N$ matrices (independent of coordinates x) and $a = 1, \ldots, N^2 - 1$. Here the spinors are multi-component spinors. In simplest quark model, they are quark doublets $q_i = \begin{pmatrix} u \\ d \end{pmatrix}$ and u, d are spinors.

Focus on infinitesimal transformations with $\alpha_a(x) \in \mathbb{R}$

$$q(x) \mapsto q'(x) = [\mathbb{1}_N + i\alpha_a(x)T_a] \, q(x), \tag{3.7.1}$$
$$\partial_\mu q \mapsto \partial_\mu [\mathbb{1}_N + i\alpha_a(x)T_a] \, q,$$
$$= (\mathbb{1} + i\alpha_a T_a) \, \partial_\mu q + i(\partial_\mu \alpha_a) T_a q(x),$$

as before kinetic part is (at first) not invariant. We need $N^2 - 1$ gauge fields to compensate this, $\mathcal{G}_\mu^a$ with $a = 1, \ldots, N^2 - 1$.

The covariant derivative is

$$\mathrm{D}_\mu = \partial_\mu + igT_a \mathcal{G}_\mu^a.$$

Then we want to construct $\mathcal{L}$ for Non-Abelian case. Following the previous recipe

$$\mathcal{L} = \bar{q} \left(i\gamma^\mu \partial_m u - m \right) q - g(\bar{q}\gamma^\mu T_a q) \mathcal{G}_\mu^a, \tag{3.7.2}$$

with $g \in \mathbb{R}$ a number independent of x.

Apply transformation 3.7.1 to kinetic term and see if it is sufficient to get invariant $\mathcal{L}$

$$\bar{q}\gamma^\mu T_a q \mapsto \bar{q} \left(\mathbb{1} - i\alpha_a T_b \right) \gamma^\mu T_a (\mathbb{1} + i\alpha_b T_b) q,$$
$$= \bar{q}\gamma^\mu T_a q - i\alpha_b \bar{q} T_b T_a \gamma^\mu q + i\alpha_b \bar{q} T_a T_b \gamma^\mu q + O(\alpha_b^2),$$
$$= \bar{q}\gamma^\mu T_a q + i\alpha_b \bar{q} [T_a, T_b] \gamma^\mu q.$$

The commutator is (summation over c is implied)

$$[T_a, T_b] = i f_{abc} T_c. \tag{3.7.3}$$

with f structure constant. Then

$$\bar{q}\gamma^\mu T_a q \mapsto \bar{q}\gamma^\mu T_a q - f_{abc} \alpha_b \bar{q}\gamma^\mu T_c q. \tag{3.7.4}$$

$\mathcal{L}$ with covariant derivative is still not invariant. It turns out that we must modify the transformation of $\mathcal{G}_\mu^a$ further

$$\mathcal{G}_\mu^a \mapsto \mathcal{G}_\mu^a - \frac{1}{g} \partial_\mu \alpha_a - f_{abc} \alpha_b \mathcal{G}_\mu^c. \tag{3.7.5}$$

So now the Lagrangian 3.7.2 is invariant.

As before we need a kinetic term.

$$\mathcal{L}_{\mathcal{G}}^{\mathrm{kin}} = -\frac{1}{4} \mathcal{G}_{\mu\nu}^a \mathcal{G}_a^{\mu\nu}. \tag{3.7.6}$$

Could it be like photon case? i.e.

$$\mathcal{G}_{\mu\nu}^a = \partial_\mu \mathcal{G}_\nu^a - \partial_\nu \mathcal{G}_\mu^a. \tag{3.7.7}$$

Apply this into the transformation of gauge field and see f_{abc} term doesn't drop out. Then we must modify to

$$\mathcal{G}^a_{\mu\nu} = \partial_\mu \mathcal{G}^a_\nu - \partial_\nu \mathcal{G}^a_\mu - g f_{abc} \mathcal{G}^b_\mu \mathcal{G}^c_\mu. \tag{3.7.8}$$

So in Abelian case, gauge field and its transformation are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

$$A_\mu \mapsto A_\mu - \frac{1}{e} \partial_\mu \alpha,$$

$$F'_{\mu\nu} = \partial_\mu \left(A_\mu + \frac{1}{e} \partial_\nu \alpha \right) - \partial_\nu \left(A_\mu + \frac{1}{e} \partial_\mu \alpha \right),$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu,$$

$$= F_{\mu\nu}.$$

$F_{\mu\nu}$ is gauge invariant. This property is specific to **U**(1)!

Now apply this to Non-Abelian case, you will find $G^a_{\mu\nu}$ is not gauge invariant, but $-\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$ is.

Connection between $\mathcal{L}$ and Feynman rules   Abelian case

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi + e \bar{\psi} \gamma^\mu \psi A^\mu.$$

Kinetic term implies propagators and interaction vertex

$$\sim\!\sim\!\sim\!\sim\!\sim = \frac{-i g^{\mu\nu}}{p^2 + i\epsilon}. \tag{3.7.9}$$

So there is no vertex of photon interacting with each other, photons rarely interfere each other. At higher order one has
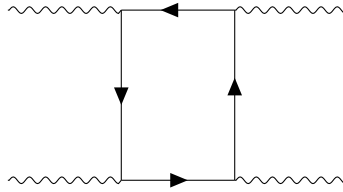

Figure 3.3: photon-photon scattering

In non-Abelian case gluon propagator, interaction vertex$(-ig\gamma^\mu (T^a)_{ij})$ Because of the term in gluon field with differentiation, gluon interacts with itself. Formal way of constructing $\mathcal{G}^a_{\mu\nu}$. The covariant derivative $D_\mu = \partial_\mu + ig T^a \mathcal{G}^a_\mu$ and $\left[T^a, T^b \right] = i f^{abc} T^c$. With some calculation

$$\left[D_\mu, D_\nu \right] = ig T_c \left[\partial_\mu \mathcal{G}^c_\nu - \partial_\nu \mathcal{G}^c_\mu - g f_{abc} \mathcal{G}^a_\mu \mathcal{G}^b_\nu \right],$$

$$= ig T_c \mathcal{G}^c_{\mu\nu}.$$

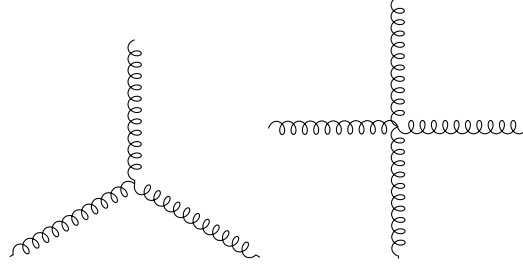It works as a general construction.

Figure 3.4: Gluon self-interactions

3.8 Spontaneous Symmetry Breaking

3.8.1 Real Field

The Lagrangian for real scalar field with ϕ^4-interaction contains two parts

$$\mathcal{L} = T - V = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V, \tag{3.8.1}$$

$$V = \frac{1}{2}\mu^2 \phi^2 + \frac{1}{4}\lambda \phi^4. \tag{3.8.2}$$

It shows mirror symmetry. We are interested in minimum energy of energy or ground state

$$E = T + V.$$

Kinetic term T containts derivative, so set ϕ as a constant to minimize V. First λ must be postive, otherwise there is no minimum, i.e. unbound from below. With $\underline{\mu^2 > 0}$, we find

$$\frac{\partial V}{\partial \phi} = \phi(\mu^2 + \lambda \phi^2) = 0.$$

We have minimum at $\phi = 0$. In other word $\langle \phi \rangle = 0$, vacuum expectation value (VEV) is zero. System is still mirror symmetric. The mass of the field

$$\left. \frac{\partial^2 V}{\partial \phi^2} \right|_{\phi = \langle \phi \rangle} = \mu^2. \tag{3.8.3}$$

Now consider the case $\underline{\mu^2 < 0}$

$$\frac{\partial V}{\partial \phi} = \phi(\mu^2 + \lambda \phi^2) \overset{!}{=} 0. \tag{3.8.4}$$

There are three solutions possible

$$\phi = 0, \quad \phi = \pm v = \pm \sqrt{\frac{-\mu^2}{\lambda}}, \tag{3.8.5}$$

$$\tag{3.8.6}$$

Checking the sign of derivative reveals that $\phi = 0$ is a maximum. Two minima means the vacuum is degenerate. By choosing one specific state, the mirror symmetry is spontaneous broken.

We choose the vacuum state to be $+v$. Split the field into two parts (it makes the physics and observable more obvious), the vacuum v and perturbation around the vacuum $\eta(x)$

$$\phi(x) = v + \eta(x),$$

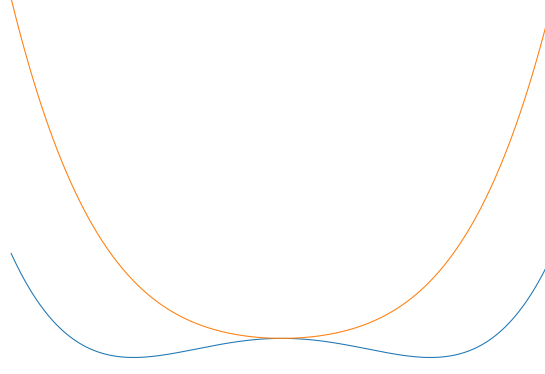$$v = \langle \phi \rangle = \sqrt{\frac{-\mu^2}{\lambda}} = \text{const.}$$

Figure 3.5: SSB for real scalar field

Since $\eta(x)$ is a perturbation, its expectation value should be zero.

$$\langle \eta(x) \rangle = 0,$$
$$\partial_\mu \phi = \partial_\mu \eta(x).$$

The kinetic term turns into

$$(\partial_\mu \phi)(\partial^\mu \phi) = (\partial_\mu \eta)(\partial^\mu \eta),$$
$$V = \frac{1}{2}\mu^2(v+\eta)^2 + \frac{1}{4}\lambda(v+\eta)^4,$$
$$= \frac{1}{2}\mu^2 v^2 + \frac{1}{2}\mu^2\eta^2 + \mu^2 v\eta + \frac{1}{4}\lambda\left[v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4\right],$$
$$= \frac{1}{2}\mu^2 v^2 + \frac{1}{4}\lambda v^2 + \eta\left(\mu^2 v + \lambda v^3\right) + \eta^2\left(\frac{1}{2}\mu^2 + \frac{6}{4}v^2\lambda\right) + \lambda v\eta^3 + \frac{1}{4}\lambda\eta^4.$$

There is no longer mirror symmetry in η. Now the mass is

$$\left.\frac{\partial^2 V}{\partial \eta^2}\right|_{\eta=0} = \mu^2 + 3v^2\lambda = \mu^2 + 3\left(\frac{-\mu^2}{\lambda}\right)\lambda = -2\mu^2.$$

Thus

$$m_\eta = \sqrt{-2\mu^2}.$$

After symmetry breaking $V(\eta)$ contains η^3 and η^4 terms, thus additional Feynman rules.

$$\sim \lambda v \qquad\qquad (3.8.7)$$

$$\sim \frac{1}{4}v \qquad\qquad (3.8.8)$$

3.8.2 Complex Field

Now we want to consider complex scalar field $\phi(x) \in \mathbb{C}$. We can also write the fields as

$$\phi(x) = \frac{1}{\sqrt{2}} (\phi_1(x) + i\phi_2(x)), \tag{3.8.9}$$

with $\phi_1, \phi_2 \in \mathbb{R}$.

Only scalar fields can get a vacuum expectation value, otherwise it violates Lorentz invariance. Vacuum has spin and then constantly interacts with all particles.

The Lagrangian must be real, $\mathcal{L} \in \mathbb{R}$

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2. \tag{3.8.10}$$

This is invariant under transformation $\phi(x) \mapsto e^{i\alpha} \phi(x), \alpha \in \mathbb{R}$. Rewrite the Lagrangian in terms of ϕ_1 and ϕ_2,

$$\mathcal{L} = \left(\partial_\mu \phi_1\right)(\partial^\mu \phi_1) + \left(\partial_\mu \phi_2\right)(\partial^\mu \phi_2) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2.$$

The last two terms are potential.

- $\mu^2 > 0$. Minimum is at $\phi_1 = \phi_2 = 0$

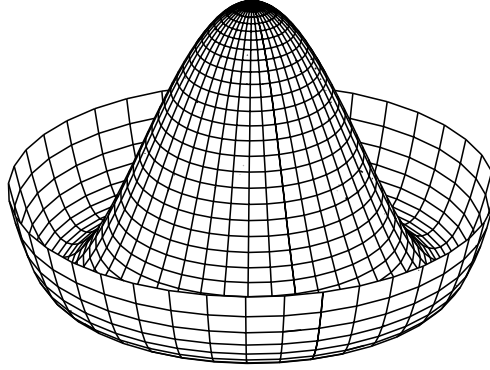- $\mu^2 < 0$. The potential has the shape of Mexican hat.

Figure 3.6: Mexican hat[3]

We have set of minima forming a circular line $\phi_1^2 + \phi_2^2 = v^2 = -\mu^2/\lambda$. As an example $\phi_1 = v$ and $\phi_2 = 0$ and $\mathbf{U}(1)$ symmetry is spontaneous broken. The choice of vacuum breaks symmetry spontaneously. Field shift

$$\phi(x) = \frac{1}{\sqrt{2}}\Big[\underbrace{v + \eta(x)}_{\phi_1} + i \underbrace{\xi(x)}_{\phi_2} \Big]. \tag{3.8.11}$$

with $\eta(x), \xi(x) \in \mathbb{R}$ and $\langle \eta \rangle = \langle \xi \rangle = 0$. Insert into $V(\phi) = V(\phi_1, \phi_2)$

$$\mathcal{L}(\eta, \xi) = \frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi) + \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \text{const} + \mu^2 \eta^2 + \text{(cubic and quartic in } \eta, \xi). \tag{3.8.12}$$

So $m_\eta^2 = -2\mu^2$ but $m_\xi^2 = 0$.

Continuous global symmetry is spontaneously broken and it leads to massless state. It is called <u>Goldstone theorem</u>. The massless state is the Goldstone boson.

3.8.3 Higgs Mechanism

Modify the transformation to

$$\phi(x) \mapsto e^{i\alpha(x)}\phi(x),$$

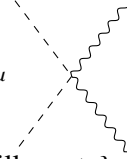with $\phi(x) \in \mathbb{C}$ and $\alpha(x) \in \mathbb{R}$.

First step is to extend $\mathcal{L}_\phi$ such that is invariant under this transformation. Introduce a spin-1 field $A_\mu(x)$ with the covariant derivative

$$D_\mu = \partial_\mu - ieA_\mu,$$

$$A_\mu \mapsto A_\mu + \frac{1}{e}\partial_\mu\alpha.$$

The new Lagrangian is

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{3.8.13}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It contains the interaction between two ϕ and two A_μ .

We want to look at spontaneous symmetry breaking of this system. We still want $\lambda > 0$, bound from below. In the case of $\mu^2 > 0$, symmetry not broken and we have scalar QED. In particle physics we do have charged scalars, such as $\pi^\pm$, $K^\pm$ and so on. We need $\mu^2 < 0$ to have spontaneous symmetry breaking. The field shift is as before

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\xi(x)), \tag{3.8.14}$$

$$\mathcal{L}(\eta,\xi) = \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta^2 + \frac{1}{2}e^2v^2A_\mu A^\mu$$
$$- evA_\mu\partial^\mu\xi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{(scalar interaction terms)}. \tag{3.8.15}$$

So $m_\eta = \sqrt{2\lambda v^2}$, $m_A = ev$ and $m_\xi = 0$.

$A_\mu\partial^\mu\xi$ seems to mix A and ξ. Massless spin 1 has two degrees of freedom. $A_\mu(x)$ has four components, but Lorentz and gauge invariance restrict to two. Photon has two polarizations. Massive spin 1 has three degree of freedom. We will show later on, ξ is actually the additional degree of freedom of A. The term $A_\mu A^\mu$ breaks the gauge invariance.

To see physics, the choice of η and ξ is not very good. Instead choose

$$\phi \mapsto \frac{1}{\sqrt{2}}(v + h(x))e^{i\Theta(x)/v}, \tag{3.8.16}$$

with $h(x) \in \mathbb{R}, \Theta(x) \in \mathbb{R}$. We interpret this as a gauge transformation with $\alpha = \Theta/v$. To rewrite gauge transformation of $A_\mu(x)$

$$A_\mu \mapsto A_\mu + \frac{1}{ev}\partial_\mu\Theta(x). \tag{3.8.17}$$

The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial h)^2 - \lambda v^2 h^2 + \frac{1}{2}e^2v^2A_\mu A^\mu - \lambda v h^3 - \frac{1}{4}\lambda h^4 + \frac{1}{2}e^2A^2h^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.8.18}$$

ξ is no longer there. This is Higgs mechanism. So we have before $2 + 2$ degree of freedom $(\eta, \xi, A^{m=0})$ to $1 + 3$ degrees of freedom $(h, A^{m\neq0})$.

Phase transition in magnetism is an example of symmetry breaking. Magnetization is temperature dependent. In the early universe the temperature is high so that $\mu^2(T) > 0$. As it cools down $\mu^2 < 0$.

3.8.4 Spontaneous Breaking of local $SU(2)$ Theory

We start with ungauged, complex scalar field

$$\mathcal{L} = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \tag{3.8.19}$$

Now ϕ contains two complex scalar fields.

$$\phi = \begin{pmatrix} \phi_\alpha \\ \phi_\beta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \tag{3.8.20}$$

with $\phi_{1,2,3,4} \in \mathbb{R}$.

Gauge transformation is

$$\phi \mapsto \phi' = e^{i\alpha_a(x)\tau_a/2}\phi. \tag{3.8.21}$$

There are only three generators of $SU(2)$, $a = 1, 2, 3$.

Introduce three gauge bosons W_μ^a, $a = 1, 2, 3$. Define the covariant derivative

$$D_\mu = \partial_\mu - ig\frac{\tau_a}{2}W_\mu^a. \tag{3.8.22}$$

Note that $T^a = \tau^a/2$.

The τ matrices are Pauli matrices

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.8.23}$$

Gauge transformation of W_a^μ is in the form of

$$W_\mu^a \mapsto W_\mu^a - \frac{1}{g}\partial_\mu\alpha^a - f^{abc}\alpha_b W_{c\mu}. \tag{3.8.24}$$

Here $f_{abc} = \epsilon_{abc}$ totally symmetric tensor in three dimension.

Lagrangian becomes

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi) - \frac{1}{4}W_{\mu\nu}^a W^{a\mu\nu} \tag{3.8.25}$$

Kinetic term for gauge fields

$$W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g(W_\mu \times W_\nu)^a \tag{3.8.26}$$

The potential

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \tag{3.8.27}$$

We are interested in the case $\lambda > 0$ and $\mu^2 < 0$. Minima are at

$$\phi^\dagger\phi = \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) = -\frac{\mu^2}{2\lambda}$$

We can make a choice for minimum $\phi_1 = \phi_2 = \phi_4 = 0$ and $\phi_3^2 = v^2 = -\mu^2/\lambda$

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{3.8.28}$$

with $v \in \mathbb{R}$.

Shift of ϕ field

$$\phi(x) \mapsto e^{i\hat{\tau}\cdot\hat{\theta}/v} \phi(x) = e^{i\hat{\tau}\cdot\hat{\theta}/v} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{3.8.29}$$

Infinitesimal expansion in $\theta(x)$

$$\begin{aligned}
\phi(x) &= \frac{1}{\sqrt{2}} \left(\mathbb{1} + i\hat{\tau}\cdot\hat{\theta}(x)/v \right) \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + i\theta_3/v & i(\theta_1 - i\theta_2)/v \\ i(\theta_1 + i\theta_2)/v & 1 - i\theta_3/v \end{pmatrix} \begin{pmatrix} 0 \\ v + k(x) \end{pmatrix}
\end{aligned}$$

Look at kinetic term in ϕ

$$\begin{aligned}
\mathcal{L}_{\text{kin}}^{\phi} &= (D_\mu \phi)^\dagger (D^\mu \phi) \\
&= \left(\partial_\mu \phi + ig\hat{\tau}\cdot\hat{W}\phi \right)^\dagger \left(\partial^\mu \phi + ig\hat{\tau}\cdot\hat{W}\phi \right)
\end{aligned}$$

compute at minimum $\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$

$$\begin{aligned}
\left| \frac{i}{2} g\hat{\tau}\cdot\hat{W}\phi \right|^2 &= \frac{g^2}{8} a \left| \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 \\
&= \frac{g^2 v^2}{8} \left| \begin{pmatrix} W_\mu^1 - iW_\mu^2 \\ -W_\mu^3 \end{pmatrix} \right|^2 \\
&= \frac{g^2 v^2}{8} \boldsymbol{W}^2 \\
&= \frac{1}{2} M^2 \boldsymbol{W}^2
\end{aligned}$$

with $M = gv/2$. We spontaneously broken $\mathbf{SU}(2)$ gauge theory. However, this is not realized in nature.

4 Standard Model

We will be looking at the gauge group $\mathbf{SU(2)_L} \times \mathbf{U(1)_Y}$. This will be spontaneously broken into $\mathbf{U(1)_{EM}}$. Strong interaction will add $\mathbf{SU(3)_C}$ and it doesn't get spontaneously broken.

4.1 Leptons

Reintroduce fermions

$$\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

$$\psi_L = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}$$

$\mathbf{SU(2)_L}$ only interacts with ψ_L.

In Chiral representation

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{4.1.1}$$

The projection operators

$$P_L = \frac{1}{2}(\mathbb{1}_2 - \gamma^5) = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.1.2}$$

$$P_R = \frac{1}{2}(\mathbb{1}_2 + \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{4.1.3}$$

with $P_L^2 = P_L$ and $P_L + P_R = \mathbb{1}$.

