Lecture 1: Random walks and quantum walks

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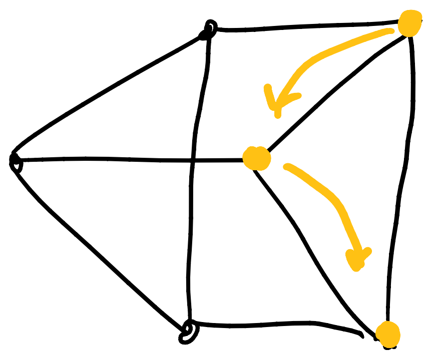
simple, regular graph $G = (V, E)$, $|V| = n$, $|E| = m$



RANDOM WALK

initial vertex $\sim p_0 \in \mathbb{R}^n$

after 1 step:

$$p_1 = P p_0,$$

where $P_{x,y} = \frac{1}{d}$ if $(x,y) \in E$

If G connected, then $\exists$ unique π s.t. $P\pi = \pi$.

"stationary distribution"

$\pi = \frac{1}{n}\underline{1}$ if regular

If G connected & non-bipartite, then

$$P^t p_0 \xrightarrow{t \to \infty} \pi, \quad \forall \text{ distr. } p_0.$$

"MIXING TIME"

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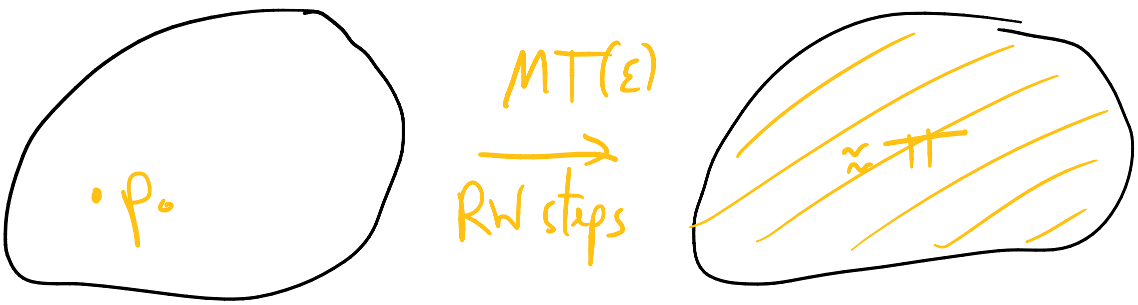
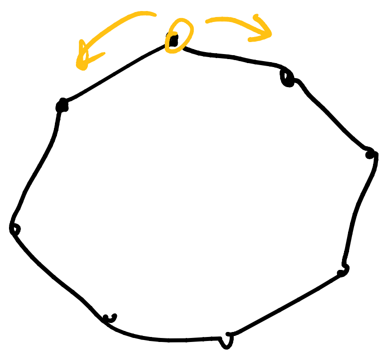
$$MT(\varepsilon) = \min\{t \mid \|P^t p_0 - \pi\|_1 \le \varepsilon, \ \forall p_0\}$$



EXAMPLE: n-cycle

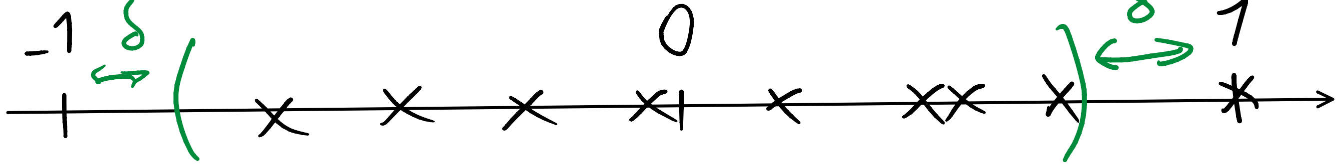
$$MT(\varepsilon) \sim n^2 \log\left(\frac{1}{\varepsilon}\right)$$



eigenv. $1 = \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n \ge -1$

MT(ε) related to "spectral gap" of P

$$\boxed{\delta = 1 - \max\{|\lambda_2|, |\lambda_n|\}}$$



Lemma: $MT(\varepsilon) \leq \frac{1}{\delta}\left(\log(n) + \log\left(\frac{1}{\varepsilon}\right)\right)$.

