Theoretical particle physics

Chenhuan Wang

October 15, 2019

Contents

0 Organisational

Tutorials: Thursday: 8-10, 10-12 Friday: 10-12, 13-15

Exam consists of four problems

- first quickies
- 2nd-4th: similar in style to homework; two will be very close to homework

One needs 50% of points from homework. May hand in pairs.

Content of the lectures

- Standard Model of particle physics
- Electroweak sector
  - gauge principle
  - Higgs mechanism
  - Yukawa interactions
  - CP-violation

Exercises

- go through basics of computing Feynman diagrams
- not to derive the formalism
- Lagrange → Feynman rules → amplitudes → cross section and decay rates (measured quantities)

Literature

- Halzen and Martin, Quarks and Leptons (a lot of basics of QCD)
- Cheng and Li (includes also quantum field theory topics CP-violation in Standard Model)
- Mark Thomson
- QFT basics
  - Peskin and Schroeder
  - M.Schwartz
  - Ryder
- Okun, Leptons and Quarks

1 Introduction

1.1 Standard Model

It is the fundamental theory of nature. There are three interactions included

- electromagnetic
- weak
- strong
- Higgs boson exchange

Electromagnetic and weak interactions can be unified into electroweak interactions.

In the Sun all these three interactions and gravity are present

- Photons reaching us clearly indicate QED's involvement.
- Neutrinos produced in weak interaction. Four protons to two protons and two neutrons (Helium). Only weak interaction can change the colour of quarks.
- Binding of Helium via strong interaction and binding energy released as kinetic energy.
- Gravity brings protons together and at high temperature to give helium.

Gauge theories (Lie groups algebras)

- EM: $U(1)_{EM}$, $U(1)_Y$
- weak: $SU(2)_L$
- strong: $SU(3)_c$

Forces in quantum theories involve exchange particles spin 1, vector bosons

- photon γ
- weak $W^{\pm}$, W^0 and Z^0 (mixture of W^0 and hyper charge) (discovered at CERN)
- strong $g^a, a = 1, \ldots, 8$ gluons (discovered at DESY)

particles with spin $\frac{1}{2}$
Leptons

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L, e_R^-;\quad \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L, \mu_R^-;\quad \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L, \tau_R^-$$

Quarks

$$\begin{pmatrix}u\\ d\end{pmatrix}_L, u_R, d_R;\quad \begin{pmatrix}c\\ s\end{pmatrix}_L, c_R, s_R;\quad \begin{pmatrix}t\\ b\end{pmatrix}_L, t_R, b_R$$

One complete generation

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L, e_R^-;\quad \begin{pmatrix}u\\ d\end{pmatrix}_L, u_R, d_R;$$

To remove any one part, then gauge theory is inconsistent. It is known as "anomaly". (AQFT)

Higgs boson h^0, spin 0
In Standard Model Higgs bosons are described by complex scalar fields $\begin{pmatrix}H^+\\ H^0\end{pmatrix}$. h^0 is the only fundamental scalar in nature, as far as we know.

1.2 Energy scales

- binding energy of atoms $1-10\text{eV}$
- binding energy of nucleons $\approx 1\text{MeV}$
- no known binding energy in particle physics
- protons and neutrons $\approx \Lambda_{QCD} \approx \mathcal{O}(100\text{MeV})$

Particles have masses

- electron $m_e = 511\,\text{keV}$
- muon $m_\mu = 105\,\text{MeV}$
- tau $m_\tau = 1.7\,\text{GeV}$
- neutrinos $m_\nu < 1\,\text{eV}$
- quarks*
  - $m_u \approx 3\,\text{MeV}$
  - $m_d \approx 5\,\text{MeV}$
  - ...
- photon $m_\gamma = 0$
- gluon $m_g = 0$
- Higgs $m_{Higgs} \approx 125\,\text{GeV}$

*mass of proton mainly comes from dynamical effect "gluon"

Colliders

- LEP $91\,\mathrm{GeV} - 200\,\mathrm{GeV}$
- Tevatron($p\bar{p}$) $800\,\mathrm{GeV} - 2\,\mathrm{TeV}$
- LHC $7\,\mathrm{TeV} - 13\,\mathrm{TeV}$

1.3 Natural units

$$\hbar = c = 1 \tag{1.3.1}$$
$$k_B = 1 \tag{1.3.2}$$

Everything expressed in term of powers of energy.