$\chi_{L,R}$ two-component Weyl spinors.

$$(\psi_L)^\dagger = (P_L\psi)^\dagger = \psi^\dagger P_L^\dagger = \psi^\dagger P_L$$
$$\overline{\psi}_L = (\psi_L)^\dagger \gamma^0 = \psi^\dagger P_L \gamma^0 = \overline{\psi} P_R$$

Introduce a doublet of left-handed particles

$$L = \begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}$$

$$\nu_L = (\psi_\nu)_L = \begin{pmatrix} \chi_{\nu L} \\ 0 \end{pmatrix}$$

$$e_L^- = (\psi_{e^-})_L = \begin{pmatrix} \chi_{e^-L} \\ 0 \end{pmatrix}$$

$$e_L = (\psi_e)_L = \frac{1}{2}(\mathbb{1} - \gamma_5)\psi_e$$

There is no right-handed neutrino in this theory (singlet).

$$R = e_R = P_R(\psi_e) = (\psi_e)_R$$

Under $\mathbf{U}(1)_Y$ the hypercharges are define as

$$Y(L) = -1 \tag{4.1.4}$$
$$Y(R) = -2 \tag{4.1.5}$$

Here each component of L has $Y = -1$. They are chosen so that

$$Q_{\mathrm{EM}} = T_L^3 + \frac{1}{2}Y$$

with $T_L^3 = \frac{1}{2}\tau^3$ a generator of $\mathbf{SU}(2)$.

$$T_L^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$T_L^3 L = \begin{pmatrix} \frac{1}{2}\nu_L \\ -\frac{1}{2}e_L^- \end{pmatrix}$$

Take a look at the examples

$$Q_{\mathrm{EM}}(\nu_L) = T_L^3(\nu_L) + \frac{1}{2}Y(\nu_L) = \frac{1}{2} - \frac{1}{2} = 0$$
$$Q_{\mathrm{EM}}(e_L^-) = T_L^3(e_L) + \frac{1}{2}Y(e_L) = -\frac{1}{2} - \frac{1}{2} = 0$$
$$Q_{\mathrm{EM}}(e_R^-) = T_L^3(e_R) + \frac{1}{2}Y(e_R) = 0 - 1 = -1$$

T_L only acts on left handed particles under $\mathbf{SU}(2)$

$$\psi_{e_R} = e^{i(0)}\psi_{e_R}$$
$$[T^a, Y] = 0$$

Normal kinetic term in Dirac theory

$$\mathcal{L}_{\mathrm{kin}} = \bar{\psi}i\gamma^\mu D_\mu (P_R + P_L)\psi$$

To split this using the anti-commutator of γ^5 and other gamma matrices, $\gamma^\mu P_L = P_R \gamma^\mu$

$$= \bar{\psi}_L i\gamma^\mu D_\mu \psi_L + \bar{\psi}_R i\gamma^\mu D_\mu \psi_R$$

Thus kinetic terms for leptons are

$$\mathcal{L}_{\mathrm{leptons}}^{\mathrm{kin}} = \bar{R}i\gamma^\mu D'_\mu R + \bar{L}i\gamma^\mu D_\mu L \tag{4.1.6}$$

$$D'_\mu = \partial_\mu - \frac{ig'}{2}YB_\mu$$
$$= \partial_\mu + ig'B_\mu \tag{4.1.7}$$

$$D_\mu = \partial_\mu - \frac{ig'}{2}YB_\mu - ig\frac{\tau^a}{2}W_\mu^a$$
$$= \partial_\mu + \frac{i}{2}g'B_\mu - ig\frac{\tau^a}{2}W_\mu^a \tag{4.1.8}$$

In electromagnetic $\mathbf{U}(1)_{\mathrm{EM}}$ the charge generator Q with $Q(e^-) = -1$. Covariant derivative

$$D_\mu = \partial_\mu - ieQA_\mu \tag{4.1.9}$$

In $\mathbf{U}(1)_Y$ the gauge boson B_μ. Field strength tensor $F^Y_{\mu\nu} = \partial_\mu B_\mu - \partial_\nu B_\mu$. Charge generator $Y/2$. Covariant derivative

$$D_\mu = \partial_\mu - \frac{ig'}{2} Y B_\mu \tag{4.1.10}$$

In Dirac Lagrangian there is a mass term $m\bar\psi\psi$.

$$\begin{aligned}
\mathcal{L}_m = m\bar\psi\psi &= m\bar\psi(P_R + P_L)\psi \\
&= m(\bar\psi P_R \psi + \bar\psi P_L \psi) \\
&= m(\bar\psi P_R^2 \psi + \bar\psi P_L^2 \psi) \\
&= m(\bar\psi_L \psi_R + \bar\psi_R \psi_L)
\end{aligned}$$

$\bar\psi\psi$ mass term is called Dirac mass term. There is only left handed neutrino, so we cannot write Dirac mass term for neutrino. Although ψ_L and ψ_R transform differently under Lorentz transformation, $\bar\psi\psi$ is still Lorentz invariant.

Recall that $\mathbf{U}(1)_{EM}$ in QED $\psi \mapsto \psi' = e^{i\alpha(x)Q}\psi$. If Q is same for left and right-handed components, $\bar\psi\psi$ is gauge invariant. If left and right have different hypercharges Y, then $\bar\psi_R\psi_L$ not $\mathbf{U}(1)_Y$ gauge invariant.

How about $\mathbf{SU}(2)_L$ gauge invariance of $\bar\psi_L\psi_R$? Under $\mathbf{SU}(2)$

$$\begin{aligned}
R &\mapsto R' = R \\
L &\mapsto L' = e^{i\alpha_a(x)\tau^a/2}L
\end{aligned}$$

So it's obvious that $\bar\psi_R\psi_L$ not $\mathbf{SU}(2)_L$ gauge invariant.

Any fermionic mass term vanishes, if we require $\mathbf{SU}(2)_L \times \mathbf{U}(1)_Y$ gauge invariance. Spontaneous symmetry breaking to let fermion gain mass.

4.2 Add scalars

Introduce a Higgs doublet

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{4.2.1}$$

The superscripts denote the charge the field carries.

From previous section $Q_{EM} = T_L^3 + \frac{1}{2}Y$, so

$$Q_{EM}(\phi^+) = +1 = \frac{1}{2} + \frac{1}{2}Y \Leftrightarrow Y = +1$$

$$Q_{EM}(\phi^0) = 0 = -\frac{1}{2} + \frac{1}{2}Y \Leftrightarrow Y = +1$$

Together $Y(\Phi) = +1$

Lagrangian

$$\begin{aligned}
\mathcal{L}_{scalar} &= (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi^\dagger\Phi) \\
D_\mu &= \partial_\mu - \frac{ig'}{2}B_\mu - \frac{ig}{2}\tau_i W^i_\mu \\
V &= \mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 \\
\Phi^\dagger &= \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}^\dagger = \begin{pmatrix} \phi^+ & \phi^0 \end{pmatrix}^* = \begin{pmatrix} \phi^- & (\phi^0)^* \end{pmatrix}
\end{aligned} \tag{4.2.2}$$

4.3 Coupling of Scalars and Fermions

Recall that simple complex scalar $\phi^0 \in \mathbb{C}$, $\mathbf{SU}(2)$ singlet. Under Lorentz transformations (per definition) $\phi^0(x) \mapsto \phi^0(x)$. $\bar{\psi}\psi$ is also Lorentz invariant. It means $\phi^0\bar{\psi}\psi$ is Lorentz invariant. This type of interaction is called Yukawa interaction.

$$\mathcal{L}_{\text{Yukawa}} = -y_e \left(\bar{R}\Phi^\dagger L + \bar{L}\Phi R \right)$$
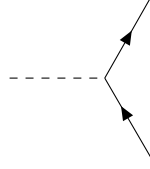
How to combine Φ, L and R?

$$Y(\Phi) = +1$$
$$Y(L) = -1$$
$$Y(R) = -2$$
$$Y(\bar{L}) = +1$$

Gauge transformation of interaction term

$$\phi\bar{\psi}\psi \mapsto \phi'\bar{\psi}'\psi' = e^{i\alpha(Q(\phi)+Q(\bar{\psi})+Q(\psi))}\phi\bar{\psi}\psi$$

Gauge invariance means the sum of hypercharges is zero. $\Phi\bar{L}R$ is $\mathbf{U}(1)_Y$ gauge invariant.
What if we want $\mathbf{SU}(2)_L \times \mathbf{U}(1)_Y$ gauge invariant.
R is $\mathbf{SU}(2)$ invariant. Φ is $\underline{2}$ under $\mathbf{SU}(2)$. L and $\bar{L}$ are $\underline{2}$ under $\mathbf{SU}(2)$. For $\mathbf{SU}(2)$ these two kinds of representations are the same $\underline{\bar{2}} = \underline{2}$.

$$\Phi\bar{L} : \underline{2} \otimes \underline{\bar{2}} = \underline{3} \oplus \underline{1}_A$$

$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. $\underline{1}$ is $\mathbf{SU}(2)$ singlet. Need antisymmetric combination of $\Phi\bar{L}$ to get $\mathbf{SU}(2)$ singlet.
In components

$$\bar{L}\Phi R = \begin{pmatrix} \bar{v}_L & \bar{e}_L^- \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R$$
$$= \bar{v}_L\phi^+ e_R + \bar{e}_L^-\phi^0 e_R$$

here $\sum Y_i = 0$

$$\bar{R}\Phi^+ L = \bar{e}_R^- \begin{pmatrix} \phi^- & (\phi^0)^* \end{pmatrix} \begin{pmatrix} v_L \\ e_L^- \end{pmatrix}$$
$$= \bar{e}_R^-\phi^- v_L + \bar{e}_R^- (\phi^0)^* e_L^-$$

The Standard Model does not contain v_R. But let's consider v_R anyway

$$Q_{\text{EM}} = T_L^3 + \frac{1}{2}Y$$
$$0 = 0 + \frac{1}{2}Y(v_R)$$

So it doesn't couple to B_μ.

How about Dirac mass term?

$$\bar{\nu}_L \nu_R + \bar{\nu}_R \nu_L$$

It is not $\mathbf{U}(1)_Y$ or $\mathbf{SU}(2)_L$ invariant. Not a huge problem since it was the same for electron before we introduce Higgs mechanism.

Cannot write interaction like $\bar{L}\nu_R\Phi$ since the hypercharge is not zero. Instead

$$\tilde{\Phi} = i\tau_2 \Phi^* = \begin{pmatrix} \phi^0 \\ -\phi^- \end{pmatrix}$$

$$Y(\tilde{\Phi}) = -1$$

$$\mathcal{L}_{\nu_L} = \bar{L}\tilde{\Phi}\nu_R + \bar{\nu}_R \tilde{\Phi}^\dagger L$$

Dirac mass term for neutrino. If ν_R exists, this is a possible mass term.

$$\bar{L}\tilde{\Phi}\nu_R = \begin{pmatrix} \bar{\nu}_L & \bar{e}_L \end{pmatrix} \begin{pmatrix} (\phi^0)^* \\ -\phi^- \end{pmatrix} \nu_R$$

$$= \bar{\nu}(\phi^0)^* \nu_R - \bar{e}_L \phi^- \nu_R$$

4.4 Spontaneous Symmetry Breaking (Mass and Mixing of Gauge Bosons)*

In the early universe $\mu = \mu(T)$ and the symmetry got broken. Consider the case $\mu^2 < 0$.

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{4.4.1}$$

with $v \in \mathbb{R}$ and $[v] = 1$. Field ϕ^0 has $Q_{EM} = 0$. $Y(\phi^0) \neq 0$ breaks the $\mathbf{U}(1)_Y$ and $T_L^3(\phi^0) \neq 0$ breaks $\mathbf{SU}(2)_L$.

As before the field shift is

$$\Phi = U^{-1}(\boldsymbol{\xi}) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix}$$

$$U(\boldsymbol{\xi}) = \exp(-i\boldsymbol{\xi} \cdot \boldsymbol{\tau}/(2v))$$

$\eta(x), \boldsymbol{\xi}(x) \in \mathbb{R}$. The transformation $U^{-1}(\boldsymbol{\xi})$ has three parameters and thus give Φ four degrees of freedom. It has the same form as $\mathbf{SU}(2)$ gauge transformation. Because of gauge symmetry, this transformation gets cancelled.

$$\Phi \mapsto \Phi' = U(\xi)\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix} \tag{4.4.2}$$

$$L \mapsto L' = U(\boldsymbol{\xi})L \tag{4.4.3}$$

$$W_\mu \mapsto W'_\mu \tag{4.4.4}$$

$$\boldsymbol{\tau} \cdot \boldsymbol{W}'_\mu = U(\boldsymbol{\xi})\left[\boldsymbol{\tau} \cdot \boldsymbol{W}_\mu - \frac{i}{g} U^{-1} \partial_\mu U\right] \tag{4.4.5}$$

---

*see also in Cheng and Li, Ch.11

Insert transformed field (4.4.2) into $\mathcal{L}$ and see the physical interpretation. Yukawa coupling

$$-y_e \bar{L} \Phi R = -y_e \begin{pmatrix} \bar{\nu}_e & \bar{e}_L \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix} e_R$$

$$= -\frac{y_e}{\sqrt{2}} \bar{e}_L (v + \eta(x)) e_R$$

Thus $m_e = y_e v / \sqrt{2}$. Knowing m_e we can fix $y_e v$. It does not predict the mass of electron, but we can accommodate. We do predict $m_\nu = 0$ though.

There is an extra term

$$-\frac{y_e}{\sqrt{2}} \bar{e}_L e_R \eta(x)$$

It indicates the coupling of Higgs is direct proportional to mass of fermion. We found $y_e \sim 10^{-6}$, so it is then not highly unlikely to observe Higgs decay into electrons at LHC. We have already seen Higgs decay into tau and bottom quarks.

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2} (\partial_\mu \eta)(\partial^\mu \eta) + V\left(\left(\frac{v + \eta}{\sqrt{2}}\right)^2\right) \tag{4.4.6}$$

Write

$$\Phi = \frac{v + \eta}{\sqrt{2}} \chi$$

$$\chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Multiply $(D_\mu \Phi)^\dagger (D^\mu \Phi)$ out but focus only on terms without derivative

$$D_\mu = \partial_\mu - \frac{i}{2} g' B_\mu - i g \frac{\tau^i}{2} W_\mu^i$$

$$\mathcal{L} \subset \frac{(v + \eta)^2}{8} \chi^\dagger \left[(g' B_\mu^\dagger + g \tau^i W_\mu^i) \cdot (g' B^\mu + g \tau^i W^{i\mu}) \right] \chi$$

Terms with one τ^1 and τ^2 vanish

$$\chi^\dagger \tau^1 \chi = 0$$

$$\chi^\dagger \tau^2 \chi = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\chi^\dagger \tau^3 \chi = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1$$

Focus on $\sim v^2$ terms, they are bilinear in gauge boson fields and thus mass terms.

$$\mathcal{L} \subset \frac{1}{8} v^2 \left[(g' B_\mu - g W_\mu^3)(g' B^\mu - g W^{3\mu}) + g^2 (W_\mu^1)^2 + g^2 (W_\mu^2)^2 \right] \tag{4.4.7}$$

with $M_{W^1}^2 = M_{W^2}^2 = g^2 v^2 / 8$.

Using the familiar formula $Q = T_3 + \frac{1}{2}Y$ for the **SU**(2) gauge boson $W^{1,2,3}$. They form a **SU**(2) triplet $T_3 = \pm 1, 0$. W^3 has zero electric charge. B_μ as **SU**(2) singlet has zero electric charge and hypercharge.

Focus on the electric neutral part

$$\mathcal{L} \subset \frac{v^2}{8} \left[g'^2 B_\mu B^\mu + g^2 W_\mu^3 W^{\mu 3} + g'g B_\mu W^{\mu 3} + g'g W_\mu^3 B^\mu \right]$$

It has $W_\mu B^\mu$ mixing terms. Rewritten in matrix form

$$= \frac{v^2}{8} \begin{pmatrix} B_\mu & W_\mu^3 \end{pmatrix} \begin{pmatrix} g'^2 & gg' \\ gg' & g^2 \end{pmatrix} \begin{pmatrix} B^\mu \\ W^{3\mu} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} Z_\mu & A_\mu \end{pmatrix} \begin{pmatrix} M_Z^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix}$$

Using the properties of 2×2 matrices

$$\det = \lambda_1 \cdot \lambda_2 \tag{4.4.8}$$

$$\text{tr} = \lambda_1 + \lambda_2 \tag{4.4.9}$$

Obviously here $\det = 0$ and $\text{tr} = g^2 + g'^2$.

Eigenvectors are

$$Z_\mu = \frac{-gW_\mu^3 + g'B_\mu}{\sqrt{g^2 + g'^2}}$$

$$A_\mu = \frac{gB_\mu + g'W_\mu^3}{\sqrt{g^2 + g'^2}} \tag{4.4.10}$$

$$M_A = 0$$

$$M_Z = \frac{v}{2}\sqrt{g^2 + g'^2} \tag{4.4.11}$$

Remember real fields have mass term in the form $\frac{1}{2}m^2\phi^2$.

Go back to W_μ^1 and W_μ^2. Define

$$W^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \pm W_\mu^2) \tag{4.4.12}$$

$$W_\mu^+ W^{-\mu} = \frac{1}{2}\left[(W_\mu^1)^2 + (W_\mu^2)^2 \right] \tag{4.4.13}$$

W is a triplet $(W^+, W^0, W^-)^T$. The superscripts denote the hypercharge (T^3). W_μ^3 has zero hypercharge and thus zero electric charge.

Mass term $\frac{1}{2}m^2(W_\mu^+ W^{-\mu} + W_\mu^- W^{\mu +})$

$$\frac{1}{2}M_W^2 = \frac{1}{8}g^2 v^2$$

$$M_W = \frac{1}{2}gv$$

Go back to leptons

$$\mathcal{L}_{\text{kin}}^{\text{leptons}} = \bar{R}i\gamma^\mu D'_\mu R + \bar{L}i\gamma^\mu D_\mu L$$

$$D'_\mu = \partial_\mu + i\frac{g'}{2}B_\mu$$

$$D_\mu = \partial_\mu + \frac{i}{2}g'B_\mu - ig\frac{\tau^i}{2}W_\mu^i$$

Use equation (4.4.10) to rewrite $B_\mu = f(A_\mu, Z_\mu)$ and $W_\mu^3 = f'(A_\mu, Z_\mu)$.
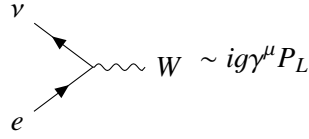
Insert that into D'_μ and D_μ and then into $\mathcal{L}_{\text{kin}}^{\text{leptons}}$. Also replace $W_\mu'^2 \mapsto W_\mu^\pm$

$$\bar{L}i\gamma^\mu D_\mu L = \bar{L}i\gamma^\mu \left(\partial_\mu + \frac{i}{2}g'B_\mu - ig\frac{\tau^i}{2}W_\mu^i\right)L$$

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

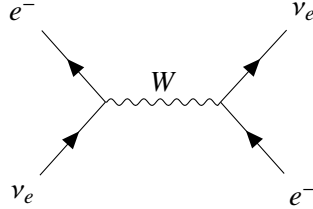$$\tau_1 W_\mu^1 + \tau^2 W_\mu^2 = \begin{pmatrix} 0 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & 0 \end{pmatrix}$$

Charged current

$$A = \bar{L}\gamma^\mu \left(\tau^1 W_\mu^1 + \tau^2 W_\mu^2\right) L$$

$$= \frac{g}{2} \begin{pmatrix} \bar{\nu}_L & \bar{e}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} 0 & \sqrt{2}W_\mu^+ \\ \sqrt{2}W_\mu^- & 0 \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

$$= \frac{g}{2} \begin{pmatrix} \bar{\nu}_L & \bar{e}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} \sqrt{2}W_\mu^+ e_L \\ \sqrt{2}W_\mu^- \nu_L \end{pmatrix}$$

$$= \frac{g}{\sqrt{2}} \left[\bar{\nu}_L W_\mu^+ \gamma^\mu e_L + \bar{e}_L W_\mu^- \gamma^\mu \nu_L\right]$$

Diagrammatically

Neutrino-electron scattering has the cross section $\sim g^4/M_W^4$
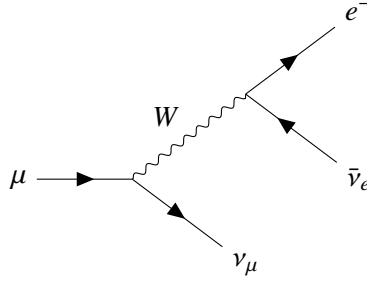

Fermi constant

$$G_F = \frac{\sqrt{2}g^2}{8M_W^2} \tag{4.4.14}$$

$\mathcal{L}_{\text{kin}}^{\text{leptons}}$ can be extended to muons. Then we can draw diagram for muon decay. Decay rate $\Gamma \sim G_F^2$.

$\tau_\mu \sim 2\,\mu\text{s}.\ G_F \sim 10^{-5}/m_\mu^2$

We have three parameters in gauge sector $(g, g', v) \leftrightarrow (\alpha_{\text{EM}}, G_{\text{F}}, G_{\text{NC}})$. We measured $\alpha_{\text{EM}} = 1/137$, G_{F} in μ-decay and G_{NC} in neutral current interactions.

Equations (4.4.10) are just basis transformation and can be parametrized with one parameter θ_W.

$$\tan(\theta_W) = \frac{g'}{g} \tag{4.4.15}$$

$$\sin(\theta_W) = \frac{g'}{\sqrt{g^2 + g'^2}} \tag{4.4.16}$$

$$\cos(\theta_W) = \frac{g}{\sqrt{g^2 + g'^2}} \tag{4.4.17}$$

Equations (4.4.10) become

$$A_\mu = \cos(\theta_W)B_\mu + \sin(\theta_W)W_\mu^3 \tag{4.4.18}$$

$$Z_\mu = \sin(\theta_W)B_\mu - \cos(\theta_W)W_\mu^3 \tag{4.4.19}$$

The transformation can be inverted

$$B_\mu = \cos(\theta_W)A_\mu + \sin(\theta_W)Z_\mu \tag{4.4.20}$$

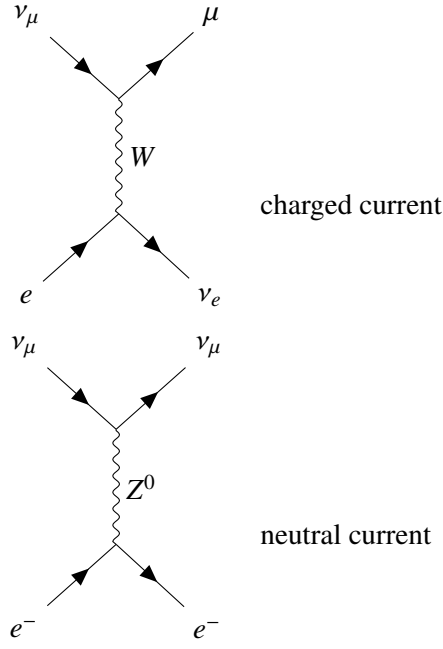$$W_\mu^3 = \sin(\theta_W)A_\mu - \cos(\theta_W)Z_\mu \tag{4.4.21}$$

The angle (parameter) θ_W is called Weinberg angle or electroweak mixing angle.

