HS.1

$$\phi_j \to \phi_j + \delta\phi_j \,, \qquad \delta\phi_j = i\alpha^a T^a_{jk}\,\phi_k$$

$$V(\phi_j + \delta\phi_j) = V(\phi_j) + \frac{\partial V(\phi_j)}{\partial \phi_j}\,\delta\phi_j + \mathcal{O}((\delta\phi_j)^2)$$

$$\overset{!}{=} V(\phi_j)$$

$$\Rightarrow \qquad \frac{\partial V(\phi_j)}{\partial \phi_j}\,\delta\phi_j = 0$$

b)

$$\frac{\partial}{\partial\phi_i}\left(\frac{\partial V(\phi_j)}{\partial\phi_j}\,\delta\phi_j\right)\Bigg|_{\phi=\langle\phi\rangle} = 0$$

$$\left[\frac{\partial V(\phi_j)}{\partial\phi_i\,\partial\phi_j}\,\delta\phi_j + \underbrace{\frac{\partial V(\phi_j)}{\partial\phi_j}}_{=0}\;\frac{\partial(\delta\phi_j)}{\delta\phi_i}\right]\Bigg|_{\phi=\langle\phi\rangle} = 0$$

$$\Rightarrow \quad m_{ij}\;\underbrace{i\alpha^a}_{\text{Non-zero}}\;T^a_{jk}\,\langle\phi_k\rangle = 0$$

$$\Rightarrow \quad m_{ij}\,T^a_{jk}\,\langle\phi_k\rangle = 0$$

c)

$$\phi_j(x) \longrightarrow \exp(i\alpha^a Q^a)\,\phi_j(x)\,\exp(-i\alpha^b Q^b)$$

$$= (1 + i\alpha^a Q^a)\,\phi_j\,(1 - i\alpha^b Q^b)$$

$$= \phi_j + \underbrace{i(\alpha^a Q^a \phi_j - \phi_j\,\alpha^b Q^b)}_{} + \mathcal{O}(\alpha^2)$$

$$= i(\alpha^a Q^a \phi_j - \phi_j\,\alpha^a Q^a)$$

$$= i\alpha^a\,[Q^a, \phi_j]$$

$$= \delta\phi_j = i\alpha^a T^a_{jk}\,\phi_k$$

$$(2) \longrightarrow \quad m_{ij}\,\langle 0|\,[Q^a, \phi_j]\,|0\rangle$$

d) $\exp(i\alpha^a Q^a)|0\rangle$

$= (1+i\alpha^a Q^a)|0\rangle$

Symmetric : $Q^a|0\rangle = 0$

Asymmetric: $Q^a|0\rangle = Q^a|0\rangle$

$\in \mathbb{C}$ (or $\mathbb{R}$)

e) $m_{ij}\langle 0|[Q^a, \phi_j]|0\rangle = 0$

- $a = 1, \ldots, m$ $\quad Q^a|0\rangle = 0 \Rightarrow m_{ij} \neq 0$
- $a = m+1, \ldots, n$ $\quad Q^a|0\rangle \neq 0 \Rightarrow m_{ij} = 0$

f) n = # of rep. of G

m = # of rep. of H

$\Rightarrow$ m are massive, $(n-m)$ are massless

H.6

$$\mathcal{L} = (D^\mu \Phi)^\dagger (D_\mu \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2$$

$$\Phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \quad , \quad D_\mu = \partial_\mu + ig\frac{\tau^a}{2} W^a_\mu$$

a) $\mathcal{L}$ is (given on sheet) SU(2) invariant

mass term

$$\sim W^a_\mu (W^a)^\mu$$

$$\longrightarrow \left(W^a_\mu - \frac{1}{g}\partial_\mu \alpha^a - \epsilon^{abc}\alpha^b W^c_\mu\right)\left(W^{a\mu} - \frac{1}{g}\partial^\mu \alpha^a - \epsilon^{amn}\alpha^m W^{n\mu}\right)$$

$$= (W^a_\mu)(W^{a\mu}) + \frac{1}{g^2}(\partial_\mu \alpha^a)(\partial^\mu \alpha^a) + \epsilon^{abc}\epsilon^{amn}\alpha^b \alpha^m W^c_\mu W^{n\mu}$$

$$\underline{\underline{- \frac{1}{g}\partial^\mu \alpha^a W^a_\mu}} - \underline{\epsilon^{amn}\alpha^m W^{n\mu} W^a_\mu}$$