$$1\,\mathrm{fm} = 1 \times 10^{-15}\,\mathrm{m} = 5\,\mathrm{GeV}^{-1}$$

2 Lorentz Transformation

2.1 Introduction

Metric (used for distance measuring)

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.1.1}$$

In string and general relativity people tend to use diag$(-, +, +, +)$.

$$p = (E, \boldsymbol{p}) \tag{2.1.2}$$

$$p^2 = E^2 - \boldsymbol{p}^2 = m^2 \tag{2.1.3}$$

Light is always light-like

$$t^2 - (x^2 + y^2 + z^2) = 0$$

Greek indices always go from 0 to 3

$$r^2 = g_{\mu\nu} r^\mu r^\nu = t^2 - \boldsymbol{r}^2 \tag{2.1.4}$$

Distance between two spacetime point is defined via

$$|r_A - r_B| = \sqrt{(r_A - r_B) \cdot (r_A - r_B)} = \sqrt{r_A^2 + r_B^2 - 2 r_A \cdot r_B} \tag{2.1.5}$$

2.2 Lorentz Transformation

Lorentz Transformation is transformation between two inertial frames moving with constant velocity $\boldsymbol{v}$ with respect to each other (boosts).

$$\begin{aligned} x &= (x_0, x_1, x_2, x_3) \\ x' &= (x'_0, x'_1, x'_2, x'_3) \\ x_0 &= ct = t; \quad c = 1 \end{aligned}$$

We define

$$\beta = \frac{v}{c} \quad \text{or} \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c} \tag{2.2.1}$$

Coordinates in these two frames are related like

$$
\begin{aligned}
x_0' &= \gamma(x_0 - \beta x_1) \\
x_1' &= \gamma(x_1 - \beta x_0) \\
x_2' &= x_2 \\
x_3' &= x_3
\end{aligned}
$$

Inverse transformation with $\beta \mapsto -\beta$
γ-factor is defined via

$$
\gamma = \frac{1}{\sqrt{1 - \boldsymbol{\beta}^2}} \tag{2.2.2}
$$

Since $|\boldsymbol{\beta}| \leq 1 \Rightarrow \gamma \geq 1$
Alternative parametrization

$$
\begin{aligned}
\beta &= \tanh(\zeta), \quad \gamma = \cosh(\zeta) \\
\gamma\beta &= \sinh(\zeta)
\end{aligned}
$$

Insert this into equation (2.2.2)

$$
\begin{aligned}
x_0' &= x_0 \cosh(\zeta) - x_1 \sinh(\zeta) \\
x_1' &= x_0 \sinh(\zeta) - x_1 \cosh(\zeta)
\end{aligned}
$$

We can turn this into matrices

$$
\begin{aligned}
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} \\
x &= \Lambda x'
\end{aligned} \tag{2.2.3}
$$

2.3 Mathematical Properties of Lorentz Transformation

Distance is invariant under Lorentz transformation

$$
s^2 = x_0^2 - x_1^2 + x_2^2 + x_3^2 = x^2 \tag{2.3.1}
$$

Lorentz transformation includes

- Rotation and boosts
- Parity $\boldsymbol{x} \mapsto -\boldsymbol{x}$
- Time reversal $t \mapsto -t$

We can also expand it with translation. It then turns to Poincare group.

2.3.1 Tensors

Define a function of original coordinates ($\alpha = 0, 1, 2, 3$)

$$x'^{\alpha} = x'^{\alpha}(x^0, x^1, x^2, x^3) \tag{2.3.2}$$

If x'^{α} transforms like

$$x'^{\alpha} = \frac{x'^{\alpha}}{x^{\beta}} x^{\beta} \tag{2.3.3}$$

it is called *contravariant*

Consider derivative $\frac{\partial}{\partial x'^{\alpha}}$

$$\frac{\partial f(x)}{\partial x'^{\alpha}} = \frac{\partial f(x)}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \tag{2.3.4}$$

We can see the x' is now in the denominator. The objects transformed like this are called *covariant*.

Consider the following generic objects: A'^{α} contravariant vector

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} \tag{2.3.5}$$

B'_{α} is covariant

$$B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} \tag{2.3.6}$$

Note (x^0, x^1, x^2, x^3) is contravariant.

The field strength tensor $F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta}$ is contravariant rank 2.

