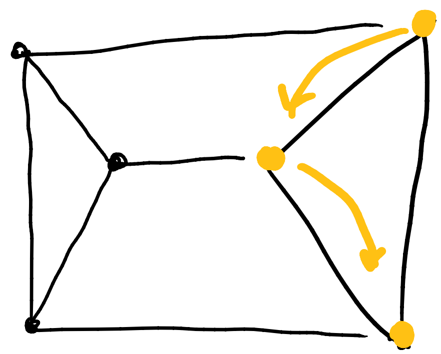
Lecture 1: Random walks and quantum walks

simple, regular graph $G = (V, E)$, $|V| = n$, $|E| = m$



initial vertex $\sim p_0 \in \mathbb{R}^n$

after 1 step:

$$p_1 = P p_0 ,$$

where $P_{x,y} = \frac{1}{d}$ if $(x,y) \in E$

If G connected, then $\exists$ unique π s.t. $P\pi = \pi$.

"stationary distribution"

$\pi = \frac{1}{n} \underline{1}$ if regular

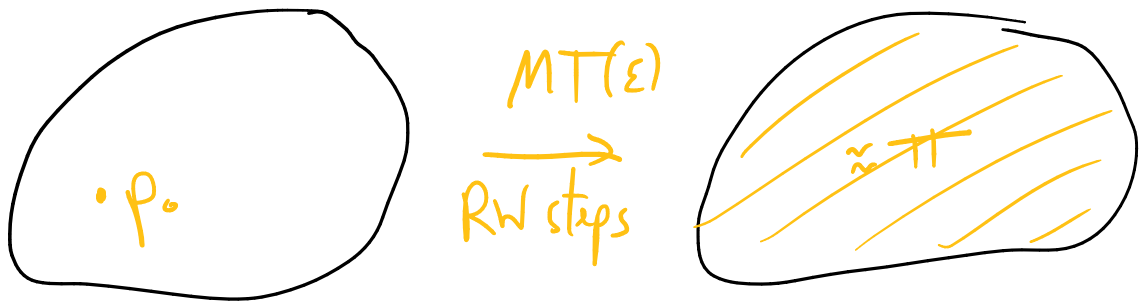
If G connected & non-bipartite, then

$$P^t p_0 \xrightarrow{t \to \infty} \pi , \quad \forall \text{ distr. } p_0 .$$

"MIXING TIME"

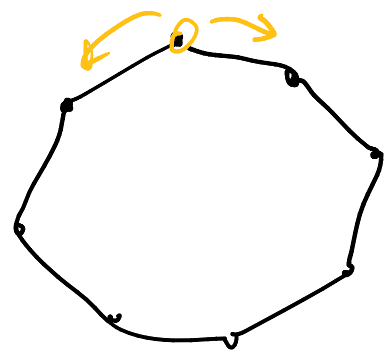
"MIXING TIME"

$$MT(\varepsilon) = \min\{t \mid \|P^t p_0 - \pi\|_1 \le \varepsilon,\ \forall p_0\}$$



EXAMPLE: n-cycle

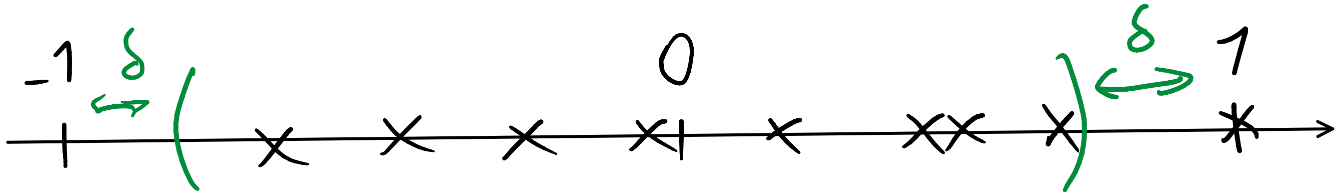
$$MT(\varepsilon) \sim n^2 \log\left(\frac{1}{\varepsilon}\right)$$



eigenv. $1 = \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n \ge -1$

MT(ε) related to "spectral gap" of P

$$\boxed{\delta = 1 - \max\{|\lambda_2|, |\lambda_n|\}}$$



Lemma: $MT(\varepsilon) \leq \frac{1}{\delta}\left(\log(n) + \log\left(\frac{1}{\varepsilon}\right)\right).$