Rewrite the $\mathcal{L}^{\text{lepton}}$ in particular neutral current and factor out A_μ, Z_μ

$$\mathcal{L}^{\text{lepton}} = A_\mu \frac{gg'}{\sqrt{g^2 + g'^2}} \left[\bar{e}_R \gamma^\mu e_R + \bar{e}_L \gamma^\mu e_L \right]$$

$$+ Z_\mu \frac{1}{2\sqrt{g^2 + g'^2}} \left[g'^2 \left(2\bar{e}_R \gamma^\mu e_R + \bar{e}_L \gamma^\mu e_L + \bar{\nu}_L \gamma^\mu \nu_L \right) - g^2 \left(\bar{e}_L \gamma^\mu e_L - \bar{\nu}_L \gamma^\mu \nu_L \right) \right]$$

First two terms are coupling of electron and photon and the coupling constant is just electric charge.

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} \approx 0.3 \tag{4.4.22}$$

Charged current interaction involves $W^\pm$. Neutral current interaction involves A_μ and Z^0_μ.

One can experimentally determine g, g', v (actually G_F, e^2 and $\sin^2(\theta_W)$)

$$e = g \sin(\theta_W) = g' \cos(\theta_W) \qquad\qquad \Rightarrow \sin^2(\theta_W) \approx 0.22$$

$$M_W^2 = \frac{1}{4} g^2 v^2 = \frac{e^2}{\sin^2(\theta_W)} \frac{1}{4\sqrt{2} G_F} \qquad\qquad \Rightarrow M_W \approx 80\,\text{GeV} \quad M_Z \approx 90\,\text{GeV}$$

More precisely one should also consider electroweak quantum corrections (loop corrections).

$$A_\mu \left[\bar{e}_R \gamma^\mu e_R + \bar{e}_L \gamma^\mu e_L \right] \frac{gg'}{\sqrt{g^2 + g'^2}}$$

$$= A_\mu \left[\bar{e}_R \gamma^\mu e_R + \bar{e}_L \gamma^\mu e_L \right] e_{\text{el}}$$

$$= A_\mu \left[\bar{e} \gamma_\mu P_R e + \bar{e} \gamma^\mu P_L e \right] e_{\text{el}}$$

$$= A_\mu (\bar{e} \gamma^\mu e)$$

It is Lorentz four vector.

For Z_μ it is different. Left-handed electrons are coupled differently from right-handed ones.

$$Z_\mu \bar{e}_R \gamma^\mu e_R \sim \frac{2g'^2}{2\sqrt{g^2 + g'^2}}$$

$$Z_\mu \bar{e}_L \gamma^\mu e_L \sim \frac{g'^2 - g^2}{2\sqrt{g^2 + g'^2}}$$

Write this in one equation

$$\frac{1}{2} Z_\mu \left[\bar{e} \gamma^\mu \left(P_R \frac{2g'^2}{\sqrt{g'^2 + g^2}} + P_L \frac{g'^2 - g^2}{\sqrt{g^2 + g'^2}} \right) \right] e_{\text{el}} \qquad\qquad (4.4.23)$$

suppress the constants into C_A and C_V

$$=\bar{e}\gamma^\mu C_V e + \bar{e}\gamma^\mu\gamma^5 C_A e \tag{4.4.24}$$

The ratio of these two coupling is

$$\frac{C_V}{C_A} = \frac{3g'^2 - g^2}{g^2 + g'^2} = -1 + 4\sin^2(\theta_W) \tag{4.4.25}$$

4.5 Quarks

Introduce a quark doublet (under $\mathbf{SU}(2)$)

$$Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \tag{4.5.1}$$

u_L and d_L are actually triplets under $\mathbf{SU}(3)_C$

$$Q_{el}(u_L) = +\frac{2}{3} \leftrightarrow Y(u_L) = \frac{1}{3}$$
$$Q_{el}(d_L) = -\frac{1}{3} \leftrightarrow Y(d_L) = \frac{1}{3}$$

Together

$$Y(Q) = +\frac{1}{3} \tag{4.5.2}$$

u_R and d_R are $\mathbf{SU}(2)$ singlets.

$$Q_{el}(u_R) = +\frac{2}{3} \leftrightarrow Y(u_L) = \frac{4}{3}$$
$$Q_{el}(d_R) = -\frac{1}{3} \leftrightarrow Y(d_L) = \frac{4}{3}$$

	$\mathbf{SU}(3)_C$	$\mathbf{SU}(2)_L$	$\mathbf{U}(1)_Y$
e_L	1	2	-1
u_R	3	1	$4/3$
d_R	3	1	$-2/3$
Q	3	2	$1/3$
Φ	1	2	1

Kinetic terms for quarks in Lagrangian

$$\mathcal{L}_{kin}^{quarks} = \bar{Q}i\gamma_\mu D^\mu Q + \bar{u}_R i\gamma^\mu D^\mu u_R + \bar{d}_R i\gamma_\mu D^\mu d_R \tag{4.5.3}$$

$$D^\mu = \partial_\mu - \frac{ig'}{2}YB_\mu - ig\frac{\tau^i W_\mu^i}{2} - ig_3 T^a G^{\mu a} \tag{4.5.4}$$

Higgs Lagrangian unchanged. Replace $(B_\mu, W_\mu^3) \mapsto (A_\mu, Z_\mu)$

Neutral current

$$\frac{1}{2}Z^\mu \frac{1}{\sqrt{g'^2 + g^2}} \left[g'^2 \left(\frac{4}{3} \bar{u}_R \gamma_\mu u_R - \frac{2}{3} \bar{d}_R \gamma_\mu d_R + \frac{1}{3} \bar{u}_L \gamma_\mu u_L + \frac{1}{3} \bar{d}_L \gamma_\mu \bar{d}_L \right) + g^2 \left(\bar{u}_L \gamma_\mu u_L - \bar{d}_L \gamma_\mu d_L \right) \right] \quad (4.5.5)$$

A_μ is vector like again with charges 2/3 (u) and $-1/3$ (d).

Neutral currents are always diagonal in quark fields.

Yukawa coupling of quarks to Higgs

$$y_d \bar{Q} \Phi d_R$$

Do the hypercharge add up to zero?

$$Y(\bar{Q}) + Y(\Phi) + Y(d_R) = -\frac{1}{3} + 1 - \frac{2}{3} = 0$$

Analogous terms with u_R does <u>not</u> work use ν_R trick $\tilde{\Phi} = i\sigma_2 \Phi^*$

$$y_u \bar{Q} \tilde{\Phi} u_R$$

Charged Current

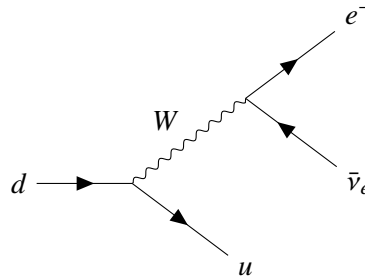$$A_\mu \left[\bar{u}_R \gamma^\mu u_e + \bar{u}_L \gamma^\mu u_L \right] e e_u = A_\mu \bar{u} \gamma^\mu u \quad (4.5.6)$$

As before, same $W^\pm$ interaction only involved P_L

$$\bar{L} \gamma^\mu W_\mu L$$
$$L = P_L L$$
$$\gamma^\mu P_L = \frac{1}{2} \left(\gamma^\mu - \gamma^\mu \gamma^5 \right)$$

γ_μ is vector interaction. $\gamma^\mu \gamma^5$ axial vector interaction. So charged current is V-A interaction. Parity is maximally violated.

Coupling g is same for leptons and quarks.

Nuclear beta decay $n \rightarrow p e^- \bar{\nu}_e$

This diagram and the one with muon have same coupling G_F. Only kinematics is different. Assuming $m_\nu = 0$ we can compare these two processes by measuring neutron lifetime and μ-lifetime. The two Fermi constants are not the same!

It should be clear how to generalize current Lagrangian to other quarks

$$Q_1 = \begin{pmatrix} u \\ d \end{pmatrix} \quad Q_2 = \begin{pmatrix} c \\ s \end{pmatrix} \quad Q_3 = \begin{pmatrix} t \\ b \end{pmatrix}$$

$$\boldsymbol{Q} = (Q_1, Q_2, Q_3)$$

$$\boldsymbol{u}_R = (u_R, c_R, t_R)$$

$$\boldsymbol{d}_R = (d_R, s_R, b_R)$$

$$\frac{4}{3}\bar{\boldsymbol{u}}_R\gamma_\mu\boldsymbol{u}_R Z^\mu = \frac{4}{3}\left[\bar{u}_R\gamma_\mu u_R + \bar{c}_R\gamma_\mu c_R + \bar{t}_R\gamma_\mu t_R\right]Z^\mu$$

Neutral current interactions are unaffected by a unitary transformation in flavour space

$$\boldsymbol{u}_R \mapsto \boldsymbol{u}'_R = A\boldsymbol{u}_R = \begin{pmatrix} u'_R \\ c'_R \\ t'_R \end{pmatrix} \tag{4.5.7}$$

Then the interaction term

$$\bar{\boldsymbol{u}}_R\gamma^\mu\boldsymbol{u}_R \mapsto \bar{\boldsymbol{u}}'_R\gamma^\mu\boldsymbol{u}'_R$$

$$= \bar{\boldsymbol{u}}_R A^\dagger\gamma^\mu A\boldsymbol{u}_R$$

$$= \bar{\boldsymbol{u}}_R\gamma^\mu\boldsymbol{u}_R$$

The Yukawa interactions

$$y^d_{ij}\bar{Q}_i\Phi d_{R_j} + y^u_{ij}\bar{Q}_j\tilde{\Phi}u_{R_j}$$

Higgs interactions $\Phi \mapsto v$

$$y^d_{ij} \cdot v = M^d_{ij}$$

It is a $n \times n$ complex matrix. n is the number of families. In Standard Model there are three families.
Charge current interactions

- leptons

$$\mathcal{L}^{\text{CC}}_{\text{leptons}} = \frac{g}{\sqrt{2}}\left[\bar{v}_L W^\dagger_\mu\gamma^\mu e_L + \bar{e}_L W^-_\mu\gamma^\mu v_L\right]$$

- quarks

$$\mathcal{L}^{\text{CC}}_{\text{quarks}} = \frac{g}{\sqrt{2}}\left[\bar{u}_L W^\dagger_\mu\gamma^\mu d_L + \bar{d}_L W^-_\mu\gamma^\mu u_L\right]$$

The matrices are not diagonal in flavour.

Focus on M_{ij}. It is not necessary symmetric. Mass terms are in form of $m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$

Theorem M_{ij} can be diagonalized by a bi-unitary transformation

$$S^\dagger M T = M_d \tag{4.5.8}$$

with S and T unitary. M_d is diagonal and has positive eigenvalues.

Proof Any matrix $M = H \cdot V$ with H hermitian and V unitary.

$$(MM^\dagger)^\dagger = MM^\dagger$$

In word, $MM^\dagger$ is hermitian. Here matrix can be diagonalized by unitary matrix S

$$M_d^2 = S^\dagger(MM^\dagger)S$$
$$M_d^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$$

S is unique up to a diagonal phase matrix.

$$F = \mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}) \tag{4.5.9}$$

Replace $S \mapsto S' = SF$

$$(SF)^\dagger(MM^\dagger)(SF)$$
$$= F^\dagger S^\dagger(MM^\dagger)SF$$
$$= F^\dagger M_d^2 F$$
$$= M_d^2$$

$$S^\dagger M T = M_d$$
$$(SF)^\dagger M T = M_d$$
$$S^\dagger M T = F M_d$$

Use the phase freedom to have entries $m_i \geq 0$.

Define $H = S M_d S^\dagger$

$$H^\dagger = (S M_d S^\dagger)^\dagger = S M_d S^\dagger = H$$

H is hermitian.

Define $V = H^{-1} M V^\dagger = M^\dagger H^{-1}$

Compute

$$VV^\dagger = H^{-1} M M^\dagger H^{-1}$$
$$= H^{-1} S M_d^2 S^\dagger H^{-1}$$
$$= H^{-1} S M_d S^\dagger S M_d S^\dagger H^{-1}$$
$$= H^{-1} H H H^{-1}$$
$$= \mathbb{1}$$

so V is unitary.

$$V = H^{-1}M$$
$$HV = M$$
$$H = MV^\dagger$$

$$S^\dagger H S = S^\dagger M V^\dagger S$$

LHS by definition is M_d

$$M_d = S^\dagger M V^\dagger S$$
$$M_d = S^\dagger M T \quad \text{with } T = V^\dagger S$$

Recall that our entries after spontaneous symmetry breaking involve

$$M_{ij}^u \bar{u}_{L_i} U_{R_j} + M_{ij}^d \bar{d}_{L_i} d_{R_j}$$
$$= \boldsymbol{u}_L \underline{M}^u \boldsymbol{u}_R + \boldsymbol{d}_L \underline{M}^d \boldsymbol{d}_R$$

Call the transformation to mass eigenstates

$$\boldsymbol{u}_L = S_u \boldsymbol{u}'_L$$
$$\boldsymbol{d}_L = S_d \boldsymbol{d}'_L$$
$$\boldsymbol{u}_R = T_u \boldsymbol{u}'_R$$
$$\boldsymbol{d}_R = T_d \boldsymbol{d}'_R$$

This transformation has no effect on neutral current interactions. (It does not live in spinor space). Charged current interactions on the other hand

$$\frac{g}{\sqrt{2}} \left[\bar{u}_L W_\mu^\dagger \gamma^\mu d_L + \bar{d}_L W_\mu^- \gamma^\mu u_L \right] + \text{terms for } c, s, t \text{ and } b$$
$$= \frac{g}{\sqrt{2}} \left[\bar{u}_L \gamma^\mu d_L W_\mu^\dagger + \bar{d}_L \gamma^\mu u_L W_\mu^- \right]$$

Charged current transforms to

$$= \frac{g}{\sqrt{2}} \left[\bar{u}'_L \gamma^\mu \left(S_u^\dagger S_d \right) d'_L W_\mu^\dagger + \bar{d}'_L \gamma^\mu \left(S_d^\dagger S_u \right) u'_L W_\mu^- \right]$$

Cabbibo-Kobayashi-Maskawa matrix

$$V_{\text{CKM}} = S_u^\dagger S_d$$
$$V_{\text{CKM}}^\dagger = S_d^\dagger S_u \tag{4.5.10}$$

It is unitary and related to CP-violation.

How many parameters are there in V_{CKM}? 3×3 complex matrix in general has 18-real parameter.

How many phases? $\bar{q} i \slashed{\partial} q$ not affected by (global) phase. Mass terms must change q_L and q_R by same phase.

First doublet

$$Q_{1L} = \begin{pmatrix} u \\ V_{11}d + V_{12}s + V_{13}b \end{pmatrix}_L \tag{4.5.11}$$

$V_{11} = (V_{\text{CKM}})_{11}$, $V_{11} = R_{11}e^{i\delta}$, $R_{11} \in \mathbb{R}$. Use the notation $u = u'e^{i\delta}$ and $V = V'e^{i\delta}$

$$Q_{1L} = e^{i\delta_1}\begin{pmatrix} u' \\ R_{11}d + V'_{12}s + V'_{13}b \end{pmatrix}$$

$$Q_{2L} = e^{i\delta_2}\begin{pmatrix} c' \\ R_{21}d + V'_{22}s + V'_{23}b \end{pmatrix}$$

$$Q_{3L} = e^{i\delta_3}\begin{pmatrix} t' \\ R_{31}d + V'_{32}s + V'_{33}b \end{pmatrix}$$

Overall phases of doublets Q_{iL} don't affect anything. Still have freedom of phase shift in s and b. Get rid of phase in V'_{12} and V'_{13}

$$V'_{22} \mapsto V''_{22} \quad V'_{23} \mapsto V''_{23} \quad \ldots$$

Started with 18 parameters, absorbed 5 phases. It ends up with 13 parameters. They are 9 real parameter, 4 phases. Normalization condition on 3 guys indicates that all states have to orthogonal. Thus 6 states. $13 - 3 - 6 = 4$ It has 4 real parameter in 3×3 case.

Then V_{CKM} is a orthogonal matrix multiplied with a phase. A orthogonal 3×3 matrix has 3 real and 1 phase parameters. This phase leads to CP-violation.

CKM matrix $n \times n$ cases. There are $2n^2$ (real) parameters. Unitarity of CKM matrix conditions, then n^2 parameters. $(2n - 1)$ phases can be removed by a redefinition of quark phases $2 \cdot 3 - 1 = 5$ ($n = 3$).

$n \times n$ orthogonal matrix has $n(n-1)/2$ real parameters. Number of phases

$$n^2 - (2n - 1) - n(n-1)/2 = n^2 - 2n + 1 - \frac{1}{2}n^2 + \frac{1}{2}n$$

$$= \frac{1}{2}n^2 - \frac{3}{2}n + 1 = \frac{1}{2}(n-1)(n-2)$$

- $n = 2$ There is no phase
- $n = 3$ 1 phase Standard Model
- $n = 4$ 3 phases

The phases lead to CP-violation.

5 Scattering Theory

5.1 Non-relativistic Perturbation Theory

Quantum states are eigenfunctions of (bare) Hamiltonians. They should be (complete) orthonormal basis for the Hilbert space.

$$H_0\phi_n = E_n\phi_n$$

$$\int \mathrm{d}^3x\, \phi_m^*\phi_n = \delta_{mn}$$

Schrödinger equation

$$(H_0 + V)\psi = i\frac{\mathrm{d}\psi}{\mathrm{d}t}$$

Make the ansatz that an arbitrary state is a superposition of of ϕ_n

$$\psi = \sum a_n(t)\phi_n(\boldsymbol{x})\mathrm{e}^{-iE_nt} \tag{5.1.1}$$

Insert it into Schrödinger equation 5.1.1

$$i\sum \frac{\mathrm{d}a_n}{\mathrm{d}t}\phi_n(\boldsymbol{x})\mathrm{e}^{-iE_nt} = \sum V(\boldsymbol{x},t)a_n(t)\phi_n(\boldsymbol{x})\mathrm{e}^{-iE_nt}$$

multiply by $\phi_f^*(\boldsymbol{x})$ and $\int \mathrm{d}^3x$

$$\frac{\mathrm{d}a_f}{\mathrm{d}t} = -i\sum_n a_n(t)\int \mathrm{d}^3x\, \phi_f^*V\phi_n \cdot \mathrm{e}^{i(E_f-E_n)t}$$

Assume this potential acts for a finite time T, between $[-T/2, T/2]$.
The initial state is $|i\rangle$ and eigenstate of H_0.

$$a_i(-T/2) = 1$$
$$a_n(-T/2) = 0 \ \text{ with } n \neq i$$

$$\frac{\mathrm{d}a_f}{\mathrm{d}t} = -ia_n(t)\int \mathrm{d}^3x\, \phi_f^*V\phi_i \cdot \mathrm{e}^{i(E_f-E_i)t} \tag{5.1.2}$$

integrate

$$a_f(t) = -i\int_{-T/2}^t \mathrm{d}t'\int \mathrm{d}^3x\, \phi_f^*V\phi_i\mathrm{e}^{i(E_f-E_i)t}\bigg|_{t=-T/2}$$

$$T_{fi} = a_f(T/2)$$

$$= -i \int_{-T/2}^{T/2} \mathrm{d}t \int \mathrm{d}^3x \left[\phi_f(\boldsymbol{x})\mathrm{e}^{-iE_f t}\right]^* V(\boldsymbol{x},t) \left[\phi_i(\boldsymbol{x})\mathrm{e}^{-iE_i t}\right]$$

$$T_{fi} = -i \int \mathrm{d}^4x\, \phi_f^*(x) V(x) \phi_i(x) \qquad (5.1.3)$$

for $a_f(t) \ll 1$.

Assume static potential $V(x) = V(\boldsymbol{x})$

$$T_{fi} = -i V_{fi} \int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{-i(E_f - E_i)t}$$

$$= -2\pi i V_{fi}\delta(E_f - E_i) \qquad (5.1.4)$$

$$V_{fi} = \int \mathrm{d}^3x\, \phi_f^*(\boldsymbol{x}) V(\boldsymbol{x}) \phi_i(\boldsymbol{x})$$

As $\Delta E \to 0$, $\Delta t = \infty$, so infinite time is needed to get from $|i\rangle \to |f\rangle$. Define transition probability per unit time

$$W = \lim_{T\to\infty} \frac{|T_{fi}|^2}{T}$$

$$= \lim_{T\to\infty} 2\pi \frac{|V_{fi}|^2}{T} \delta(E_f - E_i) \int_{-T/2}^{T/2} \mathrm{d}t\, \mathrm{e}^{i(E_f - E_i)t}$$

$$= \lim_{T\to\infty} 2\pi \frac{|V_{fi}|^2}{T} \delta(E_f - E_i) T$$

$$= 2\pi |V_{fi}|^2 \delta(E_f - E_i) \qquad (5.1.5)$$

It is only meaningful after integration over initial and final states. In collider initial state is prepared with fixed momentum $\boldsymbol{p_e}$. So no integration over initial states. For proton (hadron), it is a bit more complicated since the quarks are moving inside of proton.