Mixed is also allowed $H'^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{x^{\delta}}{\partial x'^{\beta}} H^{\delta}_{\gamma}$

Inner or scalar product

$$\begin{aligned}
B' \cdot A' &= B'_{\alpha} A'^{\alpha} \\
&= \left(\frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta}\right)\left(\frac{\partial x'^{\alpha}}{\partial x^{\gamma}} A^{\gamma}\right) \\
&= \frac{\partial x^{\beta}}{\partial x^{\gamma}} B_{\beta} A^{\gamma} \\
&= \delta^{\beta}_{\gamma} B_{\beta} A^{\gamma} = B \cdot A
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}s^2 &= \left(\mathrm{d}x^0\right)^2 - (\mathrm{d}\boldsymbol{x})^2 \\
&= (g_{\alpha\beta}\,\mathrm{d}x^{\alpha})\,\mathrm{d}x^{\beta} = \mathrm{d}x_{\beta}\,\mathrm{d}x^{\beta}
\end{aligned} \tag{2.3.7}$$

Thus we can use metric tensor to lower index $\mathrm{d}x_{\beta} = g_{\alpha\beta}\,\mathrm{d}x^{\alpha}$

$$\begin{aligned}
A^{\alpha} &= \left(A^0, \boldsymbol{A}\right) \\
A_{\alpha} &= \left(A^0, -\boldsymbol{A}\right)
\end{aligned}$$

2.4 Matrix Representation of Lorentz Transformation

2.4.1 General Properties

We have

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

Then $a \cdot b = (a, gb) = g_{\mu\nu} a^\mu b^\nu = (ga, b) = a^T gb = (ga)^T b$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \mapsto x' = \Lambda x \tag{2.4.1}$$

$$x \cdot x = x' \cdot x' = (\Lambda x)(\Lambda x) \tag{2.4.2}$$

$$\begin{aligned} g_{\mu\nu} x^\mu x^\nu &= g_{\sigma\tau} x'^\sigma x'^\tau \\ &= g_{\sigma\tau} \Lambda^\sigma{}_\mu x^\mu \Lambda^\tau{}_\nu x^\nu \\ &= g_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu x^\mu x^\nu \end{aligned}$$

Then we have the *defining* rule of Lorentz group

$$g_{\mu\nu} = g_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \tag{2.4.3}$$

$$g = \Lambda^T g \Lambda \tag{2.4.4}$$

Properties

- $|\det(\Lambda)| = 1$
- $\left|\Lambda^0{}_0\right| \geq 1$

The orthochronous Lorentz transformations Λ forms a group.

Parity does not form a group

$$\Lambda_P = \mathrm{diag}(1, -1, -1, -1) \tag{2.4.5}$$

Time reversal

$$\Lambda_T = \mathrm{diag}(-1, +1, +1, +1) \tag{2.4.6}$$

There are four classes of Lorentz transformations depending on $\left(\mathrm{sgn}(\det(\Lambda)), \mathrm{sgn}(\Lambda^0_0)\right)$

- $(+, +)$ Λ
- $(-, -)$ $\Lambda_T \Lambda$
- $(-, +)$ $\Lambda_P \Lambda$
- $(+, -)$ $\Lambda_T \Lambda_P \Lambda$

Orthochronous Λ has 6 parameters, 3 for boosts and 3 for rotations. $\Lambda^T g \Lambda = g$ is actually 16 equations. All matrices here are symmetric. Thus 6 of 16 are redundant. There are 10 independent equations. Λ has 16 entries and it has $16 - 10 = 6$ free parameters.

2.4.2 Explicit Construction

We will restrict ourselves in orthochronous Lorentz transformations. The exponential function is defines via Taylor expansion. With $L \in \mathbb{R}^{4\times 4}$

$$\Lambda = \mathrm{e}^{L} = \exp(L)$$

From linear algebra we know

$$\det(\Lambda) = \det\left(\mathrm{e}^{L}\right) = \mathrm{e}^{\mathrm{tr}(L)} \tag{2.4.7}$$

Since $\det(\Lambda) = 1$, $\mathrm{tr}(L) = 0$

$$\begin{aligned}
\Lambda^T g \Lambda &= g \\
g\Lambda^T g \Lambda &= \mathbb{1}_4 \\
g\Lambda^T g &= \Lambda^{-1} \\
\exp\left(gL^T g\right) &= \Lambda^{-1} = \exp(-L) \\
\Leftrightarrow gL^T g &= -L \\
\Leftrightarrow (gL)^T &= -gL
\end{aligned}$$

This means that gL is anti-symmetric

$$L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & L_{12} & 0 & L_{23} \\ L_{03} & L_{13} & L_{23} & 0 \end{pmatrix}$$

