Theoretical particle physics

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October 24, 2019

Contents

0 Organisational

Tutorials: Thursday: 8-10, 10-12 Friday: 10-12, 13-15

Exam consists of four problems

- first quickies
- 2nd-4th: similar in style to homework; two will be very close to homework

One needs 50% of points from homework. May hand in pairs.

Content of the lectures

- Standard Model of particle physics
- Electroweak sector
  - gauge principle
  - Higgs mechanism
  - Yukawa interactions
  - CP-violation

Exercises

- go through basics of computing Feynman diagrams
- not to derive the formalism
- Lagrange → Feynman rules → amplitudes → cross section and decay rates (measured quantities)

Literature

- Halzen and Martin, Quarks and Leptons (a lot of basics of QCD)
- Cheng and Li (includes also quantum field theory topics CP-violation in Standard Model)
- Mark Thomson
- QFT basics
  - Peskin and Schroeder
  - M.Schwartz
  - Ryder
- Okun, Leptons and Quarks

1 Introduction

1.1 Standard Model

It is the fundamental theory of nature. There are three interactions included

- electromagnetic
- weak
- strong
- Higgs boson exchange

Electromagnetic and weak interactions can be unified into electroweak interactions.

In the Sun all these three interactions and gravity are present

- Photons reaching us clearly indicate QED's involvement.
- Neutrinos produced in weak interaction. Four protons to two protons and two neutrons (Helium). Only weak interaction can change the colour of quarks.
- Binding of Helium via strong interaction and binding energy released as kinetic energy.
- Gravity brings protons together and at high temperature to give helium.

Gauge theories (Lie groups algebras)

- EM: $U(1)_{EM}$, $U(1)_Y$
- weak: $SU(2)_L$
- strong: $SU(3)_c$

Forces in quantum theories involve exchange particles spin 1, vector bosons

- photon γ
- weak $W^{\pm}$, W^0 and Z^0 (mixture of W^0 and hyper charge) (discovered at CERN)
- strong $g^a, a = 1, \ldots, 8$ gluons (discovered at DESY)

particles with spin $\frac{1}{2}$
Leptons

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, e_R^-; \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \mu_R^-; \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, \tau_R^-$$

Quarks

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, u_R, d_R; \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, c_R, s_R; \quad \begin{pmatrix} t \\ b \end{pmatrix}_L, t_R, b_R$$

One complete generation

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, e_R^-; \quad \begin{pmatrix} u \\ d \end{pmatrix}_L, u_R, d_R;$$

To remove any one part, then gauge theory is inconsistent. It is known as "anomaly". (AQFT)

Higgs boson h^0, spin 0
In Standard Model Higgs bosons are described by complex scalar fields $\begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$. h^0 is the only fundamental scalar in nature, as far as we know.

1.2 Energy scales

- binding energy of atoms $1 - 10$eV
- binding energy of nucleons $\approx$ 1MeV
- no known binding energy in particle physics
- protons and neutrons $\approx \Lambda_{QCD} \approx \mathcal{O}(100\text{MeV})$

Particles have masses

- electron $m_e = 511\,\text{keV}$
- muon $m_\mu = 105\,\text{MeV}$
- tau $m_\tau = 1.7\,\text{GeV}$
- neutrinos $m_\nu < 1\,\text{eV}$
- quarks*
  - $m_u \approx 3\,\text{MeV}$
  - $m_d \approx 5\,\text{MeV}$
  - ...
- photon $m_\gamma = 0$
- gluon $m_g = 0$
- Higgs $m_{Higgs} \approx 125\,\text{GeV}$

*mass of proton mainly comes from dynamical effect "gluon"

Colliders

- LEP 91 GeV – 200 GeV
- Tevatron($p\bar{p}$) 800 GeV – 2 TeV
- LHC 7 TeV – 13 TeV

1.3 Natural units

$$\hbar = c = 1 \tag{1.3.1}$$
$$k_B = 1 \tag{1.3.2}$$

Everything expressed in term of powers of energy.

$$1\,\mathrm{fm} = 1 \times 10^{-15}\,\mathrm{m} = 5\,\mathrm{GeV}^{-1}$$

2 Lorentz Transformation

2.1 Introduction

Metric (used for distance measuring)

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.1.1}$$

In string and general relativity people tend to use diag$(-, +, +, +)$.

$$p = (E, \boldsymbol{p}) \tag{2.1.2}$$

$$p^2 = E^2 - \boldsymbol{p}^2 = m^2 \tag{2.1.3}$$

Light is always light-like

$$t^2 - (x^2 + y^2 + z^2) = 0$$

Greek indices always go from 0 to 3

$$r^2 = g_{\mu\nu} r^\mu r^\nu = t^2 - \boldsymbol{r}^2 \tag{2.1.4}$$

Distance between two spacetime point is defined via

$$|r_A - r_B| = \sqrt{(r_A - r_B)\cdot(r_A - r_B)} = \sqrt{r_A^2 + r_B^2 - 2r_A \cdot r_B} \tag{2.1.5}$$

2.2 Lorentz Transformation

Lorentz Transformation is transformation between two inertial frames moving with constant velocity $\boldsymbol{v}$ with respect to each other (boosts).

$$\begin{aligned} x &= (x_0, x_1, x_2, x_3) \\ x' &= (x'_0, x'_1, x'_2, x'_3) \\ x_0 &= ct = t; \quad c = 1 \end{aligned}$$

We define

$$\beta = \frac{v}{c} \quad \text{or} \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c} \tag{2.2.1}$$

Coordinates in these two frames are related like

$$
\begin{aligned}
x_0' &= \gamma(x_0 - \beta x_1) \\
x_1' &= \gamma(x_1 - \beta x_0) \\
x_2' &= x_2 \\
x_3' &= x_3
\end{aligned}
$$

Inverse transformation with $\beta \mapsto -\beta$
γ-factor is defined via

$$\gamma = \frac{1}{\sqrt{1 - \boldsymbol{\beta}^2}} \tag{2.2.2}$$

Since $|\boldsymbol{\beta}| \leq 1 \Rightarrow \gamma \geq 1$
Alternative parametrization

$$
\begin{aligned}
\beta &= \tanh(\zeta), \quad \gamma = \cosh(\zeta) \\
\gamma\beta &= \sinh(\zeta)
\end{aligned}
$$

Insert this into equation (2.2.2)

$$
\begin{aligned}
x_0' &= x_0 \cosh(\zeta) - x_1 \sinh(\zeta) \\
x_1' &= x_0 \sinh(\zeta) - x_1 \cosh(\zeta)
\end{aligned}
$$

We can turn this into matrices

$$
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix}
$$

$$x = \Lambda x' \tag{2.2.3}$$

2.3 Mathematical Properties of Lorentz Transformation

Distance is invariant under Lorentz transformation

$$s^2 = x_0^2 - x_1^2 + x_2^2 + x_3^2 = x^2 \tag{2.3.1}$$

Lorentz transformation includes

- Rotation and boosts
- Parity $\boldsymbol{x} \mapsto -\boldsymbol{x}$
- Time reversal $t \mapsto -t$

We can also expand it with translation. It then turns to Poincare group.