Define the density of final states $\rho(E_f)$. <u>Fermi's Golden rule</u>

$$W_{fi} = 2\pi \int \mathrm{d}E_f\, \rho(E_f) |V_{fi}|^2 \delta(E_f - E_i)$$

$$= 2\pi |V_{fi}|^2 \rho(E_i) \qquad (5.1.6)$$

The result can be improved by considering higher order scatterings, i.e. scatter multiple times. For each order insert the result for $a_f(t)$ into formula (5.1.2) for the $a_{n\neq i}(t)$

$$\frac{\mathrm{d}a_f}{\mathrm{d}t} = -\sum a_n(t) \int \mathrm{d}\phi_f^* V\phi_n\, \mathrm{d}^3x\, \mathrm{e}^{i(E_f - E_i)t} \qquad (5.1.7)$$

replace

$$a_n(t) = -i \int_{T/2}^{t} \mathrm{d}t' \int \mathrm{d}^3x'\, \phi_n^* V\phi_i \mathrm{e}^{i(E_n - E_i)t} \qquad (5.1.8)$$

Then

$$\frac{\mathrm{d}a_f}{\mathrm{d}t} = -i \int \mathrm{d}^3x\, \phi_f^* V\phi_i \mathrm{e}^{i(E_f - E_i)t} + (-i)^2 \sum_{n\neq i} \int_{-T/2}^{t} \mathrm{d}t'\, \mathrm{e}^{i(E_n - E_i)t'} V_{fn} \mathrm{e}^{i(E_f - E_n)t}$$

Correlation to T_{fi}

$$T_{fi} = \cdots - \sum_{n \neq i} V_{fn} V_{ni} \int_{-\infty}^{+\infty} \mathrm{d}t \, \mathrm{e}^{i(E_f - E_n)t} \int_{-\infty}^{t} \mathrm{d}t' \, \mathrm{e}^{i(E_n - E_i)t}$$

regulate the last exponential by adding a small constant ϵ

$$T_{fi} = \cdots - 2\pi i \sum_{n \neq i} \frac{V_{fn} V_{ni}}{E_i - E_n + i\epsilon} \delta(E_f - E_i)$$

This is second order in perturbation theory including time dependence.

We can treat it like Feynman diagram with vertex V_{ni} and propagator $1/(E_i - E_n + i\epsilon)$.

It is not manifestly covariant. Potential is assumed static and it defines the preferred reference frame. To go to relativistic formulation, consider other particle(s), which create electromagnetic potential for first particle to scatter in and vice versa.

5.2 Scalar Electrodynamics

5.2.1 Complex scalar

$$\mathcal{L} = (\partial_m u \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi \tag{5.2.1}$$

It has the equation of motion

$$(\partial^2 + m^2)\phi = 0 \tag{5.2.2}$$

To include electromagnetism consider generalized momentum

$$p^\mu \mapsto p^\mu +_e A^\mu$$
$$i\partial^\mu \mapsto i\partial^\mu + eA^\mu$$

Insert it into equation of motion

$$(\partial^2 + m^2)\phi = -V\phi$$
$$V = -ie(\partial_\mu A^\mu + A^\mu \partial_m u) - e^2 A^2 \tag{5.2.3}$$

Fine structure constant $\alpha = e^2/(4\pi) \approx 1/137 \ll 1$. So in first order drop the e^2 term.

$$T_{fi} = -i \int \mathrm{d}^4 x \, \phi_f^* x V(x) \phi_i(x)$$
$$= -i \int \mathrm{d}^4 x \, \phi_f^*(x) ie \left[\partial_\mu A^\mu + A^\mu \partial_\mu \right] \phi_i(x)$$

integrate by parts and drop surface terms

$$= -i \int j_\mu^{fi} A^\mu \, \mathrm{d}^4 x \tag{5.2.4}$$

with the current

$$j_\mu^{fi}(x) = -ie \left[\phi_f^*(\partial_\mu \phi_i) - (\partial_\mu \phi_f)^* \phi_i \right] \tag{5.2.5}$$
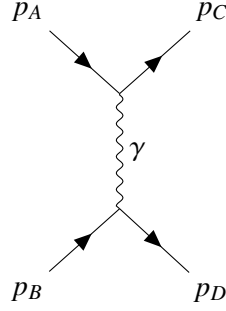
The current is conserved for free field $\partial_\mu j^\mu(x) = 0$

Write ϕ_i and ϕ_f as plane waves

$$\phi_i(x) = N_i \mathrm{e}^{-ip_i \cdot x}$$
$$j_\mu^{fi} = -eN_i N_f (p_i + p_f) \mathrm{e}^{i(p_f - p_i) \cdot x}$$

5.2.2 Scalar Charged Scattering

Consider potential for particle 1 to scatter caused by particle 2 and vice versa.

What potential A_μ does $j_\mu^{(2)}$ create for particle 1 to scatter in inhomogeneous Maxwell equation

$$\partial^2 A^\mu = j^{\mu(2)} = -e N_B N_D (p_D + p_B)^\mu e^{i(p_D - p_B)\cdot x}$$

Also consider photons as plane waves

$$\partial^2 e^{iqx} = -q^2 e^{iqx}$$

thus

$$A^\mu = -\frac{1}{q^2} j^{\mu(2)} \tag{5.2.6}$$

with $q = p_D - p_B$.

Insert into 5.2.4

$$
\begin{aligned}
T_{fi} &= -i \int j_\mu^{(1)}(x)\left(-\frac{1}{q^2}\right) j^{\mu(2)}(x)\, \mathrm{d}^4 x \\
&= -i e^2 N_A N_B N_C N_D \int \mathrm{d}^4 x\, e^{i(p_c + p_D - p_A - p_B)}(p_C + p_A)_\mu (p_D + p_B)^\mu \frac{-1}{q^2} \\
&= -i N_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_C + p_D - p_A - p_B)\mathcal{M}
\end{aligned}
$$

The invariant matrix element

$$-i\mathcal{M} = ie(p_A + p_C)^\mu \left(-i\frac{g_{\mu\nu}}{q^2}\right) ie(p_B + p_D)^\nu$$

It is symmetric in (A, C) and (B, D).

5.2.3 From Amplitude to Cross Section

In the laboratory we measure cross sections. Need to fix norm, $\phi = N e^{-ipx}$. The number density is defined by

$$\rho = 2E|N|^2 \tag{5.2.7}$$

The infinitesimal number of particles $\rho\, \mathrm{d}^3 x$ is Lorentz invariant. Norm is chosen as

$$\int \rho\, \mathrm{d}V = 2E \tag{5.2.8}$$

We are interested in Transition rate per unit time and volume.

$$W_{fi} = \frac{|T_{fi}|^2}{T \cdot V}$$

$$T_{fi} = -iN_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_c + p_D - p_A - p_B)\mathcal{M}$$

Same trick as before (5.1.5) to get rid of one of δ

$$\int d^4 x = T \cdot V$$

$$W_{fi} = \frac{1}{V^4}(2\pi)^4 \delta^{(4)}(p_C + p_D - p_A - p_B)|\mathcal{M}|^2$$

Definition of cross section

$$\sigma = \frac{W_{fi}}{\text{initial flux}} \cdot (\text{number of final states}) \tag{5.2.9}$$

Imagine particles in a box

$$(\text{number of final state particles}) = \frac{V \, d^3 p}{(2\pi)^3 2E}$$

Density of incoming particles A $2E_A/V$. The box has length $\boldsymbol{v}_A \cdot t$. Number of beam particles passing through unit area A per unit time is $\boldsymbol{v}_A \cdot t 2E_A/(Vt)$. Particle B (target). Initial flux is $|\boldsymbol{v}_A| \frac{2E_a}{v} \frac{2E_B}{V}$

$$d\sigma = \frac{V^2}{|\boldsymbol{v}_A| 2E_A 2E_B} \frac{1}{V^4}|M|^2 \frac{(2\pi)^4}{(2\pi)^4} \delta^{(4)}(p_c + p_D - p_A - p_B) \frac{d^3 p_C}{2\pi} \frac{d^3 p_D}{2\pi} V^2$$

$$= \frac{|\mathcal{M}|^2}{F} \, dQ$$

with $F = |\boldsymbol{v}_A| 2E_A 2E_B$ and $dQ = (2\pi)^4 \delta^{(4)}(p_C + p_D - p_A - p_B)\frac{d^3 p_C}{(2\pi)^3 2E_C} \frac{d^3 p_D}{(2\pi)^3 2E_D}$ and B at rest.

If both are moving $F = |\boldsymbol{v}_A - \boldsymbol{v}_B| 2E_A 2E_B = 4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}$.

In centre of mass system

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{p_f}{p_i}|M|^2 \tag{5.2.10}$$

For $e\mu$ scattering

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(\frac{3 + \cos(\theta)}{1 - \cos(\theta)}\right)$$

with θ angle between $\boldsymbol{p}_A$ and $\boldsymbol{p}_C$.

Also it includes decays, for example $n \to pe^-\bar{\nu}_e$.

Decay versus cross section   Examples are $\pi^- \to \mu^- \bar{\nu}_\mu$, $t \to W^+ b$

$$d\Gamma = \frac{1}{2E_A}|\mathcal{M}|^2 \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} \tag{5.2.11}$$

$$\Gamma = \int d\Gamma \tag{5.2.12}$$

Pion for instance has different decay modes.

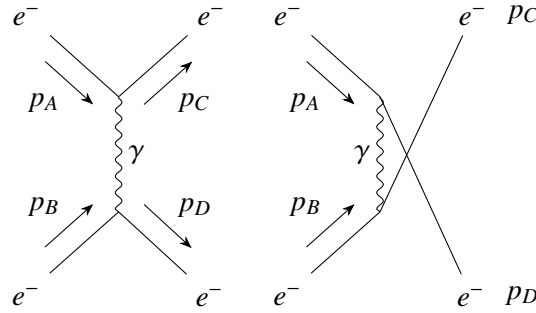$$\Gamma_{\text{tot}} = \Gamma(\pi \to \mu\nu_\mu) + \Gamma(\pi \to e\bar{\nu}_e) \tag{5.2.13}$$

Radioactive decay

$$N_A(t) = N(0)\text{e}^{-\Gamma_{\text{tot}}t} \tag{5.2.14}$$

Lifetime is defined as
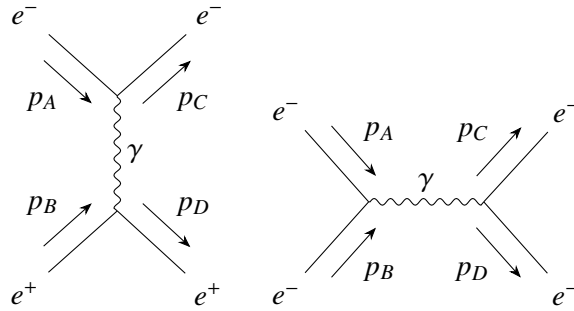
$$\tau_A = 1/\Gamma_{\text{tot}}(A) \tag{5.2.15}$$

5.3 Electron Electron scattering

One cannot differentiate electron from electron. We have to add both diagrams before squaring

$$-iM_{e^-e^-} = -ie^2\left[-\frac{(p_A + p_C)_\mu \cdot (p_B + p_D)^\mu}{(p_D - p_B)^2} - \frac{(p_A + p_D)_\mu \cdot (p_B + p_C)^\mu}{(p_C - p_B)^2}\right]$$

5.4 Electron Positron scattering

5.5 Mandelstam Variables

Only for two-to-two scattering

$$s = (p_A + p_B)^2 \tag{5.5.1}$$

$$u = (p_A - p_C)^2 \tag{5.5.2}$$

$$t = (p_A - p - D)^2 \tag{5.5.3}$$

Diagrams are often classified by the momentum of propagator. Mandelstam variables s, t and u are not independent.

$$s + u + t = \sum m_i^2 \tag{5.5.4}$$

One can set the reference frame such that

$$p_A = (E, 0, 0, E)$$
$$p_B = (E, 0, 0, -E)$$
$$p_C = (E, 0, E\sin(\theta), E\cos(\theta))$$
$$p_D = (E, 0, -E\sin(\theta), -E\cos(\theta))$$

Then

$$s = 4E^2 \tag{5.5.5}$$
$$t = -\frac{s}{2}(1 - \cos(\theta)) \tag{5.5.6}$$
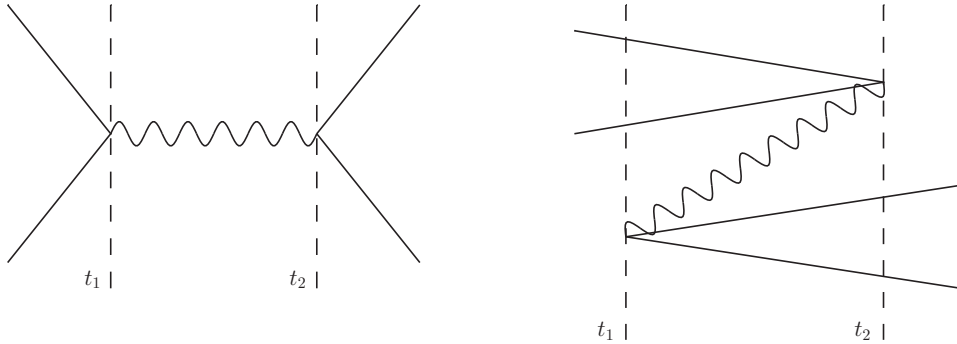$$u = -\frac{s}{2}(1 + \cos(\theta)) \tag{5.5.7}$$

5.6 Origin of the propagator

Non-relativistically

$$T_{fi}^{(2)} = -i \sum_{n \neq i} V_{fn} \frac{1}{E_i - E_n} V_{ni} 2\pi\delta(E_f - E_i) \tag{5.6.1}$$

Electron positron scattering each Feynman diagram is the sum of all possible time orderings in old fashioned perturbation theory.

Two times orderings for s-channel diagram

(one can understand the second diagram by saying that an amount of energy is borrowed to produce particle pair.)

$$\mathcal{M} \sim V_{fn} \frac{1}{E_i - E_\gamma} V_{ni} + V_{fn} \frac{1}{E_i - 2E_i - E_\gamma} V_{ni}$$
$$= V_{fn} \frac{2E_\gamma}{E_i^2 - E_\gamma^2} V_{ni}$$

$$(p_A + p_B)^2 = E_i^2 - (\boldsymbol{p}_A + \boldsymbol{p}_B)^2$$
$$E_\gamma^2 = \boldsymbol{p}^2 + m_\gamma^2$$
$$p = p_A + p_B$$
$$\frac{1}{E_i^2 - E_\gamma^2} = \frac{1}{(p_A + p_B)^2 - m_\gamma^2} = \frac{1}{q^2}$$

5.7 Electron interacting with A^μ

$$(ip_\mu \gamma^\mu - m)\psi = 0 \tag{5.7.1}$$
$$\psi = u(\boldsymbol{p})\mathrm{e}^{-ip \cdot x} \tag{5.7.2}$$

Consider $p^\mu \mapsto p^\mu + eA^\mu$

$$(\gamma^\mu p_\mu - m)\psi = (\gamma^0 V)\psi$$
$$\gamma^0 V = -e\gamma_\mu A^\mu$$

$$T_{fi} = -i \int \mathrm{d}^4 x\, \psi_f^\dagger(x) V(x) \psi_i(x)$$
$$= ie \int \mathrm{d}^4 x\, \overline{\psi}_f(x) \gamma_\mu A^\mu \psi_i$$
$$= -i \int \mathrm{d}^4 x\, j_\mu^{fi} A^\mu$$

$$j_\mu^{fi} = -e\overline{\psi}_f \gamma_\mu \psi_i = -e\overline{u}_f \gamma_\mu u_i \mathrm{e}^{i(p_f - p_i)x} \tag{5.7.3}$$

In non-relativistic case

$$j_\mu^{fi} = -e\overline{\psi}_f \gamma_\mu \psi_i = -e(p_f + p_i)_\mu \mathrm{e}^{i(p_f - p_i)x} \tag{5.7.4}$$

So we have Feynman rules for external fermions and vertex.

One can write the following identity

$$\overline{u}_f \gamma^\mu u_i = \frac{1}{2m} \overline{u}_f \left[(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu \right] u_i \tag{5.7.5}$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. First term is exactly we have before. It comes from electric charge interaction. The second part is then spin or magnetic interaction. It can be shown that $j^\mu A_\mu$ in Non-relativistic limit becomes $\boldsymbol{\sigma} \cdot \boldsymbol{B}$ from $\sigma^{\mu\nu}$ term.

5.8 Møller Scattering

$e^- e^- \to e^- e^-$

$$T_{fi} = -i \int d^4x\, j_\mu^{(1)} (-\frac{1}{q^2}) j_2^\mu(x)$$

$$= -i(-e\bar{u}_c\gamma_\mu u_A)\left(-\frac{1}{q^2}\right)(-e\bar{u}_D\gamma^\mu u_B)(2\pi)^4\delta^{(4)}(p_A + p_B - p_C - p_D)$$

with $q = p_A - p_C = p_D - p_B$

$$= -i(2\pi)^4\delta^{(4)}(p_A + p_B - p_C - p_D)\mathcal{M}$$

$$-i\mathcal{M} = (ie\bar{u}_C\gamma^\mu u_A)\left(\frac{-ig_{\mu\nu}}{q^2}\right)(ie\bar{u}_D\gamma^\nu u_B)$$

swapping order of final state electrons. Minus sign comes from electron being fermion.

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = -e^2\frac{(\bar{u}_C\gamma^\mu u_A)(\bar{u}_D\gamma_\mu u_B)}{(p_A - p_C)^2} + e^2\frac{(\bar{u}_D\gamma^\mu u_A)(\bar{u}_C\gamma_\mu u_B)}{(p_A - p_D)^2}$$

$$|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\operatorname{Re}(\mathcal{M}_1\mathcal{M}_2^*) \tag{5.8.1}$$

$$|\mathcal{M}_1|^2 = \frac{e^4}{t^2}(\bar{u}_C\gamma^\mu u_A)(\bar{u}_A\gamma^\nu u_D)(\bar{u}_D\gamma_\mu u_B)(\bar{u}_B\gamma_\nu u_D) \tag{5.8.2}$$

Most experiments are done without polarized beams. Thus we must average incoming spins.

$$\frac{1}{(2S_A + 1)(2S_B + 1)}\sum_{S_A, S_B} \tag{5.8.3}$$

We should also sum over the final states (if it will not be observed).

Shuffle the spinors and use spin sum identities

$$\sum_S u^{(S)}(p)\bar{u}^{(S)}(p) = \slashed{p} + m \tag{5.8.4}$$

we get (for the second half)

$$\operatorname{tr}\left[(\slashed{p}_D + m)\gamma^\mu(\slashed{p}_B + m)\gamma^\nu\right]$$

$$\frac{1}{4}\sum_{\text{spin}}|\mathcal{M}_1|^2 = \frac{e^4}{4t^2}\operatorname{tr}\left[(\slashed{p}_D + m)\gamma^\mu(\slashed{p}_B + m)\gamma^\nu\right]\operatorname{tr}\left[(\slashed{p}_C + m)\gamma^\mu(\slashed{p}_A + m)\gamma^\nu\right]$$

We know the trace of odd number of gamma matrices vanished. Thus

$$\frac{4e^4}{t^2}[2p_A \cdot p_D p_C \cdot p_B + 2p_A \cdot p_B e p_C \cdot p_D - 4p_A \cdot p_C p_B \cdot p_D + 4p_A \cdot p_C p_B \cdot p_D +$$

$$m^2(2p_B \cdot p_D - 4p_B \cdot p_D + 2p_A \cdot p_C - 4p_A \cdot p_C) + 4m^4]$$

In most experiments, center of mass energy is well above electron mass, thus we can set $m_e = 0$

$$= 8(u^2 + s^2)$$

$$s = 2p_A \cdot p_B = 2p_C \cdot p_D$$
$$t = -2p_A \cdot p_C$$
$$u = -2p_A \cdot p_D$$

Correspondingly

$$\frac{1}{4} \sum |\mathcal{M}_2|^2 = \frac{2e^4}{u^2}(t^2 + s^2)$$

In interference term, there is just one (long) trace.

$$\frac{1}{4} \sum 2 \operatorname{Re}(\mathcal{M}_1 \mathcal{M}_2^*) = \frac{4e^4}{tu} s^2$$

$$\frac{1}{4} \sum |\mathcal{M}|^2 = \overline{|\mathcal{M}|^2}$$
$$= 2e^4 \left[\frac{u^2 + s^2}{t^2} + \frac{t^2 + s^2}{u^2} + 2\frac{s^2}{tu} \right]$$

$e^- \mu^- \to e^- \mu^-$

$e^- e^+ \to \mu^+ \mu^-$ Change $k' \to -p$ and $s \leftrightarrow t$.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1 + \cos^2 \theta)$$

integrate over solid angle

$$= \frac{4\pi\alpha^2}{3s}$$

Cross section drops down with $1/s$. With high luminosity we can evade the problem.

5.9 Photons and Polarization Vectors

Photon field $A^\mu(x)$ satisfies Maxwell equations

$$\partial^2 A^\mu = j^\mu$$

Even in classical theory, there is gauge freedom. With Lorenz condition $\partial_\mu A^\mu = 0$ there is still some gauge freedom.

$$A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \Lambda(x) \tag{5.9.1}$$

if $\partial^2 \Lambda(x) = 0$.

   Free photon

$$\partial^2 A^\mu = 0 \tag{5.9.2}$$
$$A^\mu = \epsilon^\mu(\boldsymbol{q}) e^{-iq \cdot x} \tag{5.9.3}$$

with $\epsilon(\boldsymbol{p})$ polarization vector with $\mu = 0, 1, 2, 3$. Thus equation for polarization vector is

$$q^2 \epsilon^\mu(\boldsymbol{q}) e^{-iq\cdot x} = 0 \tag{5.9.4}$$

with $q^2 = m_\gamma^2 = 0$.

From QM, we know for spin 1 particle, $S_z = +1, 0, -1$. With Lorenz condition,

$$q_\mu \epsilon^\mu = 0$$

thus there are three independent components.

