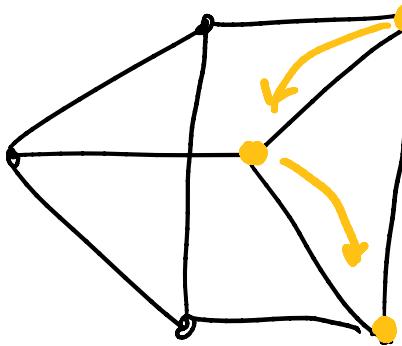


simple, regular graph $G = (V, E)$
 $|V| = n$
 $|E| = m$

**RANDOM WALK**initial vertex $\sim p_0 \in \mathbb{R}^n$

after 1 step:

$$p_1 = P p_0,$$

where $P_{x,y} = \frac{1}{d}$ if $(x,y) \in E$

If G connected, then \exists unique π

$$\text{s.t. } P\pi = \pi.$$

(**"stationary distribution"**
 $\pi = \frac{1}{n} \mathbf{1}$ if regular)

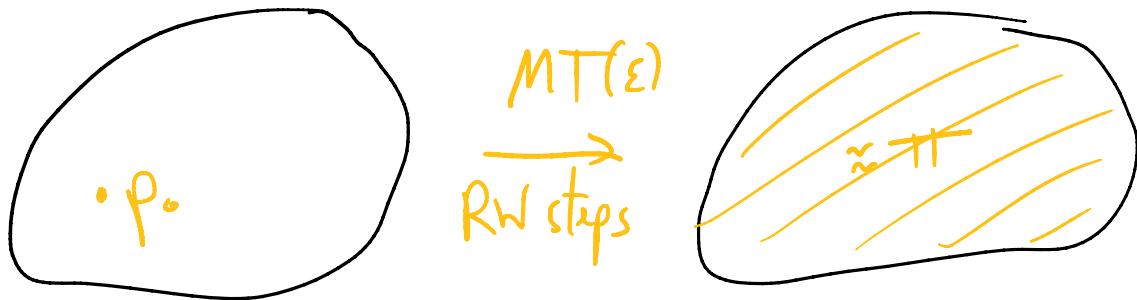
If G connected & non-bipartite, then

$$P_{p_0}^t \xrightarrow{t \rightarrow \infty} \pi, \forall \text{ distr. } p_0.$$

"MIXING TIME"

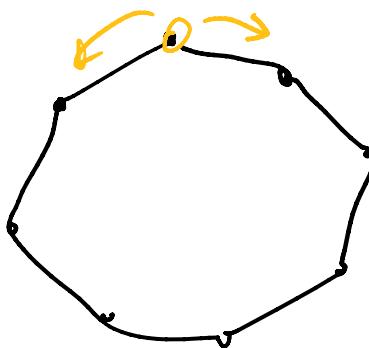
"MIXING TIME"

$$MT(\varepsilon) = \min \left\{ t \mid \|P_{p_0}^t - \pi\|_1 \leq \varepsilon, \forall p_0 \right\}$$



EXAMPLE: n -cycle

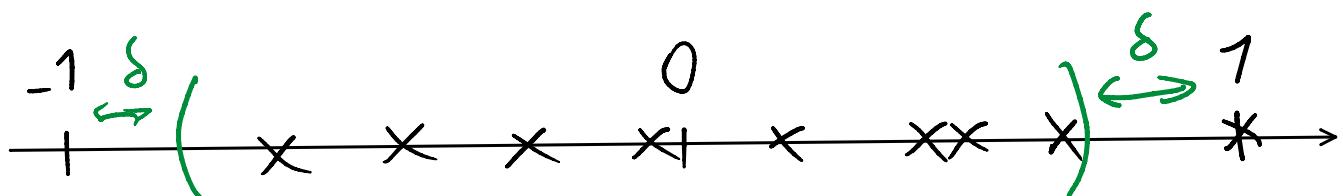
$$MT(\varepsilon) \sim n^2 \log \left(\frac{1}{\varepsilon} \right)$$



$$\text{eigenv. } 1 = \lambda_1 > \lambda_2 > \dots > \lambda_n \geq -1$$

$MT(\varepsilon)$ related to "spectral gap" of P

$$\delta = 1 - \max \{ |\lambda_2|, |\lambda_n| \}$$



Lemma: $MTR(\varepsilon) \leq \frac{1}{\delta} \left(\log(n) + \log\left(\frac{1}{\varepsilon}\right) \right)$.

