Markovian approximations of stochastic volatility models

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Hello World!

Black-Scholes and its problems

The most basic model for stock prices is the Black-Scholes model given by

$$dS_t = rS_t dt + \sigma S_t dW_t$$

with $r \in \mathbb{R}$, $\sigma \geq 0$, and W a Brownian motion.

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Solution: Make volatility σ stochastic!

Rough volatility models

Rough Bergomi model:

$$
\begin{aligned}
dS_t &= \sqrt{V_t} S_t \Big(\rho dW_t + \sqrt{1-\rho^2} dB_t \Big), \\
V_t &= V_0 \exp\left(\eta \sqrt{2H} \int_0^t (t-s)^{H-1/2} dW_s - \frac{\eta^2}{2} t^{2H} \right).
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Problem: Volatility is neither a semimartingale nor a Markov process. Hence, significant difficulty in numerical simulation.

Sample paths of Black-Scholes and rough Bergomi

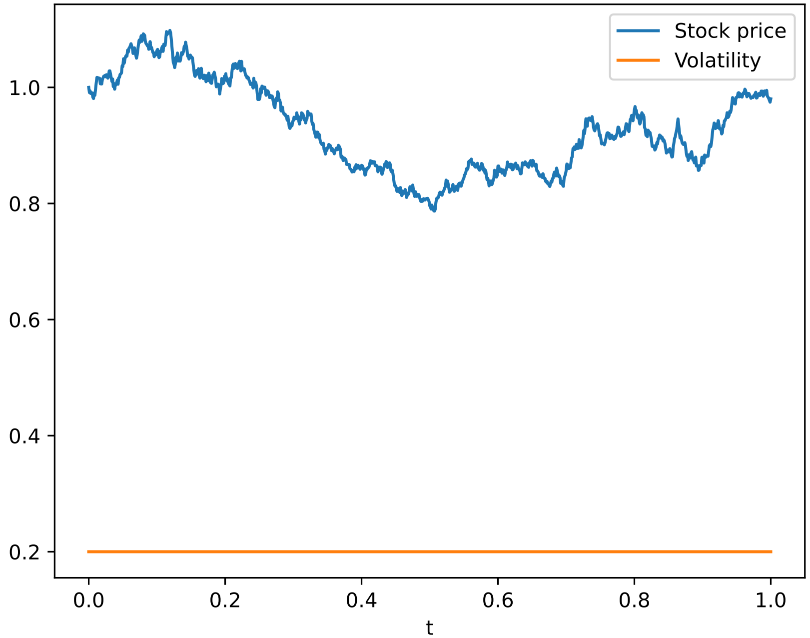

Figure: Black-Scholes

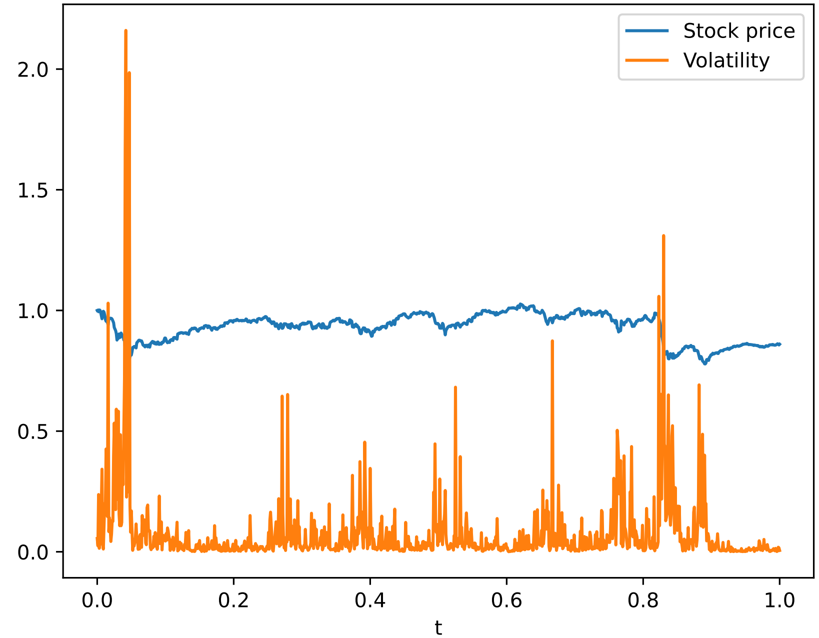

Figure: Rough Bergomi

Stochastic Volterra equations

Consider general stochastic Volterra equation

$$X_t = X_0 + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s,$$

where b, σ are Lipschitz and $K(t) = t^{H-1/2}$.