We still have constraint 5.9.1. To make it specific, choose

$$\Lambda = iae^{-iq\cdot x} \tag{5.9.5}$$

then the constraint is satisfied

$$\partial^2 \Lambda = q^2 \Lambda = 0$$

Use this specific Λ, the transformation of A_μ becomes

$$A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \Lambda \tag{5.9.6}$$
$$\epsilon e^{-iq\cdot x} \mapsto \epsilon' e^{-iq\cdot x} = \epsilon e^{-iq\cdot x} + ia(-i)q_\mu e^{-iq\cdot x} \tag{5.9.7}$$
$$\epsilon \mapsto \epsilon'_\mu = \epsilon_\mu + aq_\mu \tag{5.9.8}$$

ϵ and ϵ' describe the same photon.

Use this freedom to demand $\epsilon^{\mu=0} = 0$. Then we have Coulomb gauge

$$\boldsymbol{\epsilon} \cdot \boldsymbol{q} = 0 \tag{5.9.9}$$

Only two independent polarization vectors.

$$\boldsymbol{q} = (0, 0, q)$$
$$\epsilon_1 = (1, 0, 0)$$
$$\epsilon_2 = (0, 1, 0)$$

Photon only with $\epsilon_{1,2}$ are linear polarized.

Circularly polarized

$$\epsilon_R = -\frac{1}{\sqrt{2}}(\epsilon_1 + i\epsilon_2) \tag{5.9.10}$$

$$\epsilon_L = \frac{1}{\sqrt{2}}(\epsilon_1 - i\epsilon_2) \tag{5.9.11}$$

ϵ_R has $+1$ helicity and ϵ_L has -1.

One can show completeness relation

$$\sum_{\lambda=L,R} (\epsilon_\lambda)_i^* (\epsilon_\lambda)_j = \delta_{ij} - \hat{q}_i \hat{q}_j \tag{5.9.12}$$

with $i, j = 1, 2, 3$ and hat denotes unit vector.

5.10 Electron Propagator

In non-relativistic limit

$$T_{fi} = -2\pi i \delta(E_f - E_i) \left[\langle f|V|i \rangle + i \sum_{n \neq i} \langle f|V|n \rangle \frac{1}{E_i - E_n} \langle n|V|i \rangle + \ldots \right]$$

since $H_0 |n\rangle = E_n |n\rangle$

$$= 2\pi \delta(E_f - E_i) \langle f| \left[(-iV) + (-iV) \frac{i}{E_i - H_0} (-iV) + \ldots \right] |i\rangle$$

So we have vertex $-iV$ and propagator $-i/(E - H_0)$.

Go to relativistic case. With spinless particle, Klein-Gordon equation

$$i(\partial^2 + m^2)\phi = -iV\phi \tag{5.10.1}$$

We expect propagator

$$\frac{i}{p^2 - m^2} \tag{5.10.2}$$

For spin $\frac{1}{2}$ particle, Dirac equation in momentum space

$$(\not{p} - m)\psi = -e\gamma^\mu A_\mu \psi$$

So we expect propagator to be

$$\frac{i}{\not{p} - m} = \frac{i(\not{p} + m)}{p^2 - m^2} \tag{5.10.3}$$

Remember $\not{p} + m = \sum_{\text{spins}} u(p)\bar{u}(p)$. In fact for any particle the propagator can be written as

$$\frac{i \sum_{\text{spins}(\ldots)}}{p^2 - m^2} \tag{5.10.4}$$

5.11 Photon Propagator

Gauge freedom leads to the fact that photon propagator is not unique.

Wave equation

$$\partial^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu \tag{5.11.1}$$

$$(g^{\mu\lambda}\partial^2 - \partial^\mu \partial^\lambda)A_\lambda = j^\mu \tag{5.11.2}$$

Propagator should be inverse of this operator. In momentum space

$$(g^{\mu\lambda}q^2 - q^\mu q^\lambda)A_\lambda = j^\mu \tag{5.11.3}$$

Ansatz for inverse

$$A(q^2)q^2 g_{\nu\mu} + B(q^2)q_\nu q_\mu \tag{5.11.4}$$

Multiply back to equation

$$(A(q^2)g_{\nu\mu} + B(q^2)q_\nu q_\mu)(-g^{\mu\lambda}q^2 + q^\mu q^\lambda)$$
$$= -A(q^2)q^4\delta_\nu^\lambda + A(q^2)q^2 q_\nu q^\lambda - B(q^2)q_\nu q^\lambda q^2 + B(q^2)q^2 q_\nu q^\lambda$$
$$= -A(q^2)q^2(q^2\delta_\nu^\lambda - q_\nu q^\lambda)$$
$$\overset{!}{=} \delta_\nu^\lambda$$

It is impossible!

In Lorenz gauge $\partial_\lambda A^\lambda = 0$

$$g^{\nu\lambda}\partial^2 A_\lambda = j^\nu$$

in momentum space

$$g^{\nu\lambda}q^2 A_\lambda = j^\nu$$

It has inverse

$$\frac{-g_{\mu\nu}}{q^2} \tag{5.11.5}$$

in which $-g_{\mu\nu}$ corresponds to sum over polarization. Propagator in this form is Feynman propagator, ideal for QED calculation.

More general form

$$\left[g^{\nu\lambda}\partial^2 - \left(1 - \frac{1}{\xi}\partial^\nu\partial^\lambda\right)\right]A_\lambda = j^\nu$$
$$\left[g^{\nu\lambda}q^2 - \left(1 - \frac{1}{\xi}q^\nu q^\lambda\right)\right]\tilde{A}_\lambda = \tilde{j}^\nu$$

with $\xi \in \mathbb{R}$. It is called R_ξ-gauge. The equation now is invertible. Construct inverse

$$(-q^2 g^{\nu\lambda} + \frac{\xi-1}{\xi}q^\nu q^\lambda)(Aq^2 g_{\lambda\mu} + Bq_\lambda q_\mu)$$
$$= -q^2\delta_\mu^\nu A - Bq^2 q^\nu q_\mu + \frac{\xi-1}{\xi}Aq^2 q^\nu q_\mu + \frac{\xi-1}{\xi}Bq^2 q^\nu q_\mu$$

choose $A = -1/q^4$ and the rest should vanish.

$$= Bq^2 q^\nu q_\mu + \frac{\xi-1}{\xi}\frac{q^\nu q_\mu}{q^2} + \frac{\xi-1}{\xi}Bq^2 q^\nu q_\mu$$
$$= \frac{q^\nu q_\mu}{q^2}\left[-Bq^4 - \frac{\xi-1}{\xi} + Bq^4\frac{\xi-1}{\xi}\right]$$

define $B = \tilde{B}/q^4$

$$= \frac{q^\nu q_\mu}{q^2\xi}\left[-\tilde{B}\xi - \xi + 1 + \tilde{B}\xi - \tilde{B}\right]$$
$$= \frac{q^\nu q_\mu}{q^2\xi}[-\xi + 1 - \tilde{B}] \overset{!}{=} 0$$

Thus

$$B = \frac{1-\xi}{q^4}$$

and propagator

$$\frac{i}{q^2}\left(-g_{\mu\nu} + (1-\xi)\frac{q_\mu q_\nu}{q^2}\right) \tag{5.11.6}$$

If ξ is left general, the final answer cannot contain ξ. In QED, current conservation ensures it with $q_\mu \tilde{j}^\mu = 0$.

5.12 Massive Vector Particles

$$\frac{i}{q^2 - M^2}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M^2}\right) \tag{5.12.1}$$

5.13 Real and Virtual Photons

Real photons have only two polarization states.

$$\sum_{\lambda=L,R}(\epsilon_\lambda)_i^*(\epsilon_\lambda)_j = \delta_{ij} - \hat{q}_i\hat{q}_j \tag{5.13.1}$$

$$
\begin{aligned}
-g_{\mu\nu} &= \sum_{\lambda=1}^4 \epsilon_\mu^{(\lambda)*}\epsilon_\nu^{(\lambda)} \\
&= \sum_{\text{transverse}} (\epsilon_\mu^T)^*(\epsilon_\nu^T) + \epsilon_\mu^{L*}\epsilon_\nu^L + \epsilon_\mu^{S*}\epsilon_\nu^S
\end{aligned}
$$

Every photon is virtual in some sense. It must interact with electron during detection.

$$T_{fi} = -i\int \mathrm{d}^4x\, j_\mu^A(x)\frac{-g^{\mu\nu}}{q^2} j_\nu^B(x)$$

$q = (q^0, 0, 0, |\boldsymbol{q}|)$

$$= -i\int \mathrm{d}^4x\left[\frac{j_1^A j_1^B + j_2^A j_2^B}{q^2} + \frac{j_3^A j_3^B - j_0^A j_0^B}{q^2}\right]$$

with first half in transverse mode and second half in longitudinal and scalar mode

From current conservation

$$q^\mu j_\mu = 0 = q^0 j_0 - |\boldsymbol{q}| j_3 = 0$$

If photon is almost real $q^0 \approx |\boldsymbol{q}|$, then $j_0 \approx j_3$. In general

$$j_3 = \frac{1}{|\boldsymbol{q}|}q^0 j_0$$

Insert back into T_{fi} we get expression for general virtual photon

$$T_{fi} = -i \int d^4x \left[\frac{j_1^A j_1^B + j_2^A j_2^B}{q^2} + \left(\frac{q^0}{|\boldsymbol{q}|} \frac{j_0^A j_0^B - j_0^A j_0^B}{q^2} \right) \right]$$

$$= -i \int d^4x \left[\frac{j_1^A j_1^B + j_2^A j_2^B}{q^2} + j_0^A j_0^B \frac{q^{0^2}/|\boldsymbol{q}|^2 - 1}{q^2} \right]$$

$$= -i \int d^4x \left[\frac{j_1^A j_1^B + j_2^A j_2^B}{q^2} + j_0^A j_0^B \frac{q^2}{q^2 |\boldsymbol{q}|^2} \right]$$

$$= -i \int d^4x \left[\frac{j_1^A j_1^B + j_2^A j_2^B}{q^2} + \frac{j_0^A j_0^B}{|\boldsymbol{q}|^2} \right]$$

$$\frac{1}{|\boldsymbol{q}|^2} = \int d^3x \, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \frac{1}{4\pi|\boldsymbol{x}|} \qquad (5.13.2)$$

Fourier transform

$$T_{fi} = -i \int \cdots - i \int dt \int d^3x_1 \int d^3x_2 \frac{j_0^A(t,\boldsymbol{x}_1) j_0^B(t,\boldsymbol{x}_2)}{4\pi|\boldsymbol{x}_2 - \boldsymbol{x}_1|}$$
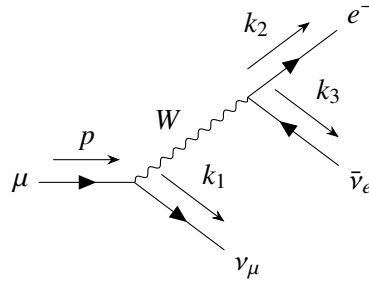
With $j_0 \sim \rho_{el}$ electric charge density. It describes instantaneous Coulomb interaction.

6 Electroweak Processes

We will consider $\mathbf{SU}(2)_L \times \mathbf{U}(1)_Y$ theory.

6.1 Muon decay

V − A interaction

$q = k_2 + k_3 = p - k_1$

$$i\mathcal{M} = \bar{u}(k_1)\frac{-ig}{\sqrt{2}}\gamma^\mu P_L u(p)\frac{-ig_{\mu\nu}}{q^2 - M_W^2}\bar{u}(k_2)\frac{-ig}{\sqrt{2}}\gamma^\nu P_L v(k_3)$$

In spinor space, Dirac equation

$$(\not{p} - m)u(p) = 0. \tag{6.1.1}$$

Propagator $M_W \sim 80\,\text{GeV}$ and $m_\mu = 100\,\text{MeV}$, i.e. $p^2 \ll M^2$. So 4-Fermi approximation is obtained.

$$\frac{1}{q^2 - M_W^2} = -\frac{1}{M_W^2}\frac{1}{1 - q^2/M_W^2} \approx -\frac{1}{M_W^2}$$

Then the coupling is effectively

$$G_F = \frac{\sqrt{2}g^2}{8M_W^2} \tag{6.1.2}$$

Note that when for example top and bottom quarks are present, this approximation is not valid any more.

Proceed with $m_e = m_{\nu_e} = m_{\nu_\mu} = 0$

$$\mathcal{M} = -\frac{4G_F}{\sqrt{2}}\bar{u}(k_1)\gamma^\mu P_L u(p)\bar{u}(k_2)\gamma_\mu P_L v(k_3)$$

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{16G_F^2}{2\cdot 2}\bar{u}(k_1)\gamma^\mu P_L u(p)\bar{u}(p)P_R\gamma^\nu u(k_1)\bar{u}(k_2)\gamma_\mu P_L v(k_3)\bar{v}(k_3)P_R\gamma_\nu u(k_2)$$

$$= 4G_F^2\,\text{tr}\Big[\not{k}_1\gamma^\mu P_L(\not{p} + m_\mu)P_R\gamma^\nu\Big]\cdot\text{tr}\Big[\not{k}_2\gamma_\mu P_L\not{k}_3 P_R\gamma_\nu\Big]$$

Term with m_μ vanishes, since $P_L P_R = 0$

Computing the two traces

$$\text{tr}\left[\not{k}_1 \gamma^\mu \not{p} \gamma^\nu P_L\right] = \frac{4}{2}\left(k_1^\mu p^\nu - k_1 p g^{\mu\nu} + k_1^\nu p^\mu\right) - \frac{4i}{2}\epsilon_{\alpha\mu\beta\nu}k_{1\alpha}p_\beta$$

$$\text{tr}\left[\not{k}_2 \gamma_\mu \not{k}_3 \gamma_\nu P_L\right] = \frac{4}{2}\left((k_2)_\mu(k_3)_\nu - k_2 k_3 g_{\mu\nu} + k_{2\nu}k_{3\mu}\right) + \frac{4i}{2}\epsilon\epsilon_{\alpha\mu\beta\nu}k_2^\alpha k_3^\beta$$

Using the fact that ϵ is totally anti-symmetric tensor, so the terms with one ϵ sum to zero. Note further

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu}{}^{\rho'\sigma'} = -2(g^{\rho\rho'}g^{\sigma\sigma'} - g^{\rho\sigma'}g^{\rho'\sigma}) \tag{6.1.3}$$

In order to use this formula

$$\sigma^{\alpha\mu\beta\nu}\sigma_{\alpha'\mu\beta'\nu}$$
$$= \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu}{}^{\alpha''\beta''}g_{\alpha'\alpha''}g_{\beta'\beta''}$$
$$= (-2)\left[g^{\alpha\alpha''}g^{\beta\beta''} - g^{\alpha\beta''}g^{\beta\alpha''}\right]g_{\alpha'\alpha''}g_{\beta'\beta''}$$
$$= (-2)\left[\delta^\alpha_{\alpha'}\delta^\beta_{\beta'} - \delta^\alpha_{\beta'}\delta^\beta_{\alpha'}\right]$$

Terms without ϵ

$$4(k_1^\mu p^\nu + k_1^\nu p^\mu - g^{\mu\nu}k_1 \cdot p)(k_{2\mu}k_{3\nu} + k_{2\nu}k_{3\mu} - k_2 \cdot k_3 g_{\mu\nu})$$
$$= \cdots = 4\left[2(k_1 \cdot k_2)(p \cdot k_3) + 2(k_1 \cdot k_3)(p \cdot k_2)\right]$$
$$= 8\left[(k_1 \cdot k_2)(p \cdot k_3) + (k_1 \cdot k_3)(p \cdot k_2)\right]$$

Then

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = 64G_F^2(k_1 \cdot k_2)(p \cdot k_3)$$

Recall

$$d\Gamma = \frac{1}{2E}\overline{|\mathcal{M}|^2}\,dQ$$

$$dQ = \frac{d^3k_1}{(2\pi)^3 2E_{k_1}}\frac{d^3k_2}{(2\pi)^3 2E_{k_2}}\frac{d^3k_3}{(2\pi)^3 2E_{k_3}}(2\pi)^4\delta^{(4)}(p - k_1 - k_2 - k_3) \tag{6.1.4}$$

One can prove (In Standard Model neutrino has no mass.)

$$\frac{d^3k_1}{(2\pi)^3 2E_{k_1}} = \int d^4k_1\,\theta(E_1)\delta(k_1^2) \tag{6.1.5}$$

Use identity of Delta function involving composite function.

$$\frac{d^3k_1}{(2\pi)^3 2E_{k_1}}\delta^{(4)}(p - k_1 - k_2 - k_3) = \int d^4k_1\,\theta(E_1)\delta(k_1^2)\delta^{(4)}(p - k_1 - k_2 - k_3)$$

$$= \theta(k_1)\delta^{k_1^2}$$

Thus

$$dQ = \frac{1}{(2\pi)^5} \frac{d^3 k_2}{2E_{k_2}} \frac{d^3 k_3}{2E_{k_3}} \theta(E - E_{k_2} - E_{k_3}) \delta((p - k_2 - k_3)^2)$$

In rest frame of muon $p \cdot k_3 = m_\mu \cdot E_3$

$$(k_1 + k_2)^2 \approx 2k_1 \cdot k_2$$

$$k_1 \cdot k_2 \approx \frac{1}{2}(k_1 + k_2)^2$$

$$= \frac{1}{2}(p - k_3)^2$$

$$= \frac{1}{2}\left[p^2 - 2p \cdot k_3 + k_3^2\right]$$

$$= \frac{1}{2}[m_\mu^2 - 2m_\mu E_3]$$

$$\frac{1}{2}\overline{|\mathcal{M}|^2} = 64G_F^2 k_1 \cdot k_2 p \cdot k_3$$

$$= 64G_F^2 m_\mu E_3[m_\mu^2 - 2m_\mu E_3]$$

Now we have computed $|\mathcal{M}|^2$, the actual difficulties are in phase space integral. For n-particle phase space, often use Monte-Carlo integration technique.

With θ the angle between electron and electron neutrino (k_2 and k_3)

$$(p - k_2 - k_3)^2 = p^2 + k_2^2 + k_3^2 - 2p \cdot k_2 - 2p \cdot k_3 + 2k_2 \cdot k_3$$

$$= m_\mu^2 - 2m_\mu E_2 - 2m_\mu E_3 + 2E_2 E_3(1 - \cos\theta)$$

$$d^3 k_2 \, d^3 k_3 = (4\pi)(2\pi)E_2^2 \, dE_2 \, E_3^2 \, dE_3 \, d\cos\theta$$

$$\delta(\cdots - 2E_2 E_3 \cos\theta) = \frac{1}{2E_2 E_3}\delta(-\cos\theta)$$

$$d\Gamma = \frac{G_F^2}{2\pi^3} dE_3 \, dE_3 \, E_3(m_\mu^2 - 2m_\mu E_3)$$

$$\frac{1}{2}m_\mu - E_2 \leq E_3 \leq \frac{1}{2}m_\mu$$

$$0 \leq E_2 \leq \frac{1}{2}m_\mu$$

Carrying the integration out

$$\frac{d\Gamma}{dE_2} = \frac{m_\mu G_F^2}{2\pi^3} \int_{\frac{1}{2}m_\mu - E_2}^{\frac{1}{2}m_\mu} dE_3 \, E_3(m_\mu - 2E_3)$$

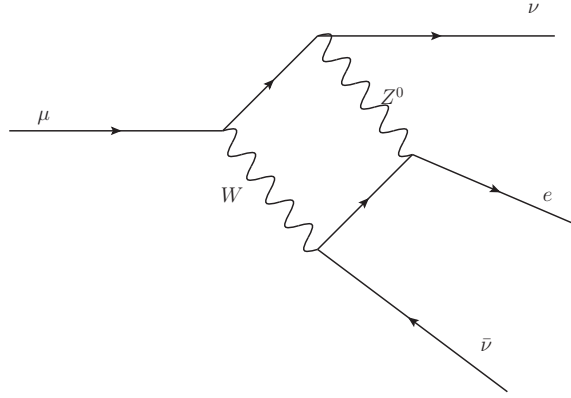$$= \frac{G_F^2}{2\pi^3}m_\mu^2 E_2^2 \left(3 - \frac{4E_2}{m_\mu}\right)$$

Finally

$$\Gamma = \frac{1}{\tau_\mu} = \int_0^{\frac{1}{2}m_\mu} dE_2 \frac{d\Gamma}{dE_2} = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

If $m_e \neq 0$

$$\Gamma = \frac{1}{\tau_\mu} = \int_0^{\frac{1}{2}m_\mu} dE_2 \frac{d\Gamma}{dE_2} = \frac{G_F^2 m_\mu^5}{192\pi^3} \left[1 - 8r^2 + 8r^6 - r^8 - 24r^4 \ln r \right]$$

with $r = m_e/m_\mu$. r enters in boundary of phase space integration.

If one tries to measure the decay width precisely, the 1-loop electroweak correction must also be included.

Neutrino can also be considered in the calculation and one finds an upper bound for neutrino mass $m_{\nu_\mu} \leq 1\,\text{keV}$.

