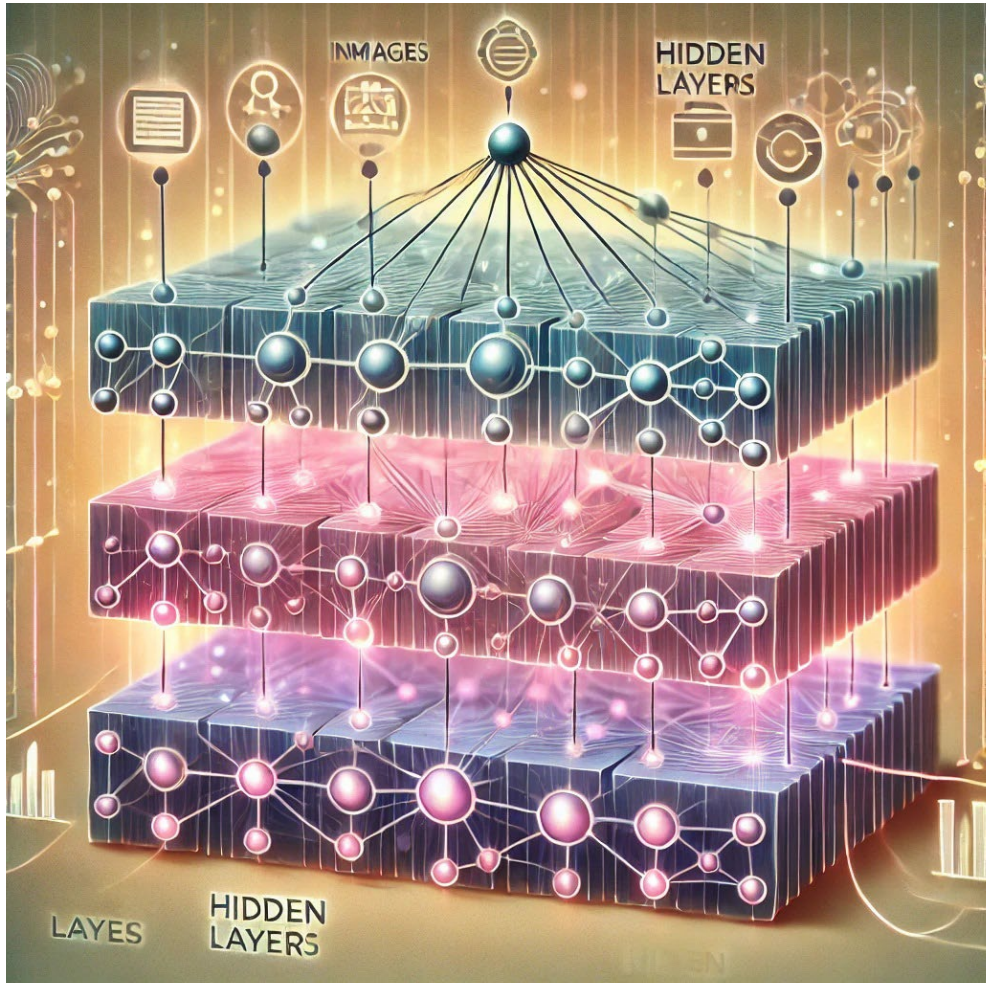
Neural Networks



AI604 Deep Learning for Computer Vision

Prof. Hyunjung Shim

Slide credit: Justin Johnson, Fei-Fei Li, Ehsan Adeli

Recap: Classification/Loss/Optimizer

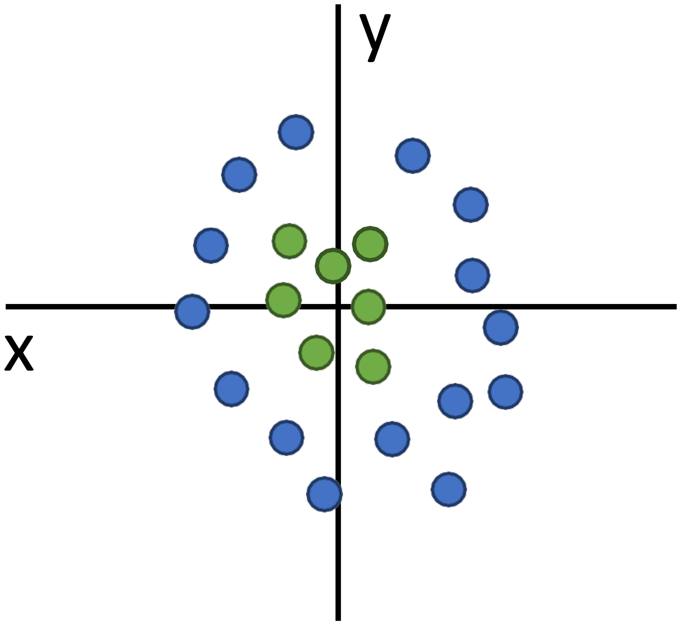
1. Use **Linear Models** for image classification problems (three perspectives)

2. Use **Loss Functions** (SVM, Softmax) to express preferences over different choices of weights

3. Use **Stochastic Gradient Descent** to minimize our loss functions and train the model

Problem: Linear Classifiers aren't that powerful

Geometric Viewpoint

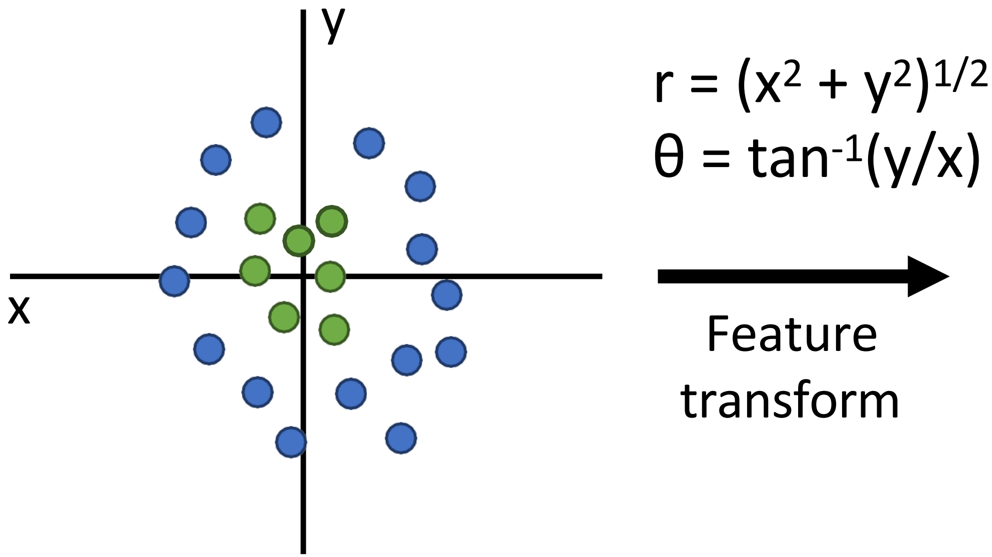


Visual Viewpoint

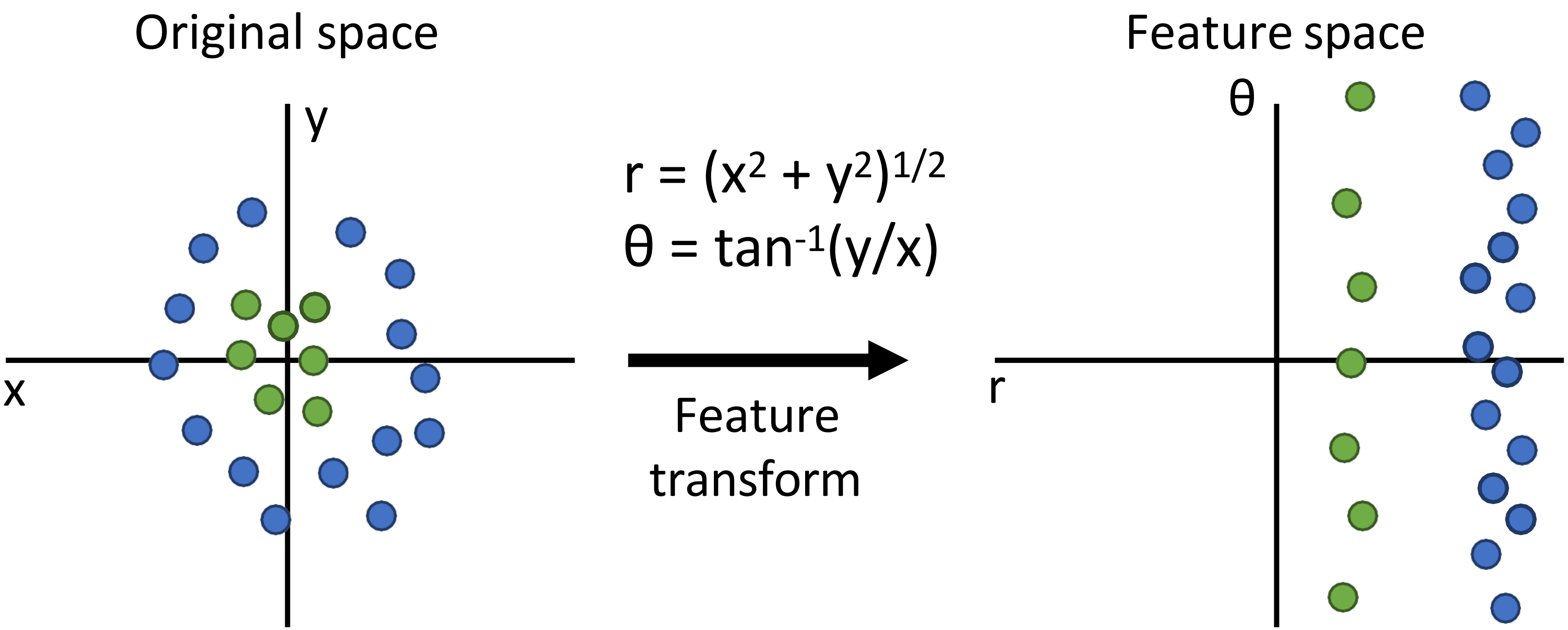
One template per class: Can't recognize different modes of a class

One solution: Feature Transforms

Original space



One solution: Feature Transforms



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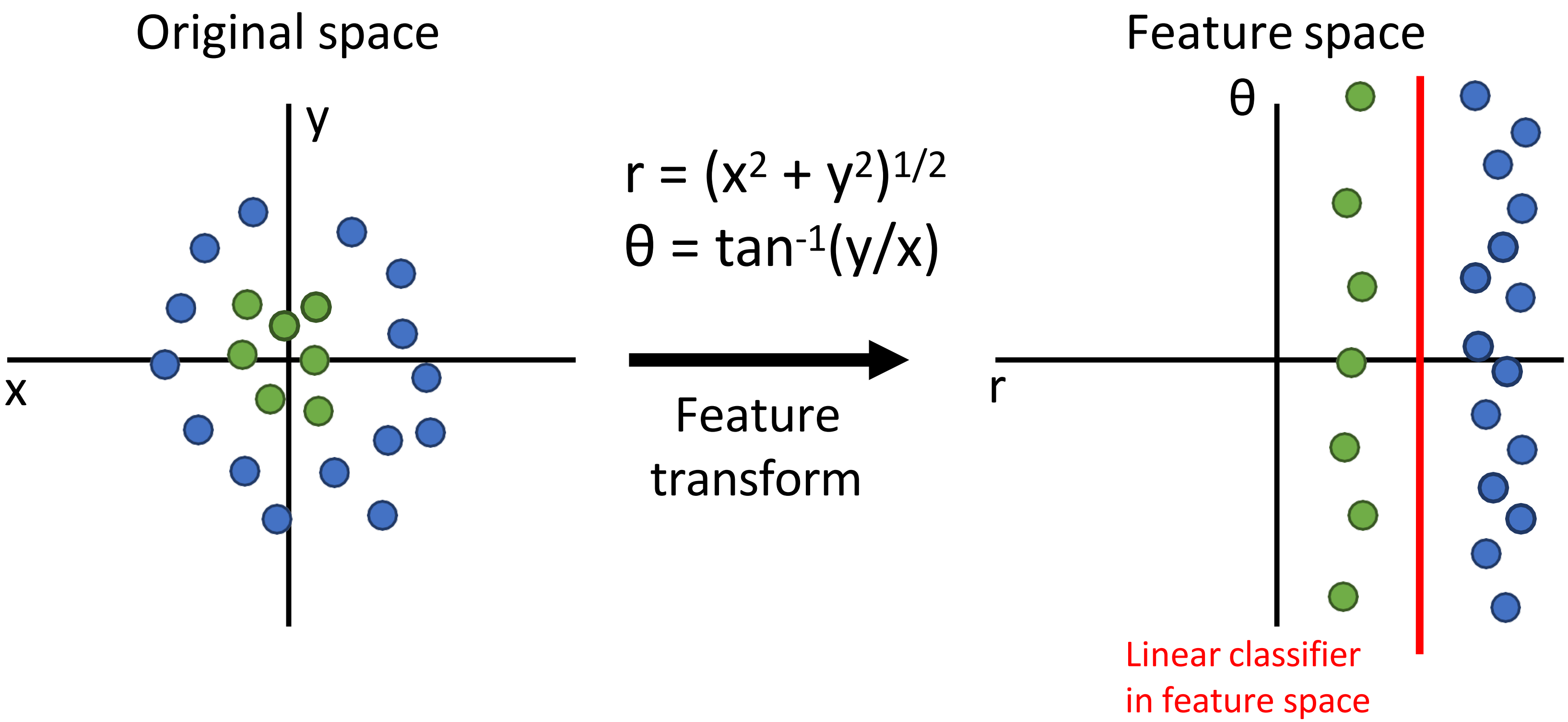


One solution: Feature Transforms

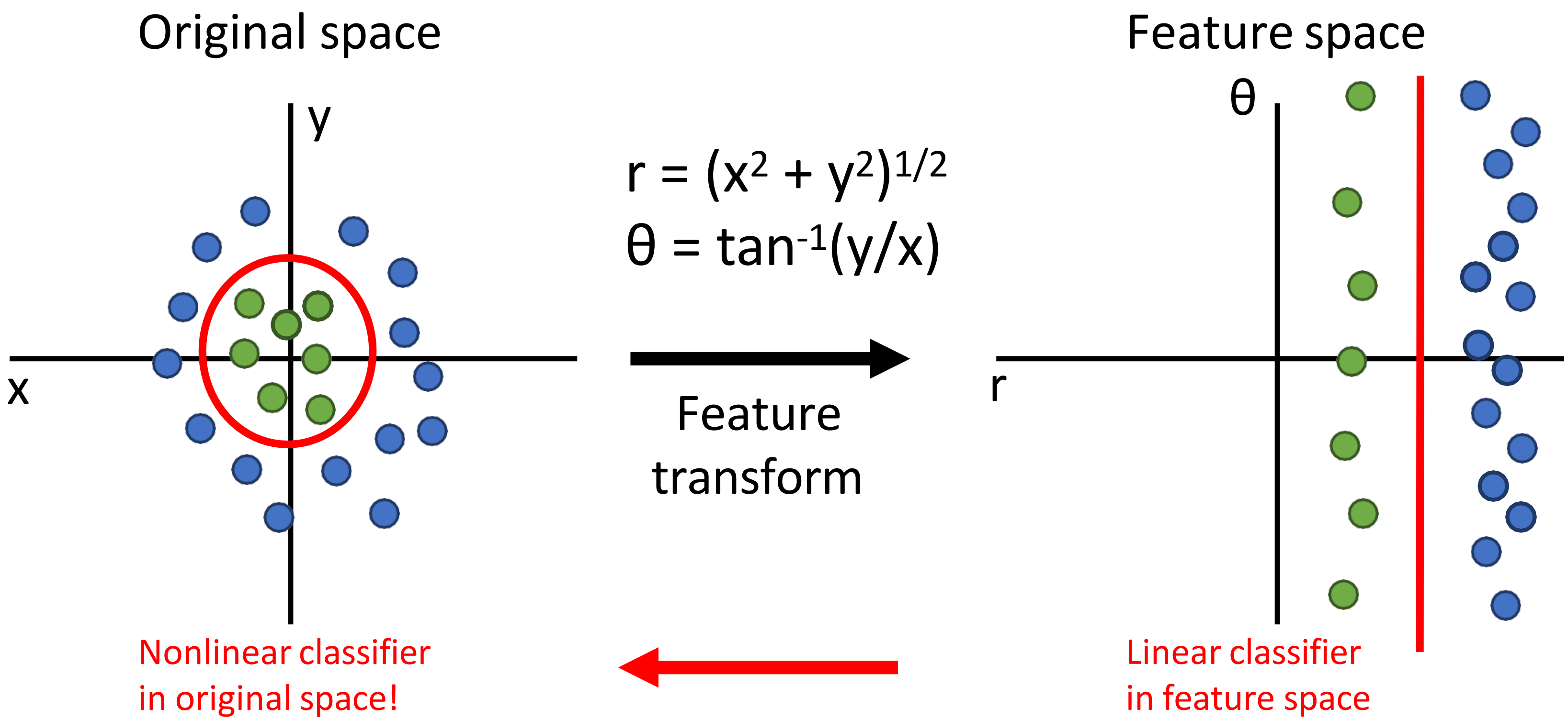
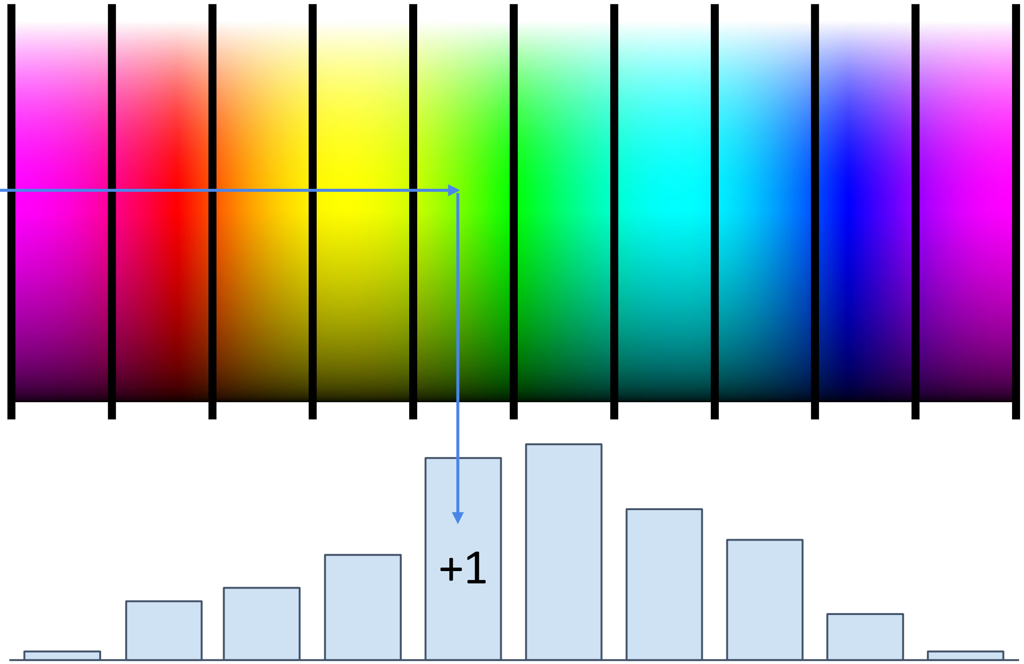


Image Features: Color Histogram



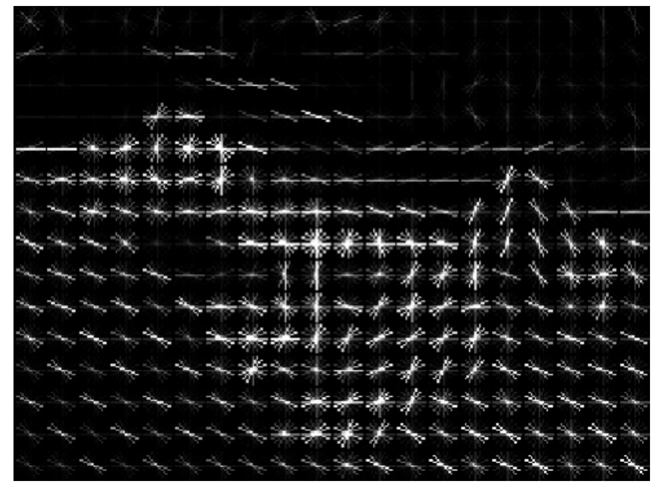
Ignores texture,
spatial positions

Image Features: Histogram of Oriented Gradients (HoG)

