# Theoretical particle physics

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# Contents

# 0 Organisational

Tutorials: Thursday: 8-10, 10-12 Friday: 10-12, 13-15

**Exam** consists of four problems

- first quickies
- 2nd-4th: similar in style to homework; two will be very close to homework

One needs 50% of points from homework. May hand in pairs.

### Content of the lectures

- Standard Model of particle physics
- Electroweak sector
  - gauge principle
  - Higgs mechanism
  - Yukawa interactions
  - CP-violation

### Exercises

- go through basics of computing Feynman diagrams
- not to derive the formalism
- Lagrange → Feynman rules → amplitudes → cross section and decay rates (measured quantities)

### Literature

- Halzen and Martin, Quarks and Leptons (a lot of basics of QCD)
- Cheng and Li (includes also quantum field theory topics CP-violation in Standard Model )
- Mark Thomson
- QFT basics
  - Peskin and Schroeder
  - M.Schwartz
  - Ryder
- Okun, Leptons and Quarks

# 1 Introduction

## 1.1 Standard Model

It is the fundamental theory of nature. There are three interactions included

- electromagnetic
- weak
- strong
- Higgs boson exchange

Electromagnetic and weak interactions can be unified into electroweak interactions.

In the Sun all these three interactions and gravity are present

- Photons reaching us clearly indicate QED's involvement.
- Neutrinos produced in weak interaction. Four protons to two protons and two neutrons (Helium). Only weak interaction can change the colour of quarks.
- Binding of Helium via strong interaction and binding energy released as kinetic energy.
- Gravity brings protons together and at high temperature to give helium.

**Gauge theories** (Lie groups algebras)

- EM: $U(1)_{EM}$, $U(1)_Y$
- weak: $SU(2)_L$
- strong: $SU(3)_c$

Forces in quantum theories involve exchange particles spin 1, vector bosons

- photon $\gamma$
- weak $W^{\pm}$, $W^0$ and $Z^0$ (mixture of $W^0$ and hyper charge) (discovered at CERN)
- strong $g^a, a = 1, \ldots, 8$ gluons (discovered at DESY)

**particles with spin $\frac{1}{2}$**
Leptons

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, e_R^-; \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \mu_R^-; \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, \tau_R^-$$

Quarks

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, u_R, d_R; \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, c_R, s_R; \quad \begin{pmatrix} t \\ b \end{pmatrix}_L, t_R, b_R$$

One complete generation

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, e_R^-; \quad \begin{pmatrix} u \\ d \end{pmatrix}_L, u_R, d_R;$$

To remove any one part, then gauge theory is inconsistent. It is known as "anomaly". (AQFT)

**Higgs boson** $h^0$, spin 0
In Standard Model Higgs bosons are described by complex scalar fields $\begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$. $h^0$ is the only fundamental scalar in nature, as far as we know.

## 1.2 Energy scales

- binding energy of atoms $1 - 10$eV
- binding energy of nucleons $\approx$ 1MeV
- no known binding energy in particle physics
- protons and neutrons $\approx \Lambda_{QCD} \approx O(100\text{MeV})$

Particles have masses

- electron $m_e = 511\,\text{keV}$
- muon $m_\mu = 105\,\text{MeV}$
- tau $m_\tau = 1.7\,\text{GeV}$
- neutrinos $m_\nu < 1\,\text{eV}$
- quarks*
  - $m_u \approx 3\,\text{MeV}$
  - $m_d \approx 5\,\text{MeV}$
  - ...
- photon $m_\gamma = 0$
- gluon $m_g = 0$
- Higgs $m_{Higgs} \approx 125\,\text{GeV}$

*mass of proton mainly comes from dynamical effect "gluon"

**Colliders**

- LEP 91 GeV − 200 GeV
- Tevatron($p\bar{p}$) 800 GeV − 2 TeV
- LHC 7 TeV − 13 TeV

## 1.3 Natural units

$$\hbar = c = 1 \tag{1.3.1}$$
$$k_B = 1 \tag{1.3.2}$$

Everything expressed in term of powers of energy.

$$1\,\mathrm{fm} = 1 \times 10^{-15}\,\mathrm{m} = 5\,\mathrm{GeV}^{-1}$$

# 2 Lorentz Transformation

## 2.1 Introduction

Metric (used for distance measuring)

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.1.1}$$

In string and general relativity people tend to use diag$(-, +, +, +)$.

$$p = (E, \boldsymbol{p}) \tag{2.1.2}$$

$$p^2 = E^2 - \boldsymbol{p}^2 = m^2 \tag{2.1.3}$$

Light is always light-like

$$t^2 - (x^2 + y^2 + z^2) = 0$$

Greek indices always go from 0 to 3

$$r^2 = g_{\mu\nu} r^\mu r^\nu = t^2 - \boldsymbol{r}^2 \tag{2.1.4}$$

Distance between two spacetime point is defined via

$$|r_A - r_B| = \sqrt{(r_A - r_B) \cdot (r_A - r_B)} = \sqrt{r_A^2 + r_B^2 - 2 r_A \cdot r_B} \tag{2.1.5}$$

## 2.2 Lorentz Transformation

*Lorentz Transformation* is transformation between two inertial frames moving with constant velocity $\boldsymbol{v}$ with respect to each other (boosts).

$$\begin{aligned} x &= (x_0, x_1, x_2, x_3) \\ x' &= (x'_0, x'_1, x'_2, x'_3) \\ x_0 &= ct = t; \quad c = 1 \end{aligned}$$

We define

$$\beta = \frac{v}{c} \quad \text{or} \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c} \tag{2.2.1}$$

Coordinates in these two frames are related like

$$
\begin{aligned}
x_0' &= \gamma(x_0 - \beta x_1) \\
x_1' &= \gamma(x_1 - \beta x_0) \\
x_2' &= x_2 \\
x_3' &= x_3
\end{aligned}
$$

Inverse transformation with $\beta \mapsto -\beta$
$\gamma$-factor is defined via

$$\gamma = \frac{1}{\sqrt{1-\boldsymbol{\beta}^2}} \tag{2.2.2}$$

Since $|\boldsymbol{\beta}| \leq 1 \Rightarrow \gamma \geq 1$
Alternative parametrization

$$
\begin{aligned}
\beta &= \tanh(\zeta), \quad \gamma = \cosh(\zeta) \\
\gamma\beta &= \sinh(\zeta)
\end{aligned}
$$

Insert this into equation (2.2.2)

$$
\begin{aligned}
x_0' &= x_0 \cosh(\zeta) - x_1 \sinh(\zeta) \\
x_1' &= x_0 \sinh(\zeta) - x_1 \cosh(\zeta)
\end{aligned}
$$

We can turn this into matrices

$$
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix}
$$

$$x = \Lambda x' \tag{2.2.3}$$

## 2.3 Mathematical Properties of Lorentz Transformation

Distance is invariant under Lorentz transformation

$$s^2 = x_0^2 - x_1^2 + x_2^2 + x_3^2 = x^2 \tag{2.3.1}$$

Lorentz transformation includes

- Rotation and boosts
- Parity $\boldsymbol{x} \mapsto -\boldsymbol{x}$
- Time reversal $t \mapsto -t$

We can also expand it with translation. It then turns to Poincare group.

### 2.3.1 Tensors

Define a function of original coordinates ($\alpha = 0, 1, 2, 3$)

$$x'^{\alpha} = x'^{\alpha}(x^0, x^1, x^2, x^3) \tag{2.3.2}$$

If $x'^{\alpha}$ transforms like

$$x'^{\alpha} = \frac{x'^{\alpha}}{x^{\beta}} x^{\beta} \tag{2.3.3}$$

it is called *contravariant*

Consider derivative $\frac{\partial}{\partial x'^{\alpha}}$

$$\frac{\partial f(x)}{\partial x'^{\alpha}} = \frac{\partial f(x)}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \tag{2.3.4}$$

We can see the $x'$ is now in the denominator. The objects transformed like this are called *covariant*.

Consider the following generic objects: $A'^{\alpha}$ contravariant vector

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} \tag{2.3.5}$$

$B'_{\alpha}$ is covariant

$$B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} \tag{2.3.6}$$

Note $(x^0, x^1, x^2, x^3)$ is contravariant.

The field strength tensor $F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta}$ is contravariant rank 2.

Mixed is also allowed $H'^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{x^{\delta}}{\partial x'^{\beta}} H^{\delta}_{\gamma}$

#### Inner or scalar product

$$\begin{aligned}
B' \cdot A' &= B'_{\alpha} A'^{\alpha} \\
&= \left(\frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta}\right)\left(\frac{\partial x'^{\alpha}}{\partial x^{\gamma}} A^{\gamma}\right) \\
&= \frac{\partial x^{\beta}}{\partial x^{\gamma}} B_{\beta} A^{\gamma} \\
&= \delta^{\beta}_{\gamma} B_{\beta} A^{\gamma} = B \cdot A
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}s^2 &= \left(\mathrm{d}x^0\right)^2 - (\mathrm{d}\boldsymbol{x})^2 \\
&= (g_{\alpha\beta}\,\mathrm{d}x^{\alpha})\,\mathrm{d}x^{\beta} = \mathrm{d}x_{\beta}\,\mathrm{d}x^{\beta}
\end{aligned} \tag{2.3.7}$$

Thus we can use metric tensor to lower index $\mathrm{d}x_{\beta} = g_{\alpha\beta}\,\mathrm{d}x^{\alpha}$

$$\begin{aligned}
A^{\alpha} &= \left(A^0, \boldsymbol{A}\right) \\
A_{\alpha} &= \left(A^0, -\boldsymbol{A}\right)
\end{aligned}$$

## 2.4 Matrix Representation of Lorentz Transformation

### 2.4.1 General Properties

We have

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

Then $a \cdot b = (a, gb) = g_{\mu\nu} a^\mu b^\nu = (ga, b) = a^T gb = (ga)^T b$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \mapsto x' = \Lambda x \tag{2.4.1}$$
$$x \cdot x = x' \cdot x' = (\Lambda x)(\Lambda x) \tag{2.4.2}$$

$$\begin{aligned} g_{\mu\nu} x^\mu x^\nu &= g_{\sigma\tau} x'^\sigma x'^\tau \\ &= g_{\sigma\tau} \Lambda^\sigma{}_\mu x^\mu \Lambda^\tau{}_\nu x^\nu \\ &= g_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu x^\mu x^\nu \end{aligned}$$

Then we have the *defining* rule of Lorentz group

$$g_{\mu\nu} = g_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \tag{2.4.3}$$
$$g = \Lambda^T g \Lambda \tag{2.4.4}$$

#### Properties

- $|\det(\Lambda)| = 1$
- $\left|\Lambda^0{}_0\right| \geq 1$

The orthochronous Lorentz transformations $\Lambda$ forms a group.

Parity does not form a group

$$\Lambda_P = \mathrm{diag}(1, -1, -1, -1) \tag{2.4.5}$$

Time reversal

$$\Lambda_T = \mathrm{diag}(-1, +1, +1, +1) \tag{2.4.6}$$

There are four classes of Lorentz transformations depending on $\left(\mathrm{sgn}(\det(\Lambda)), \mathrm{sgn}(\Lambda^0_0)\right)$

- $(+, +)$ $\Lambda$
- $(-, -)$ $\Lambda_T \Lambda$
- $(-, +)$ $\Lambda_P \Lambda$
- $(+, -)$ $\Lambda_T \Lambda_P \Lambda$

Orthochronous $\Lambda$ has 6 parameters, 3 for boosts and 3 for rotations. $\Lambda^T g \Lambda = g$ is actually 16 equations. All matrices here are symmetric. Thus 6 of 16 are redundant. There are 10 independent equations. $\Lambda$ has 16 entries and it has $16 - 10 = 6$ free parameters.

### 2.4.2 Explicit Construction

We will restrict ourselves in orthochronous Lorentz transformations. The exponential function is defines via Taylor expansion. With $L \in \mathbb{R}^{4\times 4}$

$$\Lambda = \mathrm{e}^{L} = \exp(L)$$

From linear algebra we know

$$\det(\Lambda) = \det\left(\mathrm{e}^{L}\right) = \mathrm{e}^{\mathrm{tr}(L)} \tag{2.4.7}$$

Since $\det(\Lambda) = 1$, $\mathrm{tr}(L) = 0$

$$\begin{aligned}
\Lambda^T g \Lambda &= g \\
g\Lambda^T g \Lambda &= \mathbb{1}_4 \\
g\Lambda^T g &= \Lambda^{-1} \\
\exp\left(gL^T g\right) &= \Lambda^{-1} = \exp(-L) \\
\Leftrightarrow gL^T g &= -L \\
\Leftrightarrow (gL)^T &= -gL
\end{aligned}$$

This means that $gL$ is anti-symmetric

$$L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & L_{12} & 0 & L_{23} \\ L_{03} & L_{13} & L_{23} & 0 \end{pmatrix}$$

Define six basis matrices $S_{1,2,3}$ and $K_{1,2,3}$

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$S_i$ is the generator of 3-dimensional rotations and $K_i$ is the generator of 3-dimensional boosts.

$$\begin{aligned}
\hat{n} &\in \mathbb{R}^3, \quad |\hat{n}| = 1 \\
\hat{n} \cdot \boldsymbol{S} &= n_1 S_1 + n_2 S_2 + n_3 S_3 \\
(\hat{n} \cdot \boldsymbol{S})^3 &= -\hat{n} \cdot \boldsymbol{S} \\
(\hat{n} \cdot \boldsymbol{K})^3 &= +\hat{n} \cdot \boldsymbol{S}
\end{aligned}$$

In the end

$$L = -\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\zeta} \cdot \boldsymbol{K} \quad \text{with } \boldsymbol{\omega}, \boldsymbol{\zeta} \in \mathbb{R}^3 \tag{2.4.8}$$

$$\Lambda = \exp(-\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\zeta} \cdot \boldsymbol{K}) \tag{2.4.9}$$

$\boldsymbol{\omega}$ is the axis of rotation, $|\boldsymbol{\omega}|$ is then the angle of rotation.
$\tanh|\boldsymbol{\zeta}| = \beta$ and $\frac{\boldsymbol{\zeta}}{|\zeta|}$ is the direction of boost.
We now will look at concrete examples

- $\boldsymbol{\zeta} = 0,\ \boldsymbol{\omega} = w\hat{e}_z$

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  Rotational angle is $\omega$.