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Idea: Approximate K by

$$\widehat{K}(t) = \sum_{i=1}^N w_i e^{-x_i t}$$

and solve

$$\widehat{X}_t = X_0 + \int_0^t \widehat{K}(t-s)b(\widehat{X}_s)ds + \int_0^t \widehat{K}(t-s)\sigma(\widehat{X}_s)dW_s.$$

Markovian structure

Proposition (Abi Jaber, El Euch, 2019; Alfonsi, Kebaier, 2021)

The process $\widehat{X}$ is the solution to an N-dimensional ordinary SDE.

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There exists a constant C depending only on H, b, σ and T, such that

$$\mathbb{E}|X_T - \widehat{X}_T|^2 \leq C \int_0^T |K(t) - \widehat{K}(t)|^2 dt.$$

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Question: How to choose nodes (x_i) and weights (w_i)?

Previously known convergence rates

Theorem (Alfonsi, Kebaier, 2021)

Truncate the domain of integration to $[0, L]$ *and partition* $[0, L]$ *into* N *equisized intervals. Use midpoint rule on these intervals. Then,*

$$\mathbb{E}|X_T - \widehat{X}_T|^2 \leq CN^{-2H/3}.$$

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Theorem (Harms, 2019)

Special case of fractional Brownian motion (i.e. $b \equiv 0$, $\sigma \equiv 1$): Truncate domain of integration to $[\xi_0, \xi_n]$ and partition $[\xi_0, \xi_n]$ into n geometrically spaced intervals. Use Gaussian quadrature of level m on each of these intervals. Then,

$$\mathbb{E}|X_T - \widehat{X}_T|^2 \leq C(m)n^{-2Hm/3}.$$

Improved point set

Theorem (Bayer, B., 2021)

Truncate the domain of integration to $[\xi_0, \xi_n]$ and partition $[\xi_0, \xi_n]$ into n gemetrically spaced intervals. Use Gaussian quadrature of level m on each of these intervals. Add an additional node at $x_0 = 0$. For the correct (known) choice of m, ξ_0 and ξ_n, we have, with $N = nm$,

$$\mathbb{E}|X_T - \widehat{X}_T|^2 \leq CN^{0.21} \exp\left(- \frac{2.1283}{A_H} \sqrt{N} \right),$$

where

$$A_H = \left(\frac{1}{H} + \frac{1}{3/2 - H} \right)^{1/2}.$$

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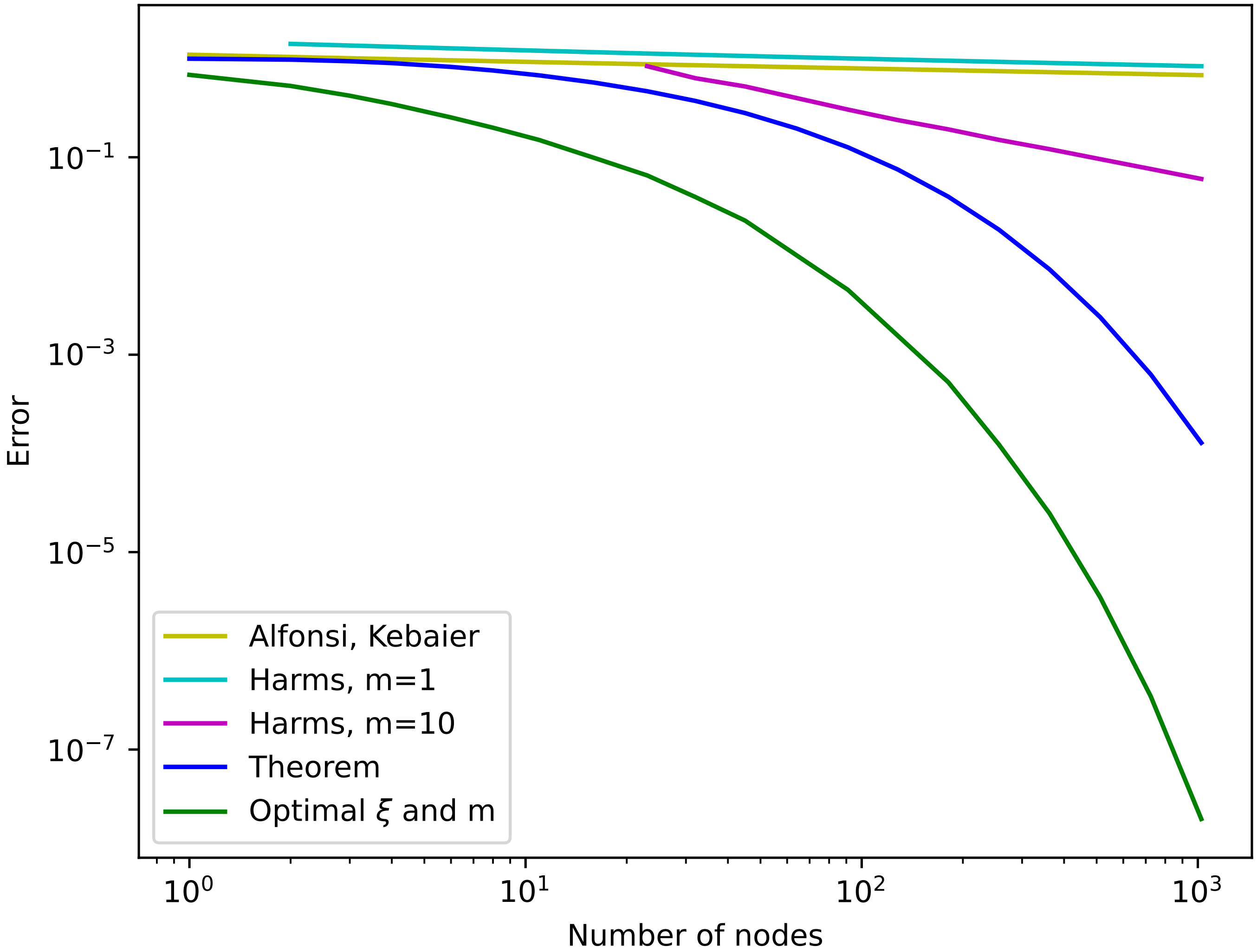
Numerical experiments suggest that we can even get