1. Compute edge direction / strength at each pixel
2. Divide image into 8x8 regions
3. Within each region compute a histogram of edge directions weighted by edge strength

Lowe, "Object recognition from local scale-invariant features", ICCV 1999
Dalal and Triggs, "Histograms of oriented gradients for human detection," CVPR 2005

Image Features: Histogram of Oriented Gradients (HoG)



1. Compute edge direction / strength at each pixel
2. Divide image into 8x8 regions
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Example: 320x240 image gets divided into 40x30 bins; 9 directions per bin; feature vector has 30*40*9 = 10,800 numbers

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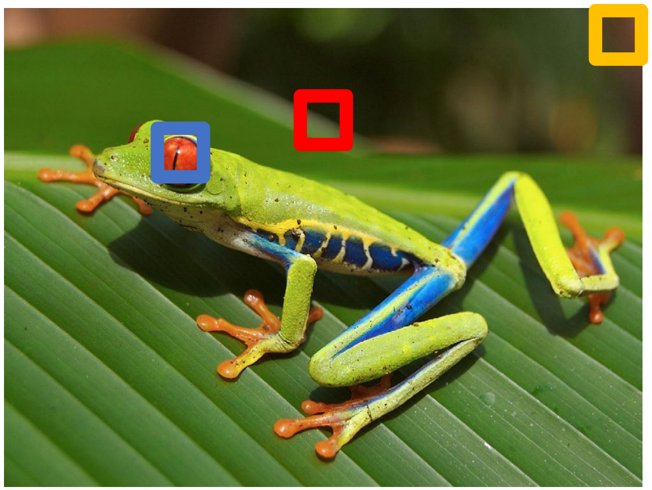
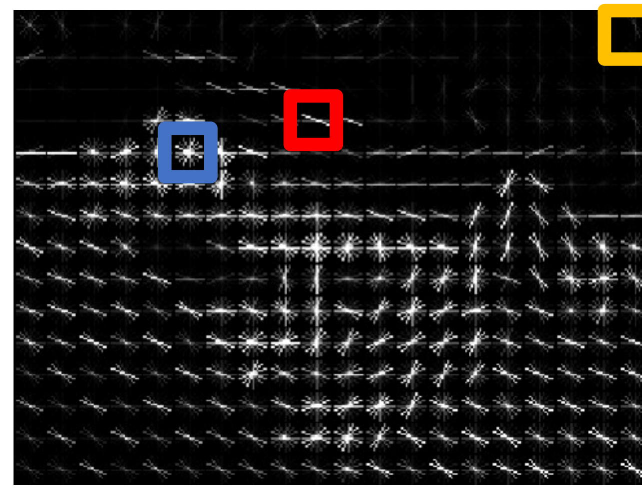
Image Features: Histogram of Oriented Gradients (HoG)



Weak edges

Strong diagonal edges

Edges in all directions



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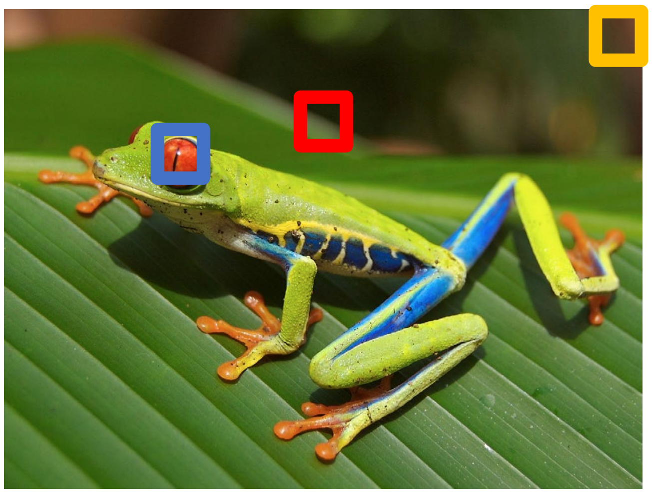

1. Compute edge direction / strength at each pixel
2. Divide image into 8x8 regions
3. Within each region compute a histogram of edge directions weighted by edge strength

Weak edges

Strong diagonal edges

Edges in all directions

Captures texture and position, robust to small image changes

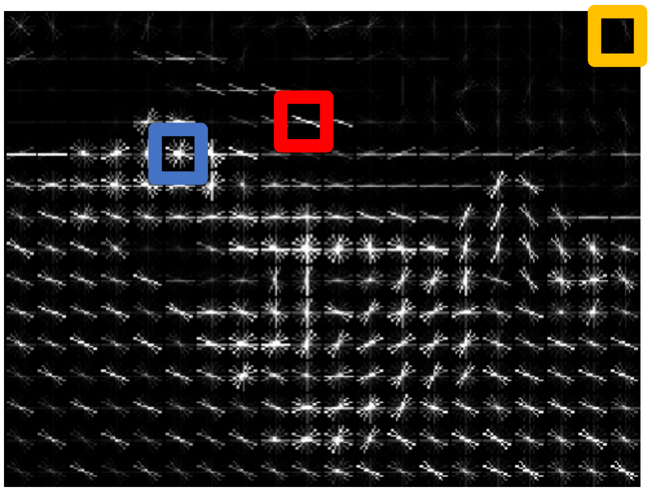

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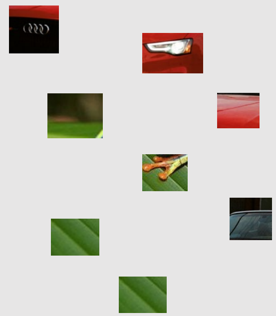
Image Features: Bag of Words (Data-Driven!)

Step 1: Build codebook

Extract random patches



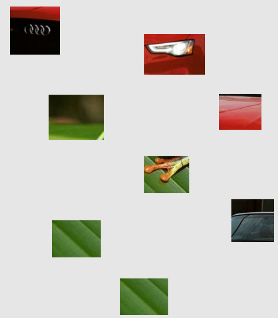
Cluster patches to form “codebook” of “visual words”

Fei-Fei and Perona, “A bayesian hierarchical model for learning natural scene categories”, CVPR 2005

Image Features: Bag of Words (Data-Driven!)

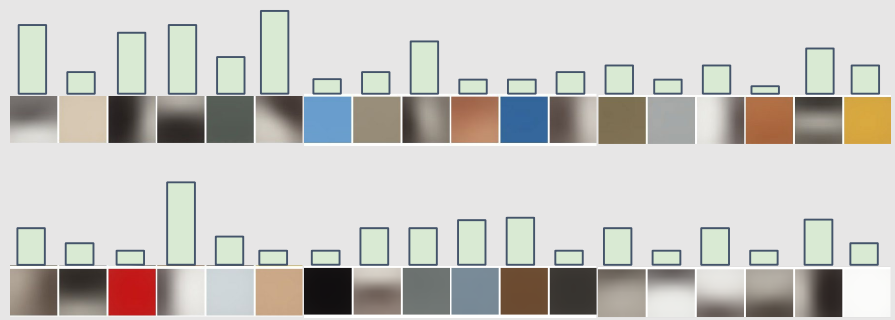
Step 1: Build codebook

Extract random patches



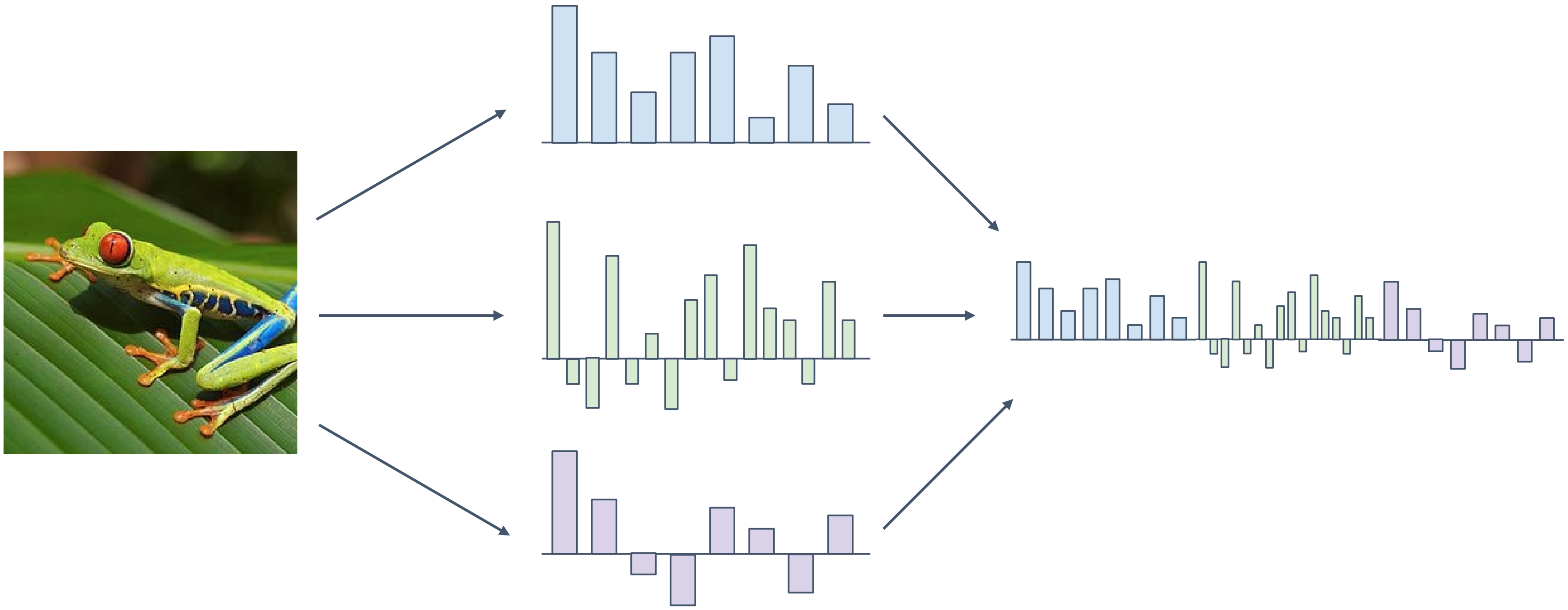
Cluster patches to form "codebook" of "visual words"

Step 2: Encode images



Fei-Fei and Perona, "A bayesian hierarchical model for learning natural scene categories", CVPR 2005

Image Features



Example: Winner of 2011 ImageNet challenge

Low-level feature extraction ≈ 10k patches per image

- SIFT: 128-dim
- color: 96-dim

reduced to 64-dim with PCA

FV extraction and compression:

- N=1,024 Gaussians, R=4 regions ⇨ 520K dim x 2
- compression: G=8, b=1 bit per dimension

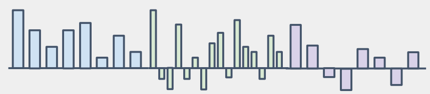
One-vs-all SVM learning with SGD

Late fusion of SIFT and color systems

F. Perronnin, J. Sánchez, “Compressed Fisher vectors for LSVRC”, PASCAL VOC / ImageNet workshop, ICCV, 2011.

Image Features

Feature Extraction



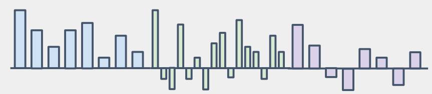
f

10 numbers giving scores for classes

training

Image Features

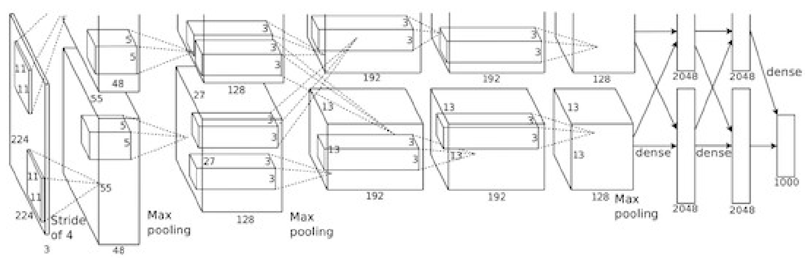
Feature Extraction



f

10 numbers giving scores for classes

training



Krizhevsky, Sutskever, and Hinton, “Imagenet classification with deep convolutional neural networks”, NIPS 2012.

10 numbers giving scores for classes

training

Neural Networks

(**Before**) Linear score function: $$f = Wx$$

$$x \in \mathbb{R}^D, W \in \mathbb{R}^{C \times D}$$

Neural Networks

(**Before**) Linear score function: $f = Wx$

(**Now**) 2-layer Neural Network $f = W_2 \max(0, W_1 x)$

$$W_2 \in \mathbb{R}^{C \times H} \quad W_1 \in \mathbb{R}^{H \times D} \quad x \in \mathbb{R}^D$$

(In practice we will usually add a learnable bias at each layer as well)

Neural Networks

(**Before**) Linear score function: $f = Wx$

(**Now**) 2-layer Neural Network $f = W_2 \max(0, W_1 x)$
or 3-layer Neural Network

$$f = W_3 \max(0, W_2 \max(0, W_1 x))$$

$$W_3 \in \mathbb{R}^{C \times H_2} \quad W_2 \in \mathbb{R}^{H_2 \times H_1} \quad W_1 \in \mathbb{R}^{H_1 \times D} \quad x \in \mathbb{R}^D$$

(In practice we will usually add a learnable bias at each layer as well)

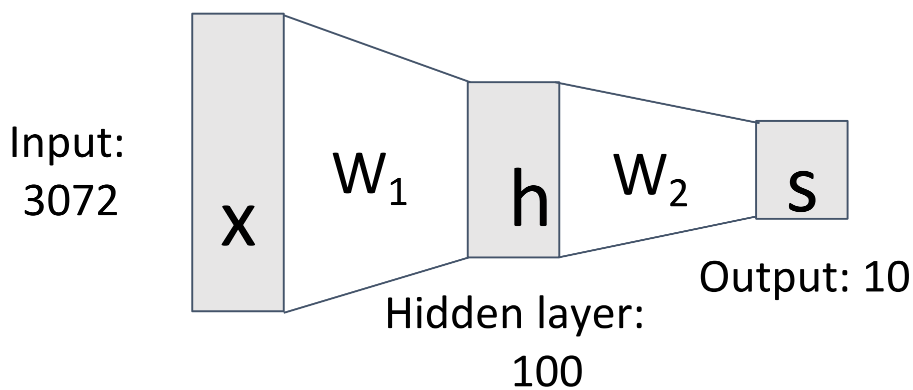
Neural Networks

(**Before**) Linear score function:

$$f = Wx$$

(**Now**) 2-layer Neural Network

$$f = W_2 \max(0, W_1 x)$$



$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

(**Before**) Linear score function:

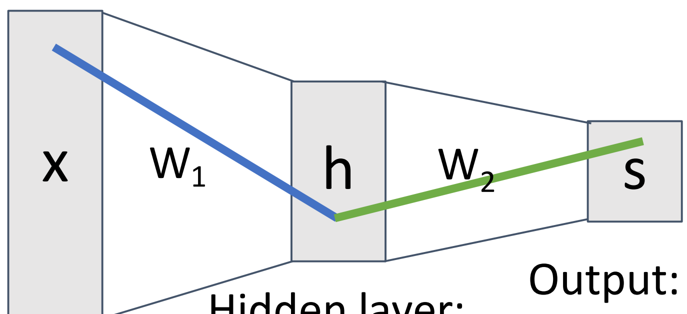
$$f = Wx$$

(**Now**) 2-layer Neural Network

$$f = W_2 \max(0, W_1 x)$$

Element (i, j) of W_1 gives the effect on h_i from x_j

Input: 3072



Hidden layer: 100

Output: 10

Element (i, j) of W_2 gives the effect on s_i from h_j

$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

(**Before**) Linear score function:

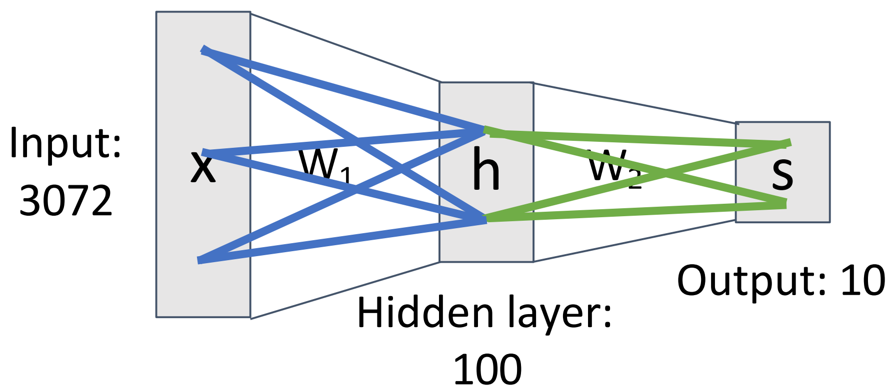
(**Now**) 2-layer Neural Network

$$f = Wx$$

$$f = W_2 \max(0, W_1 x)$$

Element (i, j) of W_1 gives the effect on h_i from x_j

All elements of x affect all elements of h



Element (i, j) of W_2 gives the effect on s_i from h_j

All elements of h affect all elements of s

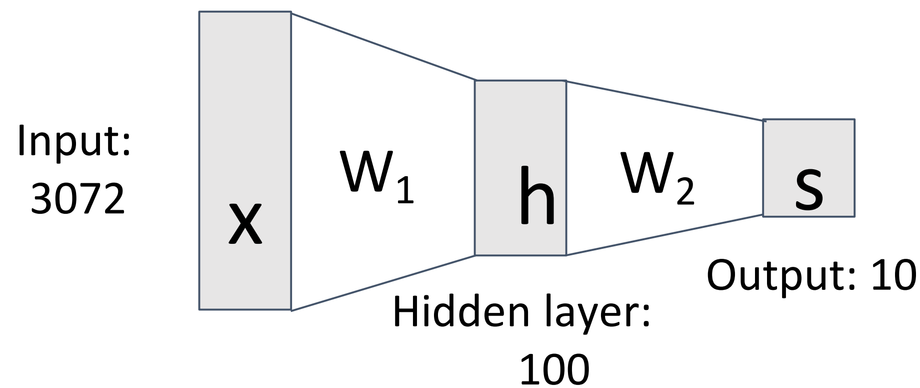
Fully-connected neural network
Also “Multi-Layer Perceptron” (MLP)

Neural Networks

(**Before**) Linear score function:

(**Now**) 2-layer Neural Network

Linear classifier: One template per class



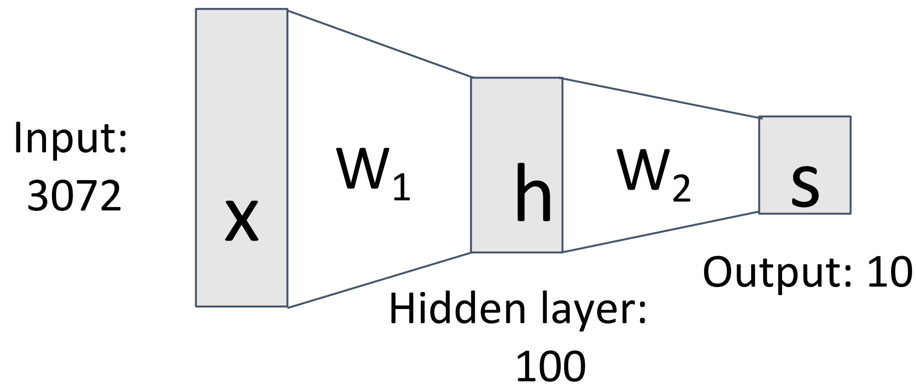
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

Neural net: first layer is bank of templates;
Second layer recombines templates

(**Before**) Linear score function:

(**Now**) 2-layer Neural Network



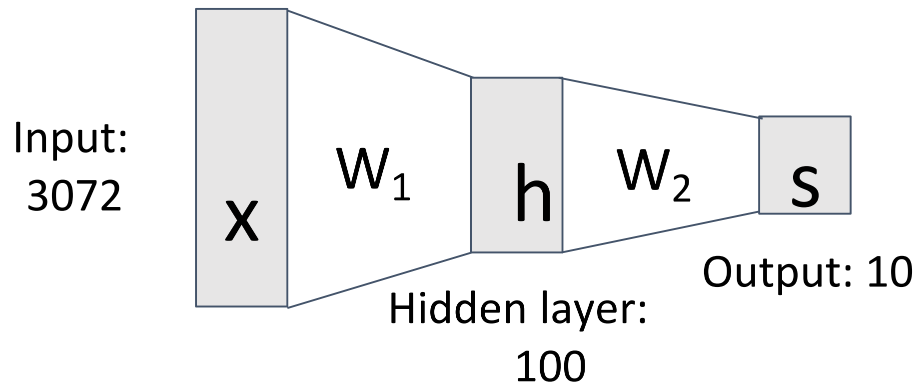
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

Can use different templates to cover multiple modes of a class!

