last lecture:

RW P $\longrightarrow$ $\begin{cases} \text{star states } |\psi_x\rangle \\ \text{QW operator } W \\ \text{Szegedy's lemma:} \end{cases}$

spectrum $\sigma(W) \Leftrightarrow \sigma(P)$

this lecture:

general Hermitian matrix H

$\downarrow$

* star states: assume $\|H\|_1 \leq 1$!

$\langle y|\perp\rangle = 0, \forall y$

$$|\psi_x\rangle = \sum \sqrt{H_{x,y}}\,|x,y\rangle + \sqrt{1-\sum |H_{x,y}|}\;|x,\perp\rangle$$

* QW operator:

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$$W = S \cdot \left(2 \sum |\psi_x\rangle\langle\psi_x| - \mathbb{1}\right)$$

$$= S \cdot U_\psi \left(2\Pi_0 - \mathbb{1}\right) U_\psi^\dagger$$

$S|x,y\rangle = |y,x\rangle$, $S|x,\bot\rangle = |x,\bot\rangle$

Exercise: Verify that $\langle\psi_y|S|\psi_x\rangle = H_{x,y}$.

* Szegedy's lemma:

$\forall$ eigenpair $\left(\lambda_j = \cos(\theta_j) \neq \pm 1,\ |v_j\rangle\right)$,

the QW operator has eigenpairs

$$\left(e^{\pm i\theta_j},\ |\phi_j^\pm\rangle\right)$$

such that

$$|\phi_j^+\rangle + |\phi_j^-\rangle$$

$$U_\psi |v_j\rangle|0\rangle = \frac{|\phi_j^+\rangle + |\phi_j^-\rangle}{\sqrt{2}}.$$

Hence, consider $|x\rangle = \sum \beta_j |v_j\rangle$:

$$W^t U_\psi |x\rangle|0\rangle$$

$$= \sum \beta_j \frac{e^{i\theta_j t}|\phi_j^+\rangle + e^{-i\theta_j t}|\phi_j^-\rangle}{\sqrt{2}}$$

$$= \sum \beta_j \left[\cos(\theta_j t)\frac{|\phi_j^+\rangle + |\phi_j^-\rangle}{\sqrt{2}} + i\sin(\theta_j t)\frac{|\phi_j^+\rangle - |\phi_j^-\rangle}{\sqrt{2}}\right]$$

such that

$$U_\psi^\dagger W^t U_\psi |x\rangle|0\rangle$$

$$= \sum \beta_j \Big[\cos(\theta_j t)|v_j\rangle|0\rangle$$

$$+ \, |\phi^+\rangle - |\phi^-\rangle \Big]$$

$$\sum \beta_j \Big[\cos(\theta_j t) |v_j\rangle |0\rangle$$

$$+ i \sin(\theta_j t) \underbrace{U_\psi^{+} \frac{|\phi_j^+\rangle - |\phi_j^-\rangle}{\sqrt{2}}}_{! \perp \mathrm{span}\{|x\rangle|0\rangle\}} \Big]$$

and $\Pi_0 U_\psi^{+} W^{t} U_\psi |x\rangle |0\rangle$

$$= \sum \beta_j \cos(\theta_j t) |v_j\rangle |0\rangle$$

$$= \sum \beta_j \underbrace{T_t(\cos(\theta_j))}_{t\text{-th Cheb. pol. of } 1^{st} \text{ kind}} |v_j\rangle |0\rangle$$

$$= \Big(\sum \beta_j T_t(\lambda_j) |v_j\rangle \Big) |0\rangle$$

$$= T_t(H) |x\rangle |0\rangle .$$

↳ QW applies "function" of H on $|x\rangle$:

1. apply $U_\psi^{+} W^{t} U_\psi$

1. apply $U_\psi^\dagger W' U_\psi$
2. measure 2^{nd} register
$\rightarrow$ if outcome "0" (prob. $\|T_t(H)|x\rangle\|_2^2$),
then state $\dfrac{T_t(H)|x\rangle}{\|T_t(H)|x\rangle\|_2}$.