2.3.1 Tensors

Define a function of original coordinates ($\alpha = 0, 1, 2, 3$)

$$x'^{\alpha} = x'^{\alpha}(x^0, x^1, x^2, x^3) \tag{2.3.2}$$

If x'^{α} transforms like

$$x'^{\alpha} = \frac{x'^{\alpha}}{x^{\beta}} x^{\beta} \tag{2.3.3}$$

it is called *contravariant*

Consider derivative $\frac{\partial}{\partial x'^{\alpha}}$

$$\frac{\partial f(x)}{\partial x'^{\alpha}} = \frac{\partial f(x)}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \tag{2.3.4}$$

We can see the x' is now in the denominator. The objects transformed like this are called *covariant*.

Consider the following generic objects: A'^{α} contravariant vector

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} \tag{2.3.5}$$

B'_{α} is covariant

$$B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} \tag{2.3.6}$$

Note (x^0, x^1, x^2, x^3) is contravariant.

The field strength tensor $F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta}$ is contravariant rank 2.

Mixed is also allowed $H'^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{x^{\delta}}{\partial x'^{\beta}} H^{\delta}_{\gamma}$

Inner or scalar product

$$\begin{aligned} B' \cdot A' &= B'_{\alpha} A'^{\alpha} \\ &= \left(\frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta}\right)\left(\frac{\partial x'^{\alpha}}{\partial x^{\gamma}} A^{\gamma}\right) \\ &= \frac{\partial x^{\beta}}{\partial x^{\gamma}} B_{\beta} A^{\gamma} \\ &= \delta^{\beta}_{\gamma} B_{\beta} A^{\gamma} = B \cdot A \end{aligned}$$

$$\begin{aligned} \mathrm{d}s^2 &= \left(\mathrm{d}x^0\right)^2 - (\mathrm{d}\boldsymbol{x})^2 \\ &= (g_{\alpha\beta}\,\mathrm{d}x^{\alpha})\,\mathrm{d}x^{\beta} = \mathrm{d}x_{\beta}\,\mathrm{d}x^{\beta} \end{aligned} \tag{2.3.7}$$

Thus we can use metric tensor to lower index $\mathrm{d}x_{\beta} = g_{\alpha\beta}\,\mathrm{d}x^{\alpha}$

$$\begin{aligned} A^{\alpha} &= \left(A^0, \boldsymbol{A}\right) \\ A_{\alpha} &= \left(A^0, -\boldsymbol{A}\right) \end{aligned}$$

2.4 Matrix Representation of Lorentz Transformation

2.4.1 General Properties

We have

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

Then $a \cdot b = (a, gb) = g_{\mu\nu} a^\mu b^\nu = (ga, b) = a^T gb = (ga)^T b$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \mapsto x' = \Lambda x \tag{2.4.1}$$

$$x \cdot x = x' \cdot x' = (\Lambda x)(\Lambda x) \tag{2.4.2}$$

$$\begin{aligned} g_{\mu\nu} x^\mu x^\nu &= g_{\sigma\tau} x'^\sigma x'^\tau \\ &= g_{\sigma\tau} \Lambda^\sigma{}_\mu x^\mu \Lambda^\tau{}_\nu x^\nu \\ &= g_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu x^\mu x^\nu \end{aligned}$$

Then we have the *defining* rule of Lorentz group

$$g_{\mu\nu} = g_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \tag{2.4.3}$$

$$g = \Lambda^T g \Lambda \tag{2.4.4}$$

Properties

- $|\det(\Lambda)| = 1$
- $\left|\Lambda^0{}_0\right| \geq 1$

The orthochronous Lorentz transformations Λ forms a group.

Parity does not form a group

$$\Lambda_P = \mathrm{diag}(1, -1, -1, -1) \tag{2.4.5}$$

Time reversal

$$\Lambda_T = \mathrm{diag}(-1, +1, +1, +1) \tag{2.4.6}$$

There are four classes of Lorentz transformations depending on $\left(\mathrm{sgn}(\det(\Lambda)), \mathrm{sgn}(\Lambda^0_0)\right)$

- $(+, +)$ Λ
- $(-, -)$ $\Lambda_T \Lambda$
- $(-, +)$ $\Lambda_P \Lambda$
- $(+, -)$ $\Lambda_T \Lambda_P \Lambda$

Orthochronous Λ has 6 parameters, 3 for boosts and 3 for rotations. $\Lambda^T g \Lambda = g$ is actually 16 equations. All matrices here are symmetric. Thus 6 of 16 are redundant. There are 10 independent equations. Λ has 16 entries and it has $16 - 10 = 6$ free parameters.

2.4.2 Explicit Construction

We will restrict ourselves in orthochronous Lorentz transformations. The exponential function is defines via Taylor expansion. With $L \in \mathbb{R}^{4\times 4}$

$$\Lambda = \mathrm{e}^L = \exp(L)$$

From linear algebra we know

$$\det(\Lambda) = \det\left(\mathrm{e}^L\right) = \mathrm{e}^{\mathrm{tr}(L)} \tag{2.4.7}$$

Since $\det(\Lambda) = 1$, $\mathrm{tr}(L) = 0$

$$\begin{aligned}
\Lambda^T g \Lambda &= g \\
g\Lambda^T g \Lambda &= \mathbb{1}_4 \\
g\Lambda^T g &= \Lambda^{-1} \\
\exp\left(gL^T g\right) &= \Lambda^{-1} = \exp(-L) \\
\Leftrightarrow gL^T g &= -L \\
\Leftrightarrow (gL)^T &= -gL
\end{aligned}$$

This means that gL is anti-symmetric

$$L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & L_{12} & 0 & L_{23} \\ L_{03} & L_{13} & L_{23} & 0 \end{pmatrix}$$

Define six basis matrices $S_{1,2,3}$ and $K_{1,2,3}$

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

S_i is the generator of 3-dimensional rotations and K_i is the generator of 3-dimensional boosts.

$$\begin{aligned}
\hat{n} \in \mathbb{R}^3, &\quad |\hat{n}| = 1 \\
\hat{n} \cdot \boldsymbol{S} &= n_1 S_1 + n_2 S_2 + n_3 S_3 \\
(\hat{n} \cdot \boldsymbol{S})^3 &= -\hat{n} \cdot \boldsymbol{S} \\
(\hat{n} \cdot \boldsymbol{K})^3 &= +\hat{n} \cdot \boldsymbol{S}
\end{aligned}$$