(Before) Linear score function:

(Now) 2-layer Neural Network



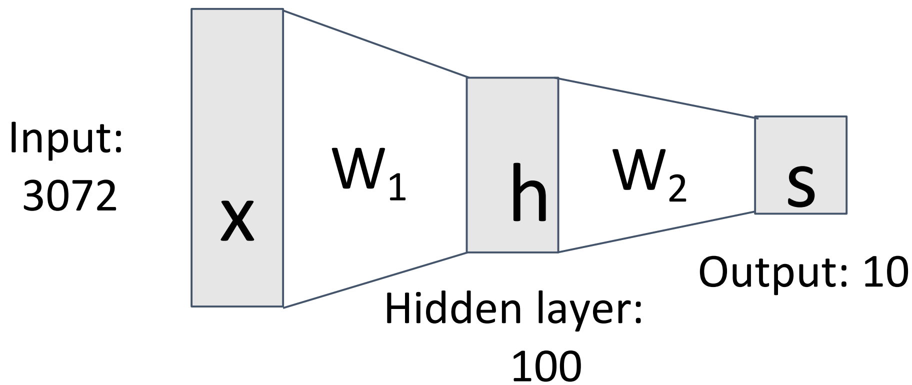
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural Networks

"Distributed representation":
Most templates not interpretable!

(**Before**) Linear score function:

(**Now**) 2-layer Neural Network



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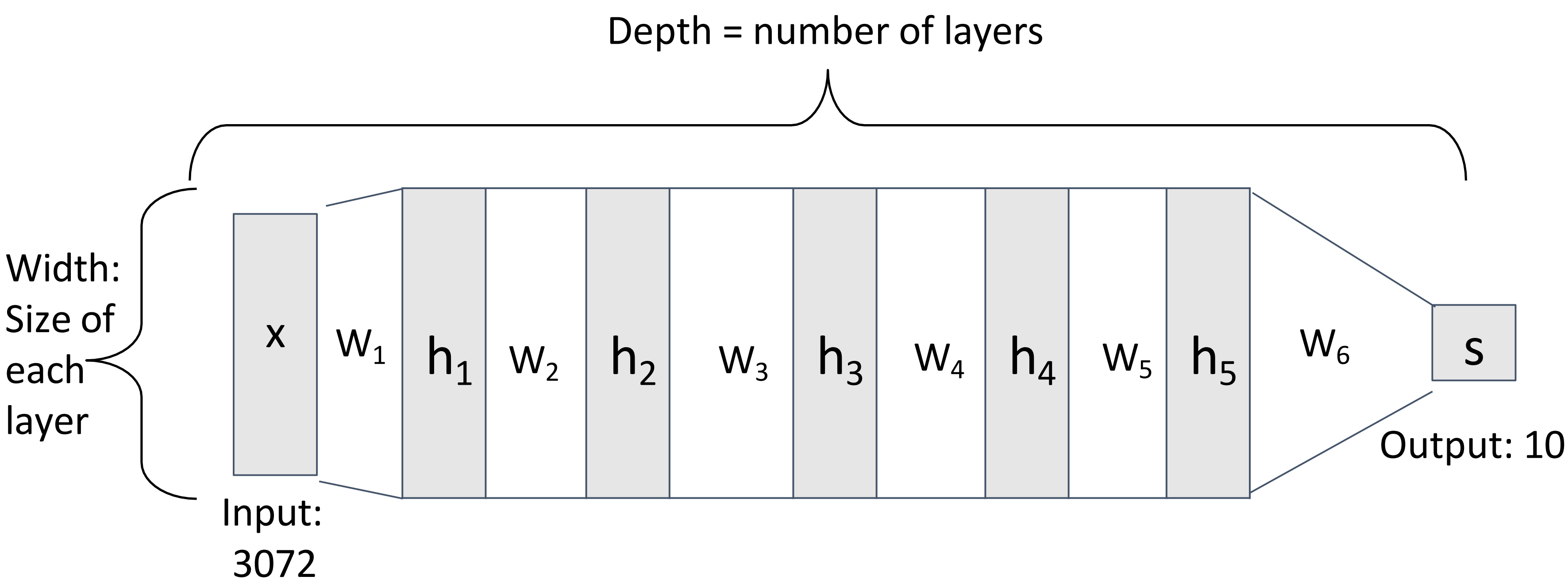
Deep Neural Networks

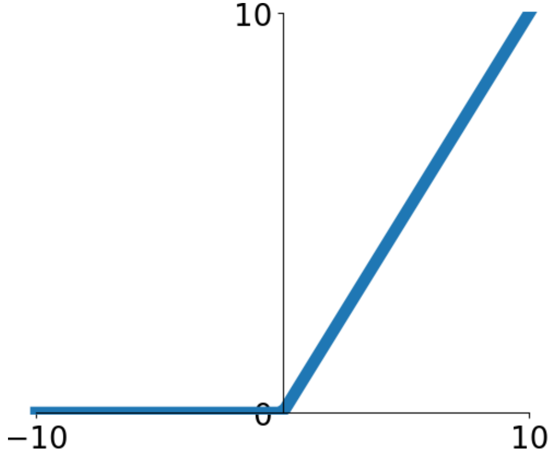


$$s = W_6 \max(0, W_6 \max(0, W_5 \max(0, W_4 \max(0, W_3 \max(0, W_2 \max(0, W_1 x))))))$$

Activation Functions

2-layer Neural Network

The function $ReLU(z) = max(0, z)$ is called "Rectified Linear Unit"



$$f = W_2 \boxed{\max(0,} W_1 x)$$

This is called the **activation function** of the neural network

Activation Functions

2-layer Neural Network

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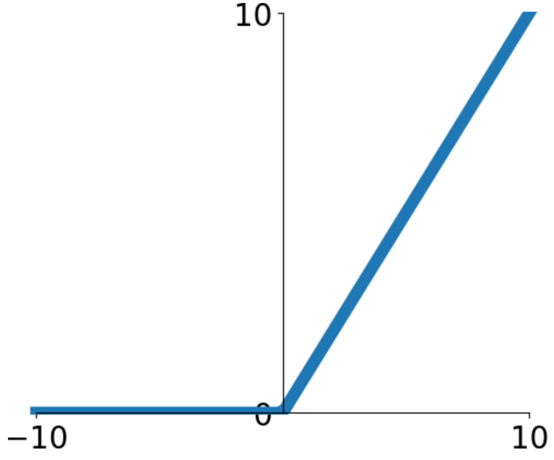

Q: What happens if we build a neural network with no activation function?

$$s = W_2 W_1 x$$

Activation Functions

2-layer Neural Network

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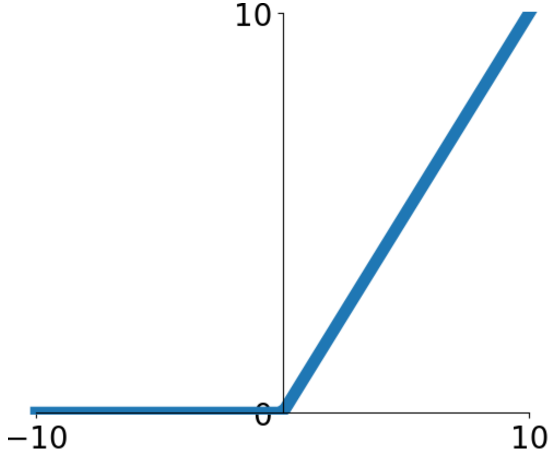


Q: What happens if we build a neural network with no activation function?

$$s = W_2 W_1 x$$

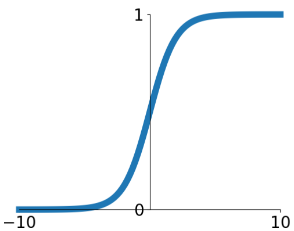
$$W_3 = W_2 W_1 \in \mathbb{R}^{C \times H} \quad s = W_3 x$$

A: We end up with a linear classifier!

Activation Functions

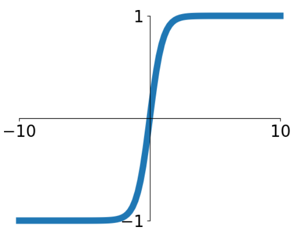
Sigmoid

$\sigma(x) = \frac{1}{1+e^{-x}}$



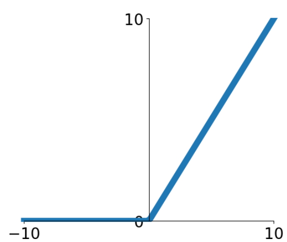
tanh

$\tanh(x)$



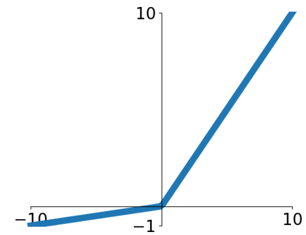
ReLU

$\max(0, x)$



Leaky ReLU

$\max(0.1x, x)$

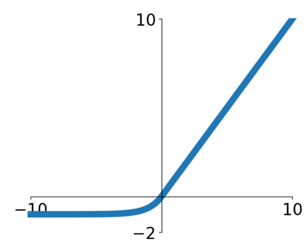


Maxout

$\max(w_1^T x + b_1, w_2^T x + b_2)$

ELU

$$\begin{cases} x & x \geq 0 \\ \alpha(e^x - 1) & x < 0 \end{cases}$$

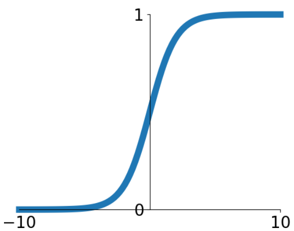


Activation Functions

ReLU is a good default choice for most problems

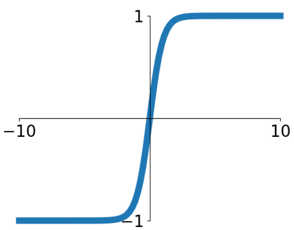
Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



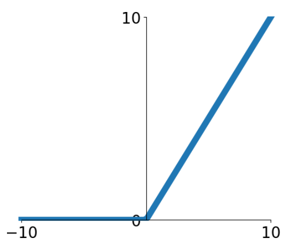
tanh

$$\tanh(x)$$



ReLU

$$\max(0, x)$$



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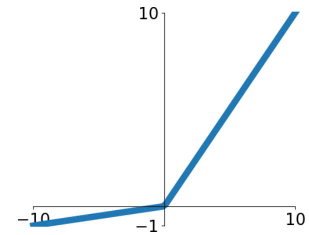

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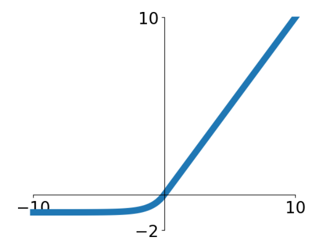
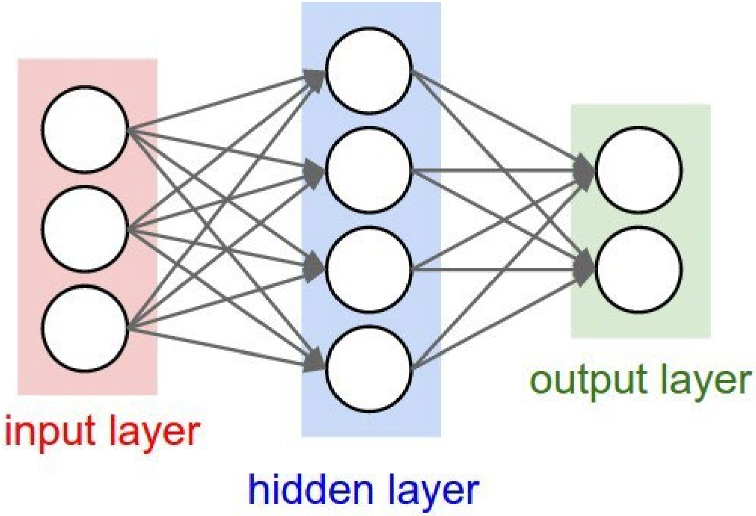


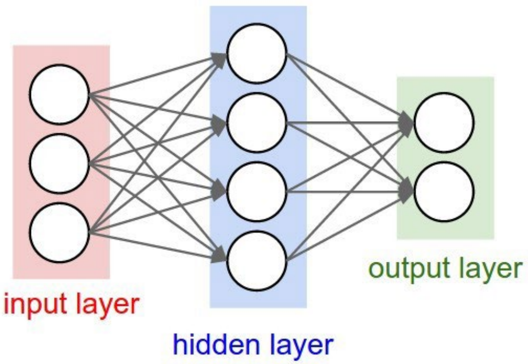
Neural Net in <20 lines!




```python
import numpy as np
from numpy.random import randn

N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
for t in range(10000):
  h = 1.0 / (1.0 + np.exp(-x.dot(w1)))
  y_pred = h.dot(w2)
  loss = np.square(y_pred - y).sum()
  dy_pred = 2.0 * (y_pred - y)
  dw2 = h.T.dot(dy_pred)
  dh = dy_pred.dot(w2.T)
  dw1 = x.T.dot(dh * h * (1 - h))
  w1 -= 1e-4 * dw1
  w2 -= 1e-4 * dw2
```

Neural Net in <20 lines!



Initialize weights and data

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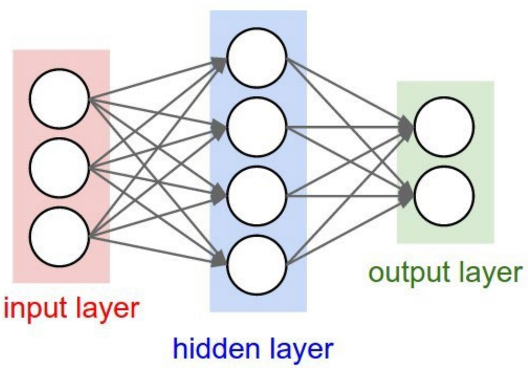
Neural Net in <20 lines!



Initialize weights and data

Compute loss (sigmoid activation, L2 loss)

```python
import numpy as np
from numpy.random import randn

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  dw1 = x.T.dot(dh * h * (1 - h))
  w1 -= 1e-4 * dw1
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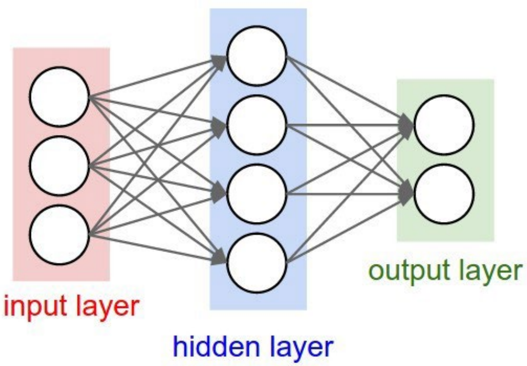
```

Neural Net in <20 lines!




```
import numpy as np
from numpy.random import randn

```

Initialize weights and data

```
N, Din, H, Dout = 64, 1000, 100, 10
x, y = randn(N, Din), randn(N, Dout)
w1, w2 = randn(Din, H), randn(H, Dout)
for t in range(10000):
```

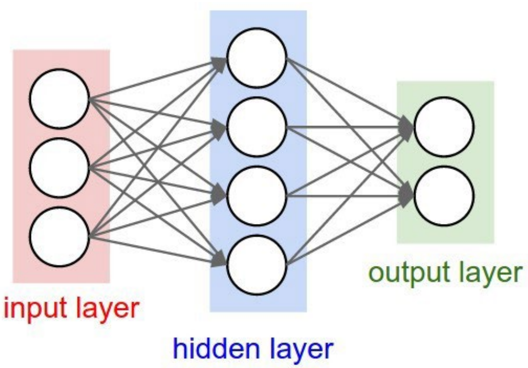
Compute loss (sigmoid activation, L2 loss)

```
  h = 1.0 / (1.0 + np.exp(-x.dot(w1)))
  y_pred = h.dot(w2)
  loss = np.square(y_pred - y).sum()
```

Compute gradients

```
  dy_pred = 2.0 * (y_pred - y)
  dw2 = h.T.dot(dy_pred)
  dh = dy_pred.dot(w2.T)
  dw1 = x.T.dot(dh * h * (1 - h))
  w1 -= 1e-4 * dw1
  w2 -= 1e-4 * dw2
```

Neural Net in <20 lines!




```
import numpy as np
from numpy.random import randn

```

Initialize weights and data

```
N, Din, H, Dout = 64, 1000, 100, 10
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for t in range(10000):
```

Compute loss (sigmoid activation, L2 loss)

```
  h = 1.0 / (1.0 + np.exp(-x.dot(w1)))
  y_pred = h.dot(w2)
  loss = np.square(y_pred - y).sum()
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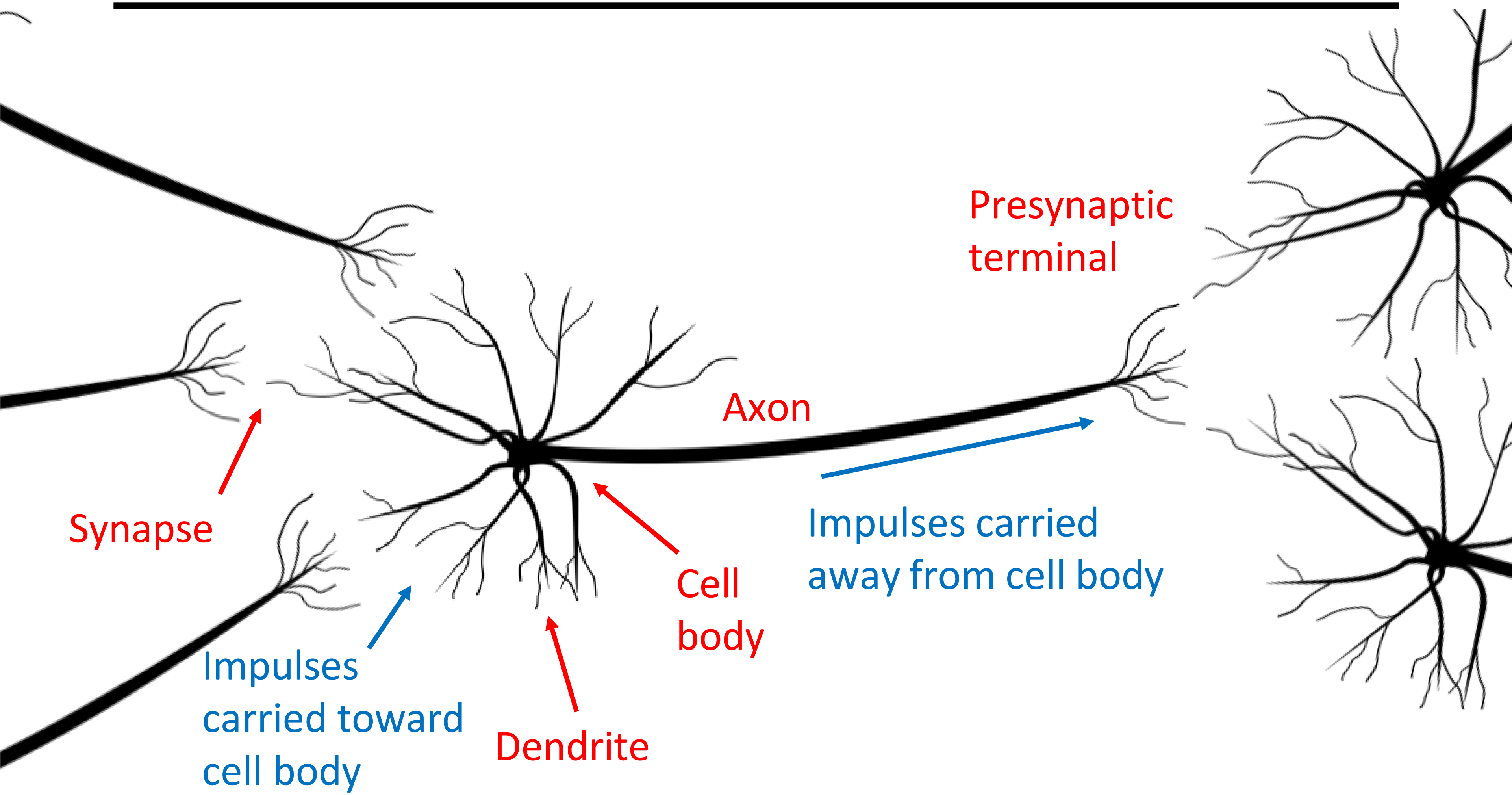
Compute gradients

