Theoretical particle physics

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Contents

0 Organisational

Tutorials: Thursday: 8-10, 10-12 Friday: 10-12, 13-15

Exam consists of four problems

- first quickies
- 2nd-4th: similar in style to homework; two will be very close to homework

One needs 50% of points from homework. May hand in pairs.

Content of the lectures

- Standard Model of particle physics
- Electroweak sector
  - gauge principle
  - Higgs mechanism
  - Yukawa interactions
  - CP-violation

Exercises

- go through basics of computing Feynman diagrams
- not to derive the formalism
- Lagrange → Feynman rules → amplitudes → cross section and decay rates (measured quantities)

Literature

- Halzen and Martin, Quarks and Leptons (a lot of basics of QCD)
- Cheng and Li (includes also quantum field theory topics CP-violation in Standard Model)
- Mark Thomson
- QFT basics
  - Peskin and Schroeder
  - M.Schwartz
  - Ryder
- Okun, Leptons and Quarks

1 Introduction

1.1 Standard Model

It is the fundamental theory of nature. There are three interactions included

- electromagnetic
- weak
- strong
- Higgs boson exchange

Electromagnetic and weak interactions can be unified into electroweak interactions.

In the Sun all these three interactions and gravity are present

- Photons reaching us clearly indicate QED's involvement.
- Neutrinos produced in weak interaction. Four protons to two protons and two neutrons (Helium). Only weak interaction can change the colour of quarks.
- Binding of Helium via strong interaction and binding energy released as kinetic energy.
- Gravity brings protons together and at high temperature to give helium.

Gauge theories (Lie groups algebras)

- EM: $U(1)_{EM}$, $U(1)_Y$
- weak: $SU(2)_L$
- strong: $SU(3)_c$

Forces in quantum theories involve exchange particles spin 1, vector bosons

- photon γ
- weak $W^{\pm}$, W^0 and Z^0 (mixture of W^0 and hyper charge) (discovered at CERN)
- strong $g^a, a = 1, \ldots, 8$ gluons (discovered at DESY)

particles with spin $\frac{1}{2}$
Leptons

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L, e_R^-;\quad \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L, \mu_R^-;\quad \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L, \tau_R^-$$

Quarks

$$\begin{pmatrix}u\\ d\end{pmatrix}_L, u_R, d_R;\quad \begin{pmatrix}c\\ s\end{pmatrix}_L, c_R, s_R;\quad \begin{pmatrix}t\\ b\end{pmatrix}_L, t_R, b_R$$

One complete generation

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L, e_R^-;\quad \begin{pmatrix}u\\ d\end{pmatrix}_L, u_R, d_R;$$

To remove any one part, then gauge theory is inconsistent. It is known as "anomaly". (AQFT)

Higgs boson h^0, spin 0
In Standard Model Higgs bosons are described by complex scalar fields $\begin{pmatrix}H^+\\ H^0\end{pmatrix}$. h^0 is the only fundamental scalar in nature, as far as we know.

1.2 Energy scales

- binding energy of atoms $1-10$eV
- binding energy of nucleons $\approx$ 1MeV
- no known binding energy in particle physics
- protons and neutrons $\approx \Lambda_{QCD} \approx \mathcal{O}(100\text{MeV})$

Particles have masses

- electron $m_e = 511\,\text{keV}$
- muon $m_\mu = 105\,\text{MeV}$
- tau $m_\tau = 1.7\,\text{GeV}$
- neutrinos $m_\nu < 1\,\text{eV}$
- quarks*
  - $m_u \approx 3\,\text{MeV}$
  - $m_d \approx 5\,\text{MeV}$
  - ...
- photon $m_\gamma = 0$
- gluon $m_g = 0$
- Higgs $m_{Higgs} \approx 125\,\text{GeV}$

*mass of proton mainly comes from dynamical effect "gluon"

Colliders

- LEP 91 GeV − 200 GeV
- Tevatron($p\bar{p}$) 800 GeV − 2 TeV
- LHC 7 TeV − 13 TeV

1.3 Natural units

$$\hbar = c = 1 \tag{1.3.1}$$
$$k_B = 1 \tag{1.3.2}$$

Everything expressed in term of powers of energy.

$$1\,\mathrm{fm} = 1 \times 10^{-15}\,\mathrm{m} = 5\,\mathrm{GeV}^{-1}$$

2 Lorentz Transformation

2.1 Introduction

Metric (used for distance measuring)

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.1.1}$$

In string and general relativity people tend to use diag$(-,+,+,+)$.

$$p = (E, \boldsymbol{p}) \tag{2.1.2}$$

$$p^2 = E^2 - \boldsymbol{p}^2 = m^2 \tag{2.1.3}$$

Light is always light-like

$$t^2 - (x^2 + y^2 + z^2) = 0$$

Greek indices always go from 0 to 3

$$r^2 = g_{\mu\nu} r^\mu r^\nu = t^2 - \boldsymbol{r}^2 \tag{2.1.4}$$

Distance between two spacetime point is defined via

$$|r_A - r_B| = \sqrt{(r_A - r_B) \cdot (r_A - r_B)} = \sqrt{r_A^2 + r_B^2 - 2 r_A \cdot r_B} \tag{2.1.5}$$

2.2 Lorentz Transformation

Lorentz Transformation is transformation between two inertial frames moving with constant velocity $\boldsymbol{v}$ with respect to each other (boosts).

$$\begin{aligned} x &= (x_0, x_1, x_2, x_3) \\ x' &= (x_0', x_1', x_2', x_3') \\ x_0 &= ct = t; \quad c = 1 \end{aligned}$$

We define

$$\beta = \frac{v}{c} \quad \text{or} \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c} \tag{2.2.1}$$

Coordinates in these two frames are related like

$$
\begin{aligned}
x_0' &= \gamma(x_0 - \beta x_1) \\
x_1' &= \gamma(x_1 - \beta x_0) \\
x_2' &= x_2 \\
x_3' &= x_3
\end{aligned}
$$

Inverse transformation with $\beta \mapsto -\beta$
γ-factor is defined via

$$
\gamma = \frac{1}{\sqrt{1 - \boldsymbol{\beta}^2}} \tag{2.2.2}
$$

Since $|\boldsymbol{\beta}| \leq 1 \Rightarrow \gamma \geq 1$
Alternative parametrization

$$
\begin{aligned}
\beta &= \tanh(\zeta), \quad \gamma = \cosh(\zeta) \\
\gamma\beta &= \sinh(\zeta)
\end{aligned}
$$

Insert this into equation (2.2.2)

$$
\begin{aligned}
x_0' &= x_0 \cosh(\zeta) - x_1 \sinh(\zeta) \\
x_1' &= x_0 \sinh(\zeta) - x_1 \cosh(\zeta)
\end{aligned}
$$

We can turn this into matrices

$$
\begin{aligned}
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} \\
x &= \Lambda x'
\end{aligned} \tag{2.2.3}
$$

2.3 Mathematical Properties of Lorentz Transformation

Distance is invariant under Lorentz transformation

$$
s^2 = x_0^2 - x_1^2 + x_2^2 + x_3^2 = x^2 \tag{2.3.1}
$$

Lorentz transformation includes

- Rotation and boosts
- Parity $\boldsymbol{x} \mapsto -\boldsymbol{x}$
- Time reversal $t \mapsto -t$

We can also expand it with translation. It then turns to Poincare group.

2.3.1 Tensors

Define a function of original coordinates ($\alpha = 0, 1, 2, 3$)

$$x'^{\alpha} = x'^{\alpha}(x^0, x^1, x^2, x^3) \tag{2.3.2}$$

If x'^{α} transforms like

$$x'^{\alpha} = \frac{x'^{\alpha}}{x^{\beta}} x^{\beta} \tag{2.3.3}$$

it is called *contravariant*
Consider derivative $\frac{\partial}{\partial x'^{\alpha}}$

$$\frac{\partial f(x)}{\partial x'^{\alpha}} = \frac{\partial f(x)}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \tag{2.3.4}$$

We can see the x' is now in the denominator. The objects transformed like this are called *covariant*.
Consider the following generic objects: A'^{α} contravariant vector

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} \tag{2.3.5}$$

B'_{α} is covariant

$$B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} \tag{2.3.6}$$

Note (x^0, x^1, x^2, x^3) is contravariant.
The field strength tensor $F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta}$ is contravariant rank 2.
Mixed is also allowed $H'^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{x^{\delta}}{\partial x'^{\beta}} H^{\delta}_{\gamma}$

Inner or scalar product

$$\begin{aligned}
B' \cdot A' &= B'_{\alpha} A'^{\alpha} \\
&= \left(\frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} \right) \left(\frac{\partial x'^{\alpha}}{\partial x^{\gamma}} A^{\gamma} \right) \\
&= \frac{\partial x^{\beta}}{\partial x^{\gamma}} B_{\beta} A^{\gamma} \\
&= \delta^{\beta}_{\gamma} B_{\beta} A^{\gamma} = B \cdot A
\end{aligned}$$

$$\begin{aligned}
ds^2 &= \left(dx^0 \right)^2 - (d\boldsymbol{x})^2 \\
&= (g_{\alpha\beta}\, dx^{\alpha})\, dx^{\beta} = dx_{\beta}\, dx^{\beta}
\end{aligned} \tag{2.3.7}$$

Thus we can use metric tensor to lower index $dx_{\beta} = g_{\alpha\beta}\, dx^{\alpha}$

$$\begin{aligned}
A^{\alpha} &= \left(A^0, \boldsymbol{A} \right) \\
A_{\alpha} &= \left(A^0, -\boldsymbol{A} \right)
\end{aligned}$$

2.4 Matrix Representation of Lorentz Transformation

2.4.1 General Properties

We have

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

Then $a \cdot b = (a, gb) = g_{\mu\nu} a^\mu b^\nu = (ga, b) = a^T gb = (ga)^T b$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \mapsto x' = \Lambda x \tag{2.4.1}$$

$$x \cdot x = x' \cdot x' = (\Lambda x)(\Lambda x) \tag{2.4.2}$$

$$\begin{aligned} g_{\mu\nu} x^\mu x^\nu &= g_{\sigma\tau} x'^\sigma x'^\tau \\ &= g_{\sigma\tau} \Lambda^\sigma{}_\mu x^\mu \Lambda^\tau{}_\nu x^\nu \\ &= g_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu x^\mu x^\nu \end{aligned}$$

Then we have the *defining* rule of Lorentz group

$$g_{\mu\nu} = g_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \tag{2.4.3}$$

$$g = \Lambda^T g \Lambda \tag{2.4.4}$$

Properties

- $|\det(\Lambda)| = 1$
- $\left|\Lambda^0{}_0\right| \geq 1$

The orthochronous Lorentz transformations Λ forms a group.

Parity does not form a group

$$\Lambda_P = \mathrm{diag}(1, -1, -1, -1) \tag{2.4.5}$$

Time reversal

$$\Lambda_T = \mathrm{diag}(-1, +1, +1, +1) \tag{2.4.6}$$

There are four classes of Lorentz transformations depending on $\left(\mathrm{sgn}(\det(\Lambda)), \mathrm{sgn}(\Lambda^0_0)\right)$

- $(+, +)$ Λ
- $(-, -)$ $\Lambda_T \Lambda$
- $(-, +)$ $\Lambda_P \Lambda$
- $(+, -)$ $\Lambda_T \Lambda_P \Lambda$

Orthochronous Λ has 6 parameters, 3 for boosts and 3 for rotations. $\Lambda^T g \Lambda = g$ is actually 16 equations. All matrices here are symmetric. Thus 6 of 16 are redundant. There are 10 independent equations. Λ has 16 entries and it has $16 - 10 = 6$ free parameters.

2.4.2 Explicit Construction

We will restrict ourselves in orthochronous Lorentz transformations. The exponential function is defines via Taylor expansion. With $L \in \mathbb{R}^{4\times 4}$

$$\Lambda = \mathrm{e}^{L} = \exp(L)$$

From linear algebra we know

$$\det(\Lambda) = \det\left(\mathrm{e}^{L}\right) = \mathrm{e}^{\mathrm{tr}(L)} \tag{2.4.7}$$

Since $\det(\Lambda) = 1$, $\mathrm{tr}(L) = 0$

$$\begin{aligned}
\Lambda^T g \Lambda &= g \\
g\Lambda^T g \Lambda &= \mathbb{1}_4 \\
g\Lambda^T g &= \Lambda^{-1} \\
\exp\left(gL^T g\right) &= \Lambda^{-1} = \exp(-L) \\
\Leftrightarrow gL^T g &= -L \\
\Leftrightarrow (gL)^T &= -gL
\end{aligned}$$

This means that gL is anti-symmetric

$$L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & L_{12} & 0 & L_{23} \\ L_{03} & L_{13} & L_{23} & 0 \end{pmatrix}$$

Define six basis matrices $S_{1,2,3}$ and $K_{1,2,3}$

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

S_i is the generator of 3-dimensional rotations and K_i is the generator of 3-dimensional boosts.

$$\begin{aligned}
\hat{n} \in \mathbb{R}^3, &\quad |\hat{n}| = 1 \\
\hat{n} \cdot \boldsymbol{S} &= n_1 S_1 + n_2 S_2 + n_3 S_3 \\
(\hat{n} \cdot \boldsymbol{S})^3 &= -\hat{n} \cdot \boldsymbol{S} \\
(\hat{n} \cdot \boldsymbol{K})^3 &= +\hat{n} \cdot \boldsymbol{S}
\end{aligned}$$

In the end

$$L = -\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\zeta} \cdot \boldsymbol{K} \quad \text{with } \boldsymbol{\omega}, \boldsymbol{\zeta} \in \mathbb{R}^3 \tag{2.4.8}$$

$$\Lambda = \exp(-\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\zeta} \cdot \boldsymbol{K}) \tag{2.4.9}$$

$\boldsymbol{\omega}$ is the axis of rotation, $|\boldsymbol{\omega}|$ is then the angle of rotation.
$\tanh|\boldsymbol{\zeta}| = \beta$ and $\frac{\boldsymbol{\zeta}}{|\zeta|}$ is the direction of boost.
We now will look at concrete examples

- $\boldsymbol{\zeta} = 0,\ \boldsymbol{\omega} = w\hat{e}_z$

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  Rotational angle is ω.

- $\boldsymbol{\omega} = 0,\ \boldsymbol{\zeta} = \zeta\hat{e}_x$

$$\Lambda = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Pure general boost $\boldsymbol{\zeta}$

$$\Lambda = \exp(-\boldsymbol{\zeta}\cdot\boldsymbol{K})$$
$$\boldsymbol{\zeta} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|}\tanh^{-1}|\boldsymbol{\beta}|, \quad \hat{\beta} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|}$$
$$\Lambda = \exp\Big(-\hat{\beta}\cdot\boldsymbol{K}\tanh^{-1}(\beta)\Big)$$

2.4.3 Algebra of generators

Consider the commutation algebra of $S_{i=1,2,3}$ and $K_{i=1,2,3}$

$$\left[S_i, S_j\right] = \epsilon_{ijk}S_k \tag{2.4.10}$$
$$\left[S_i, K_j\right] = \epsilon_{ijk}K_k \tag{2.4.11}$$
$$\left[K_i, K_j\right] = -\epsilon_{ijk}S_k \tag{2.4.12}$$

The last equation causes Thomas precession in atomic physics.
Choose a different basis

$$\boldsymbol{S}_+ = \frac{1}{2}(\boldsymbol{S} + i\boldsymbol{K})\boldsymbol{S}_- = \frac{1}{2}(\boldsymbol{S} - i\boldsymbol{K}) \tag{2.4.13}$$

Then we can calculate the algebra

$$\left[S_{+,i}, S_{+,j}\right] = i\epsilon_{ijk}S_{+,k} \tag{2.4.14}$$
$$\left[S_{-,i}, S_{-,j}\right] = i\epsilon_{ijk}S_{-,k} \tag{2.4.15}$$
$$\left[S_{+,i}, S_{-,j}\right] = 0 \tag{2.4.16}$$

In other word, the algebras are decoupled. This familiar algebra is angular momentum algebra **SU**(2).

Classification by two numbers $(j_+,\ j_-)$

$$\begin{aligned} j_+ &= 0, \frac{1}{2}, 1, \dots \\ j_- &= 0, \frac{1}{2}, 1, \dots \end{aligned}$$

Dimension = $(2j_+ + 1)(2j_- + 1)$.

A field is scalar field if $j_+ = j_- = 0$. One fundamental example of scalar field is Higgs boson. Other scalar particles are just bound states.

There are two possible states with spin $\frac{1}{2}$: $(j_+, j_-) = (\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$

$$\begin{aligned} \boldsymbol{S}_+ &= \frac{1}{2}\boldsymbol{\sigma} \\ \boldsymbol{S}_- &= 0 \\ \boldsymbol{S} &= \frac{1}{2}\boldsymbol{\sigma} \\ i\boldsymbol{K} &= \frac{1}{2}\boldsymbol{\sigma} \end{aligned}$$

for $(0, \frac{1}{2})$ there is "-"

So two types of spin $\frac{1}{2}$ fermions

$$\begin{aligned} \left(\frac{1}{2}, 0\right) &\to e^-_L \\ \left(0, \frac{1}{2}\right) &\to e^-_R \end{aligned}$$

They have different transformation law. W boson only to e^-_L but photon couples to both.

Under parity transformation $e^-_L \leftrightarrow e^-_R$. Both particles are needed for theory to be invariant under parity transformation, like EM and strong interactions.

3 Relativistic Quantum Field Theory

In non-relativistic quantum mechanics

$$
\begin{aligned}
E &= \frac{\boldsymbol{p}^2}{2m} \\
E &\mapsto i\hbar\frac{\partial}{\partial t} \\
\boldsymbol{p} &\to -i\hbar\boldsymbol{\nabla}
\end{aligned}
$$

After promoting the momentum and energy into operators in dispersion relation we have the Schrödinger equation

$$
i\frac{\partial}{\partial t}\psi + \frac{1}{2m}\boldsymbol{\nabla}^2\psi = 0 \tag{3.0.1}
$$

Density of probability is defined via

$$
\rho = |\psi|^2 = \psi\psi^* \tag{3.0.2}
$$

It obeys the continuity equation

$$
\begin{aligned}
-\frac{\partial}{\partial t}\int_V \rho \,\mathrm{d}V &= \int \boldsymbol{j}\cdot\boldsymbol{n}\,\mathrm{d}S \\
&= \int_V \boldsymbol{\nabla}\cdot\boldsymbol{j}\,\mathrm{d}V \\
\Rightarrow \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} &= 0
\end{aligned} \tag{3.0.3}
$$

Writing this explicitly

$$
\begin{aligned}
\frac{\partial\rho}{\partial t} &= \frac{\partial}{\partial t}(\psi\psi^*) \\
&= \psi\frac{\partial\psi^*}{\partial t} + \psi^*\frac{\partial\psi}{\partial t} \\
&= \frac{i}{2m}\left(\psi^*\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\psi\right) \\
\Rightarrow \boldsymbol{j} &= -\frac{i}{2m}\left(\psi^*\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\psi\right)
\end{aligned} \tag{3.0.4}
$$

If we have a plane wave state, as an example

$$
\begin{aligned}
\psi &= N\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}-iEt} \\
\boldsymbol{j} &= \frac{\boldsymbol{p}}{m}|N|^2
\end{aligned}
$$

3.1 Relativistic wave equation

Now we enter the relativistic regime

$$E^2 = \boldsymbol{p}^2 + m^2$$
$$p^\mu = (E, \boldsymbol{p}) \quad p_\mu = (E, -\boldsymbol{p})$$
$$p^2 = m^2$$

Promoting energy and momentum into operators

$$p^\mu \mapsto i\partial^\mu$$
$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial^2 t} - \nabla^2$$

We have then Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\phi(\boldsymbol{x}, t) = 0 \tag{3.1.1}$$

The current in KG-theory is conserved as well

$$j^\mu = (\rho, \boldsymbol{j}) = i\left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right) \tag{3.1.2}$$
$$\partial_\mu j^\mu = 0 \tag{3.1.3}$$

An example solution

$$\phi = N\mathrm{e}^{-ip\cdot x}$$
$$j^\mu = 2p^\mu |N|^2$$

In terms of Lorentz transformation

$$\rho \sim E$$

Energies of particles

$$E^2 = \boldsymbol{p}^2 + m^2$$
$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2}$$

It also implies negative probability

$$E > 0 \mapsto \rho > 0$$
$$E < 0 \mapsto \rho < 0$$

3.2 Feynman-Stückelberg Interpretation of negative energy states

"Electron" with $E, \boldsymbol{p}$ and charge $-e$

$$j^\mu_{e^-} = 2e|N|^2(E, \boldsymbol{p})$$

"Positron" with $E, \boldsymbol{p}$ and charge $+e$

$$j^\mu_{e^+} = 2e|N|^2(E, \boldsymbol{p}) = -2e|N|^2(-E, -\boldsymbol{p})$$

We can think of $E < 0$ solution as particle flying backwards in time or $E > 0$ anti-particle forwards in time.

In a relativistic systems we need to remember following points

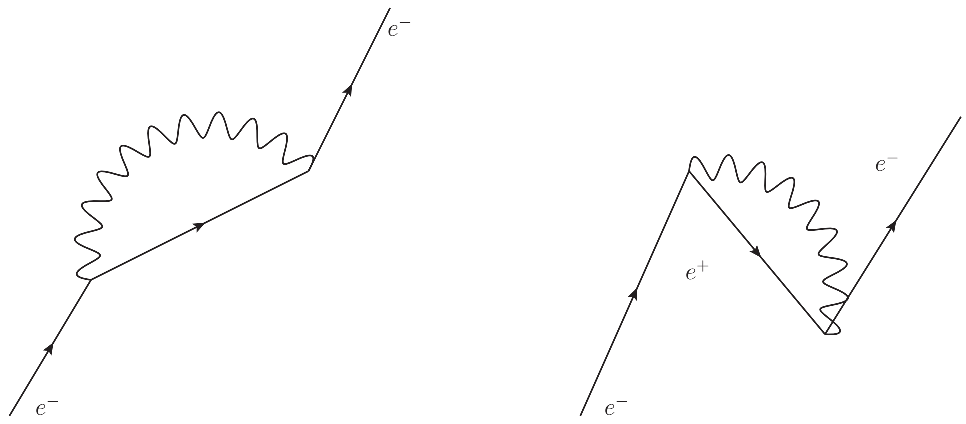


Figure 3.1: scattering process; horizontal time-axis; in the second diagram a electron positron pair is produced

- anti-particles
- particle numbers are not conserved

3.3 Electrodynamics (spin 1)

Maxwell equations are

$$\boldsymbol{E} = -\vec{\nabla}\phi - \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A} \tag{3.3.1}$$

$$\boldsymbol{B} = \vec{\nabla} \times \boldsymbol{A} \tag{3.3.2}$$

$$\vec{\nabla} \times \boldsymbol{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B} \tag{3.3.3}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{3.3.4}$$

Field strength tensor and four-potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.3.5}$$

$$A^\mu(x) = (\phi, \boldsymbol{A}) \tag{3.3.6}$$

The fields can be calculated from it

$$E^i = F^{0i} = \partial^i A^0 - \partial^0 A^i \tag{3.3.7}$$

$$B_i = -\epsilon_{ijk}\partial^i A^k = -\epsilon_{0ijk}F^{jk} \tag{3.3.8}$$

Often it is useful to use the dual tensor

$$\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\sigma\tau}F^{\sigma\tau} \tag{3.3.9}$$

$$\partial^\mu \tilde{F}_{\mu\nu} = 0 \tag{3.3.10}$$

is the second set of maxwell equations.