In the end

$$L = -\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\zeta} \cdot \boldsymbol{K} \quad \text{with } \boldsymbol{\omega}, \boldsymbol{\zeta} \in \mathbb{R}^3 \tag{2.4.8}$$

$$\Lambda = \exp(-\boldsymbol{\omega} \cdot \boldsymbol{S} - \boldsymbol{\zeta} \cdot \boldsymbol{K}) \tag{2.4.9}$$

$\boldsymbol{\omega}$ is the axis of rotation, $|\boldsymbol{\omega}|$ is then the angle of rotation.
$\tanh|\boldsymbol{\zeta}| = \beta$ and $\frac{\boldsymbol{\zeta}}{|\zeta|}$ is the direction of boost.
We now will look at concrete examples

- $\boldsymbol{\zeta} = 0, \ \boldsymbol{\omega} = w\hat{e}_z$

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  Rotational angle is ω.

- $\boldsymbol{\omega} = 0, \ \boldsymbol{\zeta} = \zeta\hat{e}_x$

$$\Lambda = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Pure general boost $\boldsymbol{\zeta}$

$$\Lambda = \exp(-\boldsymbol{\zeta}\cdot\boldsymbol{K})$$
$$\boldsymbol{\zeta} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|}\tanh^{-1}|\boldsymbol{\beta}|, \quad \hat{\beta} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|}$$
$$\Lambda = \exp\Big(-\hat{\beta}\cdot\boldsymbol{K}\tanh^{-1}(\beta)\Big)$$

2.4.3 Algebra of generators

Consider the commutation algebra of $S_{i=1,2,3}$ and $K_{i=1,2,3}$

$$\left[S_i, S_j\right] = \epsilon_{ijk}S_k \tag{2.4.10}$$
$$\left[S_i, K_j\right] = \epsilon_{ijk}K_k \tag{2.4.11}$$
$$\left[K_i, K_j\right] = -\epsilon_{ijk}S_k \tag{2.4.12}$$

The last equation causes Thomas precession in atomic physics.
Choose a different basis

$$\boldsymbol{S}_+ = \frac{1}{2}(\boldsymbol{S} + i\boldsymbol{K})\,\boldsymbol{S}_- = \frac{1}{2}(\boldsymbol{S} - i\boldsymbol{K}) \tag{2.4.13}$$

Then we can calculate the algebra

$$\left[S_{+,i}, S_{+,j}\right] = i\epsilon_{ijk}S_{+,k} \tag{2.4.14}$$
$$\left[S_{-,i}, S_{-,j}\right] = i\epsilon_{ijk}S_{-,k} \tag{2.4.15}$$
$$\left[S_{+,i}, S_{-,j}\right] = 0 \tag{2.4.16}$$

In other word, the algebras are decoupled. This familiar algebra is angular momentum algebra **SU**(2).

Classification by two numbers $(j_+,\ j_-)$

$$
\begin{aligned}
j_+ &= 0, \frac{1}{2}, 1, \dots \\
j_- &= 0, \frac{1}{2}, 1, \dots
\end{aligned}
$$

Dimension = $(2j_+ + 1)(2j_- + 1)$.

A field is scalar field if $j_+ = j_- = 0$. One fundamental example of scalar field is Higgs boson. Other scalar particles are just bound states.

There are two possible states with spin $\frac{1}{2}$: $(j_+, j_-) = (\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$

$$
\begin{aligned}
\boldsymbol{S}_+ &= \frac{1}{2}\boldsymbol{\sigma} \\
\boldsymbol{S}_- &= 0 \\
\boldsymbol{S} &= \frac{1}{2}\boldsymbol{\sigma} \\
i\boldsymbol{K} &= \frac{1}{2}\boldsymbol{\sigma}
\end{aligned}
$$

for $(0, \frac{1}{2})$ there is "-"

So two types of spin $\frac{1}{2}$ fermions

$$
\begin{aligned}
\left(\frac{1}{2}, 0\right) &\to e_L^- \\
\left(0, \frac{1}{2}\right) &\to e_R^-
\end{aligned}
$$

They have different transformation law. W boson only to e_L^- but photon couples to both.

Under parity transformation $e_L^- \leftrightarrow e_R^-$. Both particles are needed for theory to be invariant under parity transformation, like EM and strong interactions.

3 Relativistic Quantum Field Theory

In non-relativistic quantum mechanics

$$
\begin{aligned}
E &= \frac{\boldsymbol{p}^2}{2m} \\
E &\mapsto i\hbar\frac{\partial}{\partial t} \\
\boldsymbol{p} &\to -i\hbar\boldsymbol{\nabla}
\end{aligned}
$$

After promoting the momentum and energy into operators in dispersion relation we have the Schrödinger equation

$$
i\frac{\partial}{\partial t}\psi + \frac{1}{2m}\boldsymbol{\nabla}^2\psi = 0 \tag{3.0.1}
$$

Density of probability is defined via

$$
\rho = |\psi|^2 = \psi\psi^* \tag{3.0.2}
$$

It obeys the continuity equation

$$
\begin{aligned}
-\frac{\partial}{\partial t}\int_V \rho\,\mathrm{d}V &= \int \boldsymbol{j}\cdot\boldsymbol{n}\,\mathrm{d}S \\
&= \int_V \boldsymbol{\nabla}\cdot\boldsymbol{j}\,\mathrm{d}V \\
\Rightarrow \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} &= 0 \qquad (3.0.3)
\end{aligned}
$$

Writing this explicitly

$$
\begin{aligned}
\frac{\partial\rho}{\partial t} &= \frac{\partial}{\partial t}(\psi\psi^*) \\
&= \psi\frac{\partial\psi^*}{\partial t} + \psi^*\frac{\partial\psi}{\partial t} \\
&= \frac{i}{2m}\left(\psi^*\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\psi\right) \\
\Rightarrow \boldsymbol{j} &= -\frac{i}{2m}\left(\psi^*\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\psi\right) \qquad (3.0.4)
\end{aligned}
$$

If we have a plane wave state, as an example

$$
\begin{aligned}
\psi &= N\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}-iEt} \\
\boldsymbol{j} &= \frac{\boldsymbol{p}}{m}|N|^2
\end{aligned}
$$

3.1 Relativistic wave equation

Now we enter the relativistic regime

$$E^2 = \boldsymbol{p}^2 + m^2$$
$$p^\mu = (E, \boldsymbol{p}) \quad p_\mu = (E, -\boldsymbol{p})$$
$$p^2 = m^2$$