```
  dy_pred = 2.0 * (y_pred - y)
  dw2 = h.T.dot(dy_pred)
  dh = dy_pred.dot(w2.T)
  dw1 = x.T.dot(dh * h * (1 - h))
```

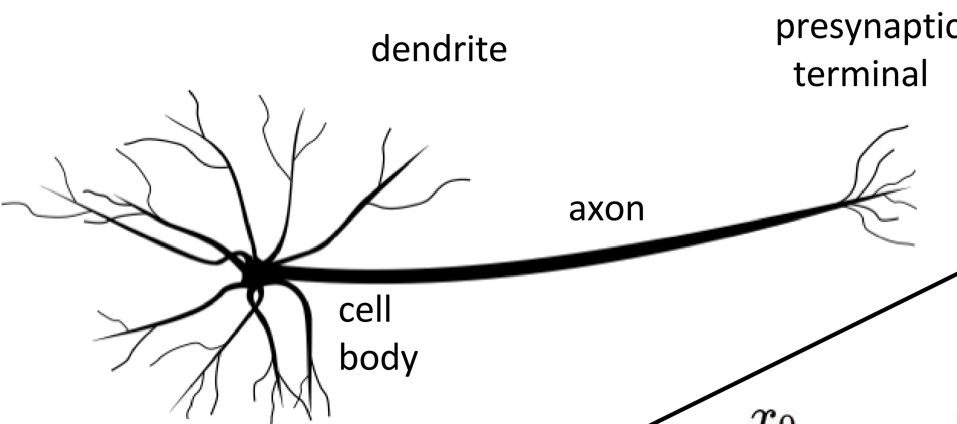
SGD step

