Theoretical particle physics

Chenhuan Wang

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Contents

0 Organisational

Tutorials: Thursday: 8-10, 10-12 Friday: 10-12, 13-15

Exam consists of four problems

- first quickies
- 2nd-4th: similar in style to homework; two will be very close to homework

One needs 50% of points from homework. May hand in pairs.

Content of the lectures

- Standard Model of particle physics
- Electroweak sector
  - gauge principle
  - Higgs mechanism
  - Yukawa interactions
  - CP-violation

Exercises

- go through basics of computing Feynman diagrams
- not to derive the formalism
- Lagrange → Feynman rules → amplitudes → cross section and decay rates (measured quantities)

Literature

- Halzen and Martin, Quarks and Leptons (a lot of basics of QCD)
- Cheng and Li (includes also quantum field theory topics CP-violation in Standard Model)
- Mark Thomson
- QFT basics
  - Peskin and Schroeder
  - M.Schwartz
  - Ryder
- Okun, Leptons and Quarks

1 Introduction

1.1 Standard Model

It is the fundamental theory of nature. There are three interactions included

- electromagnetic
- weak
- strong
- Higgs boson exchange

Electromagnetic and weak interactions can be unified into electroweak interactions.

In the Sun all these three interactions and gravity are present

- Photons reaching us clearly indicate QED's involvement.
- Neutrinos produced in weak interaction. Four protons to two protons and two neutrons (Helium). Only weak interaction can change the colour of quarks.
- Binding of Helium via strong interaction and binding energy released as kinetic energy.
- Gravity brings protons together and at high temperature to give helium.

Gauge theories (Lie groups algebras)

- EM: $U(1)_{EM}$, $U(1)_Y$
- weak: $SU(2)_L$
- strong: $SU(3)_c$

Forces in quantum theories involve exchange particles spin 1, vector bosons

- photon γ
- weak $W^{\pm}$, W^0 and Z^0 (mixture of W^0 and hyper charge) (discovered at CERN)
- strong $g^a, a = 1, \ldots, 8$ gluons (discovered at DESY)

particles with spin $\frac{1}{2}$
Leptons

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, e_R^-; \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \mu_R^-; \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, \tau_R^-$$

Quarks

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, u_R, d_R; \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, c_R, s_R; \quad \begin{pmatrix} t \\ b \end{pmatrix}_L, t_R, b_R$$

One complete generation

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, e_R^-; \quad \begin{pmatrix} u \\ d \end{pmatrix}_L, u_R, d_R;$$

To remove any one part, then gauge theory is inconsistent. It is known as "anomaly". (AQFT)

Higgs boson h^0, spin 0
In Standard Model Higgs bosons are described by complex scalar fields $\begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$. h^0 is the only fundamental scalar in nature, as far as we know.

1.2 Energy scales

- binding energy of atoms $1 - 10$eV
- binding energy of nucleons $\approx$ 1MeV
- no known binding energy in particle physics
- protons and neutrons $\approx \Lambda_{QCD} \approx \mathcal{O}(100\text{MeV})$

Particles have masses

- electron $m_e = 511\,\text{keV}$
- muon $m_\mu = 105\,\text{MeV}$
- tau $m_\tau = 1.7\,\text{GeV}$
- neutrinos $m_\nu < 1\,\text{eV}$
- quarks*
  - $m_u \approx 3\,\text{MeV}$
  - $m_d \approx 5\,\text{MeV}$
  - ...
- photon $m_\gamma = 0$
- gluon $m_g = 0$
- Higgs $m_{Higgs} \approx 125\,\text{GeV}$

*mass of proton mainly comes from dynamical effect "gluon"

Colliders

- LEP 91 GeV − 200 GeV
- Tevatron($p\bar{p}$) 800 GeV − 2 TeV
- LHC 7 TeV − 13 TeV

1.3 Natural units

$$\hbar = c = 1 \tag{1.3.1}$$
$$k_B = 1 \tag{1.3.2}$$

Everything expressed in term of powers of energy.

$$1\,\text{fm} = 1 \times 10^{-15}\,\text{m} = 5\,\text{GeV}^{-1}$$

2 Lorentz Transformation

2.1 Introduction

Metric (used for distance measuring)

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.1.1}$$

In string and general relativity people tend to use diag$(-, +, +, +)$.

$$p = (E, \boldsymbol{p}) \tag{2.1.2}$$

$$p^2 = E^2 - \boldsymbol{p}^2 = m^2 \tag{2.1.3}$$

Light is always light-like

$$t^2 - (x^2 + y^2 + z^2) = 0$$

Greek indices always go from 0 to 3

$$r^2 = g_{\mu\nu} r^\mu r^\nu = t^2 - \boldsymbol{r}^2 \tag{2.1.4}$$

Distance between two spacetime point is defined via

$$|r_A - r_B| = \sqrt{(r_A - r_B) \cdot (r_A - r_B)} = \sqrt{r_A^2 + r_B^2 - 2 r_A \cdot r_B} \tag{2.1.5}$$