The other set of two equations is

$$\partial_\nu F^{\mu\nu} = 4\pi j^\mu \tag{3.3.11}$$

$\boldsymbol{E}, \boldsymbol{B}$ are observable, $\boldsymbol{A}$ is not. A^μ is not uniquely fixed by $\boldsymbol{E}$ and $\boldsymbol{B}$. It has the following gauge symmetry

$$\tilde{A}_\mu = A_\mu + \partial_\mu \Lambda(\boldsymbol{x}, t) \tag{3.3.12}$$

Use this transformation to get

$$\partial_\mu A^\mu = 0 \tag{3.3.13}$$

Plugging it back then we have the relativistic wave equation

$$\partial_\mu \partial^\mu A^\nu = 0 \tag{3.3.14}$$

it essentially is Klein-Gordon equation with mass $m = 0$

A^μ is a vector with spin 1

$$(j_+, j_-) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

It implied it has two transverse degrees of freedom. It has spin 1 properties: $+1, 0, -1$, in which 0 mode does not exist.

3.4 Description of Fermions

Original motivation for Dirac. He wants a linear equation in E or $\frac{\partial}{\partial t}$

$$p^\mu \mapsto i\partial^\mu$$

Take the ansatz

$$\begin{aligned} i\hbar \frac{\partial}{\partial t}\psi &= H\psi \\ &= (\vec{\alpha} \cdot \boldsymbol{p} + \beta m)\psi \end{aligned}$$

but $\boldsymbol{\alpha}$ and β unknown. It still has to obey the relativistic energy relation

$$\begin{aligned} A &= (\alpha_i p_i + \beta m)(\alpha_i p_i + \beta m) \\ &\stackrel{!}{=} \boldsymbol{p}^2 + m^2 \\ &= \alpha_i \alpha_j p_i p_j + \beta^2 m^2 + \alpha_i \beta p_i m + \beta \alpha_j p_j m \end{aligned}$$

From this we demand

$$\beta^2 = 1 \tag{3.4.1}$$
$$\alpha_i^2 = 1 \tag{3.4.2}$$
$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \tag{3.4.3}$$
$$\alpha_i \beta + \beta \alpha_i = 0 \tag{3.4.4}$$

So α and β are not just numbers, but (can be proven to be) hermitian traceless matrices with eigenvalue ± 1. In addition, it only exits in even dimensions. Since α_i and β are 4×4 matrices. ψ has to be a 4-component spinor.

For parity conservation need $(\frac{1}{2}, 0) \bigoplus (0, \frac{1}{2})$ Thus

$$\begin{pmatrix} \begin{pmatrix} & \end{pmatrix}_{2\times 2} & \\ & \begin{pmatrix} & \end{pmatrix}_{2\times 2} \end{pmatrix}$$

There are different sets of α_i, β which satisfy the conditions. They are called representations.

Dirac-Pauli representation

$$\alpha_i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{3.4.5}$$

$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \tag{3.4.6}$$

with σ^i the Pauli matrices.

Weyl (chiral) representation

$$\alpha^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{3.4.7}$$

$$\beta = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \tag{3.4.8}$$

they are mainly used in high energy physics ($\mathrm{E} \gg m$).

3.4.1 Gamma Matrices

We now define 4 gamma matrices γ^μ, $\mu = 0, 1, 2, 3$

$$\gamma^\mu = (\beta, \beta\boldsymbol{\alpha}) \tag{3.4.9}$$

Note that having an index does not make it Lorentz vector.

The Clifford algebra is defined as following

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{3.4.10}$$

In Dirac-Pauli representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{3.4.11}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{3.4.12}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{3.4.13}$$

In Weyl representation

$$\gamma^0 \leftrightarrow \gamma^5$$

Rewriting the Dirac equation using γs

$$\begin{aligned}
i\partial_t\psi &= (\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m)\,\psi\\
i\partial_t\psi &= -i\boldsymbol{\alpha}\cdot\vec{\nabla}\psi+m\beta\psi\\
i\beta\partial_t\psi &= -i\beta\boldsymbol{\alpha}\cdot\vec{\nabla}\psi+m\psi\\
\left(i\gamma^\mu\partial_\mu-m\right)\psi &= 0
\end{aligned}\tag{3.4.14}$$

Conventionally we use ϕ for spin 0 particle and A_μ for spin 1.

It is convenient to also have an equation for $\psi^\dagger$. First one can show $\gamma^{\dagger\mu}=\gamma^0\gamma^\mu\gamma^0$.

- $\mu=0$: $\gamma^0=\beta$ and $\gamma^{\dagger\,0}=\gamma^0\gamma^0\gamma^0\Rightarrow\beta^2=\mathbb{1}_4$
- $\gamma^{\dagger\mu}=(\beta\alpha^k)^\dagger=(\alpha^k)^\dagger\beta^\dagger=\alpha^k\beta=\beta^2\alpha^k\beta=\beta\gamma^k\beta=\gamma^0\gamma^k\gamma^0$

$$\begin{aligned}
i\gamma^0\partial_0\psi+i\gamma^k\partial_k\partial-m\psi &= 0\\
-i\partial_0\psi^\dagger(\gamma^0)^\dagger-i(\partial_k\psi^\dagger)\gamma^{\dagger\,k}-m\psi^\dagger &= 0\\
-i\partial_0\psi^\dagger\gamma^0-i(\partial_k\psi^\dagger)\gamma^0\gamma^k\gamma^0-m\psi^\dagger &= 0
\end{aligned}$$

define $\bar{\psi}=\psi^\dagger\gamma^0$

$$\begin{aligned}
-i\partial_0\bar{\psi}\gamma^0-i\partial_\mu\bar{\psi}\gamma^\mu-m\bar{\psi} &= 0\\
i(\partial_\mu\bar{\psi})\gamma^\mu+m\bar{\psi} &= 0
\end{aligned}\tag{3.4.15}$$

3.4.2 Free Particle Solution to Dirac Equation

$$\left(i\gamma^\mu\partial_\mu-m\right)\psi=0$$

multiplying $\gamma^\nu\partial_\nu$ from left

$$\begin{aligned}
i\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\psi-m\gamma^\nu\partial_\nu\psi &= 0\\
i\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\psi+im^2\psi &= 0
\end{aligned}$$

$$\begin{aligned}
\gamma^\mu\gamma^\nu &= \frac{1}{2}\left(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\right)\\
&= \frac{1}{2}\left(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu+2g^{\mu\nu}\right)\\
&= \frac{1}{2}\left[\gamma^\mu,\gamma^\nu\right]+g^{\mu\nu}
\end{aligned}$$

The commutator is anti-symmetric and multiplying to symmetric tensor (derivatives) the term must vanish.

Each component of spinor satisfies the Klein-Gordon equation.

$$(\partial_\mu\partial^\mu+m^2)\psi_i=0\tag{3.4.16}$$

Thus we can write the solution as plane-wave

$$\psi=u(\boldsymbol{p})\mathrm{e}^{-ipx}\tag{3.4.17}$$

$u(\boldsymbol{p})$ is also a 4-component object but as function $\boldsymbol{p}$ not $\boldsymbol{x}$

Insert back into Dirac equation, then we have Dirac equation in momentum space

$$\left(\gamma^{\mu} p_{\mu} - m\right) u(\boldsymbol{p}) = 0 \tag{3.4.18}$$

Solution by considering Dirac-Pauli representation

$$\left(\not{p} - m\right) u(\boldsymbol{p}) = \begin{pmatrix} (E-m)\mathbb{1} & -\boldsymbol{p}\cdot\boldsymbol{\sigma} \\ \boldsymbol{p}\cdot\boldsymbol{\sigma} & -(E+m)\mathbb{1} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$\boldsymbol{p} = 0$ then $E = \pm m$

- $E = +m$ Two solutions

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $E = -m$

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\boldsymbol{p} = 0$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} u_B = (E-m) u_A \tag{3.4.19}$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{p} u_A = (E+m) u_B \tag{3.4.20}$$

- $E > 0$

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Ansatz $u_A^{(s)} = \chi^{(s)}$

$$u_B^{(s)} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} \quad u_A^{(s)} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)}$$
$$u(\boldsymbol{p}) = N \begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \end{pmatrix} \tag{3.4.21}$$

$E < 0$ and $u_B^{(s)} = \chi^{(s)}$

$$u(\boldsymbol{p}) = N \begin{pmatrix} -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \tag{3.4.22}$$

One can show $u^{\dagger(r)} u^{(s)} = N^2 \delta^{rs}$

Two fold degeneracy in each case. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $E > 0$ and $E < 0$. There must be another observable which commutes with H and $\boldsymbol{p}$.

$$H = \gamma^i p_i + \gamma^0 m$$

$$\boldsymbol{S} \cdot \hat{P} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} \cdot \boldsymbol{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{p} \end{pmatrix} \tag{3.4.23}$$

Helicity

$$\frac{1}{2} \boldsymbol{\sigma} \cdot \hat{p} = \frac{1}{2} \begin{pmatrix} \hat{p}_z & \hat{p}_x + i\hat{p}_y \\ \hat{p}_x - i\hat{p}_y & -\hat{p}_z \end{pmatrix} \tag{3.4.24}$$

$$\det(\boldsymbol{\sigma} \cdot \hat{p}) = -\hat{p}^2 = -1 \tag{3.4.25}$$

Determinant is the product of two eigenvalues, then

$$\begin{aligned} \lambda_1 + \lambda_2 &= 0 \\ \lambda_1 \cdot \lambda_2 &= 1 \\ \lambda_{1,2} &= \pm 1 \end{aligned}$$

Antiparticle solution $u^{(3,4)}(-\boldsymbol{p})e^{-i(-p)x} = v^{(2,1)}$

$$\left(\not{p} + m\right) v(\boldsymbol{p}) = 0 \tag{3.4.26}$$

Normalization is

$$\int \rho \, \mathrm{d}V = 2E \tag{3.4.27}$$

$$N = \sqrt{E + m} \tag{3.4.28}$$

Completeness relation (spin sums)

$$\sum u^{(s)}(p) \bar{u}^{(s)}(p) = \left(\not{p} + m\right) \tag{3.4.29}$$

$$\sum v^{(s)}(p) \bar{v}^{(s)}(p) = \left(\not{p} - m\right) \tag{3.4.30}$$

Define a projector projecting out positive and negative energy states

$$\Lambda_{\pm} = \frac{\pm \not{p} + m}{2m} \tag{3.4.31}$$

In Chiral (Weyl) representation

$$\left(\not{p} - m\right) u(\boldsymbol{p}) = \begin{pmatrix} m & p \cdot \sigma \\ \boldsymbol{k} \cdot \bar{\sigma} & -m \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix} \tag{3.4.32}$$

$\bar{\sigma} = (\sigma^0, -\boldsymbol{\sigma})$ and $\sigma^0 = \mathbb{1}_2$

Weyl equation

$$\begin{aligned} -mu_L + p \cdot \sigma u_R &= 0 \\ p \cdot \bar{\sigma} u_L - mu_R &= 0 \end{aligned} \tag{3.4.33}$$

if $m = 0$, the equations decouple from each other.

We want to construct Lagrangians involving the fields ψ, A_μ and ψ. The reason we choose Lagrangians as opposed to Hamiltonians, is that Hamiltonian H is associated with energy of system, which is not Lorentz (or relativistic) invariant. But Lagrangian density is $S = \int \mathrm{d}t = \int \mathrm{d}^4 x \, \mathcal{L}$. In natural unit, the action is dimensionless. One can in addition prove d^4x is Lorentz invariant. The fact that $\mathcal{L}$ is Lorentz invariant, means that $\mathcal{L}$ need to have at least 2 spin-$\frac{1}{2}$ fields ψ or $\bar{\psi}$, to make a spin-0.

We are interested in the objects like

$$\bar{\psi}(\gamma^\mu)\psi$$

Representation of the Lorentz group can be split into $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$. In Weyl or chiral representation, ψ_L has $\left(\frac{1}{2},0\right)$ and ψ_R $\left(0,\frac{1}{2}\right)$.

Here we are taking a different approach from the paper [1]. First define

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right] \tag{3.4.34}$$

We want to find out the transformation of $\psi = u(\boldsymbol{p})e^{-ip\cdot x}$. Note that the second factor is written in covariant form, i.e. Lorentz invairant. Consider the Dirac equation in two different frames $x' = \Lambda x$

$$i\gamma^\mu\frac{\partial\psi(x)}{\partial x^\mu} - m\psi(x) = 0 \tag{3.4.35}$$

$$i\gamma^\mu\frac{\partial\psi'(x')}{\partial x'^\mu} - m\psi'(x') = 0 \tag{3.4.36}$$

We make the ansatz that the spinor transforms with S

$$\psi'(x') = S\psi(x) \tag{3.4.37}$$

S must be independent of x. Plug 3.4.37 into 3.4.36

$$\begin{aligned} i\gamma^\mu\frac{\partial}{\partial x'^\mu}\left[S\psi(x)\right] - mS\psi(x) &= 0 \\ iS^{-1}\gamma^\mu S\frac{\partial\psi(x)}{\partial x^\nu}\underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{=\Lambda^{-1}} - m\psi(x) &= 0 \end{aligned}$$

This equation must be the same as 3.4.35, then we get

$$S^{-1}\gamma^\mu S\left[\Lambda^{-1}\right]^\nu_{\ \mu} = \gamma^\nu$$

$$S^{-1}\gamma^\mu S = \Lambda^\mu_{\ \nu}\gamma^\nu \tag{3.4.38}$$

$S_{\mu\nu}$ is anti-symmetric in μ and ν

$$S_i = \frac{1}{2}\epsilon_{ijk}S^{jk} \tag{3.4.39}$$

$$K_i = S^{0i} \tag{3.4.40}$$

$$\psi'_\alpha(x') = \left[\exp\left(-\frac{i}{2}\Theta^{\mu\nu}S_{\mu\nu}\right)\right]_\alpha^{\ \beta}\psi_\beta \tag{3.4.41}$$

$$\omega_i = \frac{1}{2}\epsilon_{ijk}\Theta_{ijk} \tag{3.4.42}$$

Lorentz transformation containing rotations and boosts, but infinitesimal version (infinitesimally different from $\mathbb{1}$)

$$\Lambda^\nu_{\ \mu} = \delta^\nu_{\ \mu} + \epsilon^\nu_{\ \mu} \tag{3.4.43}$$

We will show in the exercise the expression of transformation under boosts and rotations

$$S_L = \mathbb{1}_4 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} \tag{3.4.44}$$

$$S_L^{-1} = \mathbb{1}_4 + \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} \tag{3.4.45}$$

It satisfies $S^{-1}\gamma^\mu S = \Lambda^\mu_{\ \nu}\gamma^\nu = \gamma^\nu + \epsilon^\mu_{\ \nu}\gamma^\nu$

$$\left(\mathbb{1}_4 + \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}\right)\gamma^\mu\left(\mathbb{1}_4 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}\right) = \gamma^\mu + \frac{i}{4}\left[\sigma_{\alpha\beta}\gamma^\mu - \gamma^\mu\sigma_{\alpha\beta}\right]\epsilon^{\alpha\beta}$$

somehow

$$= \gamma^\mu + \epsilon^\mu_{\ \nu}\gamma^\nu$$

Parity transformation cannot be written in infinitesimal form. Thus it is often called (one of) discrete transformation.

$$\Lambda_{\mathrm{P}} = \mathrm{diag}(+1, -1, -1, -1) \tag{3.4.46}$$

Parity symmetry is naturally violated. Especially in weak interaction, since neutrinos are left-handed ν_L.

$$S_{\mathrm{P}}^{-1}\gamma^\mu S_{\mathrm{P}} = \Lambda^\mu_{\mathrm{P}\,\nu}\gamma^\nu \tag{3.4.47}$$

- For $\mu = 0$

$$S_{\mathrm{P}}^{-1}\gamma^0 S_{\mathrm{P}} = \gamma^0$$

- For $\mu = i$

$$S_{\mathrm{P}}^{-1}\gamma^i S_{\mathrm{P}} = -\gamma^i$$

It is easy to show $S_{\mathrm{P}} = \gamma^0$ satisfies the equations. Then the spinor transform under parity like

$$\psi'(t, -\boldsymbol{x}) = \gamma^0\psi(t, \boldsymbol{x}) \tag{3.4.48}$$

In Dirac-Pauli representation $\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad \psi' = \begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_3 \\ -\psi_4 \end{pmatrix}$$

In chiral representation $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$ and

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad \psi' = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

Recall $\bar{\psi} = \psi^\dagger\gamma^0$.

$$\bar{\psi}' = \psi'^\dagger\gamma^0 = \psi^\dagger S^\dagger\gamma^0$$

it can easily be shown using the explicit expresseion of S_L and S_L. Parity transformation is γ^0.

$$= \psi^\dagger\gamma^0 S^{-1}$$
$$= \bar{\psi} S^{-1}$$

There are 16 different bilinear $\bar{\psi}A\psi$

- $A = \mathbb{1}_4$

$$\bar{\psi}'\psi' = (\bar{\psi}S^{-1})(S\psi) = \bar{\psi}\psi$$

- $A = \gamma^\mu$

$$\bar{\psi}'\gamma^\mu\psi' = \bar{\psi}S^{-1}\gamma^\mu S\psi = \Lambda^\mu_{\ \nu}(\bar{\psi}\gamma^\nu\psi)$$

It transforms exactly like a Lorentz vector. Especially under parity

$$\bar{\psi}'\gamma^\mu\psi' = \begin{cases} \bar{\psi}'\gamma^0\psi' & \mu = 0 \\ -\bar{\psi}'\gamma^i\psi' & \mu = 1, 2, 3 \end{cases}$$

- $A = \gamma^5$

$$\bar{\psi}'\gamma^5\psi' = \bar{\psi}S_L^{-1}\gamma^5 S_L\psi$$

$\left\{\gamma^\mu, \gamma^5\right\} = 0$ and $\sigma_{\mu\nu}$ contains two γs

$$= \bar{\psi}\gamma^5\psi$$

It is a scalar under boosts and rotations. Under parity it is pseudoscalar, like pions.

$$\gamma^5 S_\mathrm{P} = \gamma^5\gamma^0 = -S_\mathrm{P}\gamma^5$$

thus

$$\bar{\psi}'\gamma^5\psi' = -\bar{\psi}'\gamma^5\psi'$$

For experiments, look it up in Introduction to HEP, Perkins

Chiral spinors (in chiral representation)

$$P_R = \frac{1}{2}\left(\mathbb{1}_4 + \gamma^5\right) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{3.4.49}$$

$$P_L = \frac{1}{2}\left(\mathbb{1}_4 - \gamma^5\right) = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.4.50}$$

If ψ written like $\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$, it is in chiral representation

$$P_L\psi = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} = \psi_L$$
$$P_R\psi = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix} = \psi_R$$

They are projectors

$$P_R^2 = \frac{1}{4}(\mathbb{1}_4 + \gamma^5)(\mathbb{1}_4 + \gamma^5) = \frac{1}{4}(\mathbb{1}_4 + 2\gamma^5 + (\gamma^5)^2) = P_R$$
$$P_L^2 = P_L$$

They also project onto complete space meaning $P_L + P_R = \mathbb{1}_4$.

Left-handed particles have spin and momentum in the opposite direction and right-handed in the same direction. In Dirac-Pauli representation in high energy limit

$$\gamma^5 u^{(5)} \approx \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{p} \end{pmatrix}$$

To show this take the free particle solution ($E > 0$)

$$u^{(s)} = N \begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E+m} \chi^{(s)} \end{pmatrix}$$

$$\gamma^5 u^{(s)} = N \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E+m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \approx N \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{p} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

In general $(\boldsymbol{\sigma} \cdot \boldsymbol{a})^2 = \boldsymbol{a}^2 \mathbb{1}$

$$= N\boldsymbol{\sigma} \cdot \hat{p} \begin{pmatrix} \chi^{(s)} \\ \boldsymbol{\sigma} \cdot \hat{p} \chi^{(s)} \end{pmatrix} = \underbrace{\boldsymbol{\sigma} \cdot \hat{p}}_{\text{helicity operator}} u^{(s)}$$

At high energy limit ($E \gg m$), γ^5 helicity operator, but not at low energy. Chirality $\psi_{L,R}$ is always a good quantum number.

Angular momentum Must be another observable which commutes with H and P (u = u($\mathbf{p}$))

$$\boldsymbol{\Sigma} \cdot \hat{p} = \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{p} \end{pmatrix} \tag{3.4.51}$$

$$H = \alpha \cdot P + \beta m \tag{3.4.52}$$

$$[H, \boldsymbol{\Sigma} \cdot \hat{p}] = 0 \tag{3.4.53}$$

Angular momentum operator is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{P} \tag{3.4.54}$$

$$\tag{3.4.55}$$

Use the relation $\left[\hat{x}_i, \hat{P}_j\right] = i\delta_{ij}$ and check the commutation

$$\begin{aligned} [H, L_1] &= [\boldsymbol{\alpha} \cdot \boldsymbol{P}, x_2 P_3 - x_3 P_2] \\ &= [\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3, x_2 P_3 - x_3 P_2] \\ &= \alpha_2 \left[P_2, x_2\right] P_3 - \alpha_3 \left[P_3, x_3\right] P_2 \\ &= -i \left(\alpha_2 P_3 - \alpha_3 P_2\right) \\ &= -i(\alpha \times P)_1 \end{aligned}$$

Thus $[H, \boldsymbol{L}] = -i(\boldsymbol{\alpha} \times P) \neq 0$. In other word, L not conserved.

But we observe

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{3.4.56}$$

$$[H, \boldsymbol{\Sigma}] = 2i(\boldsymbol{\alpha} \times P) \tag{3.4.57}$$

Thus define total angular momentum, second term describe the intrinsic angular momentum of particles

$$\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\boldsymbol{\Sigma} \tag{3.4.58}$$

and J is conserved.

Charge conjugation Equation describing fermions couple to electromagnetic current

$$\left[\gamma^\mu\left(i\partial_\mu + eA_\mu\right) - m\right]\psi = 0 \tag{3.4.59}$$

with e the electric charge of electron and $A_\mu(x)$ vector potential in electromagnetism.