Promoting energy and momentum into operators

$$p^\mu \mapsto i\partial^\mu$$
$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial^2 t} - \nabla^2$$

We have then Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\phi(\boldsymbol{x}, t) = 0 \tag{3.1.1}$$

The current in KG-theory is conserved as well

$$j^\mu = (\rho, \boldsymbol{j}) = i\left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right) \tag{3.1.2}$$
$$\partial_\mu j^\mu = 0 \tag{3.1.3}$$

An example solution

$$\phi = N\mathrm{e}^{-ip\cdot x}$$
$$j^\mu = 2p^\mu |N|^2$$

In terms of Lorentz transformation

$$\rho \sim E$$

Energies of particles

$$E^2 = \boldsymbol{p}^2 + m^2$$
$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2}$$

It also implies negative probability

$$E > 0 \mapsto \rho > 0$$
$$E < 0 \mapsto \rho < 0$$

3.2 Feynman-Stückelberg Interpretation of negative energy states

"Electron" with $E, \boldsymbol{p}$ and charge $-e$

$$j^\mu_{e^-} = 2e|N|^2(E, \boldsymbol{p})$$

"Positron" with $E, \boldsymbol{p}$ and charge $+e$

$$j^\mu_{e^+} = 2e|N|^2(E, \boldsymbol{p}) = -2e|N|^2(-E, -\boldsymbol{p})$$

We can think of $E < 0$ solution as particle flying backwards in time or $E > 0$ anti-particle forwards in time.

In a relativistic systems we need to remember following points

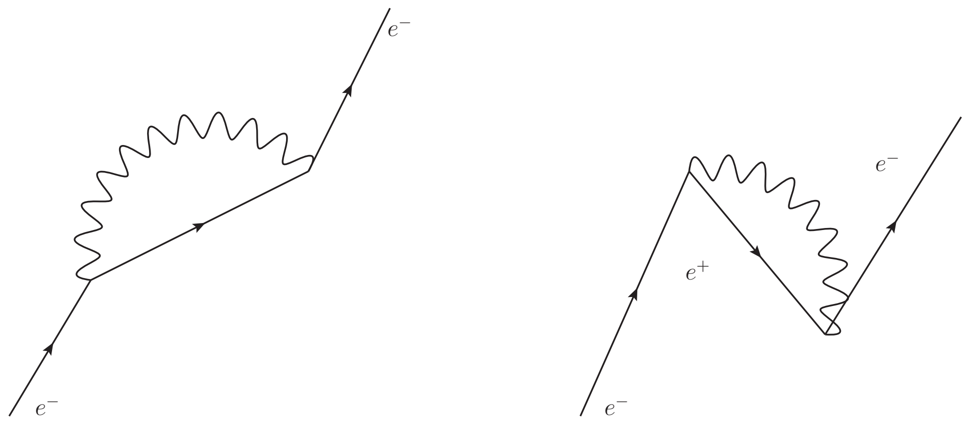

Figure 3.1: scattering process; horizontal time-axis; in the second diagram a electron positron pair is produced

- anti-particles
- particle numbers are not conserved

3.3 Electrodynamics (spin 1)

Maxwell equations are

$$\boldsymbol{E} = -\vec{\nabla}\phi - \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A} \tag{3.3.1}$$
$$\boldsymbol{B} = \vec{\nabla} \times \boldsymbol{A} \tag{3.3.2}$$
$$\vec{\nabla} \times \boldsymbol{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B} \tag{3.3.3}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{3.3.4}$$

Field strength tensor and four-potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.3.5}$$
$$A^\mu(x) = (\phi, \boldsymbol{A}) \tag{3.3.6}$$

The fields can be calculated from it

$$E^i = F^{0i} = \partial^i A^0 - \partial^0 A^i \tag{3.3.7}$$
$$B_i = -\epsilon_{ijk}\partial^i A^k = -\epsilon_{0ijk}F^{jk} \tag{3.3.8}$$

Often it is useful to use the dual tensor

$$\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\sigma\tau}F^{\sigma\tau} \tag{3.3.9}$$
$$\partial^\mu \tilde{F}_{\mu\nu} = 0 \tag{3.3.10}$$

is the second set of maxwell equations.

The other set of two equations is

$$\partial_\nu F^{\mu\nu} = 4\pi j^\mu \tag{3.3.11}$$

$\boldsymbol{E}, \boldsymbol{B}$ are observable, $\boldsymbol{A}$ is not. A^μ is not uniquely fixed by $\boldsymbol{E}$ and $\boldsymbol{B}$. It has the following gauge symmetry

$$\tilde{A}_\mu = A_\mu + \partial_\mu \Lambda(\boldsymbol{x}, t) \tag{3.3.12}$$

Use this transformation to get

$$\partial_\mu A^\mu = 0 \tag{3.3.13}$$

Plugging it back then we have the relativistic wave equation

$$\partial_\mu \partial^\mu A^\nu = 0 \tag{3.3.14}$$

it essentially is Klein-Gordon equation with mass $m = 0$
A^μ is a vector with spin 1

$$(j_+, j_-) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

It implied it has two transverse degrees of freedom. It has spin 1 properties: $+1, 0, -1$, in which 0 mode does not exist.

3.4 Description of Fermions

Original motivation for Dirac. He wants a linear equation in E or $\frac{\partial}{\partial t}$

$$p^\mu \mapsto i\partial^\mu$$

Take the ansatz

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi &= H\psi \\ &= (\vec{\alpha} \cdot \boldsymbol{p} + \beta m)\psi \end{aligned}$$

but $\boldsymbol{\alpha}$ and β unknown. It still has to obey the relativistic energy relation

$$\begin{aligned} A &= (\alpha_i p_i + \beta m)(\alpha_i p_i + \beta m) \\ &\overset{!}{=} \boldsymbol{p}^2 + m^2 \\ &= \alpha_i \alpha_j p_i p_j + \beta^2 m^2 + \alpha_i \beta p_i m + \beta \alpha_j p_j m \end{aligned}$$

From this we demand

$$\beta^2 = 1 \tag{3.4.1}$$
$$\alpha_i^2 = 1 \tag{3.4.2}$$
$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \tag{3.4.3}$$
$$\alpha_i \beta + \beta \alpha_i = 0 \tag{3.4.4}$$

So α and β are not just numbers, but (can be proven to be) hermitian traceless matrices with eigenvalue ± 1. In addition, it only exits in even dimensions. Since α_i and β are 4×4 matrices. ψ has to be a 4-component spinor.

For parity conservation need $(\frac{1}{2},0)\bigoplus(0,\frac{1}{2})$ Thus

$$\begin{pmatrix} \begin{pmatrix} & \end{pmatrix}_{2\times 2} & \\ & \begin{pmatrix} & \end{pmatrix}_{2\times 2} \end{pmatrix}$$

There are different sets of α_i,β which satisfy the conditions. They are called representations.