- $\boldsymbol{\omega} = 0,\ \boldsymbol{\zeta} = \zeta\hat{e}_x$

$$\Lambda = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Pure general boost $\boldsymbol{\zeta}$

$$\Lambda = \exp(-\boldsymbol{\zeta}\cdot\boldsymbol{K})$$
$$\boldsymbol{\zeta} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|}\tanh^{-1}|\boldsymbol{\beta}|, \quad \hat{\beta} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|}$$
$$\Lambda = \exp\Big(-\hat{\beta}\cdot\boldsymbol{K}\tanh^{-1}(\beta)\Big)$$

### 2.4.3 Algebra of generators

Consider the commutation algebra of $S_{i=1,2,3}$ and $K_{i=1,2,3}$

$$\left[S_i, S_j\right] = \epsilon_{ijk}S_k \tag{2.4.10}$$
$$\left[S_i, K_j\right] = \epsilon_{ijk}K_k \tag{2.4.11}$$
$$\left[K_i, K_j\right] = -\epsilon_{ijk}S_k \tag{2.4.12}$$

The last equation causes Thomas precession in atomic physics.
Choose a different basis

$$\boldsymbol{S}_+ = \frac{1}{2}(\boldsymbol{S} + i\boldsymbol{K})\boldsymbol{S}_- = \frac{1}{2}(\boldsymbol{S} - i\boldsymbol{K}) \tag{2.4.13}$$

Then we can calculate the algebra

$$\left[S_{+,i}, S_{+,j}\right] = i\epsilon_{ijk}S_{+,k} \tag{2.4.14}$$
$$\left[S_{-,i}, S_{-,j}\right] = i\epsilon_{ijk}S_{-,k} \tag{2.4.15}$$
$$\left[S_{+,i}, S_{-,j}\right] = 0 \tag{2.4.16}$$

In other word, the algebras are decoupled. This familiar algebra is angular momentum algebra **SU**(2).

Classification by two numbers $(j_+,\ j_-)$

$$j_+ = 0, \frac{1}{2}, 1, \dots$$
$$j_- = 0, \frac{1}{2}, 1, \dots$$

Dimension = $(2j_+ + 1)(2j_- + 1)$.

A field is scalar field if $j_+ = j_- = 0$. One fundamental example of scalar field is Higgs boson. Other scalar particles are just bound states.

There are two possible states with spin $\frac{1}{2}$: $(j_+, j_-) = (\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$

$$\begin{aligned} \boldsymbol{S}_+ &= \frac{1}{2}\boldsymbol{\sigma} \\ \boldsymbol{S}_- &= 0 \\ \boldsymbol{S} &= \frac{1}{2}\boldsymbol{\sigma} \\ i\boldsymbol{K} &= \frac{1}{2}\boldsymbol{\sigma} \end{aligned}$$

for $(0, \frac{1}{2})$ there is "-"

So two types of spin $\frac{1}{2}$ fermions

$$\left(\frac{1}{2}, 0\right) \to e^-_L$$
$$\left(0, \frac{1}{2}\right) \to e^-_R$$

They have different transformation law. W boson only to $e^-_L$ but photon couples to both.

Under parity transformation $e^-_L \leftrightarrow e^-_R$. Both particles are needed for theory to be invariant under parity transformation, like EM and strong interactions.

# 3 Relativistic Quantum Field Theory

In non-relativistic quantum mechanics

$$
\begin{aligned}
E &= \frac{\boldsymbol{p}^2}{2m} \\
E &\mapsto i\hbar\frac{\partial}{\partial t} \\
\boldsymbol{p} &\to -i\hbar\boldsymbol{\nabla}
\end{aligned}
$$

After promoting the momentum and energy into operators in dispersion relation we have the Schrödinger equation

$$
i\frac{\partial}{\partial t}\psi + \frac{1}{2m}\boldsymbol{\nabla}^2\psi = 0 \tag{3.0.1}
$$

Density of probability is defined via

$$
\rho = |\psi|^2 = \psi\psi^* \tag{3.0.2}
$$

It obeys the continuity equation

$$
\begin{aligned}
-\frac{\partial}{\partial t}\int_V \rho \, \mathrm{d}V &= \int \boldsymbol{j}\cdot\boldsymbol{n}\, \mathrm{d}S \\
&= \int_V \boldsymbol{\nabla}\cdot\boldsymbol{j}\, \mathrm{d}V \\
\Rightarrow \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} &= 0 \qquad (3.0.3)
\end{aligned}
$$

Writing this explicitly

$$
\begin{aligned}
\frac{\partial\rho}{\partial t} &= \frac{\partial}{\partial t}\left(\psi\psi^*\right) \\
&= \psi\frac{\partial\psi^*}{\partial t} + \psi^*\frac{\partial\psi}{\partial t} \\
&= \frac{i}{2m}\left(\psi^*\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\psi\right) \\
\Rightarrow \boldsymbol{j} &= -\frac{i}{2m}\left(\psi^*\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\psi\right) \qquad (3.0.4)
\end{aligned}
$$

If we have a plane wave state, as an example

$$
\begin{aligned}
\psi &= N\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}-iEt} \\
\boldsymbol{j} &= \frac{\boldsymbol{p}}{m}|N|^2
\end{aligned}
$$

## 3.1 Relativistic wave equation

Now we enter the relativistic regime

$$
\begin{aligned}
E^2 &= \boldsymbol{p}^2 + m^2 \\
p^\mu &= (E, \boldsymbol{p}) \quad p_\mu = (E, -\boldsymbol{p}) \\
p^2 &= m^2
\end{aligned}
$$

Promoting energy and momentum into operators

$$
\begin{aligned}
p^\mu &\mapsto i\partial^\mu \\
\partial_\mu \partial^\mu &= \frac{\partial^2}{\partial^2 t} - \nabla^2
\end{aligned}
$$

We have then Klein-Gordon equation

$$
(\partial_\mu \partial^\mu + m^2)\phi(\boldsymbol{x}, t) = 0 \tag{3.1.1}
$$

The current in KG-theory is conserved as well

$$
j^\mu = (\rho, \boldsymbol{j}) = i\left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right) \tag{3.1.2}
$$

$$
\partial_\mu j^\mu = 0 \tag{3.1.3}
$$

An example solution

$$
\begin{aligned}
\phi &= N\mathrm{e}^{-ip\cdot x} \\
j^\mu &= 2p^\mu |N|^2
\end{aligned}
$$

In terms of Lorentz transformation

$$
\rho \sim E
$$

Energies of particles

$$
\begin{aligned}
E^2 &= \boldsymbol{p}^2 + m^2 \\
E &= \pm\sqrt{\boldsymbol{p}^2 + m^2}
\end{aligned}
$$

It also implies negative probability

$$
\begin{aligned}
E > 0 &\mapsto \rho > 0 \\
E < 0 &\mapsto \rho < 0
\end{aligned}
$$

## 3.2 Feynman-Stückelberg Interpretation of negative energy states

"Electron" with $E, \boldsymbol{p}$ and charge $-e$

$$
j^\mu_{e^-} = 2e|N|^2(E, \boldsymbol{p})
$$

"Positron" with $E, \boldsymbol{p}$ and charge $+e$

$$
j^\mu_{e^+} = 2e|N|^2(E, \boldsymbol{p}) = -2e|N|^2(-E, -\boldsymbol{p})
$$

We can think of $E < 0$ solution as particle flying backwards in time or $E > 0$ anti-particle forwards in time.

In a relativistic systems we need to remember following points

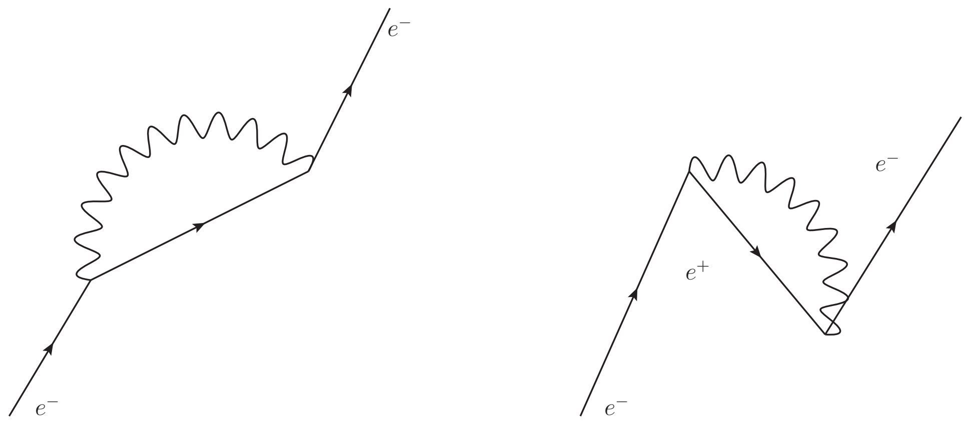



Figure 3.1: scattering process; horizontal time-axis; in the second diagram a electron positron pair is produced

- anti-particles
- particle numbers are not conserved

## 3.3 Electrodynamics (spin 1)

Maxwell equations are

$$\boldsymbol{E} = -\vec{\nabla}\phi - \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A} \tag{3.3.1}$$

$$\boldsymbol{B} = \vec{\nabla} \times \boldsymbol{A} \tag{3.3.2}$$

$$\vec{\nabla} \times \boldsymbol{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B} \tag{3.3.3}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{3.3.4}$$

Field strength tensor and four-potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.3.5}$$

$$A^\mu(x) = (\phi, \boldsymbol{A}) \tag{3.3.6}$$

The fields can be calculated from it

$$E^i = F^{0i} = \partial^i A^0 - \partial^0 A^i \tag{3.3.7}$$

$$B_i = -\epsilon_{ijk}\partial^i A^k = -\epsilon_{0ijk}F^{jk} \tag{3.3.8}$$

Often it is useful to use the dual tensor

$$\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\sigma\tau}F^{\sigma\tau} \tag{3.3.9}$$

$$\partial^\mu \tilde{F}_{\mu\nu} = 0 \tag{3.3.10}$$

is the second set of maxwell equations.

The other set of two equations is

$$\partial_\nu F^{\mu\nu} = 4\pi j^\mu \tag{3.3.11}$$

$\boldsymbol{E}, \boldsymbol{B}$ are observable, $\boldsymbol{A}$ is not. $A^\mu$ is not uniquely fixed by $\boldsymbol{E}$ and $\boldsymbol{B}$. It has the following gauge symmetry

$$\tilde{A}_\mu = A_\mu + \partial_\mu \Lambda(\boldsymbol{x}, t) \tag{3.3.12}$$

Use this transformation to get

$$\partial_\mu A^\mu = 0 \tag{3.3.13}$$

Plugging it back then we have the relativistic wave equation

$$\partial_\mu \partial^\mu A^\nu = 0 \tag{3.3.14}$$

it essentially is Klein-Gordon equation with mass $m = 0$

$A^\mu$ is a vector with spin 1

$$(j_+, j_-) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

It implied it has two transverse degrees of freedom. It has spin 1 properties: $+1, 0, -1$, in which 0 mode does not exist.

## 3.4 Description of Fermions

Original motivation for Dirac. He wants a linear equation in $E$ or $\frac{\partial}{\partial t}$

$$p^\mu \mapsto i\partial^\mu$$

Take the ansatz

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi &= H\psi \\ &= (\vec{\alpha} \cdot \boldsymbol{p} + \beta m)\psi \end{aligned}$$

but $\boldsymbol{\alpha}$ and $\beta$ unknown. It still has to obey the relativistic energy relation

$$\begin{aligned} A &= (\alpha_i p_i + \beta m)(\alpha_i p_i + \beta m) \\ &\stackrel{!}{=} \boldsymbol{p}^2 + m^2 \\ &= \alpha_i \alpha_j p_i p_j + \beta^2 m^2 + \alpha_i \beta p_i m + \beta \alpha_j p_j m \end{aligned}$$

From this we demand

$$\beta^2 = 1 \tag{3.4.1}$$

$$\alpha_i^2 = 1 \tag{3.4.2}$$

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \tag{3.4.3}$$

$$\alpha_i \beta + \beta \alpha_i = 0 \tag{3.4.4}$$

So $\alpha$ and $\beta$ are not just numbers, but (can be proven to be) hermitian traceless matrices with eigenvalue $\pm 1$. In addition, it only exits in even dimensions. Since $\alpha_i$ and $\beta$ are $4 \times 4$ matrices. $\psi$ has to be a 4-component spinor.

