Circuits, QFT, Grover: exercises

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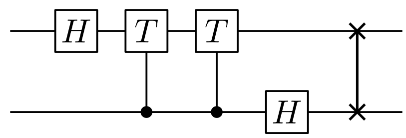
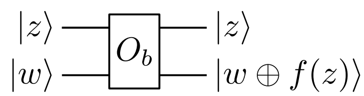
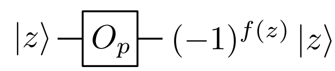
Exercise 1 (QFT)**.** What does F_2, the QFT on 1 qubit, correspond to? Consider the following circuit, where the last operation denotes swapping of the two qubits.



Show that this circuit corresponds to F_4, the QFT on 2 qubits.

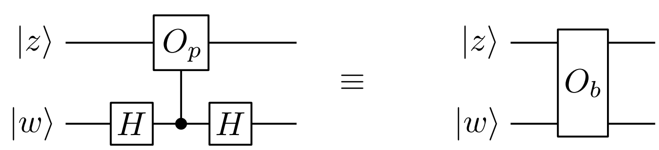
Exercise 2 (Oracles)**.** We described a bit oracle O_b and a phase oracle O_p for accessing a function $f : \{0,1\}^n \to \{0,1\}$. They are defined as follows, with $z \in \{0,1\}^n$ and $w \in \{0,1\}$:



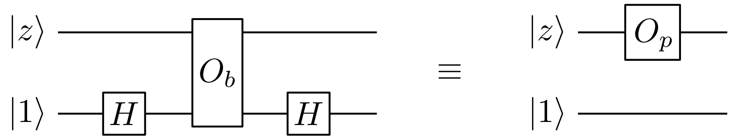


We can show that both oracles are equivalent in a sense.

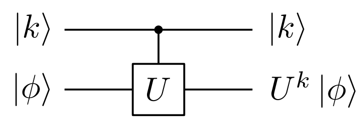
- Show that the phase oracle can simulate the bit oracle:



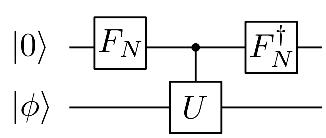
- Show that the bit oracle can simulate the phase oracle:



Exercise 3 (Quantum phase estimation)**.** Assume access to a unitary U and eigenvector $|\phi\rangle$ such that $U|\phi\rangle = e^{2\pi i\theta}|\phi\rangle$ for some $\theta \in [0,1)$. To avoid approximation issues, we assume that $N\theta$ is an integer for some $N = 2^n$. Consider the controlled version of U, represented by the following circuit:



where now $k \in \{0,1,\ldots,N-1\}$. The circuit for quantum phase estimation is the following:



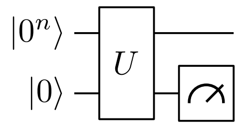
Show that we can learn θ from the output of this circuit.

Exercise 4 (Amplitude amplification)**.** A useful variation on Grover's algorithm is called *amplitude amplification*. Assume that we have access to a unitary U such that

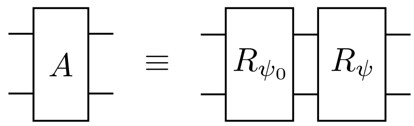
$$U\left|0^n\right\rangle\left|0\right\rangle = \left|\psi\right\rangle = \sqrt{p}\left|\psi_1\right\rangle\left|1\right\rangle + \sqrt{1-p}\left|\psi_0\right\rangle\left|0\right\rangle,$$

and we would like to prepare the "marked" state $\left|\psi_1\right\rangle$.

- The following circuit presents a simple solution. What is its success probability?

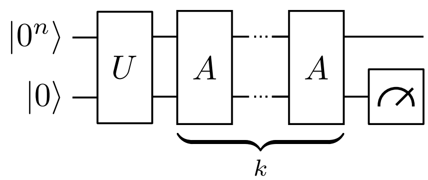


Amplitude amplification improves on this. Consider the amplitude amplification operator:



with reflections $R_\psi = 2\left|\psi\right\rangle\left\langle\psi\right| - I$ and $R_{\psi_0} = 2\left|\psi_0, 0\right\rangle\left\langle\psi_0, 0\right| - I$.