For positron the charge is the opposite

$$\left[\gamma^\mu\left(i\partial_\mu - eA_\mu\right) - m\right]\psi^{\mathrm{C}} = 0 \tag{3.4.60}$$

ψ^{C} is the charge conjugate of ψ. We want to know the relation between $\psi \leftrightarrow \psi^{\mathrm{C}}$. Take complex conjugate of equation 3.4.59

$$\left[-(\gamma^\mu)^*\left(i\partial_\mu - eA_\mu\right) - m\right]\psi^* = 0 \tag{3.4.61}$$

We postulate a matrix C so that $\psi^{\mathrm{C}} = C\gamma^0\psi^*$ and multiply equation 3.4.61 by $C\gamma^0$ from left.

$$C\gamma^0\left[-(\gamma^\mu)^*\left(i\partial_\mu - eA_\mu\right) - m\right]\psi^* = 0$$

It must be the same as equation 3.4.60. Thus

$$\begin{aligned}
-(C\gamma^0)(\gamma^\mu)^* &= \gamma^\mu C\gamma^0 \\
-C(\gamma^\mu)^{\mathrm{T}} &= \gamma^\mu C \\
C\gamma^{\mu T}C^{-1} &= -\gamma^\mu \\
C &= i\gamma^0\gamma^2
\end{aligned} \tag{3.4.62}$$

Then we find

$$\psi^{\mathrm{C}} = C\gamma^0\psi^* = C\bar{\psi}^{\mathrm{T}} \tag{3.4.63}$$

It would be interesting to compare $P_L(\psi^{\mathrm{C}})$ and $(P_L\psi)^{\mathrm{C}}$

Continuity equation for ψ

$$\rho = \psi^\dagger\psi = \bar{\psi}\gamma^0\psi \tag{3.4.64}$$

Multiply $\bar{\psi}$ from left to dirac equation and ψ from right to conjugated Dirac equation

$$\begin{aligned}
\bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0, &\qquad i(\partial_\mu\bar{\psi})\gamma^\mu\psi + m\bar{\psi}\psi = 0 \\
\bar{\psi}\gamma^\mu(\partial_\mu\psi) + (\partial_\mu\bar{\psi})\gamma^\mu\psi &= 0 \\
\partial_\mu(\bar{\psi}\gamma^\mu\psi) &= 0 \\
\partial_\mu j^\mu &= 0
\end{aligned}$$

3.5 Classical Field Theory

Reminder

Lagrangian and action

$$L = L(q_i, \dot{q}_i, t) \qquad S = \int \mathrm{d}t\, L$$

Variantion principle

$$\delta S = 0$$

is equivalent to Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

Going to field theory, we transform $q(t) \to \phi(x)$ and $L \to \mathcal{L}$

$$\begin{aligned}\mathcal{L} &= \mathcal{L}\left(\phi, \frac{\partial \phi}{\partial x^\mu}, x_\mu\right)\\ S &= \int \mathrm{d}^4 x\, \mathcal{L}(\phi, \frac{\partial \phi}{\partial x^\mu}, x_\mu)\end{aligned}$$

Using the variation principle

$$\begin{aligned}\delta S &= \sum_i \frac{\mathrm{d}S}{\mathrm{d}\alpha_i}\delta\alpha_i\\ &= \sum_i \int \mathrm{d}^4 x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial \alpha_i}\delta\alpha_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial(\partial_\mu\phi)}{\partial \alpha_i}\delta\alpha_i \right\}\\ &= \sum_i \int \mathrm{d}^4 x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial \alpha_i} - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial \phi}{\partial \alpha_i}\right)\right\}\delta\alpha_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial \phi}{\partial \alpha_i}\delta\alpha_i \bigg|_{x_0}^{x_1}\end{aligned}$$

Then we have our new Euler-Lagrange equations for continuous fields

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \tag{3.5.1}$$

Conservation Laws and Noether Theorem Reminder: Under transformation $t \mapsto t' = t + \delta b$, system stays invariant if $\delta_T L = \frac{\mathrm{d}}{\mathrm{d}t}(\delta\Omega)$.

$\delta\Omega = 0$ if L invariant

$$\delta_T L = L(q, \dot{q}, t + \delta b) - L(q, \dot{q}, t) = \frac{\partial L}{\partial t}\delta b$$

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}\\ &= \frac{\partial L}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}\\ &= \frac{\partial L}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}}\dot{q}\right]\end{aligned}$$

System invariant if

$$\frac{\partial L}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}(\delta\Omega) \tag{3.5.2}$$

if L not dependent explicitly on t, $\mathrm{d}L\,/\,\mathrm{d}t = 0$

If $\partial L/\partial t = 0$, the conserved quantity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L - \frac{\partial L}{\partial \dot{q}}\dot{q}\right] = 0$$
$$L - \frac{\partial L}{\partial \dot{q}}\dot{q} = -H = L - \dot{q}p$$

using legendre transformation and $\partial L/\partial \dot{q}$ the general momentum.

Analogously for fields

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\mu} &= \left(\frac{\partial \mathcal{L}}{\partial \phi}\right)\frac{\partial \phi}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\partial_\rho(\partial_\mu \phi) \\
&= \left(\partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\right)\partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\partial_\rho \partial_\mu \phi \\
\partial_\mu \mathcal{L} &= \partial_\rho\left[\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\partial_\mu \phi\right] \\
&\Rightarrow \partial_\rho\left[\mathcal{L}\delta^\rho_\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\partial_\mu \phi\right] = 0
\end{aligned}$$

this is energy momentum tensor T^ρ_μ.

Examples Real scalar field, $\phi(x) \in \mathbb{R}$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \tag{3.5.3}$$

Since $L = T - V$, we interpret first term as kinetic and second as potential, which has minimum at $\phi = 0$ Since action is dimensionless, $[\phi] = 1$. Take Euler-Lagrange

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

put together

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0$$

This is just Klein-Gordon equation. In Feynman rules we have the propagator and mass term as interaction.

One can also have complex scalar field $\phi(x) \in \mathbb{C}$ and ϕ, ϕ^* are independent fields.

Example Fermionic field

$$\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi \tag{3.5.4}$$

$[\psi] = \frac{3}{2}$. $\bar{\psi}$ and ψ are independent fields. Solve Euler-Lagrange we get familiar Dirac equation.

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} i\gamma_\mu \delta^\mu \psi - m\psi = 0 \tag{3.5.5}$$

Example Spin 1 field $A_\mu(x)$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \tag{3.5.6}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.5.7}$$

Equation of motion can be computed

$$\frac{\partial\mathcal{L}}{\partial(\partial_\rho A_\sigma)} = \frac{1}{4}4\delta_{\rho\mu}\delta_{\nu\sigma}F^{\mu\nu} = -F^{\rho\sigma}$$

$$\frac{\partial\mathcal{L}}{\partial A_\sigma} = -j^\sigma$$

then we have the maxwell equations

$$-\partial_\rho F^{\rho\sigma} + j^\sigma = 0$$

Using Lorenz gauge we can get

$$\partial_\mu\partial^\mu A^\sigma = j^\sigma$$

In principle we can add a term $A_\mu A^\mu = A^2$, which is Lorentz invariant by construction.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2}m^2 A_\mu A^\mu \tag{3.5.8}$$

so each component satisfies Klein-Gordon equation for $j^\mu = 0$. Later we will see j^μ corresponds to an interaction, e.g. $e\bar{\psi}\gamma^\mu\psi$.

3.6 Gauge Theories

Hermann Weyl first considered the gauge theory (*Eichtheorie*). We start with Dirac Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$$

If we add a phase $\psi(x) \mapsto \psi'(x) = \mathrm{e}^{i\alpha}\psi(x)$ with $\alpha \in \mathbb{R}$ and constant,

$$\mathcal{L}' = i\bar{\psi}'\gamma_\mu\partial^\mu\psi' - m\bar{\psi}'\psi'$$

since α is just a number, it commutes with derivative and gamma matrices.

$$= i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$$
$$= \mathcal{L}$$

The Lagrangian is invariant under this transformation. It is a continuous symmetry and one can apply Noether theorem and compute the Noether current.

The set of transformation $\left\{\mathrm{e}^{i\alpha}\right\}_{\alpha\in\mathbb{R}}$ forms a $\mathbf{U}(1)$ (gauge) group. If the parameter does not depend on coordinates, i.e. constant, it is called global transformation.

Now consider the case $\alpha = \alpha(x)$, not constant but a continuous function. In other word, we allow a local change. The transformation of mass term is

$$\bar{\psi}'\psi' = \mathrm{e}^{-i\alpha(x)}\bar{\psi}\mathrm{e}^{i\alpha(x)}\psi$$
$$= \bar{\psi}\psi$$

Kinetic term transforms like

$$\begin{aligned}\bar{\psi}'\gamma_\mu\partial^\mu\psi' &= \mathrm{e}^{-i\alpha(x)}\bar{\psi}\gamma_\mu\partial^\mu\left(\mathrm{e}^{i\alpha(x)}\psi\right)\\ &= \bar{\psi}\gamma_\mu\left(\partial^\mu + i\partial^\mu\alpha(x)\right)\psi\end{aligned}$$

But we can still demand local invariance and we need to modify $\mathcal{L}$ by introducing a term $\bar{\psi}\gamma_\mu A^\mu(x)\psi$ in Lagrangian.

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + \bar{\psi}\gamma_\mu A^\mu\psi \tag{3.6.1}$$

We also need a transformation property for $A_\mu(x)$ to get Lagrangian invariant.

$$A^\mu \mapsto A'^\mu = A^\mu + \partial^\mu\alpha(x) \tag{3.6.2}$$

then

$$\begin{aligned}\bar{\psi}'\gamma_\mu A'^\mu\psi' &= \bar{\psi}\mathrm{e}^{-i\alpha(x)}\gamma_\mu\left(A^\mu + \partial^\mu\alpha(x)\right)\cdot\mathrm{e}^{i\alpha(x)}\psi\\ &= \bar{\psi}\gamma_\mu\left(A^\mu + \partial^\mu\alpha(x)\right)\psi\end{aligned}$$

The second term cancels the extra terms with $\partial^\mu\alpha(x)$ because of local transformation.

Introduce a charge $e\bar{\psi}\gamma_\mu A^\mu\psi = j_\mu A^\mu$

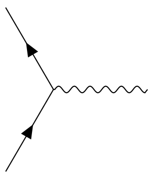
$$\begin{aligned}\mathcal{L} &= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + e\bar{\psi}\gamma_\mu A^\mu\psi\\ &= i\bar{\psi}\gamma^\mu\mathrm{D}_\mu\psi - m\bar{\psi}\psi\end{aligned} \tag{3.6.3}$$

with $\mathrm{D}_\mu = \partial_\mu - ieA_\mu$ covariant derivative and $(\mathrm{D}_\mu\psi)' = \mathrm{e}^{i\alpha(x)}\mathrm{D}_\mu\psi$.

Gauge field A_μ is introduced into Lagrangian and a kinetic term is needed to obtain a wave equation is needed

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + e\bar{\psi}\gamma_\mu A^\mu\psi \tag{3.6.4}$$

thus we have an additional Feynman rule



$$= -ie\gamma^\mu \tag{3.6.5}$$

By demanding invariance under local phase transformation, we introduce $A_\mu(x)$ and an interaction term in Lagrangian. We call A_μ a gauge field, meaning that it is not physical observable. The gauge group is an abelian group (commutative).

One can also check a mass term for gauge field $m^2A_\mu A^\mu$ is not invariant under $U(1)$ gauge. So gauge or local principle excludes photon mass term. Experimentally we are able to measure the value of ξ by looking at the electromagnetic interaction. If photon has mass, the Coulomb force has the following form

$$F \sim \frac{1}{4\pi\epsilon}\frac{q_1q_2}{r^{2+\xi}}$$

Calculation of loop diagrams like 3.2 leads to most precisely computed object, anomalous magnetic moment of electron. In fact this calculation is so accurate, people use this to define the electron charge e.

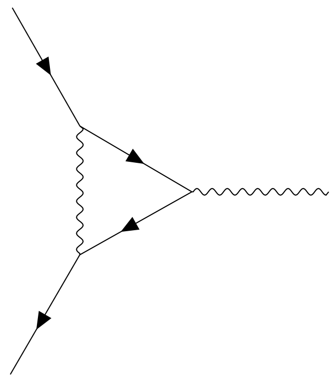


Figure 3.2: Anomalous magnetic dipole moment

3.7 Non-Abelian Gauge Theories

We will restrict ourselves in $\mathbf{SU}(N)$ groups, special unitary group of dimension N. It has N^2-1 generators. Weak interaction is described by $\mathbf{SU}(2)$ and strong interaction $\mathbf{SU}(3)$.

$$\mathcal{L}_0 = \bar{q}_j(x)(i\gamma^\mu\partial_\mu - m)q_j$$

with $j = 1, \dots, N$

$$q(x) \mapsto q'(x) = \mathrm{e}^{i\alpha_a(x)T_a}q(x)$$

with T_a a $N \times N$ matrices (independent of coordinates x) and $a = 1, \dots, N^2-1$. Here fields are quark doublet $q_i = \begin{pmatrix} u \\ d \end{pmatrix}$ and u, d are spinors.

Focus on infinitesimal transformations, $\alpha_a(x) \in \mathbb{R}$

$$\begin{aligned} q(x) &\mapsto q'(x) = [\mathbb{1}_N + i\alpha_a(x)T_a]\, q(x) \\ \partial_\mu q &\mapsto \partial_\mu [\mathbb{1}_N + i\alpha_a(x)T_a]\, q \\ &= (\mathbb{1} + i\alpha_a T_a)\, \partial_\mu q + i(\partial_\mu\alpha_a)T_a q(x) \end{aligned} \tag{3.7.1}$$

as before it is (at first) not invariant. We need N^2-1 gauge fields to compensate this, $\mathcal{G}_\mu^a$ with $a = 1, \dots, N^2-1$.

The covariant derivative is

$$\mathrm{D}_\mu = \partial_\mu + igT_a\mathcal{G}_\mu^a$$

then to construct $\mathcal{L}$ for Non-Abelian case. Following the previous recipe

$$\mathcal{L} = \bar{q}\,(i\gamma^\mu\partial_m u - m)\, q - g(\bar{q}\gamma^\mu T_a q)\mathcal{G}_\mu^a \tag{3.7.2}$$

with $g \in \mathbb{R}$ a number independent of x.

Apply transformation 3.7.1 to kinetic term and see if it is sufficient to get invariant $\mathcal{L}$

$$\begin{aligned} \bar{q}\gamma^\mu T_a q \mapsto& \bar{q}\,(\mathbb{1} - i\alpha_a T_b)\,\gamma^\mu T_a(\mathbb{1} + i\alpha_b T_b)q \\ &= \bar{q}\gamma^\mu T_a q - i\alpha_b\bar{q}T_bT_a\gamma^\mu q + i\alpha_b\bar{q}T_aT_b\gamma^\mu q + O(\alpha_b^2) \\ &= \bar{q}\gamma^\mu T_a q + i\alpha_b\bar{q}[T_a, T_b]\gamma^\mu q \end{aligned}$$

The commutator is (summation over c is implied)

$$[T_a, T_b] = if_{abc}T_c \tag{3.7.3}$$

with f structure constant. Then

$$\bar{q}\gamma^\mu T_a q \mapsto \bar{q}\gamma^\mu T_a q - f_{abc}\alpha_b \bar{q}\gamma^\mu T_c q \tag{3.7.4}$$

$\mathcal{L}$ with covariant derivative is still not invariant. It turns out that we must modify the transformation of $\mathcal{G}_\mu^a$

$$\mathcal{G}_\mu^a \mapsto \mathcal{G}_\mu^a - \frac{1}{g}\partial_\mu \alpha_a - f_{abc}\alpha_b \mathcal{G}_\mu^c \tag{3.7.5}$$

so now the Lagrangian 3.7.2 is invariant.

As before we need a kinetic term.

$$\mathcal{L}_{\mathcal{G}}^{\text{kin}} = -\frac{1}{4}\mathcal{G}_{\mu\nu}^a \mathcal{G}_a^{\mu\nu} \tag{3.7.6}$$

could it be like photon case, i.e.

$$\mathcal{G}_{\mu\nu}^a = \partial_\mu \mathcal{G}_\nu^a - \partial_\nu \mathcal{G}_\mu^a \tag{3.7.7}$$

apply this into the transformation of gauge field and see f_{abc} term doesn't drop out. Then we must modify to

$$\mathcal{G}_{\mu\nu}^a = \partial_\mu \mathcal{G}_\nu^a - \partial_\nu \mathcal{G}_\mu^a - gf_{abc}\mathcal{G}_\mu^b \mathcal{G}_\mu^c \tag{3.7.8}$$

So in Abelian case

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
A_\mu &\mapsto A_\mu - \frac{1}{e}\partial_\mu \alpha \\
F'_{\mu\nu} &= \partial_\mu \left(A_\mu + \frac{1}{e}\partial_\nu \alpha\right) - \partial_\nu \left(A_\mu + \frac{1}{e}\partial_\mu \alpha\right) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= F_{\mu\nu}
\end{aligned}$$

$F_{\mu\nu}$ is gauge invariant. This property is specific to $\mathbf{U}(1)$.

Now apply this to Non-Abelian case, you will find $G_{\mu\nu}^a$ is not gauge invariant, but $-\frac{1}{4}G_{\mu\nu}^a G_a^{\mu\nu}$ is.

Connection between $\mathcal{L}$ and Feynman rules Abelian case

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi + e\bar{\psi}\gamma^\mu \psi A^\mu$$

Kinetic term implies propagators and interaction vertex.

$$= \frac{-ig^{\mu\nu}}{p^2 + i\epsilon} \tag{3.7.9}$$

So there is no vertex of photon interacting with each other, photons rarely interfere each other. At higher order one has

In non-Abelian case gluon propagator, interaction vertex$(-ig\gamma^\mu (T^a)_{ij})$ Because of the term in gluon field with differentiation, gluon interacts with itself. Formal way of constructing $\mathcal{G}_{\mu\nu}^a$. The covariant derivative $D_\mu = \partial_\mu + igT^a \mathcal{G}_\mu^a$ and $\left[T^a, T^b\right] = if^{abc}T^c$. With some calculation

$$\begin{aligned}
\left[D_\mu, D_\nu\right] &= igT_c\left[\partial_\mu \mathcal{G}_\nu^c - \partial_\nu \mathcal{G}_\mu^c - gf_{abc}\mathcal{G}_\mu^a \mathcal{G}_\nu^b\right] \\
&= igT_c \mathcal{G}_{\mu\nu}^c
\end{aligned}$$

It works as a general construction.

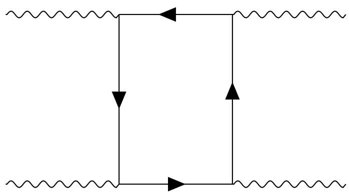


Figure 3.3: photon-photon scattering

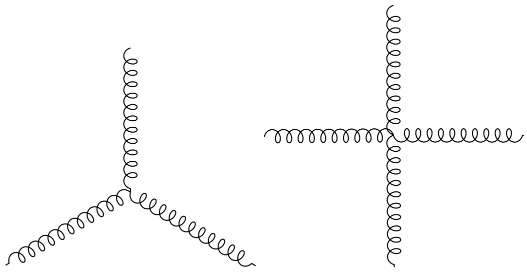


Figure 3.4: Gluon self-interactions

3.8 Spontaneous Symmetry Breaking

3.8.1 Real Field

$$\mathcal{L} = T - V = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V \tag{3.8.1}$$

$$V = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4 \tag{3.8.2}$$

It shows mirror symmetry. We are interested in minimum energy of energy or ground state.

$$E = T + V$$

Kinetic term T containts derivative, so set ϕ as a constant to minimize V. First λ must be postive, otherwise there is no minimum, i.e. unbound from below. With $\underline{\mu^2 > 0}$, we find

$$\frac{\partial V}{\partial \phi} = \phi(\mu^2 + \lambda\phi^2) = 0$$

We have minimum at $\phi = 0$. In other word $\langle\phi\rangle = 0$, vacuum expectation value (VEV) is zero. System is still mirror symmetric. The mass of the field

$$\left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi=\langle\phi\rangle} = \mu^2 \tag{3.8.3}$$

Now consider the case $\underline{\mu^2 < 0}$

$$\frac{\partial V}{\partial \phi} = \phi(\mu^2 + \lambda\phi^2) \stackrel{!}{=} 0 \tag{3.8.4}$$

there are three solutions possible

$$\phi = 0; \quad \phi = \pm v = \pm\sqrt{\frac{-\mu^2}{\lambda}} \tag{3.8.5}$$

$$\frac{\partial V}{\partial \phi} = \phi(\mu^2 + \lambda\phi^2) < 0$$

it means $\phi = 0$ is a maximum. Two minima means the vacuum is degenerate. By choosing one specific state, the mirror symmetry is spontaneous broken.

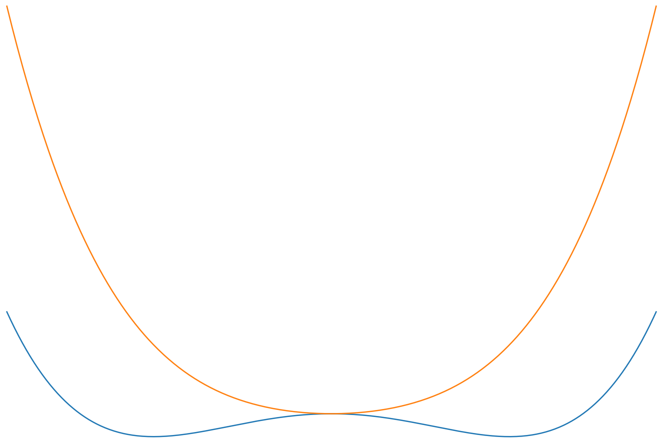


Figure 3.5: SSB for real scalar field

We choose the vacuum state to be $+v$. Split the field into two parts (it makes the physics and observable more obvious), the vacuum v and perturbation around the vacuum η

$$\begin{aligned}
\phi(x) &= v + \eta(x) \\
v = \langle \phi \rangle &= \sqrt{\frac{-\mu^2}{\lambda}} = \text{const} \\
\langle \eta(x) \rangle &= 0 \\
\partial_\mu \phi &= \partial_\mu \eta
\end{aligned}$$

The kinetic term turns into

$$\begin{aligned}
(\partial_\mu \phi)(\partial^\mu \phi) &= (\partial_\mu \eta)(\partial^\mu \eta) \\
V &= \frac{1}{2}\mu^2(v+\eta)^2 + \frac{1}{4}\lambda(v+\eta)^4 \\
&= \frac{1}{2}\mu^2 v^2 + \frac{1}{2}\mu^2\eta^2 + \mu^2 v\eta + \frac{1}{4}\lambda\left[v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4\right] \\
&= \frac{1}{2}\mu^2 v^2 + \frac{1}{4}\lambda v^2 + \eta\left(\mu^2 v + \lambda v^3\right) + \eta^2\left(\frac{1}{2}\mu^2 + \frac{6}{4}v^2\lambda\right) + \lambda v\eta^3 + \frac{1}{4}\lambda\eta^4
\end{aligned}$$

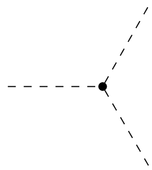
since $v = \sqrt{-\mu^2/\lambda}$, $\mu^2 v + \lambda v^3 = v(\mu^2 + \lambda v^2) = 0$. There is no longer mirror symmetry in η. Now the mass is

$$\left.\frac{\partial^2 V}{\partial \eta^2}\right|_{\eta=0} = \mu^2 + 3v^2\lambda = \mu^2 + 3\left(\frac{-\mu^2}{\lambda}\right)\lambda = -2\mu^2$$

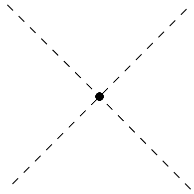
thus

$$m_\eta = \sqrt{-2\mu^2}$$

After symmetry breaking $V(\eta)$ contains η^3 and η^4 terms, thus additional Feynman rules.



$$\sim \lambda v \tag{3.8.6}$$



$$\sim \frac{1}{4} v \tag{3.8.7}$$

3.8.2 Complex Field

Now we want to consider complex scalar field $\phi(x) \in \mathbb{C}$. We can also write the fields as

$$\phi(x) = \frac{1}{\sqrt{2}} \left(\phi_1(x) + i\phi_2(x)\right) \tag{3.8.8}$$

with $\phi_1, \phi_2 \in \mathbb{R}$.