Proof:

spectrum $P : \begin{cases} 1/\sqrt{n}, & \lambda_1 = 1 \\ v_j, & j = 2, \dots, n \end{cases}$

$$\text{s.t. } P = \sum_{j=1}^n \lambda_j v_j v_j^T = \Pi I^T + \sum_{j>1} \lambda_j v_j v_j^T$$

$$P_0 = \sum (v_j^T p_0) v_j = \Pi + \sum_{j>1} (v_j^T p_0) v_j$$

and

$$P^T p_0 - \Pi = \sum_{j=2}^n \lambda_j^T v_j (v_j^T p_0)$$

$$\hookrightarrow \|P^T p_0 - \Pi\|_2 = \sqrt{\sum_{j=2}^n |\lambda_j|^2 |v_j^T p_0|^2} \quad (\text{since } \|p_0\|_2 \leq 1)$$

$$\leq (1-\delta)^t \sqrt{\sum |v_j^T p_0|^2} \quad (\text{using that } (1-\delta)^t \leq \varepsilon)$$

$$\leq (1-\delta)^t \leq \varepsilon' \quad (\text{using that } (1-\delta)^t \leq \varepsilon')$$

$$\text{if } t \geq \frac{1}{\delta} \log\left(\frac{1}{\varepsilon'}\right)$$

Now Cauchy-Schwarz: \dots \dots

Now, Cauchy-Schwarz:

$$\|P^t p_0 - \pi\|_1 \leq \|1\|_2 \|1x1\|_2$$

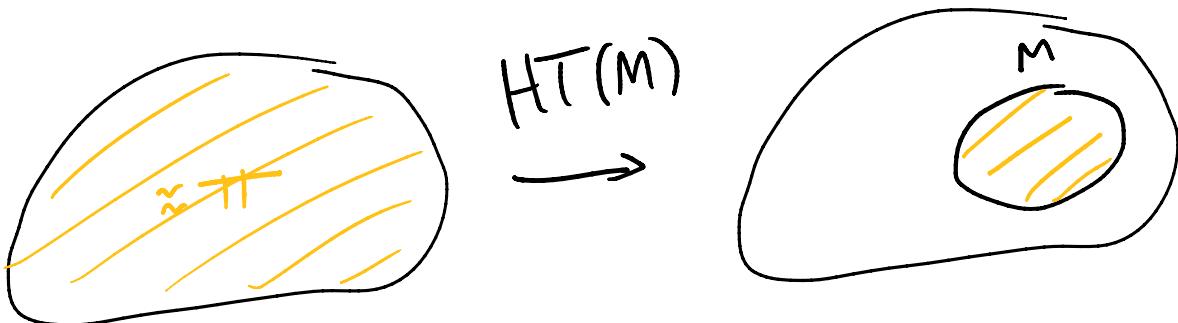
$$\leq \sqrt{n} \|P^t p_0 - \pi\|_2 \leq \sqrt{n} \varepsilon' \leq \varepsilon$$

if $\varepsilon' \leq \frac{\varepsilon}{\sqrt{n}}$

and so $t \geq \frac{1}{\delta} \left(\log\left(\frac{1}{\varepsilon}\right) + \log(n) \right)$. \square

Conversely: $MT(\varepsilon) \in \Omega\left(\frac{1}{\delta} \log\left(\frac{1}{\varepsilon}\right)\right)$