Proof:

spectrum P : $\begin{cases} \mathbb{1}/\sqrt{n}, & \lambda_1 = 1 \\ v_j, & j = 2, \dots, n \end{cases}$

s.t. $P = \sum_{j=1}^{n} \lambda_j v_j v_j^T = \pi \mathbb{1}^T + \sum_{j>1} \lambda_j v_j v_j^T$

$$p_0 = \sum (v_j^T p_0) v_j = \pi + \sum_{j>1} (v_j^T p_0) v_j$$

and

$$P^t p_0 - \pi = \sum_{j=2}^{n} \lambda_j^t v_j (v_j^T p_0)$$

$$\hookrightarrow \| P^t p_0 - \pi \|_2 = \sqrt{\sum_{j=2}^{n} |\lambda_j|^{2t} |v_j^T p_0|^2}$$

$$\leq (1-\delta)^t \sqrt{\sum |v_j^T p_0|^2}$$ (since $\|p_0\|_2 \leq 1$)

$$\leq (1-\delta)^t \leq \varepsilon'$$ (using that $(1-\delta)^{\frac{1}{\delta}} \leq 1/e$)

if $t \geq \frac{1}{\delta} \log\left(\frac{1}{\varepsilon'}\right)$

Now Cauchy-Schwarz:

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$$\|P^t p_0 - \pi\|_1$$

$\hookrightarrow \|x\|_1 = \langle 1, |x| \rangle \le \|1\|_2 \cdot \| |x| \|_2$

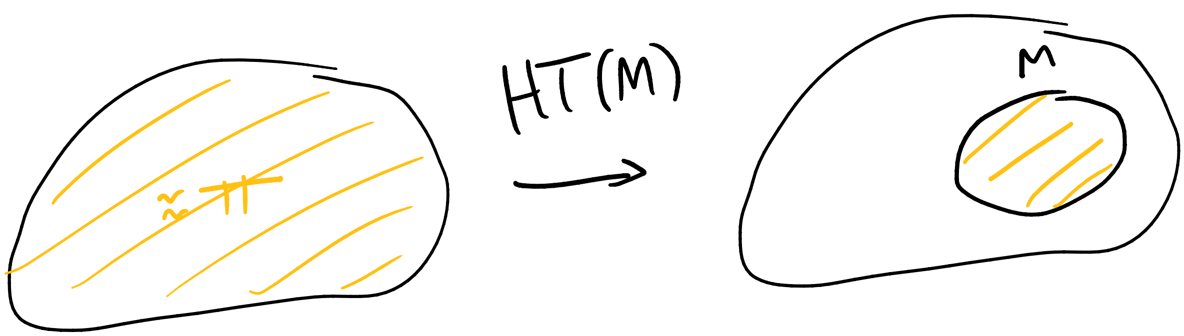
$$\le \sqrt{n}\,\|P^t p_0 - \pi\|_2 \le \sqrt{n}\,\varepsilon' \le \varepsilon$$

if $\varepsilon' \le \frac{\varepsilon}{\sqrt{n}}$

and so $t \ge \frac{1}{\delta}\left(\log\left(\frac{1}{\varepsilon}\right) + \log(n)\right)$. $\square$

Conversely: $MT(\varepsilon) \in \Omega\left(\frac{1}{\delta}\log\left(\frac{1}{\varepsilon}\right)\right)$

"HITTING TIME"



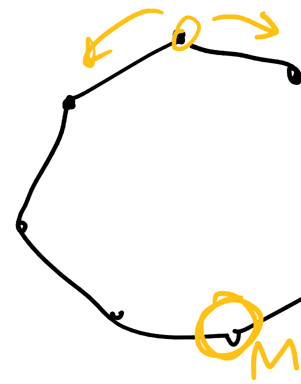
for $M \subseteq V$,

$$HT(M) = \mathbb{E}[\#\text{steps to hit } M \mid p_0 = \pi]$$

For n-cycle

E.g., n-cycle

$HT(M) \sim n^2$



<u>Lemma</u>: $HT(M) \in \tilde{O}\left(\frac{1}{\delta} \frac{1}{\pi(M)}\right)$.