Only scalar fields can get a vacuum expectation value, otherwise it violates Lorentz invariance. Vacuum has spin and then constantly interacts with all particles.

The Lagrangian must be real, $\mathcal{L} \in \mathbb{R}$

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - \mu^2 \phi^* \phi - \lambda(\phi^* \phi)^2 \tag{3.8.9}$$

This is invariant under transformation $\phi(x) \mapsto e^{i\alpha}\phi(x), \alpha \in \mathbb{R}$. Rewrite the Lagrangian with ϕ_1 and ϕ_2

$$\mathcal{L} = \left(\partial_\mu \phi_1\right)(\partial^\mu \phi_1) + \left(\partial_\mu \phi_2\right)(\partial^\mu \phi_2) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2$$

The last two terms are $-V(\phi)$

- $\mu^2 > 0$. Minimum is at $\phi_1 = \phi_2 = 0$
- $\mu^2 < 0$. The potential has the shape of Mexican hat. We have set of minima forming a circular

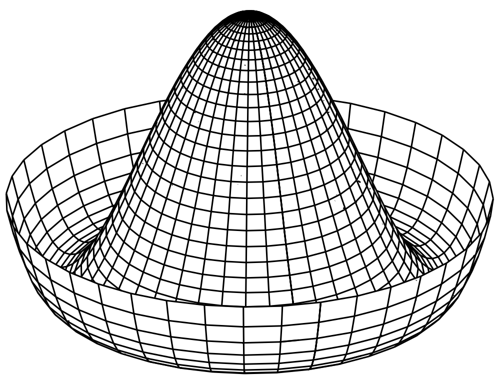


Figure 3.6: Mexican hat[2]

line $\phi_1^2 + \phi_2^2 = v^2 = -\mu^2/\lambda$. As an example $\phi_1 = v$ and $\phi_2 = 0$ and $\mathbf{U}(1)$ symmetry is spontaneous broken. The choice of vacuum breaks symmetry spontaneously. Field shift

$$\phi(x) = \frac{1}{\sqrt{2}}\Big[\underbrace{v + \eta(x)}_{\phi_1} + i\underbrace{\xi(x)}_{\phi_2}\Big] \tag{3.8.10}$$

with $\eta(x), \xi(x) \in \mathbb{R}$ and $\langle\eta\rangle = \langle\xi\rangle = 0$. Insert into $V(\phi) = V(\phi_1, \phi_2)$

$$\mathcal{L}(\eta,\xi) = \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \text{const} + \mu^2\eta^2 + (\text{cubic and quartic in } \eta, \xi) \tag{3.8.11}$$

so $m_\eta^2 = -2\mu^2$ but $m_\xi^2 = 0$

Continuous global symmetry is spontaneously broken and it leads to massless state. It is called Goldstone theorem. The massless state is the Goldstone boson.

3.8.3 Higgs Mechanism

Modify the transformation to

$$\phi(x) \mapsto e^{i\alpha(x)}\phi(x)$$

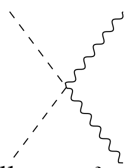
with $\phi(x) \in \mathbb{C}$ and $\alpha(x) \in \mathbb{R}$.

First step is to extend $\mathcal{L}_\phi$ such that is invariant under this transformation. Introduce a spin-1 field $A_\mu(x)$ with the covariant derivative

$$\begin{aligned} D_\mu &= \partial_\mu - ieA_\mu \\ A_\mu &\mapsto A_\mu + \frac{1}{e}\partial_\mu\alpha \end{aligned}$$

The new Lagrangian

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{3.8.12}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It contains the interaction between two ϕ and two A_μ 

We want to look at spontaneous symmetry breaking of this system. We still want $\lambda > 0$, bound from below. In the case of $\mu^2 > 0$, symmetry not broken and we have scalar QED. In particle physics we do have charged scalars, such as $\pi^\pm$, $K^\pm$ and so on. We need $\mu^2 < 0$ to have spontaneous symmetry breaking. The field shift is as before

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\xi(x)) \tag{3.8.13}$$

$$\begin{aligned} \mathcal{L}(\eta,\xi) = &\frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta^2 + \frac{1}{2}e^2v^2A_\mu A^\mu \\ &- evA_\mu\partial^\mu\xi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\text{scalar interaction temrs}) \end{aligned} \tag{3.8.14}$$

So $m_\eta = \sqrt{2\lambda v^2}$, $m_A = ev$ and $m_\xi = 0$.

$A_\mu \partial^\mu \xi$ seems to mix A and ξ. Massless spin 1 has two degrees of freedom. $A_\mu(x)$ has four components, but Lorentz and gauge invariance restrict to two. Photon has two polarizations. Massive spin 1 has three degree of freedom. We will show later on, ξ is actually the additional degree of freedom of A. The term $A_\mu A^\mu$ breaks the gauge invariance.

To see physics, the choice of η and ξ is not very good. Instead choose

$$\phi \mapsto \frac{1}{\sqrt{2}}(v + h(x))\,\mathrm{e}^{i\Theta(x)/v} \tag{3.8.15}$$

with $h(x) \in \mathbb{R}, \Theta(x) \in \mathbb{R}$. We interpret this as a gauge transformation with $\alpha = \Theta/v$. To rewrite gauge transformation of $A_\mu(x)$

$$A_\mu \mapsto A_\mu + \frac{1}{ev}\partial_\mu \Theta(x) \tag{3.8.16}$$

The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial h)^2 - \lambda v^2 h^2 + \frac{1}{2}e^2 v^2 A_\mu A^\mu - \lambda v h^3 - \frac{1}{4}\lambda h^4 + \frac{1}{2}e^2 A^2 h^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3.8.17}$$

ξ is no longer there. This is Higgs mechanism. So we have before $2+2$ degree of freedom $(\eta, \xi, A^{m=0})$ to $1+3$ degrees of freedom $(h, A^{m\neq 0})$.

Phase transition in magnetism is an example of symmetry breaking. Magnetization is temperature dependent. In the early universe the temperature is high so that $\mu^2(T) > 0$. As it cools down $\mu^2 < 0$.

3.8.4 Spontaneous Breaking of local $\mathbf{SU}(2)$ Theory

We start with ungauged, complex scalar field

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 \tag{3.8.18}$$

Now ϕ contains two complex scalar fields.

$$\phi = \begin{pmatrix} \phi_\alpha \\ \phi_\beta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{3.8.19}$$

with $\phi_{1,2,3,4} \in \mathbb{R}$.

Gauge transformation

$$\phi \mapsto \phi' = \mathrm{e}^{i\alpha_a(x)\tau_a/2}\phi \tag{3.8.20}$$

There are only three generators of $\mathbf{SU}(2)$, $a = 1, 2, 3$.

Introduce three gauge bosons W_μ^a, $a = 1, 2, 3$. Define the covariant derivative

$$D_\mu = \partial_\mu - ig\frac{\tau_a}{2}W_\mu^a \tag{3.8.21}$$

Note $T^a = \tau^a/2$.

The τ matrices are Pauli matrices

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.8.22}$$

Have gauge transformation of W_a^μ

$$W_\mu^a \mapsto W_\mu^a - \frac{1}{g}\partial_\mu \alpha^a - f^{abc}\alpha_b W_{c\mu} \tag{3.8.23}$$

Here $f_{abc} = \epsilon_{abc}$ totally symmetric tensor in three dimension.

Lagrangian becomes

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi) - \frac{1}{4}W^a_{\mu\nu}W^{a\,\mu\nu} \tag{3.8.24}$$

Kinetic term for gauge fields

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g(W_\mu \times W_\nu)^a \tag{3.8.25}$$

The potential

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \tag{3.8.26}$$

We are interested in the case $\lambda > 0$ and $\mu^2 < 0$. Minima are at

$$\phi^\dagger\phi = \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) = -\frac{\mu^2}{2\lambda}$$

We can make a choice for minimum $\phi_1 = \phi_2 = \phi_4 = 0$ and $\phi_3^2 = v^2 = -\mu^2/\lambda$

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} \tag{3.8.27}$$

with $v \in \mathbb{R}$.

Shift of ϕ field

$$\phi(x) \mapsto \mathrm{e}^{i\hat{\tau}\cdot\hat{\theta}/v}\phi(x) = \mathrm{e}^{i\hat{\tau}\cdot\hat{\theta}/v}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v + h(x)\end{pmatrix} \tag{3.8.28}$$

Infinitesimal expansion in $\theta(x)$

$$\begin{aligned}
\phi(x) &= \frac{1}{\sqrt{2}}\left(\mathbb{1} + i\hat{\tau}\cdot\hat{\theta}(x)/v\right)\begin{pmatrix}0\\ v + h(x)\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}1 + i\theta_3/v & i(\theta_1 - i\theta_2)/v \\ i(\theta_1 + i\theta_2)/v & 1 - i\theta_3/v\end{pmatrix}\begin{pmatrix}0\\ v + k(x)\end{pmatrix}
\end{aligned}$$

Look at kinetic term in ϕ

$$\begin{aligned}
\mathcal{L}^\phi_{\text{kin}} &= (D_\mu\phi)^\dagger(D^\mu\phi) \\
&= \left(\partial_\mu\phi + ig\hat{\tau}\cdot\hat{W}\phi\right)^\dagger\left(\partial^\mu\phi + ig\hat{\tau}\cdot\hat{W}\phi\right)
\end{aligned}$$

compute at minimum $\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}$

$$\begin{aligned}
\left|\frac{i}{2}g\hat{\tau}\cdot\hat{W}\phi\right|^2 &= \frac{g^2}{8}a\left|\begin{pmatrix}W^3_\mu & W^1_\mu - iW^2_\mu \\ W^1_\mu + iW^2_\mu & -W^3_\mu\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}\right|^2 \\
&= \frac{g^2v^2}{8}\left|\begin{pmatrix}W^1_\mu - iW^2_\mu \\ -W^3_\mu\end{pmatrix}\right|^2 \\
&= \frac{g^2v^2}{8}\boldsymbol{W}^2 \\
&= \frac{1}{2}M^2\boldsymbol{W}^2
\end{aligned}$$

with $M = gv/2$. We spontaneously broken $\mathbf{SU}(2)$ gauge theory. However, this is not realized in nature.

4 Standard Model

We will be looking at the gauge group $\mathbf{SU}(2)_\mathrm{L} \times \mathbf{U}(1)_\mathrm{Y}$. This will be spontaneously broken into $\mathbf{U}(1)_\mathrm{EM}$. Strong interaction will add $\mathbf{SU}(3)_\mathrm{C}$ and it doesn't get spontaneously broken.

4.1 Leptons

Reintroduce fermions

$$\psi = \begin{pmatrix} \chi_\mathrm{L} \\ \chi_\mathrm{R} \end{pmatrix}$$
$$\psi_\mathrm{L} = \begin{pmatrix} \chi_\mathrm{L} \\ 0 \end{pmatrix}$$

$\mathbf{SU}(2)_\mathrm{L}$ only interacts with ψ_L.

In Chiral representation

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{4.1.1}$$

The projection operators

$$P_L = \frac{1}{2}(\mathbb{1}_2 - \gamma^5) = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.1.2}$$
$$P_R = \frac{1}{2}(\mathbb{1}_2 + \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{4.1.3}$$

with $P_L^2 = P_L$ and $P_L + P_R = \mathbb{1}$.

$\chi_\mathrm{L,R}$ two-component Weyl spinors.

$$(\psi_\mathrm{L})^\dagger = (P_\mathrm{L}\psi)^\dagger = \psi^\dagger P_\mathrm{L}^\dagger = \psi^\dagger P_\mathrm{L}$$
$$\overline{\psi}_\mathrm{L} = (\psi_L)^\dagger \gamma^0 = \psi^\dagger P_L \gamma^0 = \overline{\psi} P_R$$

Introduce a doublet of left-handed particles

$$L = \begin{pmatrix} \nu_\mathrm{L} \\ e^-_\mathrm{L} \end{pmatrix}$$
$$\nu_\mathrm{L} = (\psi_\nu)_\mathrm{L} = \begin{pmatrix} \chi_{\nu \mathrm{L}} \\ 0 \end{pmatrix}$$
$$e^-_\mathrm{L} = (\psi_{e^-})_\mathrm{L} = \begin{pmatrix} \chi_{e^- \mathrm{L}} \\ 0 \end{pmatrix}$$
$$e_\mathrm{L} = (\psi_e)_\mathrm{L} = \frac{1}{2}(\mathbb{1} - \gamma_5)\,\psi_e$$

There is no right-handed neutrino in this theory (singlet).

$$R = e_\mathrm{R} = P_\mathrm{R}(\psi_e) = (\psi_e)_\mathrm{R}$$

Under $\mathbf{U}(1)_\mathrm{Y}$ the hypercharges are define as

$$Y(L) = -1 \tag{4.1.4}$$
$$Y(R) = -2 \tag{4.1.5}$$

Here each component of L has $Y = -1$. They are chosen so that

$$Q_\mathrm{EM} = T^3_\mathrm{L} + \frac{1}{2}Y$$

with $T^3_\mathrm{L} = \frac{1}{2}\tau^3$ a generator of $\mathbf{SU}(2)$.

$$T^3_L = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$
$$T^3_L L = \begin{pmatrix} \frac{1}{2}\nu_L \\ -\frac{1}{2}e^-_L \end{pmatrix}$$

Take a look at the examples

$$Q_\mathrm{EM}(\nu_\mathrm{L}) = T^3_\mathrm{L}(\nu_\mathrm{L}) + \frac{1}{2}Y(\nu_\mathrm{L}) = \frac{1}{2} - \frac{1}{2} = 0$$
$$Q_\mathrm{EM}(e^-_\mathrm{L}) = T^3_\mathrm{L}(e_\mathrm{L}) + \frac{1}{2}Y(e_\mathrm{L}) = -\frac{1}{2} - \frac{1}{2} = 0$$
$$Q_\mathrm{EM}(e^-_\mathrm{R}) = T^3_\mathrm{L}(e_\mathrm{R}) + \frac{1}{2}Y(e_\mathrm{R}) = 0 - 1 = -1$$

T_L only acts on left handed particles under $\mathbf{SU}(2)$

$$\psi_{e_R} = e^{i(0)}\psi_{e_R}$$
$$[T^a, Y] = 0$$

Normal kinetic term in Dirac theory

$$\mathcal{L}_\mathrm{kin} = \bar{\psi}i\gamma^\mu D_\mu(P_R + P_L)\psi$$

To split this using the anti-commutator of γ^5 and other gamma matrices, $\gamma^\mu P_L = P_R\gamma^\mu$

$$= \bar{\psi}_L i\gamma^\mu D_\mu\psi_L + \bar{\psi}_R i\gamma^\mu D_\mu\psi_R$$

Thus kinetic terms for leptons are

$$\mathcal{L}^\mathrm{kin}_\mathrm{leptons} = \bar{R}i\gamma^\mu D'_\mu R + \bar{L}i\gamma^\mu D_\mu L \tag{4.1.6}$$

$$\begin{aligned} D'_\mu &= \partial_\mu - \frac{ig'}{2}YB_\mu \\ &= \partial_\mu + ig'B_\mu \end{aligned} \tag{4.1.7}$$

$$\begin{aligned} D_\mu &= \partial_\mu - \frac{ig'}{2}YB_\mu - ig\frac{\tau^a}{2}W^a_\mu \\ &= \partial_\mu + \frac{i}{2}g'B_\mu - ig\frac{\tau^a}{2}W^a_\mu \end{aligned} \tag{4.1.8}$$

In electromagnetic $\mathbf{U}(1)_\mathrm{EM}$ the charge generator Q with $Q(e^-) = -1$. Covariant derivative

$$D_\mu = \partial_\mu - ieQA_\mu \tag{4.1.9}$$

In $\mathbf{U}(1)_\mathrm{Y}$ the gauge boson B_μ. Field strength tensor $F^{\mathrm{Y}}_{\mu\nu} = \partial_\mu B_\mu - \partial_\nu B_\mu$. Charge generator $Y/2$. Covariant derivative

$$D_\mu = \partial_\mu - \frac{ig'}{2} Y B_\mu \tag{4.1.10}$$

In Dirac Lagrangian there is a mass term $m\bar{\psi}\psi$.

$$\begin{aligned}
\mathcal{L}_m &= m\bar{\psi}\psi = m\bar{\psi}(P_R + P_L)\psi \\
&= m(\bar{\psi}P_R\psi + \bar{\psi}P_L\psi) \\
&= m(\bar{\psi}P_R^2\psi + \bar{\psi}P_L^2\psi) \\
&= m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)
\end{aligned}$$

$\bar{\psi}\psi$ mass term is called Dirac mass term. There is only left handed neutrino, so we cannot write Dirac mass term for neutrino. Although ψ_L and ψ_R transform differently under Lorentz transformation, $\bar{\psi}\psi$ is still Lorentz invariant.

Recall that $\mathbf{U}(1)_\mathrm{EM}$ in QED $\psi \mapsto \psi' = \mathrm{e}^{i\alpha(x)Q}\psi$. If Q is same for left and right-handed components, $\bar{\psi}\psi$ is gauge invariant. If left and right have different hypercharges Y, then $\bar{\psi}_R\psi_L$ not $\mathbf{U}(1)_Y$ gauge invariant.

How about $\mathbf{SU}(2)_\mathrm{L}$ gauge invariance of $\bar{\psi}_L\psi_R$? Under $\mathbf{SU}(2)$

$$\begin{aligned}
R &\mapsto R' = R \\
L &\mapsto L' = \mathrm{e}^{i\alpha_a(x)\tau^a/2} L
\end{aligned}$$

So it's obvious that $\bar{\psi}_R\psi_L$ not $\mathbf{SU}(2)_\mathrm{L}$ gauge invariant.

Any fermionic mass term vanishes, if we require $\mathbf{SU}(2)_\mathrm{L} \times \mathbf{U}(1)_\mathrm{Y}$ gauge invariance. Spontaneous symmetry breaking to let fermion gain mass.

4.2 Add scalars

Introduce a Higgs doublet

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{4.2.1}$$

The superscripts denote the charge the field carries.

From previous section $Q_\mathrm{EM} = T_L^3 + \frac{1}{2}Y$, so

$$\begin{aligned}
Q_\mathrm{EM}(\phi^+) &= +1 = \frac{1}{2} + \frac{1}{2}Y \Leftrightarrow Y = +1 \\
Q_\mathrm{EM}(\phi^0) &= 0 = -\frac{1}{2} + \frac{1}{2}Y \Leftrightarrow Y = +1
\end{aligned}$$

Together $Y(\Phi) = +1$

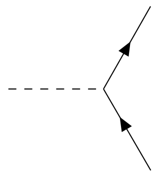
Lagrangian

$$\begin{aligned}
\mathcal{L}_\mathrm{scalar} &= (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi^\dagger\Phi) \\
D_\mu &= \partial_\mu - \frac{ig'}{2}B_\mu - \frac{ig}{2}\tau_i W_\mu^i \\
V &= \mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 \\
\Phi^\dagger &= \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}^\dagger = \begin{pmatrix} \phi^+ & \phi^0 \end{pmatrix}^* = \begin{pmatrix} \phi^- & (\phi^0)^* \end{pmatrix}
\end{aligned} \tag{4.2.2}$$

4.3 Coupling of Scalars and Fermions

Recall that simple complex scalar $\phi^0 \in \mathbb{C}$, $\mathbf{SU}(2)$ singlet. Under Lorentz transformations (per definition) $\phi^0(x) \mapsto \phi^0(x)$. $\bar{\psi}\psi$ is also Lorentz invariant. It means $\phi^0\bar{\psi}\psi$ is Lorentz invariant. This type of interaction is called Yukawa interaction.

$$\mathcal{L}_{\text{Yukawa}} = -y_e\left(\overline{R}\Phi^\dagger L + \overline{L}\Phi R\right)$$



How to combine Φ, L and R?

$$\begin{aligned} Y(\Phi) &= +1 \\ Y(L) &= -1 \\ Y(R) &= -2 \\ Y(\bar{L}) &= +1 \end{aligned}$$

Gauge transformation of interaction term

$$\phi\bar{\psi}\psi \mapsto \phi'\bar{\psi}'\psi' = \mathrm{e}^{i\alpha(Q(\phi)+Q(\bar{\psi})+Q(\psi))}\phi\bar{\psi}\psi$$

Gauge invariance means the sum of hypercharges is zero. $\Phi\bar{L}R$ is $\mathbf{U}(1)_Y$ gauge invariant.

What if we want $\mathbf{SU}(2)_{\mathrm{L}} \times \mathbf{U}(1)_{\mathrm{Y}}$ gauge invariant.

R is $\mathbf{SU}(2)$ invariant. Φ is $\underline{2}$ under $\mathbf{SU}(2)$. L and $\bar{L}$ are $\underline{2}$ under $\mathbf{SU}(2)$. For $\mathbf{SU}(2)$ these two kinds of representations are the same $\bar{\underline{2}} = \underline{2}$.

$$\Phi\bar{L} : \underline{2} \otimes \bar{\underline{2}} = \underline{3} \oplus \underline{1}_A$$

$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. $\underline{1}$ is $\mathbf{SU}(2)$ singlet. Need antisymmetric combination of $\Phi\bar{L}$ to get $\mathbf{SU}(2)$ singlet.