HAMILTONIAN SIMULATION

"quantum system" described by **Hamiltonian H**:

initial state $|\psi(0)\rangle$,

Schrödinger's equation

Schrödinger's equation

$$\boxed{i \frac{\partial}{\partial t} |\chi(t)\rangle = H |\chi(t)\rangle}$$

→ <u>solution:</u> $|\chi(t)\rangle = e^{-iHt} |\chi(t)\rangle$,

where $e^{-iHt} = \sum_{k \geq 0} \frac{(iHt)^k}{k!}$.

= unitary operator!

"Hamiltonian simulation":

"other" fct. of H

given "access" to H, implement e^{iHt}.

→ nontrivial, relevant & actual problem!

ALGORITHM: using QWs + phase estimation

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! similar to QW reflection,

function $f(x) = \mathrm{sgn}(x-1)$: $f(H) = \sum f(\lambda_i) |v_i\rangle\langle v_i|$

$$= -|\pi\rangle\langle\pi| + (\mathbb{1} - |\pi\rangle\langle\pi|)$$

1. map to star subspace:

$$|\psi\rangle|0\rangle|0\rangle \xrightarrow{U_\psi} \sum_j \beta_j \frac{|\phi_j^+\rangle + |\phi_j^-\rangle}{\sqrt{2}} |0\rangle$$

2. run phase estimation (precision $\hat{\varepsilon}$):

$$\xrightarrow{U_P} \sum_j \beta_j \frac{|\phi_j^+\rangle|\tilde{\theta}_j\rangle + |\phi_j^-\rangle|-\tilde{\theta}_j\rangle}{\sqrt{2}}$$

3. add phase $e^{i\cos(\theta)t}$ conditioned on phase reg. $|\theta\rangle$:

$$\xrightarrow{R} \sum \beta_j e^{i\cos(\tilde{\theta}_j)t} |\phi_j^+\rangle|\tilde{\theta}_j\rangle + |\phi_j^-\rangle|-\tilde{\theta}_j\rangle$$

$$\xmapsto{R} \sum_j \beta_j e^{i\cos(\tilde\theta_j)t} \frac{|\phi_j^+\rangle|\tilde\theta_j\rangle + |\phi_j^-\rangle|-\tilde\theta_j\rangle}{\sqrt{2}}$$

4. uncompute phase estimate:

$$\xmapsto{U_\ell^\dagger} \sum_j \beta_j e^{i\cos(\tilde\theta_j)t} \frac{|\phi_j^+\rangle + |\phi_j^-\rangle}{\sqrt{2}} |0\rangle$$

5. map back to original state space:

$$\xmapsto{U_v^\dagger} \sum_j \beta_j e^{i\cos(\tilde\theta_j)t} |v_j\rangle|0\rangle|0\rangle.$$

TOTAL COST $= \tilde{O}(1/\hat\varepsilon)$ $\in \tilde{O}(t/\varepsilon)$

CORRECTNESS:

Show that if $\hat\varepsilon \le \varepsilon/t$, then

$$\| \sum \quad e^{i\cos(\tilde\theta_j)t} \quad\quad e^{iHt} |\psi\rangle \| \le \varepsilon$$

$$\left\| \sum \beta_j e^{i\cos(\tilde{\theta}_j)t} |v_j\rangle - e^{iHt}|\psi\rangle \right\|_2 \leq \varepsilon.$$

$$\hookrightarrow \text{"} \quad = \sum |\beta_j|^2 \cdot \underbrace{\left| e^{i\cos(\tilde{\theta}_j)t} - e^{i\cos(\theta_j)t} \right|}$$

$$= \left| 1 - e^{i(\cos(\theta_j) - \cos(\tilde{\theta}_j))t} \right|$$

if $\hat{\varepsilon} t << 1$

$$\leq \left| 1 - e^{i(\cos(\theta_j + \hat{\varepsilon}) - \cos(\theta_j))t} \right|$$

$$\leq \left| 1 - e^{i\hat{\varepsilon}t} \right|$$

$$= \left| 1 - 1 + i\hat{\varepsilon}t + O(\hat{\varepsilon}^2 t^2) \right|$$

$$\in O(\hat{\varepsilon}t) \in O(\varepsilon).$$

QUANTUM LINEAR SYSTEM SOLVING

SYSTEM SOLVING

Given: invertible, Hermitian (w.l.o.g.) matrix A
vector b

find x s.t. $\boxed{Ax = b}$.