<u>Proof</u>: (sketch)

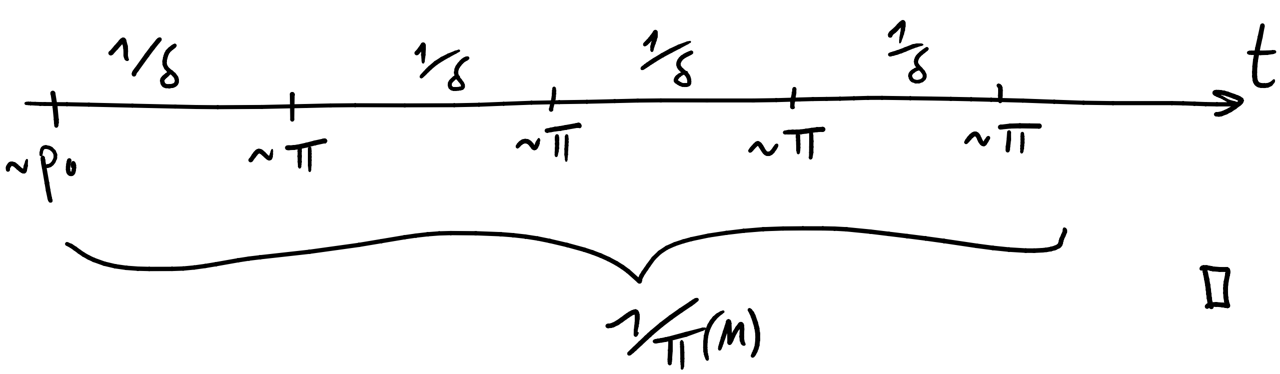
(i) # samples from π to hit M

$$= 1/\pi(M)$$

(ii) time for 1 sample

$$= MT(\varepsilon) \quad (\varepsilon \leq \pi(M)/2 \text{ suffices})$$

$$\in \tilde{O}\left(\frac{1}{\delta}\right)$$



□

Claim: Quantum walks find marked el. in $\tilde{O}\left(\frac{1}{\sqrt{\varepsilon\delta}}\right)$ steps.

— break —

QUANTUM WALKS

key RW concepts:

- stationary distr. π
- spectral gap δ

will encounter quantum variants

state space:

QW over edges

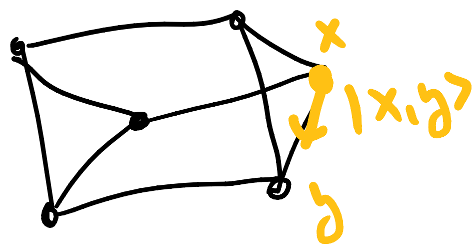
$$|\psi\rangle = \sum_{(x,y)\in E} \alpha_{xy} |x, y\rangle$$

"next node"

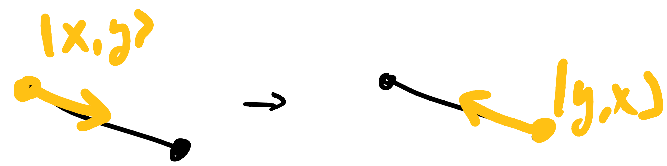
$$|\psi\rangle = \sum_{(x,y)\in E} \alpha_{xy} |x,y\rangle$$

"current node" "next node"