Define six basis matrices $S_{1,2,3}$ and $K_{1,2,3}$

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

S_i is the generator of 3-dimensional rotations and K_i is the generator of 3-dimensional boosts.

$$\begin{aligned}
&\hat{n} \in \mathbb{R}^3, \quad |\hat{n}| = 1 \\
&\hat{n} \cdot \boldsymbol{S} = n_1 S_1 + n_2 S_2 + n_3 S_3 \\
&(\hat{n} \cdot \boldsymbol{S})^3 = -\hat{n} \cdot \boldsymbol{S} \\
&(\hat{n} \cdot \boldsymbol{K})^3 = +\hat{n} \cdot \boldsymbol{S}
\end{aligned}$$

In the end

$$L = -\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\zeta} \cdot \boldsymbol{K} \quad \text{with } \boldsymbol{\omega}, \boldsymbol{\zeta} \in \mathbb{R}^3 \tag{2.4.8}$$

$$\Lambda = \exp(-\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\zeta} \cdot \boldsymbol{K}) \tag{2.4.9}$$

$\boldsymbol{\omega}$ is the axis of rotation, $|\boldsymbol{\omega}|$ is then the angle of rotation.
$\tanh|\boldsymbol{\zeta}| = \beta$ and $\frac{\boldsymbol{\zeta}}{|\zeta|}$ is the direction of boost.
We now will look at concrete examples

- $\boldsymbol{\zeta} = 0,\ \boldsymbol{\omega} = w\hat{e}_z$

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  Rotational angle is ω.

- $\boldsymbol{\omega} = 0,\ \boldsymbol{\zeta} = \zeta\hat{e}_x$

$$\Lambda = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Pure general boost $\boldsymbol{\zeta}$

$$\Lambda = \exp(-\boldsymbol{\zeta}\cdot\boldsymbol{K})$$
$$\boldsymbol{\zeta} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|}\tanh^{-1}|\boldsymbol{\beta}|, \quad \hat{\beta} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|}$$
$$\Lambda = \exp\Big(-\hat{\beta}\cdot\boldsymbol{K}\tanh^{-1}(\beta)\Big)$$

2.4.3 Algebra of generators

Consider the commutation algebra of $S_{i=1,2,3}$ and $K_{i=1,2,3}$

$$\left[S_i, S_j\right] = \epsilon_{ijk}S_k \tag{2.4.10}$$
$$\left[S_i, K_j\right] = \epsilon_{ijk}K_k \tag{2.4.11}$$
$$\left[K_i, K_j\right] = -\epsilon_{ijk}S_k \tag{2.4.12}$$

The last equation causes Thomas precession in atomic physics.
Choose a different basis

$$\boldsymbol{S}_+ = \frac{1}{2}(\boldsymbol{S} + i\boldsymbol{K})\,\boldsymbol{S}_- = \frac{1}{2}(\boldsymbol{S} - i\boldsymbol{K}) \tag{2.4.13}$$

Then we can calculate the algebra

$$\left[S_{+,i}, S_{+,j}\right] = i\epsilon_{ijk}S_{+,k} \tag{2.4.14}$$
$$\left[S_{-,i}, S_{-,j}\right] = i\epsilon_{ijk}S_{-,k} \tag{2.4.15}$$
$$\left[S_{+,i}, S_{-,j}\right] = 0 \tag{2.4.16}$$

In other word, the algebras are decoupled. This familiar algebra is angular momentum algebra $\mathbf{SU}(2)$.

Classification by two numbers $(j_+,\ j_-)$

$$
\begin{aligned}
j_+ &= 0, \frac{1}{2}, 1, \dots \\
j_- &= 0, \frac{1}{2}, 1, \dots
\end{aligned}
$$

Dimension = $(2j_+ + 1)(2j_- + 1)$.

A field is scalar field if $j_+ = j_- = 0$. One fundamental example of scalar field is Higgs boson. Other scalar particles are just bound states.

There are two possible states with spin $\frac{1}{2}$: $(j_+, j_-) = (\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$

$$
\begin{aligned}
\boldsymbol{S}_+ &= \frac{1}{2}\boldsymbol{\sigma} \\
\boldsymbol{S}_- &= 0 \\
\boldsymbol{S} &= \frac{1}{2}\boldsymbol{\sigma} \\
i\boldsymbol{K} &= \frac{1}{2}\boldsymbol{\sigma}
\end{aligned}
$$

for $(0, \frac{1}{2})$ there is "-"

So two types of spin $\frac{1}{2}$ fermions

$$
\begin{aligned}
\left(\frac{1}{2}, 0\right) &\rightarrow e_L^- \\
\left(0, \frac{1}{2}\right) &\rightarrow e_R^-
\end{aligned}
$$

They have different transformation law. W boson only to e_L^- but photon couples to both.

