AQAlg: Advanced Quantum Algorithms **2025 - 2026**

Exercises 3: Adiabatic quantum computation

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Exercise 1 (Pauli basis)**.** Recall the unitary Pauli matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- Show that the Pauli basis $\{I, X, Y, Z\}$ forms a basis for the complex 2-by-2 matrices. I.e., any $A \in \mathbb{C}^{2\times 2}$ can be expanded as $A = \alpha_1 I + \alpha_x X + \alpha_y Y + \alpha_z Z$.
- Argue that this implies that the n-qubit Pauli basis $\{I, X, Y, Z\}^{\otimes n}$ forms a basis for the 2^n-by-2^n matrices.
- Show that if $A \in \mathbb{C}^{2^n \times 2^n}$ is Hermitian, then its coefficients in the Pauli basis are real.

Exercise 2 (Max cut)**.** Finding the maximum cut in a graph is a canonical problem in combinatorial optimization. For a graph G with vertex set $[n]$ and (symmetric) edge set $E \subseteq [n]^2$, a maximum cut is described by a subset $Z \subset [n]$ that cuts a maximum number of edges (edges crossing from Z to Z^c). Equivalently, it maximizes the cut function

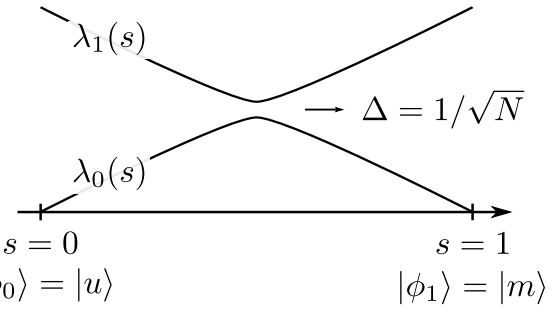
$$c(A) = \sum_{(i,j)\in E} I_{i\in A, j\notin A}.$$

- Identify a subset A with the indicator $a \in \{0,1\}^n$ ($i \in A \Leftrightarrow a_i = 1$). Express the cut function $c(A)$ as a degree-2 polynomial h in a.
- Rewrite the cost Hamiltonian $H_1 = -\sum_{z\in\{0,1\}^n} c(z) \left|z\right\rangle \left\langle z\right|$ in terms of identity and Pauli-Z matrices.

Exercise 3 (Adiabatic Grover algorithm and avoided crossing)**.** Consider the unstructured search problem over the set $[N] = \{0,1\}^n$, and assume that there is a single (unknown) marked element $m \in \{0,1\}^n$. We can solve this problem using the adiabatic optimization algorithm with cost function $h(z) = 0$ if $z = m$ and $h(z) = 1$ otherwise. Letting $|u\rangle = \frac{1}{\sqrt{N}} \sum_{x\in\{0,1\}^n} |x\rangle$, we can use the Hamiltonians

$$H_0 = I - \left|u\right\rangle \left\langle u\right| \quad \text{and} \quad H_1 = I - \left|m\right\rangle \left\langle m\right|,$$

with ground states $|\phi_0\rangle = |u\rangle$ and $|\phi_1\rangle = |m\rangle$, respectively. Similar to Grover, the Hamiltonian $H(s) = (1-s)H_0 + sH_1$ only acts nontrivially in the subspace spanned by $|u_0\rangle = \frac{1}{\sqrt{N-1}} \sum_{x\neq m} |x\rangle$ and $|m\rangle$. Calculate the nontrivial eigenvalues of $H(s)$, and show that they behave as in the following figure.[1]



[1] While this proves that the gap is inverse polynomial, and so the adiabatic algorithm solves unstructured search in poly(n), it doesn't directly recover the quadratic Grover speedup. A more refined adiabatic algorithm does (Roland-Cerf, "Quantum search by local adiabatic evolution", Physical Review A, 2002).