In components

$$\begin{aligned} \overline{L}\Phi R &= \begin{pmatrix} \overline{\nu}_L & \overline{e_L^-} \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R \\ &= \overline{\nu}_L\phi^+ e_R + \overline{e_L^-}\phi^0 e_R \end{aligned}$$

here $\sum Y_i = 0$

$$\begin{aligned} \overline{R}\Phi^+ L &= \overline{e_R^-}\begin{pmatrix} \phi^- & \left(\phi^0\right)^* \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix} \\ &= \overline{e_R^-}\phi^- \nu_L + \overline{e_R^-}\left(\phi^0\right)^* e_L^- \end{aligned}$$

The Standard Model does not contain ν_R. But let's consider ν_R anyway

$$\begin{aligned} Q_{\text{EM}} &= T_L^3 + \frac{1}{2}Y \\ 0 &= 0 + \frac{1}{2}Y(\nu_R) \end{aligned}$$

So it doesn't couple to B_μ.

How about Dirac mass term?

$$\overline{\nu}_L \nu_R + \overline{\nu}_R \nu_L$$

It is not $\mathbf{U}(1)_\mathrm{Y}$ or $\mathbf{SU}(2)_\mathrm{L}$ invariant. Not a huge problem since it was the same for electron before we introduce Higgs mechanism.

Cannot write interaction like $\overline{L}\nu_R\Phi$ since the hypercharge is not zero. Instead

$$\begin{aligned}\tilde{\Phi} &= i\tau_2\Phi^* = \begin{pmatrix}\phi^0 \\ -\phi^-\end{pmatrix} \\ Y(\tilde{\Phi}) &= -1 \\ \mathcal{L}_{\nu_L} &= \overline{L}\tilde{\Phi}\nu_R + \overline{\nu}_R\tilde{\Phi}^\dagger L\end{aligned}$$

Dirac mass term for neutrino. If ν_R exists, this is a possible mass term.

$$\begin{aligned}\overline{L}\tilde{\Phi}\nu_R &= \begin{pmatrix}\overline{\nu}_L & \overline{e_L^-}\end{pmatrix}\begin{pmatrix}(\phi^0)^* \\ -\phi^-\end{pmatrix}\nu_R \\ &= \overline{\nu}(\phi^0)^*\nu_R - \overline{e_L^-}\phi^-\nu_R\end{aligned}$$

4.4 Spontaneous Symmetry Breaking (Mass and Mixing of Gauge Bosons)*

In the early universe $\mu = \mu(T)$ and the symmetry got broken. Consider the case $\mu^2 < 0$.

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ v\end{pmatrix} \tag{4.4.1}$$

with $v \in \mathbb{R}$ and $[v] = 1$. Field ϕ^0 has $Q_\mathrm{EM} = 0$. $Y(\phi^0) \neq 0$ breaks the $\mathbf{U}(1)_Y$ and $T_L^3(\phi^0) \neq 0$ breaks $\mathbf{SU}(2)_L$.

As before the field shift is

$$\begin{aligned}\Phi &= U^{-1}(\boldsymbol{\xi})\frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ v + \eta(x)\end{pmatrix} \\ U(\boldsymbol{\xi}) &= \exp(-i\boldsymbol{\xi}\cdot\boldsymbol{\tau}/(2v))\end{aligned}$$

$\eta(x), \boldsymbol{\xi}(x) \in \mathbb{R}$. The transformation $U^{-1}(\boldsymbol{\xi})$ has three parameters and thus give Φ four degrees of freedom. It has the same form as $\mathbf{SU}(2)$ gauge transformation. Because of gauge symmetry, this transformation gets cancelled.

$$\Phi \mapsto \Phi' = U(\xi)\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ v + \eta(x)\end{pmatrix} \tag{4.4.2}$$

$$L \mapsto L' = U(\boldsymbol{\xi})L \tag{4.4.3}$$

$$W_\mu \mapsto W'_\mu \tag{4.4.4}$$

$$\boldsymbol{\tau}\cdot\boldsymbol{W}'_\mu = U(\boldsymbol{\xi})\left[\boldsymbol{\tau}\cdot\boldsymbol{W}_\mu - \frac{i}{g}U^{-1}\partial_\mu U\right] \tag{4.4.5}$$

*see also in Cheng and Li, Ch.11

Insert transformed field (4.4.2) into $\mathcal{L}$ and see the physical interpretation. Yukawa coupling

$$
\begin{aligned}
-y_e \overline{L}\Phi R &= -y_e \begin{pmatrix} \overline{\nu}_e & \overline{e_L} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+\eta(x) \end{pmatrix} e_R \\
&= -\frac{y_e}{\sqrt{2}} \overline{e_L}(v+\eta(x)) e_R
\end{aligned}
$$

Thus $m_e = y_e v/\sqrt{2}$. Knowing m_e we can fix $y_e v$. It does not predict the mass of electron, but we can accommodate. We do predict $m_\nu = 0$ though.

There is an extra term

$$
-\frac{y_e}{\sqrt{2}} \overline{e_L} e_R \eta(x)
$$

It indicates the coupling of Higgs is direct proportional to mass of fermion. We found $y_e \sim 10^{-6}$, so it is then not highly unlikely to observe Higgs decay into electrons at LHC. We have already seen Higgs decay into tau and bottom quarks.

$$
\mathcal{L}_{\text{scalar}} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + V\left(\left(\frac{v+\eta}{\sqrt{2}}\right)^2\right) \tag{4.4.6}
$$

Write

$$
\begin{aligned}
\Phi &= \frac{v+\eta}{\sqrt{2}} \chi \\
\chi &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

Multiply $(D_\mu \Phi)^\dagger (D^\mu \Phi)$ out but focus only on terms without derivative

$$
\begin{aligned}
D_\mu &= \partial_\mu - \frac{i}{2} g' B_\mu - ig \frac{\tau^i}{2} W_\mu^i \\
\mathcal{L} &\subset \frac{(v+\eta)^2}{8} \chi^\dagger \left[(g' B_\mu^\dagger + g\tau^i W_\mu^i) \cdot (g' B^\mu + g\tau^i W^{i\mu}) \right] \chi
\end{aligned}
$$

Terms with one τ^1 and τ^2 vanish

$$
\begin{aligned}
\chi^\dagger \tau^1 \chi &= 0 \\
\chi^\dagger \tau^2 \chi &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \\
\chi^\dagger \tau^3 \chi &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1
\end{aligned}
$$

Focus on $\sim v^2$ terms, they are bilinear in gauge boson fields and thus mass terms.

$$
\mathcal{L} \subset \frac{1}{8} v^2 \left[(g' B_\mu - g W_\mu^3)(g' B^\mu - g W^{3\mu}) + g^2 (W_\mu^1)^2 + g^2 (W_\mu^2)^2 \right] \tag{4.4.7}
$$

with $M_{W^1}^2 = M_{W^2}^2 = g^2 v^2/8$.

Using the familiar formula $Q = T_3 + \frac{1}{2}Y$ for the $\mathbf{SU}(2)$ gauge boson $W^{1,2,3}$. They form a $\mathbf{SU}(2)$ triplet $T_3 = \pm 1, 0$. W^3 has zero electric charge. B_μ as $\mathbf{SU}(2)$ singlet has zero electric charge and hypercharge.

Focus on the electric neutral part

$$\mathcal{L} \subset \frac{v^2}{8}\left[g'^2 B_\mu B^\mu + g^2 W^3_\mu W^{\mu 3} + g' g B_\mu W^{\mu 3} + g' g W^3_\mu B^\mu\right]$$

It has $W_\mu B^\mu$ mixing terms. Rewritten in matrix form

$$\begin{aligned} &= \frac{v^2}{8}\begin{pmatrix} B_\mu & W^3_\mu \end{pmatrix}\begin{pmatrix} g'^2 & gg' \\ gg' & g^2 \end{pmatrix}\begin{pmatrix} B^\mu \\ W^{3\mu} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} Z_\mu & A_\mu \end{pmatrix}\begin{pmatrix} M_Z^2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix} \end{aligned}$$

Using the properties of 2×2 matrices

$$\det = \lambda_1 \cdot \lambda_2 \tag{4.4.8}$$

$$\mathrm{tr} = \lambda_1 + \lambda_2 \tag{4.4.9}$$

Obviously here $\det = 0$ and $\mathrm{tr} = g^2 + g'^2$.

Eigenvectors are

$$\begin{aligned} Z_\mu &= \frac{-gW^3_\mu + g' B_\mu}{\sqrt{g^2 + g'^2}} \\ A_\mu &= \frac{gB_\mu + g' W^3_\mu}{\sqrt{g^2 + g'^2}} \end{aligned} \tag{4.4.10}$$

$$\begin{aligned} M_A &= 0 \\ M_Z &= \frac{v}{2}\sqrt{g^2 + g'^2} \end{aligned} \tag{4.4.11}$$

Remember real fields have mass term in the form $\frac{1}{2}m^2\phi^2$.

Go back to W^1_μ and W^2_μ. Define

$$W^\pm = \frac{1}{\sqrt{2}}(W^1_\mu \pm W^2_\mu) \tag{4.4.12}$$

$$W^+_\mu W^{-\mu} = \frac{1}{2}\left[(W^1_\mu)^2 + (W^2_\mu)^2\right] \tag{4.4.13}$$

$\boldsymbol{W}$ is a triplet $(W^+, W^0, W^-)^T$. The superscripts denote the hypercharge (T^3). W^3_μ has zero hypercharge and thus zero electric charge.

Mass term $\frac{1}{2}m^2(W^+_\mu W^{-\mu} + W^-_\mu W^{\mu +})$

$$\begin{aligned} \frac{1}{2}M_W^2 &= \frac{1}{8}g^2 v^2 \\ M_W &= \frac{1}{2}gv \end{aligned}$$

Go back to leptons

$$\begin{aligned} \mathcal{L}^{\text{leptons}}_{\text{kin}} &= \overline{R} i\gamma^\mu D'_\mu R + \overline{L} i\gamma^\mu D_\mu L \\ D'_\mu &= \partial_\mu + i\frac{g'}{2}B_\mu \\ D_\mu &= \partial_\mu + \frac{i}{2}g' B_\mu - ig\frac{\tau^i}{2}W^i_\mu \end{aligned}$$

Use equation (4.4.10) to rewrite $B_\mu = f(A_\mu, Z_\mu)$ and $W^3_\mu = f'(A_\mu, Z_\mu)$.

Insert that into D'_μ and D_μ and then into $\mathcal{L}^{\text{leptons}}_{\text{kin}}$. Also replace $W'^{\,2}_\mu \mapsto W^\pm_\mu$

$$\overline{L} i\gamma^\mu D_\mu L = \overline{L} i\gamma^\mu \left(\partial_\mu + \frac{i}{2} g' B_\mu - ig\frac{\tau^i}{2} W^i_\mu\right) L$$

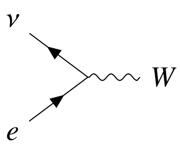
$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\tau_1 W^1_\mu + \tau^2 W^2_\mu = \begin{pmatrix} 0 & W^1_\mu - iW^2_\mu \\ W_\mu + iW^2_\mu & 0 \end{pmatrix}$$

Charged current

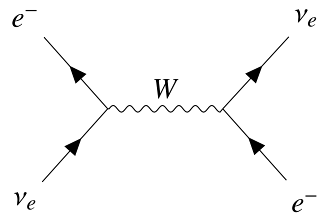
$$\begin{aligned} A &= \overline{L}\gamma^\mu \left(\tau^1 W^1_\mu + \tau^2 W^2_\mu\right) L \\ &= \frac{g}{2} \begin{pmatrix} \overline{\nu}_L & \overline{e}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} 0 & \sqrt{2} W^+_\mu \\ \sqrt{2} W^-_\mu & 0 \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\ &= \frac{g}{2} \begin{pmatrix} \overline{\nu}_L & \overline{e}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} \sqrt{2} W^+_\mu e_L \\ \sqrt{2} W^-_\mu \nu_L \end{pmatrix} \\ &= \frac{g}{\sqrt{2}} \left[\overline{\nu}_L W^+_\mu \gamma^\mu e_L + \overline{e}_L W^-_\mu \gamma^\mu \nu_L\right] \end{aligned}$$

Diagrammatically



$\sim ig\gamma^\mu P_L$

Neutrino-electron scattering has the cross section $\sim g^4/M_W^4$

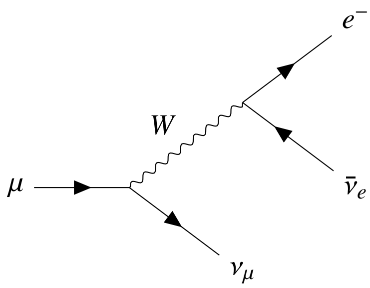


Fermi constant

$$G_F = \frac{\sqrt{2} g^2}{8 M_W^2} \tag{4.4.14}$$

$\mathcal{L}^{\text{leptons}}_{\text{kin}}$ can be extended to muons. Then we can draw diagram for muon decay. Decay rate $\Gamma \sim G_F^2$.

$\tau_\mu \sim 2\,\mu\text{s}$. $G_F \sim 10^{-5}/m_\mu^2$



We have three parameters in gauge sector $(g, g', v) \leftrightarrow (\alpha_{\text{EM}}, G_{\text{F}}, G_{\text{NC}})$. We measured $\alpha_{\text{EM}} = 1/137$, G_{F} in μ-decay and G_{NC} in neutral current interactions.

Equations (4.4.10) are just basis transformation and can be parametrized with one parameter θ_W.

$$\tan(\theta_W) = \frac{g'}{g} \tag{4.4.15}$$

$$\sin(\theta_W) = \frac{g'}{\sqrt{g^2 + g'^2}} \tag{4.4.16}$$

$$\cos(\theta_W) = \frac{g}{\sqrt{g^2 + g'^2}} \tag{4.4.17}$$

Equations (4.4.10) become

$$A_\mu = \cos(\theta_W) B_\mu + \sin(\theta_W) W_\mu^3 \tag{4.4.18}$$

$$Z_\mu = \sin(\theta_W) B_\mu - \cos(\theta_W) W_\mu^3 \tag{4.4.19}$$

The transformation can be inverted

$$B_\mu = \cos(\theta_W) A_\mu + \sin(\theta_W) Z_\mu \tag{4.4.20}$$

$$W_\mu^3 = \sin(\theta_W) A_\mu - \cos(\theta_W) Z_\mu \tag{4.4.21}$$

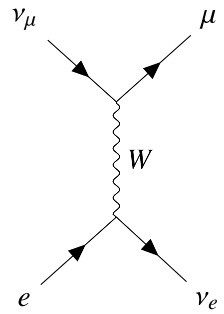
The angle (parameter) θ_W is called Weinberg angle or electroweak mixing angle.

Rewrite the $\mathcal{L}^{\text{lepton}}$ in particular neutral current and factor out A_μ, Z_μ

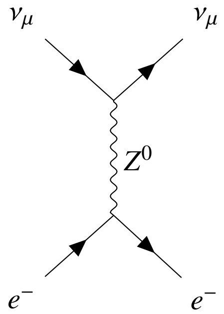
$$\begin{aligned}\mathcal{L}^{\text{lepton}} &= A_\mu \frac{gg'}{\sqrt{g^2 + g'^2}} \left[\bar{e}_R \gamma^\mu e_R + \bar{e}_L \gamma^\mu e_L\right] \\ &+ Z_\mu \frac{1}{2\sqrt{g^2 + g'^2}} \left[g'^2 \left(2\bar{e}_R \gamma^\mu e_R + \bar{e}_L \gamma^\mu e_L + \bar{\nu}_L \gamma^\mu \nu_L\right) - g^2 \left(\bar{e}_L \gamma^\mu e_L - \bar{\nu}_L \gamma^\mu \nu_L\right)\right]\end{aligned}$$

First two terms are coupling of electron and photon and the coupling constant is just electric charge.

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} \approx 0.3 \tag{4.4.22}$$



charged current



neutral current

Charged current interaction involves $W^\pm$. Neutral current interaction involves A_μ and Z^0_μ.

One can experimentally determine g, g', v (actually G_F, e^2 and $\sin^2(\theta_W)$)

$$e = g\sin(\theta_W) = g'\cos(\theta_W) \qquad \Rightarrow \sin^2(\theta_W) \approx 0.22$$

$$M_W^2 = \frac{1}{4}g^2v^2 = \frac{e^2}{\sin^2(\theta_W)}\frac{1}{4\sqrt{2}G_F} \qquad \Rightarrow M_W \approx 80\,\mathrm{GeV} \quad M_Z \approx 90\,\mathrm{GeV}$$

More precisely one should also consider electroweak quantum corrections (loop corrections).

$$\begin{aligned}
&A_\mu\left[\bar{e}_R\gamma^\mu e_R + \bar{e}_L\gamma^\mu e_L\right]\frac{gg'}{\sqrt{g^2+g'^2}}\\
=&A_\mu\left[\bar{e}_R\gamma^\mu e_R + \bar{e}_L\gamma^\mu e_L\right]e_{\mathrm{el}}\\
=&A_\mu\left[\bar{e}\gamma_\mu P_R e + \bar{e}\gamma^\mu P_L e\right]e_{\mathrm{el}}\\
=&A_\mu(\bar{e}\gamma^\mu e)
\end{aligned}$$

It is Lorentz four vector.

For Z_μ it is different. Left-handed electrons are coupled differently from right-handed ones.

$$\begin{aligned}
Z_\mu\bar{e}_R\gamma^\mu e_R &\sim \frac{2g'^2}{2\sqrt{g^2+g'^2}}\\
Z_\mu\bar{e}_L\gamma^\mu e_L &\sim \frac{g'^2-g^2}{2\sqrt{g^2+g'^2}}
\end{aligned}$$

Write this in one equation

$$\frac{1}{2}Z_\mu\left[\bar{e}\gamma^\mu\left(P_R\frac{2g'^2}{\sqrt{g'^2+g^2}} + P_L\frac{g'^2-g^2}{\sqrt{g^2+g'^2}}\right)\right]e_{\mathrm{el}} \tag{4.4.23}$$

suppress the constants into C_A and C_V

$$=\bar{e}\gamma^{\mu}C_V e + \bar{e}\gamma^{\mu}\gamma^5 C_A e \tag{4.4.24}$$

The ratio of these two coupling is

$$\frac{C_V}{C_A} = \frac{3g'^2 - g^2}{g^2 + g'^2} = -1 + 4\sin^2(\theta_W) \tag{4.4.25}$$

4.5 Quarks

Introduce a quark doublet (under $\mathbf{SU}(2)$)

$$Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \tag{4.5.1}$$

u_L and d_L are actually triplets under $\mathbf{SU}(3)_\mathrm{C}$

$$Q_\mathrm{el}(u_L) = +\frac{2}{3} \leftrightarrow Y(u_L) = \frac{1}{3}$$
$$Q_\mathrm{el}(d_L) = -\frac{1}{3} \leftrightarrow Y(d_L) = \frac{1}{3}$$

Together

$$Y(Q) = +\frac{1}{3} \tag{4.5.2}$$

u_R and d_R are $\mathbf{SU}(2)$ singlets.

$$Q_\mathrm{el}(u_R) = +\frac{2}{3} \leftrightarrow Y(u_L) = \frac{4}{3}$$
$$Q_\mathrm{el}(d_R) = -\frac{1}{3} \leftrightarrow Y(d_L) = \frac{4}{3}$$

	$\mathbf{SU}(3)_\mathrm{C}$	$\mathbf{SU}(2)_\mathrm{L}$	$\mathbf{U}(1)_\mathrm{Y}$
e_L	1	2	-1
u_R	3	1	$4/3$
d_R	3	1	$-2/3$
Q	3	2	$1/3$
Φ	1	2	1

Kinetic terms for quarks in Lagrangian

$$\mathcal{L}_\mathrm{kin}^\mathrm{quarks} = \bar{Q}i\gamma_\mu D^\mu Q + \bar{u}_R i\gamma^\mu D^\mu u_R + \bar{d}_R i\gamma_\mu D^\mu d_R \tag{4.5.3}$$

$$D^\mu = \partial_\mu - \frac{ig'}{2}YB_\mu - ig\frac{\tau^i W_\mu^i}{2} - ig_3 T^a \mathcal{G}^{\mu a} \tag{4.5.4}$$

Higgs Lagrangian unchanged. Replace $(B_\mu, W_\mu^3) \mapsto (A_\mu, Z_\mu)$

Neutral current

$$\frac{1}{2}Z^{\mu}\frac{1}{\sqrt{g'^2+g^2}}\left[g'^2\left(\frac{4}{3}\bar{u}_R\gamma_\mu u_R-\frac{2}{3}\bar{d}_R\gamma_\mu d_R+\frac{1}{3}\bar{u}_L\gamma_\mu u_L+\frac{1}{3}\bar{d}_L\gamma_\mu \bar{d}_L\right)+g^2\left(\bar{u}_L\gamma_\mu u_L-\bar{d}_L\gamma_\mu d_L\right)\right] \tag{4.5.5}$$

A_μ is vector like again with charges $2/3$ (u) and $-1/3$ (d).

Neutral currents are always diagonal in quark fields.

Yukawa coupling of quarks to Higgs

$$y_d\bar{Q}\Phi d_R$$

Do the hypercharge add up to zero?

$$Y(\bar{Q})+Y(\Phi)+Y(d_R)=-\frac{1}{3}+1-\frac{2}{3}=0$$

Analogous terms with u_R does <u>not</u> work use ν_R trick $\tilde{\Phi}=i\sigma_2\Phi^*$

$$y_u\bar{Q}\tilde{\Phi}u_R$$

Charged Current

$$A_\mu\left[\bar{u}_R\gamma^\mu u_e+\bar{u}_L\gamma^\mu u_L\right]ee_u=A_\mu\bar{u}\gamma^\mu u \tag{4.5.6}$$

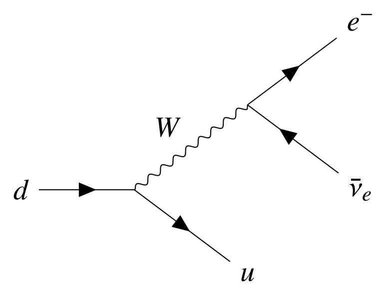
As before, same $W^\pm$ interaction only involved P_L

$$\begin{aligned}\bar{L}\gamma^\mu W_\mu L\\ L=P_L L\\ \gamma^\mu P_L=\frac{1}{2}\left(\gamma^\mu-\gamma^\mu\gamma^5\right)\end{aligned}$$

γ_μ is vector interaction. $\gamma^\mu\gamma^5$ axial vector interaction. So charged current is V-A interaction. Parity is maximally violated.

Coupling g is same for leptons and quarks.

Nuclear beta decay $n\to pe^-\bar{\nu}_e$



This diagram and the one with muon have same coupling G_F. Only kinematics is different. Assuming $m_\nu=0$ we can compare these two processes by measuring neutron lifetime and μ-lifetime. The two Fermi constants are not the same!