"HITTING TIME"



for $M \subseteq V$,

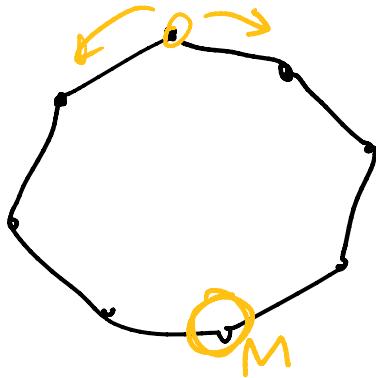
$$HT(M) = \mathbb{E} [\# \text{steps to hit } M \mid p_0 = \pi]$$

$T = n - m - 1$



E.g., n -cycle

$$HT(M) \sim n^2$$



Lemma: $HT(M) \in \tilde{O}\left(\frac{1}{\delta} \frac{1}{\pi(M)}\right)$.

Proof: (sketch)

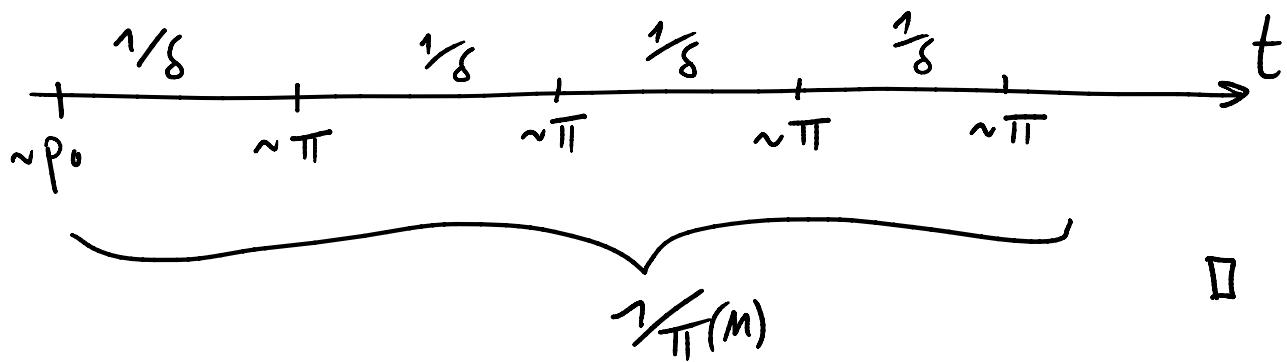
(i) # Samples from π to hit M

$$= \frac{1}{\pi(M)}$$

(ii) time for 1 sample

$$= MT(\varepsilon) \quad (\varepsilon \leq \pi(M)/2 \text{ suffices})$$

$$\in \tilde{O}\left(\frac{1}{\delta}\right)$$



Claim: Quantum walks find marked el.
in $\tilde{O}(\frac{1}{\sqrt{\epsilon \delta}})$ steps.

~~~~~ break ~~~~~

## QUANTUM WALKS

key RW concepts :

- stationary distr.  $\pi$
  - spectral gap  $\delta$
- will encounter quantum variants

State space :

$$|y\rangle = \sum_{(x,y) \in E} \alpha_{xy} |x, y\rangle$$

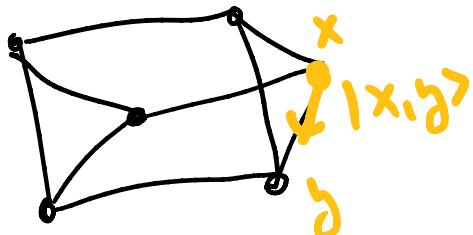
QW over  
edges

"next node"

$$|\psi\rangle = \sum_{(x,y) \in E} |x,y\rangle$$

"next node"

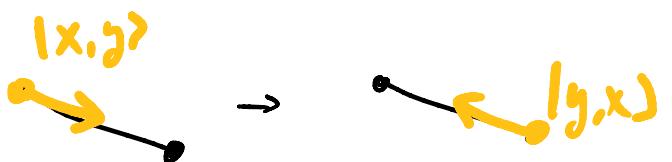
3-reg. h



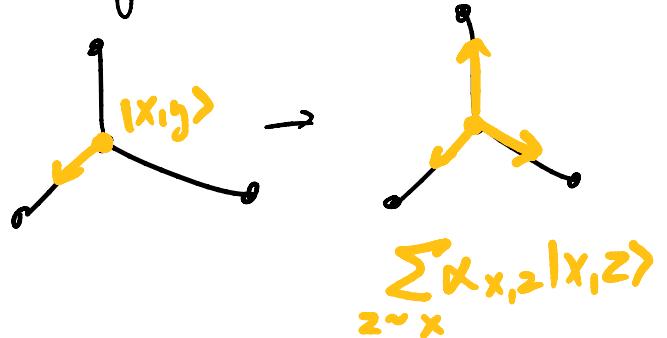
"current node"

2 operations:

(i) "SWAP"  $S|x,y\rangle = |y,x\rangle$  = "step"



(ii) "COIN TOSS"



defined using "star states"

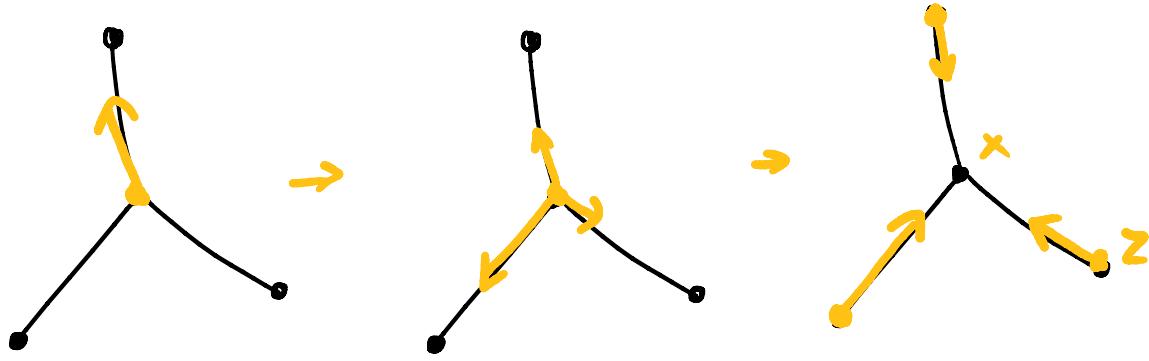
$$|\psi_x\rangle = \frac{1}{\sqrt{d}} \sum_{y \sim x} |x,y\rangle$$

$$C(P) = 2 \left( \underbrace{\sum_x |\psi_x \rangle \langle \psi_x|}_{\text{proj. onto span } \{|\psi_x\rangle | x \in V\}} \right) - 1$$

proj. onto span  $\{|\psi_x\rangle | x \in V\}$

