H1.1 $\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{3!}\phi^3$

a) $[\mathcal{L}] = d, \quad [\partial^\mu \phi] = d/2 \quad [\phi] = d/2 - 1$

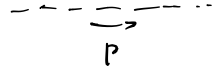
$$\Rightarrow [\lambda] + 3[\phi] = d$$

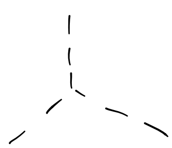
$$[\lambda] = d - 3\left(\frac{d}{2} - 1\right)$$

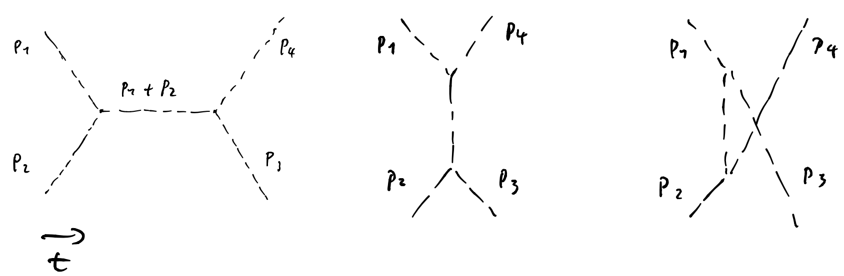
$$= d - \frac{3}{2}d + 3$$

$$= 3 - \frac{1}{2}d$$

$$d = 4, \quad \rightarrow [\lambda] = 1$$

b)  $= \frac{i}{p^2 - m^2 + i\varepsilon}$

 $= -i\lambda$

c) 

d) $i\mathcal{M}(s) = (-i\lambda)^2 \frac{i}{s - m^2 + i\varepsilon} = -i\frac{\lambda^2}{s - m^2 + i\varepsilon}$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{64\pi^2 s} \sum_{s,t,u} |\mathcal{M}|^2$$

$$= \frac{1}{64\pi^2 s} \lambda^4 \left(\frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2}\right)^2$$

e) $s = (p_1 + p_2)^2 = (p_1 + p_2)_\mu (p_1 + p_2)^\mu$ → written in covariant form, automatically Lorentz-invariant

$$s + t + u = (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2$$

$$= 6m^2 + 2p_1 \cdot p_2 - 2p_1 \cdot p_3 - 2p_1 - p_4$$

$$= 6m^2 + 2p_1 \cdot \underbrace{(p_2 - p_3 - p_4)}_{= -p_1} = 4m^2$$

H1.2

a)

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu , \quad x'_\mu = \Lambda_\mu{}^\nu x_\nu$$

$$x'^\mu x'_\mu = g_{\mu\alpha} x'^\mu x'^\alpha$$
$$= g_{\mu\alpha} \Lambda^\mu{}_\lambda x^\lambda \Lambda^\alpha{}_\sigma x^\sigma$$
$$= g_{\lambda\sigma} x^\lambda x^\sigma$$
$$= x_\sigma x^\sigma$$

b)

$$\det(g) = \det(\Lambda^T g \Lambda)$$
$$= \det(g) \det(\Lambda^T) \det(\Lambda) , \quad \det(A^T) = \det(A)$$

$$\Rightarrow \det(\Lambda)^2 = 1 \quad \Rightarrow |\det\Lambda| = 1$$

$$g_{\mu\nu} = g_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu ,$$

$\mu = \nu = 0$

$$g_{00} = g_{\sigma\tau} \Lambda^\sigma{}_0 \Lambda^\tau{}_0 = (\Lambda^0{}_0)^2 - \underbrace{\sum_i (\Lambda^i{}_0)^2}_{\geq 0} \overset{!}{=} 1$$

$$\Rightarrow |\Lambda^0{}_0| \geq 1$$

c)

- Λ, $(+,+)$ $\quad$ $(\mathrm{sgn}(\det\Lambda), \mathrm{sgn}(\Lambda^0{}_0))$
- $\Lambda_T \Lambda$ $(-,-)$
- $\Lambda_P \Lambda$ $(-,+)$ $\quad$ Only Λ has identity
- $\Lambda_T \Lambda_P \Lambda$ $(+,-)$