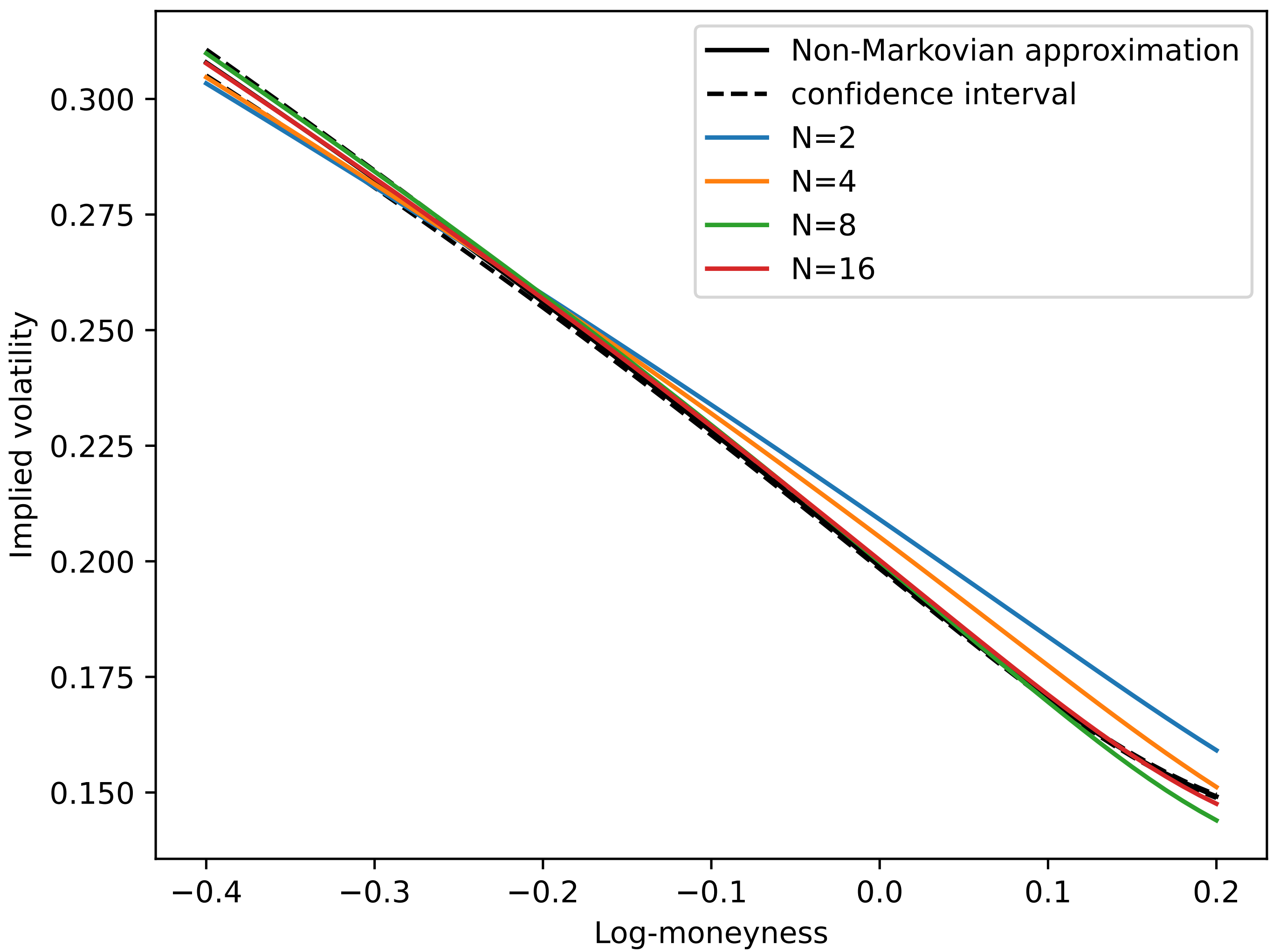
$$\mathbb{E}|X_T - \widehat{X}_T|^2 \leq C \exp\left(-\frac{3.6}{A_H}\sqrt{N}\right)$$