For parity conservation need $(\frac{1}{2}, 0) \bigoplus (0, \frac{1}{2})$ Thus

$$\begin{pmatrix} \begin{pmatrix} & \end{pmatrix}_{2\times 2} & \\ & \begin{pmatrix} & \end{pmatrix}_{2\times 2} \end{pmatrix}$$

There are different sets of $\alpha_i, \beta$ which satisfy the conditions. They are called representations.

Dirac-Pauli representation

$$\alpha_i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{3.4.5}$$

$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \tag{3.4.6}$$

with $\sigma^i$ the Pauli matrices.

Weyl (chiral) representation

$$\alpha^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{3.4.7}$$

$$\beta = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \tag{3.4.8}$$

they are mainly used in high energy physics ($\mathrm{E} \gg m$).

### 3.4.1 Gamma Matrices

We now define 4 gamma matrices $\gamma^\mu$, $\mu = 0, 1, 2, 3$

$$\gamma^\mu = (\beta, \beta\boldsymbol{\alpha}) \tag{3.4.9}$$

Note that having an index does not make it Lorentz vector.

The Clifford algebra is defined as following

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{3.4.10}$$

In Dirac-Pauli representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{3.4.11}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{3.4.12}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{3.4.13}$$

In Weyl representation

$$\gamma^0 \leftrightarrow \gamma^5$$

Rewriting the Dirac equation using $\gamma$s

$$
\begin{aligned}
i\partial_t\psi &= (\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m)\,\psi \\
i\partial_t\psi &= -i\boldsymbol{\alpha}\cdot\vec{\nabla}\psi + m\beta\psi \\
i\beta\partial_t\psi &= -i\beta\boldsymbol{\alpha}\cdot\vec{\nabla}\psi + m\psi \\
\left(i\gamma^\mu\partial_\mu - m\right)\psi &= 0
\end{aligned} \tag{3.4.14}
$$

Conventionally we use $\phi$ for spin 0 particle and $A_\mu$ for spin 1.

It is convenient to also have an equation for $\psi^\dagger$. First one can show $\gamma^{\dagger\mu} = \gamma^0\gamma^\mu\gamma^0$.

- $\mu = 0$: $\gamma^0 = \beta$ and $\gamma^{\dagger\,0} = \gamma^0\gamma^0\gamma^0 \Rightarrow \beta^2 = \mathbb{1}_4$
- $\gamma^{\dagger\mu} = (\beta\alpha^k)^\dagger = (\alpha^k)^\dagger\beta^\dagger = \alpha^k\beta = \beta^2\alpha^k\beta = \beta\gamma^k\beta = \gamma^0\gamma^k\gamma^0$

$$
\begin{aligned}
i\gamma^0\partial_0\psi + i\gamma^k\partial_k\partial - m\psi &= 0 \\
-i\partial_0\psi^\dagger(\gamma^0)^\dagger - i(\partial_k\psi^\dagger)\gamma^{\dagger\,k} - m\psi^\dagger &= 0 \\
-i\partial_0\psi^\dagger\gamma^0 - i(\partial_k\psi^\dagger)\gamma^0\gamma^k\gamma^0 - m\psi^\dagger &= 0
\end{aligned}
$$

define $\bar{\psi} = \psi^\dagger\gamma^0$

$$
\begin{aligned}
-i\partial_0\bar{\psi}\gamma^0 - i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} &= 0 \\
i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} &= 0
\end{aligned} \tag{3.4.15}
$$

### 3.4.2 Free Particle Solution to Dirac Equation

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0$$

multiplying $\gamma^\nu\partial_\nu$ from left

$$
\begin{aligned}
i\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\psi - m\gamma^\nu\partial_\nu\psi &= 0 \\
i\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\psi + im^2\psi &= 0
\end{aligned}
$$

$$
\begin{aligned}
\gamma^\mu\gamma^\nu &= \frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) \\
&= \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu + 2g^{\mu\nu}\right) \\
&= \frac{1}{2}\left[\gamma^\mu,\gamma^\nu\right] + g^{\mu\nu}
\end{aligned}
$$

The commutator is anti-symmetric and multiplying to symmetric tensor (derivatives) the term must vanish.

Each component of spinor satisfies the Klein-Gordon equation.

$$(\partial_\mu\partial^\mu + m^2)\psi_i = 0 \tag{3.4.16}$$

Thus we can write the solution as plane-wave

$$\psi = u(\boldsymbol{p})\mathrm{e}^{-ipx} \tag{3.4.17}$$

$u(\boldsymbol{p})$ is also a 4-component object but as function $\boldsymbol{p}$ not $\boldsymbol{x}$

Insert back into Dirac equation, then we have Dirac equation in momentum space

$$\left(\gamma^\mu p_\mu - m\right) u(\boldsymbol{p}) = 0 \tag{3.4.18}$$

Solution by considering Dirac-Pauli representation

$$\left(\not{p} - m\right) u(\boldsymbol{p}) = \begin{pmatrix} (E-m)\mathbb{1} & -\boldsymbol{p}\cdot\boldsymbol{\sigma} \\ \boldsymbol{p}\cdot\boldsymbol{\sigma} & -(E+m)\mathbb{1} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$\boldsymbol{p} = 0$ then $E = \pm m$

- $E = +m$ Two solutions

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $E = -m$

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\boldsymbol{p} = 0$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} u_B = (E-m) u_A \tag{3.4.19}$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{p} u_A = (E+m) u_B \tag{3.4.20}$$

- $E > 0$

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Ansatz $u_A^{(s)} = \chi^{(s)}$

$$u_B^{(s)} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} \quad u_A^{(s)} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)}$$
$$u(\boldsymbol{p}) = N \begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \end{pmatrix} \tag{3.4.21}$$

$E < 0$ and $u_B^{(s)} = \chi^{(s)}$

$$u(\boldsymbol{p}) = N \begin{pmatrix} -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \tag{3.4.22}$$

One can show $u^{\dagger(r)} u^{(s)} = N^2 \delta^{rs}$

Two fold degeneracy in each case. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $E > 0$ and $E < 0$. There must be another observable which commutes with $H$ and $\boldsymbol{p}$.

$$H = \gamma^i p_i + \gamma^0 m$$

$$\boldsymbol{S}\cdot\hat{P}=\frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{p} & 0\\ 0 & \boldsymbol{\sigma}\cdot\hat{p}\end{pmatrix} \tag{3.4.23}$$

Helicity

$$\frac{1}{2}\boldsymbol{\sigma}\cdot\hat{p}=\frac{1}{2}\begin{pmatrix}\hat{p}_z & \hat{p}_x+i\hat{p}_y\\ \hat{p}_x-i\hat{p}_y & -\hat{p}_z\end{pmatrix} \tag{3.4.24}$$

$$\det(\boldsymbol{\sigma}\cdot\hat{p})=-\hat{p}^2=-1 \tag{3.4.25}$$

Determinant is the product of two eigenvalues, then

$$\begin{aligned}\lambda_1+\lambda_2&=0\\ \lambda_1\cdot\lambda_2&=1\\ \lambda_{1,2}&=\pm1\end{aligned}$$

Antiparticle solution $u^{(3,4)}(-\boldsymbol{p})e^{-i(-p)x}=v^{(2,1)}$

$$\left(\not{p}+m\right)v(\boldsymbol{p})=0 \tag{3.4.26}$$

Normalization is

$$\int\rho\,\mathrm{d}V=2E \tag{3.4.27}$$

$$N=\sqrt{E+m} \tag{3.4.28}$$

Completeness relation (spin sums)

$$\sum u^{(s)}(p)\bar{u}^{(s)}(p)=\left(\not{p}+m\right) \tag{3.4.29}$$

$$\sum v^{(s)}(p)\bar{v}^{(s)}(p)=\left(\not{p}-m\right) \tag{3.4.30}$$

Define a projector projecting out positive and negative energy states

$$\Lambda_\pm=\frac{\pm\not{p}+m}{2m} \tag{3.4.31}$$

In Chiral (Weyl) representation

$$\left(\not{p}-m\right)u(\boldsymbol{p})=\begin{pmatrix}m & p\cdot\sigma\\ \boldsymbol{k}\cdot\bar{\sigma} & -m\end{pmatrix}\begin{pmatrix}u_L\\ u_R\end{pmatrix} \tag{3.4.32}$$

$\bar{\sigma}=(\sigma^0,-\boldsymbol{\sigma})$ and $\sigma^0=\mathbb{1}_2$

Weyl equation

$$\begin{aligned}-mu_L+p\cdot\sigma u_R&=0\\ p\cdot\bar{\sigma}u_L-mu_R&=0\end{aligned} \tag{3.4.33}$$

if $m=0$, the equations decouple from each other.

We want to construct Lagrangians involving the fields $\psi$, $A_\mu$ and $\psi$. The reason we choose Lagrangians as opposed to Hamiltonians, is that Hamiltonian $H$ is associated with energy of system, which is not Lorentz (or relativistic) invariant. But Lagrangian density is $S=\int\mathrm{d}t=\int\mathrm{d}^4x\,\mathcal{L}$. In natural unit, the action is dimensionless. One can in addition prove $\mathrm{d}^4x$ is Lorentz invariant. The fact that $\mathcal{L}$ is Lorentz invariant, means that $\mathcal{L}$ need to have at least 2 spin-$\frac{1}{2}$ fields $\psi$ or $\bar{\psi}$, to make a spin-0.

We are interested in the objects like

$$\bar{\psi}(\gamma^\mu)\psi$$

Representation of the Lorentz group can be split into $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$. In Weyl or chiral representation, $\psi_L$ has $\left(\frac{1}{2},0\right)$ and $\psi_R$ $\left(0,\frac{1}{2}\right)$.

Here we are taking a different approach from the paper [1]. First define

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] \tag{3.4.34}$$

We want to find out the transformation of $\psi = u(\boldsymbol{p})e^{-ip\cdot x}$. Note that the second factor is written in covariant form, i.e. Lorentz invairant. Consider the Dirac equation in two different frames $x' = \Lambda x$

$$i\gamma^\mu \frac{\partial\psi(x)}{\partial x^\mu} - m\psi(x) = 0 \tag{3.4.35}$$

$$i\gamma^\mu \frac{\partial\psi'(x')}{\partial x'^\mu} - m\psi'(x') = 0 \tag{3.4.36}$$

We make the ansatz that the spinor transforms with $S$

$$\psi'(x') = S\psi(x) \tag{3.4.37}$$

$S$ must be independent of $x$. Plug 3.4.37 into 3.4.36

$$\begin{aligned} i\gamma^\mu \frac{\partial}{\partial x'^\mu}\left[S\psi(x)\right] - mS\psi(x) &= 0 \\ iS^{-1}\gamma^\mu S\frac{\partial\psi(x)}{\partial x^\nu}\underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{=\Lambda^{-1}} - m\psi(x) &= 0 \end{aligned}$$

This equation must be the same as 3.4.35, then we get

$$S^{-1}\gamma^\mu S\left[\Lambda^{-1}\right]^\nu_{\ \mu} = \gamma^\nu$$

$$S^{-1}\gamma^\mu S = \Lambda^\mu_{\ \nu}\gamma^\nu \tag{3.4.38}$$

$S_{\mu\nu}$ is anti-symmetric in $\mu$ and $\nu$

$$S_i = \frac{1}{2}\epsilon_{ijk}S^{jk} \tag{3.4.39}$$

$$K_i = S^{0i} \tag{3.4.40}$$

$$\psi'_\alpha(x') = \left[\exp\left(-\frac{i}{2}\Theta^{\mu\nu}S_{\mu\nu}\right)\right]^\beta_\alpha \psi_\beta \tag{3.4.41}$$

$$\omega_i = \frac{1}{2}\epsilon_{ijk}\Theta_{ijk} \tag{3.4.42}$$

Lorentz transformation containing rotations and boosts, but infinitesimal version (infinitesimally different from $\mathbb{1}$)

$$\Lambda^\nu_{\ \mu} = \delta^\nu_{\ \mu} + \epsilon^\nu_{\ \mu} \tag{3.4.43}$$

We will show in the exercise the expression of transformation under boosts and rotations

$$S_L = \mathbb{1}_4 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} \tag{3.4.44}$$

$$S_L^{-1} = \mathbb{1}_4 + \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} \tag{3.4.45}$$

It satisfies $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu = \gamma^\nu + \epsilon^\mu{}_\nu\gamma^\nu$

$$\left(\mathbb{1}_4 + \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}\right)\gamma^\mu\left(\mathbb{1}_4 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}\right) = \gamma^\mu + \frac{i}{4}\left[\sigma_{\alpha\beta}\gamma^\mu - \gamma^\mu\sigma_{\alpha\beta}\right]\epsilon^{\alpha\beta}$$

somehow

$$= \gamma^\mu + \epsilon^\mu{}_\nu\gamma^\nu$$

Parity transformation cannot be written in infinitesimal form. Thus it is often called (one of) discrete transformation.

$$\Lambda_{\mathrm{P}} = \mathrm{diag}(+1, -1, -1, -1) \tag{3.4.46}$$

Parity symmetry is naturally violated. Especially in weak interaction, since neutrinos are left-handed $\nu_L$.