Timreversal $\quad$ parity

d) $\Lambda = \exp(L)$, $g = \Lambda^T g \Lambda$

$$\Rightarrow \quad \Lambda^{-1} = g \Lambda^T g$$

$$e^{-L} = e^{g L^T g}$$

$$-L = g L^T g$$

$$\Rightarrow \quad L^T = -gLg$$

$$\text{or} \quad g L^T = -Lg$$

gL is skew-symmetric

$$\Rightarrow \quad L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & L_{12} & 0 & L_{23} \\ L_{03} & L_{22} & L_{23} & 0 \end{pmatrix}$$

only 6 independent entries

e) $$\Lambda_{xy} = \exp(-\omega S_3) = \sum_{n=0}^{\infty} \frac{1}{n!} (-\omega S_3)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^n$$

$$\begin{pmatrix} 0 & & & \\ & 0 & \omega & \\ & -\omega & 0 & \\ & & & 0 \end{pmatrix}^2 = -\omega^2 \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \quad A^0 = \mathbb{1}_4$$

$$= \begin{pmatrix} 1 & & & \\ & 1-\frac{\omega^2}{2}+\cdots & \omega+\cdots & \\ & -\omega+\cdots & 1-\frac{\omega^2}{2}+\cdots & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & \cos\omega & \sin\omega & \\ & -\sin\omega & \cos\omega & \\ & & & 1 \end{pmatrix}$$

f) $$\Lambda_x = \exp(-\zeta K_1) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[-\zeta \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \right]^n$$

$$\left(\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \right.$$

$$= \begin{pmatrix} 1 - \frac{1}{2}(-\zeta)^2 + \dots & -\zeta + \dots & & \\ -\zeta + \dots & 1 - \frac{1}{2}(-\zeta)^2 + \dots & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1 + \frac{\beta^2}{2} + \frac{3}{8}\beta^4 + \mathcal{O}(\beta^6)$$

$$-\beta\gamma = -\beta - \frac{\beta^2}{3} + \dots$$

g) S_i, K_i are generators of Lorent group

w_i, ζ_i are the parameters

$\uparrow$ angle of rotation

$\nwarrow$ boost parameter

H1.3 a)

$$[S_i^+, S_j^+] = \frac{1}{4} [S_i + ik_i , S_j + ik_j]$$

$$= \frac{1}{4} \Big(\underbrace{[S_i, S_j]}_{\epsilon_{ijk} S_k} - \underbrace{[k_i, k_j]}_{-\epsilon_{ijk} S_k} + i \underbrace{[S_i, k_j]}_{\epsilon_{ijk} k_k} + i \underbrace{[k_i, S_j]}_{\substack{-\epsilon_{jik} k_k \\ = \epsilon_{ijk} k_k}} \Big)$$

$$= \frac{1}{2} (\epsilon_{ijk} S_k + i \epsilon_{ijk} k_k)$$

$$= \epsilon_{ijk} S_k^+$$

$$[S_i^-, S_j^-] = \frac{1}{4} [S_i - ik_i , S_j - ik_j]$$

$$= \frac{1}{4} \Big(\underbrace{[S_i, S_j]}_{\epsilon_{ijk} S_k} - \underbrace{[k_i, k_j]}_{-\epsilon_{ijk} S_k} - i \underbrace{[S_i, k_j]}_{\epsilon_{ijk} k_k} - i \underbrace{[k_i, S_j]}_{\substack{-\epsilon_{jik} k_k \\ = \epsilon_{ijk} k_k}} \Big)$$

$$= \epsilon_{ijk} S_k^-$$

$$[S_i^+, S_j^-] = \frac{1}{4} [S_i + ik_i , S_j - ik_j]$$

$$= \frac{1}{4} \Big(\underbrace{[S_i, S_j]}_{\cancel{\epsilon_{ijk} S_k}} + \underbrace{[k_i, k_j]}_{\cancel{-\epsilon_{ijk} S_k}} - i \underbrace{[S_i, k_j]}_{\cancel{\epsilon_{ijk} k_k}} + i \underbrace{[k_i, S_j]}_{\substack{-\epsilon_{jik} k_k \\ = \cancel{\epsilon_{ijk} k_k}}} \Big)$$

$$= 0$$

b) classifying by two numbers j_-, j_+