Proof: (EXERCISE)

spectrum P: $\begin{cases} \underline{1}/\sqrt{n}, & \lambda_1 = 1 \\ v_j, & j = 2, \dots, n \end{cases}$

$$\text{s.t. } P = \sum_{j=1}^{n} \lambda_j v_j v_j^T = \pi \mathbb{1}^T + \sum_{j>1} \lambda_j v_j v_j^T$$

$$p_0 = \sum (v_j^T p_0) v_j = \pi + \sum_{j>1} (v_j^T p_0) v_j$$

and

$$P^t p_0 - \pi = \sum_{j=2}^{n} \lambda_j^t v_j (v_j^T p_0)$$

$$\hookrightarrow \|P^t p_0 - \pi\|_2 = \sqrt{\sum_{j=2}^{n} |\lambda_j|^{2t} |v_j^T p_0|^2}$$

$$\leq (1-\delta)^t \sqrt{\sum |v_j^T p_0|^2}$$ (since $\|p_0\|_2 \leq 1$)

$$\leq (1-\delta)^t \leq \varepsilon'$$ (using that $(1-\delta)^{\frac{1}{\delta}} \leq 1/e$)

$$\text{if } t \geq \frac{1}{\delta} \log\left(\frac{1}{\varepsilon'}\right)$$

Now Cauchy-Schwarz:

Now, Cauchy-Schwarz:

↳ $\|x\|_1 = \langle 1, |x| \rangle \le \|1\|_2 \cdot \||x|\|_2$

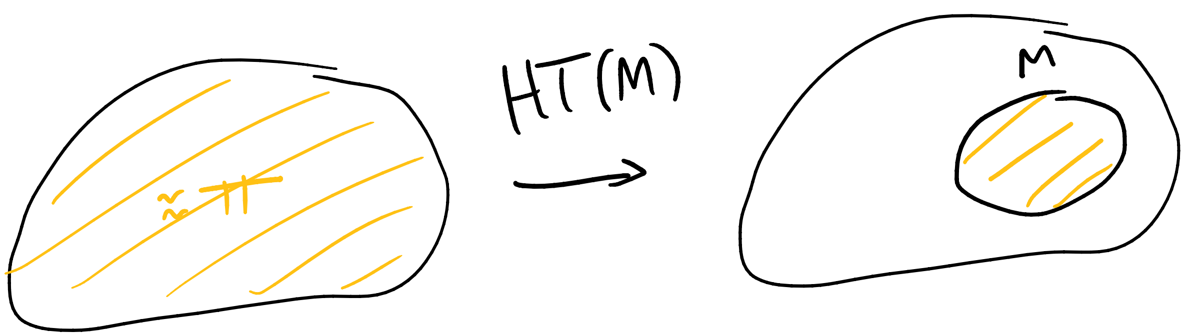
$$\|P^t p_0 - \pi\|_1 \le \sqrt{n}\,\|P^t p_0 - \pi\|_2 \le \sqrt{n}\,\varepsilon' \le \varepsilon$$

if $\varepsilon' \le \frac{\varepsilon}{\sqrt{n}}$

and so $t \geq \frac{1}{\delta}\left(\log\left(\frac{1}{\varepsilon}\right) + \log(n)\right)$. □

Conversely: $MT(\varepsilon) \in \Omega\left(\frac{1}{\delta}\log\left(\frac{1}{\varepsilon}\right)\right)$

"HITTING TIME"

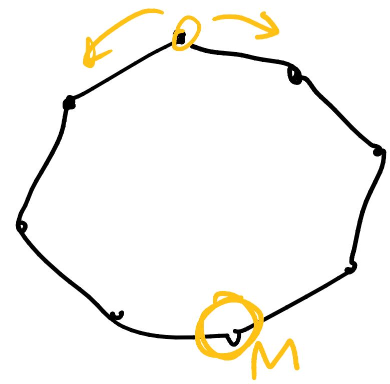


for $M \subseteq V$,

$$HT(M) = \mathbb{E}[\#\text{steps to hit } M \mid p_0 = \pi]$$

E.g., n-cycle

$HT(M) \sim n^2$



Lemma: $HT(M) \in \tilde{O}\left(\frac{1}{\delta}\,\frac{1}{\pi(M)}\right)$.