```
  w1 -= 1e-4 * dw1
  w2 -= 1e-4 * dw2
```

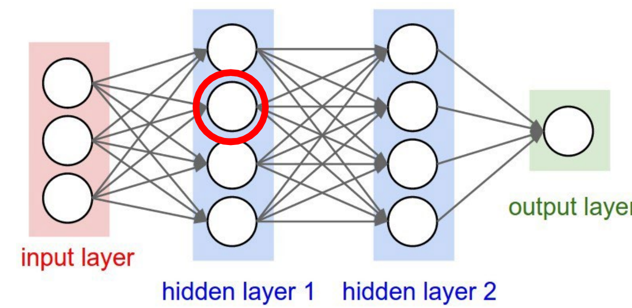
Our brains are made of Neurons

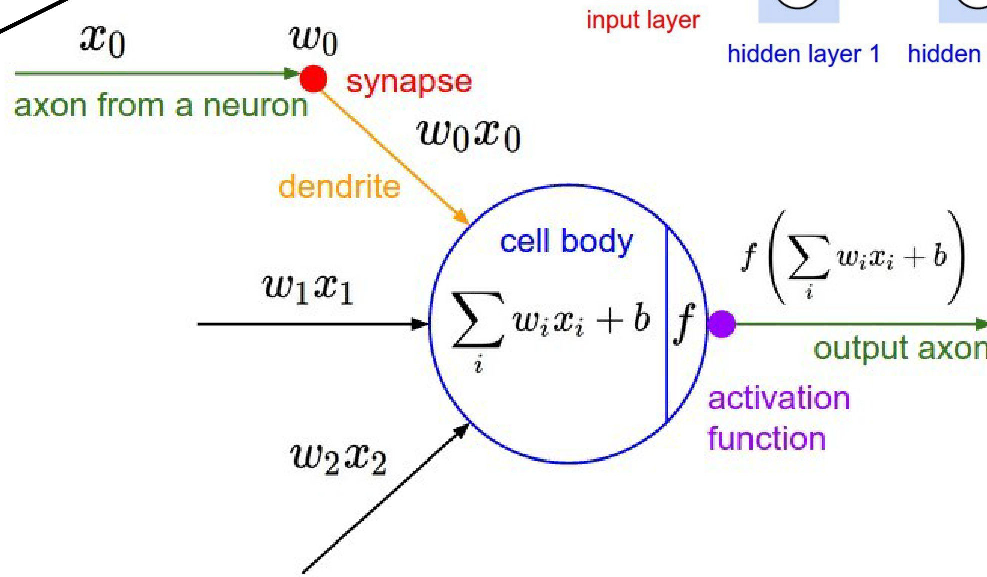


Biological Neuron

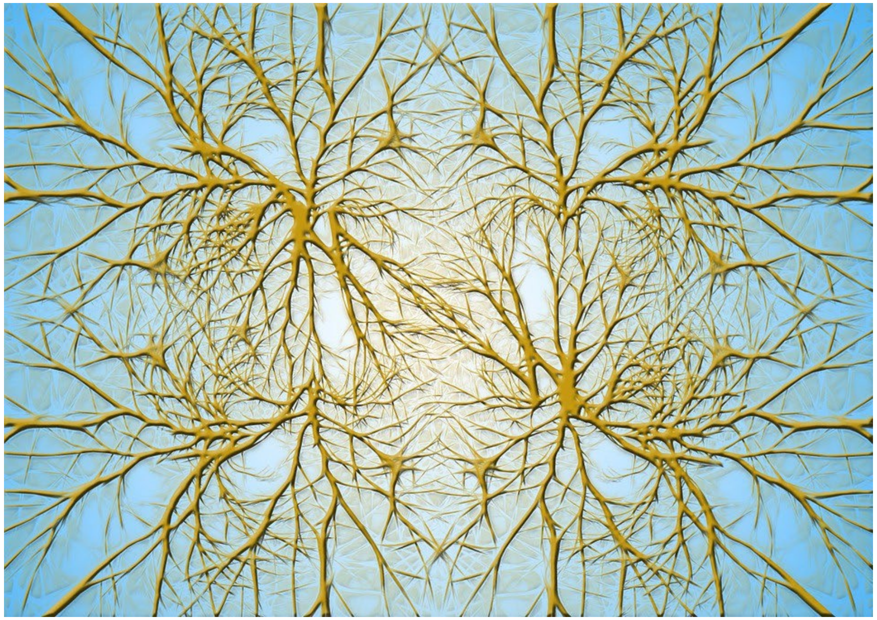


Artificial Neuron

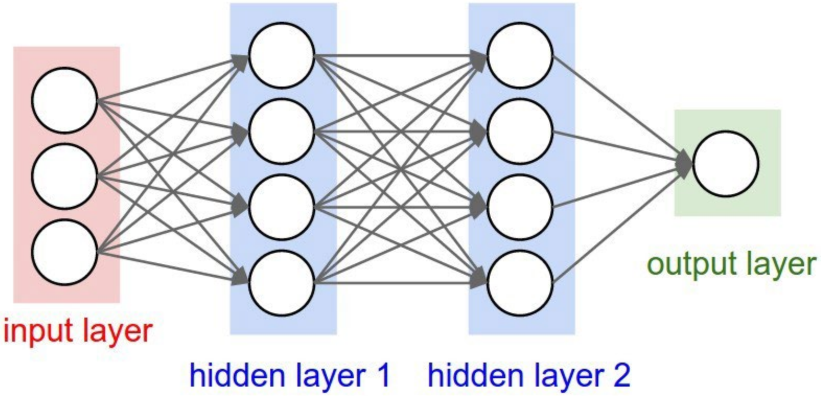




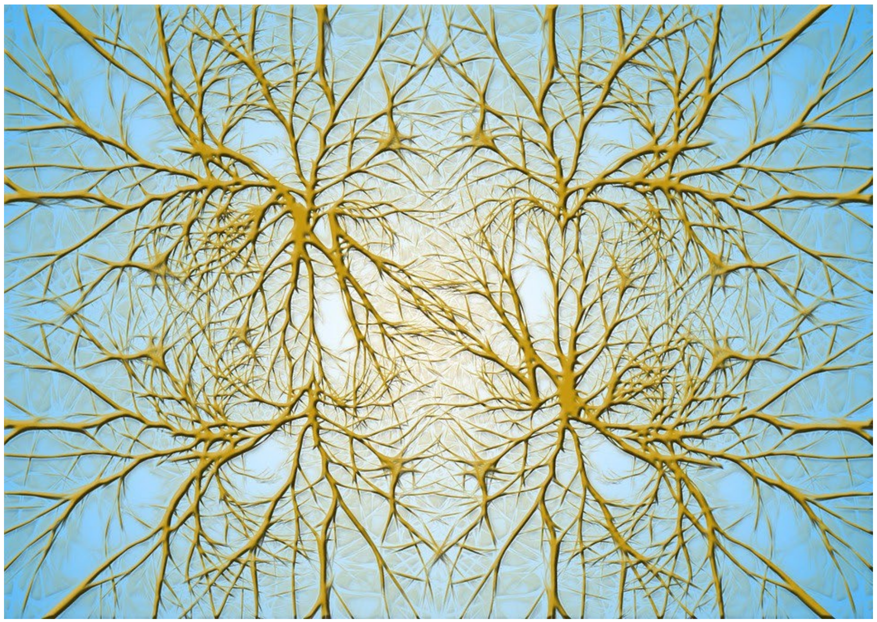
Biological Neurons:
Complex connectivity patterns



Neurons in a neural network:
Organized into regular layers for computational efficiency

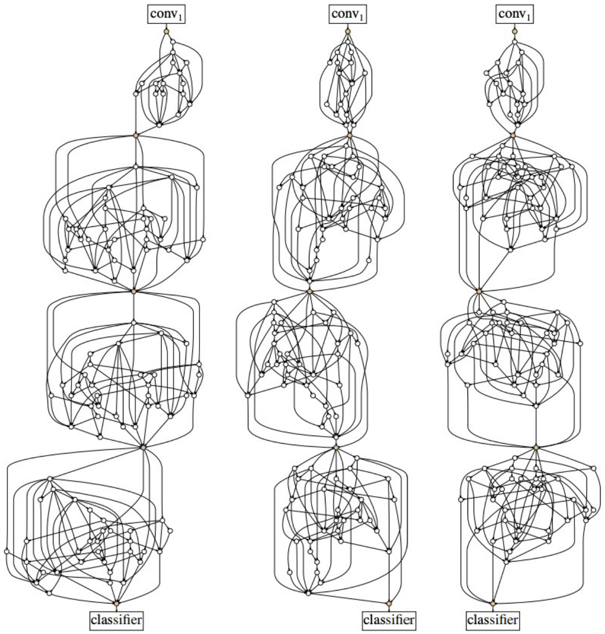


Biological Neurons:
Complex connectivity patterns



This image is CC0 Public Domain

But neural networks with random connections can work too!



Xie et al, "Exploring Randomly Wired Neural Networks for Image Recognition", ICCV 2019

Be very careful with brain analogies!

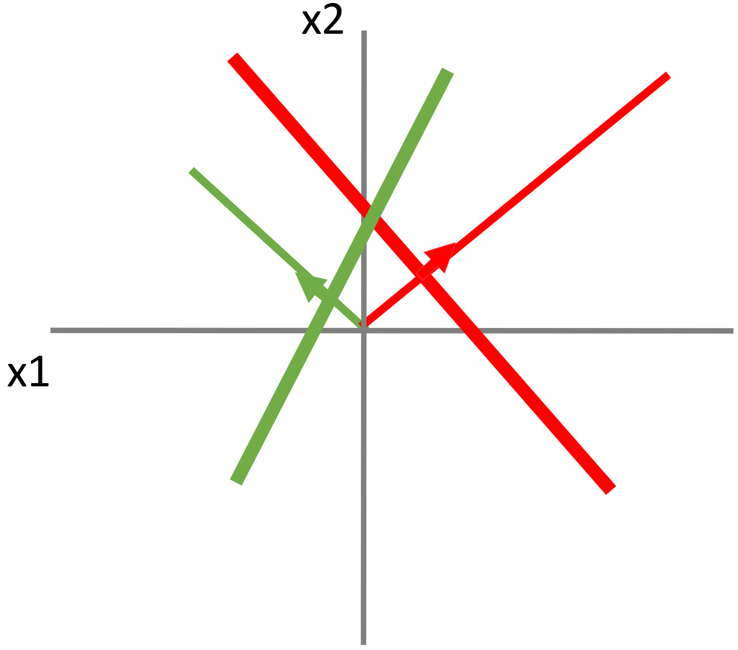
Biological Neurons:

- Many different types
- Dendrites can perform complex non-linear computations
- Synapses are not a single weight but a complex non-linear dynamical system
- Rate code may not be adequate

[Dendritic Computation. London and Hausser]

Space Warping

Consider a linear transform: h = Wx
Where x, h are both 2-dimensional



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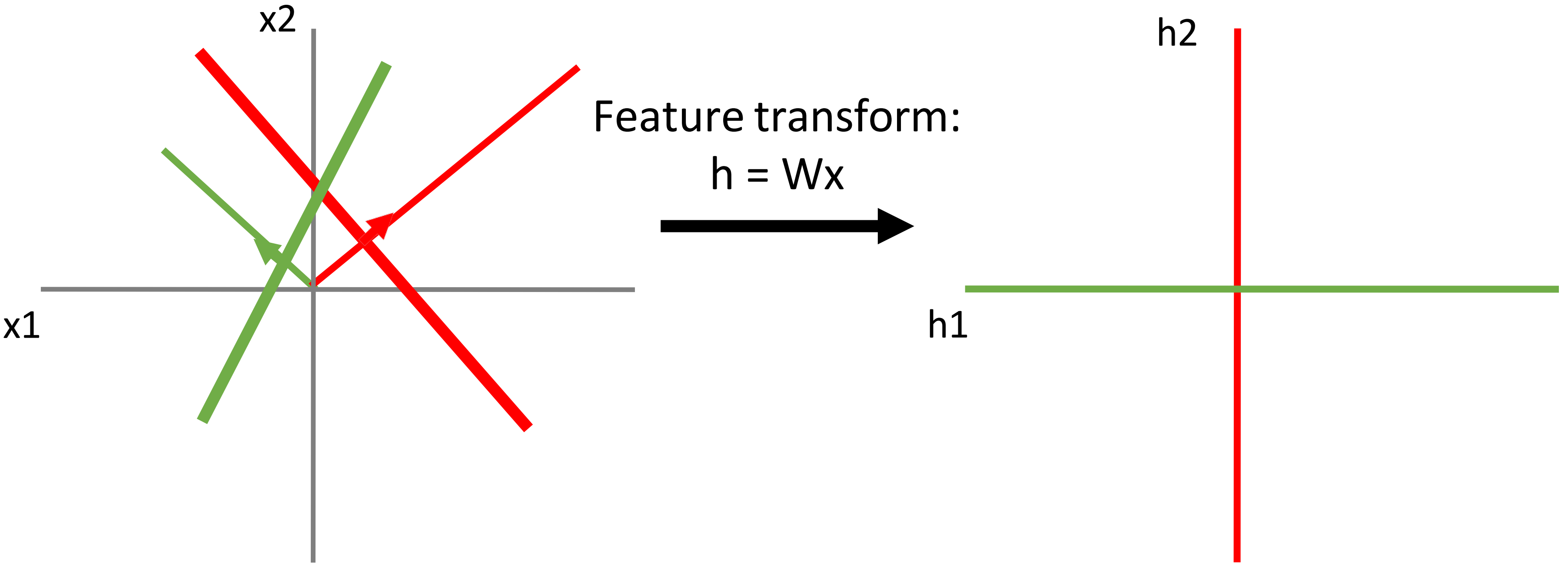

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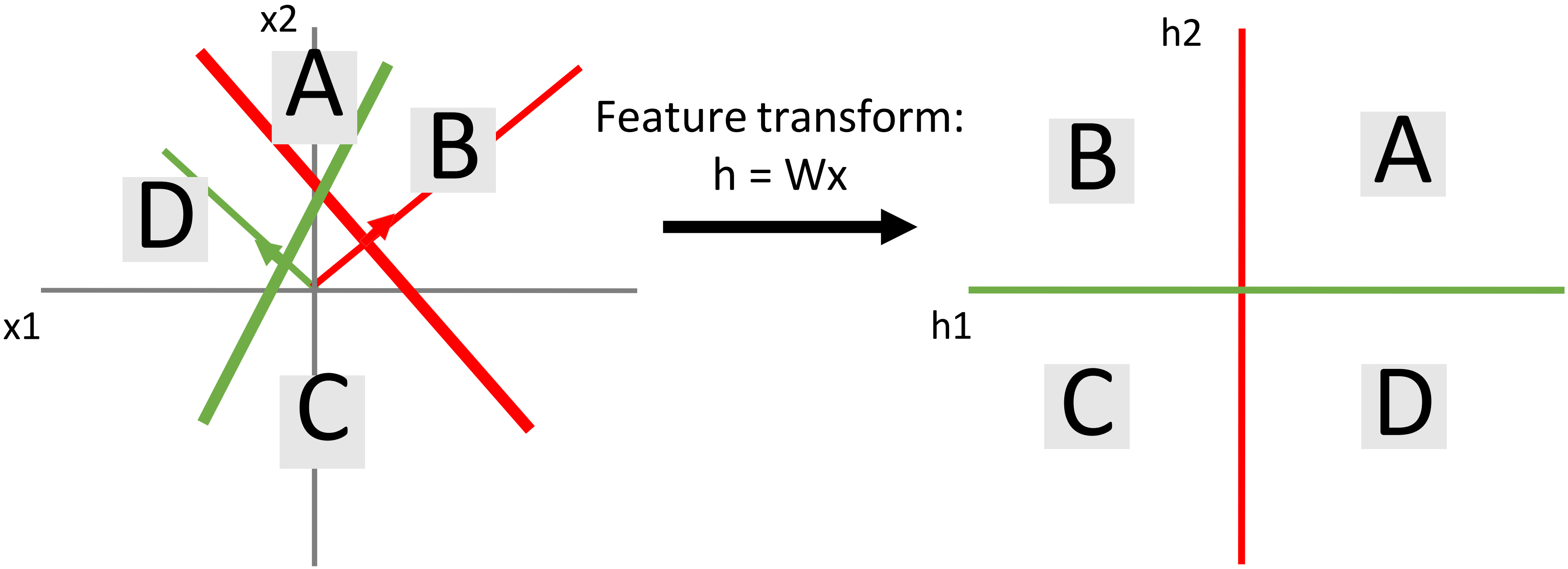
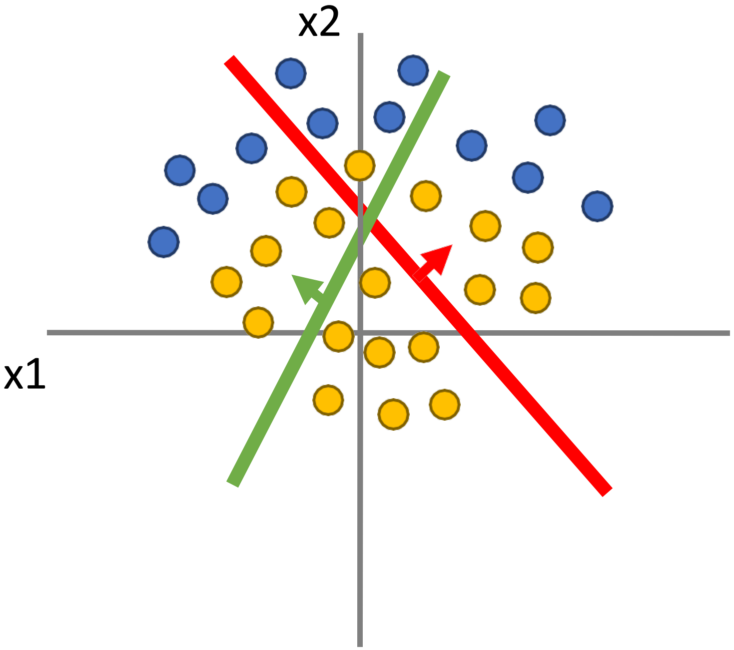


Space Warping

Points not linearly separable in original space



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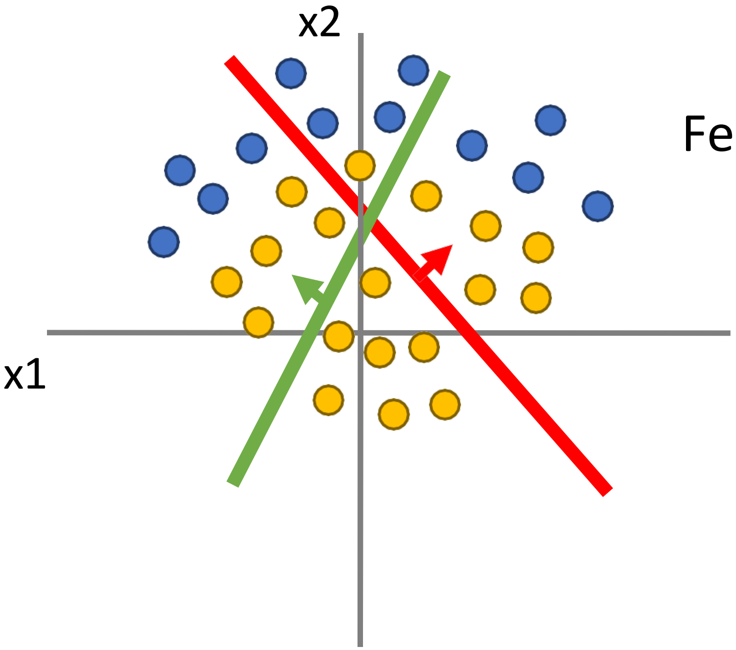
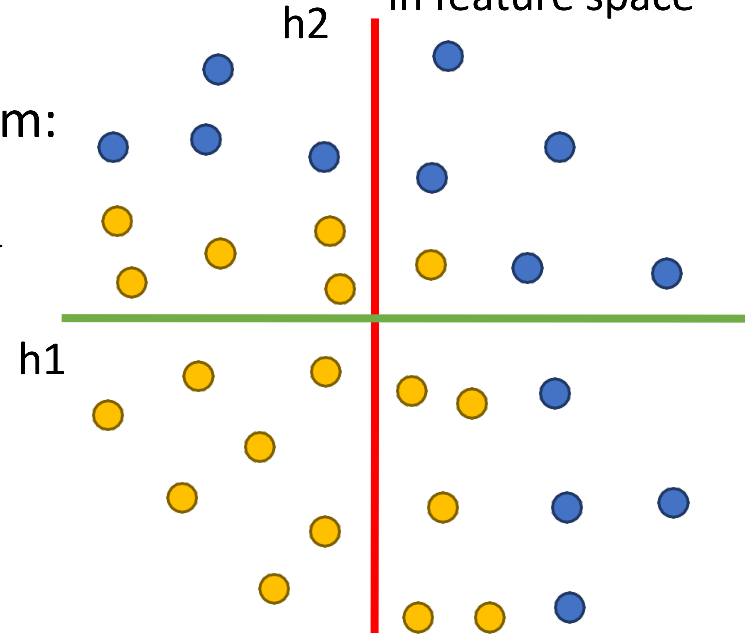


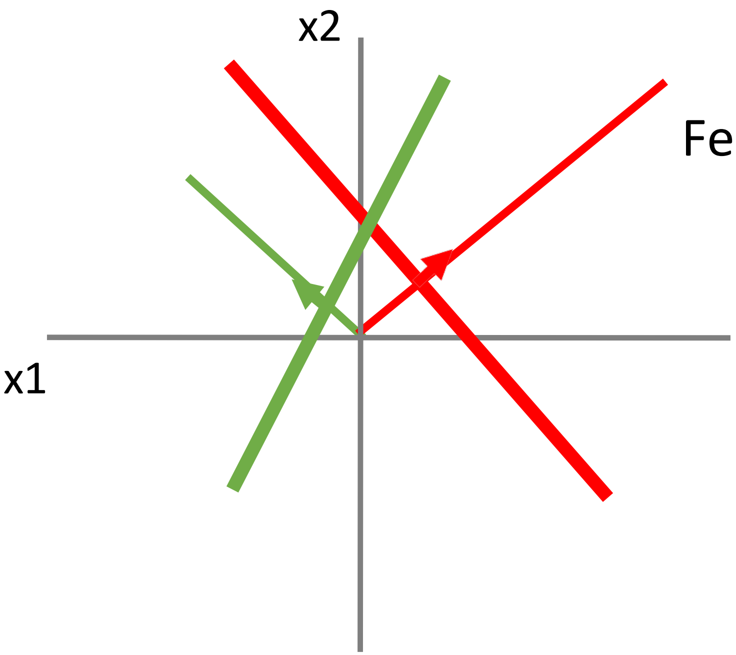
Feature transform:
h = Wx

Not linearly separable in feature space

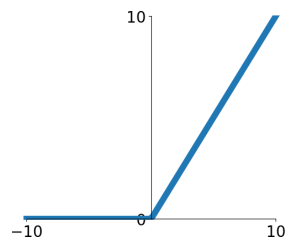


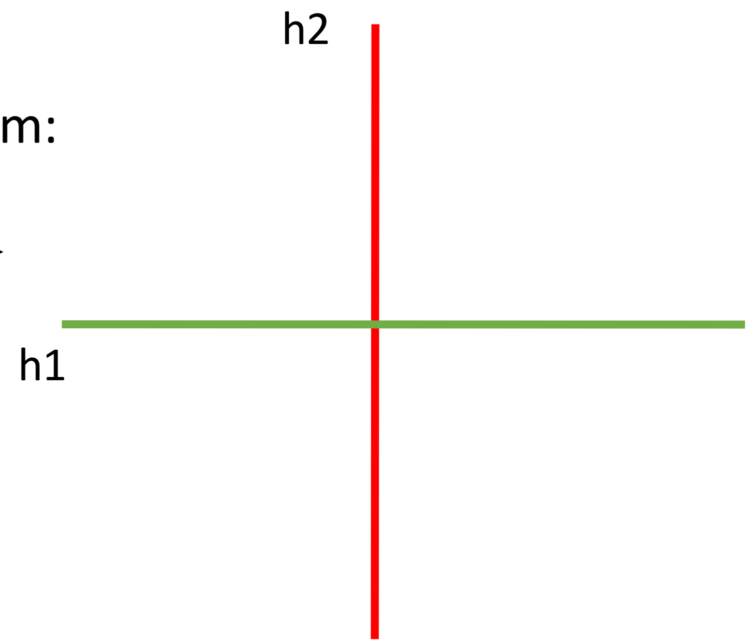
Space Warping

Consider a neural net hidden layer:
h = ReLU(Wx) = max(0, Wx)
Where x, h are both 2-dimensional



Feature transform:
h = ReLU(Wx)





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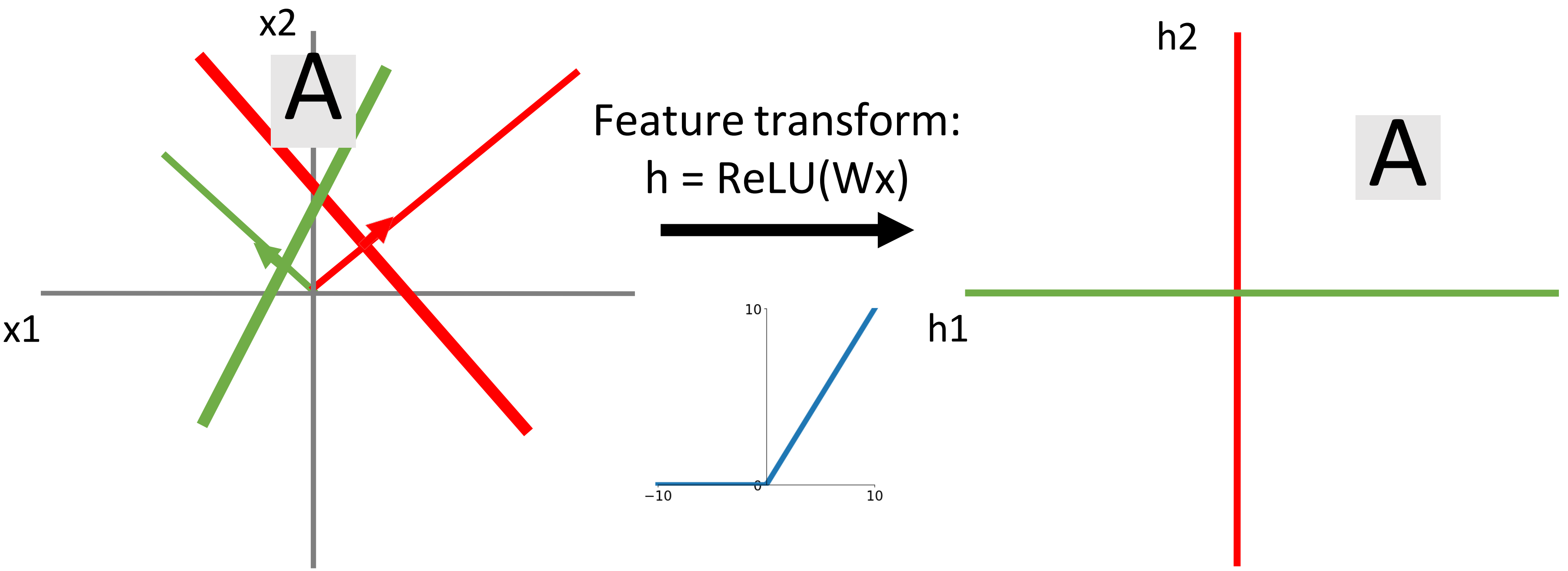

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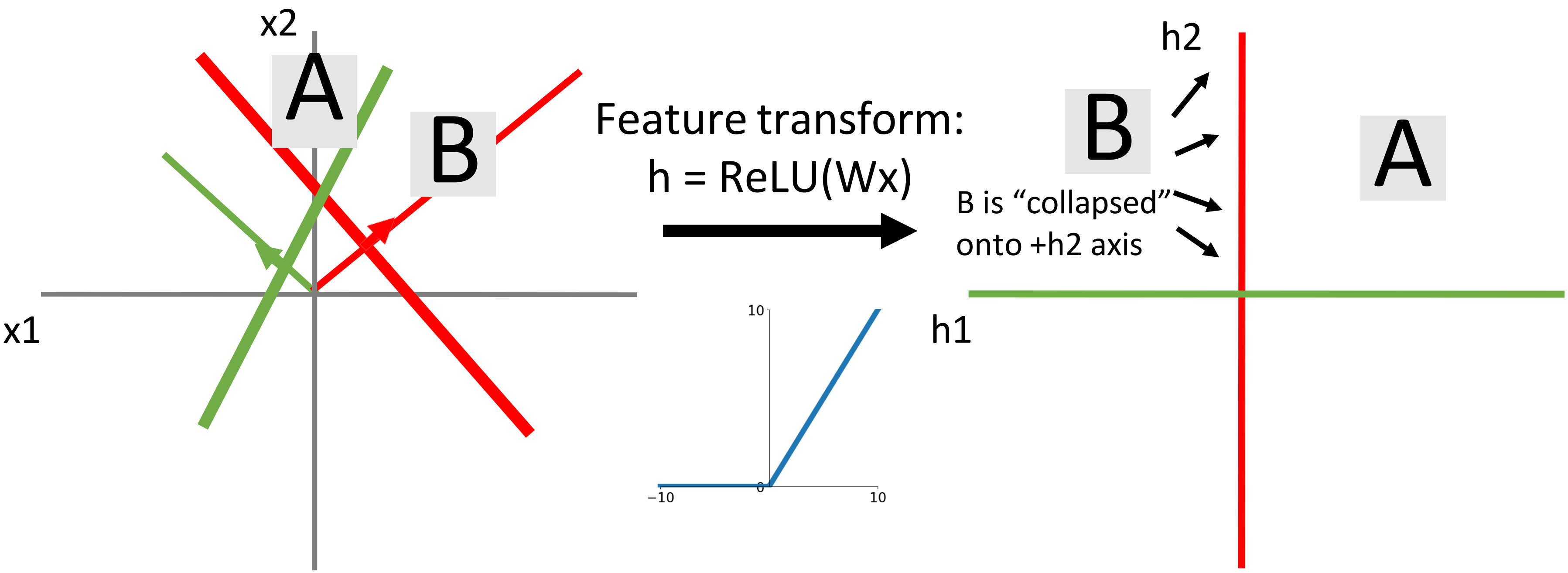

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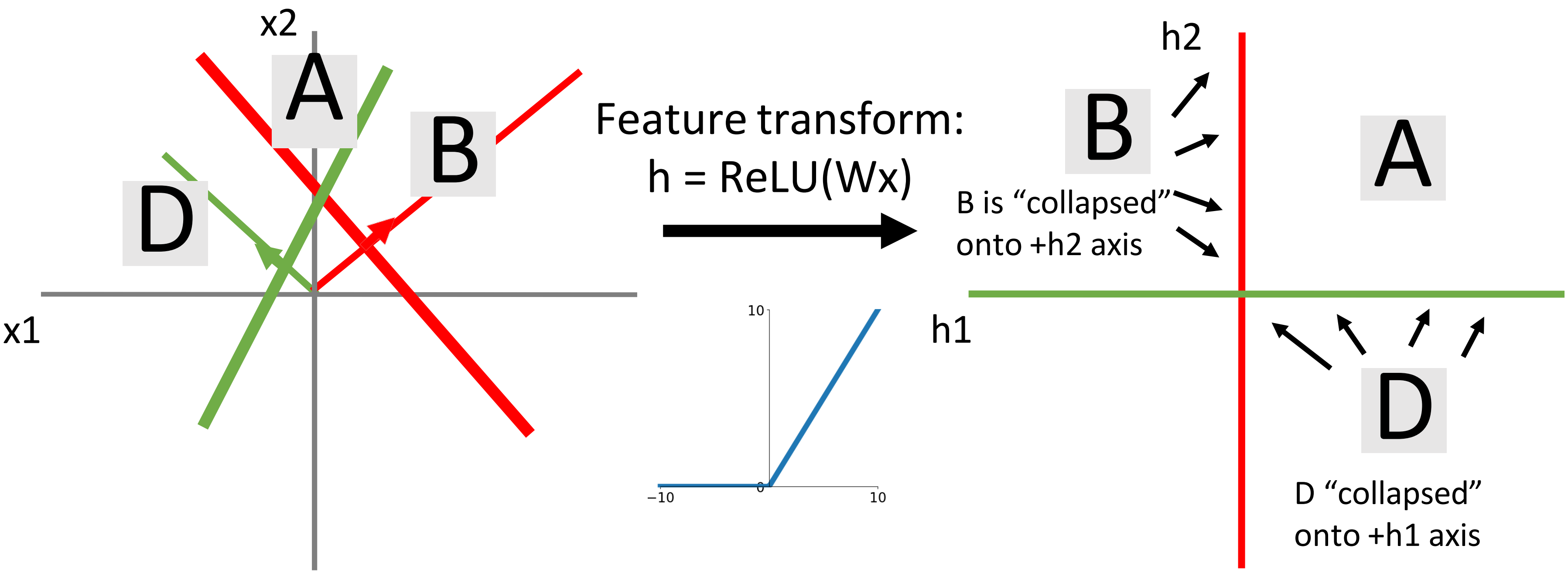

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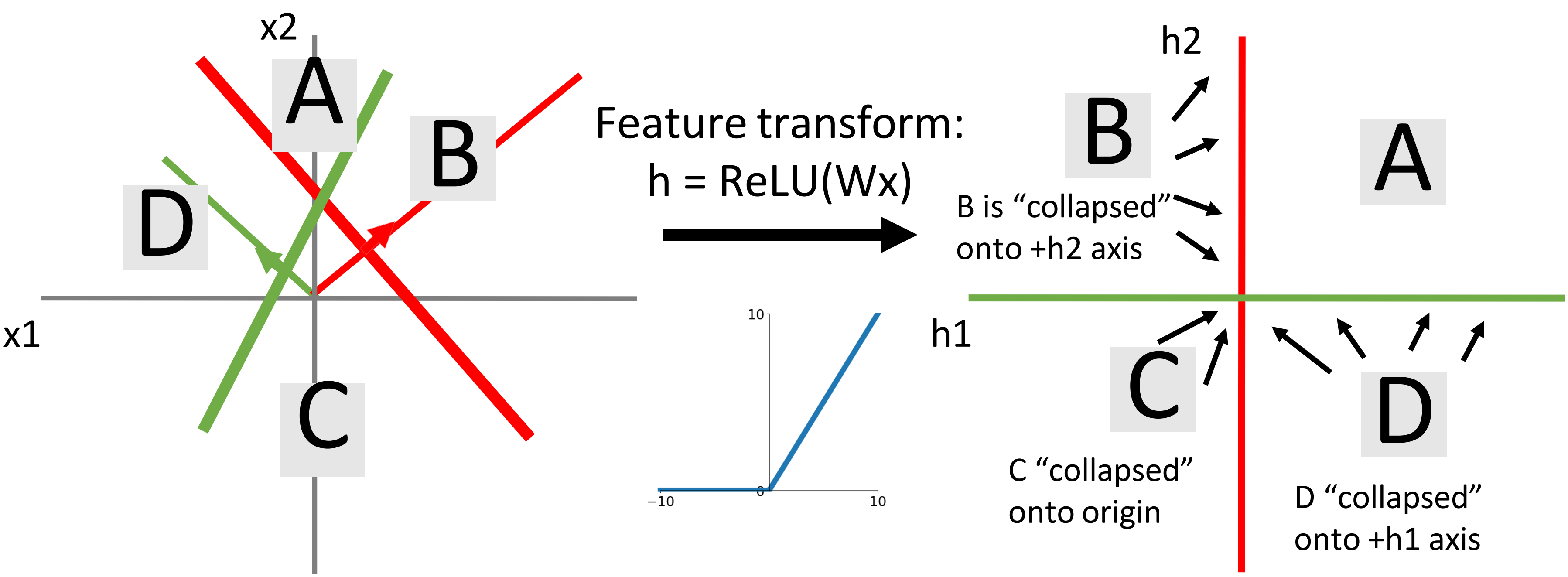


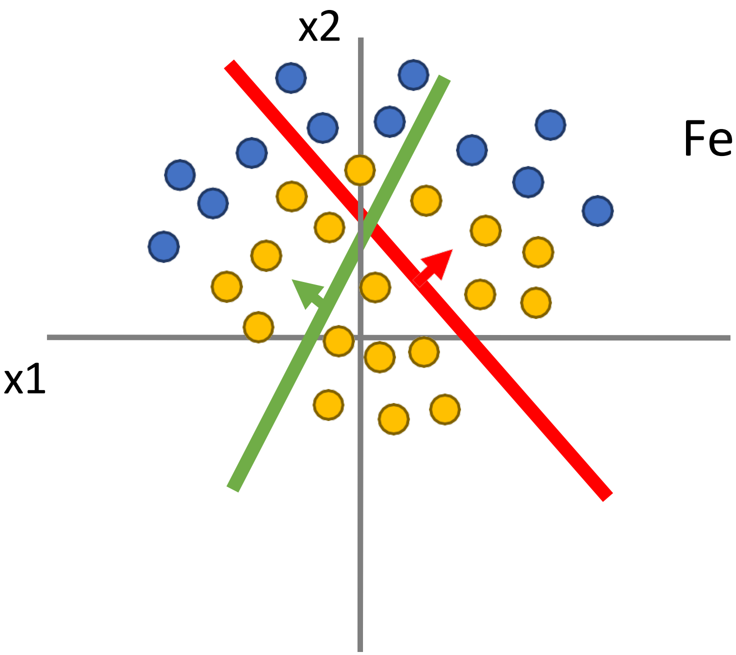
Space Warping

Consider a neural net hidden layer:

h = ReLU(Wx) = max(0, Wx)

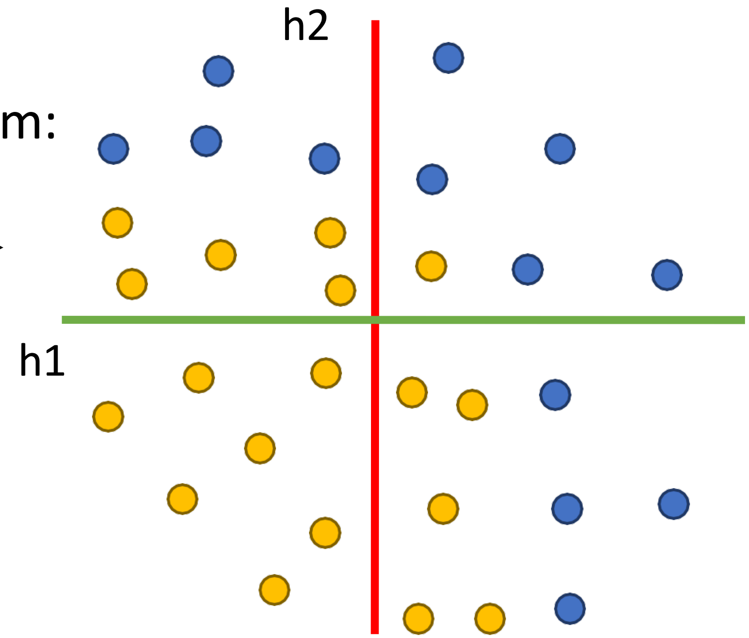
Where x, h are both 2-dimensional

Points not linearly separable in original space



Feature transform:

h = Wx



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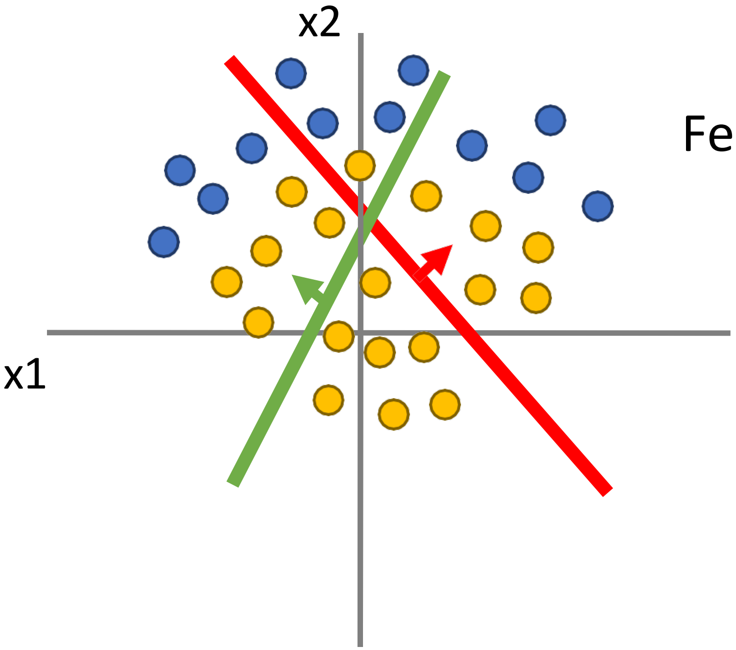
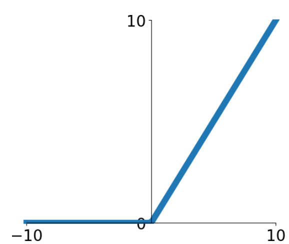
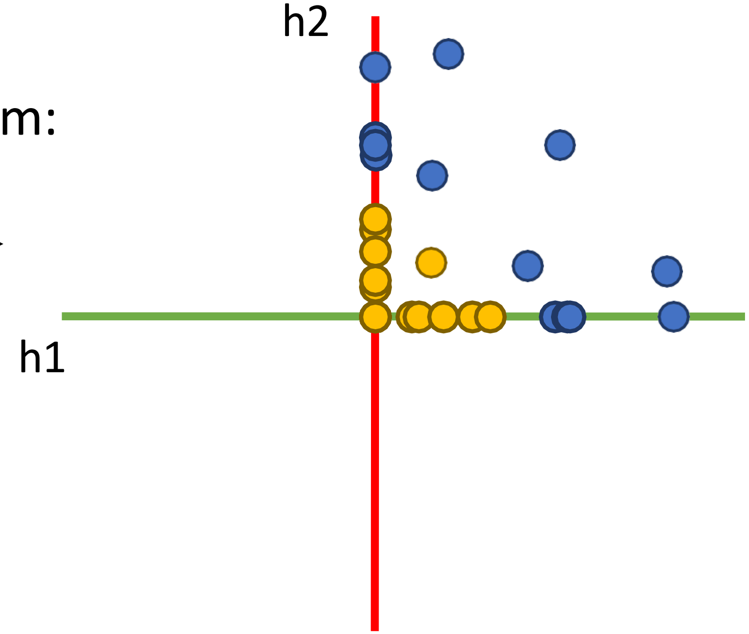


Feature transform:
h = ReLU(Wx)





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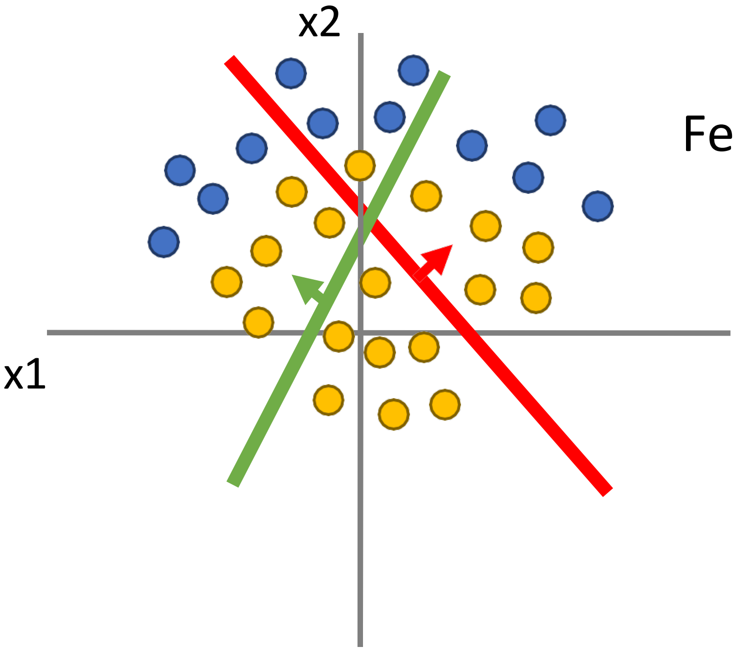
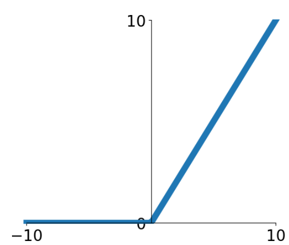
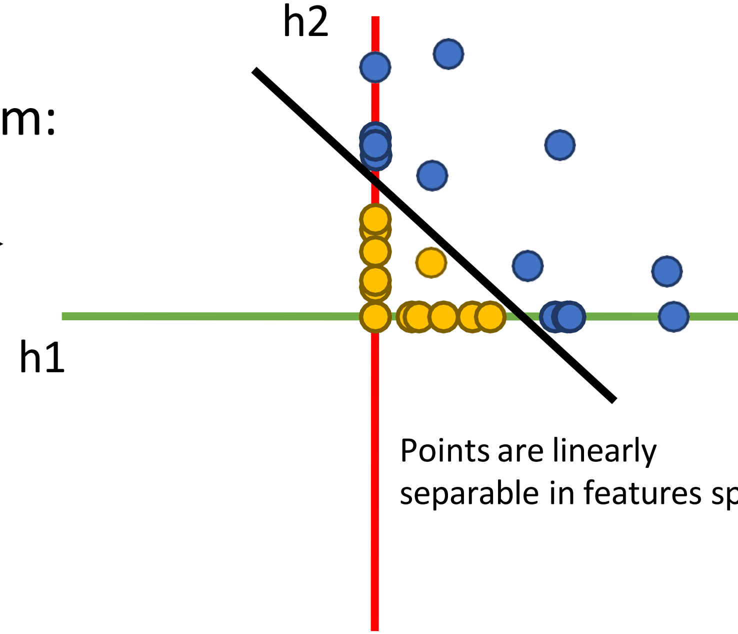


Feature transform:
h = ReLU(Wx)





Points are linearly separable in features space!

Space Warping

Consider a neural net hidden layer:
h = ReLU(Wx) = max(0, Wx)
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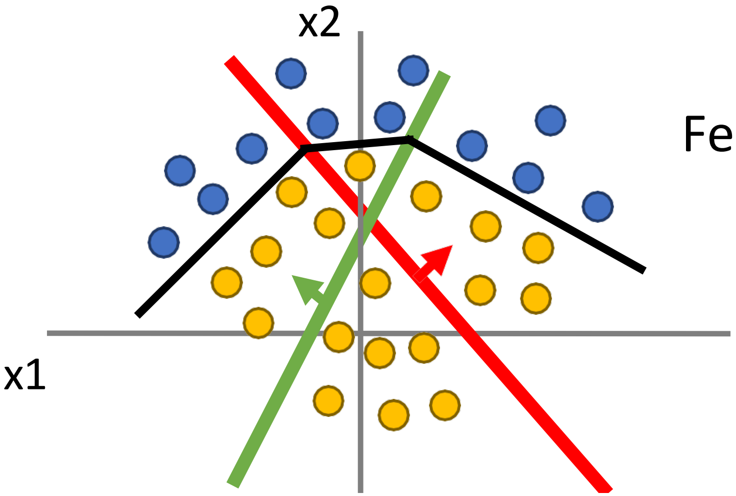
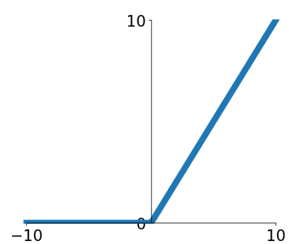
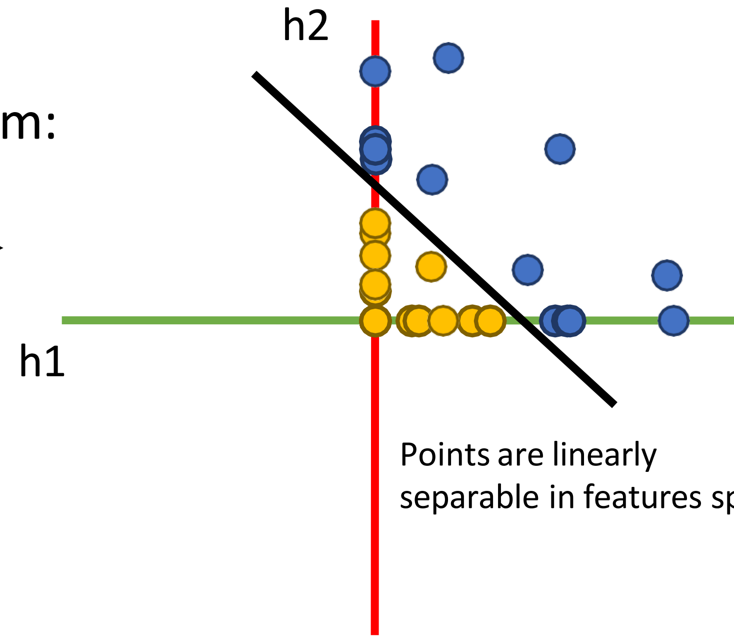
Points not linearly separable in original space