$$S_{\mathrm{P}}^{-1}\gamma^\mu S_{\mathrm{P}} = \Lambda_{\mathrm{P}}{}^\mu{}_\nu\gamma^\nu \tag{3.4.47}$$

- For $\mu = 0$

$$S_{\mathrm{P}}^{-1}\gamma^0 S_{\mathrm{P}} = \gamma^0$$

- For $\mu = i$

$$S_{\mathrm{P}}^{-1}\gamma^i S_{\mathrm{P}} = -\gamma^i$$

It is easy to show $S_{\mathrm{P}} = \gamma^0$ satisfies the equations. Then the spinor transform under parity like

$$\psi'(t, -\boldsymbol{x}) = \gamma^0\psi(t, \boldsymbol{x}) \tag{3.4.48}$$

In Dirac-Pauli representation $\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad \psi' = \begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_3 \\ -\psi_4 \end{pmatrix}$$

In chiral representation $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$ and

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad \psi' = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

Recall $\bar{\psi} = \psi^\dagger\gamma^0$.

$$\bar{\psi}' = \psi'^\dagger\gamma^0 = \psi^\dagger S^\dagger\gamma^0$$

it can easily be shown using the explicit expresseion of $S_L$ and $S_L$. Parity transformation is $\gamma^0$.

$$= \psi^\dagger\gamma^0 S^{-1}$$

$$= \bar{\psi}S^{-1}$$

There are 16 different bilinear $\bar{\psi}A\psi$

- $A = \mathbb{1}_4$

$$\bar{\psi}'\psi' = (\bar{\psi}S^{-1})(S\psi) = \bar{\psi}\psi$$

- $A = \gamma^\mu$

$$\bar{\psi}'\gamma^\mu\psi' = \bar{\psi}S^{-1}\gamma^\mu S\psi = \Lambda^\mu_{\ \nu}(\bar{\psi}\gamma^\nu\psi)$$

It transforms exactly like a Lorentz vector. Especially under parity

$$\bar{\psi}'\gamma^\mu\psi' = \begin{cases} \bar{\psi}'\gamma^0\psi' & \mu = 0 \\ -\bar{\psi}'\gamma^i\psi' & \mu = 1,2,3 \end{cases}$$

- $A = \gamma^5$

$$\bar{\psi}'\gamma^5\psi' = \bar{\psi}S_L^{-1}\gamma^5 S_L\psi$$

$\left\{\gamma^\mu, \gamma^5\right\} = 0$ and $\sigma_{\mu\nu}$ contains two $\gamma$s

$$= \bar{\psi}\gamma^5\psi$$

It is a scalar under boosts and rotations. Under parity it is pseudoscalar, like pions.

$$\gamma^5 S_{\mathrm{P}} = \gamma^5\gamma^0 = -S_{\mathrm{P}}\gamma^5$$

thus

$$\bar{\psi}'\gamma^5\psi' = -\bar{\psi}'\gamma^5\psi'$$

For experiments, look it up in Introduction to HEP, Perkins

Chiral spinors (in chiral representation)

$$P_R = \frac{1}{2}\left(\mathbb{1}_4 + \gamma^5\right) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{3.4.49}$$

$$P_L = \frac{1}{2}\left(\mathbb{1}_4 - \gamma^5\right) = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.4.50}$$

If $\psi$ written like $\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$, it is in chiral representation

$$P_L\psi = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} = \psi_L$$
$$P_R\psi = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix} = \psi_R$$

They are projectors

$$P_R^2 = \frac{1}{4}(\mathbb{1}_4 + \gamma^5)(\mathbb{1}_4 + \gamma^5) = \frac{1}{4}(\mathbb{1}_4 + 2\gamma^5 + (\gamma^5)^2) = P_R$$
$$P_L^2 = P_L$$

They also project onto complete space meaning $P_L + P_R = \mathbb{1}_4$.

Left-handed particles have spin and momentum in the opposite direction and right-handed in the same direction. In Dirac-Pauli representation in high energy limit

$$\gamma^5 u^{(5)} \approx \begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{p} \end{pmatrix}$$

To show this take the free particle solution ($E > 0$)

$$u^{(s)} = N \begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \end{pmatrix}$$

$$\gamma^5 u^{(s)} = N \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \approx N \begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{p}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

In general $(\boldsymbol{\sigma}\cdot\boldsymbol{a})^2 = \boldsymbol{a}^2\mathbb{1}$

$$= N\boldsymbol{\sigma}\cdot\hat{p} \begin{pmatrix} \chi^{(s)} \\ \boldsymbol{\sigma}\cdot\hat{p}\chi^{(s)} \end{pmatrix} = \underbrace{\boldsymbol{\sigma}\cdot\hat{p}}_{\text{helicity operator}} \quad u^{(s)}$$

At high energy limit ($E \gg m$), $\gamma^5$ helicity operator, but not at low energy. Chirality $\psi_{L,R}$ is always a good quantum number.

**Angular momentum** Must be another observable which commutes with $H$ and $P$ (u = u(**p**))

$$\boldsymbol{\Sigma}\cdot\hat{p} = \begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{p} \end{pmatrix} \tag{3.4.51}$$

$$H = \alpha\cdot P + \beta m \tag{3.4.52}$$

$$[H, \boldsymbol{\Sigma}\cdot\hat{p}] = 0 \tag{3.4.53}$$

Angular momentum operator is defined as

$$\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{P} \tag{3.4.54}$$

$$\tag{3.4.55}$$

Use the relation $\left[\hat{x}_i, \hat{P}_j\right] = i\delta_{ij}$ and check the commutation

$$\begin{aligned} [H, L_1] &= [\boldsymbol{\alpha}\cdot\boldsymbol{P}, x_2P_3 - x_3P_2] \\ &= [\alpha_1P_1 + \alpha_2P_2 + \alpha_3P_3, x_2P_3 - x_3P_2] \\ &= \alpha_2\,[P_2, x_2]\,P_3 - \alpha_3\,[P_3, x_3]\,P_2 \\ &= -i\,(\alpha_2P_3 - \alpha_3P_2) \\ &= -i(\alpha\times P)_1 \end{aligned}$$

Thus $[H, \boldsymbol{L}] = -i(\boldsymbol{\alpha}\times P) \neq 0$. In other word, $L$ not conserved.

But we observe

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{3.4.56}$$

$$[H, \boldsymbol{\Sigma}] = 2i(\boldsymbol{\alpha}\times P) \tag{3.4.57}$$

Thus define total angular momentum, second term describe the intrinsic angular momentum of particles

$$\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\boldsymbol{\Sigma} \tag{3.4.58}$$

and $J$ is conserved.

**Charge conjugation** Equation describing fermions couple to electromagnetic current

$$\left[\gamma^\mu\left(i\partial_\mu + eA_\mu\right) - m\right]\psi = 0 \tag{3.4.59}$$

with $e$ the electric charge of electron and $A_\mu(x)$ vector potential in electromagnetism.

For positron the charge is the opposite

$$\left[\gamma^\mu\left(i\partial_\mu - eA_\mu\right) - m\right]\psi^{\mathrm{C}} = 0 \tag{3.4.60}$$

$\psi^{\mathrm{C}}$ is the charge conjugate of $\psi$. We want to know the relation between $\psi \leftrightarrow \psi^{\mathrm{C}}$. Take complex conjugate of equation 3.4.59

$$\left[-(\gamma^\mu)^*\left(i\partial_\mu - eA_\mu\right) - m\right]\psi^* = 0 \tag{3.4.61}$$

We postulate a matrix $C$ so that $\psi^{\mathrm{C}} = C\gamma^0\psi^*$ and multiply equation 3.4.61 by $C\gamma^0$ from left.

$$C\gamma^0\left[-(\gamma^\mu)^*\left(i\partial_\mu - eA_\mu\right) - m\right]\psi^* = 0$$

It must be the same as equation 3.4.60. Thus

$$\begin{aligned}
-(C\gamma^0)(\gamma^\mu)^* &= \gamma^\mu C\gamma^0 \\
-C(\gamma^\mu)^{\mathrm{T}} &= \gamma^\mu C \\
C\gamma^{\mu T}C^{-1} &= -\gamma^\mu \\
C &= i\gamma^0\gamma^2
\end{aligned} \tag{3.4.62}$$

Then we find

$$\psi^{\mathrm{C}} = C\gamma^0\psi^* = C\bar{\psi}^{\mathrm{T}} \tag{3.4.63}$$

It would be interesting to compare $P_L(\psi^{\mathrm{C}})$ and $(P_L\psi)^{\mathrm{C}}$

**Continuity equation for $\psi$**

$$\rho = \psi^\dagger\psi = \bar{\psi}\gamma^0\psi \tag{3.4.64}$$

Multiply $\bar{\psi}$ from left to dirac equation and $\psi$ from right to conjugated Dirac equation

$$\begin{aligned}
\bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi &= 0, \qquad i(\partial_\mu\bar{\psi})\gamma^\mu\psi + m\bar{\psi}\psi = 0 \\
\bar{\psi}\gamma^\mu(\partial_\mu\psi) + (\partial_\mu\bar{\psi})\gamma^\mu\psi &= 0 \\
\partial_\mu(\bar{\psi}\gamma^\mu\psi) &= 0 \\
\partial_\mu j^\mu &= 0
\end{aligned}$$

## 3.5 Classical Field Theory

### Reminder

Lagrangian and action

$$L = L(q_i, \dot{q}_i, t) \qquad S = \int \mathrm{d}t\, L$$

Variantion principle

$$\delta S = 0$$

is equivalent to Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

Going to field theory, we transform $q(t) \to \phi(x)$ and $L \to \mathcal{L}$

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}\left(\phi, \frac{\partial \phi}{\partial x^\mu}, x_\mu\right) \\
S &= \int \mathrm{d}^4 x\, \mathcal{L}(\phi, \frac{\partial \phi}{\partial x^\mu}, x_\mu)
\end{aligned}$$

Using the variation principle

$$\begin{aligned}
\delta S &= \sum_i \frac{\mathrm{d}S}{\mathrm{d}\alpha_i}\delta\alpha_i \\
&= \sum_i \int \mathrm{d}^4 x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial \alpha_i}\delta\alpha_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial(\partial_\mu \phi)}{\partial \alpha_i}\delta\alpha_i \right\} \\
&= \sum_i \int \mathrm{d}^4 x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial \alpha_i} - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial \phi}{\partial \alpha_i}\right)\right\}\delta\alpha_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial \phi}{\partial \alpha_i}\delta\alpha_i\bigg|_{x_0}^{x_1}
\end{aligned}$$

Then we have our new Euler-Lagrange equations for continuous fields

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \tag{3.5.1}$$

**Conservation Laws and Noether Theorem** Reminder: Under transformation $t \mapsto t' = t + \delta b$, system stays invariant if $\delta_T L = \frac{\mathrm{d}}{\mathrm{d}t}(\delta\Omega)$.

$\delta\Omega = 0$ if $L$ invariant

$$\delta_T L = L(q, \dot{q}, t + \delta b) - L(q, \dot{q}, t) = \frac{\partial L}{\partial t}\delta b$$

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \\
&= \frac{\partial L}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \\
&= \frac{\partial L}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}}\dot{q}\right]
\end{aligned}$$

System invariant if

$$\frac{\partial L}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}(\delta\Omega) \tag{3.5.2}$$

if $L$ not dependent explicitly on $t$, $\mathrm{d}L\,/\,\mathrm{d}t = 0$

If $\partial L/\partial t = 0$, the conserved quantity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L - \frac{\partial L}{\partial \dot{q}}\dot{q}\right] = 0$$
$$L - \frac{\partial L}{\partial \dot{q}}\dot{q} = -H = L - \dot{q}p$$

using legendre transformation and $\partial L/\partial \dot{q}$ the general momentum.

Analogously for fields

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x^\mu} &= \left(\frac{\partial \mathcal{L}}{\partial \phi}\right)\frac{\partial \phi}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\partial_\rho(\partial_\mu \phi) \\ &= \left(\partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\right)\partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\partial_\rho \partial_\mu \phi \\ \partial_\mu \mathcal{L} &= \partial_\rho\left[\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\partial_\mu \phi\right] \\ &\Rightarrow \partial_\rho\left[\mathcal{L}\delta^\rho_\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\partial_\mu \phi\right] = 0\end{aligned}$$

this is energy momentum tensor $T^\rho_\mu$.

**Examples** Real scalar field, $\phi(x) \in \mathbb{R}$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \tag{3.5.3}$$

Since $L = T - V$, we interpret first term as kinetic and second as potential, which has minimum at $\phi = 0$ Since action is dimensionless, $[\phi] = 1$. Take Euler-Lagrange

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

put together

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0$$

This is just Klein-Gordon equation. In Feynman rules we have the propagator and mass term as interaction.

One can also have complex scalar field $\phi(x) \in \mathbb{C}$ and $\phi$, $\phi^*$ are independent fields.

**Example** Fermionic field

$$\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi \tag{3.5.4}$$

$[\psi] = \frac{3}{2}$. $\bar{\psi}$ and $\psi$ are independent fields. Solve Euler-Lagrange we get familiar Dirac equation.

