AQAlg: Advanced Quantum Algorithms **2022 - 2023**

Exercises 7: Random walks and quantum walks

Lecturer: Simon Apers *Teaching assistant: Samuel Bouaziz-Ermann*[1]

Exercise 1. In this exercise we will prove that

$$\mathrm{MT}(\epsilon) \leq \frac{1}{\delta}\left(\frac{1}{2}\log(n) + \log\left(\frac{1}{\epsilon}\right)\right),$$

or equivalently

$$\|P^t p_0 - \pi\|_1 \leq \epsilon$$

for any initial distribution p_0 if $t \geq \frac{1}{\delta}\left(\frac{1}{2}\log(n) + \log\left(\frac{1}{\epsilon}\right)\right)$.

- We first bound the 2-norm distance. Consider the spectral decomposition

  $$P = \sum_{j\geq 1} \lambda_j v_j v_j^T,$$

  with $v_1 = \frac{1}{\sqrt{n}}\sum_x e_x$ the uniform eigenvalue-1 eigenvector of P, and v_j for $2 \leq j \leq n$ the remaining eigenvectors with corresponding eigenvalues $-1+\delta < \lambda_j < 1-\delta$. Show that

  $$\|P^t p_0 - \pi\|_2 \leq \epsilon$$

  if $t \geq \frac{1}{\delta}\log\frac{1}{\epsilon}$.

- From this bound we get a bound for the 1-norm distance. Use Cauchy-Schwarz[2] to show that

  $$\|P^t p_0 - \pi\|_1 \leq \sqrt{n}\|P^t p_0 - \pi\|_2.$$

  From this, deduce that

  $$\|P^t p_0 - \pi\|_1 \leq \epsilon$$

  if $t \geq \frac{1}{\delta}\left(\frac{1}{2}\log(n) + \log\left(\frac{1}{\epsilon}\right)\right)$.

Conversely, one can (but we won't) prove that $\mathrm{MT}(\epsilon) \in \Omega\left(\frac{1}{\delta}\log\left(\frac{1}{\epsilon}\right)\right)$, so that this bound is tight up to the $\log(n)$-factor.

Exercise 2. Show that the quantum walk $W(P) = S\cdot C(P)$ is a product of a reflection around (i) the symmetric subspace $\mathrm{span}\{|x,y\rangle + |y,x\rangle \mid (x,y)\in E\}$ and (ii) the star subspace $\mathrm{span}\{|\psi_x\rangle \mid x \in V\}$.

Use Lemma 1 from the lecture notes to show that the invariant subspace of the quantum walk operator $W(P)$ is spanned by

$$\{|\pi\rangle\} \cup \{|f\rangle \mid f \text{ a closed flow}\},$$

where $|f\rangle = \sum_{(x,y)\in E} f(x,y)\,|x,y\rangle$ and f is a *closed flow* if $f(x,y) = -f(y,x)$ and $\sum_y f(x,y) = 0$ for all $x, y \in V$.

Exercise 3. Show that the quantum walk spectral gap $\Delta = \arccos(\lambda_2)$ is quadratically larger than the random walk spectral gap: $\Delta \geq \sqrt{2\delta}$. Use that $\lambda_2 \leq 1-\delta$ and $\cos x \geq 1 - x^2/2$.

[1] samuel.bouaziz-ermann@lip6.fr

[2] For any $v, w \in \mathbb{R}^n$ it holds that $\langle v, w\rangle \leq \|v\|_2\|w\|_2$.

Exercise 4 (Extra)**.** Here we prove that the hitting time

$$\mathrm{HT}(M) \in \widetilde{O}\left(\frac{1}{\delta}\frac{1}{\pi(M)}\right).$$

- Prove that $\|p - \pi\|_1 \leq \epsilon$ implies that $p(M) \geq \pi(M) - \epsilon$.
- Let $\epsilon_M = \pi(M)/2$. Argue that after $\mathrm{MT}(\epsilon_M) \in \widetilde{O}(1/\delta)$ steps the probability of hitting an element in M is at least $\pi(M)/2$.
- Use this to prove the bound on the expected hitting time.