Linear classifier in feature space gives nonlinear classifier in original space

Feature transform:
h = ReLU(Wx)

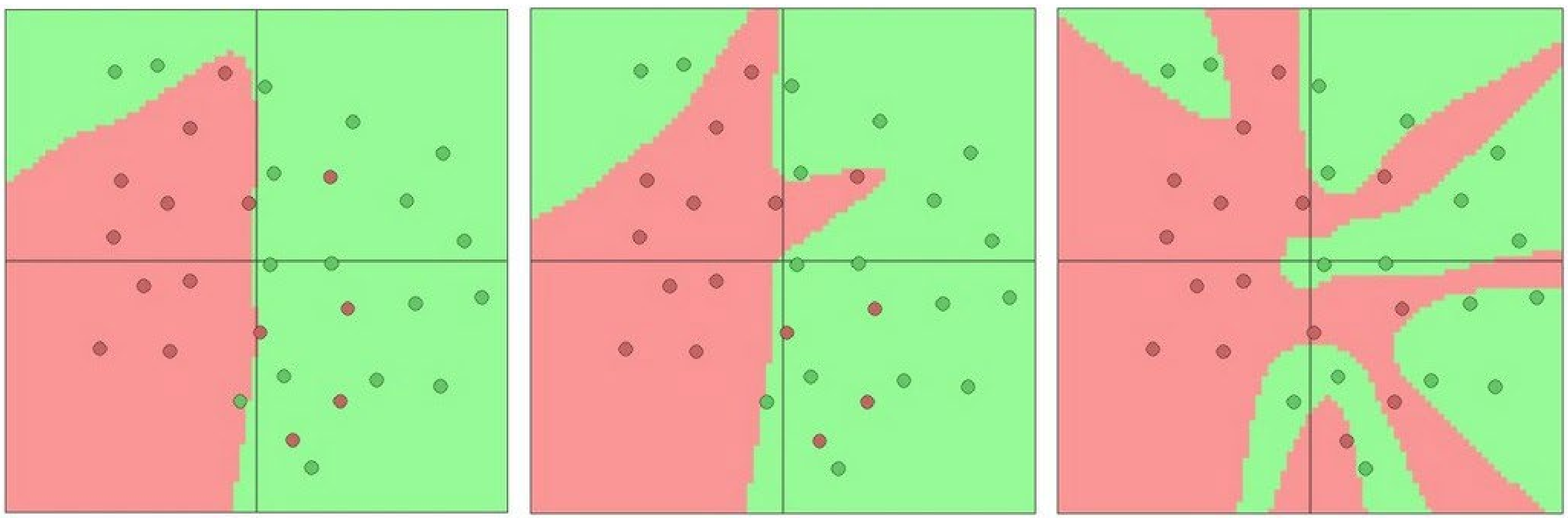




Points are linearly separable in features space!

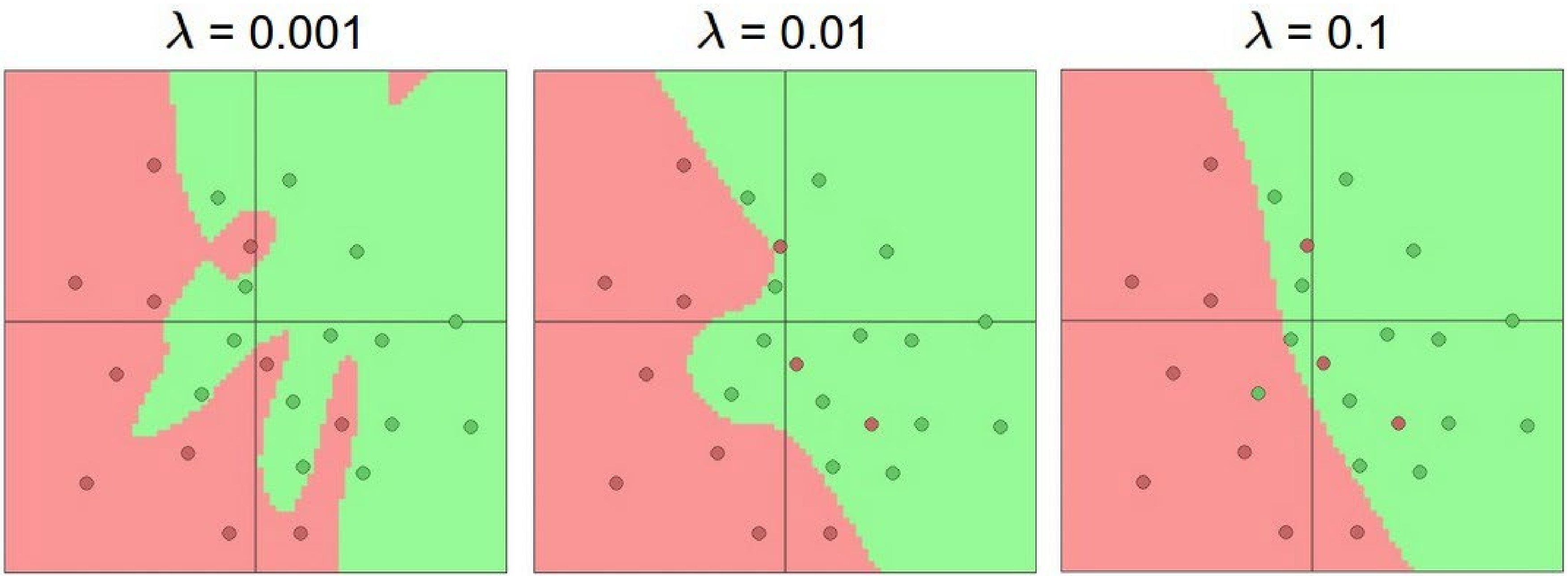
Setting the number of layers and their sizes

3 hidden units | 6 hidden units | 20 hidden units



More hidden units = more capacity

Don’t regularize with size; instead use stronger L2



(Web demo with ConvNetJS: http://cs.stanford.edu/people/karpathy/convnetjs/demo/classify2d.html)

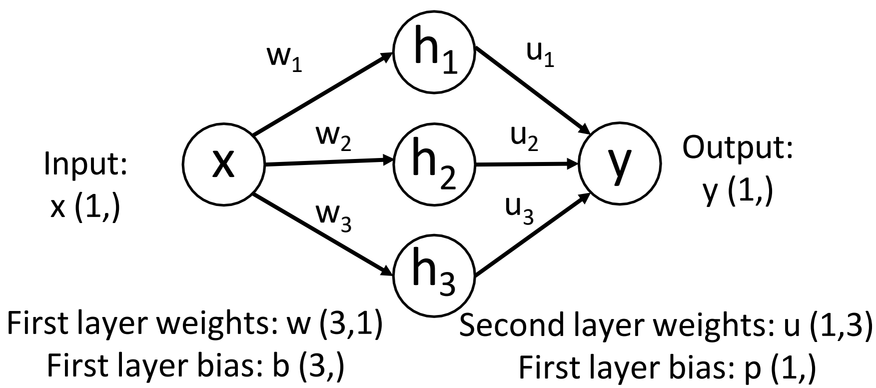
Universal Approximation

A neural network with one hidden layer can approximate any function f: R^N -> R^M with arbitrary precision*

*Many technical conditions: Only holds on compact subsets of R^N; function must be continuous; need to define “arbitrary precision”; etc

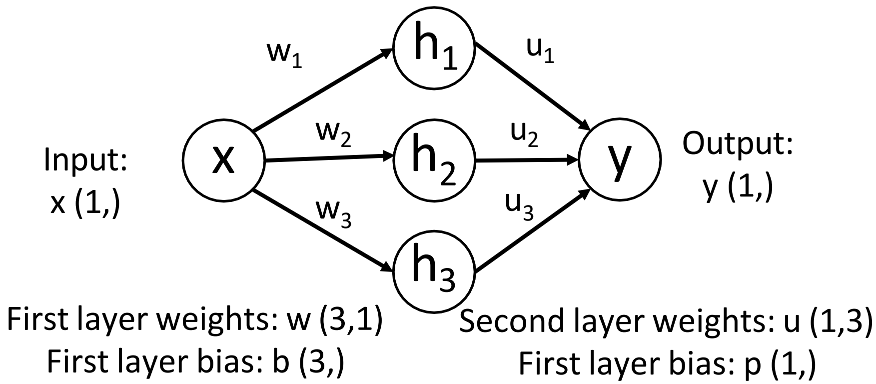
Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



Universal Approximation

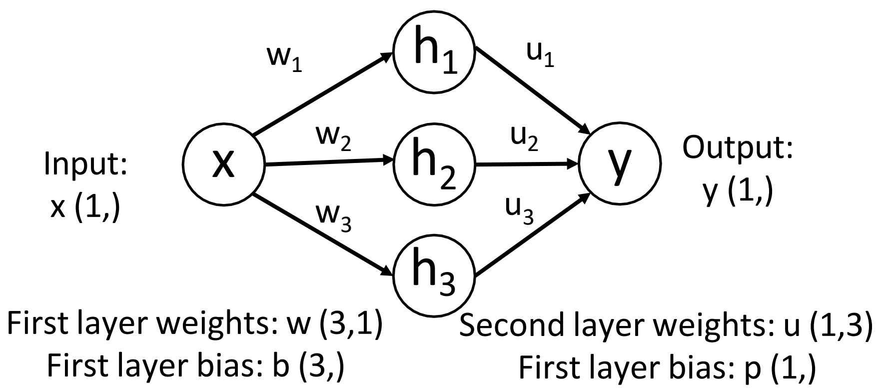
Example: Approximating a function f: R -> R with a two-layer ReLU network



h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



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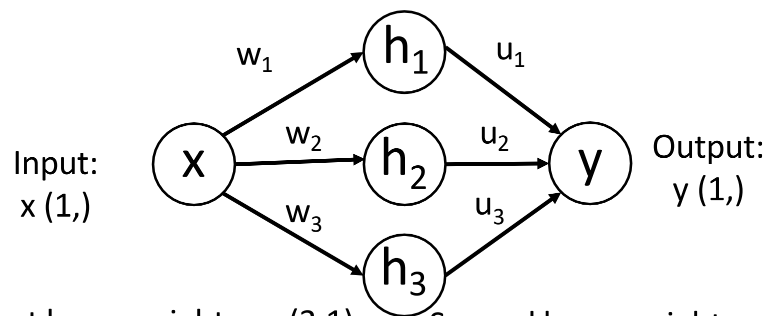
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



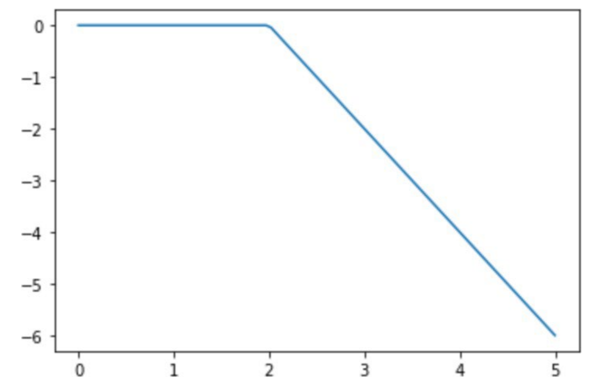
First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

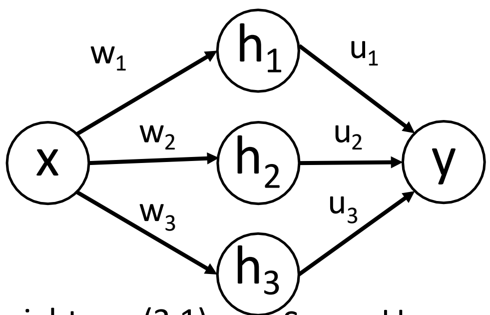
Output is a sum of shifted, scaled ReLUs:



Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

First layer weights: w (3,1)
First layer bias: b (3,)

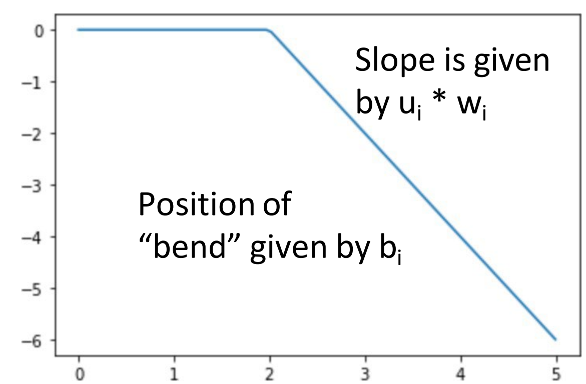
Second layer weights: u (1,3)
First layer bias: p (1,)

h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
  + u2 * max(0, w2 * x + b2)
  + u3 * max(0, w3 * x + b3)
  + p

Output is a sum of shifted, scaled ReLUs:

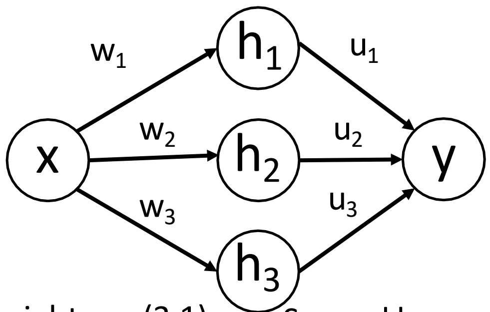
Flip left / right based on sign of w_i



Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)

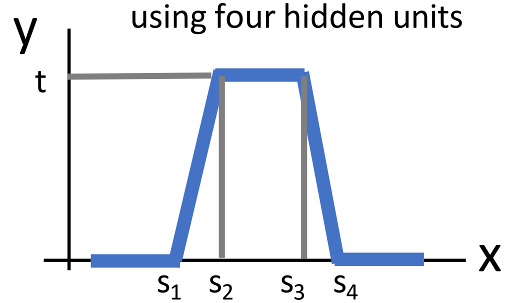


Output:
y (1,)

First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

We can build a "bump function" using four hidden units



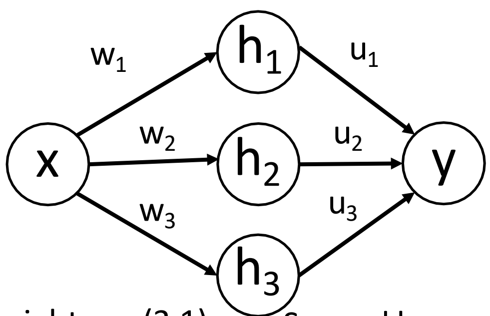
h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)

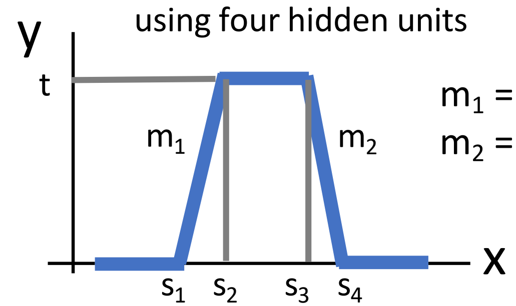


Output:
y (1,)

First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

We can build a "bump function" using four hidden units



$m_1 = t / (s_2 - s_1)$
$m_2 = t / (s_4 - s_3)$

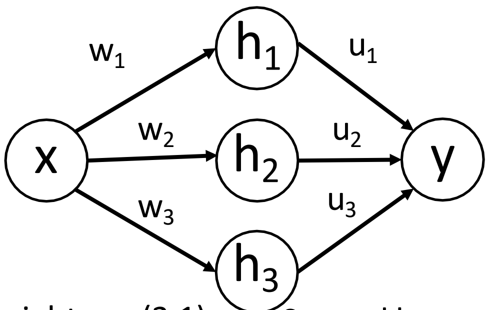
h1 = max(0, w1 * x + b1)
h2 = max(0, w2 * x + b2)
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
+ u2 * max(0, w2 * x + b2)
+ u3 * max(0, w3 * x + b3)
+ p

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



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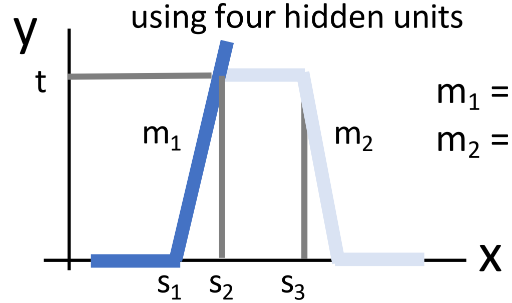
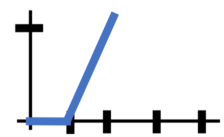
h3 = max(0, w3 * x + b3)
y = u1 * h1 + u2 * h2 + u3 * h3 + p

y = u1 * max(0, w1 * x + b1)
 + u2 * max(0, w2 * x + b2)
 + u3 * max(0, w3 * x + b3)
 + p

We can build a "bump function" using four hidden units



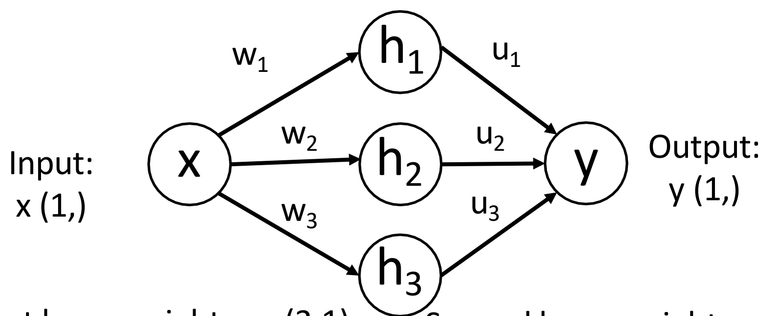
$m_1 = t / (s_2 - s_1)$
$m_2 = t / (s_4 - s_3)$



m_1 * max(0, $x - s_1$)

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

h1 = max(0, w1 * x + b1)
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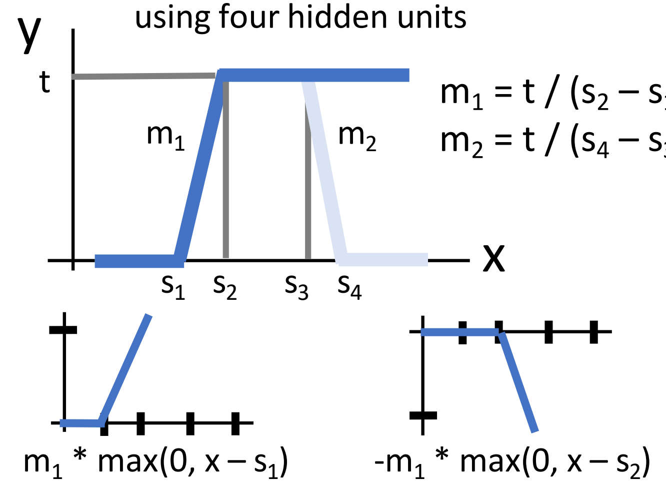