It should be clear how to generalize current Lagrangian to other quarks

$$Q_1 = \begin{pmatrix} u \\ d \end{pmatrix} \quad Q_2 = \begin{pmatrix} c \\ s \end{pmatrix} \quad Q_3 = \begin{pmatrix} t \\ b \end{pmatrix}$$
$$\boldsymbol{Q} = (Q_1, Q_2, Q_3)$$
$$\boldsymbol{u}_R = (u_R, c_R, t_R)$$
$$\boldsymbol{d}_R = (d_R, s_R, b_R)$$

$$\frac{4}{3}\bar{\boldsymbol{u}}_R\gamma_\mu\boldsymbol{u}_R Z^\mu = \frac{4}{3}\left[\bar{u}_R\gamma_\mu u_R + \bar{c}_R\gamma_\mu c_R + \bar{t}_R\gamma_\mu t_R\right] Z^\mu$$

Neutral current interactions are unaffected by a unitary transformation in flavour space

$$\boldsymbol{u}_R \mapsto \boldsymbol{u}'_R = A\boldsymbol{u}_R = \begin{pmatrix} u'_R \\ c'_R \\ t'_R \end{pmatrix} \tag{4.5.7}$$

Then the interaction term

$$\begin{aligned}\bar{\boldsymbol{u}}_R\gamma^\mu\boldsymbol{u}_R \mapsto \bar{\boldsymbol{u}}'_R\gamma^\mu\boldsymbol{u}'_R \\ = \bar{\boldsymbol{u}}_R A^\dagger\gamma^\mu A\boldsymbol{u}_R \\ = \bar{\boldsymbol{u}}_R\gamma^\mu\boldsymbol{u}_R\end{aligned}$$

The Yukawa interactions

$$y^d_{ij}\bar{Q}_i\Phi d_{R_j} + y^u_{ij}\bar{Q}_j\tilde{\Phi}u_{R_j}$$

Higgs interactions $\Phi \mapsto v$

$$y^d_{ij}\cdot v = M^d_{ij}$$

It is a $n \times n$ complex matrix. n is the number of families. In Standard Model there are three families.
Charge current interactions

- leptons

$$\mathcal{L}^{\mathrm{CC}}_{\mathrm{leptons}} = \frac{g}{\sqrt{2}}\left[\bar{\nu}_L W^\dagger_\mu\gamma^\mu e_L + \bar{e}_L W^-_\mu\gamma^\mu\nu_L\right]$$

- quarks

$$\mathcal{L}^{\mathrm{CC}}_{\mathrm{quarks}} = \frac{g}{\sqrt{2}}\left[\bar{u}_L W^\dagger_\mu\gamma^\mu d_L + \bar{d}_L W^-_\mu\gamma^\mu u_L\right]$$

The matrices are not diagonal in flavour.

Focus on M_{ij}. It is not necessary symmetric. Mass terms are in form of $m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$

Theorem M_{ij} can be diagonalized by a bi-unitary transformation

$$S^\dagger M T = M_d \tag{4.5.8}$$

with S and T unitary. M_d is diagonal and has positive eigenvalues.
Proof Any matrix $M = H \cdot V$ with H hermitian and V unitary.

$$(MM^\dagger)^\dagger = MM^\dagger$$

In word, $MM^\dagger$ is hermitian. Here matrix can be diagonalized by unitary matrix S

$$\begin{aligned} M_d^2 &= S^\dagger (MM^\dagger) S \\ M_d^2 &= \mathrm{diag}(m_1^2, m_2^2, m_3^2) \end{aligned}$$

S is unique up to a diagonal phase matrix.

$$F = \mathrm{diag}(\mathrm{e}^{i\phi_1}, \mathrm{e}^{i\phi_2}, \mathrm{e}^{i\phi_3}) \tag{4.5.9}$$

Replace $S \mapsto S' = SF$

$$\begin{aligned} &(SF)^\dagger (MM^\dagger)(SF) \\ =&F^\dagger S^\dagger (MM^\dagger) SF \\ =&F^\dagger M_d^2 F \\ =&M_d^2 \end{aligned}$$

$$\begin{aligned} S^\dagger MT &= M_d \\ (SF)^\dagger MT &= M_d \\ S^\dagger MT &= FM_d \end{aligned}$$

Use the phase freedom to have entries $m_i \geq 0$.
Define $H = SM_dS^\dagger$

$$H^\dagger = (SM_dS^\dagger)^\dagger = SM_dS^\dagger = H$$

H is hermitian.
Define $V = H^{-1}M$ $V^\dagger = M^\dagger H^{-1}$
Compute

$$\begin{aligned} VV^\dagger &= H^{-1}MM^\dagger H^{-1} \\ &= H^{-1}SM_d^2S^\dagger H^{-1} \\ &= H^{-1}SM_dS^\dagger SM_dS^\dagger H^{-1} \\ &= H^{-1}HHH^{-1} \\ &= \mathbb{1} \end{aligned}$$

so V is unitary.

$$
\begin{aligned}
V &= H^{-1}M \\
HV &= M \\
H &= MV^{\dagger}
\end{aligned}
$$

$$S^{\dagger}HS = S^{\dagger}MV^{\dagger}S$$

LHS by definition is M_d

$$
\begin{aligned}
M_d &= S^{\dagger}MV^{\dagger}S \\
M_d &= S^{\dagger}MT \quad \text{with } T = V^{\dagger}S
\end{aligned}
$$

Recall that our entries after spontaneous symmetry breaking involve

$$
\begin{aligned}
&M^{u}_{ij}\bar{u}_{L_i}U_{R_j} + M^{d}_{ij}\bar{d}_{L_i}d_{R_j} \\
=&\boldsymbol{u}_L\underline{M}^u\boldsymbol{u}_R + \boldsymbol{d}_L\underline{M}^d\boldsymbol{d}_R
\end{aligned}
$$

Call the transformation to mass eigenstates

$$
\begin{aligned}
\boldsymbol{u}_L &= S_u\boldsymbol{u}'_L \\
\boldsymbol{d}_L &= S_d\boldsymbol{d}'_L \\
\boldsymbol{u}_R &= T_u\boldsymbol{u}'_R \\
\boldsymbol{d}_R &= T_d\boldsymbol{d}'_R
\end{aligned}
$$

This transformation has no effect on neutral current interactions. (It does not live in spinor space). Charged current interactions on the other hand

$$
\begin{aligned}
&\frac{g}{\sqrt{2}}\left[\bar{u}_L W^{\dagger}_{\mu}\gamma^{\mu}d_L + \bar{d}_L W^{-}_{\mu}\gamma^{\mu}u_L\right] + \text{terms for } c,\ s,\ t \text{ and } b \\
=&\frac{g}{\sqrt{2}}\left[\bar{\boldsymbol{u}}_L\gamma^{\mu}\boldsymbol{d}_L W^{\dagger}_{\mu} + \bar{\boldsymbol{d}}_L\gamma^{\mu}\boldsymbol{u}_L W^{-}_{\mu}\right]
\end{aligned}
$$

Charged current transforms to

$$
=\frac{g}{\sqrt{2}}\left[\bar{\boldsymbol{u}}'_L\gamma^{\mu}\left(S^{\dagger}_uS_d\right)\boldsymbol{d}'_L W^{\dagger}_{\mu} + \bar{\boldsymbol{d}}'_L\gamma^{\mu}\left(S^{\dagger}_dS_u\right)\boldsymbol{u}'_L W^{-}_{\mu}\right]
$$

Cabbibo-Kobayashi-Maskawa matrix

$$
\begin{aligned}
V_{\text{CKM}} &= S^{\dagger}_uS_d \\
V^{\dagger}_{\text{CKM}} &= S^{\dagger}_dS_u
\end{aligned}
\tag{4.5.10}
$$

It is unitary and related to CP-violation.

How many parameters are there in V_{CKM}? 3×3 complex matrix in general has 18-real parameter.

How many phases? $\bar{q}i\partial\!\!\!/ q$ not affected by (global) phase. Mass terms must change q_L and q_R by same phase.

First doublet

$$
Q_{1L} = \begin{pmatrix} u \\ V_{11}d + V_{12}s + V_{13}b \end{pmatrix}_L \tag{4.5.11}
$$

$V_{11} = (V_{\text{CKM}})_{11}$, $V_{11} = R_{11}\mathrm{e}^{i\delta}$, $R_{11} \in \mathbb{R}$. Use the notation $u = u'\mathrm{e}^{i\delta}$ and $V = V'\mathrm{e}^{i\delta}$

$$Q_{1L} = \mathrm{e}^{i\delta_1}\begin{pmatrix} u' \\ R_{11}d + V'_{12}s + V'_{13}b \end{pmatrix}$$
$$Q_{2L} = \mathrm{e}^{i\delta_2}\begin{pmatrix} c' \\ R_{21}d + V'_{22}s + V'_{23}b \end{pmatrix}$$
$$Q_{3L} = \mathrm{e}^{i\delta_3}\begin{pmatrix} t' \\ R_{31}d + V'_{32}s + V'_{33}b \end{pmatrix}$$

Overall phases of doublets Q_{iL} don't affect anything. Still have freedom of phase shift in s and b. Get rid of phase in V'_{12} and V'_{13}

$$V'_{22} \mapsto V''_{22} \quad V'_{23} \mapsto V''_{23} \quad \dots$$

Started with 18 parameters, absorbed 5 phases. It ends up with 13 parameters. They are 9 real parameter, 4 phases. Normalization condition on 3 guys indicates that all states have to orthogonal. Thus 6 states. $13 - 3 - 6 = 4$ It has 4 real parameter in 3×3 case.

Then V_{CKM} is a orthogonal matrix multiplied with a phase. A orthogonal 3×3 matrix has 3 real and 1 phase parameters. This phase leads to CP-violation.

CKM matrix $n \times n$ cases. There are $2n^2$ (real) parameters. Unitarity of CKM matrix conditions, then n^2 parameters. $(2n - 1)$ phases can be removed by a redefinition of quark phases $2 \cdot 3 - 1 = 5$ $(n = 3)$.

$n \times n$ orthogonal matrix has $n(n-1)/2$ real parameters. Number of phases

$$\begin{aligned} n^2 - (2n-1) - n(n-1)/2 = n^2 - 2n + 1 - \frac{1}{2}n^2 + \frac{1}{2}n \\ = \frac{1}{2}n^2 - \frac{3}{2}n + 1 = \frac{1}{2}(n-1)(n-2) \end{aligned}$$

- $n = 2$ There is no phase
- $n = 3$ 1 phase Standard Model
- $n = 4$ 3 phases

The phases lead to CP-violation.

5 Scattering Theory

5.1 Non-relativistic Perturbation Theory

Quantum states are eigenfunctions of (bare) Hamiltonians. They should be (complete) orthonormal basis for the Hilbert space.

$$H_0\phi_n = E_n\phi_n$$
$$\int \mathrm{d}^3x\, \phi_m^*\phi_n = \delta_{mn}$$

Schrödinger equation

$$(H_0 + V)\psi = i\frac{\mathrm{d}\psi}{\mathrm{d}t}$$

Make the ansatz that an arbitrary state is a superposition of of ϕ_n

$$\psi = \sum a_n(t)\phi_n(\boldsymbol{x})\mathrm{e}^{-iE_nt} \tag{5.1.1}$$

Insert it into Schrödinger equation 5.1.1

$$i\sum \frac{\mathrm{d}a_n}{\mathrm{d}t}\phi_n(\boldsymbol{x})\mathrm{e}^{-iE_nt} = \sum V(\boldsymbol{x},t)a_n(t)\phi_n(\boldsymbol{x})\mathrm{e}^{-iE_nt}$$

multiply by $\phi_f^*(\boldsymbol{x})$ and $\int \mathrm{d}^3x$

$$\frac{\mathrm{d}a_f}{\mathrm{d}t} = -i\sum_n a_n(t)\int \mathrm{d}^3x\,\phi_f^*V\phi_n\cdot \mathrm{e}^{i(E_f-E_n)t}$$

Assume this potential acts for a finite time T, between $[-T/2, T/2]$.
The initial state is $|i\rangle$ and eigenstate of H_0.

$$a_i(-T/2) = 1$$
$$a_n(-T/2) = 0 \;\text{ with } n \neq i$$

$$\frac{\mathrm{d}a_f}{\mathrm{d}t} = -ia_n(t)\int \mathrm{d}^3x\,\phi_f^*V\phi_i\cdot \mathrm{e}^{i(E_f-E_i)t} \tag{5.1.2}$$

integrate

$$a_f(t) = -i\int_{-T/2}^{t}\mathrm{d}t'\int \mathrm{d}^3x\,\phi_f^*V\phi_i\mathrm{e}^{i(E_f-E_i)t}\bigg|_{t=-T/2}$$

$$
\begin{aligned}
T_{fi} &= a_f(T/2) \\
&= -i\int_{-T/2}^{T/2} \mathrm{d}t \int \mathrm{d}^3x \left[\phi_f(\boldsymbol{x})\mathrm{e}^{-iE_f t}\right]^* V(\boldsymbol{x},t)\left[\phi_i(\boldsymbol{x})\mathrm{e}^{-iE_i t}\right] \\
T_{fi} &= -i\int \mathrm{d}^4x\, \phi_f^*(x)V(x)\phi_i(x)
\end{aligned}
\tag{5.1.3}
$$

for $a_f(t) \ll 1$.

Assume static potential $V(x) = V(\boldsymbol{x})$

$$
\begin{aligned}
T_{fi} &= -iV_{fi}\int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{-i(E_f - E_i)t} \\
&= -2\pi i V_{fi}\delta(E_f - E_i) \\
V_{fi} &= \int \mathrm{d}^3x\, \phi_f^*(\boldsymbol{x})V(\boldsymbol{x})\phi_i(\boldsymbol{x})
\end{aligned}
\tag{5.1.4}
$$

As $\Delta E \to 0$, $\Delta t = \infty$, so infinite time is needed to get from $|i\rangle \to |f\rangle$. Define transition probability per unit time

$$
\begin{aligned}
W &= \lim_{T\to\infty} \frac{|T_{fi}|^2}{T} \\
&= \lim_{T\to\infty} 2\pi\frac{|V_{fi}|^2}{T}\delta(E_f - E_i)\int_{-T/2}^{T/2} \mathrm{d}t\, \mathrm{e}^{i(E_f - E_i)t} \\
&= \lim_{T\to\infty} 2\pi\frac{|V_{fi}|^2}{T}\delta(E_f - E_i)T \\
&= 2\pi|V_{fi}|^2\delta(E_f - E_i)
\end{aligned}
\tag{5.1.5}
$$

It is only meaningful after integration over initial and final states. In collider initial state is prepared with fixed momentum $\boldsymbol{p_e}$. So no integration over initial states. For proton (hadron), it is a bit more complicated since the quarks are moving inside of proton.

Define the density of final states $\rho(E_f)$. Fermi's Golden rule

$$
\begin{aligned}
W_{fi} &= 2\pi\int \mathrm{d}E_f\, \rho(E_f)|V_{fi}|^2\delta(E_f - E_i) \\
&= 2\pi|V_{fi}|^2\rho(E_i)
\end{aligned}
\tag{5.1.6}
$$

The result can be improved by considering higher order scatterings, i.e. scatter multiple times. For each order insert the result for $a_f(t)$ into formula (5.1.2) for the $a_{n\neq i}(t)$

$$
\frac{\mathrm{d}a_f}{\mathrm{d}t} = -\sum a_n(t)\int \mathrm{d}\phi_f^* V\phi_n\, \mathrm{d}^3x\, \mathrm{e}^{i(E_f - E_i)t}
\tag{5.1.7}
$$

replace

$$
a_n(t) = -i\int_{T/2}^{t} \mathrm{d}t' \int \mathrm{d}^3x'\, \phi_n^* V\phi_i \mathrm{e}^{i(E_n - E_i)t}
\tag{5.1.8}
$$

Then

$$
\frac{\mathrm{d}a_f}{\mathrm{d}t} = -i\int \mathrm{d}^3x\, \phi_f^* V\phi_i \mathrm{e}^{i(E_f - E_i)t} + (-i)^2\sum_{n\neq i}\int_{-T/2}^{t} \mathrm{d}t'\, \mathrm{e}^{i(E_n - E_i)t'} V_{fn}\mathrm{e}^{i(E_f - E_n)t}
$$

Correlation to T_{fi}

$$T_{fi} = \cdots - \sum_{n \neq i} V_{fn} V_{ni} \int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{i(E_f - E_n)t} \int_{-\infty}^{t} \mathrm{d}t'\, \mathrm{e}^{i(E_n - E_i)t}$$

regulate the last exponential by adding a small constant ϵ

$$T_{fi} = \cdots - 2\pi i \sum_{n \neq i} \frac{V_{fn} V_{ni}}{E_i - E_n + i\epsilon} \delta(E_f - E_i)$$

This is second order in perturbation theory including time dependence.

We can treat it like Feynman diagram with vertex V_{ni} and propagator $1/(E_i - E_n + i\epsilon)$.

It is not manifestly covariant. Potential is assumed static and it defines the preferred reference frame. To go to relativistic formulation, consider other particle(s), which create electromagnetic potential for first particle to scatter in and vice versa.

5.2 Scalar Electrodynamics

5.2.1 Complex scalar

$$\mathcal{L} = (\partial_m u\phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi \tag{5.2.1}$$

It has the equation of motion

$$(\partial^2 + m^2)\phi = 0 \tag{5.2.2}$$

To include electromagnetism consider generalized momentum

$$\begin{aligned} p^\mu &\mapsto p^\mu +_e A^\mu \\ i\partial^\mu &\mapsto i\partial^\mu + eA^\mu \end{aligned}$$

Insert it into equation of motion

$$\begin{aligned} (\partial^2 + m^2)\phi &= -V\phi \\ V &= -ie(\partial_\mu A^\mu + A^\mu \partial_m u) - e^2 A^2 \end{aligned} \tag{5.2.3}$$

Fine structure constant $\alpha = e^2/(4\pi) \approx 1/137 \ll 1$. So in first order drop the e^2 term.

$$\begin{aligned} T_{fi} &= -i \int \mathrm{d}^4 x\, \phi_f^* x V(x) \phi_i(x) \\ &= -i \int \mathrm{d}^4 x\, \phi_f^*(x) ie \left[\partial_\mu A^\mu + A^\mu \partial_\mu\right] \phi_i(x) \end{aligned}$$

integrate by parts and drop surface terms

$$= -i \int j_\mu^{fi} A^\mu\, \mathrm{d}^4 x \tag{5.2.4}$$

with the current

$$j_\mu^{fi}(x) = -ie\left[\phi_f^*(\partial_\mu \phi_i) - (\partial_\mu \phi_f)^* \phi_i\right] \tag{5.2.5}$$

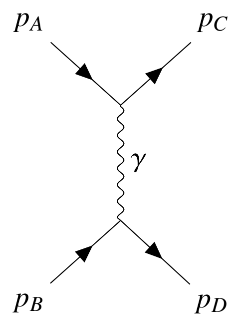
The current is conserved for free field $\partial_\mu j^\mu(x) = 0$

Write ϕ_i and ϕ_f as plane waves

$$\begin{aligned} \phi_i(x) &= N_i \mathrm{e}^{-ip_i \cdot x} \\ j_\mu^{fi} &= -eN_i N_f (p_i + p_f) \mathrm{e}^{i(p_f - p_i)\cdot x} \end{aligned}$$

5.2.2 Scalar Charged Scattering

Consider potential for particle 1 to scatter caused by particle 2 and vice versa.



What potential A_μ does $j_\mu^{(2)}$ create for particle 1 to scatter in inhomogeneous Maxwell equation

$$\partial^2 A^\mu = j^{\mu(2)} = -eN_B N_D (p_D + p_B)^\mu \mathrm{e}^{i(p_D - p_B)\cdot x}$$

Also consider photons as plane waves

$$\partial^2 \mathrm{e}^{iqx} = -q^2 \mathrm{e}^{iqx}$$

thus

$$A^\mu = -\frac{1}{q^2} j^{\mu(2)} \tag{5.2.6}$$

with $q = p_D - p_B$.

Insert into 5.2.4

$$\begin{aligned} T_{fi} &= -i \int j_\mu^{(1)}(x) \left(-\frac{1}{q^2}\right) j^{\mu(2)}(x)\, \mathrm{d}^4 x \\ &= -ie^2 N_A N_B N_C N_D \int \mathrm{d}^4 x\, \mathrm{e}^{i(p_c + p_D - p_A - p_B)} (p_C + p_A)_\mu (p_D + p_B)^\mu \frac{-1}{q^2} \\ &= -iN_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_C + p_D - p_A - p_B) \mathcal{M} \end{aligned}$$

The invariant matrix element

$$-i\mathcal{M} = ie(p_A + p_C)^\mu \left(-i\frac{g_{\mu\nu}}{q^2}\right) ie(p_B + p_D)^\nu$$

It is symmetric in (A, C) and (B, D).

5.2.3 From Amplitude to Cross Section

In the laboratory we measure cross sections. Need to fix norm, $\phi = N\mathrm{e}^{-ipx}$. The number density is defined by

$$\rho = 2E|N|^2 \tag{5.2.7}$$

The infinitesimal number of particles $\rho\, \mathrm{d}^3 x$ is Lorentz invariant. Norm is chosen as

$$\int \rho\, \mathrm{d}V = 2E \tag{5.2.8}$$

We are interested in Transition rate per unit time and volume.

$$W_{fi} = \frac{|T_{fi}|^2}{T \cdot V}$$
$$T_{fi} = -iN_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_c + p_D - p_A - p_B)\mathcal{M}$$

Same trick as before (5.1.5) to get rid of one of δ

$$\int \mathrm{d}^4 x = T \cdot V$$
$$W_{fi} = \frac{1}{V^4}(2\pi)^4 \delta^{(4)}(p_C + p_D - p_A - p_B)|\mathcal{M}|^2$$

Definition of cross section

$$\sigma = \frac{W_{fi}}{\text{initial flux}} \cdot (\text{number of final states}) \tag{5.2.9}$$

Imagine particles in a box

$$(\text{number of final state particles}) = \frac{V\,\mathrm{d}^3 p}{(2\pi)^3 2E}$$

Density of incoming particles A $2E_A/V$. The box has length $\boldsymbol{v}_A \cdot t$. Number of beam particles passing through unit area A per unit time is $\boldsymbol{v}_A \cdot t 2E_A/(Vt)$. Particle B (target). Initial flux is $|\boldsymbol{v}_A|\frac{2E_a}{v}\frac{2E_B}{V}$

$$\begin{aligned} \mathrm{d}\sigma &= \frac{V^2}{|\boldsymbol{v}_A| 2E_A 2E_B}\frac{1}{V^4}|M|^2 \frac{(2\pi)^4}{(2\pi)^4}\delta^{(4)}(p_c + p_D - p_A - p_B)\frac{\mathrm{d}^3 p_C}{2\pi}\frac{\mathrm{d}^3 p_D}{2\pi}V^2 \\ &= \frac{|\mathcal{M}|^2}{F}\,\mathrm{d}Q \end{aligned}$$

with $F = |\boldsymbol{v}_A| 2E_A 2E_B$ and $\mathrm{d}Q = (2\pi)^4 \delta^{(4)}(p_C + p_D - p_A - p_B)\frac{\mathrm{d}^3 p_C}{(2\pi)^3 2E_C}\frac{\mathrm{d}^3 p_D}{(2\pi)^3 2E_D}$and B at rest.