3-reg. G

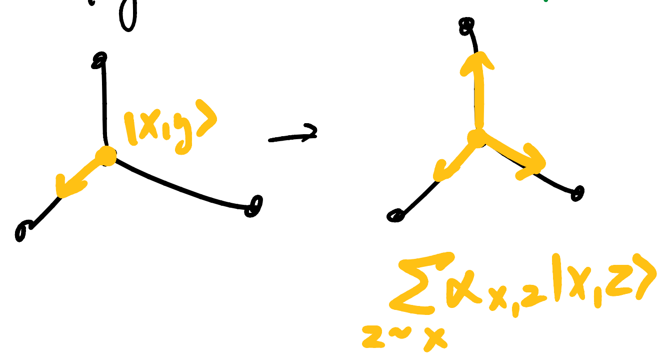


2 operations:



(i) "SWAP" $S|x,y\rangle = |y,x\rangle$ = "step"

(ii) "COIN TOSS"



↓

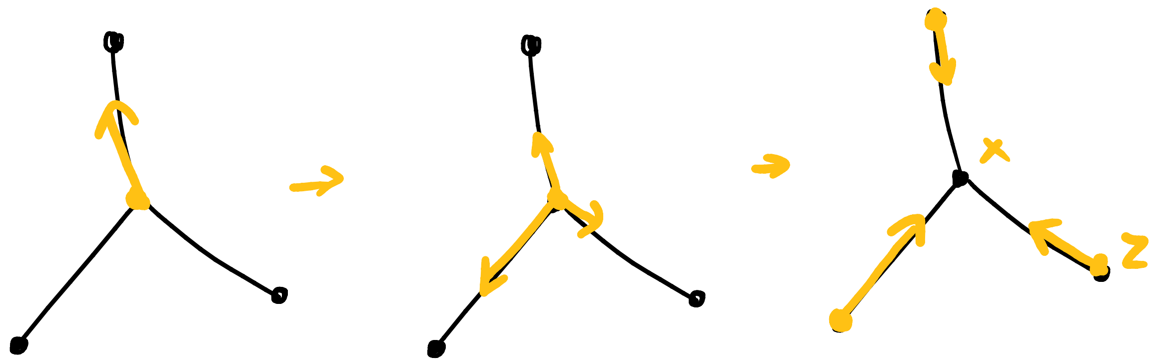
defined using "star states"

$$|\psi_x\rangle = \frac{1}{\sqrt{d}} \sum_{y\sim x} |x,y\rangle$$

$$C(P) = 2\Big(\underbrace{\sum_x |\psi_x\rangle\langle\psi_x|}_{\text{proj. onto span}_{\mathbb{C}}\{|\psi_x\rangle \,|\, x\in V\}}\Big) - \mathbb{1}$$

$$= \mathrm{ref}\left\{ \mathrm{span}_{\mathbb{C}} \langle |\psi_x\rangle \mid x \in V \rangle \right\}$$

→ Quantum walk operator:



$|x,y\rangle$ → $C(P)\,|x,y\rangle = \sum_{z\sim x} \alpha_{xz} |x,z\rangle$ → $W(P)\,|x,y\rangle = \sum_{z\sim x} \alpha_{xz} |z,x\rangle$

<u>Lemma</u>: $|\pi\rangle = \frac{1}{\sqrt{n}} \sum_x |\psi_x\rangle = \frac{1}{\sqrt{nd}} \sum_{(x,y)\in E} |x,y\rangle$

is a stationary state of $W(P)$.

<u>Proof</u>: $W(P) = S \cdot C(P)$

and $C(P)\,|\pi\rangle$

$$= \left(2\left(\sum |\psi_x\rangle\langle\psi_x|\right) - \mathbb{1}\right) \frac{1}{\sqrt{n}} \sum_{x'} |\psi_{x'}\rangle$$

$$= \left(2\left(\sum_x |\psi_x\rangle\langle\psi_x|\right) - \mathbb{1}\right)\frac{1}{\sqrt{n}}\sum_{x'}|\psi_{x'}\rangle$$

$$= \frac{1}{\sqrt{n}}\sum_x |\psi_x\rangle \quad \left(\text{using that } \langle\psi_x|\psi_y\rangle = \delta_{x,y}\right)$$

$$S|\pi\rangle$$

$$= S\frac{1}{\sqrt{nd}}\sum_{(x,y)\in E}|x,y\rangle$$

$$= \frac{1}{\sqrt{nd}}\sum_{(x,y)\in E}|y,x\rangle = |\pi\rangle,$$

hence $W(P)|\pi\rangle = S\cdot C(P)|\pi\rangle = S|\pi\rangle = |\pi\rangle$. $\square$