Proof: (sketch)

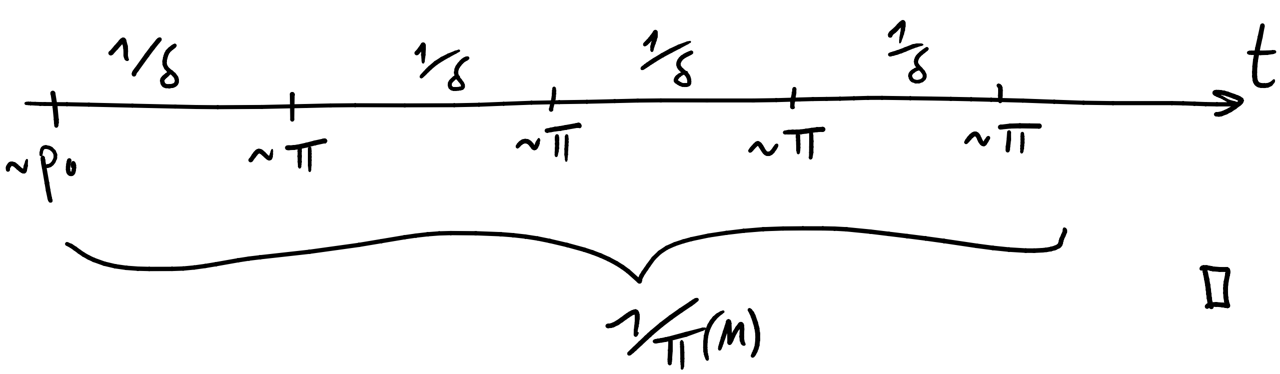
(i) # samples from π to hit M

$$= 1/\pi(M)$$

(ii) time for 1 sample

$$= MT(\varepsilon) \quad (\varepsilon \leq \pi(M)/2 \text{ suffices})$$

$$\in \tilde{O}\left(\frac{1}{\delta}\right)$$



∎

Claim: Quantum walks find marked el. in $\tilde{O}\left(1/\sqrt{\varepsilon\delta}\right)$ steps.

~~~ break ~~~

QUANTUM WALKS

key RW concepts:

- stationary distr. π
- spectral gap δ

will encounter quantum variants
~~~

state space:

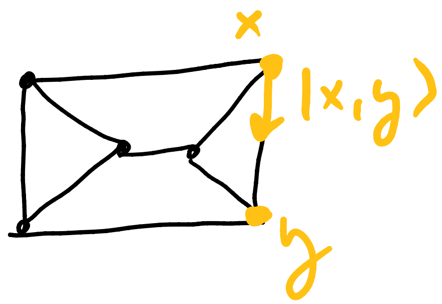
$$|\psi\rangle = \sum_{(x,y)\in E} \alpha_{xy} |x,y\rangle$$

QW over edges

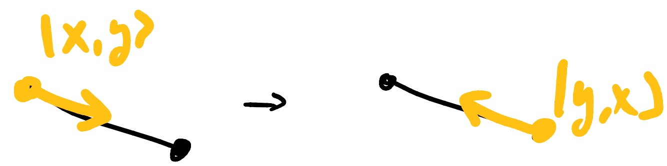
"current node"

"next node"

3-reg. G

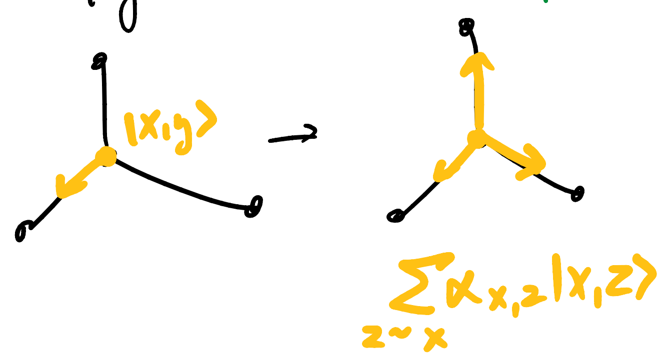


2 operations:



(i) "SWAP" $S|x,y\rangle = |y,x\rangle$ = "step"

(ii) "COIN TOSS"



↓

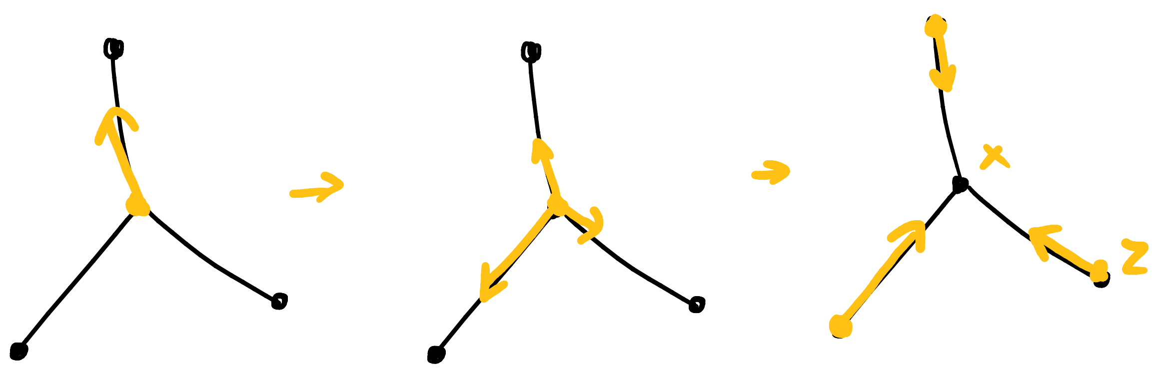
defined using "star states"

$$|\psi_x\rangle = \frac{1}{\sqrt{d}} \sum_{y\sim x} |x,y\rangle$$

$$C(P) = 2\Big(\underbrace{\sum_x |\psi_x\rangle\langle\psi_x|}_{\text{proj. onto } \mathrm{span}_{\mathbb{C}}\{|\psi_x\rangle \mid x\in V\}}\Big) - \mathbb{1}$$

$$= \mathrm{ref}\left\{ \mathrm{span}_{\mathbb{C}} \langle |\psi_x\rangle \mid x \in V \rangle \right\}$$

$\rightarrow$ Quantum walk operator:



$|x,y\rangle$ $\qquad$ $C(P)\,|x,y\rangle = \sum_{z\sim x} \alpha_{xz} |x,z\rangle$ $\qquad$ $W(P)\,|x,y\rangle = \sum_{z\sim x} \alpha_{xz} |z,x\rangle$

<u>Lemma</u>: $|\pi\rangle = \frac{1}{\sqrt{n}} \sum_x |\psi_x\rangle = \frac{1}{\sqrt{nd}} \sum_{(x,y)\in E} |x,y\rangle$

is a stationary state of $W(P)$.

is a stationary state of $W(P)$.

Proof: $W(P) = S \cdot C(P)$

and $C(P)|\pi\rangle$

$$= \left(2\left(\sum_x |\psi_x\rangle\langle\psi_x|\right) - \mathbb{1}\right)\frac{1}{\sqrt{n}}\sum_{x'}|\psi_{x'}\rangle$$

$$= \frac{1}{\sqrt{n}}\sum_x |\psi_x\rangle \quad \left(\text{using that } \langle\psi_x|\psi_y\rangle = \delta_{x,y}\right)$$

$S|\pi\rangle$

$$= S\frac{1}{\sqrt{nd}}\sum_{(x,y)\in E}|x,y\rangle$$

$$= \frac{1}{\sqrt{nd}}\sum_{(x,y)\in E}|y,x\rangle = |\pi\rangle,$$

hence $W(P)|\pi\rangle = S \cdot C(P)|\pi\rangle = S|\pi\rangle = |\pi\rangle$. $\square$

!not only stationary state:

$W(P) = S \cdot C(P)$

$W(P) = S \; C$

reflection around $\text{span}\{|xy\rangle + |yx\rangle \mid (x,y) \in E\}$

reflection around $\text{span}\{|\psi_x\rangle \mid x \in V\}$

$\rightarrow$ QW = product of two reflections!