2.2 Lorentz Transformation

Lorentz Transformation is transformation between two inertial frames moving with constant velocity $\boldsymbol{v}$ with respect to each other (boosts).

$$x = (x_0, x_1, x_2, x_3)$$
$$x' = (x_0', x_1', x_2', x_3')$$
$$x_0 = ct = t; \quad c = 1$$

We define

$$\beta = \frac{v}{c} \quad \text{or} \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c} \tag{2.2.1}$$

Coordinates in these two frames are related like

$$
\begin{aligned}
x_0' &= \gamma(x_0 - \beta x_1) \\
x_1' &= \gamma(x_1 - \beta x_0) \\
x_2' &= x_2 \\
x_3' &= x_3
\end{aligned}
$$

Inverse transformation with $\beta \mapsto -\beta$
γ-factor is defined via

$$
\gamma = \frac{1}{\sqrt{1 - \boldsymbol{\beta}^2}} \tag{2.2.2}
$$

Since $|\boldsymbol{\beta}| \leq 1 \Rightarrow \gamma \geq 1$
Alternative parametrization

$$
\begin{aligned}
\beta &= \tanh(\zeta), \quad \gamma = \cosh(\zeta) \\
\gamma\beta &= \sinh(\zeta)
\end{aligned}
$$

Insert this into equation (2.2.2)

$$
\begin{aligned}
x_0' &= x_0 \cosh(\zeta) - x_1 \sinh(\zeta) \\
x_1' &= x_0 \sinh(\zeta) - x_1 \cosh(\zeta)
\end{aligned}
$$

We can turn this into matrices

$$
\begin{aligned}
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} \\
x &= \Lambda x'
\end{aligned} \tag{2.2.3}
$$

2.3 Mathematical Properties of Lorentz Transformation

Distance is invariant under Lorentz transformation

$$
s^2 = x_0^2 - x_1^2 + x_2^2 + x_3^2 = x^2 \tag{2.3.1}
$$

Lorentz transformation includes

- Rotation and boosts
- Parity $\boldsymbol{x} \mapsto -\boldsymbol{x}$
- Time reversal $t \mapsto -t$

We can also expand it with translation. It then turns to Poincare group.

2.3.1 Tensors

Define a function of original coordinates ($\alpha = 0, 1, 2, 3$)

$$x'^{\alpha} = x'^{\alpha}(x^0, x^1, x^2, x^3) \tag{2.3.2}$$

If x'^{α} transforms like

$$x'^{\alpha} = \frac{x'^{\alpha}}{x^{\beta}} x^{\beta} \tag{2.3.3}$$

it is called *contravariant*

Consider derivative $\frac{\partial}{\partial x'^{\alpha}}$

$$\frac{\partial f(x)}{\partial x'^{\alpha}} = \frac{\partial f(x)}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \tag{2.3.4}$$

We can see the x' is now in the denominator. The objects transformed like this are called *covariant*.

Consider the following generic objects: A'^{α} contravariant vector

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} \tag{2.3.5}$$

B'_{α} is covariant

$$B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} \tag{2.3.6}$$

Note (x^0, x^1, x^2, x^3) is contravariant.

The field strength tensor $F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta}$ is contravariant rank 2.

Mixed is also allowed $H'^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{x^{\delta}}{\partial x'^{\beta}} H^{\delta}_{\gamma}$

Inner or scalar product

$$\begin{aligned} B' \cdot A' &= B'_{\alpha} A'^{\alpha} \\ &= \left(\frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} \right) \left(\frac{\partial x'^{\alpha}}{\partial x^{\gamma}} A^{\gamma} \right) \\ &= \frac{\partial x^{\beta}}{\partial x^{\gamma}} B_{\beta} A^{\gamma} \\ &= \delta^{\beta}_{\gamma} B_{\beta} A^{\gamma} = B \cdot A \end{aligned}$$

$$\begin{aligned} \mathrm{d}s^2 &= \left(\mathrm{d}x^0\right)^2 - (\mathrm{d}\boldsymbol{x})^2 \\ &= (g_{\alpha\beta}\, \mathrm{d}x^{\alpha})\, \mathrm{d}x^{\beta} = \mathrm{d}x_{\beta}\, \mathrm{d}x^{\beta} \end{aligned} \tag{2.3.7}$$

Thus we can use metric tensor to lower index $\mathrm{d}x_{\beta} = g_{\alpha\beta}\, \mathrm{d}x^{\alpha}$

$$\begin{aligned} A^{\alpha} &= \left(A^0, \boldsymbol{A}\right) \\ A_{\alpha} &= \left(A^0, -\boldsymbol{A}\right) \end{aligned}$$