- What is the success probability of the following circuit?



- Write reflection R_ψ using U_ψ and R_0

Exercise 5 (Quantum approximate counting)**.** Check that the amplitude amplification operator A has eigenvectors and corresponding eigenvalues

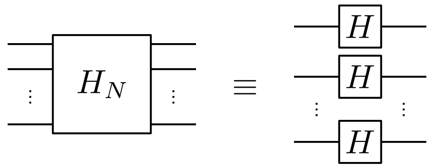
$$\left|\psi_\pm\right\rangle = \frac{\left|\psi_1, 1\right\rangle \pm i\left|\psi_0, 0\right\rangle}{\sqrt{2}}, \qquad \lambda_\pm = e^{\pm 2i\theta},$$

with θ such that $\sin(\theta) = \sqrt{p}$. Use quantum phase estimation on the initial state

$$\left|\psi\right\rangle = \frac{-i}{\sqrt{2}}\left(e^{i\theta}\left|\psi_+\right\rangle - e^{-i\theta}\left|\psi_-\right\rangle\right).$$

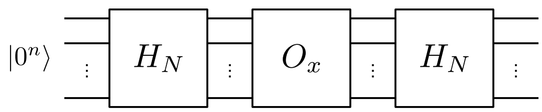
to estimate θ (and hence p).

Exercise 6 (Hadamard transform). A variation on the quantum Fourier transform is the Hadamard transform H_N for $N = 2^n$. It is defined by $H_N = H^{\otimes n}$, which corresponds to the circuit



- What is $H_N |0^n\rangle$ equal to?
- What is $H_N |k\rangle = H_N |k_1 \dots k_n\rangle$ equal to? Use the inner product $j \cdot k = \sum_\ell j_\ell k_\ell$.[1]

Exercise 7 (Bernstein-Vazirani algorithm). Consider a string $x \in \{0,1\}^N$, for $N = 2^n$, that is determined by some unknown $a \in \{0,1\}^n$ such that $x_i = (i \cdot a) \pmod 2$. We can access the string through a "phase oracle" $O_x |i\rangle = (-1)^{x_i} |i\rangle$. What is the output of the following circuit?



Exercise 8 (Factoring reduction (optional)). Here we walk through Shor's reduction from factoring to period finding. Recall that we are given an n-bit integer N such that $2^{n-1} \leq N < 2^n$, and we wish to find a (nontrivial) factor of N. Without loss of generality, we can assume that N is odd and not a prime power. Why?[2]

Now pick $x \in \{2, \dots, N-1\}$ uniformly at random. If $\gcd(N, x) > 1$ then we can run Euclid's algorithm to find a factor. Hence, assume that N and x are coprime, and consider the series

$$x^0 = 1 \pmod N, \qquad x \pmod N, \qquad x^2 \pmod N, \qquad \dots$$

Since N and x are coprime, there does not exist s such that $x^s = 0 \pmod N$. Show that this implies that the series must have a period $r \leq N$ for which $x^r = 1 \pmod N$. It is precisely this factor that is calculated using quantum period finding.

One can show (not in this exercise!) that, with probability at least $1/2$ over the choice of x, the period r will be even and both $x^{r/2} + 1$ and $x^{r/2} - 1$ are not multiples of N. Use $x^r = 1 \pmod N$ to show that this implies that both $x^{r/2} + 1$ and $x^{r/2} - 1$ must share a (nontrivial) factor with N. Once we computed r, we can then find these factors by computing $\gcd(x^{r/2} \pm 1, N)$.

[1] Hint: show that $H |k_\ell\rangle = \frac{1}{\sqrt{2}} \sum_{j_\ell=0}^{1} (-1)^{j_\ell k_\ell} |j_\ell\rangle$.
[2] Hint: if $N = p^k$ for some prime $p \geq 2$ then necessarily $k \leq n$.