> **Lemma**: Consider projectors Π_1, Π_2.
> Operator $(2\Pi_2 - \mathbb{1})(2\Pi_1 - \mathbb{1})$ has inv. subspace
> $$(\ker(\Pi_1) \cap \ker(\Pi_2)) \cup (\ker(\Pi_1)^\perp \cap \ker(\Pi_2)^\perp).$$

QW operator: $W(P) = (2\Pi_+ - \mathbb{1})(2\Pi_* - \mathbb{1})$

where $$\Pi_+ = \sum_{(x,y) \in E} \frac{1}{2} (|xy\rangle + |yx\rangle)(\langle xy| + \langle yx|)$$

$$\Pi_* = \sum |\psi_x\rangle\langle\psi_x|$$

$x \in V$

$\rightarrow \ker(\Pi_+)^\perp \cap \ker(\Pi_*)^\perp \rightarrow \in \mathrm{span}\{|\psi_x\rangle\}$

$$= \mathrm{span}\left\{ \sum_{x \in V} \alpha_x |\psi_x\rangle \;\middle|\; \alpha_x = \alpha_y \ \forall (x,y) \in E \right\}$$

$$= \mathrm{span}\{|\pi\rangle\}$$

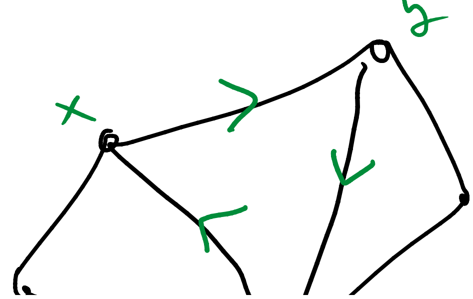
$\rightarrow \ker(\Pi_+) \cap \ker(\Pi_*) \nearrow \perp \mathrm{span}\{|\psi_x\rangle\}$

$$= \mathrm{span}\{ |f\rangle \mid f \text{ "closed flow"} \}$$

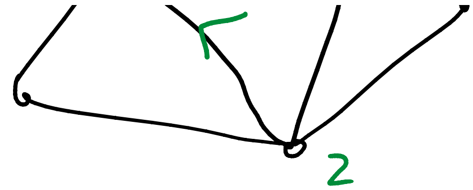
$$\hookrightarrow |f\rangle = \sum_{(x,y) \in E} f(x,y) |x,y\rangle$$

$$\text{s.t.} \begin{cases} f(x,y) = -f(y,x) \\ \sum_{y \sim x} f(x,y) = 0 = \langle \psi_x | f \rangle \end{cases}$$

$\Leftrightarrow f$ closed flow



$$|f\rangle = |xy\rangle - |yx\rangle + |yz\rangle - |zy\rangle + |zx\rangle - |xz\rangle$$