Dirac-Pauli representation

$$\alpha_i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{3.4.5}$$

$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \tag{3.4.6}$$

with σ^i the Pauli matrices.

Weyl (chiral) representation

$$\alpha^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{3.4.7}$$

$$\beta = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \tag{3.4.8}$$

they are mainly used in high energy physics ($\mathrm{E} \gg m$).

3.4.1 Gamma Matrices

We now define 4 gamma matrices γ^μ, $\mu = 0,1,2,3$

$$\gamma^\mu = (\beta, \beta\boldsymbol{\alpha}) \tag{3.4.9}$$

Note that having an index does not make it Lorentz vector.

The Clifford algebra is defined as following

$$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{3.4.10}$$

In Dirac-Pauli representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{3.4.11}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{3.4.12}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{3.4.13}$$

In Weyl representation

$$\gamma^0 \leftrightarrow \gamma^5$$

Rewriting the Dirac equation using γs

$$\begin{aligned}
i\partial_t\psi &= (\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m)\,\psi \\
i\partial_t\psi &= -i\boldsymbol{\alpha}\cdot\vec{\nabla}\psi + m\beta\psi \\
i\beta\partial_t\psi &= -i\beta\boldsymbol{\alpha}\cdot\vec{\nabla}\psi + m\psi \\
\left(i\gamma^\mu\partial_\mu - m\right)\psi &= 0 \qquad (3.4.14)
\end{aligned}$$

Conventionally we use ϕ for spin 0 particle and A_μ for spin 1.

It is convenient to also have an equation for $\psi^\dagger$. First one can show $\gamma^{\dagger\mu} = \gamma^0\gamma^\mu\gamma^0$.

- $\mu = 0$: $\gamma^0 = \beta$ and $\gamma^{\dagger\,0} = \gamma^0\gamma^0\gamma^0 \Rightarrow \beta^2 = \mathbb{1}_4$
- $\gamma^{\dagger\mu} = (\beta\alpha^k)^\dagger = (\alpha^k)^\dagger\beta^\dagger = \alpha^k\beta = \beta^2\alpha^k\beta = \beta\gamma^k\beta = \gamma^0\gamma^k\gamma^0$

$$\begin{aligned}
i\gamma^0\partial_0\psi + i\gamma^k\partial_k\partial - m\psi &= 0 \\
-i\partial_0\psi^\dagger(\gamma^0)^\dagger - i(\partial_k\psi^\dagger)\gamma^{\dagger\,k} - m\psi^\dagger &= 0 \\
-i\partial_0\psi^\dagger\gamma^0 - i(\partial_k\psi^\dagger)\gamma^0\gamma^k\gamma^0 - m\psi^\dagger &= 0
\end{aligned}$$

define $\bar{\psi} = \psi^\dagger\gamma^0$

$$\begin{aligned}
-i\partial_0\bar{\psi}\gamma^0 - i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} &= 0 \\
i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} &= 0 \qquad (3.4.15)
\end{aligned}$$

3.4.2 Free Particle Solution to Dirac Equation

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0$$

multiplying $\gamma^\nu\partial_\nu$ from left

$$\begin{aligned}
i\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\psi - m\gamma^\nu\partial_\nu\psi &= 0 \\
i\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\psi + im^2\psi &= 0
\end{aligned}$$

$$\begin{aligned}
\gamma^\mu\gamma^\nu &= \frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) \\
&= \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu + 2g^{\mu\nu}\right) \\
&= \frac{1}{2}\left[\gamma^\mu,\gamma^\nu\right] + g^{\mu\nu}
\end{aligned}$$

The commutator is anti-symmetric and multiplying to symmetric tensor (derivatives) the term must vanish.

Each component of spinor satisfies the Klein-Gordon equation.

$$(\partial_\mu\partial^\mu + m^2)\psi_i = 0 \qquad (3.4.16)$$

Thus we can write the solution as plane-wave

$$\psi = u(\boldsymbol{p})\mathrm{e}^{-ipx} \qquad (3.4.17)$$

$u(\boldsymbol{p})$ is also a 4-component object but as function $\boldsymbol{p}$ not $\boldsymbol{x}$

Insert back into Dirac equation, then we have Dirac equation in momentum space

$$\left(\gamma^\mu p_\mu - m\right) u(\boldsymbol{p}) = 0 \tag{3.4.18}$$

Solution by considering Dirac-Pauli representation

$$\left(\not{p} - m\right) u(\boldsymbol{p}) = \begin{pmatrix} (E-m)\mathbb{1} & -\boldsymbol{p}\cdot\boldsymbol{\sigma} \\ \boldsymbol{p}\cdot\boldsymbol{\sigma} & -(E+m)\mathbb{1} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$\boldsymbol{p} = 0$ then $E = \pm m$

- $E = +m$ Two solutions

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $E = -m$

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\boldsymbol{p} = 0$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} u_B = (E-m) u_A \tag{3.4.19}$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{p} u_A = (E+m) u_B \tag{3.4.20}$$

- $E > 0$

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Ansatz $u_A^{(s)} = \chi^{(s)}$

$$u_B^{(s)} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} \quad u_A^{(s)} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)}$$
$$u(\boldsymbol{p}) = N \begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \end{pmatrix} \tag{3.4.21}$$

$E < 0$ and $u_B^{(s)} = \chi^{(s)}$

$$u(\boldsymbol{p}) = N \begin{pmatrix} -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \tag{3.4.22}$$

One can show $u^{\dagger(r)} u^{(s)} = N^2 \delta^{rs}$

Two fold degeneracy in each case. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $E > 0$ and $E < 0$. There must be another observable which commutes with H and $\boldsymbol{p}$.

$$H = \gamma^i p_i + \gamma^0 m$$

$$\boldsymbol{S}\cdot\hat{P} = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{p} & 0\\ 0 & \boldsymbol{\sigma}\cdot\hat{p}\end{pmatrix} \tag{3.4.23}$$

Helicity

$$\frac{1}{2}\boldsymbol{\sigma}\cdot\hat{p} = \frac{1}{2}\begin{pmatrix}\hat{p}_z & \hat{p}_x + i\hat{p}_y\\ \hat{p}_x - i\hat{p}_y & -\hat{p}_z\end{pmatrix} \tag{3.4.24}$$

$$\det(\boldsymbol{\sigma}\cdot\hat{p}) = -\hat{p}^2 = -1 \tag{3.4.25}$$

Determinant is the product of two eigenvalues, then

$$\begin{aligned}\lambda_1 + \lambda_2 &= 0\\ \lambda_1\cdot\lambda_2 &= 1\\ \lambda_{1,2} &= \pm 1\end{aligned}$$