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} i\gamma_\mu \delta^\mu \psi - m\psi = 0 \tag{3.5.5}$$

**Example** Spin 1 field $A_\mu(x)$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \tag{3.5.6}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.5.7}$$

Equation of motion can be computed

$$\frac{\partial\mathcal{L}}{\partial(\partial_\rho A_\sigma)} = \frac{1}{4}4\delta_{\rho\mu}\delta_{\nu\sigma}F^{\mu\nu} = -F^{\rho\sigma}$$
$$\frac{\partial\mathcal{L}}{\partial A_\sigma} = -j^\sigma$$

then we have the maxwell equations

$$-\partial_\rho F^{\rho\sigma} + j^\sigma = 0$$

Using Lorenz gauge we can get

$$\partial_\mu\partial^\mu A^\sigma = j^\sigma$$

In principle we can add a term $A_\mu A^\mu = A^2$, which is Lorentz invariant by construction.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu + \frac{1}{2}m^2 A_\mu A^\mu \tag{3.5.8}$$

so each component satisfies Klein-Gordon equation for $j^\mu = 0$. Later we will see $j^\mu$ corresponds to an interaction, e.g. $e\bar{\psi}\gamma^\mu\psi$.

## 3.6 Gauge Theories

Hermann Weyl first considered the gauge theory (*Eichtheorie*). We start with Dirac Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$$

If we add a phase $\psi(x) \mapsto \psi'(x) = \mathrm{e}^{i\alpha}\psi(x)$ with $\alpha \in \mathbb{R}$ and constant,

$$\mathcal{L}' = i\bar{\psi}'\gamma_\mu\partial^\mu\psi' - m\bar{\psi}'\psi'$$

since $\alpha$ is just a number, it commutes with derivative and gamma matrices.

$$= i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi$$
$$= \mathcal{L}$$

The Lagrangian is invariant under this transformation. It is a continuous symmetry and one can apply Noether theorem and compute the Noether current.

The set of transformation $\left\{\mathrm{e}^{i\alpha}\right\}_{\alpha\in\mathbb{R}}$ forms a $\mathbf{U}(1)$ (gauge) group. If the parameter does not depend on coordinates, i.e. constant, it is called global transformation.

Now consider the case $\alpha = \alpha(x)$, not constant but a continuous function. In other word, we allow a local change. The transformation of mass term is

$$\bar{\psi}'\psi' = \mathrm{e}^{-i\alpha(x)}\bar{\psi}\mathrm{e}^{i\alpha(x)}\psi$$
$$= \bar{\psi}\psi$$

Kinetic term transforms like

$$\begin{aligned}\bar{\psi}'\gamma_\mu\partial^\mu\psi' &= \mathrm{e}^{-i\alpha(x)}\bar{\psi}\gamma_\mu\partial^\mu\left(\mathrm{e}^{i\alpha(x)}\psi\right)\\ &= \bar{\psi}\gamma_\mu\left(\partial^\mu + i\partial^\mu\alpha(x)\right)\psi\end{aligned}$$

But we can still demand local invariance and we need to modify $\mathcal{L}$ by introducing a term $\bar{\psi}\gamma_\mu A^\mu(x)\psi$ in Lagrangian.

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + \bar{\psi}\gamma_\mu A^\mu\psi \tag{3.6.1}$$

We also need a transformation property for $A_\mu(x)$ to get Lagrangian invariant.

$$A^\mu \mapsto A'^\mu = A^\mu + \partial^\mu\alpha(x) \tag{3.6.2}$$

then

$$\begin{aligned}\bar{\psi}'\gamma_\mu A'^\mu\psi' &= \bar{\psi}\mathrm{e}^{-i\alpha(x)}\gamma_\mu\left(A^\mu + \partial^\mu\alpha(x)\right)\cdot\mathrm{e}^{i\alpha(x)}\psi\\ &= \bar{\psi}\gamma_\mu\left(A^\mu + \partial^\mu\alpha(x)\right)\psi\end{aligned}$$

The second term cancels the extra terms with $\partial^\mu\alpha(x)$ because of local transformation.

Introduce a charge $e\bar{\psi}\gamma_\mu A^\mu\psi = j_\mu A^\mu$

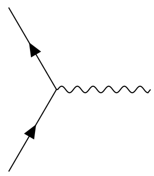
$$\begin{aligned}\mathcal{L} &= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + e\bar{\psi}\gamma_\mu A^\mu\psi\\ &= i\bar{\psi}\gamma^\mu\mathrm{D}_\mu\psi - m\bar{\psi}\psi\end{aligned} \tag{3.6.3}$$

with $\mathrm{D}_\mu = \partial_\mu - ieA_\mu$ covariant derivative and $(\mathrm{D}_\mu\psi)' = \mathrm{e}^{i\alpha(x)}\mathrm{D}_\mu\psi$.

Gauge field $A_\mu$ is introduced into Lagrangian and a kinetic term is needed to obtain a wave equation is needed

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + e\bar{\psi}\gamma_\mu A^\mu\psi \tag{3.6.4}$$

thus we have an additional Feynman rule



$$= -ie\gamma^\mu \tag{3.6.5}$$

By demanding invariance under local phase transformation, we introduce $A_\mu(x)$ and an interaction term in Lagrangian. We call $A_\mu$ a gauge field, meaning that it is not physical observable. The gauge group is an abelian group (commutative).

One can also check a mass term for gauge field $m^2A_\mu A^\mu$ is not invariant under $U(1)$ gauge. So gauge or local principle excludes photon mass term. Experimentally we are able to measure the value of $\xi$ by looking at the electromagnetic interaction. If photon has mass, the Coulomb force has the following form

$$F \sim \frac{1}{4\pi\epsilon}\frac{q_1q_2}{r^{2+\xi}}$$

Calculation of loop diagrams like 3.2 leads to most precisely computed object, anomalous magnetic moment of electron. In fact this calculation is so accurate, people use this to define the electron charge $e$.

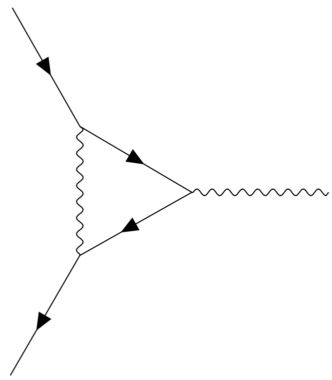


Figure 3.2: Anomalous magnetic dipole moment

## 3.7 Non-Abelian Gauge Theories

We will restrict ourselves in $\mathbf{SU}(N)$ groups, special unitary group of dimension $N$. It has $N^2-1$ generators. Weak interaction is described by $\mathbf{SU}(2)$ and strong interaction $\mathbf{SU}(3)$.

$$\mathcal{L}_0 = \bar{q}_j(x)(i\gamma^\mu\partial_\mu - m)q_j$$

with $j = 1, \ldots, N$

$$q(x) \mapsto q'(x) = \mathrm{e}^{i\alpha_a(x)T_a}q(x)$$

with $T_a$ a $N \times N$ matrices (independent of coordinates $x$) and $a = 1, \ldots, N^2 - 1$. Here fields are quark doublet $q_i = \begin{pmatrix} u \\ d \end{pmatrix}$ and $u$, $d$ are spinors.

Focus on infinitesimal transformations, $\alpha_a(x) \in \mathbb{R}$

$$\begin{aligned} q(x) &\mapsto q'(x) = [\mathbb{1}_N + i\alpha_a(x)T_a]\, q(x) \\ \partial_\mu q &\mapsto \partial_\mu [\mathbb{1}_N + i\alpha_a(x)T_a]\, q \\ &= (\mathbb{1} + i\alpha_a T_a)\, \partial_\mu q + i(\partial_\mu\alpha_a)T_a q(x) \end{aligned} \tag{3.7.1}$$

as before it is (at first) not invariant. We need $N^2 - 1$ gauge fields to compensate this, $\mathcal{G}^a_\mu$ with $a = 1, \ldots, N^2 - 1$.

The covariant derivative is

$$\mathrm{D}_\mu = \partial_\mu + igT_a\mathcal{G}^a_\mu$$

then to construct $\mathcal{L}$ for Non-Abelian case. Following the previous recipe

$$\mathcal{L} = \bar{q}\,(i\gamma^\mu\partial_m u - m)\, q - g(\bar{q}\gamma^\mu T_a q)\mathcal{G}^a_\mu \tag{3.7.2}$$

with $g \in \mathbb{R}$ a number independent of $x$.

Apply transformation 3.7.1 to kinetic term and see if it is sufficient to get invariant $\mathcal{L}$

$$\begin{aligned} \bar{q}\gamma^\mu T_a q \mapsto &\bar{q}\,(\mathbb{1} - i\alpha_a T_b)\,\gamma^\mu T_a(\mathbb{1} + i\alpha_b T_b)q \\ &= \bar{q}\gamma^\mu T_a q - i\alpha_b\bar{q}T_bT_a\gamma^\mu q + i\alpha_b\bar{q}T_aT_b\gamma^\mu q + O(\alpha_b^2) \\ &= \bar{q}\gamma^\mu T_a q + i\alpha_b\bar{q}[T_a, T_b]\gamma^\mu q \end{aligned}$$

The commutator is (summation over $c$ is implied)

$$[T_a, T_b] = i f_{abc} T_c \tag{3.7.3}$$

with $f$ structure constant. Then

$$\bar{q}\gamma^\mu T_a q \mapsto \bar{q}\gamma^\mu T_a q - f_{abc}\alpha_b \bar{q}\gamma^\mu T_c q \tag{3.7.4}$$

$\mathcal{L}$ with covariant derivative is still not invariant. It turns out that we must modify the transformation of $\mathcal{G}^a_\mu$

$$\mathcal{G}^a_\mu \mapsto \mathcal{G}^a_\mu - \frac{1}{g}\partial_\mu \alpha_a - f_{abc}\alpha_b \mathcal{G}^c_\mu \tag{3.7.5}$$

so now the Lagrangian 3.7.2 is invariant.

As before we need a kinetic term.

$$\mathcal{L}^{\text{kin}}_{\mathcal{G}} = -\frac{1}{4}\mathcal{G}^a_{\mu\nu}\mathcal{G}^{\mu\nu}_a \tag{3.7.6}$$

could it be like photon case, i.e.

$$\mathcal{G}^a_{\mu\nu} = \partial_\mu \mathcal{G}^a_\nu - \partial_\nu \mathcal{G}^a_\mu \tag{3.7.7}$$

apply this into the transformation of gauge field and see $f_{abc}$ term doesn't drop out. Then we must modify to

$$\mathcal{G}^a_{\mu\nu} = \partial_\mu \mathcal{G}^a_\nu - \partial_\nu \mathcal{G}^a_\mu - g f_{abc}\mathcal{G}^b_\mu \mathcal{G}^c_\mu \tag{3.7.8}$$

So in Abelian case

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
A_\mu &\mapsto A_\mu - \frac{1}{e}\partial_\mu \alpha \\
F'_{\mu\nu} &= \partial_\mu \left(A_\mu + \frac{1}{e}\partial_\nu \alpha\right) - \partial_\nu \left(A_\mu + \frac{1}{e}\partial_\mu \alpha\right) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= F_{\mu\nu}
\end{aligned}$$

$F_{\mu\nu}$ is gauge invariant. This property is specific to $\mathbf{U}(1)$.

Now apply this to Non-Abelian case, you will find $G^a_{\mu\nu}$ is not gauge invariant, but $-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a$ is.

Connection between $\mathcal{L}$ and Feynman rules Abelian case

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi + e\bar{\psi}\gamma^\mu \psi A^\mu$$

Kinetic term implies propagators and interaction vertex.



$$= \frac{-ig^{\mu\nu}}{p^2 + i\epsilon} \tag{3.7.9}$$

So there is no vertex of photon interacting with each other, photons rarely interfere each other. At higher order one has

In non-Abelian case gluon propagator, interaction vertex($-ig\gamma^\mu (T^a)_{ij}$) Because of the term in gluon field with differentiation, gluon interacts with itself. Formal way of constructing $\mathcal{G}^a_{\mu\nu}$. The covariant derivative $D_\mu = \partial_\mu + igT^a\mathcal{G}^a_\mu$ and $\left[T^a, T^b\right] = if^{abc}T^c$. With some calculation

$$\begin{aligned}
\left[D_\mu, D_\nu\right] &= igT_c\left[\partial_\mu \mathcal{G}^c_\nu - \partial_\nu \mathcal{G}^c_\mu - g f_{abc}\mathcal{G}^a_\mu \mathcal{G}^b_\nu\right] \\
&= igT_c\mathcal{G}^c_{\mu\nu}
\end{aligned}$$

It works as a general construction.