If both are moving $F = |\boldsymbol{v}_A - \boldsymbol{v}_B| 2E_A 2E_B = 4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}$.

In centre of mass system

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\frac{p_f}{p_i}|M|^2 \tag{5.2.10}$$

For $e\mu$ scattering

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\left(\frac{3 + \cos(\theta)}{1 - \cos(\theta)}\right)$$

with θ angle between $\boldsymbol{p}_A$ and $\boldsymbol{p}_C$.

Also it includes decays, for example $n \to p e^- \bar{\nu}_e$.

Decay versus cross section Examples are $\pi^- \to \mu^- \bar{\nu}_\mu$, $t \to W^+ b$

$$\mathrm{d}\Gamma = \frac{1}{2E_A}|\mathcal{M}|^2 \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} \tag{5.2.11}$$
$$\Gamma = \int \mathrm{d}\Gamma \tag{5.2.12}$$

Pion for instance has different decay modes.

$$\Gamma_{\text{tot}} = \Gamma(\pi \to \mu\nu_\mu) + \Gamma(\pi \to e\bar{\nu}_e) \tag{5.2.13}$$

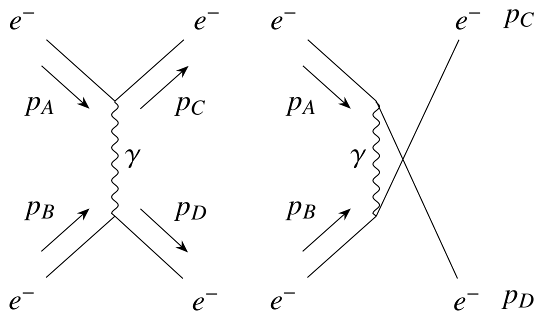
Radioactive decay

$$N_A(t) = N(0)\mathrm{e}^{-\Gamma_{\text{tot}} t} \tag{5.2.14}$$

Lifetime is defined as

$$\tau_A = 1/\Gamma_{\text{tot}}(A) \tag{5.2.15}$$

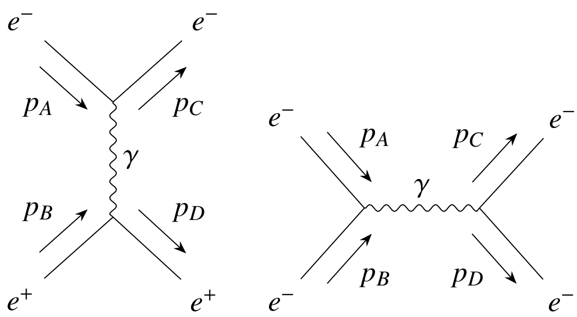
5.3 Electron Electron scattering



One cannot differentiate electron from electron. We have to add both diagrams before squaring

$$-iM_{e^-e^-} = -ie^2\left[-\frac{(p_A + p_C)_\mu \cdot (p_B + p_D)^\mu}{(p_D - p_B)^2} - \frac{(p_A + p_D)_\mu \cdot (p_B + p_C)^\mu}{(p_C - p_B)^2}\right]$$

5.4 Electron Positron scattering



5.5 Mandelstam Variables

Only for two-to-two scattering

$$s = (p_A + p_B)^2 \tag{5.5.1}$$

$$u = (p_A - p_C)^2 \tag{5.5.2}$$

$$t = (p_A - p - D)^2 \tag{5.5.3}$$

Diagrams are often classified by the momentum of propagator. Mandelstam variables s, t and u are not independent.

$$s + u + t = \sum m_i^2 \tag{5.5.4}$$

One can set the reference frame such that

$$\begin{aligned} p_A &= (E, 0, 0, E) \\ p_B &= (E, 0, 0, -E) \\ p_C &= (E, 0, E\sin(\theta), E\cos(\theta)) \\ p_D &= (E, 0, -E\sin(\theta), -E\cos(\theta)) \end{aligned}$$

Then

$$s = 4E^2 \tag{5.5.5}$$

$$t = -\frac{s}{2}(1 - \cos(\theta)) \tag{5.5.6}$$

$$u = -\frac{s}{2}(1 + \cos(\theta)) \tag{5.5.7}$$

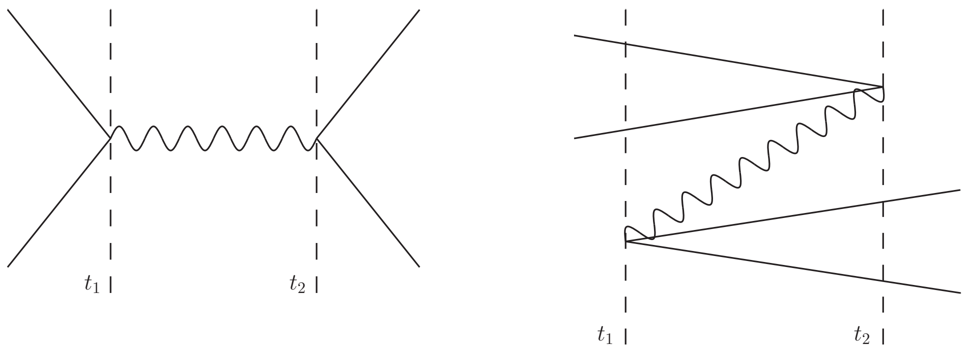
5.6 Origin of the propagator

Non-relativistically

$$T_{fi}^{(2)} = -i \sum_{n \neq i} V_{fn} \frac{1}{E_i - E_n} V_{ni} 2\pi\delta(E_f - E_i) \tag{5.6.1}$$

Electron positron scattering each Feynman diagram is the sum of all possible time orderings in old fashioned perturbation theory.

Two times orderings for s-channel diagram



(one can understand the second diagram by saying that an amount of energy is borrowed to produce particle pair.)

$$\begin{aligned} \mathcal{M} &\sim V_{fn} \frac{1}{E_i - E\gamma} V_{ni} + V_{fn} \frac{1}{E_i - 2E_i - E_\gamma} V_{ni} \\ &= V_{fn} \frac{2E_\gamma}{E_i^2 - E_\gamma^2} V_{ni} \end{aligned}$$

$$
\begin{aligned}
(p_A + p_B)^2 &= E_i^2 - (\boldsymbol{p}_A + \boldsymbol{p}_B)^2 \\
E_\gamma^2 &= \boldsymbol{p}^2 + m_\gamma^2 \\
p &= p_A + p_B \\
\frac{1}{E_i^2 - E_\gamma^2} &= \frac{1}{(p_A + p_B)^2 - m_\gamma^2} = \frac{1}{q^2}
\end{aligned}
$$

5.7 Electron interacting with A^μ

$$(ip_\mu \gamma^\mu - m)\psi = 0 \tag{5.7.1}$$

$$\psi = u(\boldsymbol{p})\mathrm{e}^{-ip\cdot x} \tag{5.7.2}$$

Consider $p^\mu \mapsto p^\mu + eA^\mu$

$$
\begin{aligned}
(\gamma^\mu p_\mu - m)\psi &= (\gamma^0 V)\psi \\
\gamma^0 V &= -e\gamma_\mu A^\mu
\end{aligned}
$$

$$
\begin{aligned}
T_{fi} &= -i \int \mathrm{d}^4 x\, \psi_f^\dagger(x) V(x) \psi_i(x) \\
&= ie \int \mathrm{d}^4 x\, \overline{\psi}_f(x) \gamma_\mu A^\mu \psi_i \\
&= -i \int \mathrm{d}^4 x\, j_\mu^{fi} A^\mu
\end{aligned}
$$

$$j_\mu^{fi} = -e\bar{\psi}_f \gamma_\mu \psi_i = -e\bar{u}_f \gamma_\mu u_i \mathrm{e}^{i(p_f - p_i)x} \tag{5.7.3}$$

In non-relativistic case

$$j_\mu^{fi} = -e\bar{\psi}_f \gamma_\mu \psi_i = -e(p_f + p_i)_\mu \mathrm{e}^{i(p_f - p_i)x} \tag{5.7.4}$$

So we have Feynman rules for external fermions and vertex.

One can write the following identity

$$\bar{u}_f \gamma^\mu u_i = \frac{1}{2m} \bar{u}_f \left[(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu \right] u_i \tag{5.7.5}$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. First term is exactly we have before. It comes from electric charge interaction. The second part is then spin or magnetic interaction. It can be shown that $j^\mu A_\mu$ in Non-relativistic limit becomes $\boldsymbol{\sigma} \cdot \boldsymbol{B}$ from $\sigma^{\mu\nu}$ term.

5.8 Møller Scattering

$e^- e^- \to e^- e^-$

$$T_{fi} = -i \int \mathrm{d}^4 x\, j_\mu^{(1)} (-\frac{1}{q^2}) j_2^\mu(x)$$
$$= -i(-e\bar{u}_c \gamma_\mu u_A)\left(-\frac{1}{q^2}\right)(-e\bar{u}_D \gamma^\mu u_B)(2\pi)^4 \delta^{(4)}(p_A + p_B - p_C - p_D)$$

with $q = p_A - p_C = p_D - p_B$

$$= -i(2\pi)^4 \delta^{(4)}(p_A + p_B - p_C - p_D)\mathcal{M}$$

$$-i\mathcal{M} = (ie\bar{u}_C \gamma^\mu u_A)\left(\frac{-ig_{\mu\nu}}{q^2}\right)(ie\bar{u}_D \gamma^\nu u_B)$$

swapping order of final state electrons. Minus sign comes from electron being fermion.

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = -e^2 \frac{(\bar{u}_C \gamma^\mu u_A)(\bar{u}_D \gamma_\mu u_B)}{(p_A - p_C)^2} + e^2 \frac{(\bar{u}_D \gamma^\mu u_A)(\bar{u}_C \gamma_\mu u_B)}{(p_A - p_D)^2}$$

$$|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\,\mathrm{Re}(\mathcal{M}_1 \mathcal{M}_2^*) \tag{5.8.1}$$
$$|\mathcal{M}_1|^2 = \frac{e^4}{t^2}(\bar{u}_C \gamma^\mu u_A)(\bar{u}_A \gamma^\nu u_D)(\bar{u}_D \gamma_\mu u_B)(\bar{u}_B \gamma_\nu u_D) \tag{5.8.2}$$

Most experiments are done without polarized beams. Thus we must average incoming spins.

$$\frac{1}{(2S_A + 1)(2S_B + 1)} \sum_{S_A, S_B} \tag{5.8.3}$$

We should also sum over the final states (if it will not be observed).

Shuffle the spinors and use spin sum identities

$$\sum_S u^{(S)}(p) \bar{u}^{(S)}(p) = \not{p} + m \tag{5.8.4}$$

we get (for the second half)

$$\mathrm{tr}\Big[(\not{p}_D + m)\gamma^\mu(\not{p}_B + m)\gamma^\nu\Big]$$

$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}_1|^2 = \frac{e^4}{4t^2}\,\mathrm{tr}\Big[(\not{p}_D + m)\gamma^\mu(\not{p}_B + m)\gamma^\nu\Big]\,\mathrm{tr}\Big[(\not{p}_C + m)\gamma^\mu(\not{p}_A + m)\gamma^\nu\Big]$$

We know the trace of odd number of gamma matrices vanished. Thus

$$\frac{4e^4}{t^2}[2p_A \cdot p_D p_C \cdot p_B + 2p_A \cdot p_B e p_C \cdot p_D - 4p_A \cdot p_C p_B \cdot p_D + 4p_A \cdot p_C p_B \cdot p_D +$$
$$m^2(2p_B \cdot p_D - 4p_B \cdot p_D + 2p_A \cdot p_C - 4p_A \cdot p_C) + 4m^4]$$

In most experiments, center of mass energy is well above electron mass, thus we can set $m_e = 0$

$$= 8(u^2 + s^2)$$

$$\begin{aligned} s &= 2p_A \cdot p_B = 2p_C \cdot p_D \\ t &= -2p_A \cdot p_C \\ u &= -2p_A \cdot p_D \end{aligned}$$

Correspondingly

$$\frac{1}{4}\sum |\mathcal{M}_2|^2 = \frac{2e^4}{u^2}(t^2 + s^2)$$

In interference term, there is just one (long) trace.

$$\frac{1}{4}\sum 2\,\mathrm{Re}(\mathcal{M}_1\mathcal{M}_2^*) = \frac{4e^4}{tu}s^2$$

$$\begin{aligned} \frac{1}{4}\sum |\mathcal{M}|^2 &= \overline{|\mathcal{M}|^2} \\ &= 2e^4\left[\frac{u^2+s^2}{t^2} + \frac{t^2+s^2}{u^2} + 2\frac{s^2}{tu}\right] \end{aligned}$$

$e^-\mu^- \to e^-\mu^-$
$e^-e^+ \to \mu^+\mu^-$ Change $k' \to -p$ and $s \leftrightarrow t$.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}(1 + \cos^2\theta)$$

integrate over solid angle

$$= \frac{4\pi\alpha^2}{3s}$$

Cross section drops down with $1/s$. With high luminosity we can evade the problem.

5.9 Photons and Polarization Vectors

Photon field $A^\mu(x)$ satisfies Maxwell equations

$$\partial^2 A^\mu = j^\mu$$

Even in classical theory, there is gauge freedom. With Lorenz condition $\partial_\mu A^\mu = 0$ there is still some gauge freedom.

$$A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \Lambda(x) \tag{5.9.1}$$

if $\partial^2\Lambda(x) = 0$.

Free photon

$$\partial^2 A^\mu = 0 \tag{5.9.2}$$

$$A^\mu = \epsilon^\mu(\boldsymbol{q})\mathrm{e}^{-iq\cdot x} \tag{5.9.3}$$

with $\epsilon(\boldsymbol{p})$ polarization vector with $\mu = 0, 1, 2, 3$. Thus equation for polarization vector is

$$q^2 \epsilon^\mu(\boldsymbol{q}) \mathrm{e}^{-iq\cdot x} = 0 \tag{5.9.4}$$

with $q^2 = m_\gamma^2 = 0$.

From QM, we know for spin 1 particle, $S_z = +1, 0, -1$. With Lorenz condition,

$$q_\mu \epsilon^\mu = 0$$

thus there are three independent components.

We still have constraint 5.9.1. To make it specific, choose

$$\Lambda = ia\mathrm{e}^{-iq\cdot x} \tag{5.9.5}$$

then the constraint is satisfied

$$\partial^2 \Lambda = q^2 \Lambda = 0$$

Use this specific Λ, the transformation of A_μ becomes

$$A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \Lambda \tag{5.9.6}$$

$$\epsilon \mathrm{e}^{-iq\cdot x} \mapsto \epsilon' \mathrm{e}^{-iq\cdot x} = \epsilon \mathrm{e}^{-iq\cdot x} + ia(-i) q_\mu \mathrm{e}^{-iq\cdot x} \tag{5.9.7}$$

$$\epsilon \mapsto \epsilon'_\mu = \epsilon_\mu + a q_\mu \tag{5.9.8}$$

ϵ and ϵ' describe the same photon.

Use this freedom to demand $\epsilon^{\mu=0} = 0$. Then we have Coulomb gauge

$$\boldsymbol{\epsilon} \cdot \boldsymbol{q} = 0 \tag{5.9.9}$$

Only two independent polarization vectors.

$$\begin{aligned} \boldsymbol{q} &= (0, 0, q) \\ \epsilon_1 &= (1, 0, 0) \\ \epsilon_2 &= (0, 1, 0) \end{aligned}$$

Photon only with $\epsilon_{1,2}$ are linear polarized.

Circularly polarized

$$\epsilon_R = -\frac{1}{\sqrt{2}}(\epsilon_1 + i\epsilon_2) \tag{5.9.10}$$

$$\epsilon_L = \frac{1}{\sqrt{2}}(\epsilon_1 - i\epsilon_2) \tag{5.9.11}$$

ϵ_R has $+1$ helicity and ϵ_L has -1.

One can show completeness relation

$$\sum_{\lambda = L,R} (\epsilon_\lambda)_i^* (\epsilon_\lambda)_j = \delta_{ij} - \hat{q}_i \hat{q}_j \tag{5.9.12}$$

with $i, j = 1, 2, 3$ and hat denotes unit vector.

5.10 Electron Propagator

In non-relativistic limit

$$T_{fi} = -2\pi i\delta(E_f - E_i)\left[\langle f|V|i\rangle + i\sum_{n\neq i}\langle f|V|n\rangle\,\frac{1}{E_i - E_n}\,\langle n|V|i\rangle + \dots\right]$$

since $H_0\,|n\rangle = E_n\,|n\rangle$

$$= 2\pi\delta(E_f - E_i)\,\langle f|\left[(-iV) + (-iV)\frac{i}{E_i - H_0}(-iV) + \dots\right]|i\rangle$$

So we have vertex $-iV$ and propagator $-i/(E - H_0)$.

Go to relativistic case. With spinless particle, Klein-Gordon equation

$$i(\partial^2 + m^2)\phi = -iV\phi \tag{5.10.1}$$

We expect propagator

$$\frac{i}{p^2 - m^2} \tag{5.10.2}$$

For spin $\frac{1}{2}$ particle, Dirac equation in momentum space

$$(\not{p} - m)\psi = -e\gamma^\mu A_\mu\psi$$

So we expect propagator to be

$$\frac{i}{\not{p} - m} = \frac{i(\not{p} + m)}{p^2 - m^2} \tag{5.10.3}$$

Remember $\not{p} + m = \sum_{\text{spins}} u(p)\bar{u}(p)$. In fact for any particle the propagator can be written as

$$\frac{i\sum_{\text{spins}(\dots)}}{p^2 - m^2} \tag{5.10.4}$$

5.11 Photon Propagator

Gauge freedom leads to the fact that photon propagator is not unique.

Wave equation

$$\partial^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu \tag{5.11.1}$$

$$(g^{\mu\lambda}\partial^2 - \partial^\mu\partial^\lambda)A_\lambda = j^\mu \tag{5.11.2}$$

Propagator should be inverse of this operator. In momentum space

$$(g^{\mu\lambda}q^2 - q^\mu q^\lambda)A_\lambda = j^\mu \tag{5.11.3}$$

Ansatz for inverse

$$A(q^2)q^2 g_{\nu\mu} + B(q^2)q_\nu q_\mu \tag{5.11.4}$$

Multiply back to equation

$$
\begin{aligned}
&(A(q^2)g_{\nu\mu} + B(q^2)q_\nu q_\mu)(-g^{\mu\lambda}q^2 + q^\mu q^\lambda) \\
&= -A(q^2)q^4\delta^\lambda_\nu + A(q^2)q^2 q_\nu q^\lambda - B(q^2)q_\nu q^\lambda q^2 + B(q^2)q^2 q_\nu q^\lambda \\
&= -A(q^2)q^2(q^2\delta^\lambda_\nu - q_\nu q^\lambda) \\
&\stackrel{!}{=} \delta^\lambda_\nu
\end{aligned}
$$

It is impossible!

In Lorenz gauge $\partial_\lambda A^\lambda = 0$

$$g^{\nu\lambda}\partial^2 A_\lambda = j^\nu$$

in momentum space

$$g^{\nu\lambda}q^2 A_\lambda = j^\nu$$

It has inverse

$$\frac{-g_{\mu\nu}}{q^2} \tag{5.11.5}$$

in which $-g_{\mu\nu}$ corresponds to sum over polarization. Propagator in this form is Feynman propagator, ideal for QED calculation.

More general form

$$
\begin{aligned}
\left[g^{\nu\lambda}\partial^2 - \left(1 - \frac{1}{\xi}\partial^\nu\partial^\lambda\right)\right]A_\lambda &= j^\nu \\
\left[g^{\nu\lambda}q^2 - \left(1 - \frac{1}{\xi}q^\nu q^\lambda\right)\right]\tilde{A}_\lambda &= \tilde{j}^\nu
\end{aligned}
$$

with $\xi \in \mathbb{R}$. It is called R_ξ-gauge. The equation now is invertible. Construct inverse

$$
\begin{aligned}
&(-q^2 g^{\nu\lambda} + \frac{\xi - 1}{\xi}q^\nu q^\lambda)(Aq^2 g_{\lambda\mu} + Bq_\lambda q_\mu) \\
&= -q^2\delta^\nu_\mu A - Bq^2 q^\nu q_\mu + \frac{\xi - 1}{\xi}Aq^2 q^\nu q_\mu + \frac{\xi - 1}{\xi}Bq^2 q^\nu q_\mu
\end{aligned}
$$

choose $A = -1/q^4$ and the rest should vanish.

$$
\begin{aligned}
&= Bq^2 q^\nu q_\mu + \frac{\xi - 1}{\xi}\frac{q^\nu q_\mu}{q^2} + \frac{\xi - 1}{\xi}Bq^2 q^\nu q_\mu \\
&= \frac{q^\nu q_\mu}{q^2}\left[-Bq^4 - \frac{\xi - 1}{\xi} + Bq^4\frac{\xi - 1}{\xi}\right]
\end{aligned}
$$

define $B = \tilde{B}/q^4$

$$
\begin{aligned}
&= \frac{q^\nu q_\mu}{q^2\xi}\left[-\tilde{B}\xi - \xi + 1 + \tilde{B}\xi - \tilde{B}\right] \\
&= \frac{q^\nu q_\mu}{q^2\xi}[-\xi + 1 - \tilde{B}] && \stackrel{!}{=} 0
\end{aligned}
$$

Thus

$$B = \frac{1 - \xi}{q^4}$$

and propagator

$$\frac{i}{q^2}\left(-g_{\mu\nu} + (1-\xi)\frac{q_\mu q_\nu}{q^2}\right) \tag{5.11.6}$$

If ξ is left general, the final answer cannot contain ξ. In QED, current conservation ensures it with $q_\mu \tilde{j}^\mu = 0$.