Under parity transformation $e_L^- \leftrightarrow e_R^-$. Both particles are needed for theory to be invariant under parity transformation, like EM and strong interactions.

3 Relativistic Quantum Field Theory

In non-relativistic quantum mechanics

$$
\begin{aligned}
E &= \frac{\boldsymbol{p}^2}{2m} \\
E &\mapsto i\hbar\frac{\partial}{\partial t} \\
\boldsymbol{p} &\to -i\hbar\boldsymbol{\nabla}
\end{aligned}
$$

After promoting the momentum and energy into operators in dispersion relation we have the Schrödinger equation

$$
i\frac{\partial}{\partial t}\psi + \frac{1}{2m}\boldsymbol{\nabla}^2\psi = 0 \tag{3.0.1}
$$

Density of probability is defined via

$$
\rho = |\psi|^2 = \psi\psi^* \tag{3.0.2}
$$

It obeys the continuity equation

$$
\begin{aligned}
-\frac{\partial}{\partial t}\int_V \rho \,\mathrm{d}V &= \int \boldsymbol{j}\cdot\boldsymbol{n}\,\mathrm{d}S \\
&= \int_V \boldsymbol{\nabla}\cdot\boldsymbol{j}\,\mathrm{d}V \\
\Rightarrow \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} &= 0
\end{aligned} \tag{3.0.3}
$$

Writing this explicitly

$$
\begin{aligned}
\frac{\partial\rho}{\partial t} &= \frac{\partial}{\partial t}(\psi\psi^*) \\
&= \psi\frac{\partial\psi^*}{\partial t} + \psi^*\frac{\partial\psi}{\partial t} \\
&= \frac{i}{2m}\left(\psi^*\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\psi\right) \\
\Rightarrow \boldsymbol{j} &= -\frac{i}{2m}\left(\psi^*\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\psi\right)
\end{aligned} \tag{3.0.4}
$$

If we have a plane wave state, as an example

$$
\begin{aligned}
\psi &= N\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}-iEt} \\
\boldsymbol{j} &= \frac{\boldsymbol{p}}{m}|N|^2
\end{aligned}
$$

3.1 Relativistic wave equation

Now we enter the relativistic regime

$$E^2 = \boldsymbol{p}^2 + m^2$$
$$p^\mu = (E, \boldsymbol{p}) \quad p_\mu = (E, -\boldsymbol{p})$$
$$p^2 = m^2$$

Promoting energy and momentum into operators

$$p^\mu \mapsto i\partial^\mu$$
$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial^2 t} - \nabla^2$$

We have then Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\phi(\boldsymbol{x}, t) = 0 \tag{3.1.1}$$

The current in KG-theory is conserved as well

$$j^\mu = (\rho, \boldsymbol{j}) = i\left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right) \tag{3.1.2}$$
$$\partial_\mu j^\mu = 0 \tag{3.1.3}$$

An example solution

$$\phi = N\mathrm{e}^{-ip\cdot x}$$
$$j^\mu = 2p^\mu |N|^2$$

In terms of Lorentz transformation

$$\rho \sim E$$

Energies of particles

$$E^2 = \boldsymbol{p}^2 + m^2$$
$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2}$$

It also implies negative probability

$$E > 0 \mapsto \rho > 0$$
$$E < 0 \mapsto \rho < 0$$

3.2 Feynman-Stückelberg Interpretation of negative energy states

"Electron" with $E, \boldsymbol{p}$ and charge $-e$

$$j^\mu_{e^-} = 2e|N|^2(E, \boldsymbol{p})$$

"Positron" with $E, \boldsymbol{p}$ and charge $+e$

$$j^\mu_{e^+} = 2e|N|^2(E, \boldsymbol{p}) = -2e|N|^2(-E, -\boldsymbol{p})$$

We can think of $E < 0$ solution as particle flying backwards in time or $E > 0$ anti-particle forwards in time.