$$x = A^{-1} b = f(A)\, b, \quad \text{with } f(x) = 1/x.$$

Assume: $\|A\|_1 \leq 1$
eigenvalues $\frac{1}{\kappa} \leq |\lambda_j| \leq 1$

"condition number"

Q. ALGORITHM:

- define star states + QW based on A

$\downarrow$ U_ψ $\quad$ $\downarrow$ W

- assume we can prepare

$$|b\rangle = \frac{1}{\|b\|_2} \sum_i b(i)|i\rangle$$

$$\left(= \sum \beta_j |v_j\rangle \right)$$

eig. $\lambda_j = \cos(\theta_j)$

Naive:

$$|b\rangle|0\rangle|0\rangle \xmapsto{U_\varphi U_\psi} \sum_j \beta_j \frac{|\phi_j^+\rangle|\tilde{\theta}_j\rangle + |\phi_j^-\rangle|-\tilde{\theta}_j\rangle}{\sqrt{2}}$$

$$\longmapsto \sum_j \beta_j \underbrace{\frac{1}{\cos(\theta_j)}}_{>1 \ \lightning} \frac{|\phi_j^+\rangle|\tilde{\theta}_j\rangle + |\phi_j^-\rangle|-\tilde{\theta}_j\rangle}{\sqrt{2}}$$

Trick: $\lambda = \cos(\theta) \geq \frac{1}{\kappa}$

so $\frac{1}{\cos(\theta)} \leq \kappa$

↳ add ancilla$\overset{\text{qubit}}{\checkmark}$ and map "conditioned rotation"

$$|\theta\rangle|0\rangle \xrightarrow{R} |\theta\rangle\left(\underbrace{\frac{1}{\kappa\cos(\theta)}}_{\leq 1}|0\rangle + \sqrt{1-\frac{1}{\kappa^2\cos(\theta)^2}}\,|1\rangle\right)$$

ALGORITHM:

$$|b\rangle|0\rangle^{\otimes 3} \xrightarrow{U_2 U_1} \sum_j \beta_j \frac{|\phi_j^+\rangle|\tilde{\theta}_j\rangle + |\phi_j^-\rangle|-\tilde{\theta}_j\rangle}{\sqrt{2}}|0\rangle$$

$$\xrightarrow{R} \sum_j \beta_j \frac{|\phi_j^+\rangle|\tilde{\theta}_j\rangle + |\phi_j^-\rangle|-\tilde{\theta}_j\rangle}{\sqrt{2}}\left(\frac{1}{\kappa\cos(\tilde{\theta}_j)}|0\rangle + \sqrt{1-\frac{1}{\kappa^2\cos(\tilde{\theta}_j)^2}}\,|1\rangle\right)$$

$$\xrightarrow{U_\psi^\dagger U_P^\dagger} \sum_j \beta_j |v_j\rangle |0\rangle|0\rangle \left(\frac{1}{\kappa \cos(\tilde{\theta}_j)} |0\rangle + \sqrt{1 - \frac{1}{\kappa^2 \cos(\tilde{\theta}_j)^2}} |1\rangle \right)$$

$$= \underbrace{\left(\sum_j \frac{1}{\kappa} \frac{\beta_j}{\cos(\tilde{\theta}_j)} |v_j\rangle \right)}_{\approx \frac{1}{\kappa} A^{-1}|b\rangle} |0\rangle^{\otimes 3} + \quad \ldots |0\rangle^{\otimes 2} |1\rangle$$

COST $\sim \frac{1}{\hat{\varepsilon}} \sim \frac{\kappa}{\varepsilon}$ for ε-approximation

TOTAL COMPLEXITY:

$$\frac{1}{\| \frac{1}{\kappa} A^{-1} |b\rangle \|} \cdot \frac{\kappa}{\varepsilon} \leq \frac{\kappa^2}{\varepsilon}.$$