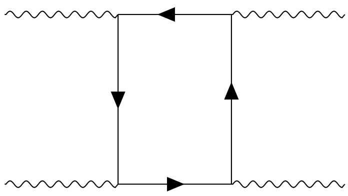


Figure 3.3: photon-photon scattering

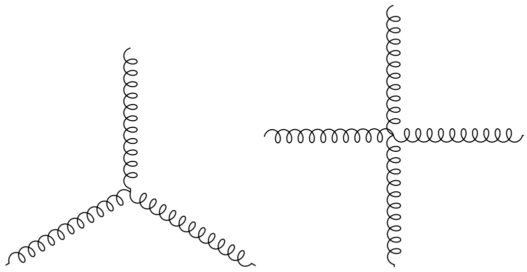


Figure 3.4: Gluon self-interactions

## 3.8 Spontaneous Symmetry Breaking

### 3.8.1 Real Field

$$\mathcal{L} = T - V = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V \tag{3.8.1}$$

$$V = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4 \tag{3.8.2}$$

It shows mirror symmetry. We are interested in minimum energy of energy or ground state.

$$E = T + V$$

Kinetic term $T$ containts derivative, so set $\phi$ as a constant to minimize $V$. First $\lambda$ must be postive, otherwise there is no minimum, i.e. unbound from below. With $\underline{\mu^2 > 0}$, we find

$$\frac{\partial V}{\partial \phi} = \phi(\mu^2 + \lambda\phi^2) = 0$$

We have minimum at $\phi = 0$. In other word $\langle\phi\rangle = 0$, vacuum expectation value (VEV) is zero. System is still mirror symmetric. The mass of the field

$$\left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi=\langle\phi\rangle} = \mu^2 \tag{3.8.3}$$

Now consider the case $\underline{\mu^2 < 0}$

$$\frac{\partial V}{\partial \phi} = \phi(\mu^2 + \lambda\phi^2) \stackrel{!}{=} 0 \tag{3.8.4}$$

there are three solutions possible

$$\phi = 0; \quad \phi = \pm v = \pm\sqrt{\frac{-\mu^2}{\lambda}} \tag{3.8.5}$$

$$\frac{\partial V}{\partial \phi} = \phi(\mu^2 + \lambda\phi^2) < 0$$

it means $\phi = 0$ is a maximum. Two minima means the vacuum is degenerate. By choosing one specific state, the mirror symmetry is spontaneous broken.

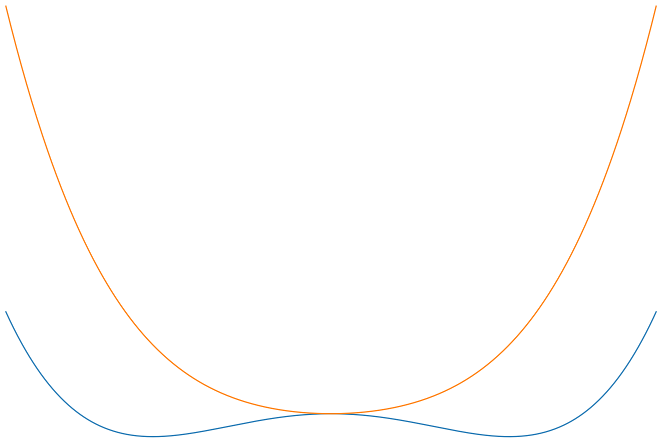


Figure 3.5: SSB for real scalar field

We choose the vacuum state to be $+v$. Split the field into two parts (it makes the physics and observable more obvious), the vacuum $v$ and perturbation around the vacuum $\eta$

$$\begin{aligned}
\phi(x) &= v + \eta(x) \\
v = \langle \phi \rangle &= \sqrt{\frac{-\mu^2}{\lambda}} = \text{const} \\
\langle \eta(x) \rangle &= 0 \\
\partial_\mu \phi &= \partial_\mu \eta
\end{aligned}$$

The kinetic term turns into

$$\begin{aligned}
(\partial_\mu \phi)(\partial^\mu \phi) &= (\partial_\mu \eta)(\partial^\mu \eta) \\
V &= \frac{1}{2}\mu^2 (v+\eta)^2 + \frac{1}{4}\lambda (v+\eta)^4 \\
&= \frac{1}{2}\mu^2 v^2 + \frac{1}{2}\mu^2 \eta^2 + \mu^2 v \eta + \frac{1}{4}\lambda \left[v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4\right] \\
&= \frac{1}{2}\mu^2 v^2 + \frac{1}{4}\lambda v^2 + \eta\left(\mu^2 v + \lambda v^3\right) + \eta^2\left(\frac{1}{2}\mu^2 + \frac{6}{4}v^2\lambda\right) + \lambda v \eta^3 + \frac{1}{4}\lambda\eta^4
\end{aligned}$$

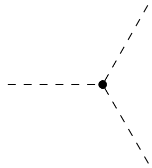
since $v = \sqrt{-\mu^2/\lambda}$, $\mu^2 v + \lambda v^3 = v(\mu^2 + \lambda v^2) = 0$. There is no longer mirror symmetry in $\eta$. Now the mass is

$$\left.\frac{\partial^2 V}{\partial \eta^2}\right|_{\eta=0} = \mu^2 + 3v^2\lambda = \mu^2 + 3\left(\frac{-\mu^2}{\lambda}\right)\lambda = -2\mu^2$$

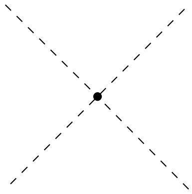
thus

$$m_\eta = \sqrt{-2\mu^2}$$

After symmetry breaking $V(\eta)$ contains $\eta^3$ and $\eta^4$ terms, thus additional Feynman rules.



$$\sim \lambda v \tag{3.8.6}$$



$$\sim \frac{1}{4} v \tag{3.8.7}$$

### 3.8.2 Complex Field

Now we want to consider complex scalar field $\phi(x) \in \mathbb{C}$. We can also write the fields as

$$\phi(x) = \frac{1}{\sqrt{2}} \left(\phi_1(x) + i\phi_2(x)\right) \tag{3.8.8}$$

with $\phi_1, \phi_2 \in \mathbb{R}$.

Only scalar fields can get a vacuum expectation value, otherwise it violates Lorentz invariance. Vacuum has spin and then constantly interacts with all particles.

The Lagrangian must be real, $\mathcal{L} \in \mathbb{R}$

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - \mu^2 \phi^* \phi - \lambda(\phi^* \phi)^2 \tag{3.8.9}$$

This is invariant under transformation $\phi(x) \mapsto e^{i\alpha}\phi(x), \alpha \in \mathbb{R}$. Rewrite the Lagrangian with $\phi_1$ and $\phi_2$

$$\mathcal{L} = \left(\partial_\mu \phi_1\right)(\partial^\mu \phi_1) + \left(\partial_\mu \phi_2\right)(\partial^\mu \phi_2) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2$$

The last two terms are $-V(\phi)$

- $\mu^2 > 0$. Minimum is at $\phi_1 = \phi_2 = 0$
- $\mu^2 < 0$. The potential has the shape of Mexican hat. We have set of minima forming a circular

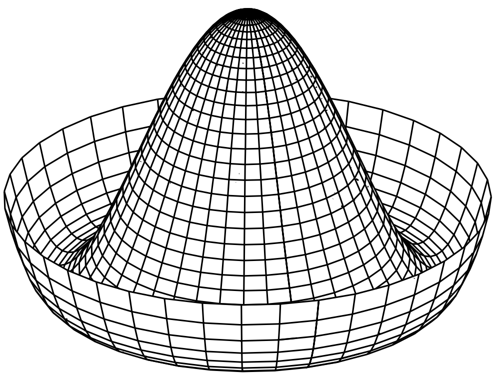


Figure 3.6: Mexican hat[2]

line $\phi_1^2 + \phi_2^2 = v^2 = -\mu^2/\lambda$. As an example $\phi_1 = v$ and $\phi_2 = 0$ and $\mathbf{U}(1)$ symmetry is spontaneous broken. The choice of vacuum breaks symmetry spontaneously. Field shift

$$\phi(x) = \frac{1}{\sqrt{2}}\Big[\underbrace{v + \eta(x)}_{\phi_1} + i\underbrace{\xi(x)}_{\phi_2}\Big] \tag{3.8.10}$$

with $\eta(x), \xi(x) \in \mathbb{R}$ and $\langle\eta\rangle = \langle\xi\rangle = 0$. Insert into $V(\phi) = V(\phi_1, \phi_2)$

$$\mathcal{L}(\eta,\xi) = \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \text{const} + \mu^2\eta^2 + (\text{cubic and quartic in } \eta, \xi) \tag{3.8.11}$$

so $m_\eta^2 = -2\mu^2$ but $m_\xi^2 = 0$

Continuous global symmetry is spontaneously broken and it leads to massless state. It is called Goldstone theorem. The massless state is the Goldstone boson.

### 3.8.3 Higgs Mechanism

Modify the transformation to

$$\phi(x) \mapsto \mathrm{e}^{i\alpha(x)}\phi(x)$$

with $\phi(x) \in \mathbb{C}$ and $\alpha(x) \in \mathbb{R}$.

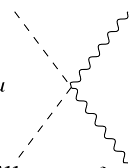
First step is to extend $\mathcal{L}_\phi$ such that is invariant under this transformation. Introduce a spin-1 field $A_\mu(x)$ with the covariant derivative

$$D_\mu = \partial_\mu - ieA_\mu$$
$$A_\mu \mapsto A_\mu + \frac{1}{e}\partial_\mu\alpha$$

The new Lagrangian

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{3.8.12}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It contains the interaction between two $\phi$ and two $A_\mu$



We want to look at spontaneous symmetry breaking of this system. We still want $\lambda > 0$, bound from below. In the case of $\mu^2 > 0$, symmetry not broken and we have scalar QED. In particle physics we do have charged scalars, such as $\pi^\pm$, $K^\pm$ and so on. We need $\mu^2 < 0$ to have spontaneous symmetry breaking. The field shift is as before

$$\phi(x) = \frac{1}{\sqrt{2}}\left(v + \eta(x) + i\xi(x)\right) \tag{3.8.13}$$

$$\begin{aligned}\mathcal{L}(\eta,\xi) = {} & \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta^2 + \frac{1}{2}e^2v^2A_\mu A^\mu \\ & - evA_\mu\partial^\mu\xi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\text{scalar interaction temrs})\end{aligned} \tag{3.8.14}$$

So $m_\eta = \sqrt{2\lambda v^2}$, $m_A = ev$ and $m_\xi = 0$.

$A_\mu \partial^\mu \xi$ seems to mix $A$ and $\xi$. Massless spin 1 has two degrees of freedom. $A_\mu(x)$ has four components, but Lorentz and gauge invariance restrict to two. Photon has two polarizations. Massive spin 1 has three degree of freedom. We will show later on, $\xi$ is actually the additional degree of freedom of $A$. The term $A_\mu A^\mu$ breaks the gauge invariance.

To see physics, the choice of $\eta$ and $\xi$ is not very good. Instead choose

$$\phi \mapsto \frac{1}{\sqrt{2}} (v + h(x))\, \mathrm{e}^{i\Theta(x)/v} \tag{3.8.15}$$

with $h(x) \in \mathbb{R}, \Theta(x) \in \mathbb{R}$. We interpret this as a gauge transformation with $\alpha = \Theta/v$. To rewrite gauge transformation of $A_\mu(x)$

$$A_\mu \mapsto A_\mu + \frac{1}{ev} \partial_\mu \Theta(x) \tag{3.8.16}$$

The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial h)^2 - \lambda v^2 h^2 + \frac{1}{2} e^2 v^2 A_\mu A^\mu - \lambda v h^3 - \frac{1}{4}\lambda h^4 + \frac{1}{2} e^2 A^2 h^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{3.8.17}$$

$\xi$ is no longer there. This is Higgs mechanism. So we have before $2 + 2$ degree of freedom ($\eta$, $\xi$, $A^{m=0}$) to $1 + 3$ degrees of freedom ($h$, $A^{m\neq 0}$).

Phase transition in magnetism is an example of symmetry breaking. Magnetization is temperature dependent. In the early universe the temperature is high so that $\mu^2(T) > 0$. As it cools down $\mu^2 < 0$.

### 3.8.4 Spontaneous Breaking of local $\mathbf{SU}(2)$ Theory

We start with ungauged, complex scalar field

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \tag{3.8.18}$$

Now $\phi$ contains two complex scalar fields.

$$\phi = \begin{pmatrix} \phi_\alpha \\ \phi_\beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{3.8.19}$$

with $\phi_{1,2,3,4} \in \mathbb{R}$.

Gauge transformation

$$\phi \mapsto \phi' = \mathrm{e}^{i\alpha_a(x)\tau_a/2} \phi \tag{3.8.20}$$

There are only three generators of $\mathbf{SU}(2)$, $a = 1, 2, 3$.

Introduce three gauge bosons $W^a_\mu$, $a = 1, 2, 3$. Define the covariant derivative

$$D_\mu = \partial_\mu - ig \frac{\tau_a}{2} W^a_\mu \tag{3.8.21}$$

Note $T^a = \tau^a/2$.

The $\tau$ matrices are Pauli matrices

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.8.22}$$

Have gauge transformation of $W^\mu_a$

$$W^a_\mu \mapsto W^a_\mu - \frac{1}{g} \partial_\mu \alpha^a - f^{abc} \alpha_b W_{c\,\mu} \tag{3.8.23}$$

Here $f_{abc} = \epsilon_{abc}$ totally symmetric tensor in three dimension.

Lagrangian becomes

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi) - \frac{1}{4}W^a_{\mu\nu}W^{a\,\mu\nu} \tag{3.8.24}$$

Kinetic term for gauge fields

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g(W_\mu \times W_\nu)^a \tag{3.8.25}$$

The potential

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \tag{3.8.26}$$

We are interested in the case $\lambda > 0$ and $\mu^2 < 0$. Minima are at

$$\phi^\dagger\phi = \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) = -\frac{\mu^2}{2\lambda}$$

We can make a choice for minimum $\phi_1 = \phi_2 = \phi_4 = 0$ and $\phi_3^2 = v^2 = -\mu^2/\lambda$

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} \tag{3.8.27}$$

with $v \in \mathbb{R}$.