2.4 Matrix Representation of Lorentz Transformation

2.4.1 General Properties

We have

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

Then $a \cdot b = (a, gb) = g_{\mu\nu} a^\mu b^\nu = (ga, b) = a^T g b = (ga)^T b$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \mapsto x' = \Lambda x \tag{2.4.1}$$

$$x \cdot x = x' \cdot x' = (\Lambda x)(\Lambda x) \tag{2.4.2}$$

$$\begin{aligned} g_{\mu\nu} x^\mu x^\nu &= g_{\sigma\tau} x'^\sigma x'^\tau \\ &= g_{\sigma\tau} \Lambda^\sigma{}_\mu x^\mu \Lambda^\tau{}_\nu x^\nu \\ &= g_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu x^\mu x^\nu \end{aligned}$$

Then we have the *defining* rule of Lorentz group

$$g_{\mu\nu} = g_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \tag{2.4.3}$$

$$g = \Lambda^T g \Lambda \tag{2.4.4}$$

Properties

- $|\det(\Lambda)| = 1$
- $\left|\Lambda^0{}_0\right| \geq 1$

The orthochronous Lorentz transformations Λ forms a group.

Parity does not form a group

$$\Lambda_P = \mathrm{diag}(1, -1, -1, -1) \tag{2.4.5}$$

Time reversal

$$\Lambda_T = \mathrm{diag}(-1, +1, +1, +1) \tag{2.4.6}$$

There are four classes of Lorentz transformations depending on $\left(\mathrm{sgn}(\det(\Lambda)), \mathrm{sgn}(\Lambda^0_0)\right)$

- $(+, +)$ Λ
- $(-, -)$ $\Lambda_T \Lambda$
- $(-, +)$ $\Lambda_P \Lambda$
- $(+, -)$ $\Lambda_T \Lambda_P \Lambda$

Orthochronous Λ has 6 parameters, 3 for boosts and 3 for rotations. $\Lambda^T g \Lambda = g$ is actually 16 equations. All matrices here are symmetric. Thus 6 of 16 are redundant. There are 10 independent equations. Λ has 16 entries and it has $16 - 10 = 6$ free parameters.

2.4.2 Explicit Construction

We will restrict ourselves in orthochronous Lorentz transformations. The exponential function is defines via Taylor expansion. With $L \in \mathbb{R}^{4\times 4}$

$$\Lambda = \mathrm{e}^L = \exp(L)$$

From linear algebra we know

$$\det(\Lambda) = \det\left(\mathrm{e}^L\right) = \mathrm{e}^{\mathrm{tr}(L)} \tag{2.4.7}$$

Since $\det(\Lambda) = 1$, $\mathrm{tr}(L) = 0$

$$\begin{aligned}
\Lambda^T g \Lambda &= g \\
g\Lambda^T g \Lambda &= \mathbb{1}_4 \\
g\Lambda^T g &= \Lambda^{-1} \\
\exp\left(gL^T g\right) &= \Lambda^{-1} = \exp(-L) \\
\Leftrightarrow gL^T g &= -L \\
\Leftrightarrow (gL)^T &= -gL
\end{aligned}$$

This means that gL is anti-symmetric

$$L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & L_{12} & 0 & L_{23} \\ L_{03} & L_{13} & L_{23} & 0 \end{pmatrix}$$

Define six basis matrices $S_{1,2,3}$ and $K_{1,2,3}$

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

S_i is the generator of 3-dimensional rotations and K_i is the generator of 3-dimensional boosts.

$$\begin{aligned}
\hat{n} \in \mathbb{R}^3, \quad |\hat{n}| &= 1 \\
\hat{n} \cdot \boldsymbol{S} &= n_1 S_1 + n_2 S_2 + n_3 S_3 \\
(\hat{n} \cdot \boldsymbol{S})^3 &= -\hat{n} \cdot \boldsymbol{S} \\
(\hat{n} \cdot \boldsymbol{K})^3 &= +\hat{n} \cdot \boldsymbol{S}
\end{aligned}$$

In the end

$$L = -\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\zeta} \cdot \boldsymbol{K} \quad \text{with } \boldsymbol{\omega}, \boldsymbol{\zeta} \in \mathbb{R}^3 \tag{2.4.8}$$

$$\Lambda = \exp(-\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\zeta} \cdot \boldsymbol{K}) \tag{2.4.9}$$

We now will look at concrete examples

- $\boldsymbol{\zeta} = 0,\ \boldsymbol{\omega} = w\hat{e}_z$

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  Rotational angle is ω.

- $\boldsymbol{\omega} = 0,\ \boldsymbol{\zeta} = \zeta\hat{e}_x$

$$\Lambda = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we want to calculate the algebra $\left[S_i, K_j\right]$ and $\left[S_i, S_j\right]$