Antiparticle solution $u^{(3,4)}(-\boldsymbol{p})e^{-i(-p)x} = v^{(2,1)}$

$$\left(\not{p} + m\right)v(\boldsymbol{p}) = 0 \tag{3.4.26}$$

Normalization is

$$\int \rho\,\mathrm{d}V = 2E \tag{3.4.27}$$

$$N = \sqrt{E+m} \tag{3.4.28}$$

Completeness relation (spin sums)

$$\sum u^{(s)}(p)\bar{u}^{(s)}(p) = \left(\not{p} + m\right) \tag{3.4.29}$$

$$\sum v^{(s)}(p)\bar{v}^{(s)}(p) = \left(\not{p} - m\right) \tag{3.4.30}$$

Define a projector projecting out positive and negative energy states

$$\Lambda_{\pm} = \frac{\pm\not{p} + m}{2m} \tag{3.4.31}$$

In Chiral (Weyl) representation

$$\left(\not{p} - m\right)u(\boldsymbol{p}) = \begin{pmatrix} m & p\cdot\sigma\\ \boldsymbol{k}\cdot\bar{\sigma} & -m\end{pmatrix}\begin{pmatrix}u_L\\ u_R\end{pmatrix} \tag{3.4.32}$$

$\bar{\sigma} = (\sigma^0, -\boldsymbol{\sigma})$ and $\sigma^0 = \mathbb{1}_2$

Weyl equation

$$\begin{aligned}-mu_L + p\cdot\sigma u_R &= 0\\ p\cdot\bar{\sigma}u_L - mu_R &= 0\end{aligned} \tag{3.4.33}$$

if $m = 0$, the equations decouple from each other.

We want to construct Lagrangians involving the fields ψ, A_μ and ψ. The reason we choose Lagrangians as opposed to Hamiltonians, is that Hamiltonian H is associated with energy of system, which is not Lorentz (or relativistic) invariant. But Lagrangian density is $S = \int \mathrm{d}t = \int \mathrm{d}^4x\,\mathcal{L}$. In natural unit, the action is dimensionless. One can in addition prove d^4x is Lorentz invariant. The fact that $\mathcal{L}$ is Lorentz invariant, means that $\mathcal{L}$ need to have at least 2 spin-$\frac{1}{2}$ fields ψ or $\bar{\psi}$, to make a spin-0.

We are interested in the objects like

$$\bar{\psi}(\gamma^\mu)\psi$$

Representation of the Lorentz group can be split into $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$. In Weyl or chiral representation, ψ_L has $\left(\frac{1}{2},0\right)$ and ψ_R $\left(0,\frac{1}{2}\right)$.

Here we are taking a different approach from the paper [1]. First define

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right] \tag{3.4.34}$$

We want to find out the transformation of $\psi = u(\boldsymbol{p})e^{-ip\cdot x}$. Note that the second factor is written in covariant form, i.e. Lorentz invairant. Consider the Dirac equation in two different frames $x' = \Lambda x$

$$i\gamma^\mu\frac{\partial\psi(x)}{\partial x^\mu} - m\psi(x) = 0 \tag{3.4.35}$$

$$i\gamma^\mu\frac{\partial\psi'(x')}{\partial x'^\mu} - m\psi'(x') = 0 \tag{3.4.36}$$

We make the ansatz that the spinor transforms with S

$$\psi'(x') = S\psi(x) \tag{3.4.37}$$

S must be independent of x. Plug 3.4.37 into 3.4.36

$$\begin{aligned} i\gamma^\mu\frac{\partial}{\partial x'^\mu}\left[S\psi(x)\right] - mS\psi(x) &= 0 \\ iS^{-1}\gamma^\mu S\frac{\partial\psi(x)}{\partial x^\nu}\underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{=\Lambda^{-1}} - m\psi(x) &= 0 \end{aligned}$$

This equation must be the same as 3.4.35, then we get

$$S^{-1}\gamma^\mu S\left[\Lambda^{-1}\right]^\nu_{\ \mu} = \gamma^\nu$$

$$S^{-1}\gamma^\mu S = \Lambda^\mu_{\ \nu}\gamma^\nu \tag{3.4.38}$$

$S_{\mu\nu}$ is anti-symmetric in μ and ν

$$S_i = \frac{1}{2}\epsilon_{ijk}S^{jk} \tag{3.4.39}$$

$$K_i = S^{0i} \tag{3.4.40}$$

$$\psi'_\alpha(x') = \left[\exp\left(-\frac{i}{2}\Theta^{\mu\nu}S_{\mu\nu}\right)\right]^\beta_\alpha\psi_\beta \tag{3.4.41}$$

$$\omega_i = \frac{1}{2}\epsilon_{ijk}\theta_{ijk} \tag{3.4.42}$$

Lorentz transformation containing rotations and boosts, but infinitesimal version (infinitesimally different from $\mathbb{1}$)

$$\Lambda^\nu_{\ \mu} = \delta^\nu_{\ \mu} + \epsilon^\nu_{\ \mu} \tag{3.4.43}$$

We will show in the exercise the expression of transformation under boosts and rotations

$$S_L = \mathbb{1}_4 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} \tag{3.4.44}$$

$$S_L^{-1} = \mathbb{1}_4 + \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} \tag{3.4.45}$$

It satisfies $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu = \gamma^\nu + \epsilon^\mu{}_\nu\gamma^\nu$

$$\left(\mathbb{1}_4 + \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}\right)\gamma^\mu\left(\mathbb{1}_4 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}\right) = \gamma^\mu + \frac{i}{4}\left[\sigma_{\alpha\beta}\gamma^\mu - \gamma^\mu\sigma_{\alpha\beta}\right]\epsilon^{\alpha\beta}$$

somehow

$$= \gamma^\mu + \epsilon^\mu{}_\nu\gamma^\nu$$

Parity transformation cannot be written in infinitesimal form. Thus it is often called (one of) discrete transformation.

$$\Lambda_{\mathrm{P}} = \mathrm{diag}(+1, -1, -1, -1) \tag{3.4.46}$$

Parity symmetry is naturally violated. Especially in weak interaction, since neutrinos are left-handed ν_L.

$$S_{\mathrm{P}}^{-1}\gamma^\mu S_{\mathrm{P}} = \Lambda_{\mathrm{P}}{}^\mu{}_\nu\gamma^\nu \tag{3.4.47}$$