Shift of $\phi$ field

$$\phi(x) \mapsto \mathrm{e}^{i\hat{\tau}\cdot\hat{\theta}/v}\phi(x) = \mathrm{e}^{i\hat{\tau}\cdot\hat{\theta}/v}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v + h(x)\end{pmatrix} \tag{3.8.28}$$

Infinitesimal expansion in $\theta(x)$

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2}}\left(\mathbb{1} + i\hat{\tau}\cdot\hat{\theta}(x)/v\right)\begin{pmatrix}0\\ v + h(x)\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 + i\theta_3/v & i(\theta_1 - i\theta_2)/v\\ i(\theta_1 + i\theta_2)/v & 1 - i\theta_3/v\end{pmatrix}\begin{pmatrix}0\\ v + k(x)\end{pmatrix}\end{aligned}$$

Look at kinetic term in $\phi$

$$\begin{aligned}\mathcal{L}^\phi_{\mathrm{kin}} &= (D_\mu\phi)^\dagger(D^\mu\phi)\\ &= \left(\partial_\mu\phi + ig\hat{\tau}\cdot\hat{W}\phi\right)^\dagger\left(\partial^\mu\phi + ig\hat{\tau}\cdot\hat{W}\phi\right)\end{aligned}$$

compute at minimum $\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}$

$$\begin{aligned}\left|\frac{i}{2}g\hat{\tau}\cdot\hat{W}\phi\right|^2 &= \frac{g^2}{8}a\left|\begin{pmatrix}W^3_\mu & W^1_\mu - iW^2_\mu\\ W^1_\mu + iW^2_\mu & -W^3_\mu\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}\right|^2\\ &= \frac{g^2v^2}{8}\left|\begin{pmatrix}W^1_\mu - iW^2_\mu\\ -W^3_\mu\end{pmatrix}\right|^2\\ &= \frac{g^2v^2}{8}\boldsymbol{W}^2\\ &= \frac{1}{2}M^2\boldsymbol{W}^2\end{aligned}$$

with $M = gv/2$. We spontaneously broken $\mathbf{SU}(2)$ gauge theory. However, this is not realized in nature.

# 4 Standard Model

We will be looking at the gauge group $\mathbf{SU}(2)_\mathrm{L} \times \mathbf{U}(1)_\mathrm{Y}$. This will be spontaneously broken into $\mathbf{U}(1)_\mathrm{EM}$. Strong interaction will add $\mathbf{SU}(3)_\mathrm{C}$ and it doesn't get spontaneously broken.

## 4.1 Leptons

Reintroduce fermions

$$\psi = \begin{pmatrix} \chi_\mathrm{L} \\ \chi_\mathrm{R} \end{pmatrix}$$
$$\psi_\mathrm{L} = \begin{pmatrix} \chi_\mathrm{L} \\ 0 \end{pmatrix}$$

$\mathbf{SU}(2)_\mathrm{L}$ only interacts with $\psi_\mathrm{L}$.

In Chiral representation

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{4.1.1}$$

The projection operators

$$P_L = \frac{1}{2}(\mathbb{1}_2 - \gamma^5) = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.1.2}$$

$$P_R = \frac{1}{2}(\mathbb{1}_2 + \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{4.1.3}$$

with $P_L^2 = P_L$ and $P_L + P_R = \mathbb{1}$.

$\chi_\mathrm{L,R}$ two-component Weyl spinors.

$$(\psi_\mathrm{L})^\dagger = (P_\mathrm{L}\psi)^\dagger = \psi^\dagger P_\mathrm{L}^\dagger = \psi^\dagger P_\mathrm{L}$$
$$\overline{\psi}_\mathrm{L} = (\psi_L)^\dagger \gamma^0 = \psi^\dagger P_L \gamma^0 = \overline{\psi} P_R$$

Introduce a doublet of left-handed particles

$$L = \begin{pmatrix} \nu_\mathrm{L} \\ e^-_\mathrm{L} \end{pmatrix}$$
$$\nu_\mathrm{L} = (\psi_\nu)_\mathrm{L} = \begin{pmatrix} \chi_{\nu\mathrm{L}} \\ 0 \end{pmatrix}$$
$$e^-_\mathrm{L} = (\psi_{e^-})_\mathrm{L} = \begin{pmatrix} \chi_{e^-\mathrm{L}} \\ 0 \end{pmatrix}$$
$$e_\mathrm{L} = (\psi_e)_\mathrm{L} = \frac{1}{2}\left(\mathbb{1} - \gamma_5\right)\psi_e$$

There is no right-handed neutrino in this theory (singlet).

$$R = e_\mathrm{R} = P_\mathrm{R}(\psi_e) = (\psi_e)_\mathrm{R}$$

Under $\mathbf{U}(1)_{\mathrm{Y}}$ the hypercharges are define as

$$Y(L) = -1 \tag{4.1.4}$$
$$Y(R) = -2 \tag{4.1.5}$$

Here each component of $L$ has $Y = -1$. They are chosen so that

$$Q_{\mathrm{EM}} = T^3_{\mathrm{L}} + \frac{1}{2}Y$$

with $T^3_{\mathrm{L}} = \frac{1}{2}\tau^3$ a generator of $\mathbf{SU}(2)$.

$$T^3_L = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$
$$T^3_L L = \begin{pmatrix} \frac{1}{2}\nu_L \\ -\frac{1}{2}e^-_L \end{pmatrix}$$

Take a look at the examples

$$Q_{\mathrm{EM}}(\nu_{\mathrm{L}}) = T^3_{\mathrm{L}}(\nu_{\mathrm{L}}) + \frac{1}{2}Y(\nu_{\mathrm{L}}) = \frac{1}{2} - \frac{1}{2} = 0$$
$$Q_{\mathrm{EM}}(e^-_{\mathrm{L}}) = T^3_{\mathrm{L}}(e_{\mathrm{L}}) + \frac{1}{2}Y(e_{\mathrm{L}}) = -\frac{1}{2} - \frac{1}{2} = 0$$
$$Q_{\mathrm{EM}}(e^-_{\mathrm{R}}) = T^3_{\mathrm{L}}(e_{\mathrm{R}}) + \frac{1}{2}Y(e_{\mathrm{R}}) = 0 - 1 = -1$$

$T_L$ only acts on left handed particles under $\mathbf{SU}(2)$

$$\psi_{e_R} = e^{i(0)}\psi_{e_R}$$
$$[T^a, Y] = 0$$

Normal kinetic term in Dirac theory

$$\mathcal{L}_{\mathrm{kin}} = \bar{\psi}i\gamma^\mu D_\mu(P_R + P_L)\psi$$

To split this using the anti-commutator of $\gamma^5$ and other gamma matrices, $\gamma^\mu P_L = P_R\gamma^\mu$

$$= \bar{\psi}_L i\gamma^\mu D_\mu\psi_L + \bar{\psi}_R i\gamma^\mu D_\mu\psi_R$$

Thus kinetic terms for leptons are

$$\mathcal{L}^{\mathrm{kin}}_{\mathrm{leptons}} = \bar{R}i\gamma^\mu D'_\mu R + \bar{L}i\gamma^\mu D_\mu L \tag{4.1.6}$$
$$\begin{aligned} D'_\mu &= \partial_\mu - \frac{ig'}{2}YB_\mu \\ &= \partial_\mu + ig'B_\mu \end{aligned} \tag{4.1.7}$$
$$\begin{aligned} D_\mu &= \partial_\mu - \frac{ig'}{2}YB_\mu - ig\frac{\tau^a}{2}W^a_\mu \\ &= \partial_\mu + \frac{i}{2}g'B_\mu - ig\frac{\tau^a}{2}W^a_\mu \end{aligned} \tag{4.1.8}$$

In electromagnetic $\mathbf{U}(1)_{\mathrm{EM}}$ the charge generator $Q$ with $Q(e^-) = -1$. Covariant derivative

$$D_\mu = \partial_\mu - ieQA_\mu \tag{4.1.9}$$

In $\mathbf{U}(1)_{\mathrm{Y}}$ the gauge boson $B_\mu$. Field strength tensor $F^{\mathrm{Y}}_{\mu\nu} = \partial_\mu B_\mu - \partial_\nu B_\mu$. Charge generator $Y/2$. Covariant derivative

$$D_\mu = \partial_\mu - \frac{ig'}{2} Y B_\mu \tag{4.1.10}$$

In Dirac Lagrangian there is a mass term $m\bar{\psi}\psi$.

$$\begin{aligned}
\mathcal{L}_m &= m\bar{\psi}\psi = m\bar{\psi}(P_R + P_L)\psi \\
&= m(\bar{\psi}P_R\psi + \bar{\psi}P_L\psi) \\
&= m(\bar{\psi}P_R^2\psi + \bar{\psi}P_L^2\psi) \\
&= m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)
\end{aligned}$$

$\bar{\psi}\psi$ mass term is called Dirac mass term. There is only left handed neutrino, so we cannot write Dirac mass term for neutrino. Although $\psi_L$ and $\psi_R$ transform differently under Lorentz transformation, $\bar{\psi}\psi$ is still Lorentz invariant.

Recall that $\mathbf{U}(1)_{\mathrm{EM}}$ in QED $\psi \mapsto \psi' = \mathrm{e}^{i\alpha(x)Q}\psi$. If $Q$ is same for left and right-handed components, $\bar{\psi}\psi$ is gauge invariant. If left and right have different hypercharges $Y$, then $\bar{\psi}_R\psi_L$ not $\mathbf{U}(1)_Y$ gauge invariant.

How about $\mathbf{SU}(2)_{\mathrm{L}}$ gauge invariance of $\bar{\psi}_L\psi_R$? Under $\mathbf{SU}(2)$

$$\begin{aligned}
R &\mapsto R' = R \\
L &\mapsto L' = \mathrm{e}^{i\alpha_a(x)\tau^a/2} L
\end{aligned}$$

So it's obvious that $\bar{\psi}_R\psi_L$ not $\mathbf{SU}(2)_{\mathrm{L}}$ gauge invariant.

Any fermionic mass term vanishes, if we require $\mathbf{SU}(2)_{\mathrm{L}} \times \mathbf{U}(1)_{\mathrm{Y}}$ gauge invariance. Spontaneous symmetry breaking to let fermion gain mass.

## 4.2 Add scalars

Introduce a Higgs doublet

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{4.2.1}$$

The superscripts denote the charge the field carries.

From previous section $Q_{\mathrm{EM}} = T_L^3 + \frac{1}{2}Y$, so

$$\begin{aligned}
Q_{\mathrm{EM}}(\phi^+) = +1 = \frac{1}{2} + \frac{1}{2}Y \Leftrightarrow Y = +1 \\
Q_{\mathrm{EM}}(\phi^0) = 0 = -\frac{1}{2} + \frac{1}{2}Y \Leftrightarrow Y = +1
\end{aligned}$$

Together $Y(\Phi) = +1$

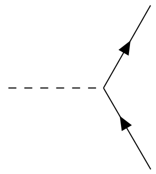
Lagrangian

$$\begin{aligned}
\mathcal{L}_{\mathrm{scalar}} &= (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi^\dagger\Phi) \\
D_\mu &= \partial_\mu - \frac{ig'}{2}B_\mu - \frac{ig}{2}\tau_i W_\mu^i \\
V &= \mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 \\
\Phi^\dagger &= \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}^\dagger = \begin{pmatrix} \phi^+ & \phi^0 \end{pmatrix}^* = \begin{pmatrix} \phi^- & (\phi^0)^* \end{pmatrix}
\end{aligned} \tag{4.2.2}$$

## 4.3 Coupling of Scalars and Fermions

Recall that simple complex scalar $\phi^0 \in \mathbb{C}$, $\mathbf{SU}(2)$ singlet. Under Lorentz transformations (per definition) $\phi^0(x) \mapsto \phi^0(x)$. $\bar{\psi}\psi$ is also Lorentz invariant. It means $\phi^0\bar{\psi}\psi$ is Lorentz invariant. This type of interaction is called Yukawa interaction.

$$\mathcal{L}_{\text{Yukawa}} = -y_e \left(\overline{R}\Phi^\dagger L + \overline{L}\Phi R\right)$$



How to combine $\Phi$, $L$ and $R$?

$$\begin{aligned}
Y(\Phi) &= +1 \\
Y(L) &= -1 \\
Y(R) &= -2 \\
Y(\bar{L}) &= +1
\end{aligned}$$

Gauge transformation of interaction term

$$\phi\bar{\psi}\psi \mapsto \phi'\bar{\psi}'\psi' = \mathrm{e}^{i\alpha(Q(\phi)+Q(\bar{\psi})+Q(\psi))}\phi\bar{\psi}\psi$$

Gauge invariance means the sum of hypercharges is zero. $\Phi\bar{L}R$ is $\mathbf{U}(1)_Y$ gauge invariant.

What if we want $\mathbf{SU}(2)_\mathrm{L} \times \mathbf{U}(1)_\mathrm{Y}$ gauge invariant.