$+ |zx\rangle - |xz\rangle$

? rest of the spectrum

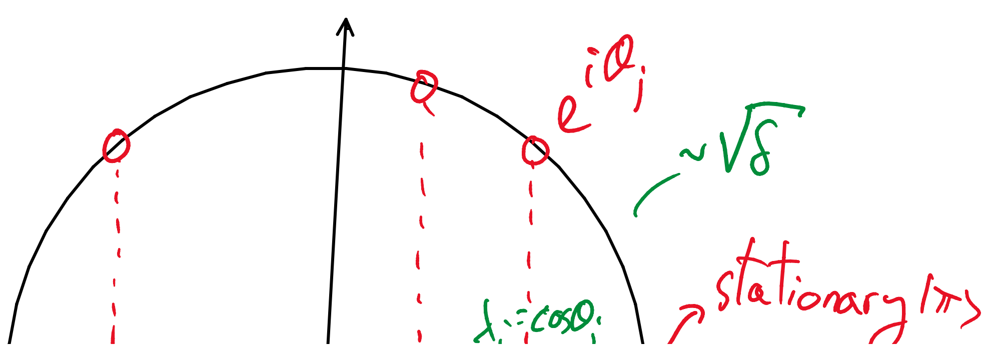
* (-1)-subspace is

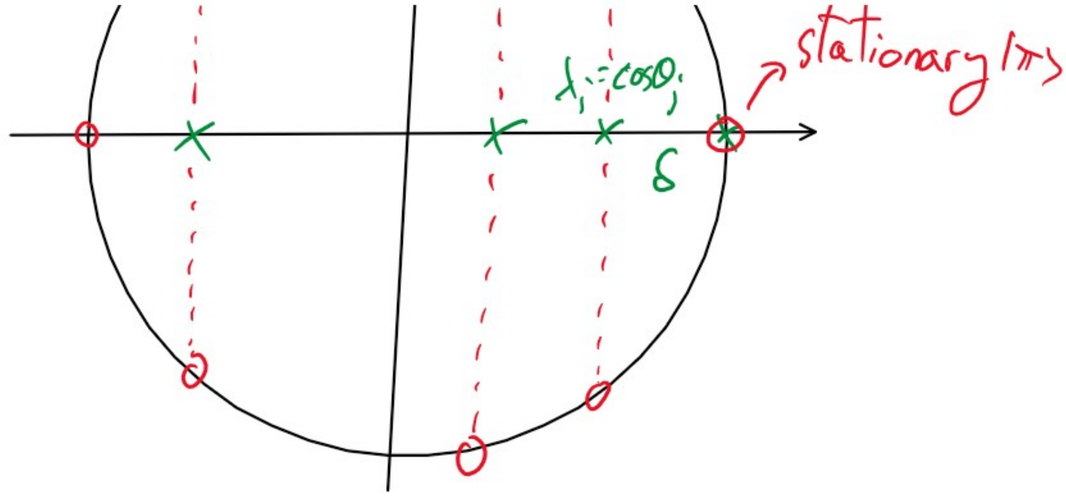
$$\left(\ker(\Pi_1) \cap \ker(\Pi_2)^\perp\right) \cup \left(\ker(\Pi_1)^\perp \cap \ker(\Pi_2)\right).$$

* Szegedy's spectral lemma

x eigenvalues P

o eigenvalues W





I.e., for every eigenpair of P

$$\left(\lambda_j = \cos\theta_j,\ |v_j\rangle = \sum v_j(x)|x\rangle\right),$$

W has eigenpairs

$$\left(e^{i\theta_j},\ |\phi_j^+\rangle\right),\ \left(e^{-i\theta_j},\ |\phi_j^-\rangle\right)$$

s.t. $\displaystyle \sum v_j(x)|\psi_x\rangle = \frac{|\phi_j^+\rangle + |\phi_j^-\rangle}{\sqrt{2}}$.

OBSERVATION 1:

any state $\sum \alpha_x |\psi_x\rangle$ in star subspace

is in $\mathrm{span}\{|\pi\rangle, |\phi_1^+\rangle, |\phi_1^-\rangle, \ldots, |\phi_{n-1}^+\rangle, |\phi_{n-1}^-\rangle\}$.

is in span$\{|\pi\rangle, |\psi_1\rangle, \ldots$

→ we characterized "relevant subspace"

OBSERVATION 2:

"phase gap" $\Delta = \min\{\theta_j \mid \theta_j > 0\}$

$$= \arccos(\lambda_2)$$

$$\geq \arccos(1-\delta)$$

RW sp-gap

Exercise: prove that $\Delta \in \Omega(\sqrt{\delta})$.

(hint: use that $\cos(x) \geq 1 - \frac{x^2}{2}$)

↳ solution: $x \geq \sqrt{2(1-\cos(x))}$

$$\Rightarrow \arccos(y) \geq \sqrt{2(1-y)}$$

$$\Rightarrow \arccos(1-\delta) \geq \sqrt{2\delta}.$$

→ quadratically larger gap!