AQAlg: Advanced Quantum Algorithms **2023 - 2024**

Exercises 3: Adiabatic quantum computation and QAOA

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Exercise 1 (Pauli basis)**.** Recall the unitary Pauli matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- Show that the Pauli basis $\{I, X, Y, Z\}$ forms a basis for the complex 2-by-2 matrices. I.e., any $A \in \mathbb{C}^{2\times 2}$ can be expanded as $A = \alpha_1 I + \alpha_x X + \alpha_y Y + \alpha_z Z$.
- Argue that this implies that the n-qubit Pauli basis $\{I, X, Y, Z\}^{\otimes n}$ forms a basis for the 2^n-by-2^n matrices.
- Show that if $A \in \mathbb{C}^{2^n \times 2^n}$ is Hermitian, then its coefficients in the Pauli basis are real.

Exercise 2 (Max cut)**.** Finding the maximum cut in a graph is a canonical problem in combinatorial optimization. For a graph G with vertex set $[n]$ and (symmetric) edge set $E \subseteq [n]^2$, a maximum cut is described by a subset $Z \subset [n]$ that cuts a maximum number of edges (edges crossing from Z to Z^c). Equivalently, it maximizes the cut function

$$c(A) = \sum_{(i,j)\in E} I_{i\in A, j\notin A}.$$

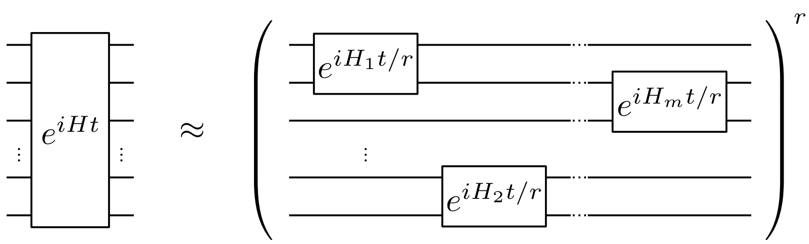
- Identify a subset A with the indicator $a \in \{0,1\}^n$ ($i \in A \Leftrightarrow a_i = 1$). Express the cut function $c(A)$ as a degree-2 polynomial h in a.
- Rewrite the cost Hamiltonian $H_1 = \sum_{z\in\{0,1\}^n} h(z) \left|z\right\rangle \left\langle z\right|$ in terms of identity and Pauli-Z matrices.

Exercise 3 (Hamiltonian simulation)**.** We saw how a circuit model computation can be encoded into an adiabatic computation. Here we describe how an adiabatic evolution (more precisely, evolution by a Hamiltonian) can be simulated in the circuit model.

From the first exercise, we know that any Hamiltonian H can be expanded as a sum of simpler Hamiltonians, $H = \sum_{j=1}^m H_j$. Here we focus on 2-local interactions, in which case H consists of $m \in O(n^2)$ 2-qubit terms, and we assume access to the 2-qubit gates $e^{i\gamma H_j}$ for $\gamma > 0$. If the H_j's do not commute, then we do not have that $e^{iHt} = e^{iH_1 t} \dots e^{iH_m t}$. However, by the Lie product formula, we do have that

$$e^{iHt} = \lim_{r\to\infty} \left(e^{iH_1 t/r} e^{iH_2 t/r} \dots e^{iH_m t/r} \right)^r.$$

For finite r, this corresponds to a quantum circuit with mr gates of the form $e^{iH_j t/r}$.



Here we bound the error incurred for finite r.

- Assume $\|A_1\|, \|A_2\| \leq 1$. Use the Taylor series $e^B = I + B + O(\|B\|^2)$ for $\|B\| \leq 1$ to show that
$$e^{A_1+A_2} = e_1^A e_2^A + E, \quad \|E\| \in O(\|A_1\|^2 + \|A_2\|^2). \tag{1}$$

- Assume $\|A_1\|, \ldots, \|A_m\| \leq \delta \leq 1/m$. Use (1) to show that
$$e^{A_1+\cdots+A_m} = e^{A_1} \ldots e^{A_m} + E_m, \quad \|E_m\| \in O(m^2\delta^2). \tag{2}$$

- Assume $\|H_1\|, \|H_2\| \leq 1$. For $r \geq t$, argue that
$$e^{i(H_1+H_2)t} = \left(e^{iH_1 t/r} e^{iH_2 t/r}\right)^r + E_r, \quad \|E_r\| \in O(t^2/r).$$

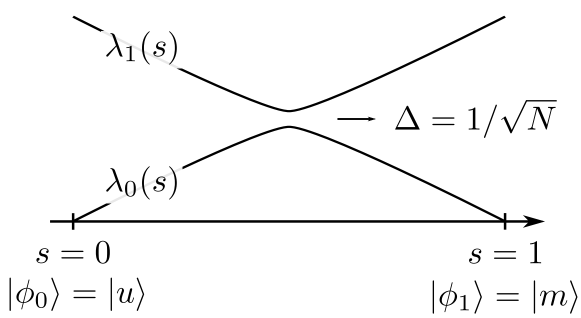
- Assume $\|H_1\|, \ldots, \|H_m\| \leq 1$. For $r \geq mt$, argue that
$$e^{i(H_1+\cdots+H_m)t} = \left(e^{iH_1 t/r} \ldots e^{iH_m t/r}\right)^r + E_r, \quad \|E_r\| \in O(t^2m^2/r).$$

Picking $r \in O(t^2m^2/\epsilon)$, it follows that we can ϵ-approximate the Hamiltonian evolution e^{iHt} using $mr \in O(t^2m^3/\epsilon)$ 2-qubit gates.

Exercise 4 (Adiabatic Grover algorithm (optional))**.** Consider the unstructured search problem over the set $[N] = \{0,1\}^n$, and assume that there is a single (unknown) marked element $m \in \{0,1\}^n$. We can solve this problem using the adiabatic optimization algorithm with cost function $h(z) = 0$ if $z = m$ and $h(z) = 1$ otherwise. Letting $|u\rangle = \frac{1}{\sqrt{N}} \sum_{x\in\{0,1\}^n} |x\rangle$, we can use the Hamiltonians

$$H_0 = I - |u\rangle\langle u| \quad \text{and} \quad H_1 = I - |m\rangle\langle m|,$$

with ground states $|\phi_0\rangle = |u\rangle$ and $|\phi_1\rangle = |m\rangle$, respectively. Similar to Grover, the Hamiltonian $H(s) = (1-s)H_0 + sH_1$ only acts nontrivially in the subspace spanned by $|u_0\rangle = \frac{1}{\sqrt{N-1}} \sum_{x\neq m} |x\rangle$ and $|m\rangle$. Calculate the nontrivial eigenvalues of $H(s)$, and show that they behave as in the following figure.[1]



References

[RC02] Jérémie Roland and Nicolas J Cerf. Quantum search by local adiabatic evolution. *Physical Review A*, 65(4):042308, 2002.

[1] While this proves that the gap is inverse polynomial, and so the adiabatic algorithm solves unstructured search in poly(n), it doesn't directly recover the quadratic Grover speedup. A more refined adiabatic algorithm does [RC02].