Mass dimension of Γ is $+1$. Using dimensional argument, since $[G_F] = -2$, the decay width $\Gamma \sim G_F^2 m_\mu^5$. Grand Unified Theory postulates the decay of proton, $p \rightarrow e^+\pi^0$. Following $M_{X,Y} \sim 10 \times 10^{16}\,\text{GeV}$, we estimate the decay width of proton $\Gamma \sim \frac{g_{\text{SU(5)}}^4}{M_X^4} M_p^5$.
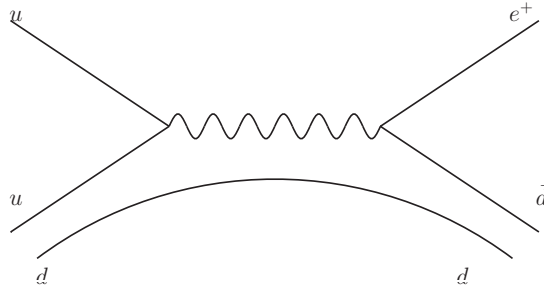
Polarized Muon Decay In the above computation, the muon polarizations are summed over. Assume the polarization is fixed

Also the spin sum relation enables us the computation. How to compute decay of polarized particle then? Recall that u, $\bar{u}$, v and $\bar{v}$ are independent solutions to Dirac equation.

$$u^{(s)} = \sqrt{E+m} \begin{pmatrix} \chi^{(s)} \\ \frac{\sigma \cdot p}{E+m} \chi^{(s)} \end{pmatrix} \tag{6.1.6}$$

To get chiral-left part of $u^{(s)} P_L u^{(s)}(\boldsymbol{p})$. Can also define two projection operators

$$\Lambda_+ = \frac{1}{2m}(\not{p} + m) \tag{6.1.7}$$

$$\Lambda_- = \frac{1}{2m}(-\not{p} + m) \tag{6.1.8}$$

They are indeed projection operators

$$\Lambda_+ + \Lambda_- = \mathbb{1}_4$$

$$(\Lambda_+)^2 = \frac{1}{4m^2}\left(\not{p}\not{p} + 2m\not{p} + m^2\right) = \frac{1}{4m^2}2m(\not{p} + m) = \Lambda_+$$

$$(\Lambda_-)^2 = \Lambda_-$$

Let $\sum_{r=1}^{4} a_r u^{(r)}$ be an arbitrary spinor

$$\Lambda_+ = \sum_{r=1}^{4} a_r u^{(r)}$$

$$= \sum_{r=1}^{4} a_r \left(\sum_{s=1}^{2} \frac{u^{(s)}\bar{u}^{(s)}}{2m}\right) u^{(r)}$$

use $\bar{u}^{(s)}u^{(r)} = 2m\delta^{rs}$

$$= \sum_{r=1}^{2} a_r u^{(r)}$$

arbitrary $E > 0$ spinor. Thus Λ_+ projects to $E > 0$ states and Λ_- projects $E < 0$ states.

Particle at rest, spin $\boldsymbol{s}$ and $|\boldsymbol{s}| = 1$. Write a four-vector $s^\mu = (0, \boldsymbol{s})$. At rest $p^\mu = (m, 0)$ and $(s \cdot p) = 0$.
Starting from the s^μ, can compute s'^μ in any frame by a Lorentz boost.

$$s^0 = \frac{\boldsymbol{p} \cdot \boldsymbol{\xi}}{m}$$

$$s^i = \xi^i + \frac{(\boldsymbol{p} \cdot \boldsymbol{\xi})}{m(m + E)} p^i$$

$\boldsymbol{\xi}$ denotes the direction of spin and $|\boldsymbol{\xi}| = 1$

Compute $s \cdot p$ in new frame

$$s \cdot p = \frac{\boldsymbol{p} \cdot \boldsymbol{\xi}}{m}E - \boldsymbol{p} \cdot \boldsymbol{\xi}$$

$$= \frac{\boldsymbol{p} \cdot \boldsymbol{\xi} \boldsymbol{p}^2}{m(m + E)}$$

$$= \boldsymbol{p} \cdot \boldsymbol{\xi}\left[\frac{E}{m} - 1 - \frac{\boldsymbol{p}^2}{m(m + E)}\right]$$

$$= \frac{\boldsymbol{p} \cdot \boldsymbol{\xi}}{m(m + E)}\left[E(m + E) - m(m + E) - \boldsymbol{p}^2\right]$$

$$= 0$$

Define two projection operators

$$\Sigma_\pm = \frac{1}{2}\left(\mathbb{1}_4 \pm \gamma^5 \not{s}\right)$$

$$\Sigma_+ + \Sigma_- = \mathbb{1}_4$$

$$(\Sigma_-)^2 = \frac{1}{4}(\mathbb{1} - \gamma^5 \not{s})(\mathbb{1} - \gamma^5 \not{s})$$

$$= \frac{1}{4}\left[\mathbb{1} - 2\gamma^5 \not{s} + \gamma^5 \not{s}\gamma^5 \not{s}\right]$$

$$= \frac{1}{4}[2\mathbb{1} - 2\gamma^5 \not{s}]$$

$$= \Sigma_-$$

$$(\Sigma_+)^2 = \Sigma_+$$

What do $\Sigma_\pm$ project out? In rest frame $\Sigma_- = \frac{1}{2}(\mathbb{1} - \gamma^5 \gamma^i s^i)$. Choose $\boldsymbol{s} = \boldsymbol{e}_3 = \boldsymbol{e}_z$.

$$\Sigma_- = \frac{1}{2}(\mathbb{1} - \gamma^5 \gamma^3)$$

$$= \frac{1}{2}\left[\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} - \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} \mathbb{1}_2 - \sigma^3 & 0 \\ 0 & \mathbb{1}_2 + \sigma^3 \end{pmatrix}$$

so project out helicity states.

$$u^k(\boldsymbol{p}, s)\bar{u}_i(\boldsymbol{p}, s) = \frac{1}{2}\left[(\not{p} + m)(\mathbb{1} - \gamma^5 \not{s})\right]^k_i \tag{6.1.9}$$

or in computation

$$(\not{p} + m) \mapsto \frac{1}{2}(\not{p} + m_\mu)(\mathbb{1} - \gamma^5 \not{s}_\mu) \tag{6.1.10}$$

Turns out we can skip computation. Replace $p_\alpha \mapsto p_\alpha - m s_\alpha$ in final answer

$$\mathrm{tr}\left[\dots P_R(\not{p} + m_\mu)(\mathbb{1} - \gamma^5 \not{s})\gamma_\alpha P_R \dots\right]$$

$$= \mathrm{tr}\left[\dots P_R(\not{p} - m_\mu\gamma^5 \not{s})\gamma_\alpha P_R \dots\right]$$

$$= \mathrm{tr}\left[\dots P_R(\not{p} - m\not{s})\gamma_\alpha P_R\right]$$

with

$$\boldsymbol{\xi} \cdot \frac{\boldsymbol{p}_e}{|\boldsymbol{p}_e|} = \cos\theta$$

In the end

$$\frac{\mathrm{d}\Gamma}{\Gamma} = \frac{1}{2}\left(1 - \frac{1}{3}\cos\theta\right)\mathrm{d}\cos\theta \tag{6.1.11}$$

6.2 Nuclear β-decay

$^{14}O \to {}^{14}N^* e^+ \nu_e \ \beta^+$-emitter $n \to p e^- \bar{\nu}_e$
  Non-relativistic limit of Pauli-Dirac spinor

$$u^{(s)}(\boldsymbol{p}) = \sqrt{E+m} \begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \end{pmatrix}$$

$$\to \sqrt{2m} \begin{pmatrix} \chi^{(s)} \\ 0 \end{pmatrix}$$

$$\bar{u}^{(s)} = \sqrt{2m} \begin{pmatrix} x^{(s)} & 0 \end{pmatrix}$$

$$\bar{\psi}_n \gamma_\mu \frac{1}{2} \left(\mathbb{1} - \gamma^5 \right) \psi_p(x)$$

γ^5 matrice has no effect and vanishes

$$= \frac{1}{2}\bar{\psi}_n(x)\gamma_\mu \psi_p(x)$$

$$= \frac{1}{2}\psi_n(x)\gamma^0 \gamma_\mu \psi_p(x)$$

$$\to \frac{1}{2}\psi_n^\dagger \psi_p$$

$\mu \to i = 1, 2, 3$

$$\gamma^0 \gamma^i = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

It is off-diagonal.
  Thus in the non-relativistic limit

$$\psi_n{}^\dagger \gamma^0 \gamma^i \psi_p \to 0$$

Typical $\Delta E = O(10\,\text{MeV})$ or so. Thus we can make the approximation $\frac{1}{q^2 - M_W^2} \to -\frac{1}{M_W^2}$.
  Consider transition amplitude in x-space

$$T_{fi} = -i\frac{4G_F}{\sqrt{2}} \int \mathrm{d}^4 x \, J_\mu^{(n)\dagger} J_{(x)}^{(e)\mu}$$

$$= -i\frac{4G_F}{\sqrt{2}} \int \mathrm{d}^4 x \left[\bar{\psi}_n(x)\gamma_\mu P_L \psi_p(x) \right] \left[\bar{\psi}_\nu(x)\gamma^\mu P_L \psi_e(x) \right]$$

Usually, consider x-integral immediately, $\to u, \bar{u}, v, \bar{v}$. Defs for now, more convenient due to non-relativistic nature.
  Since $J^P = 0^+$ for initial and final states ^{14}O and $^{14}N^*$, no parity change and γ^5 has odd parity. Thus $\bar{\psi}_n \gamma^\mu \gamma^5 \psi_p$ doesn't contribute.
  Assume nuclear wave function is unchanged by transition. $\Delta E \simeq 2\,\text{MeV} \ll m_p, m_n; m_O, m_N$.
  Use non-relativistic spinors for p and n

$$\bar{\psi}_n(x)\gamma_\mu \frac{1}{2}(\mathbb{1} - \gamma^5)\psi_p(x)$$

drop γ^5 and $\mu = 0$

$$\rightarrow \frac{1}{2}\psi_n^\dagger(x)\psi_p(x)$$

$$T_{fi} \simeq -i\frac{G_F}{\sqrt{2}}\int \mathrm{d}^4x \left[\psi_n^\dagger(x)\psi_p(x)\right]\left[\bar{\psi}_\nu(x)\gamma^0(1-\gamma^5)\psi_e(x)\right]$$

The wave functions for proton and neutron vanish outside of nuclei, thus the integral is effectively bounded inside of the radius of nuclei.

$E_\nu, E_e = O(1\,\mathrm{MeV})$. Thus the wavelength $\simeq 1 \times 10^{-13}\,\mathrm{m}$. So the wave functions of neutrino and electron are constant over the radius of nuclei.

$$T_{fi} = -i\frac{G_F}{\sqrt{2}}\left[\bar{u}(p_\nu)\gamma^0(1-\gamma^5)v(p_e)\right]\int \mathrm{d}^4x\,\psi_n^\dagger(x)\psi_p(x)e^{-i(p_\nu+p_e)\cdot x}$$

$e^{i(p_\nu+p_e)\cdot x} \simeq 1$ over nucleus.

Nuclear wave function doesn't change

$$\int \mathrm{d}^4x\,\psi_n^\dagger(x)\psi_p(x) \rightarrow \int \mathrm{d}^4x\,\psi_n^\dagger(x)\psi_n(x) \simeq 2m_N$$

$$T_{fi} = -i(2\pi)^4\delta^{(4)}(p_p - p_n - p_e - p_\nu)\mathcal{M} \tag{6.2.1}$$

$$\mathcal{M} = \frac{G_F}{\sqrt{2}}\bar{u}(p_\nu)\gamma^0(1-\gamma^5)v(p_e)(2m_N)(\mathrm{Iso}) \tag{6.2.2}$$

with Iso $= 2/\sqrt{2}$.

Total decay rate

$$\mathrm{d}\Gamma = G_F^2\left[\sum_{\mathrm{spins}}\left|\bar{u}(p_\nu)\gamma^0(1-\gamma^5)v(p_e)\right|^2\right](\mathrm{Iso})^2\frac{\mathrm{d}^3p_e}{(2\pi)^3 2E_e}\frac{\mathrm{d}^3p_\nu}{(2\pi)^3 2E_\nu}\delta(E_0 - E_e - E_\nu)$$

where $\delta^{(3)}$ was used to get rid of 3rd final state part n.

$$\sum_{\mathrm{spins}}\left|\bar{u}(p_\nu)\gamma^0(1-\gamma^5)v(p_e)\right|^2$$
$$= \mathrm{tr}\left[\slashed{p}_\nu\gamma^0(1-\gamma^5)\slashed{p}_e\gamma_0(1-\gamma^5)\right]$$
$$= 2\,\mathrm{tr}\left[\slashed{p}_\nu\gamma^0\slashed{p}_e\gamma^0(1-\gamma^5)\right]$$

γ^5 vanished due to symmetry argument

$$= 8\left[E_\nu E_e - p_\nu p_e + E_\nu E_e\right]$$
$$= 8\left[E_\nu E_e + \boldsymbol{p}_\nu \cdot \boldsymbol{p}_e\right]$$
$$= 8E_e E_\nu(1 + v_e\cos\theta)$$

$v_e \simeq 1$ for $E_e \simeq 1\,\mathrm{MeV}$.

$$d\Gamma = \frac{2G_F^2}{(2\pi)^5}(1 + \cos\theta)\left[2\pi\, d\cos\theta\, E_e^2\, dE_e\right]\left[4\pi E_\nu^2\, dE_\nu\right]\delta(E_0 - E_e - E_\nu)$$

$$\frac{d\Gamma}{dE_e} = \frac{4G_F^2}{(2\pi)^3}E_e^2(E_0 - E_e)^2\int d\cos\theta\,(1 + \cos\theta)$$

$$= \frac{G_F^2}{\pi^3}E_e^2(E_0 - E_e)^2$$

After integral

$$\Gamma = \frac{1}{\tau} = \frac{G_F^2 E_0^5}{30\pi^3}(\text{Iso})(\cos\theta_c)^2 \tag{6.2.3}$$

Insert Cabbibo angle corresponding to quark transition. Combining with μ-decay to determine $\cos\theta_c$.

6.3 Neutrinos

Three-body decay has continuous E_e-spectrum.

$$\pi^- \to \begin{cases} \mu^-\bar\nu_\mu \\ e^-\bar\nu_e \end{cases}$$

are just two-body decay. Only discrete spectrum.

$$E_\mu = \frac{1}{2M_\pi}\left(M_\pi^2 + M_\mu^2\right)$$

Redo the computation with $m_\nu \neq 0$

$$\frac{d\Gamma}{dE_e} = \frac{G_F^2\cos^2\theta_c I}{(2\pi)^3}|\boldsymbol{p}_e|^2 E_e(E_0 - E_e)\sqrt{(E_0 - E_e)^2 m_\nu^2}$$

usually write $Q = E_0 - E_e$ and $T_e = E_e - m_e$ kinetic energy of electron. F is the Coulomb interaction.

$$n(E)\,dE = \frac{G_f^2\cos^2\theta_c I^2}{(2\pi)^3}F(z, R, E)|\boldsymbol{p}_e|E_e(Q - T_e)\sqrt{(Q - T_e)^2 - m_\nu^2}\,dE_e$$

$$K(E) = \sqrt{\frac{n(E)}{F(z, R, E)|\boldsymbol{p}_e|E_e}} \propto Q - T_e$$

In Kurie plot, the end point depends on neutrino mass $(T_e)_{\max} = Q - m_\nu$.

Only small number of events are near endpoint. Want to measure out endpoint distribution to determine neutrino mass.

For $m_\nu = 0$, endpoint at Q. Taylor expand $n(E)$ around $n(Q)$

$$n(E) = n(Q) - \frac{1}{2}n'(Q)(\Delta E) + \frac{(\Delta E)^2}{6}n''(Q) + \dots$$

$$\int_{Q-\Delta E}^{Q} dE\, n(E) = n(Q)\Delta E - (\Delta E)^2 n'(Q) + \frac{(\Delta E)^3}{6}n''(Q) + \dots$$

$n(T_e) \sim (Q - T_e)^2$ for $m_\nu = 0$
Fraction of events near endpoint

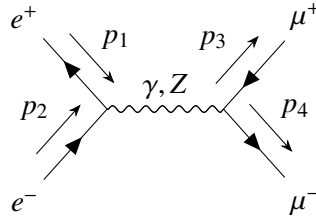$$\frac{\int_{Q-\Delta E}^{Q} \mathrm{d}E\, n(E)}{\int_0^Q \mathrm{d}E\, n(E)} \propto \left(\frac{\Delta E}{Q}\right)^3$$

Want Q small to get a large as possible fraction of events near endpoint. Choose tritium, $Q = 18.6\,\mathrm{keV}$

$$\left(\frac{\Delta E}{Q}\right)^3 \simeq 1.5 \times 10^{-13}$$

6.4 Electron-positron scattering

with γ and Z^0 interference.

Using appropriate Feynman rules, the matrix element of photon process reads

$$\mathcal{M}_\gamma = -\frac{e^2}{q^2}\,(\bar{\mu}\gamma^\nu \mu)\,(\bar{e}\gamma_\nu e)\,,$$

$$= -\frac{e^2}{q^2}\,[\bar{\mu}_L \gamma^\nu \mu_L + \bar{\mu}_R \gamma^\nu \mu_R]\,[\bar{e}_L \gamma_\nu e_L + \bar{e}_R \gamma_\nu e_R]\,.$$

In the last step, $\mathbb{1} = P_L + P_R = P_L^2 + P_R^2$ is inserted. Note that $\bar{\mu} = v(p_3)$ is the spinor for the anti-particle.
Neglecting all masses, the matrix element of Z process reads

$$\mathcal{M}_Z = -\frac{g^2}{4\cos^2\theta_W}\,\left[\bar{\mu}\gamma^\nu(c_V^\mu - c_A^\mu \gamma^5)\mu\right]\frac{g_{\nu\sigma} - q_\nu q_\sigma/q^2}{q^2 - M_Z^2 + i\Gamma_Z M_Z}\,\left[\bar{e}\gamma^\sigma(c_V^e - c_A^e \gamma^5)e\right].$$

In fact, the superscript e and μ superfluous because of electron-muon universality.

The Z momentum according to momentum conservation is $q = k_{e^-} + k_{e^+}$. In electron current, by using the Dirac equation and ignoring the electron mass one finds the second term in the vector boson propagator does not contribute

$$\bar{e}(k_{e^-})\,(k\!\!\!/_{e^-} + k\!\!\!/_{e^+})\,(c_V - c_A\gamma^5)e(k_{e^+}),$$

$$= k\!\!\!/_{e^+}(c_A - c_V\gamma^5)e(k_{e^+}),$$

$$= (c_A + c_V\gamma^5)k\!\!\!/_{e^+}e(k_{e^+}),$$

$$= 0.$$

Using the vector and axial coefficients, the left- and right-handed coefficients can be found

$$c_V = -\frac{1}{2} + 2\sin^2\theta_w, \tag{6.4.1}$$

$$c_A = -\frac{1}{2}, \tag{6.4.2}$$

$$c_R = c_V - c_A = 2\sin^2\theta_w, \tag{6.4.3}$$

$$c_L = c_V + c_A = -1 + 2\sin^2\theta_w. \tag{6.4.4}$$

The matrix element becomes

$$\mathcal{M}_Z = \underbrace{\frac{-\sqrt{2}G_F M_Z^2}{s - M_Z^2 + i\Gamma_Z M_Z}}_{\kappa} \left[c_R(\bar{\mu}_R\gamma^\nu\mu_R) + c_L(\bar{\mu}_L\gamma^\nu\mu_L)\right]\left[c_R(\bar{e}_R\gamma_\nu e_R) + c_L(\bar{e}_L\gamma_\nu e_L)\right],$$

where we have used the relations

$$s = q^2,$$

$$M_W = M_Z\cos\theta_w,$$

$$g^2 = \frac{8}{\sqrt{2}}G_F M_Z^2 \cos^2\theta_w.$$

In evaluation of $|\mathcal{M}|^2$, there are four kinds of traces.