In a relativistic systems we need to remember following points

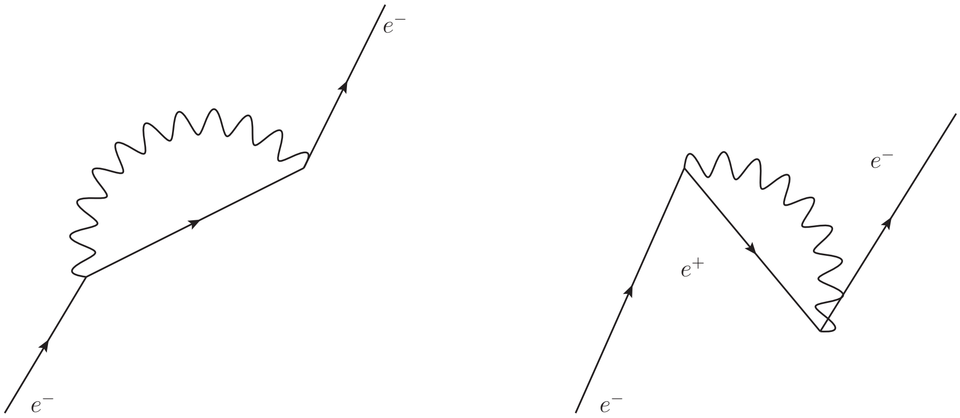

Figure 3.1: scattering process; horizontal time-axis; in the second diagram a electron positron pair is produced

- anti-particles
- particle numbers are not conserved

3.3 Electrodynamics (spin 1)

Maxwell equations are

$$\boldsymbol{E} = -\vec{\nabla}\phi - \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A} \tag{3.3.1}$$

$$\boldsymbol{B} = \vec{\nabla} \times \boldsymbol{A} \tag{3.3.2}$$

$$\vec{\nabla} \times \boldsymbol{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B} \tag{3.3.3}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{3.3.4}$$

Field strength tensor and four-potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.3.5}$$

$$A^\mu(x) = (\phi, \boldsymbol{A}) \tag{3.3.6}$$

The fields can be calculated from it

$$E^i = F^{0i} = \partial^i A^0 - \partial^0 A^i \tag{3.3.7}$$

$$B_i = -\epsilon_{ijk}\partial^i A^k = -\epsilon_{0ijk}F^{jk} \tag{3.3.8}$$

Often it is useful to use the dual tensor

$$\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\sigma\tau}F^{\sigma\tau} \tag{3.3.9}$$

$$\partial^\mu \tilde{F}_{\mu\nu} = 0 \tag{3.3.10}$$

is the second set of maxwell equations.

The other set of two equations is

$$\partial_\nu F^{\mu\nu} = 4\pi j^\mu \tag{3.3.11}$$

$\boldsymbol{E}, \boldsymbol{B}$ are observable, $\boldsymbol{A}$ is not. A^μ is not uniquely fixed by $\boldsymbol{E}$ and $\boldsymbol{B}$. It has the following gauge symmetry

$$\tilde{A}_\mu = A_\mu + \partial_\mu \Lambda(\boldsymbol{x}, t) \tag{3.3.12}$$

Use this transformation to get

$$\partial_\mu A^\mu = 0 \tag{3.3.13}$$

Plugging it back then we have the relativistic wave equation

$$\partial_\mu \partial^\mu A^\nu = 0 \tag{3.3.14}$$

it essentially is Klein-Gordon equation with mass $m = 0$

A^μ is a vector with spin 1

$$(j_+, j_-) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

It implied it has two transverse degrees of freedom. It has spin 1 properties: $+1, 0, -1$, in which 0 mode does not exist.

3.4 Description of Fermions

Original motivation for Dirac. He wants a linear equation in E or $\frac{\partial}{\partial t}$

$$p^\mu \mapsto i\partial^\mu$$

Take the ansatz

$$\begin{aligned} i\hbar \frac{\partial}{\partial t}\psi &= H\psi \\ &= (\vec{\alpha} \cdot \boldsymbol{p} + \beta m)\psi \end{aligned}$$

but $\boldsymbol{\alpha}$ and β unknown. It still has to obey the relativistic energy relation

$$\begin{aligned} A &= (\alpha_i p_i + \beta m)(\alpha_i p_i + \beta m) \\ &\overset{!}{=} \boldsymbol{p}^2 + m^2 \\ &= \alpha_i \alpha_j p_i p_j + \beta^2 m^2 + \alpha_i \beta p_i m + \beta \alpha_j p_j m \end{aligned}$$

From this we demand

$$\beta^2 = 1 \tag{3.4.1}$$

$$\alpha_i^2 = 1 \tag{3.4.2}$$

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \tag{3.4.3}$$

$$\alpha_i \beta + \beta \alpha_i = 0 \tag{3.4.4}$$

So α and β are not just numbers, but (can be proven to be) hermitian traceless matrices with eigenvalue ± 1. In addition, it only exits in even dimensions. Since α_i and β are 4×4 matrices. ψ has to be a 4-component spinor.