$$= \text{ref} \left\{ \text{span} \left\{ |x\rangle \mid x \in V \right\} \right\}$$

→ Quantum walk operator:



$|x,y\rangle$

$C(P)|x,y\rangle$

$$= \sum_{z \sim x} \alpha_{xz} |x,z\rangle$$

$W(P)|x,y\rangle$

$$= \sum_{z \sim x} \alpha_{xz} |z,x\rangle$$

Lemma:  $|\pi\rangle = \frac{1}{\sqrt{n}} \sum_x |\psi_x\rangle = \frac{1}{\sqrt{nd}} \sum_{(x,y) \in E} |x,y\rangle$   
is a stationary state of  $W(P)$ .

Proof:  $W(P) = S \cdot C(P)$

and  $C(P)|\pi\rangle$

$$= \left( \sum_x (S|\psi_x\rangle \langle \psi_x|) - 1 \right) \frac{1}{\sqrt{n}} \sum_x |\psi_x\rangle$$

$$\begin{aligned}
 &= \left( 2 \left( \sum_x |\psi_x \rangle \langle \psi_x| \right) - 1 \right) \frac{1}{\sqrt{n}} \sum_x |\psi_x\rangle \\
 &= \frac{1}{\sqrt{n}} \sum_x |\psi_x\rangle \quad (\text{using that } \langle \psi_x | \psi_y \rangle = \delta_{x,y})
 \end{aligned}$$

$$\begin{aligned}
 &\langle \pi | \pi \rangle \\
 &= \langle \pi | \frac{1}{\sqrt{n}} \sum_{(x,y) \in E} |x,y\rangle \\
 &= \frac{1}{\sqrt{n}} \sum_{\substack{(x,y) \\ \in E}} |y,x\rangle = |\pi\rangle,
 \end{aligned}$$

hence  $W(P)|\pi\rangle = S \cdot C(P)|\pi\rangle = S|\pi\rangle = |\pi\rangle$ .  $\square$

! not only stationary state:

$$W(P) = S \cdot C(P)$$

$$\begin{array}{ccc}
 \text{reflection around} & & \text{reflection around} \\
 \text{span} \{ |xy\rangle + |yx\rangle \mid (x,y) \in E \} & & \text{span} \{ |\psi_x\rangle \mid x \in V \}
 \end{array}$$

$\rightarrow W_{\text{hr}} = \text{product of two reflections!}$