$R$ is $\mathbf{SU}(2)$ invariant. $\Phi$ is $\underline{2}$ under $\mathbf{SU}(2)$. $L$ and $\bar{L}$ are $\underline{2}$ under $\mathbf{SU}(2)$. For $\mathbf{SU}(2)$ these two kinds of representations are the same $\bar{\underline{2}} = \underline{2}$.

$$\Phi\bar{L} : \underline{2} \otimes \bar{\underline{2}} = \underline{3} \oplus \underline{1}_A$$

$\Phi = \begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix} = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$. $\underline{1}$ is $\mathbf{SU}(2)$ singlet. Need antisymmetric combination of $\Phi\bar{L}$ to get $\mathbf{SU}(2)$ singlet.

In components

$$\begin{aligned}
\overline{L}\Phi R &= \begin{pmatrix}\overline{\nu}_L & \overline{e_L^-}\end{pmatrix}\begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix} e_R \\
&= \overline{\nu}_L\phi^+ e_R + \overline{e_L^-}\phi^0 e_R
\end{aligned}$$

here $\sum Y_i = 0$

$$\begin{aligned}
\overline{R}\Phi^+ L &= \overline{e_R^-}\begin{pmatrix}\phi^- & \left(\phi^0\right)^*\end{pmatrix}\begin{pmatrix}\nu_L \\ e_L^-\end{pmatrix} \\
&= \overline{e_R^-}\phi^-\nu_L + \overline{e_R^-}\left(\phi^0\right)^* e_L^-
\end{aligned}$$

The Standard Model does not contain $\nu_R$. But let's consider $\nu_R$ anyway

$$\begin{aligned}
Q_{\text{EM}} &= T_L^3 + \frac{1}{2}Y \\
0 &= 0 + \frac{1}{2}Y(\nu_R)
\end{aligned}$$

So it doesn't couple to $B_\mu$.

How about Dirac mass term?

$$\overline{\nu}_L \nu_R + \overline{\nu}_R \nu_L$$

It is not $\mathbf{U}(1)_\mathrm{Y}$ or $\mathbf{SU}(2)_\mathrm{L}$ invariant. Not a huge problem since it was the same for electron before we introduce Higgs mechanism.

Cannot write interaction like $\overline{L}\nu_R\Phi$ since the hypercharge is not zero. Instead

$$\begin{aligned}
\tilde{\Phi} &= i\tau_2\Phi^* = \begin{pmatrix} \phi^0 \\ -\phi^- \end{pmatrix} \\
Y(\tilde{\Phi}) &= -1 \\
\mathcal{L}_{\nu_L} &= \overline{L}\tilde{\Phi}\nu_R + \overline{\nu}_R\tilde{\Phi}^\dagger L
\end{aligned}$$

Dirac mass term for neutrino. If $\nu_R$ exists, this is a possible mass term.

$$\begin{aligned}
\overline{L}\tilde{\Phi}\nu_R &= \begin{pmatrix} \overline{\nu}_L & \overline{e_L} \end{pmatrix} \begin{pmatrix} (\phi^0)^* \\ -\phi^- \end{pmatrix} \nu_R \\
&= \overline{\nu}(\phi^0)^*\nu_R - \overline{e_L}\phi^-\nu_R
\end{aligned}$$

## 4.4 Spontaneous Symmetry Breaking (Mass and Mixing of Gauge Bosons)

In the early universe $\mu = \mu(T)$ and the symmetry got broken. Consider the case $\mu^2 < 0$.

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{4.4.1}$$

with $v \in \mathbb{R}$ and $[v] = 1$. Field $\phi^0$ has $Q_\mathrm{EM} = 0$. $Y(\phi^0) \neq 0$ breaks the $\mathbf{U}(1)_Y$ and $T_L^3(\phi^0) \neq 0$ breaks $\mathbf{SU}(2)_L$.

As before the field shift is

$$\begin{aligned}
\Phi &= U^{-1}(\boldsymbol{\xi})\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix} \\
U(\boldsymbol{\xi}) &= \exp(-i\boldsymbol{\xi} \cdot \boldsymbol{\tau}/(2v))
\end{aligned}$$

$\eta(x), \boldsymbol{\xi}(x) \in \mathbb{R}$

$$\Phi \mapsto \Phi' = U(\xi)\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + \eta \end{pmatrix} \tag{4.4.2}$$

$$L \mapsto L' = U(\boldsymbol{\xi})L \tag{4.4.3}$$

$$W_\mu \mapsto W'_\mu \tag{4.4.4}$$

$$\boldsymbol{\tau} \cdot \boldsymbol{W}'_\mu = U(\boldsymbol{\xi})\left[\boldsymbol{\tau} \cdot \boldsymbol{W}_\mu - \frac{i}{g}U^{-1}\partial_\mu U\right] \tag{4.4.5}$$

Insert transformation (4.4.2) into $\mathcal{L}$ and see the physical interpretation. Yukawa coupling

$$\begin{aligned}
-y_e\overline{L}\Phi R &= -y_e\begin{pmatrix} \overline{\nu}_e & \overline{e_L} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix} e_R \\
&= -\frac{y_e}{\sqrt{2}}\overline{e_L}(v + \eta(x))e_R
\end{aligned}$$

Thus $m_e = y_e v/\sqrt{2}$. Know $m_e$ we can fix $y_e v$. It does not predict the mass of electron, but we can accommodate. But we do predict $m_\nu = 0$.

There is an extra term

$$-\frac{y_e}{\sqrt{2}}\overline{\overline{e}_L} e_R \eta(x)$$

So the coupling of Higgs is direct proportional to mass of fermion. $y_e \sim 10^{-6}$, it is then not highly unlikely to observe Higgs decay into electrons at LHC. We have already seen Higgs decay into tau and bottom quarks.

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + V\left(\left(\frac{v+\eta}{\sqrt{2}}\right)^2\right) \tag{4.4.6}$$

Write

$$\Phi = \frac{v+\eta}{\sqrt{2}}\xi$$
$$\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Multiply $(D_\mu \Phi)^\dagger (D^\mu \Phi)$ out but focus only on terms without derivative

$$\begin{aligned}
D_\mu &= \partial_\mu - \frac{i}{2} g' B_\mu - ig\frac{\tau^i}{2} W^i_\mu \\
\mathcal{L} &\subset \frac{(v+\eta)^2}{8}\chi^\dagger \left[(g' B^\dagger_\mu + g\tau^i W^i_\mu)\cdot(g' B^\mu + g\tau^i W^{i\mu})\right]\chi \\
\chi^\dagger \tau^3 \chi &= \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 \\
\chi^\dagger \tau^2 \chi &= \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \\
\chi^\dagger \tau^1 \chi &= 0
\end{aligned}$$

Focus on $v^2$ terms, they are bilinear in gauge boson fields $\to$ mass terms.

$$\mathcal{L} \subset \frac{1}{8}v^2\left[(g' B_\mu - gW^3_\mu)(g' B^\mu - gW^{3\mu}) + g^2 (W^1_\mu)^2 + g^2 (W^2_\mu)^2\right] \tag{4.4.7}$$

with $M^2_{W^1} = M^2_{W^2} = g^2 v^2/8$.

Using the familiar formula $Q = T_e + \frac{1}{2}Y$ for the $\mathbf{SU}(2)$ gauge boson $W^{1,2,3}$. They form a triplet $T_3 = \pm 1, 0$. Doublet $T_3 = \pm\frac{1}{2}$. Then $W_3$ has zero electric charge. $B_\mu$ also has zero electric charge. Focus on the electric neutral part

$$\mathcal{L} \subset \frac{v^2}{8}\left[g'^2 B_\mu B^\mu + g^2 W^3_\mu W^{\mu 3} + g' g B_\mu W^{\mu 3} + g' g W^3_\mu B^\mu\right]$$

It has $W_\mu B^\mu$ mixing terms. Rewritten in matrix form

$$= \frac{v^2}{8}\begin{pmatrix} B_\mu & W^3_\mu \end{pmatrix}\begin{pmatrix} g'^2 & gg' \\ gg' & g^2 \end{pmatrix}\begin{pmatrix} B^\mu \\ W^{3\mu} \end{pmatrix}$$

The $2 \times 2$ matrix is mass matrix squared. Want to diagonalize this. Eigenvalues are (mass)$^2$. For $2 \times 2$ matrix,

$$\det = \lambda_1 \cdot \lambda_2 \tag{4.4.8}$$
$$\mathrm{tr} = \lambda_1 + \lambda_2 \tag{4.4.9}$$

Obviously here $\det = 0$ and $\mathrm{tr} = g^2 + g'^2$.

Eigenvectors are

$$\begin{aligned} Z_\mu &= \frac{-gW^3_\mu + g' B_\mu}{\sqrt{g^2 + g'^2}} \\ A_\mu &= \frac{gB_\mu + g' W^3_\mu}{\sqrt{g^2 + g'^2}} \end{aligned} \tag{4.4.10}$$

$$m^2(A_\mu) = 0$$
$$M_Z = \frac{v}{2}\sqrt{g^2 + g'^2}$$

Remember real fields have mass term $\frac{1}{2}m^2 A^\mu A_\mu$.

Go back to $W^1_\mu$ and $W^2_\mu$. Define

$$W^\pm = \frac{1}{\sqrt{2}}(W^1_\mu \pm W^2_\mu) \tag{4.4.11}$$
$$W^+_\mu W^{-\mu} = \frac{1}{2}\left[(W^1_\mu)^2 + (W^2_\mu)^2\right] \tag{4.4.12}$$

$\boldsymbol{W}$ is a triplet $(W^+, W^0, W^-)^T$. The superscripts denote the electric charge again.

Mass term $\frac{1}{2}m^2(W^+_\mu W^{-\mu} + W^-_\mu W^{\mu+})$

$$\frac{1}{2}M_W^2 = \frac{1}{8}g^2 v^2$$
$$M_W = \frac{1}{2}gv$$

Go back to leptons

$$\mathcal{L}^{\text{leptons}}_{\text{kin}} = \overline{R} i\gamma^\mu D'_\mu R + \overline{L} i\gamma^\mu D_\mu L$$
$$D'_\mu = \partial_\mu + i\frac{g'}{2}B_\mu$$
$$D_\mu = \partial_\mu + \frac{i}{2}g' B_\mu - ig\frac{\tau^i}{2}W^i_\mu$$

Use equation (4.4.10) to rewrite $B_\mu = f(A_\mu, Z_\mu)$ and $W^3_\mu = f'(A_\mu, Z_\mu)$.

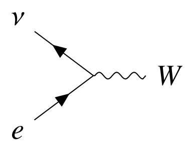
Insert that into $D'_\mu$ and $D_\mu$ and then into $\mathcal{L}^{\text{leptons}}_{\text{kin}}$. Also replace $W'^2_\mu \mapsto W^\pm_\mu$

$$\overline{L} i\gamma^\mu D_\mu L = \overline{L} i\gamma^\mu \left(\partial_\mu + \frac{i}{2}g' B_\mu - ig\frac{\tau^i}{2}W^i_\mu\right) L$$
$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\tau_1 W^1_\mu + \tau^2 W^2_\mu = \begin{pmatrix} 0 & W^1_\mu - iW^2_\mu \\ W_\mu + iW^2_\mu & 0 \end{pmatrix}$$

Charged current

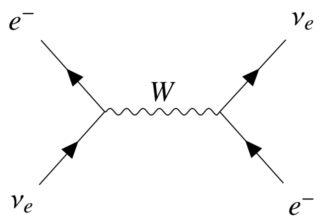
$$
\begin{aligned}
A &= \overline{L}\gamma^\mu \left(\tau^1 W^1_\mu + \tau^2 W^2_\mu\right) L \\
&= \frac{g}{2}\begin{pmatrix}\overline{\nu}_L & \overline{e}_L\end{pmatrix}\gamma^\mu \begin{pmatrix} 0 & \sqrt{2}W^+_\mu \\ \sqrt{2}W^-_\mu & 0 \end{pmatrix}\begin{pmatrix}\nu_L \\ e_L\end{pmatrix} \\
&= \frac{g}{2}\begin{pmatrix}\overline{\nu}_L & \overline{e}_L\end{pmatrix}\gamma^\mu \begin{pmatrix} \sqrt{2}W^+_\mu e_L \\ \sqrt{2}W^-_\mu \nu_L \end{pmatrix} \\
&= \frac{g}{\sqrt{2}}\left[\overline{\nu}_L W^+_\mu \gamma^\mu e_L + \overline{e}_L W^-_\mu \gamma^\mu \nu_L\right]
\end{aligned}
$$

Diagrammatically




$\sim ig\gamma^\mu P_L$

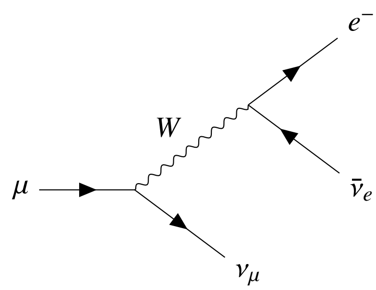
Neutrino-electron scattering has the cross section $\sim g^4/M_W^4$




Fermi constant

$$G_F = \frac{\sqrt{2}g^2}{8M_W^2} \tag{4.4.13}$$

$\mathcal{L}^{\text{leptons}}_{\text{kin}}$ can be extended to muons. Then we can draw diagram for muon decay. Decay rate $\Gamma \sim G_F^2$. $\tau_\mu \sim 2\,\mu\text{s}$. $G_F \sim \frac{10^{-5}}{m_\mu^2}$



# Bibliography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