1.

$$\sum v(p_1)_L\gamma^\nu u(p_2)_L\bar{u}(p_2)_L\gamma^\mu v(p_1)_L$$

$$= \frac{1}{4}\operatorname{tr}\left[\gamma^\nu(\mathbb{1} - \gamma_5)\not{p}_2\gamma^\mu(\mathbb{1} - \gamma^5)\not{p}_1\right]$$

$$= 2\left[(p_2)^\nu(p_1)^\mu - (p_1\cdot p_2)g^{\nu\mu} + (p_2)^\mu(p_1)^\nu - i(p_2)_\alpha(p_1)_\beta\epsilon^{\nu\alpha\mu\beta}\right]$$

2.

$$\sum \bar{u}(p_4)_L\gamma^\nu v(p_3)_L\bar{v}(p_3)_L\gamma^\mu u(p_4)_L$$

$$= 2\left[(p_3)^\nu(p_4)^\mu - (p_3\cdot p_4)g^{\mu\nu} + (p_3)^\mu(p_4)^\nu - i(p_3)_\alpha(p_4)_\beta\epsilon^{\nu\alpha\mu\beta}\right]$$

3.

$$\sum \bar{u}(p_4)_R\gamma^\nu v(p_3)_R\bar{v}(p_3)_R\gamma^\mu u(p_4)_R$$

$$= 2\left[(p_3)^\nu(p_4)^\mu - (p_3\cdot p_4)g^{\mu\nu} + (p_3)^\mu(p_4)^\nu + i(p_3)_\alpha(p_4)_\beta\epsilon^{\nu\alpha\mu\beta}\right]$$

4.

$$\sum \bar{v}(p_1)_R\gamma^\nu u(p_1)_R\bar{v}(p_2)_R\gamma^\mu u(p_1)_R$$

$$= 2\left[(p_2)^\nu(p_1)^\mu - (p_2\cdot p_1)g^{\mu\nu} + (p_2)^\mu(p_1)^\nu + i(p_2)_\alpha(p_1)_\beta\epsilon^{\nu\alpha\mu\beta}\right]$$

Consider the polarized interaction $e_L^- e_R^+ \rightarrow \mu_L^- \mu_R^+$

1. QED

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1 + \cos\theta)^2$$

2.

$$\mathcal{M}_Z \subset \kappa \left[c_L^e \bar{v}(p_1)_L \gamma^\nu u(p_2)_L \right] \left[c_L^\mu \bar{u}(p_4)_L \gamma_\nu v(p_3)_L \right]$$
$$\text{tr}[\ldots] \cdot \text{tr}[\ldots] = 16(p_1 \cdot p_4)(p_2 \cdot p_3)$$

with

$$p_1 = (E, E\hat{\pmb{e}}_z), \quad p_2 = (E, -E\hat{\pmb{e}}_z), \quad p_3 = (E, E\hat{\pmb{e}}_1), \quad p_4 = (E, -E\hat{\pmb{e}}_1),$$
$$\hat{\pmb{e}}_1 = (0, \sin\theta, \cos\theta)^t, \quad \hat{\pmb{e}}_z = (0, 0, 1)^t,$$
$$(p_1 \cdot p_3) = (p_2 \cdot p_4) = E^2(1 - \cos\theta), \quad (p_2 \cdot p_3) = (p_1 \cdot p_4) = E^2(1 + \cos\theta).$$

$$\frac{1}{4} \sum |\mathcal{M}_{Z1}|^2 = 4|\kappa|^2 (c_L^e)^2 (c_L^\mu)^2 E^4 (1 + \cos\theta)^2$$
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} \left| \frac{s}{e^2} \kappa c_L^e c_L^\mu \right|^2 (1 + \cos\theta)^2$$

3.

$$\mathcal{M}_\gamma^* \mathcal{M}_Z + \mathcal{M}_\gamma \mathcal{M}_Z^* = 2\,\text{Re}(\mathcal{M}_\gamma^* \mathcal{M}_Z)$$
$$\frac{1}{4} \sum 2\,\text{Re}(\mathcal{M}_\gamma^* \mathcal{M}_Z) = 8\,\text{Re}(\kappa \frac{e^2}{s} c_L^e c_L^\mu) E^4 (1 + \cos\theta)^2$$

In total,

$$\frac{d}{d\Omega} \sigma(e_L^- e_R^+ \to \mu_L^- \mu_R^+) = \frac{\alpha^2}{4s} \left(1 + \left| \frac{s}{e^2} \kappa c_L^e c_L^\mu \right|^2 + 2\,\text{Re}(\kappa \frac{s}{e^2}) c_L^e c_L^\mu \right) \cdot (1 + \cos\theta)^2,$$
$$= \frac{\alpha^2}{4s} \left| 1 + \frac{s}{e^2} \kappa c_L^e c_L^\mu \right|^2 (1 + \cos\theta)^2 \tag{6.4.5}$$

Analogously,

$$\frac{d}{d\Omega} \sigma(e_L^- e_R^+ \to \mu_R^- \mu_L^+) = \frac{\alpha^2}{4s} \left| 1 + \frac{s}{e^2} \kappa c_L^e c_R^\mu \right|^2 (1 - \cos\theta)^2 \tag{6.4.6}$$

$$\frac{d}{d\Omega} \sigma(e_R^- e_L^+ \to \mu_L^- \mu_R^+) = \frac{\alpha^2}{4s} \left| 1 + \frac{s}{e^2} \kappa c_R^e c_L^\mu \right|^2 (1 - \cos\theta)^2 \tag{6.4.7}$$

$$\frac{d}{d\Omega} \sigma(e_R^- e_L^+ \to \mu_R^- \mu_L^+) = \frac{\alpha^2}{4s} \left| 1 + \frac{s}{e^2} \kappa c_R^e c_R^\mu \right|^2 (1 + \cos\theta)^2 \tag{6.4.8}$$

Only consider QED

$$\sum_{\text{helicity}} \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} \left(1 + \cos^2\theta \right)$$

Consider QED with Z^0

$$\sum_{\text{helicity}} \frac{d\sigma}{d\Omega} = \frac{\alpha}{4s}\left[A_0(1 + \cos^2\theta) + A_1\cos\theta\right]$$

$$r = \frac{s}{e^2}\kappa$$

$$A_0 = 1 + \frac{1}{4}|r|^2(c_L^2 + c_R^2)^2 + \frac{1}{2}(c_L + c_R)^2\,\text{Re}(r)$$

$$= 1 + 2\,\text{Re}(r)c_V^2 + |r|^2(c_V^2 + c_A^2)^2$$

$$A_1 = (c_L - c_R)^2\left(\text{Re}(r) + \frac{|r|^2}{2}(c_L + c_R)^2\right) \neq 0$$

$$= 4\,\text{Re}(r)c_A^2 + 8|r|^2 c_V^2 c_A^2$$

In order the test the parity violation, define the quantity to evaluate the difference between forward and backward scattering (?)

$$A_{\text{FB}} = \frac{F - B}{F + B}$$

$$F = \int_0^{2\pi} d\rho \int_0^1 d\cos\theta\,\frac{d\sigma}{d\Omega}$$

$$= \frac{\pi\alpha^2}{2s}\left(\frac{4}{3}A_0 + \frac{1}{2}A_1\right)$$

$$B = \int_0^{2\pi} \int_{-1}^0 d\cos\theta\,\frac{d\sigma}{d\Omega}$$

$$= \frac{\pi\alpha^2}{2s}\left(\frac{4}{3}A_0 - \frac{1}{2}A_1\right)$$

$$A_{\text{FB}} = \frac{3}{8}\frac{A_1}{A_0}$$

Petra $\sqrt{s} = 34\,\text{GeV}$ and $\frac{s}{m_Z^2} \approx 0.14 \ll 1$, $s \ll m_Z^2$.

$$\Gamma = \left(\frac{s}{e^2}\right)\frac{\sqrt{2}G_F m_Z^2}{s - m_Z^2 + im_Z\Gamma_Z} \mapsto -\sqrt{2}G_F\frac{s}{e^2}$$

In the limit, $|r| \ll 1$

$$A_0 = 1 + O(r)$$

$$A_1 = 4c_A^2\,\text{Re}(r)$$

$$A_{\text{FB}} = -\frac{3}{\sqrt{2}}\frac{G_F}{e^2}c_A^2 s$$

In Standard Model , $c_A = -\frac{1}{2}A_{\text{FB}}$.

We have worked in the limit $s \ll m_Z^2$. Let's consider $s \sim m_Z^2$, so it can go on-shell.

$$|r|^2 = \frac{2G_F^2 m_Z^4}{(s - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \left(\frac{s}{e^2}\right)^2$$

$$\mathrm{Re}(r) = \frac{2G_F m_Z^2 (s - m_Z^2)}{(s - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \left(\frac{s}{e^2}\right) \approx 0$$

$$A_0 = 1 + |r|^2 (c_V^2 + c_A^2) \approx |r|^2 (c_V^2 + c_A^2)^2$$

$$A_1 = 8|r|^2 c_V^2 c_A^2$$

So if $\sqrt{s} \approx 90\,\mathrm{GeV}$ and $\Gamma_Z = 2.5\,\mathrm{GeV}$, then $|r|^2 = \mathcal{O}\left(10^3\right)$.

$$\sigma = \int_0^{2\pi} d\rho \int_{-1}^{1} d\cos\theta \left[A_0(1 + \cos^2\theta) + A_1 \cos\theta\right]$$

$$= \frac{4\pi\alpha^2}{3s} A_0$$

$$e = g \sin\theta_W$$

$$G_F = \frac{\sqrt{2}g^2}{8m_W^2}$$

$$\sigma = \frac{g^4}{192\pi \cos^4\theta_W} \frac{s(c_V^2 + c_A^2)^2}{(s - m_Z^2)^2 + m_Z^2\Gamma_Z^2}$$

$$\Gamma(Z \to l^+ l^-) = \frac{g^2}{48\pi \cos^2\theta_W} m_Z \left((c_V^l)^2 + (c_A^l)^2\right)$$

$$\sigma(s) = \frac{12\pi}{m_Z^2} \Gamma(Z \to e^+ e^-) \Gamma(Z \to \mu^+\mu^-) \frac{s}{(s - m_Z^2)^2 + m_Z^2\Gamma_Z^2}$$

Breit-Wigner
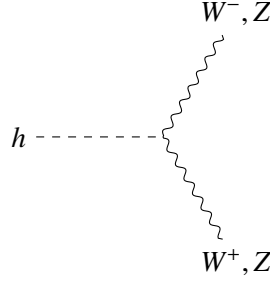
1. EM α^2 $e = g \sin\theta_W$

2. $G_F = \frac{\sqrt{2}}{8} \frac{g^2}{m_W^2}$

3. $m_Z^2 = \frac{m_W^2}{1 - \sin^2\theta_W}$

$$\sin^2\theta_W(1 - \sin^2\theta_W) = \frac{\sqrt{2}}{8}\frac{e^2}{G_F m_Z}$$

6.5 Higgs Decay

$$iM = \frac{2im_V^2}{V}\epsilon^*(p_1) \cdot \epsilon^*(p_2)$$

$$|M|^2 = \frac{4m_V^2}{V^2}(\epsilon^* \cdot \epsilon^*)(\epsilon \cdot \epsilon)$$

with $V = W, Z$

First calculate the unpolarized process

$$\sum \epsilon(p_1)_\mu \epsilon(p_1)_\nu = -g_{\mu\nu} + \frac{1}{m_V^2}(p_1)_\mu(p_1)_\nu$$

$$\sum |M|^2 = \frac{4m_V^4}{V^2}\left[2 + \frac{1}{m_V^4}(p_1 \cdot p_2)^2\right]$$

Choose the rest frame of Higgs

$$q = (m_h, 0), p_1 = (E, \boldsymbol{p}), p_2 = (E, -\boldsymbol{p}),$$

$$|\boldsymbol{p}|^2 = E^2 - m_h^2, p_1 \cdot p_2 = E^2 + |\boldsymbol{p}|^2 = \frac{m_h^2}{2} - m_V^2$$

with the variable $\lambda_V = \left(\frac{m_V}{m_h}\right)^2$

$$\sum |M|^2 = \sqrt{2}G_F m_h^4(1 + 12\lambda_V^4 - 4\lambda_V^2)$$

$$\Gamma(h \to VV)_{\text{unpol}} = \frac{\sqrt{2}G_F}{16\pi} \frac{m_h^3}{1 + \delta_{12}} \sqrt{1 - 4\lambda_V}\left(1 - 4\lambda_V + 12\lambda_V^2\right)$$

1. $\lambda = 0$

$$\epsilon(p_1, \lambda = 0) = \frac{1}{m_W}\begin{pmatrix} |\boldsymbol{p}| \\ 0 \\ 0 \\ E \end{pmatrix}, \quad \epsilon(p_2, \lambda = 0) = \frac{1}{m_W}\begin{pmatrix} |\boldsymbol{p}| \\ 0 \\ 0 \\ -E \end{pmatrix}$$

$$|M(\lambda = 0)|^2 = \frac{4m_W^4}{V}(\epsilon(0, p_1) \cdot \epsilon(0, p_2))^2$$

$$= \sqrt{2}G_F m_h^4(1 - 2\lambda_W)^2$$

2. $\lambda_1 = \lambda_2 = +1$

$$\epsilon(p_1, \lambda = 1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad \epsilon(p_\lambda = 1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}$$

$$\epsilon^*(p_1) \cdot \epsilon^*(p_2) = \epsilon(p_1) \cdot \epsilon(p_2) = -1$$

$$\Gamma(h \to W_+ W_+) = \frac{\sqrt{2} G_F}{16\pi} m_h^3 \lambda_W^4 \sqrt{1 - 4\lambda_W}$$

The sign in the subscript denotes the helicity

3. $\lambda_1 = \lambda_2 = -1$

$$\Gamma(h \to W_+ W_+) = \Gamma(h \to W_- W_-)$$

4. $\lambda_1 = 0, \lambda_2 = \pm 1$

$$\epsilon^*(p_1, \lambda = 0) \cdot \epsilon(p_2, \lambda = \pm 1)^* = 0$$

5. $\lambda_1 = \pm 1, \lambda_2 = \mp 1$

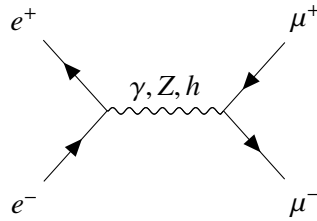$$\epsilon^*(p_1, \lambda = \pm 1) \cdot \epsilon(p_2, \lambda = \mp 1)^* = 0$$
$$\Gamma(h \to W_\pm W_\mp, W_\mp W_\pm) = 0$$

$$\frac{\Gamma(h \to W_L W_L)}{\Gamma(h \to W_\pm W_\pm)} = \frac{(1 - 2\lambda_W)^2}{\lambda_W^4} \approx \frac{1}{\lambda_W^4} \gg 1 \qquad (6.5.1)$$
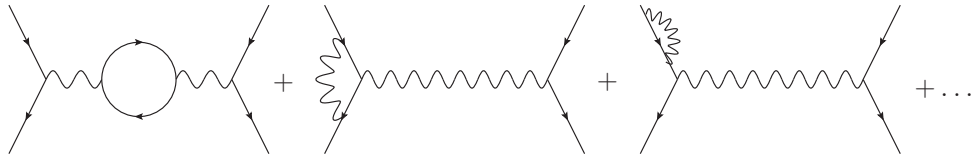
6.6 Higgs Decay into Gluons

6.6.1 Loop diagrams

In double slit experiment of QM, we have seen the interference of two paths. Also seen in in this course, e.g. $e^+ e^- \to \mu^+ \mu^-$

Again because the coupling of Higgs and electron is so weak, it does not really impact the total cross section.
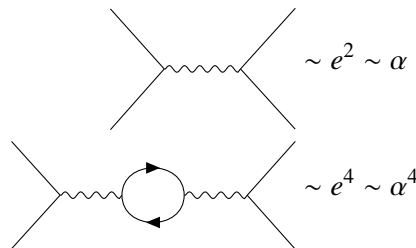
Now what about the loop diagram

All of these are indeed allowed in the Standard Model! So the total amplitude is sum of diagrams of all orders

$$i\mathcal{M} = i\mathcal{M}^{(0)} + i\mathcal{M}^{(1)} + i\mathcal{M}^{(2)} + \ldots$$

It is a infinite series. So when do/can we cut it off?

$$\sim e^2 \sim \alpha$$

$$\sim e^4 \sim \alpha^4$$

It is a perturbation series in α. In QED, $\alpha_{\mathrm{QED}} \approx 1/137$. So we can cut off series and say the result if valid up to certain order.

There are a couple of new Feynman rules calculating loop diagrams

- integrate over undetermined momenta in loop

- a closed fermion loop gives a factor of (-1) and a trace of gamma matrices!

When do we need to calculate loops?

- experiments demand more precision/accuracy, e.g. anomalous magnetic moment

- if there is no tree-level diagrams

- it gives new insight into QFT

$$\sim \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{tr}\big(\gamma^\nu (\slashed{k} + m)\gamma^\mu (\slashed{k} - \slashed{p} - m)\big)(\ldots)}{(k^2 - m^2)((k-p)^2 - m^2)(\ldots)}$$

$$\overset{k \mapsto \infty}{\to} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k^2}{k^4}$$

It is quadratically divergent! In QFT, the infinities actually always go way in observables! This procedure is called renormalization.

6.6.2 QCD

The Standard Model contains three gauge group before symmetry breaking SM = $\mathbf{SU(3)_C} \times \mathbf{SU(2)_L} \times \mathbf{U(1)_Y} \rightarrow \mathbf{SU(3)_C} \times \mathbf{U_Y}$.

	$\mathbf{SU(2)_L}$	$\mathbf{SU(3)_C}$
matter fields transform	in fundamental= 2	in fundamental= 3
generator	$T^a = \frac{\sigma^a}{2}$	$\tau^a = \frac{\lambda^a}{2}$
gauge boson	$W^i_\mu, i = 1, 2, 3$	$g^a, a = 1, \ldots, 8$

In general $\mathrm{tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}$.
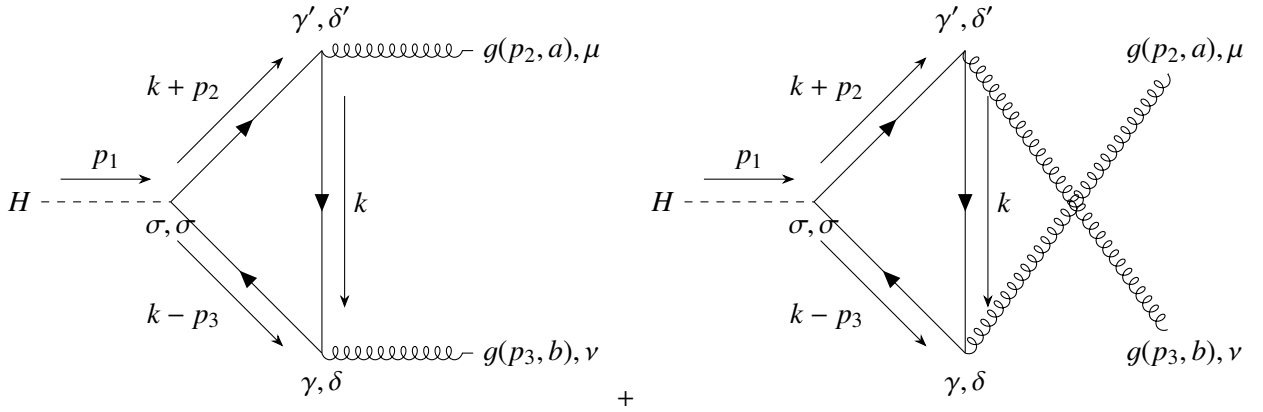
The propagators are

$$j \xrightarrow{\;\;p\;\;} i \;=\; \frac{i\delta^{ij}}{\not{p} - m + i\epsilon},$$

$$\mu, a \xrightarrow{\;\;p\;\;} \nu, b \;=\; \frac{-ig^{\mu\nu}}{p^2 + i\epsilon}\delta^{ab}.$$

There are further interesting features of QCD. Asymptotic freedom means that QCD coupling constant decreases as energy scale goes up. The theory is asymptotically free. Confinement means colored particles never observed in isolation. They always from color singlet states (mesons and baryons).

6.6.3 $H \rightarrow gg$ process

Production of Higgs boson at LHC mainly relies on gluon fusion and vector boson fusion. Consider the process $H(p_1) \rightarrow g(p_2)g(p_3)$. It produces 87% Higgs at LHC. There is no tree-level diagrams!

$$+$$

$$i\mathcal{M}_1 = (-i)\frac{m}{V}(-1)g_s^2\epsilon^a_\mu\epsilon^b_\nu(T^a)_{\delta'\gamma'}(T^b)_{\gamma\delta}\delta_{\delta\delta'}\delta_{\gamma'\sigma}\delta_{\gamma\sigma} \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{tr}\left[i\gamma^\mu i(\not{k} + \not{p}_2 + m)i(\not{k} - \not{p}_3 + m)i\gamma^\nu i(\not{k} + m)\right]}{(k^2 - m^2)((k + p_2)^2 - m^2)((k - p_3)^2 - m^2)}$$

$$\text{color factor} = T^a_{\delta'\gamma'}T^b_{\gamma\delta}\delta_{\delta\delta'}\delta_{\gamma'\delta}\delta_{\gamma\sigma} = \mathrm{tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}$$

$$\mathrm{tr}[\ldots] = 4m\left(p_3^\mu p_2^\nu + 4k^\mu k^\nu - 2k^\mu p_3^\nu + 2p_2^\mu k^\nu - p_2^\mu p_3^\nu + g^{\mu\nu}(m^2 - p_2 \cdot p_3) - g^{\mu\nu}k^2\right)$$

$$= 4mN^{\mu\nu}$$

With the Feynman parameter formula

$$\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\frac{2}{[Ax+By+Cz]^3}, \tag{6.6.1}$$

the denominator is then

$$(k^2 - m^2)x + (k^2 + p_2^2 - m^2 + 2k \cdot p_2)y + (k^2 + p_3^2 - m^2 - 2k \cdot p_3)z$$
$$= (k^2 - m^2)(x+y+z) + 2k \cdot p_2 y - 2k \cdot p_3 z$$
$$= (k + p_2 y - p_3 z)^2 - a^2$$

with $a^2 = m^2 - 2(\dots)$.

Now we have

$$I^{\mu\nu} = \int \frac{d^4 x}{(2\pi)^4} \int_0^1 dy \int_0^{1-y} dz = \frac{8mN^{\mu\nu}}{[(k+p_2 y - p_3 z)^2 - a^2]^3}$$

shift variable $k \rightarrow k + p_2 y + p_3 z$

$$= \int \frac{d^4 k}{(2\pi)^4} \int_0^1 dy \int_0^{1-y} dz \frac{8mN'^{\mu\nu}}{(k^2 - a^2)^3}$$
$$N'^{\mu\nu} = 4k^\mu k^\nu - g^{\mu\nu}k^2 + p_3^\mu p_2^\nu(1 - 4yz) + p_2^\mu p_3^\nu(-1 - 4yz + 2y + 2z)$$
$$+ p_3^\mu p_3^\nu(4z^2 - 2z) + p_2^\mu p_2^\nu(4y^2 - 2y) + g^{\mu\nu}(m^2 - p_2 \cdot p_3 + 2p_2 \cdot p_3 yz)$$

In the last step, terms linear in k^μ drop out because of symmetry of the integral.

Divergence structure of $I^{\mu\nu}$ for large k is

$$\sim \int \frac{d^4 k\, k^2}{k^6}.$$

To solve this, to work in $D = 4 - 2\epsilon$ dimensions instead of 4 and take $\epsilon \mapsto 0$ in the end (dimensional regularization)! Define

$$I(D, \alpha, \beta, a^2) = \int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^\alpha}{(k^2 - a^2)^\beta}$$
$$= \frac{i}{4\pi^{D/2}}(a^2)^{D/2}(-a^2)^{\alpha-\beta}\frac{\Gamma(\beta - \alpha - D/2)\Gamma(\alpha + D/2)}{\Gamma(\beta)\Gamma(D/2)}$$

All terms in $I^{\mu\nu}$ are in this form, except

$$\int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^\alpha k^\mu k^\nu}{(k^2 - a^2)^\beta} = I_1^{\mu\nu}$$

Contract it with $g_{\mu\nu}$ and in D dimension $g_{\mu\nu}g^{\mu\nu} = 0$

$$I_1^{\mu\nu} = \frac{g^{\mu\nu}}{D}I(D, \alpha + 1, \beta, a^2) \tag{6.6.2}$$

For the parts that do not contain k

$$I(4, 0, 3, a^2) = \frac{-i}{32\pi^2}\frac{1}{a^2}$$

Divergent parts

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^D} \frac{4k^\mu k^\nu - g^{\mu\nu}k^2}{(k^2 - a^2)^3} = \left(\frac{4}{D} - 1\right) g^{\mu\nu} I(D, 1, 3, a^2)$$

$$= \left(\frac{4}{D} - 1\right) g^{\mu\nu} \frac{i(a^2)^{D/2}(-a^2)^{-2}}{(4\pi)^{D/2}} \frac{\Gamma(2 - D/2)\Gamma(1 + D/2)}{\Gamma(3)\Gamma(D/2)}$$

$$= \left(\frac{4}{D} - 1\right) g_{\mu\nu} \frac{i}{(4\pi)^{D/2}} (a^2)^{D/2}(-a^2)^{-2} \left(\frac{D}{4}\right) \Gamma(2 - D/2)$$

using properties of Γ-function

$$= \frac{\epsilon}{2} g^{\mu\nu} \frac{i}{(4\pi)^2} \frac{a^2}{a^2} \left(\frac{1}{\epsilon} - \gamma_E + O(\epsilon)\right)$$

$$= \frac{ig^{\mu\nu}}{32\pi^2}$$

It is finite!