- For $\mu = 0$

$$S_{\mathrm{P}}^{-1}\gamma^0 S_{\mathrm{P}} = \gamma^0$$

- For $\mu = i$

$$S_{\mathrm{P}}^{-1}\gamma^i S_{\mathrm{P}} = -\gamma^i$$

It is easy to show $S_{\mathrm{P}} = \gamma^0$ satisfies the equations. Then the spinor transform under parity like

$$\psi'(t, -\boldsymbol{x}) = \gamma^0\psi(t, \boldsymbol{x}) \tag{3.4.48}$$

In Dirac-Pauli representation $\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad \psi' = \begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_3 \\ -\psi_4 \end{pmatrix}$$

In chiral representation $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$ and

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad \psi' = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

Recall $\bar{\psi} = \psi^\dagger\gamma^0$.

$$\bar{\psi}' = \psi'^\dagger\gamma^0 = \psi^\dagger S^\dagger\gamma^0$$

it can easily be shown using the explicit expresseion of S_L and S_L. Parity transformation is γ^0.

$$= \psi^\dagger\gamma^0 S^{-1}$$

$$= \bar{\psi}S^{-1}$$

There are 16 different bilinear $\bar{\psi}A\psi$

- $A = \mathbb{1}_4$

$$\bar{\psi}'\psi' = (\bar{\psi}S^{-1})(S\psi) = \bar{\psi}\psi$$

- $A = \gamma^\mu$

$$\bar{\psi}'\gamma^\mu\psi' = \bar{\psi}S^{-1}\gamma^\mu S\psi = \Lambda^\mu_{\ \nu}(\bar{\psi}\gamma^\nu\psi)$$

It transforms exactly like a Lorentz vector. Especially under parity

$$\bar{\psi}'\gamma^\mu\psi' = \begin{cases} \bar{\psi}'\gamma^0\psi' & \mu = 0 \\ -\bar{\psi}'\gamma^i\psi' & \mu = 1,2,3 \end{cases}$$

- $A = \gamma^5$

$$\bar{\psi}'\gamma^5\psi' = \bar{\psi}S_L^{-1}\gamma^5 S_L\psi$$

$\left\{\gamma^\mu, \gamma^5\right\} = 0$ and $\sigma_{\mu\nu}$ contains two γs

$$= \bar{\psi}\gamma^5\psi$$

It is a scalar under boosts and rotations. Under parity it is pseudoscalar, like pions.

$$\gamma^5 S_{\mathrm{P}} = \gamma^5\gamma^0 = -S_{\mathrm{P}}\gamma^5$$

thus

$$\bar{\psi}'\gamma^5\psi' = -\bar{\psi}'\gamma^5\psi'$$

For experiments, look it up in Introduction to HEP, Perkins

Chiral spinors (in chiral representation)

$$P_R = \frac{1}{2}\left(\mathbb{1}_4 + \gamma^5\right) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{3.4.49}$$

$$P_L = \frac{1}{2}\left(\mathbb{1}_4 - \gamma^5\right) = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.4.50}$$

If ψ written like $\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$, it is in chiral representation

$$P_L\psi = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} = \psi_L$$

$$P_R\psi = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix} = \psi_R$$

They are projectors

$$P_R^2 = \frac{1}{4}(\mathbb{1}_4 + \gamma^5)(\mathbb{1}_4 + \gamma^5) = \frac{1}{4}(\mathbb{1}_4 + 2\gamma^5 + (\gamma^5)^2) = P_R$$
$$P_L^2 = P_L$$

They also project onto complete space meaning $P_L + P_R = \mathbb{1}_4$.

Left-handed particles have spin and momentum in the opposite direction and right-handed in the same direction. In Dirac-Pauli representation in high energy limit

$$\gamma^5 u^{(5)} \approx \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{p} \end{pmatrix}$$

To show this take the free particle solution ($E > 0$)

$$\begin{aligned} u^{(s)} &= N \begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E+m} \chi^{(s)} \end{pmatrix} \\ \gamma^5 u^{(s)} &= N \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E+m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \approx N \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{p} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix} \end{aligned}$$

In general $(\boldsymbol{\sigma} \cdot \boldsymbol{a})^2 = \boldsymbol{a}^2 \mathbb{1}$

$$= N \boldsymbol{\sigma} \cdot \hat{p} \begin{pmatrix} \chi^{(s)} \\ \boldsymbol{\sigma} \cdot \hat{p} \chi^{(s)} \end{pmatrix} = \underbrace{\boldsymbol{\sigma} \cdot \hat{p}}_{\text{helicity operator}} u^{(s)}$$

At high energy limit ($E \gg m$), γ^5 helicity operator, but not at low energy. Chirality $\psi_{L,R}$ is always a good quantum number.

Angular momentum Must be another observable which commutes with H and P (u = u($\mathbf{p}$))

$$\boldsymbol{\Sigma} \cdot \hat{p} = \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{p} \end{pmatrix} \tag{3.4.51}$$

$$H = \alpha \cdot P + \beta m \tag{3.4.52}$$

$$[H, \boldsymbol{\Sigma} \cdot \hat{p}] = 0 \tag{3.4.53}$$

Angular momentum operator is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{P} \tag{3.4.54}$$

$$\tag{3.4.55}$$

Use the relation $\left[\hat{x}_i, \hat{P}_j\right] = i\delta_{ij}$ and check the commutation

$$\begin{aligned} [H, L_1] &= [\boldsymbol{\alpha} \cdot \boldsymbol{P}, x_2 P_3 - x_3 P_2] \\ &= [\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3, x_2 P_3 - x_3 P_2] \\ &= \alpha_2 [P_2, x_2] P_3 - \alpha_3 [P_3, x_3] P_2 \\ &= -i (\alpha_2 P_3 - \alpha_3 P_2) \\ &= -i(\alpha \times P)_1 \end{aligned}$$

Thus $[H, \boldsymbol{L}] = -i(\boldsymbol{\alpha} \times P) \neq 0$. In other word, L not conserved.

But we observe

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{3.4.56}$$

$$[H, \boldsymbol{\Sigma}] = 2i(\boldsymbol{\alpha} \times P) \tag{3.4.57}$$

Thus define total angular momentum, second term describe the intrinsic angular momentum of particles

$$\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\boldsymbol{\Sigma} \tag{3.4.58}$$

and J is conserved.

Charge conjugation Equation describing fermions couple to electromagnetic current

$$\left[\gamma^\mu\left(i\partial_\mu + eA_\mu\right) - m\right]\psi = 0 \tag{3.4.59}$$

with e the electric charge of electron and $A_\mu(x)$ vector potential in electromagnetism.