For parity conservation need $(\frac{1}{2}, 0) \bigoplus (0, \frac{1}{2})$ Thus

$$\begin{pmatrix} \begin{pmatrix} & \end{pmatrix}_{2\times2} & \\ & \begin{pmatrix} & \end{pmatrix}_{2\times2} \end{pmatrix}$$

There are different sets of α_i, β which satisfy the conditions. They are called representations.

Dirac-Pauli representation

$$\alpha_i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{3.4.5}$$

$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \tag{3.4.6}$$

with σ^i the Pauli matrices.

Weyl (chiral) representation

$$\alpha^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{3.4.7}$$

$$\beta = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \tag{3.4.8}$$

they are mainly used in high energy physics ($\mathrm{E} \gg m$).

3.4.1 Gamma Matrices

We now define 4 gamma matrices γ^μ, $\mu = 0, 1, 2, 3$

$$\gamma^\mu = (\beta, \beta\boldsymbol{\alpha}) \tag{3.4.9}$$

Note that having an index does not make it Lorentz vector.

The Clifford algebra is defined as following

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{3.4.10}$$

In Dirac-Pauli representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{3.4.11}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{3.4.12}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{3.4.13}$$

In Weyl representation

$$\gamma^0 \leftrightarrow \gamma^5$$

Rewriting the Dirac equation using γs

$$
\begin{aligned}
i\partial_t\psi &= (\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m)\,\psi \\
i\partial_t\psi &= -i\boldsymbol{\alpha}\cdot\vec{\nabla}\psi + m\beta\psi \\
i\beta\partial_t\psi &= -i\beta\boldsymbol{\alpha}\cdot\vec{\nabla}\psi + m\psi \\
\left(i\gamma^\mu\partial_\mu - m\right)\psi &= 0
\end{aligned}
\tag{3.4.14}
$$

Conventionally we use ϕ for spin 0 particle and A_μ for spin 1.

It is convenient to also have an equation for $\psi^\dagger$. First one can show $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

- $\mu = 0$: $\gamma^0 = \beta$ and $\gamma^{0\dagger} = \gamma^0\gamma^0\gamma^0 \Rightarrow \beta^2 = \mathbb{1}_4$
- $\gamma^{\mu\dagger} = (\beta\alpha^k)^\dagger = (\alpha^k)^\dagger\beta^\dagger = \alpha^k\beta = \beta^2\alpha^k\beta = \beta\gamma^k\beta = \gamma^0\gamma^k\gamma^0$

$$
\begin{aligned}
i\gamma^0\partial_0\psi + i\gamma^k\partial_k\partial - m\psi &= 0 \\
-i\partial_0\psi^\dagger(\gamma^0)^\dagger - i(\partial_k\psi^\dagger)\gamma^{k\dagger} - m\psi^\dagger &= 0 \\
-i\partial_0\psi^\dagger\gamma^0 - i(\partial_k\psi^\dagger)\gamma^0\gamma^k\gamma^0 - m\psi^\dagger &= 0
\end{aligned}
$$

define $\bar{\psi} = \psi^\dagger\gamma^0$

$$
\begin{aligned}
-i\partial_0\bar{\psi}\gamma^0 - i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} &= 0 \\
i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} &= 0
\end{aligned}
\tag{3.4.15}
$$

3.4.2 Free Particle Solution to Dirac Equation

$$
\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0
$$

multiplying $\gamma^\nu\partial_\nu$ from left

$$
\begin{aligned}
i\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\psi - m\gamma^\nu\partial_\nu\psi &= 0 \\
i\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\psi + im^2\psi &= 0
\end{aligned}
$$

$$
\begin{aligned}
\gamma^\mu\gamma^\nu &= \frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) \\
&= \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu + 2g^{\mu\nu}\right) \\
&= \frac{1}{2}\left[\gamma^\mu,\gamma^\nu\right] + g^{\mu\nu}
\end{aligned}
$$

The commutator is anti-symmetric and multiplying to symmetric tensor (derivatives) the term must vanish.