Plug into $I^{\mu\nu}$ and use $2p_2 \cdot p_3 = (p_2 + p_3)^2 = m_h^2$. On-shell gluons $\epsilon_\mu^{a/b} p_{2/3}^\mu = 0$

$$\mathcal{M}_1 = \frac{-g_s^2}{2} \delta^{ab} \frac{\frac{1}{2}m_h^2 g^{\mu\nu} - p_3^\mu p_2^\nu}{4\pi^2} I(\frac{m_h^2}{m^2}) \epsilon_\mu^a \epsilon_\nu^b \tag{6.6.3}$$

$$|\mathcal{M}|^2 = 4|M_1|^2 \tag{6.6.4}$$

$$\overline{|\mathcal{M}|}^2 = \frac{g_s^4 m_h^4}{4v^2\pi^4} \sum_{\text{quarks}} \left| I\left(\frac{m_h^2}{m^2}\right) \right|^2 \tag{6.6.5}$$
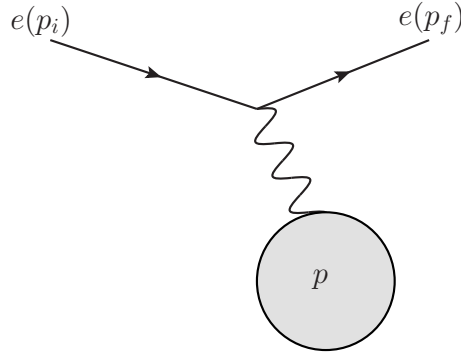
where $I \to 0$ if $m_h \gg m$ and $\to 1$ if $m \gg m_h$.

$$\Gamma(H \to gg) = \frac{G_F m_h^3}{4\sqrt{2}\pi} \left(\frac{\alpha_s}{\pi}\right)^2 |I|^2 \tag{6.6.6}$$

7 QCD and Proton Structure

7.1 Form Factor

We use photon of momentum $q = k_i - k_f$ to look at proton.

With energies $E_\gamma, \lambda_\gamma \lessgtr r_p$, we can observe proton as a whole or roughly proton structure.

Proton couples to electric charge and we want to know then what is the electric charge distribution of proton $\rho(\boldsymbol{x})$. First we make a couple of assumptions

- Non-realistic proton so that it has static charge and is unmovable

- To this scattering process corresponds a cross section (in lab frame)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{point}} \cdot |F(\boldsymbol{q})|^2, \tag{7.1.1}$$

with $\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{point}}$ the differential cross section of Rutherford scattering and $F(\boldsymbol{q})$ the form factor.

Normalization charge distribution

$$\int \mathrm{d}^3 x \, \rho(\boldsymbol{x}) = 1,$$

and Ze overall charge.

Electromagnetic field due to $Ze\rho(\boldsymbol{x})$ and resulting four-potential is $A^\mu = (\rho, 0)$. There is no magnetic field $\boldsymbol{B} = 0$, since the charge is at rest. Maxwell equations are

$$\partial^2 A^\mu = j^\mu.$$

with $j^\mu = (\rho, 0)$.

The field is static $\phi(\boldsymbol{x}, t) = \phi(\boldsymbol{x})$

$$\nabla^2 \phi = -Ze\rho(\boldsymbol{x}).$$

The transition amplitude is

$$T_{fi} = -i \int d^4x \, j_\mu^{fi} A^\mu,$$

with electron current

$$j_\mu^{fi} = -e \bar{u}_f \gamma_\mu u_i e^{i(p_f - p_i) \cdot x}.$$

Then

$$T_{fi} = -i2\pi\delta(E_f - E_i)(-e\bar{u}_f \gamma_0 u_i) \int d^3x \, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \phi(\boldsymbol{x}),$$

$$\int d^3x \, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \underbrace{\nabla^2 \phi(\boldsymbol{x})}_{-Ze\rho(\boldsymbol{x})} = -|\boldsymbol{q}|^2 \int d^3x \, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \phi(\boldsymbol{x}),$$

$$\int d^3x \, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \phi(\boldsymbol{x}) = -\frac{1}{|\boldsymbol{q}|^2} \int d^3x \, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \nabla^2 \phi(\boldsymbol{x}),$$

$$= -\frac{1}{|\boldsymbol{q}|^2} \int d^3x \, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} (-Ze)\rho(\boldsymbol{x}),$$

This is just Fourier transformation of $\rho(\boldsymbol{x})$. The form factor is defined as

$$F(\boldsymbol{q}) = \int d^3x \, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \rho(\boldsymbol{x}). \tag{7.1.2}$$

We are looking at scattering off of static charge distribution $AB \to CD$

$$T_{fi} = -iN_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_C + p_D - p_A - p_B)\mathcal{M},$$

$$W_{fi} = \frac{|T_{fi}|^2}{TV},$$

$$= \frac{(2\pi)^4}{V^4} \delta^4(p_c + p_D - p_A - p_B)|\mathcal{M}|^2.$$

Cross section is defined by

$$\sigma = \frac{W_{fi}}{\text{initial flux}} (\text{number of final states}). \tag{7.1.3}$$

We find

$$d\sigma = \frac{|T_{fi}|^2}{T} \frac{d^3k_f}{(2\pi)^3 2E_f} \left(\frac{1}{v2E_i}\right), \tag{7.1.4}$$

in which

$$d^3k_f \, \delta(E_f - E_i),$$
$$= |\boldsymbol{k}_f|^2 \, d|\boldsymbol{k}_k| \, d\Omega_f \, \delta(E_f - E_i),$$
$$= |\boldsymbol{k}_f| E_f \, dE_f \, \delta(E_f - E_i) \, d\Omega_f,$$
$$= |\boldsymbol{k}_f| E_{i=f} \, d\Omega_f.$$

where

$$E^2 = k^2 + m^2,$$
$$2E \, dE = 2k \, dk,$$
$$k \, dk = E \, dE.$$

The spinor structure in final (differential) cross section is

$$\frac{1}{2} \sum_{\text{spin}} \left| \bar{u}_f \gamma_0 u_i \right|^2 = 4E^2 \left(1 - v^2 \sin^2 \frac{\theta}{2} \right),$$

with θ the scattering angle of electron.

Expressing the momentum transfer $q^2 = (k_i - k_f)^2$ in terms of E, θ, the cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{(Z\alpha)^2 E^2}{4k^4 \sin^4 \frac{\theta}{2}} \left(1 - v^2 \sin^2 \frac{\theta}{2} \right) |F(\boldsymbol{q})|^2, \tag{7.1.5}$$

$$= \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right)_{\text{point}} |F(\boldsymbol{q})|^2. \tag{7.1.6}$$

According to the definition $F(0) = 1$.

With long wavelength or small $|\boldsymbol{q}|$, the exponential is to be expanded

$$e^{i\boldsymbol{q}\cdot\boldsymbol{x}} = 1 + i\boldsymbol{q}\cdot\boldsymbol{x} - \frac{1}{2}(\boldsymbol{q}\cdot\boldsymbol{x})^2 + \dots,$$

$$F(\boldsymbol{q}) = \int \mathrm{d}^3 x\, \rho(\boldsymbol{x}) \left(1 + i\boldsymbol{q}\cdot\boldsymbol{x} - \frac{1}{2}(\boldsymbol{q}\cdot\boldsymbol{x})^2 + \dots \right),$$

$$= 1 - \frac{1}{6}|\boldsymbol{q}|^2 \left\langle r^2 \right\rangle + \dots,$$

assuming $\rho(\boldsymbol{x}) = \rho(|\boldsymbol{x}|)$. First order term drops out because of symmetry consideration. $\left\langle r^2 \right\rangle$ is the mean square radius of charge distribution.

7.2 Electron-proton scattering, Proton Form Factor

Real proton is not static, it can recoil also has spin. Consider again $e\mu$ scattering

$$\left. \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right|_{\text{lab}} = \frac{\alpha^2}{4E^4 \sin^4 \frac{\theta}{2}} \frac{E'}{E} \left[\cos^2 \frac{\theta}{2} - \frac{q^2}{2m_\mu^2} \sin^2 \frac{\theta}{2} \right]. \tag{7.2.1}$$

The E'/E is indeed the recoil factor

$$\frac{E'}{E} = \frac{1}{1 + \frac{2E}{m_\mu} \sin^2 \frac{\theta}{2}}.$$

How did calculation proceed?

$$T_{fi} = -i \int \mathrm{d}^4 x\, j_\mu \left(-\frac{1}{q^2} \right) J_p^\mu,$$

$$j^\mu = -e\bar{u}(k')\gamma^\mu u(k) e^{i(k'-k)\cdot x},$$

$$J_p^\mu = e\bar{u}(p') [\dots]^\mu u(p) e^{i(p'-p)\cdot x}.$$

J_p^μ must be a Lorentz four-vector in the form of

$$[\dots]^\mu = F_1(q^2)\gamma^\mu + \frac{\kappa}{2m_p} F_2(q^2) i\sigma^{\mu\nu} q_\nu.$$

In principle, can also write a term with γ^5, it leads to parity violation but EM-sector doesn't violate parity. So why is this expression the most general form? More in general, one can have

$$J^\mu = e\bar{u}(p')\left[\gamma^\mu K_1 + i\sigma^{\mu\nu}(p'-p)_\nu K_2 + i\sigma^{\mu\nu}(p'+p)_\nu K_3 + (p'-p)^\mu K_4 + (p'+p)^\mu K_5\right]u(p)e^{iq\cdot x}.$$

Recall Gordon identity

$$\bar{u}_f\gamma^\mu u_i = \frac{1}{2m}\bar{u}_f\left[(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu\right]u_i. \tag{7.2.2}$$

With this identity, we can get rid of term containing K_5. For some reasons, K_3 term drops out also (second Gordon identity)

$$J^\mu = e\bar{u}(p)\left[\gamma^\mu F_1 + \frac{i\kappa}{2m_p}F_2\sigma^{\mu\nu}q_\nu + q^\mu F_3\right]u(p)e^{iq\cdot x}.$$

Applying current conservation in momentum space $q_\mu J^\mu = 0$. Then

$$q_\mu J^\mu = e\bar{u}(p')\left[\slashed{q}F_1 + \frac{i\kappa}{2m_p}F_2 q_\mu\sigma^{\mu\nu}q_\nu + q^2 F_3\right]u(p) = 0$$

The second term drops out using symmetry argument (in $\mu\nu$). The first term drops out using Dirac equation twice. Thus $F_3 = 0$.

Finally, we arrive at the same expression

$$J^\mu = e\bar{u}(p')\left[\gamma^\mu F_1 + \frac{i\kappa}{2m_p}\sigma^{\mu\nu}q_\nu F_2\right]u(p)e^{iqx}.$$

Then

$$\mathcal{M} \sim \left(\frac{-1}{q^2}\right)j_\mu J_p^\mu.$$

For long wavelength $q \to 0$, particle of charge $+e$ and magnetic moment $\frac{(1+\kappa)e}{2m_p}$

$$F_1(0) = 1, \quad F_2(0) = 1, \quad \kappa \approx 1.79.$$

We can do the same experiment on neutron

$$F_1(0) = 0, \quad F_2(0) = 1, \quad \kappa \approx -1.91.$$

The full differential cross section reads

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{lab}} = \frac{\alpha^2}{4E^2\sin^2\frac{\theta}{2}}\frac{E'}{E}\left[(F_1^2 - \frac{\kappa^2 q^2}{4m_p^2}F_2^2)\cos^2\frac{\theta}{2} - \frac{q^2}{2m_p^2}(F_1+\kappa F_2)^2\sin^2\frac{\theta}{2}\right]. \tag{7.2.3}$$

One can redefine the quantities as

$$G_E = F_1 + \frac{\kappa q^2}{4m_p^2}F_2, \tag{7.2.4}$$

$$G_M = F_1 + \kappa F_2. \tag{7.2.5}$$

With $\tau = -\frac{q^2}{4m_p^2}$, the cross section can be rewritten as

$$\frac{d\sigma}{d\Omega}\bigg|_{\text{lab}} = \frac{\alpha^2}{4E^2 \sin^2 \frac{\theta}{2}} \frac{E'}{E} \left[\frac{G_E^2 + \tau G_M^2}{1 + \tau} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right]. \tag{7.2.6}$$

Experimentally

$$G_E(q^2) \approx \left(1 - \frac{q^2}{0.71} \right)^{-2}$$

with q^2 in units of GeV^2. The radius can be determined

$$\langle r^2 \rangle = 6 \left(\frac{dG_F(q^2)}{dq^2} \right)\bigg|_{q^2=0} = (0.81 \times 10^{-13}\,\text{cm})^2.$$

There was discrepancy in determination of proton charge radius, but recently resolved. For more information, check [2].

7.3 Inelastic Proton-electron Scattering

To increase the energy of exchange photon $q^2 = (k - k')^2$, we can increase the energy of the electron Smaller wavelength means that we can probe substructure of proton. At some point, the quark structure is resolved.
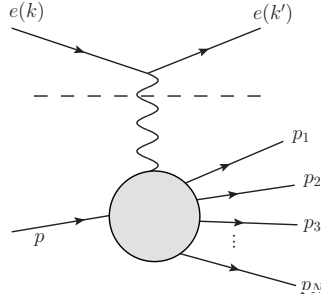
At higher energy proton an be excited and/or break up. They are not the same particle as before any more. Thus it is inelastic scattering. Denoting the momentum of outgoing particle p_i, then the invariant mass is

$$w^2 = \left(\sum_{i=1}^{N} p_i \right)^2.$$

Simplest case is Δ^+-resonance: $e^- p \rightarrow e^- \Delta^+ \rightarrow e^- p\pi^0$. Effectively it is $\gamma^* p \rightarrow \Delta^+ \rightarrow p\pi^0$. This might be relevant in cosmic rays. Partly because of supernova explosion, there are lots of energetic proton going in all directions. Cosmic microwave background provides constant photon distribution to enable this reaction. In this process, the proton loses $\sim 20\%$ of its energy. Since $M_{\Delta^+} = 1.232\,\text{GeV}$, proton with typical energy $E_p \gtrsim 10 \times 10^{20}$ eV can take part in multiple this kind of reactions. But there is no known source of proton $E_p > 10 \times 10^{20}$ eV in our galaxy.

Mean free path of a proton with $E \gtrsim 10 \times 10^{20}$ eV is almost about $50 Mpc$. Next high energy source is further away. Measure energy spectrum of cosmic rays hitting Earth. Should not see any proton above 10×10^{20} eV. It is called GZK-cutoff.

On the Earth, measuring the proton energy is hard because of the shower it produces entering the atmosphere. There were several groups claiming to detect abnormally high energy protons. But the puzzle has been resolved by Pierre Auger experiment.

Below the dashed line there is no longer a single fermion. We can just use $\bar{u}_p \Gamma u_p$.

In $e\mu$-scattering

$$d\sigma \sim L^e_{\mu\nu} (L^{\text{muon}})^{\mu\nu}.$$

Here

$$d\sigma \sim L^e_{\mu\nu} (H^{\text{had}})^{\mu\nu} = L^e_{\mu\nu} W^{\mu\nu}.$$

Write most general tensor for $W^{\mu\nu}$ with $g^{\mu\nu}, p^\mu, q^\mu$. No gamma matrices, since we have separated the fermion parts. Make the ansatz

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{m_p^2} p^\mu p^\nu + \frac{W_4}{m_p^2} q^\mu q^\nu + \frac{W_5}{m_p^2} (p^\mu q^\nu + q^\mu p^\nu). \tag{7.3.1}$$

It is symmetric in $\mu\nu$, because $L^e_{\mu\nu}$ is symmetric in $\mu\nu$ (otherwise it vanishes). W_3 is reserved for parity violation. $W_{1,2}$ are not dimensionless, $[W_{1,2}] = -1$.

Current conservation reads

$$q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0.$$

Plug the general expression in

$$q_\mu W^{\mu\nu} = -W_1 q^\nu + \frac{W_2}{m_p^2} q \cdot p p^\nu + \frac{W_5}{m_p^2} \left(p \cdot q q^\nu + q^2 p^\nu \right) + \frac{W_4}{m_p^2} q^2 q^\nu = 0,$$

$$= q^\nu \left(-W_1 + \frac{W_4}{m_p^2} q^2 + \frac{W_5}{m_p^2} p \cdot q \right) + p^\nu \left(\frac{W_2}{m_p^2} p \cdot q + q^2 \frac{W_5}{m_p^2} \right) = 0.$$

Two parts have to vanish individually. From first part we get

$$W_5 = -\frac{p \cdot q}{q^2} W_2.$$

Second part leads to

$$-W_1 + \frac{W_4}{m_p^2} q^2 + \frac{W_5}{m_p^2} p \cdot q = 0,$$

$$W_4 = \left(\frac{p \cdot q}{q^2} \right)^2 W_2 + \frac{m_p^2}{q^2} W_1.$$

The result from first part shall be used here.

So

$$W^{\mu\nu} = W_1 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{W_2}{m_p^2} \left(p^\mu - \frac{p \cdot q}{q^2} q^\mu \right) \left(p^\nu - \frac{p \cdot q}{q^2} q^\nu \right).$$

Define

$$\nu = \frac{p \cdot q}{m_p}. \tag{7.3.2}$$

then W_1 and W_2 can be expresses by ν and q^2

$$W_1 = W_1(q^2, \nu), \quad W_2 = W_2(q^2, \nu).$$

By momentum conservation $W^2 = (p + q)^2 = m_p^2 + 2m_p\nu + q^2$.
Define

$$x = -\frac{q^2}{2p \cdot q} = -\frac{q^2}{2m_p\nu}, \tag{7.3.3}$$

$$y = \frac{p \cdot q}{p \cdot k}. \tag{7.3.4}$$

They are dimensionless.

In $ep \to eX$, $0 \le x \le 1$ and $0 \le y \le 1$. Proton is at rest, $p = (m_p, 0)$.

$$q = k - k', \quad q^0 = E - E'.$$

Thus

$$y = \frac{m_p(E - E')}{m_p E} = \frac{E - E'}{E},$$

$$\nu = \frac{p \cdot q}{m_p} = \frac{m_p(E - E')}{m_p} = E - E'.$$

Now, lets get back to cross section

$$L^{e\,\mu\nu}W_{\mu\nu} = 4W_1(k \cdot k') + \frac{2W_2}{m_p^2}\left[2(p \cdot k)(p \cdot k') - m_p^2(k \cdot k')\right],$$

go to the lab frame

$$= 4EE'\left[\cos^2\frac{\theta}{2}W_2(\nu, q^2) + \sin^2\frac{\theta}{2}2W_1(\nu, q^2)\right].$$

The cross section is then

$$d\sigma = \frac{1}{4\left[(k \cdot p)^2 - m_e^2 m_p^2\right]}\left[\frac{e^4}{q^4}L^{e\,\mu\nu}W_{\mu\nu}4\pi m_p\right]\frac{d^3k'}{2E'(2\pi)^3}$$

$$\left.\frac{d\sigma}{dE'\,d\Omega}\right| = \frac{\alpha^2}{4E^2\sin^2\frac{\theta}{2}}\left\{W_2(\nu, q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu, q^2)\sin^2\frac{\theta}{2}\right\}$$

Measurement of $W_{1,2}$ to learn about high energy ep-interaction. We can use interaction of electron and point-like particle (e.g. muon) and compare it to the measured $W_{1,2}$. It should go to in the limit where the "quarks" are resolved.

Summary

$$\frac{d\sigma}{dE'\,d\Omega} = \frac{4\alpha^2 E^2}{q^2} \cdot A \tag{7.3.5}$$

- Electron-muon scattering

$$A_{e\mu \to e\mu} = \left(\cos^2 \frac{\theta}{2} - \frac{q^2}{2m^2}\sin^2 \frac{\theta}{2}\right)\delta(\nu + \frac{q^2}{m}) \tag{7.3.6}$$

so ν and q^2 are not independent.

- Electron-proton scattering with low energy

$$A_{ep \to ep} = \left(\frac{G_E^2 + \tau G_M^2}{1 + \tau}\cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2}\right)\delta(\nu + \frac{q^2}{2m}) \tag{7.3.7}$$

with $\tau = -\frac{q^2}{4m_p^2}$.

- Electron-proton scattering with high energy

$$A_{ep \to eX} = W_2(\nu, q^2)\cos^2 \frac{\theta}{2} + 2W_1(\nu, q^2)\sin^2 \frac{\theta}{2} \tag{7.3.8}$$

In (a) and (b), we can integrate δ-function to get differential cross section.

Bibliography

[1] Herbi K. Dreiner, Howard E. Haber, and Stephen P. Martin. "Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry". In: *Physics Reports* 494.1-2 (Sept. 2010), pp. 1–196. ISSN: 0370-1573. DOI: `10.1016/j.physrep.2010.05.002`. URL: `http://dx.doi.org/10.1016/j.physrep.2010.05.002`.

[2] Hans-Werner Hammer and Ulf-G. MeiSSner. *The proton radius: From a puzzle to precision*. 2019. arXiv: `1912.03881 [hep-ph]`.

[3] *Spontaneous symmetry breaking*. URL: `https://en.wikipedia.org/wiki/Spontaneous_symmetry_breaking`.