AQAlg: Advanced Quantum Algorithms **2023 - 2024**

Exercises 7: Random walks and quantum walks

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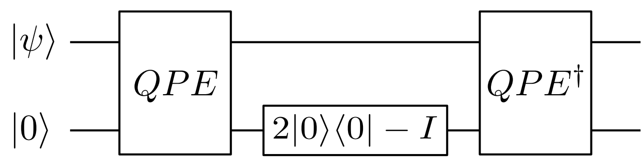
Exercise 1 (Reflecting around $|\pi\rangle$)**.** We can use a quantum walk W to reflect around the quantum state $|\pi\rangle$, i.e., implement the map

$$|\psi\rangle = \alpha\,|\pi\rangle + \beta\,|\pi^{\perp}\rangle \longrightarrow (2\,|\pi\rangle\,\langle\pi| - I)\,|\psi\rangle = \alpha\,|\pi\rangle - \beta\,|\pi^{\perp}\rangle\,.$$

Assume that $|\psi\rangle = \alpha\,|\pi\rangle + \sum_j \beta_j\,|v_j\rangle$ such that $W\,|\pi\rangle = |\pi\rangle$ and $W\,|v_j\rangle = e^{i2\pi\theta_j}\,|v_j\rangle$ with $1/2 > |\theta_j| > \Delta > 0$. We call Δ the *spectral gap of the quantum walk*. Analyze the following circuit (where QPE represents quantum phase estimation on W to precision $\Delta/2$). How many calls does it make to the quantum walk operator?



Exercise 2 (Quantum walk search)**.** We can implement Grover search using quantum walks by considering the initial state $|\pi\rangle$, a marked element $|\psi_m\rangle$ (for $m \in V$), and the Grover operator

$$(2\,|\pi\rangle\,\langle\pi| - I)(2\,|\psi_m\rangle\,\langle\psi_m| - I).$$

How many times do we have to apply the Grover operator on $|\pi\rangle$ to have a good overlap with $|\psi_M\rangle$? Argue that this yields a quantum walk algorithm for finding a marked element that makes $O\left(\frac{1}{\Delta\sqrt{\pi(M)}}\right)$ calls to a quantum walk.[1]

Exercise 3. In this exercise we will prove that

$$\mathrm{MT}(\epsilon) \leq \frac{1}{\delta}\left(\frac{1}{2}\log(n) + \log\left(\frac{1}{\epsilon}\right)\right),$$

or equivalently

$$\|P^t p_0 - \pi\|_1 \leq \epsilon$$

for any initial distribution p_0 if $t \geq \frac{1}{\delta}\left(\frac{1}{2}\log(n) + \log\left(\frac{1}{\epsilon}\right)\right)$.

- We first bound the 2-norm distance. Consider the spectral decomposition

  $$P = \sum_{j\geq 1} \lambda_j v_j v_j^T,$$

  with $v_1 = \frac{1}{\sqrt{n}}\sum_x e_x$ the uniform eigenvalue-1 eigenvector of P, and v_j for $2 \leq j \leq n$ the remaining eigenvectors with corresponding eigenvalues $-1+\delta < \lambda_j < 1-\delta$. Show that

  $$\|P^t p_0 - \pi\|_2 \leq \epsilon$$

  if $t \geq \frac{1}{\delta}\log\frac{1}{\epsilon}$.

[1]Szegedy (FOCS'04) showed that $\Delta \in \Omega(\sqrt{\delta})$ so this becomes $O(1/\sqrt{\delta\pi(M)})$, quadratically improving over the upper bound $O(1/(\delta\pi(m)))$ on the classical hitting time of a random walk.

- From this bound we get a bound for the 1-norm distance. Use Cauchy-Schwarz[2] to show that

  $$\|P^t p_0 - \pi\|_1 \leq \sqrt{n}\|P^t p_0 - \pi\|_2.$$

  From this, deduce that

  $$\|P^t p_0 - \pi\|_1 \leq \epsilon$$

  if $t \geq \frac{1}{\delta}\left(\frac{1}{2}\log(n) + \log\left(\frac{1}{\epsilon}\right)\right)$.

Conversely, one can (but we won't) prove that $\mathrm{MT}(\epsilon) \in \Omega\left(\frac{1}{\delta}\log\left(\frac{1}{\epsilon}\right)\right)$, so that this bound is tight up to the $\log(n)$-factor.

Exercise 4 (Extra)**.** Here we prove that the hitting time

$$\mathrm{HT}(M) \in \widetilde{O}\left(\frac{1}{\delta}\frac{1}{\pi(M)}\right).$$

- Prove that $\|p - \pi\|_1 \leq \epsilon$ implies that $p(M) \geq \pi(M) - \epsilon$.
- Let $\epsilon_M = \pi(M)/2$. Argue that after $\mathrm{MT}(\epsilon_M) \in \widetilde{O}(1/\delta)$ steps the probability of hitting an element in M is at least $\pi(M)/2$.
- Use this to prove the bound on the expected hitting time.

[2]For any $v, w \in \mathbb{R}^n$ it holds that $\langle v, w\rangle \leq \|v\|_2\|w\|_2$.