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Universal Approximation

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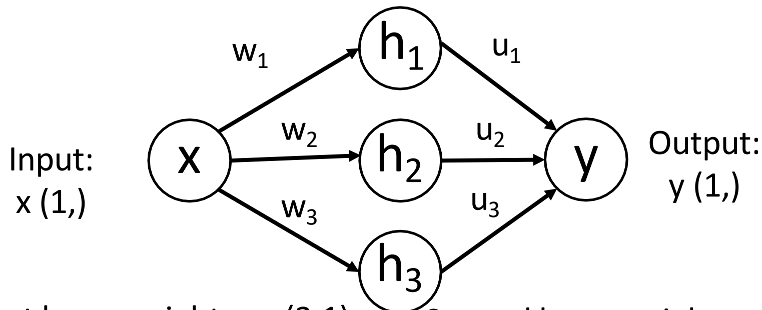

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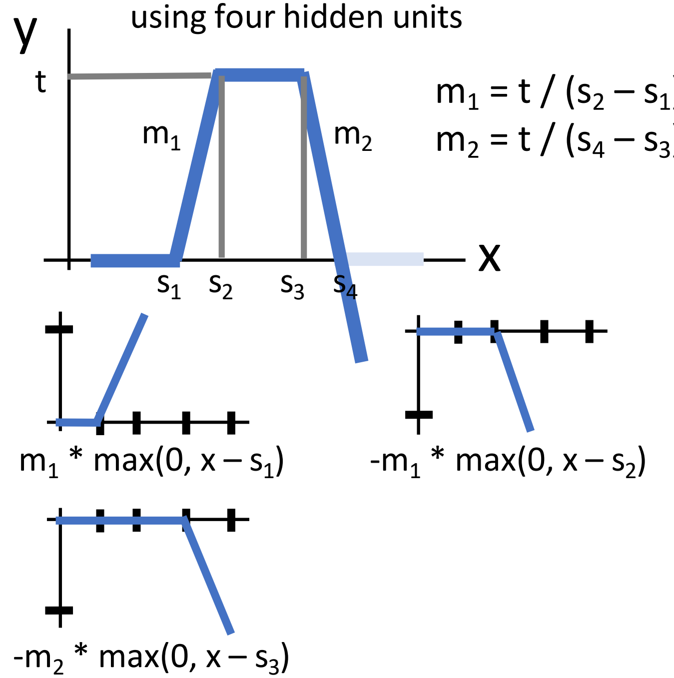
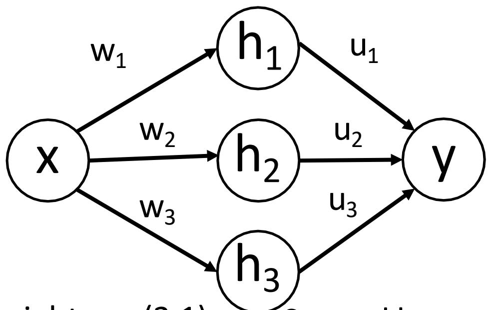


Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)



Output:
y (1,)

First layer weights: w (3,1)
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Second layer weights: u (1,3)
First layer bias: p (1,)

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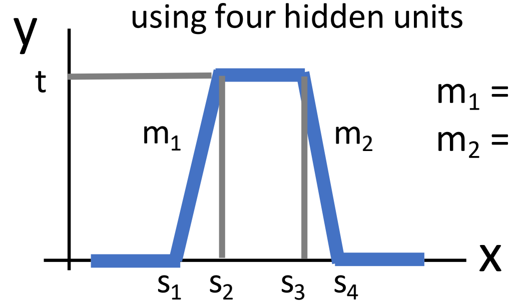

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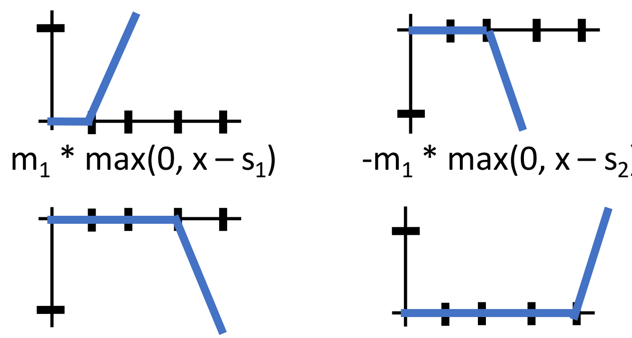
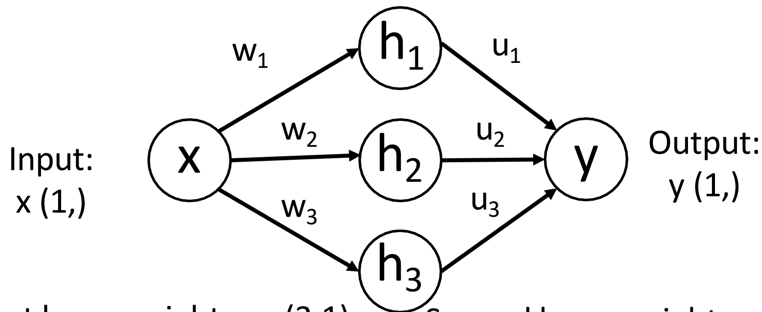


$m_1 * \max(0, x - s_1)$ $-m_1 * \max(0, x - s_2)$

$-m_2 * \max(0, x - s_3)$ $m_2 * \max(0, x - s_4)$

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network



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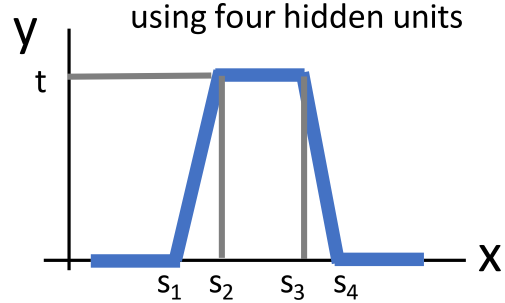
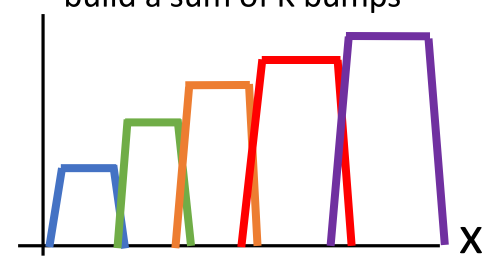
We can build a "bump function" using four hidden units



With 4K hidden units we can build a sum of K bumps



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Universal Approximation

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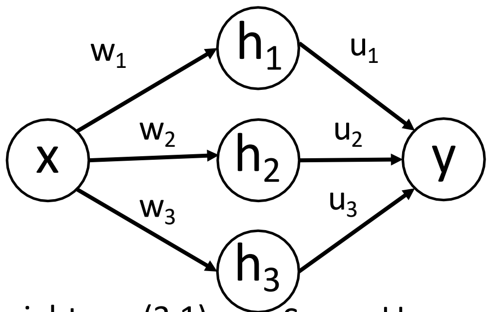

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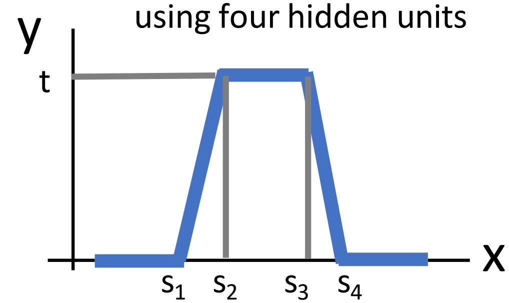

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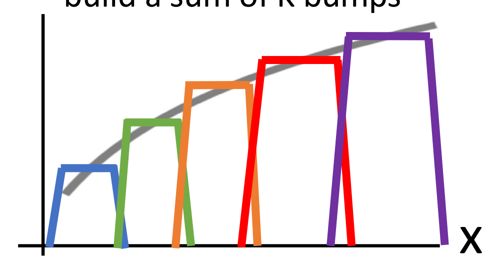
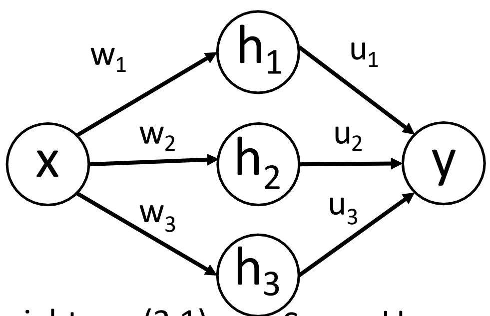


Approximate functions with bumps!

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

Input:
x (1,)

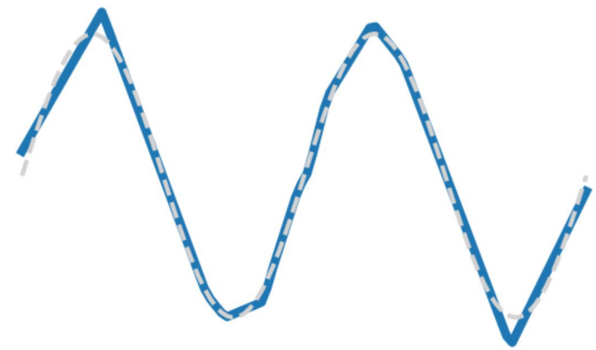


Output:
y (1,)

First layer weights: w (3,1)
First layer bias: b (3,)

Second layer weights: u (1,3)
First layer bias: p (1,)

Reality check: Networks don't really learn bumps!



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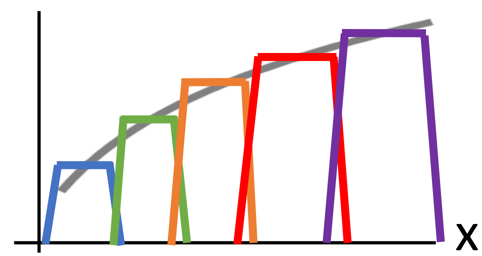
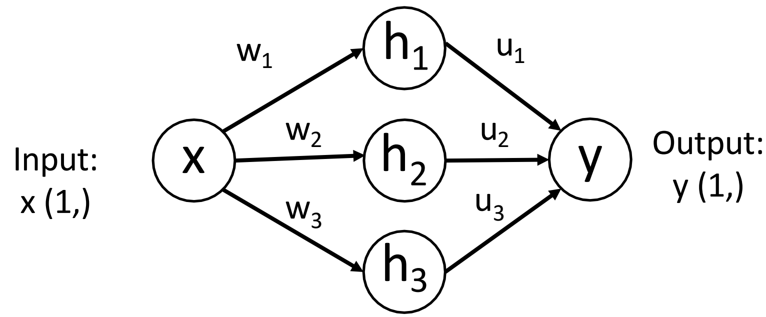


Approximate functions with bumps!

Universal Approximation

Example: Approximating a function f: R -> R with a two-layer ReLU network

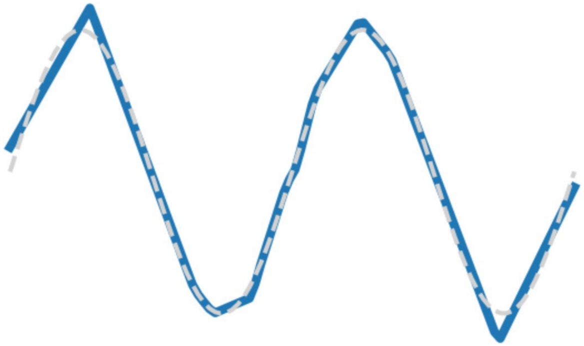


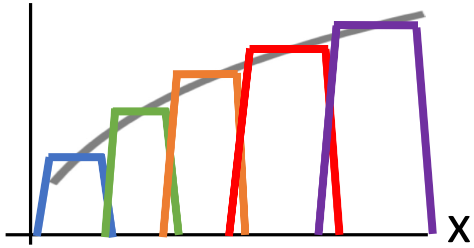
Universal approximation tells us:
- Neural nets can represent any function

Universal approximation DOES NOT tell us:
- Whether we can actually learn any function with SGD
- How much data we need to learn a function

Remember: kNN is also a universal approximator!

Reality check: Networks don't really learn bumps!





Approximate functions with bumps!

Convex Functions

A function $f : X \subseteq \mathbb{R}^N \to \mathbb{R}$ is **convex** if for all $x_1, x_2 \in X, t \in [0,1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

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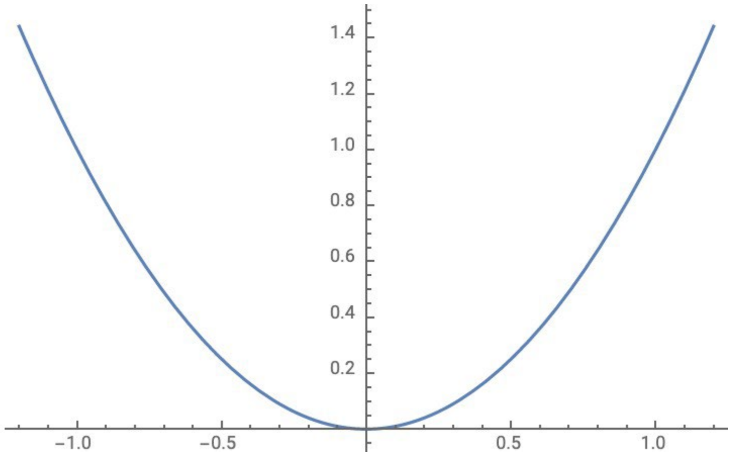
Example: $f(x) = x^2$ is convex:



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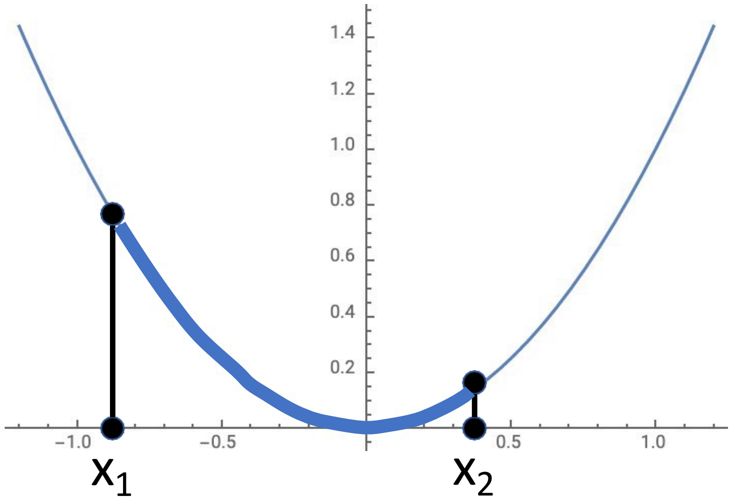

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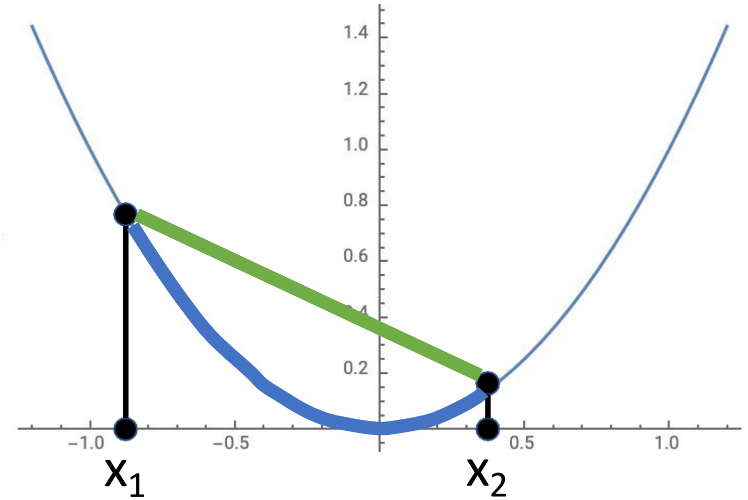

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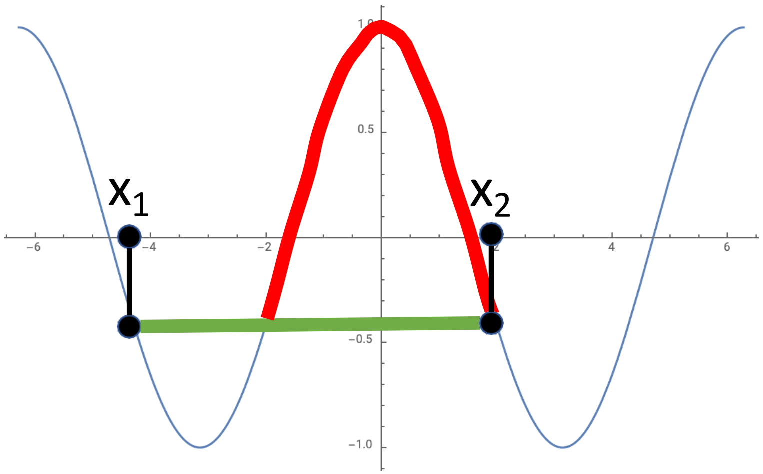
$$\boxed{f(tx_1 + (1-t)x_2)} \leq \boxed{tf(x_1) + (1-t)f(x_2)}$$

Example: $f(x) = \cos(x)$
is <u>not</u> convex:



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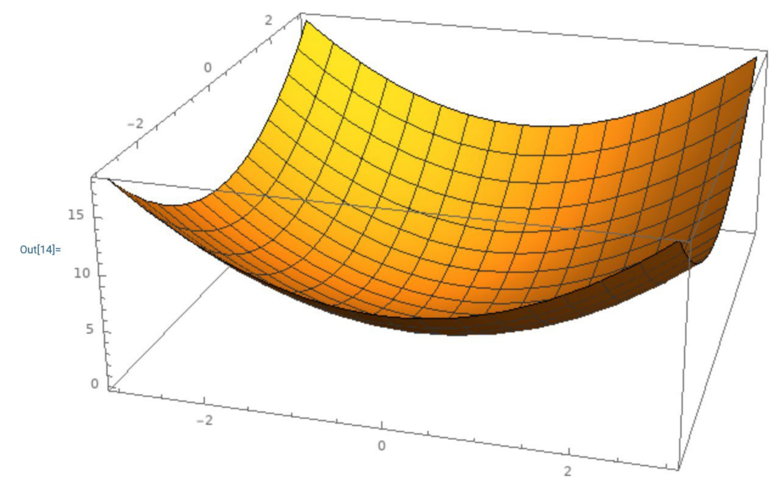