For positron the charge is the opposite

$$\left[\gamma^\mu\left(i\partial_\mu - eA_\mu\right) - m\right]\psi^{\mathrm{C}} = 0 \tag{3.4.60}$$

ψ^{C} is the charge conjugate of ψ. We want to know the relation between $\psi \leftrightarrow \psi^{\mathrm{C}}$. Take complex conjugate of equation 3.4.59

$$\left[-(\gamma^\mu)^*\left(i\partial_\mu - eA_\mu\right) - m\right]\psi^* = 0 \tag{3.4.61}$$

We postulate a matrix C so that $\psi^{\mathrm{C}} = C\gamma^0\psi^*$ and multiply equation 3.4.61 by $C\gamma^0$ from left.

$$C\gamma^0\left[-(\gamma^\mu)^*\left(i\partial_\mu - eA_\mu\right) - m\right]\psi^* = 0$$

It must be the same as equation 3.4.60. Thus

$$\begin{aligned} -(C\gamma^0)(\gamma^\mu)^* &= \gamma^\mu C\gamma^0 \\ -C(\gamma^\mu)^{\mathrm{T}} &= \gamma^\mu C \\ C\gamma^{\mu T}C^{-1} &= -\gamma^\mu \\ C &= i\gamma^0\gamma^2 \end{aligned} \tag{3.4.62}$$

Then we find

$$\psi^{\mathrm{C}} = C\gamma^0\psi^* = C\bar{\psi}^{\mathrm{T}} \tag{3.4.63}$$

It would be interesting to compare $P_L(\psi^{\mathrm{C}})$ and $(P_L\psi)^{\mathrm{C}}$

Continuity equation for ψ

$$\rho = \psi^\dagger\psi = \bar{\psi}\gamma^0\psi \tag{3.4.64}$$

Multiply $\bar{\psi}$ from left to dirac equation and ψ from right to conjugated Dirac equation

$$\begin{aligned} \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi &= 0, \qquad i(\partial_\mu\bar{\psi})\gamma^\mu\psi + m\bar{\psi}\psi = 0 \\ \bar{\psi}\gamma^\mu(\partial_\mu\psi) + (\partial_\mu\bar{\psi})\gamma^\mu\psi &= 0 \\ \partial_\mu(\bar{\psi}\gamma^\mu\psi) &= 0 \\ \partial_\mu j^\mu &= 0 \end{aligned}$$

3.5 Classical Field Theory

Reminder

Lagrangian and action

$$L = L(q_i, \dot{q}_i, t) \qquad S = \int \mathrm{d}t\, L$$

Variantion principle

$$\delta S = 0$$

is equivalent to Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

Going to field theory, we transform $q(t) \to \phi(x)$ and $L \to \mathcal{L}$

$$\begin{aligned}\mathcal{L} &= \mathcal{L}\left(\phi, \frac{\partial \phi}{\partial x^\mu}, x_\mu\right) \\ S &= \int \mathrm{d}^4x\, \mathcal{L}(\phi, \frac{\partial \phi}{\partial x^\mu}, x_\mu)\end{aligned}$$

Using the variation principle

$$\begin{aligned}\delta S &= \sum_i \frac{\mathrm{d}S}{\mathrm{d}\alpha_i}\delta\alpha_i \\ &= \sum_i \int \mathrm{d}^4x \left\{\frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial \alpha_i}\delta\alpha_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial(\partial_\mu \phi)}{\partial \alpha_i}\delta\alpha_i\right\} \\ &= \sum_i \int \mathrm{d}^4x \left\{\frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial \alpha_i} - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial \phi}{\partial \alpha_i}\right)\right\}\delta\alpha_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial \phi}{\partial \alpha_i}\delta\alpha_i\Bigg|_{x_0}^{x_1}\end{aligned}$$

Then we have our new Euler-Lagrange equations for continuous fields

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \tag{3.5.1}$$

Conservation Laws and Noether Theorem Reminder: Under transformation $t \mapsto t' = t + \delta b$, system stays invariant if $\delta_T L = \frac{\mathrm{d}}{\mathrm{d}t}(\delta\Omega)$.

$\delta\Omega = 0$ if L invariant

$$\delta_T L = L(q, \dot{q}, t + \delta b) - L(q, \dot{q}, t) = \frac{\partial L}{\partial t}\delta b$$

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \\ &= \frac{\partial L}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \\ &= \frac{\partial L}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}}\dot{q}\right]\end{aligned}$$

System invariant if

$$\frac{\partial L}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}(\delta\Omega) \tag{3.5.2}$$

if L not dependent explicitly on t, $\mathrm{d}L\,/\,\mathrm{d}t = 0$

If $\partial L/\partial t = 0$, the conserved quantity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L - \frac{\partial L}{\partial \dot{q}}\dot{q}\right] = 0$$
$$L - \frac{\partial L}{\partial \dot{q}}\dot{q} = -H = L - \dot{q}p$$

using legendre transformation and $\partial L/\partial \dot{q}$ the general momentum.

Analogously for fields

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\mu} &= \left(\frac{\partial \mathcal{L}}{\partial \phi}\right)\frac{\partial \phi}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\partial_\rho(\partial_\mu \phi) \\
&= \left(\partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\right)\partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\partial_\rho \partial_\mu \phi \\
\partial_\mu \mathcal{L} &= \partial_\rho\left[\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\partial_\mu \phi\right] \\
&\Rightarrow \partial_\rho\left[\mathcal{L}\delta^\rho_\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\partial_\mu \phi\right] = 0
\end{aligned}$$

this is energy momentum tensor T^ρ_μ.

Examples Real scalar field, $\phi(x) \in \mathbb{R}$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \tag{3.5.3}$$

Since $L = T - V$, we interpret first term as kinetic and second as potential, which has minimum at $\phi = 0$ Since action is dimensionless, $[\phi] = 1$. Take Euler-Lagrange

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

put together

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0$$

This is just Klein-Gordon equation. In Feynman rules we have the propagator and mass term as interaction

Example Fermionic field

$$\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi \tag{3.5.4}$$

$[\psi] = \frac{3}{2}$. $\bar{\psi}$ and ψ are independent fields. Solve Euler-Lagrange we get familiar Dirac equation.

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} i\gamma_\mu \delta^\mu \psi - m\psi = 0 \tag{3.5.5}$$

Example Spin 1 field $A_\mu(x)$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \tag{3.5.6}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.5.7}$$

Equation of motion can be computed

$$\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)} = \frac{1}{4} 4 \delta_{\rho\nu} \delta_{\mu\sigma} F^{\mu\nu} = -F^{\sigma\rho}$$

$$\frac{\partial \mathcal{L}}{\partial A_\sigma} = -j^\sigma$$

then we have the maxwell equations

$$-\partial_\rho F^{\sigma\rho} + j^\sigma = 0$$

use Lorenz gauge we can get

$$\partial_\mu \partial^\mu A^\sigma = j^\sigma$$

Bibliography

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