$\rightarrow QW$  = product of two reflections!

Lemma: Consider projectors  $\Pi_1, \Pi_2$ .

Operator  $(2\Pi_2 - 1)(2\Pi_1 - 1)$  has inv. subspace

$$(\ker(\Pi_1) \cap \ker(\Pi_2)) \cup (\ker(\Pi_1)^+ \cap \ker(\Pi_2)^+).$$

$QW$  operator:  $W(P) = (2\Pi_+ - 1)(2\Pi_* - 1)$

where  $\Pi_+ = \sum_{(x,y) \in E} \frac{1}{2} (|xy\rangle + |yx\rangle) \langle xy| + \langle yx|$

$$\Pi_* = \sum_{x \in V} |\psi_x \times \psi_x|$$

$$\rightarrow \ker(\Pi_+)^+ \cap \ker(\Pi_*)^+ \leftarrow \text{span}\{|k\rangle\}$$

$$= \text{span} \left\{ \sum_{x \in V} \alpha_x |\psi_x\rangle \mid \alpha_x = \alpha_y \forall (x,y) \in E \right\}$$

$$= \text{span} \{ |\pi\rangle \}$$

, ..., |n>

$$= \text{Span} \{ | \pi \rangle \}$$

$$\rightarrow \ker(\pi_+) \cap \ker(\pi_*) \xrightarrow{\text{+ Span} \{ | \pi_x \rangle \}}$$

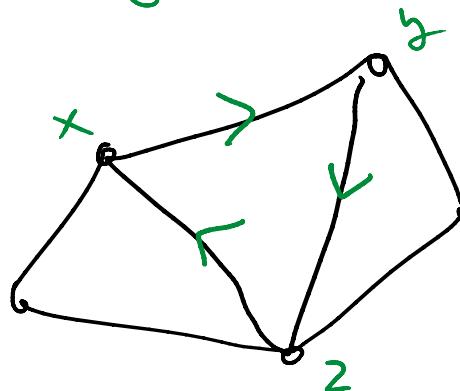
$$= \text{Span} \{ | f \rangle \mid f \text{ "closed flow"} \}$$

$$\hookrightarrow | f \rangle = \sum_{\substack{(x,y) \\ \in E}} f(x,y) | x,y \rangle$$

$$\text{s.t. } \begin{cases} f(x,y) = -f(y,x) \\ \sum_{y \sim x} f(x,y) = 0 = \langle \pi_x | f \rangle \end{cases}$$

$$\sum_{y \sim x} f(x,y) = 0 = \langle \pi_x | f \rangle$$

$\Rightarrow f$  closed flow



$$| f \rangle = | xy \rangle - | yx \rangle + | yz \rangle - | zy \rangle + | zx \rangle - | xz \rangle$$

? rest of the spectrum

\* (-1)-subspace is

$$(\ker(\pi_1) \cap \ker(\pi_2)^+) \cup (\ker(\pi_1)^\perp \cap \ker(\pi_2)).$$

\* Szegedy's spectral lemma