A function $f : X \subseteq \mathbb{R}^N \to \mathbb{R}$ is **convex** if for all $x_1, x_2 \in X, t \in [0,1]$

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \qquad ,$$

Intuition: A convex function is a (multidimensional) bowl



*Many technical details! See e.g. IOE 661 / MATH 663

Convex Functions

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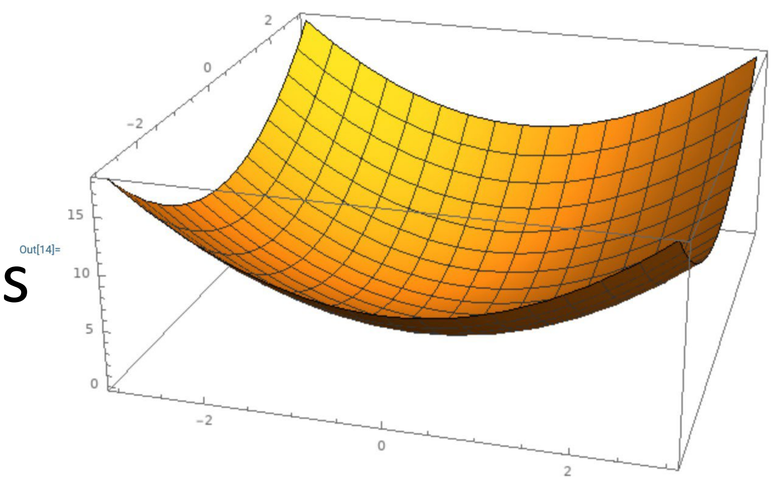

Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum***

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Linear classifiers optimize a **convex function!**

$$s = f(x; W) = Wx$$

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$ Softmax

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$ SVM

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)$$

R(W) = L2 or L1 regularization

Convex Functions

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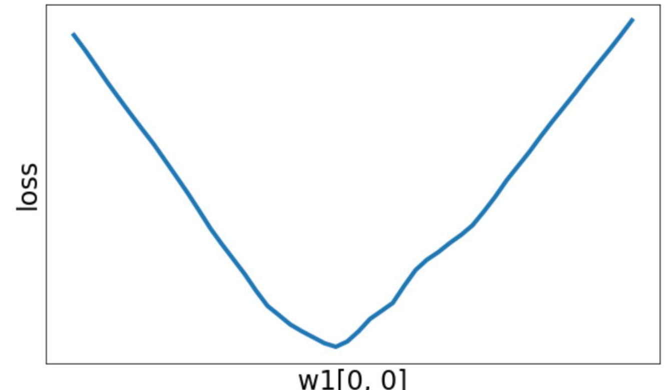
Neural net losses sometimes look convex-ish:



1D slice of loss landscape for a 4-layer ReLU network with 10 input features, 32 units per hidden layer, 10 categories, with softmax loss

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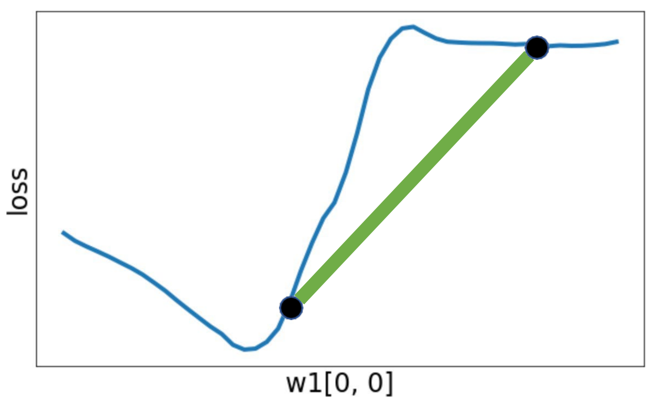
Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum***

But often clearly nonconvex:



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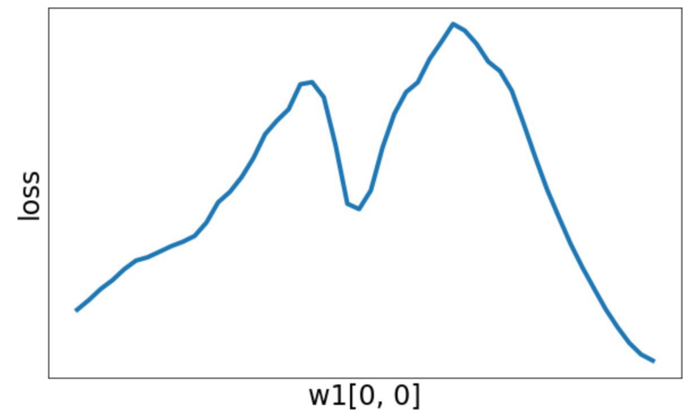
With local minima:



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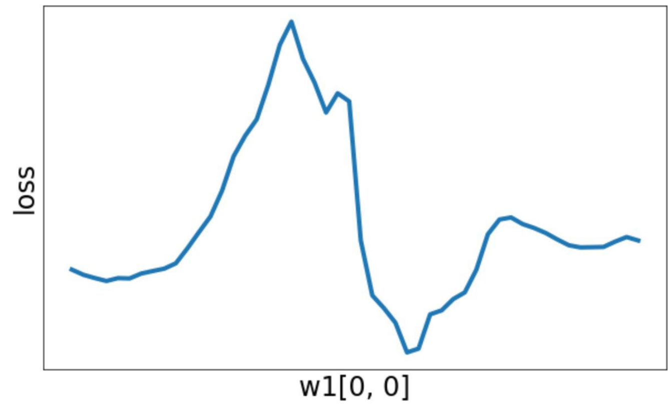
Can get very wild!



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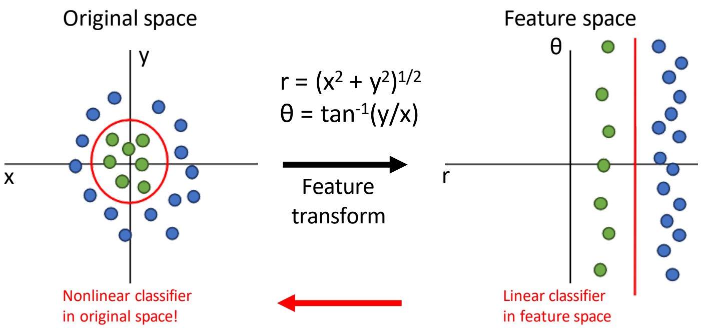
Most neural networks need **nonconvex optimization**

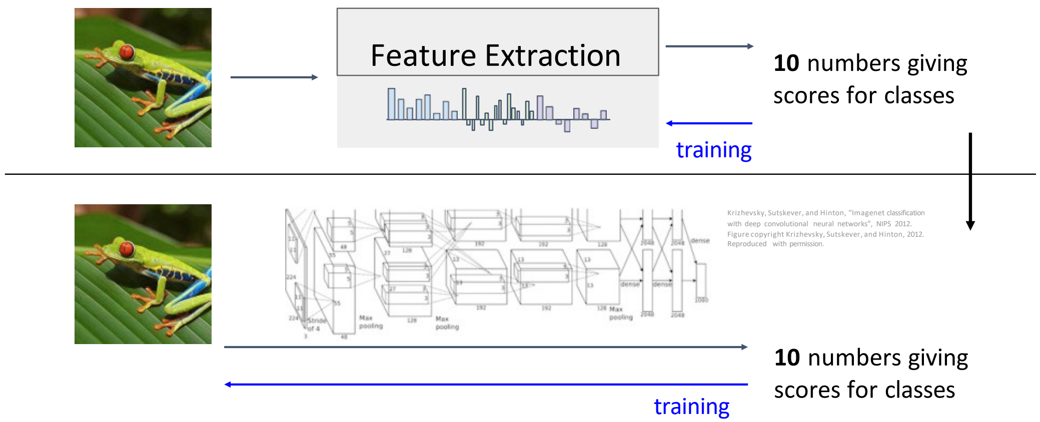
- Few or no guarantees about convergence
- Empirically it seems to work anyway
- Active area of research

Summary

Feature transform + Linear classifier allows nonlinear decision boundaries



Neural Networks as learnable feature transforms

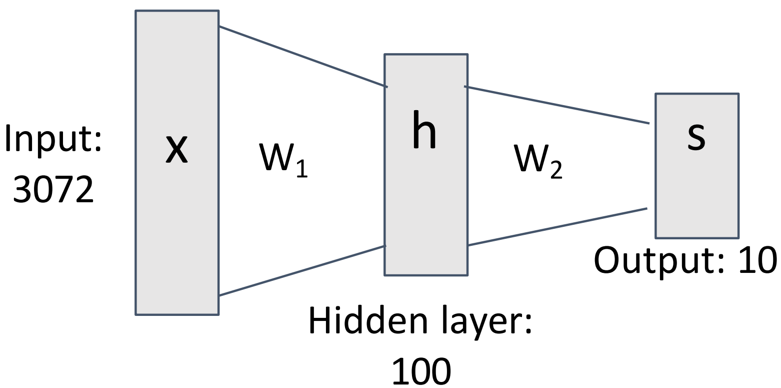


Krizhevsky, Sutskever, and Hinton, “Imagenet classification with deep convolutional neural networks”, NIPS 2012. Figure copyright Krizhevsky, Sutskever, and Hinton, 2012. Reproduced with permission.

Summary

From linear classifiers to fully-connected networks

$$f = W_2 \max(0, W_1 x)$$



Linear classifier: One template per class

Neural networks: Many reusable templates

Summary

From linear classifiers to
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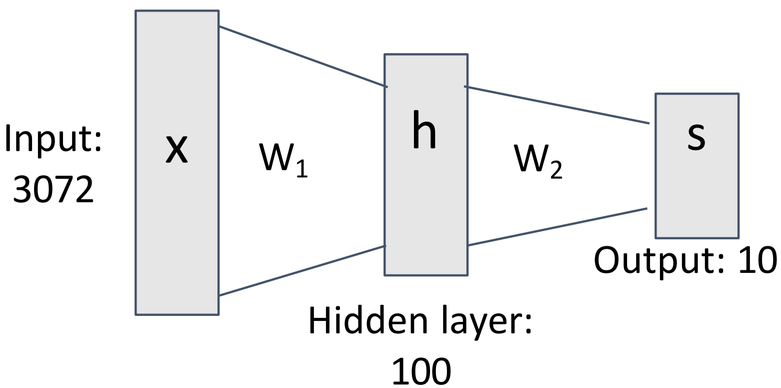
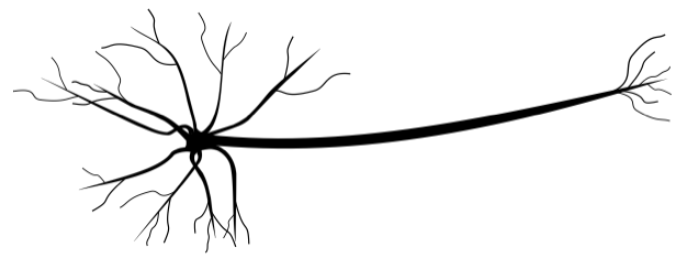


Neural networks loosely inspired by biological
neurons but be careful with analogies



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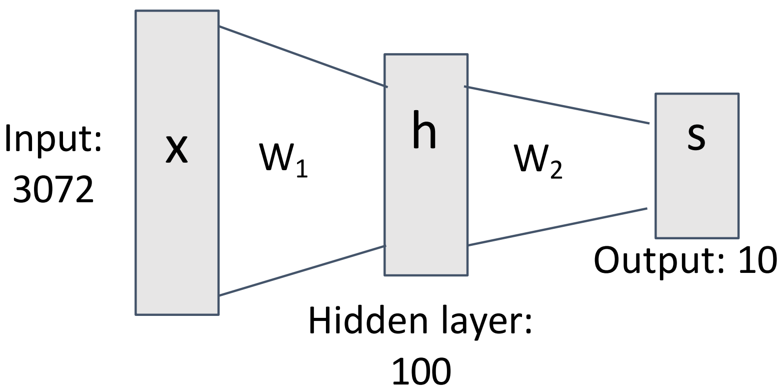
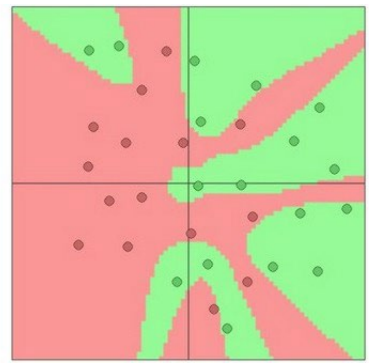
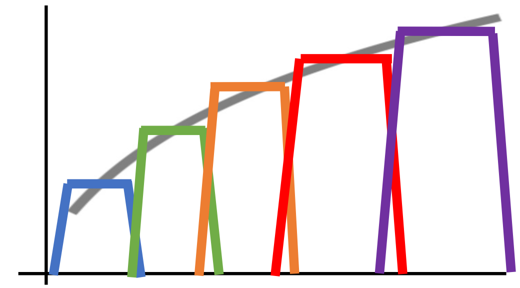


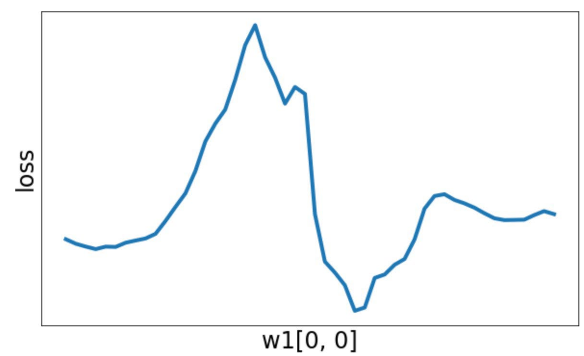
Space Warping



Universal Approximation



Nonconvex



Problem: How to compute gradients?

$$s = f(x; W_1, W_2) = W_2 \max(0, W_1 x)$$ Nonlinear score function

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$ SVM Loss on predictions

$$R(W) = \sum_k W_k^2$$ Regularization

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + \lambda R(W_1) + \lambda R(W_2)$$ Total loss: data loss + regularization

If we can compute $\frac{\partial L}{\partial W_1}, \frac{\partial L}{\partial W_2}$ n we can learn W_1 and W_2