Each component of spinor satisfies the Klein-Gordon equation.

$$
(\partial_\mu\partial^\mu + m^2)\psi_i = 0 \tag{3.4.16}
$$

Thus we can write the solution as plane-wave

$$
\psi = u(\boldsymbol{p})\mathrm{e}^{-ipx} \tag{3.4.17}
$$

$u(\boldsymbol{p})$ is also a 4-component object but as function $\boldsymbol{p}$ not $\boldsymbol{x}$

Insert in back into Dirac equation, then we have Dirac equation in momentum space

$$\left(\gamma^{\mu} p_{\mu} - m\right) u(\boldsymbol{u}) = 0 \tag{3.4.18}$$

Solution by considering Dirac-Pauli representation

$$\left(\not{p} - m\right) u(\boldsymbol{p}) = \begin{pmatrix} (E-m)\mathbb{1} & -\boldsymbol{p}\cdot\boldsymbol{\sigma} \\ \boldsymbol{p}\cdot\boldsymbol{\sigma} & -(E+m)\mathbb{1} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$\boldsymbol{p} = 0$ then $E = \pm m$

- $E = +m$ Two solutions

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $E = -m$

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\boldsymbol{p} = 0$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} u_B = (E-m) u_A \tag{3.4.19}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} u_A = (E+m) u_B \tag{3.4.20}$$

- $E > 0$

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Ansatz $u_A^{(s)} = \chi^{(s)}$

$$u_B^{(s)} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} \quad u_A^{(s)} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)}$$

$$u(\boldsymbol{p}) = N \begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \end{pmatrix} \tag{3.4.21}$$

$E < 0$ and $u_B^{(s)} = \chi^{(s)}$

$$u(\boldsymbol{p}) = N \begin{pmatrix} -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \tag{3.4.22}$$

One can show $u^{(r)\dagger} u^{(s)} = N^2 \delta^{rs}$

Two fold degeneracy in each case. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $E > 0$ and $E < 0$. There must be another observable which commutes with H and $\boldsymbol{p}$.

$$H = \gamma^i p_i + \gamma^0 m$$

$$\boldsymbol{S} \cdot \hat{P} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} \cdot \boldsymbol{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{p} \end{pmatrix} \tag{3.4.23}$$

Helicity

$$\frac{1}{2}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}} = \frac{1}{2}\begin{pmatrix} \hat{p}_z & \hat{p}_x + i\hat{p}_y \\ \hat{p}_x - i\hat{p}_y & -\hat{p}_z \end{pmatrix} \tag{3.4.24}$$

$$\det(\boldsymbol{\sigma} \cdot \hat{p}) = -\hat{p}^2 = -1 \tag{3.4.25}$$

Determinant is the product of two eigenvalues, then

$$\lambda_1 + \lambda_2 = 0$$
$$\lambda_1 \cdot \lambda_2 = 1$$
$$\lambda_{1,2} = \pm 1$$

Antiparticle solution $u^{(3,4)}(-\boldsymbol{p})e^{-i(-p)x} = v^{(2,1)}$

$$\left(\not{p} + m\right) v(\boldsymbol{p}) = 0 \tag{3.4.26}$$

Normalization is

$$\int \rho \, \mathrm{d}V = 2E \tag{3.4.27}$$

$$N = \sqrt{E + m} \tag{3.4.28}$$

Completeness relation (spin sums)

$$\sum u^{(s)}(p)\bar{u}^{(s)}(p) = \left(\not{p} + m\right) \tag{3.4.29}$$

$$\sum v^{(s)}(p)\bar{v}^{(s)}(p) = \left(\not{p} - m\right) \tag{3.4.30}$$

Define a projector projecting out positive and negative energy states

$$\Lambda_{\pm} = \frac{\pm \not{p} + m}{2m} \tag{3.4.31}$$

In Chiral (Weyl) representation

$$\left(\not{p} - m\right) u(\boldsymbol{p}) = \begin{pmatrix} m & p \cdot \sigma \\ \boldsymbol{k} \cdot \bar{\sigma} & -m \end{pmatrix}\begin{pmatrix} u_L \\ u_R \end{pmatrix} \tag{3.4.32}$$

$\bar{\sigma} = (\sigma^0, -\boldsymbol{\sigma})$ and $\sigma^0 = \mathbb{1}_2$

Weyl equation

$$-mu_L + p \cdot \sigma u_R = 0 \tag{3.4.33}$$

$$p \cdot \bar{\sigma} u_L - mu_R = 0 \tag{3.4.34}$$

if $m = 0$, the equations decouple.