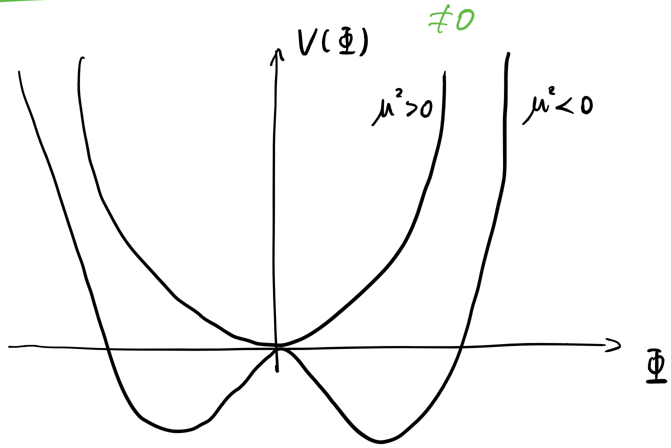
$$\underline{\underline{- \frac{1}{g}\partial_\mu \alpha^a W^{a\mu}}} + \frac{1}{g}\partial_\mu \alpha^a \epsilon^{amn}\alpha^m W^{n\alpha}$$

$$\underline{- \epsilon^{abc}\alpha^b W^c_\mu W^{a\mu}} + \frac{1}{g}\partial^\mu \alpha^a \epsilon^{abc}\alpha^b W^c_\mu$$

$$= (W^a)^2 + \frac{1}{g^2}(\partial\alpha^a)^2 + (\alpha^b W^c)^2 - \alpha^b \alpha^c W^c_\mu W^{b\mu} - \frac{2}{g}(\partial\alpha \cdot W)$$
$$- 2\epsilon^{abc}\alpha^b (W^c \cdot W^a) + \frac{2}{g}\epsilon^{abc}\partial_\mu \alpha^a \cdot \alpha^b W^{c\mu}$$

$\neq 0$

b)



$\mu^2 < 0$

$\Rightarrow$ SSB

c) $$(\Phi^+ \Phi) = (\phi_1 - i\phi_2,\ \phi_3 + i\phi_4)\begin{pmatrix}\phi_1 + i\phi_2\\ \phi_3 + i\phi_4\end{pmatrix}$$

$$= \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2$$

In order for V to minimize

$$\frac{\partial V}{\partial \Phi} = \mu^2 \Phi^+ + 2\lambda(\Phi^+\Phi)\Phi^+ \overset{!}{=} 0$$

$$\frac{\partial V}{\partial \Phi^+} = \mu^2 \Phi + 2\lambda(\Phi^+\Phi)\Phi \overset{!}{=} 0$$

$$\Rightarrow \mu^2 + 2\lambda(\Phi^+\Phi) \overset{!}{=} 0$$

d) $$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\theta_2(x) & + i\,\theta_1(x)\\ v + h(x) & - i\,\theta_3(x)\end{pmatrix}$$

$$\Phi^+\Phi = \frac{1}{2}(\theta_2 - i\theta_1,\ v + h + i\theta_3)\begin{pmatrix}\theta_2 + i\theta_1\\ v + h - i\theta_3\end{pmatrix}$$

$$= \frac{1}{2}\left[\theta_2^2 + \theta_1^2 + (v+h)^2 + \theta_3^2\right]$$

$$D_\mu \Phi = \frac{1}{\sqrt{2}}\left(\partial_\mu + ig\frac{\tau^a}{2}W_\mu^a\right)\begin{pmatrix}\theta_2 + i\theta_1\\ v + h - i\theta_3\end{pmatrix}$$

- $a = 1$

$$\frac{ig}{2}W_\mu^1\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\theta_2 + i\theta_1\\ v + h - i\theta_3\end{pmatrix} = +\frac{ig}{2}W_\mu^1\begin{pmatrix}v + h - i\theta_3\\ \theta_2 + i\theta_1\end{pmatrix}$$

- $a = 2$

$$-\frac{g}{2\sqrt{2}}W_\mu^2\begin{pmatrix}-v - h + i\theta_3\\ \theta_2 + i\theta_1\end{pmatrix}$$

- $a = 3$

$$+\frac{ig}{2\sqrt{2}}W_\mu^3\begin{pmatrix}\theta_2 + i\theta_1\\ -v - h + i\theta_3\end{pmatrix}$$

$$(D_\mu\Phi) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}\partial_\mu\theta_2 + i\partial_\mu\theta_1\\ \partial_\mu h - i\partial_\mu\theta_3\end{pmatrix} + \frac{ig}{2}W_\mu^1\begin{pmatrix}v + h - i\theta_3\\ \theta_2 + i\theta_1\end{pmatrix}\right.$$
$$\left. - \frac{g}{2}W_\mu^2\begin{pmatrix}-v - h + i\theta_3\\ \theta_2 + i\theta_1\end{pmatrix} + \frac{ig}{2}W_\mu^3\begin{pmatrix}\theta_2 + i\theta_1\\ -v - h + i\theta_3\end{pmatrix}\right]$$

$$\mathcal{L} = (D^\mu \Phi)^\dagger (D_\mu \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2$$