! not only stationary state:

$$W(P) = S \cdot C(P)$$

reflection around span$\{|xy\rangle + |yx\rangle \mid (x,y)\in E\}$

reflection around span$\{|\psi_x\rangle \mid x \in V\}$

$\rightarrow$ QW = product of two reflections!

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> **Lemma:** Consider projectors Π_1, Π_2.
> Operator $(2\Pi_2 - \mathbb{1})(2\Pi_1 - \mathbb{1})$ has inv. subspace
> $$\left(\ker(\Pi_1) \cap \ker(\Pi_2)\right) \cup \left(\ker(\Pi_1)^{\perp} \cap \ker(\Pi_2)^{\perp}\right).$$

QW operator: $W(P) = (2\Pi_+ - \mathbb{1})(2\Pi_* - \mathbb{1})$

where $\Pi_+ = \sum_{(x,y) \in E} \frac{1}{2} (|xy\rangle + |yx\rangle)(\langle xy| + \langle yx|)$

$$\Pi_* = \sum_{x \in V} |\psi_x\rangle\langle\psi_x|$$

$$\rightarrow \ker(\Pi_+)^{\perp} \cap \ker(\Pi_*)^{\perp} \rightarrow \in \operatorname{span}\{|\psi_x\rangle\}$$

$$= \operatorname{span}\left\{ \sum_{x \in V} \alpha_x |\psi_x\rangle \,\middle|\, \alpha_x = \alpha_y \;\forall (x,y) \in E \right\}$$

$$= \operatorname{span}\{|\pi\rangle\}$$

$\cup \operatorname{span}\{|k\rangle\}$

$= \operatorname{span}\{|\pi\rangle\}$

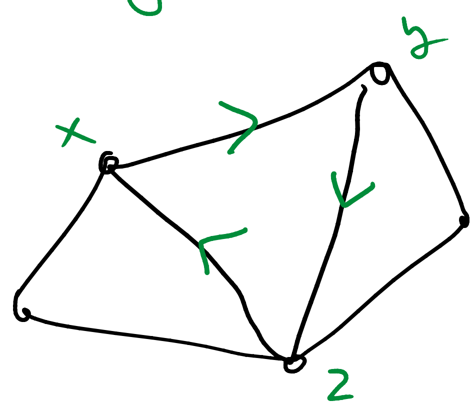
$\rightarrow \ker(\Pi_+) \cap \ker(\Pi_*)$ $\nearrow \perp \operatorname{span}\{|\psi_x\rangle\}$

$= \operatorname{span}\{|f\rangle \mid f \text{ "closed flow"}\}$

$\hookrightarrow |f\rangle = \sum_{(x,y) \in E} f(x,y)\,|x,y\rangle$

s.t. $\begin{cases} f(x,y) = -f(y,x) \\ \sum_{y \sim x} f(x,y) = 0 = \langle \psi_x | f \rangle \end{cases}$