5.12 Massive Vector Particles

$$\frac{i}{q^2 - M^2}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M^2}\right) \tag{5.12.1}$$

5.13 Real and Virtual Photons

Real photons have only two polarization states.

$$\sum_{\lambda=L,R} (\epsilon_\lambda)_i^* (\epsilon_\lambda)_j = \delta_{ij} - \hat{q}_i \hat{q}_j \tag{5.13.1}$$

$$\begin{aligned} -g_{\mu\nu} &= \sum_{\lambda=1}^{4} \epsilon_\mu^{(\lambda)*} \epsilon_\nu^{(\lambda)} \\ &= \sum_{\text{transverse}} (\epsilon_\mu^T)^* (\epsilon_\nu^T) + \epsilon_\mu^{L*}\epsilon_\nu^L + \epsilon_\mu^{S*}\epsilon_\nu^S \end{aligned}$$

Every photon is virtual in some sense. It must interact with electron during detection.

$$T_{fi} = -i\int \mathrm{d}^4x\, j_\mu^A(x) \frac{-g^{\mu\nu}}{q^2} j_\nu^B(x)$$

$q = (q^0, 0, 0, |\boldsymbol{q}|)$

$$= -i\int \mathrm{d}^4x \left[\frac{j_1^A j_1^B + j_2^A j_2^B}{q^2} + \frac{j_3^A j_3^B - j_0^A j_0^B}{q^2}\right]$$

with first half in transverse mode and second half in longitudinal and scalar mode

From current conservation

$$q^\mu j_\mu = 0 = q^0 j_0 - |\boldsymbol{q}| j_3 = 0$$

If photon is almost real $q^0 \approx |\boldsymbol{q}|$, then $j_0 \approx j_3$. In general

$$j_3 = \frac{1}{|\boldsymbol{q}|} q^0 j_0$$

Insert back into T_{fi} we get expression for general virtual photon

$$
\begin{aligned}
T_{fi} &= -i \int \mathrm{d}^4 x \left[\frac{j_1^A j_1^B + j_2^A j_2^B}{q^2} + \left(\frac{q^0}{|\boldsymbol{q}|} \frac{j_0^A j_0^B - j_0^A j_0^B}{q^2} \right) \right] \\
&= -i \int \mathrm{d}^4 x \left[\frac{j_1^A j_1^B + j_2^A j_2^B}{q^2} + j_0^A j_0^B \frac{{q^0}^2/|\boldsymbol{q}|^2 - 1}{q^2} \right] \\
&= -i \int \mathrm{d}^4 x \left[\frac{j_1^A j_1^B + j_2^A j_2^B}{q^2} + j_0^A j_0^B \frac{q^2}{q^2 |\boldsymbol{q}|^2} \right] \\
&= -i \int \mathrm{d}^4 x \left[\frac{j_1^A j_1^B + j_2^A j_2^B}{q^2} + \frac{j_0^A j_0^B}{|\boldsymbol{q}|^2} \right]
\end{aligned}
$$

$$
\frac{1}{|\boldsymbol{q}|^2} = \int \mathrm{d}^3 x \, \mathrm{e}^{i \boldsymbol{q} \cdot \boldsymbol{x}} \frac{1}{4\pi |\boldsymbol{x}|} \tag{5.13.2}
$$

Fourier transform

$$
T_{fi} = -i \int \cdots - i \int \mathrm{d}t \int \mathrm{d}^3 x_1 \int \mathrm{d}^3 x_2 \, \frac{j_0^A(t, \boldsymbol{x}_1) j_0^B(t, \boldsymbol{x}_2)}{4\pi |\boldsymbol{x}_2 - \boldsymbol{x}_1|}
$$

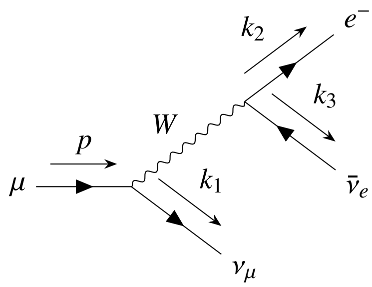
With $j_0 \sim \rho_{\mathrm{el}}$ electric charge density. It describes instantaneous Coulomb interaction.

6 Electroweak Processes

We will consider $\mathbf{SU}(2)_\mathrm{L} \times \mathbf{U}(1)_\mathrm{Y}$ theory.

6.1 Muon decay

$V - A$ interaction



$q = k_2 + k_3 = p - k_1$

$$i\mathcal{M} = \overline{u}(k_1)\frac{-ig}{\sqrt{2}}\gamma^\mu P_L u(p)\frac{-ig_{\mu\nu}}{q^2 - M_W^2}\overline{u}(k_2)\frac{-ig}{\sqrt{2}}\gamma^\nu P_L v(k_3)$$

In spinor space, Dirac equation

$$(\not{p} - m)u(p) = 0. \tag{6.1.1}$$

Propagator $M_W \sim 80\,\mathrm{GeV}$ and $m_\mu = 100\,\mathrm{MeV}$, i.e. $p^2 \ll M^2$. So 4-Fermi approximation is obtained.

$$\frac{1}{q^2 - M_W^2} = -\frac{1}{M_W^2}\frac{1}{1 - q^2/M_W^2} \approx -\frac{1}{M_W^2}$$

Then the coupling is effectively

$$G_F = \frac{\sqrt{2}g^2}{8M_W^2} \tag{6.1.2}$$

Note that when for example top and bottom quarks are present, this approximation is not valid any more.
Proceed with $m_e = m_{\nu_e} = m_{\nu_\mu} = 0$

$$\mathcal{M} = -\frac{4G_F}{\sqrt{2}}\overline{u}(k_1)\gamma^\mu P_L u(p)\overline{u}(k_2)\gamma_\mu P_L v(k_3)$$

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{16G_F^2}{2\cdot 2}\overline{u}(k_1)\gamma^\mu P_L u(p)\overline{u}(p)P_R\gamma^\nu u(k_1)\overline{u}(k_2)\gamma_\mu P_L v(k_3)\overline{v}(k_3)P_R\gamma_\nu u(k_2)$$

$$= 4G_F^2\,\mathrm{tr}\Big[\not{k}_1\gamma^\mu P_L(\not{p} + m_\mu)P_R\gamma^\nu\Big]\cdot\mathrm{tr}\Big[\not{k}_2\gamma_\mu P_L\not{k}_3 P_R\gamma_\nu\Big]$$

Term with m_μ vanishes, since $P_L P_R = 0$

Computing the two traces

$$\begin{aligned}\mathrm{tr}\Big[\not{k}_1\gamma^\mu \not{p}\gamma^\nu P_L\Big] &= \frac{4}{2}\left(k_1^\mu p^\nu - k_1 p g^{\mu\nu} + k_1^\nu p^\mu\right) - \frac{4i}{2}\epsilon_{\alpha\mu\beta\nu}k_{1\alpha}p_\beta \\ \mathrm{tr}\Big[\not{k}_2\gamma_\mu \not{k}_3\gamma_\nu P_L\Big] &= \frac{4}{2}\left((k_2)_\mu (k_3)_\nu - k_2 k_3 g_{\mu\nu} + k_{2\nu}k_{3\mu}\right) + \frac{4i}{2}\epsilon\epsilon_{\alpha\mu\beta\nu}k_2^\alpha k_3^\beta\end{aligned}$$

Using the fact that ϵ is totally anti-symmetric tensor, so the terms with one ϵ sum to zero. Note further

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu}{}^{\rho'\sigma'} = -2(g^{\rho\rho'}g^{\sigma\sigma'} - g^{\rho\sigma'}g^{\rho'\sigma}) \tag{6.1.3}$$

In order to use this formula

$$\begin{aligned}&\sigma^{\alpha\mu\beta\nu}\sigma_{\alpha'\mu\beta'\nu} \\ &= \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu}{}^{\alpha''\beta''}g_{\alpha'\alpha''}g_{\beta'\beta''} \\ &= (-2)\left[g^{\alpha\alpha''}g^{\beta\beta''} - g^{\alpha\beta''}g^{\beta\alpha''}\right]g_{\alpha'\alpha''}g_{\beta'\beta''} \\ &= (-2)\left[\delta^\alpha_{\alpha'}\delta^\beta_{\beta'} - \delta^\alpha_{\beta'}\delta^\beta_{\alpha'}\right]\end{aligned}$$

Terms without ϵ

$$\begin{aligned}&4(k_1^\mu p^\nu + k_1^\nu p^\mu - g^{\mu\nu}k_1\cdot p)(k_{2\mu}k_{3\nu} + k_{2\nu}k_{3\mu} - k_2\cdot k_3 g_{\mu\nu}) \\ &= \cdots = 4\left[2(k_1\cdot k_2)(p\cdot k_3) + 2(k_1\cdot k_3)(p\cdot k_2)\right] \\ &= 8\left[(k_1\cdot k_2)(p\cdot k_3) + (k_1\cdot k_3)(p\cdot k_2)\right]\end{aligned}$$

Then

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = 64G_F^2(k_1\cdot k_2)(p\cdot k_3)$$

Recall

$$\begin{aligned}\mathrm{d}\Gamma &= \frac{1}{2E}\overline{|\mathcal{M}|^2}\,\mathrm{d}Q \\ \mathrm{d}Q &= \frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_{k_1}}\frac{\mathrm{d}^3k_2}{(2\pi)^3 2E_{k_2}}\frac{\mathrm{d}^3k_3}{(2\pi)^3 2E_{k_3}}(2\pi)^4\delta^{(4)}(p-k_1-k_2-k_3)\end{aligned} \tag{6.1.4}$$

One can prove (In Standard Model neutrino has no mass.)

$$\frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_{k_1}} = \int \mathrm{d}^4k_1\,\theta(E_1)\delta(k_1^2) \tag{6.1.5}$$

Use identity of Delta function involving composite function.

$$\begin{aligned}\frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_{k_1}}\delta^{(4)}(p-k_1-k_2-k_3) &= \int \mathrm{d}^4k_1\,\theta(E_1)\delta(k_1^2)\delta^{(4)}(p-k_1-k_2-k_3) \\ &= \theta(k_1)\delta^{k_1^2}\end{aligned}$$

Thus

$$\mathrm{d}Q = \frac{1}{(2\pi)^5}\frac{\mathrm{d}^3k_2}{2E_{k_2}}\frac{\mathrm{d}^3k_3}{2E_{k_3}}\theta(E - E_{k_2} - E_{k_3})\delta((p - k_2 - k_3)^2)$$

In rest frame of muon $p \cdot k_3 = m_\mu \cdot E_3$

$$\begin{aligned}(k_1 + k_2)^2 &\approx 2k_1 \cdot k_2 \\ k_1 \cdot k_2 &\approx \frac{1}{2}(k_1 + k_2)^2 \\ &= \frac{1}{2}(p - k_3)^2 \\ &= \frac{1}{2}\left[p^2 - 2p \cdot k_3 + k_3^2\right] \\ &= \frac{1}{2}[m_\mu^2 - 2m_\mu E_3]\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\overline{|\mathcal{M}|^2} &= 64G_F^2 k_1 \cdot k_2 p \cdot k_3 \\ &= 64G_F^2 m_\mu E_3[m_\mu^2 - 2m_\mu E_3]\end{aligned}$$

Now we have computed $|\mathcal{M}|^2$, the actual difficulties are in phase space integral. For n-particle phase space, often use Monte-Carlo integration technique.

With θ the angle between electron and electron neutrino (k_2 and k_3)

$$\begin{aligned}(p - k_2 - k_3)^2 &= p^2 + k_2^2 + k_3^2 - 2p \cdot k_2 - 2p \cdot k_3 + 2k_2 \cdot k_3 \\ &= m_\mu^2 - 2m_\mu E_2 - 2m_\mu E_3 + 2E_2E_3(1 - \cos\theta)\end{aligned}$$

$$\mathrm{d}^3k_2\,\mathrm{d}^3k_3 = (4\pi)(2\pi)E_2^2\,\mathrm{d}E_2\,E_3^2\,\mathrm{d}E_3\,\mathrm{d}\cos\theta$$

$$\delta(\cdots - 2E_2E_3\cos\theta) = \frac{1}{2E_2E_3}\delta(-\cos\theta)$$

$$\mathrm{d}\Gamma = \frac{G_F^2}{2\pi^3}\,\mathrm{d}E_3\,\mathrm{d}E_3\,E_3(m_\mu^2 - 2m_\mu E_3)$$

$$\begin{aligned}\frac{1}{2}m_\mu - E_2 \le E_3 &\le \frac{1}{2}m_\mu \\ 0 \le E_2 &\le \frac{1}{2}m_\mu\end{aligned}$$

Carrying the integration out

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\mathrm{d}E_2} &= \frac{m_\mu G_F^2}{2\pi^3}\int_{\frac{1}{2}m_\mu - E_2}^{\frac{1}{2}m_\mu}\mathrm{d}E_3\,E_3(m_\mu - 2E_3) \\ &= \frac{G_F^2}{2\pi^3}m_\mu^2E_2^2\left(3 - \frac{4E_2}{m_\mu}\right)\end{aligned}$$

Finally

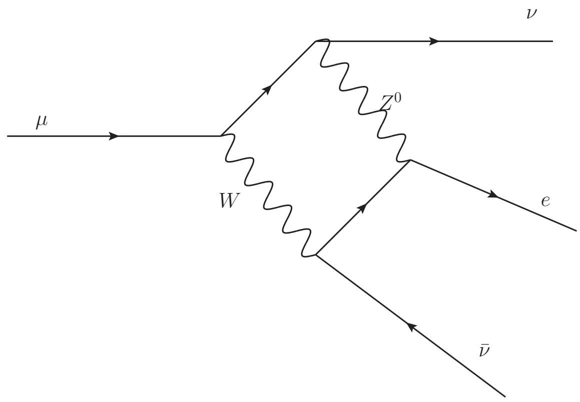
$$\Gamma = \frac{1}{\tau_\mu} = \int_0^{\frac{1}{2}m_\mu} \mathrm{d}E_2 \, \frac{\mathrm{d}\Gamma}{\mathrm{d}E_2} = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

If $m_e \neq 0$

$$\Gamma = \frac{1}{\tau_\mu} = \int_0^{\frac{1}{2}m_\mu} \mathrm{d}E_2 \, \frac{\mathrm{d}\Gamma}{\mathrm{d}E_2} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[1 - 8r^2 + 8r^6 - r^8 - 24r^4 \ln r\right]$$

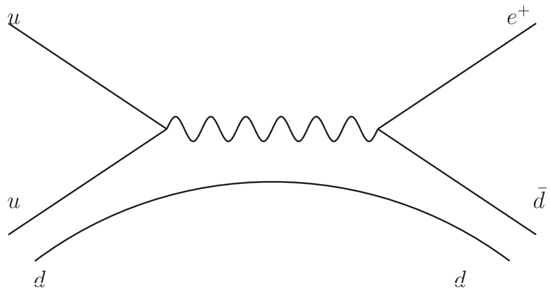
with $r = m_e/m_\mu$. r enters in boundary of phase space integration.

If one tries to measure the decay width precisely, the 1-loop electroweak correction must also be included.



Neutrino can also be considered in the calculation and one finds an upper bound for neutrino mass $m_{\nu_\mu} \leq 1\,\mathrm{keV}$.

Mass dimension of Γ is $+1$. Using dimensional argument, since $[G_F] = -2$, the decay width $\Gamma \sim G_F^2 m_\mu^5$. Grand Unified Theory postulates the decay of proton, $p \to e^+\pi^0$. Following $M_{X,Y} \sim 10 \times 10^{16}\,\mathrm{GeV}$, we estimate the decay width of proton $\Gamma \sim \frac{g^4_{\mathrm{SU(5)}}}{M_X^4} M_p^5$.



Polarized Muon Decay In the above computation, the muon polarizations are summed over. Assume the polarization is fixed

Also the spin sum relation enables us the computation. How to compute decay of polarized particle then? Recall that u, $\bar{u}$, v and $\bar{v}$ are independent solutions to Dirac equation.

$$u^{(s)} = \sqrt{E+m}\begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \end{pmatrix} \tag{6.1.6}$$

To get chiral-left part of $u^{(s)}$ $P_L u^{(s)}(\boldsymbol{p})$. Can also define two projection operators

$$\Lambda_+ = \frac{1}{2m}(\not{p} + m) \tag{6.1.7}$$

$$\Lambda_- = \frac{1}{2m}(-\not{p} + m) \tag{6.1.8}$$

They are indeed projection operators

$$\begin{aligned}
\Lambda_+ + \Lambda_- &= \mathbb{1}_4 \\
(\Lambda_+)^2 &= \frac{1}{4m^2}\left(\not{p}\not{p} + 2m\not{p} + m^2\right) = \frac{1}{4m^2}2m(\not{p} + m) = \Lambda_+ \\
(\Lambda_-)^2 &= \Lambda_-
\end{aligned}$$

Let $\sum_{r=1}^{4} a_r u^{(r)}$ be an arbitrary spinor

$$\begin{aligned}
\Lambda_+ &= \sum_{r=1}^{4} a_r u^{(r)} \\
&= \sum_{r=1}^{4} a_r \left(\sum_{s=1}^{2} \frac{u^{(s)}\bar{u}^{(s)}}{2m} \right) u^{(r)}
\end{aligned}$$

use $\bar{u}^{(s)} u^{(r)} = 2m\delta^{rs}$

$$= \sum_{r=1}^{2} a_r u^{(r)}$$

arbitrary $E > 0$ spinor. Thus Λ_+ projects to $E > 0$ states and Λ_- projects $E < 0$ states.

Particle at rest, spin $\boldsymbol{s}$ and $|\boldsymbol{s}| = 1$. Write a four-vector $s^\mu = (0, \boldsymbol{s})$. At rest $p^\mu = (m, 0)$ and $(s \cdot p) = 0$.

Starting from the s^μ, can compute s'^μ in any frame by a Lorentz boost.

$$\begin{aligned}
s^0 &= \frac{\boldsymbol{p} \cdot \boldsymbol{\xi}}{m} \\
s^i &= \xi^i + \frac{(\boldsymbol{p} \cdot \boldsymbol{\xi})}{m(m + E)} p^i
\end{aligned}$$

$\boldsymbol{\xi}$ denotes the direction of spin and $|\boldsymbol{\xi}| = 1$

Compute $s \cdot p$ in new frame

$$\begin{aligned}
s \cdot p &= \frac{\boldsymbol{p} \cdot \boldsymbol{\xi}}{m} E - \boldsymbol{p} \cdot \boldsymbol{\xi} \\
&= \frac{\boldsymbol{p} \cdot \boldsymbol{\xi} \boldsymbol{p}^2}{m(m + E)} \\
&= \boldsymbol{p} \cdot \boldsymbol{\xi} \left[\frac{E}{m} - 1 - \frac{\boldsymbol{p}^2}{m(m + E)} \right] \\
&= \frac{\boldsymbol{p} \cdot \boldsymbol{\xi}}{m(m + E)} \left[E(m + E) - m(m + E) - \boldsymbol{p}^2 \right] \\
&= 0
\end{aligned}$$

Define two projection operators

$$\begin{aligned}
\Sigma_\pm &= \frac{1}{2}\left(\mathbb{1}_4 \pm \gamma^5 \not{s}\right) \\
\Sigma_+ + \Sigma_- &= \mathbb{1}_4 \\
(\Sigma_-)^2 &= \frac{1}{4}(\mathbb{1} - \gamma^5 \not{s})(\mathbb{1} - \gamma^5 \not{s}) \\
&= \frac{1}{4}\left[\mathbb{1} - 2\gamma^5 \not{s} + \gamma^5 \not{s} \gamma^5 \not{s}\right] \\
&= \frac{1}{4}[2\mathbb{1} - 2\gamma^5 \not{s}] \\
&= \Sigma_- \\
(\Sigma_+)^2 &= \Sigma_+
\end{aligned}$$

What do $\Sigma_\pm$ project out? In rest frame $\Sigma_- = \frac{1}{2}(\mathbb{1} - \gamma^5\gamma^i s^i)$. Choose $\boldsymbol{s} = \boldsymbol{e}_3 = \boldsymbol{e}_z$.

$$\begin{aligned}
\Sigma_- &= \frac{1}{2}(\mathbb{1} - \gamma^5\gamma^3) \\
&= \frac{1}{2}\left[\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} - \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}\begin{pmatrix} \mathbb{1}_2 - \sigma^3 & 0 \\ 0 & \mathbb{1}_2 + \sigma^3 \end{pmatrix}
\end{aligned}$$

so project out helicity states.

$$u^k(\boldsymbol{p}, s)\bar{u}_i(\boldsymbol{p}, s) = \frac{1}{2}\left[(\not{p} + m)(\mathbb{1} - \gamma^5 \not{s})\right]^k_i \tag{6.1.9}$$

or in computation

$$(\not{p} + m) \mapsto \frac{1}{2}(\not{p} + m_\mu)(\mathbb{1} - \gamma^5 \not{s}_\mu) \tag{6.1.10}$$

Turns out we can skip computation. Replace $p_\alpha \mapsto p_\alpha - m s_\alpha$ in final answer

$$\begin{aligned}
&\operatorname{tr}\Big[\dots P_R(\not{p} + m_\mu)(\mathbb{1} - \gamma^5 \not{s})\gamma_\alpha P_R \dots\Big] \\
&= \operatorname{tr}\Big[\dots P_R(\not{p} - m_\mu \gamma^5 \not{s})\gamma_\alpha P_R \dots\Big] \\
&= \operatorname{tr}\Big[\dots P_R(\not{p} - m\not{s})\gamma_\alpha P_R\Big]
\end{aligned}$$

with

$$\boldsymbol{\xi} \cdot \frac{\boldsymbol{p}_e}{|\boldsymbol{p}_e|} = \cos\theta$$

In the end

$$\frac{\mathrm{d}\Gamma}{\Gamma} = \frac{1}{2}\left(1 - \frac{1}{3}\cos\theta\right)\mathrm{d}\cos\theta \tag{6.1.11}$$

Nuclear β-decay $^{14}O \to {}^{14}N^* e^+ \nu_e$ β^+-emitter $n \to p e^- \bar{\nu}_e$

Non-relativistic limit of Pauli-Dirac spinor

$$
\begin{aligned}
u^{(s)}(\boldsymbol{p}) &= \sqrt{E+m}\begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \end{pmatrix} \\
&\to \sqrt{2m}\begin{pmatrix} \chi^{(s)} \\ 0 \end{pmatrix} \\
\bar{u}^{(s)} &= \sqrt{2m}\begin{pmatrix} x^{(s)} & 0 \end{pmatrix}
\end{aligned}
$$

$$
\bar{\psi}_n \gamma_\mu \frac{1}{2}\left(\mathbb{1} - \gamma^5\right)\psi_p(x)
$$

γ^5 matrice has no effect and vanishes

$$
\begin{aligned}
&= \frac{1}{2}\bar{\psi}_n(x)\gamma_\mu\psi_p(x) \\
&= \frac{1}{2}\psi_n(x)\gamma^0\gamma_\mu\psi_p(x) \\
&\to \frac{1}{2}\psi_n^\dagger\psi_p
\end{aligned}
$$

$\mu \to i = 1, 2, 3$

$$
\begin{aligned}
\gamma^0\gamma^i &= \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}
\end{aligned}
$$

It is off-diagonal.

Thus in the non-relativistic limit

$$
{\psi_n}^\dagger \gamma^0 \gamma^i \psi_p \to 0
$$

Bibliography

[1] Herbi K. Dreiner, Howard E. Haber, and Stephen P. Martin. "Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry". In: *Physics Reports* 494.1-2 (Sept. 2010), pp. 1–196. ISSN: 0370-1573. DOI: 10.1016/j.physrep.2010.05.002. URL: http://dx.doi.org/10.1016/j.physrep.2010.05.002.

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