$$= \frac{1}{2} \left| \partial_\mu \theta_2 - i \partial_\mu \theta_1 + \frac{ig}{2} W_\mu^1 (v+h-i\theta_3) - \frac{g}{2} W_\mu^2 (-v-h+i\theta_3) + \frac{ig}{2} W_\mu^3 (\theta_2 + i\theta_1) \right|^2$$

$$+ \frac{1}{2} \left| \partial_\mu h - i\partial_\mu \theta_3 + \frac{ig}{2} W_\mu^1 (\theta_2 + i\theta_1) - \frac{g}{2} W_\mu^2 (\theta_2 + i\theta_1) + \frac{ig}{2} W_\mu^3 (-v-h+i\theta_3) \right|^2$$

$$- \mu^2 \cdot \frac{1}{2} (\theta_1^2 + \theta_2^2 + \theta_3^2 + (v+h)^2) - \frac{\lambda}{4} (\theta_1^2 + \theta_2^2 + \theta_3^2 + (v+h)^2)^2$$

contains $\sim h^2$ $\longrightarrow$ mass term for h

mass term for W^i

Focus on θ_i^k terms

$$- \frac{\mu^2}{2} (\theta_1^2 + \theta_2^2 + \theta_3^2) - \frac{\lambda}{4} \left[(\theta_1^2 + \theta_2^2)^2 + (\theta_3^2 + (v+h)^2)^2 + 2(\theta_1^2 + \theta_2^2)(\theta_3^2 + (v+h)^2) \right]$$

$$= \underbrace{- \frac{\mu^2}{2} (\theta_1^2 + \theta_2^2 + \theta_3^2) - \frac{v^2 \lambda}{2} (\theta_3^2 + \theta_1^2 + \theta_2^2)}_{=0, \text{ since } v^2 = \frac{-\mu^2}{\lambda}} + (\text{interaction})$$

$\rightarrow$ DOF: $4 \ (\phi_{1,2,3,4})$

$\longrightarrow 4 \ (\theta_{1,2,3}, h)$

e) $$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \theta_2(x) + i\theta_1(x) \\ v + h(x) - i\theta_3(x) \end{pmatrix} \overset{?}{=} \frac{1}{\sqrt{2}} e^{i\theta^a(x) \tau^a / v} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

$$\sqrt{2} \cdot RHS = (1 + i\theta^a(x) \tau^a / v) \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} + \frac{1}{v} \underbrace{\begin{pmatrix} i\theta_3 & i\theta_1 + \theta_2 \\ i\theta_1 - \theta_2 & -i\theta_3 \end{pmatrix} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}}_{= \begin{pmatrix} (i\theta_1 + \theta_2)(v + h(x)) \\ -i\theta_3 (v + h(x)) \end{pmatrix}}$$

$$= \begin{pmatrix} i\theta_1 + \theta_2 + \frac{h(x)}{v} (i\theta_1 + i\theta_2) \\ v + h(x) - i\theta_3 - i\frac{h(x)}{v} \theta_3 \end{pmatrix}$$, since h, θ_i are perturbations ignore higher order

$$= \begin{pmatrix} \theta_2 + i\theta_1 \\ v + h(x) - i\theta_3 \end{pmatrix}$$

f) $\Phi = \frac{1}{\sqrt{2}} e^{i\theta^a(x)\tau^a/v} \begin{pmatrix} 0 \\ v+h(x) \end{pmatrix}$, $\Phi \to e^{i\alpha^a(x)\tau^a/2}\Phi$

$$\longrightarrow \frac{1}{\sqrt{2}} \exp\Big(\underbrace{i\theta^a(x)\tau^a/v + i\alpha^a(x)\tau^a/2}_{\overset{!}{=}0} \Big) \begin{pmatrix} 0 \\ v+h(x) \end{pmatrix}$$

$\Rightarrow \quad \theta^a(x) = -\alpha^a(x)$

so no $\theta(x)$ in Φ'

g) Mass of W:

$$\left| ig \frac{\tau^a}{2} W^a \Phi \right|^2$$

$$= \frac{g^2}{8} \left| \begin{pmatrix} W^3 & W^1 - iW^3 \\ W^1 + iW^3 & -W^3 \end{pmatrix} \begin{pmatrix} 0 \\ v+h \end{pmatrix} \right|^2$$

$$= \frac{g^2}{8} \left| \begin{pmatrix} (W^1 - iW^3)(v+h) \\ -W^3(v+h) \end{pmatrix} \right|^2$$

interactions

$$= \frac{g^2}{8} \left[(W^1)^2 + (W^3)^2 \right] (v+h)^2 + (W^3)^2 (v+h)^2$$

$$= \frac{g^2 v^2}{8} W^2 + (\text{interactions})$$

$$\Rightarrow \quad m_W^2 = 2\frac{g^2 v^2}{8}$$

$$\Rightarrow \quad m_W = \frac{gv}{2}$$

$\rightarrow$ DOF: 3 (from W) + 1 (h(x))