$\Leftrightarrow f$ closed flow



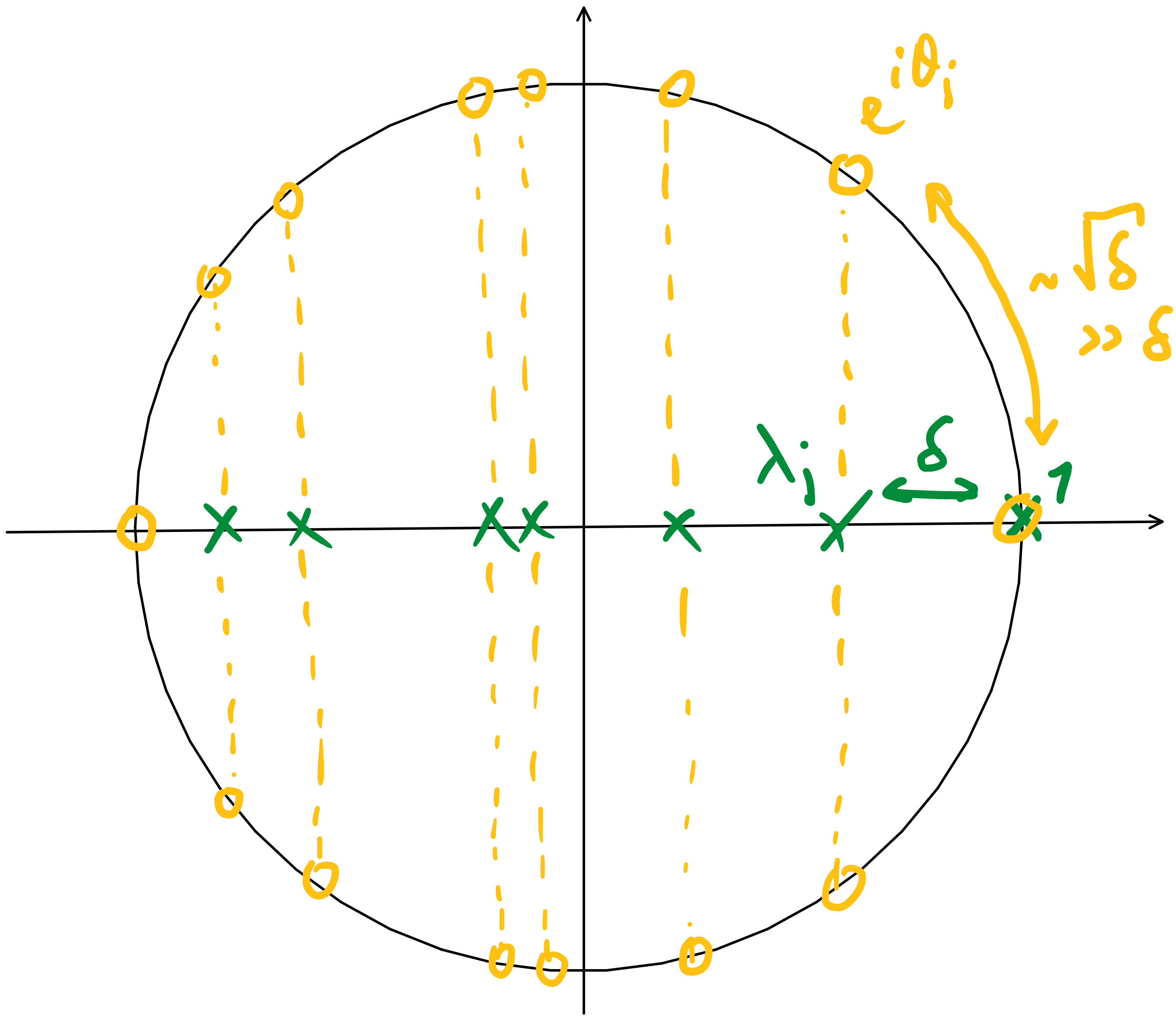
$|f\rangle = |xy\rangle - |yx\rangle$
$+ |yz\rangle - |zy\rangle$
$+ |zx\rangle - |xz\rangle$

? rest of the spectrum

* (-1)-subspace is

$$\left(\ker(\Pi_1) \cap \ker(\Pi_2)^{\perp}\right) \cup \left(\ker(\Pi_1)^{\perp} \cap \ker(\Pi_2)\right).$$

* **Szegedy's spectral lemma:**



$e^{i\theta_j}$
$\sim \sqrt{\delta}$
$\gg \delta$
λ_j
δ
1