using a smarter choice of m, ξ_0 and ξ_n.

Numerics for fractional Brownian motion with $H = 0.1$



Rough Bergomi implied volatility smile with $H = 0.07$



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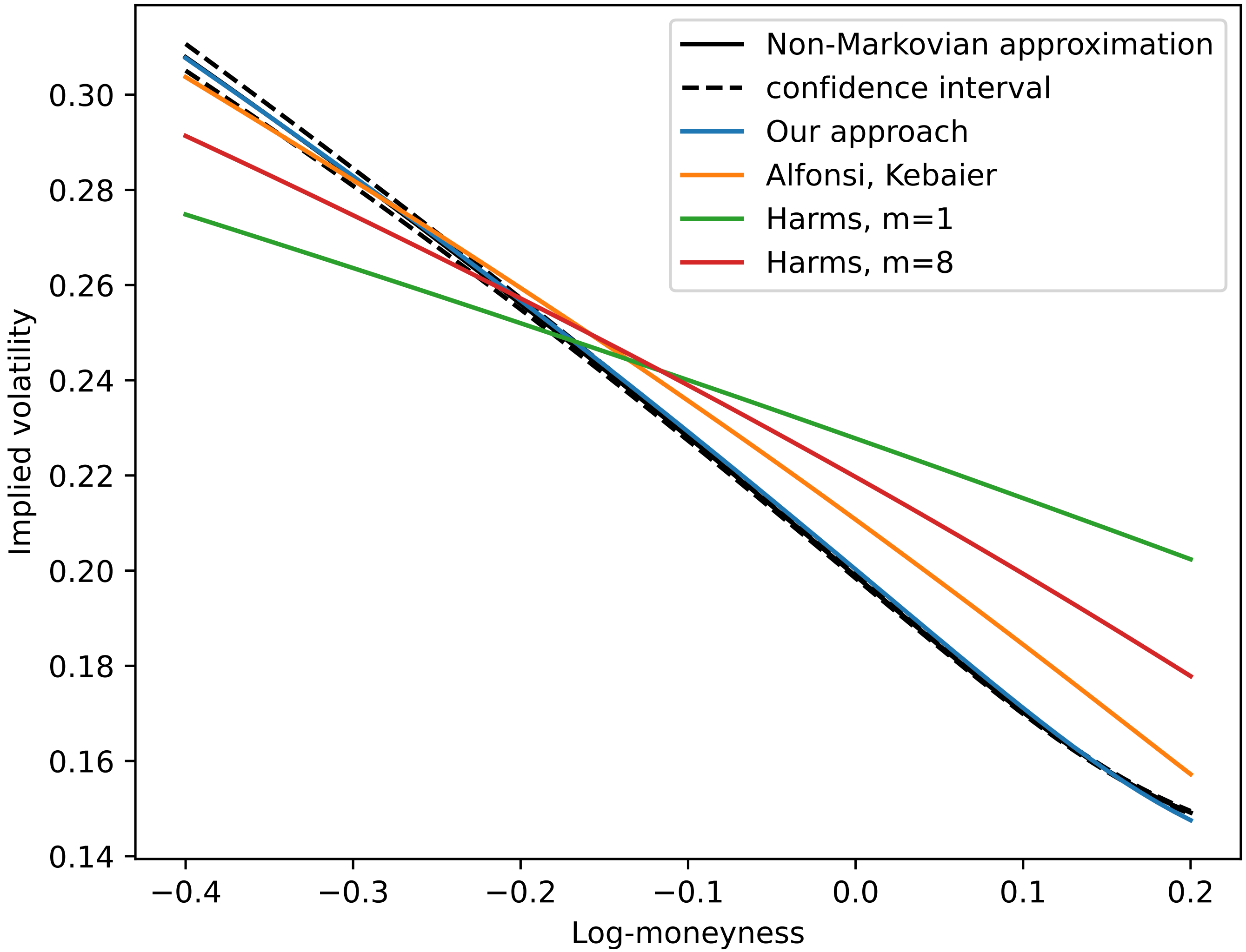
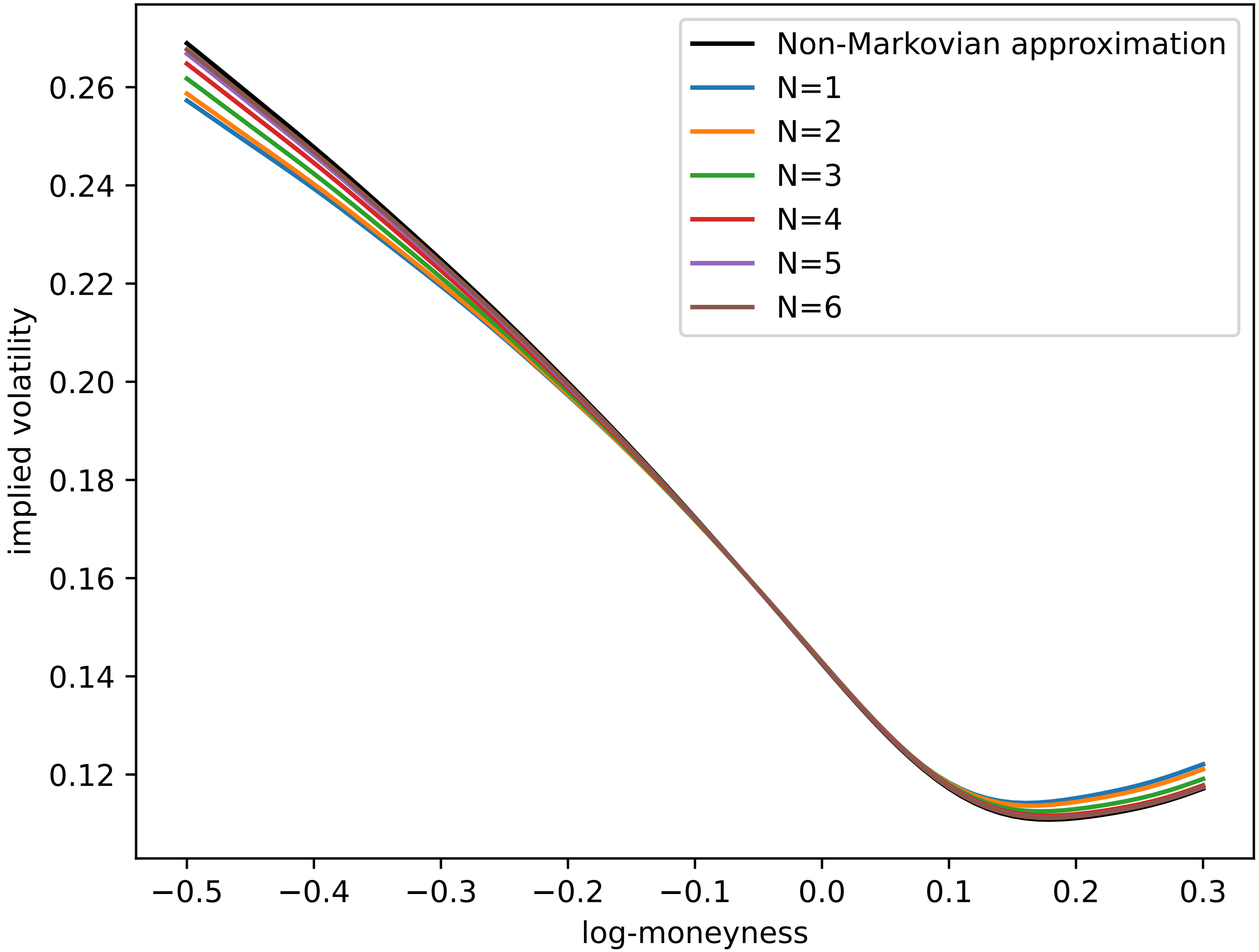


Rough Heston implied volatility smile with $H = 0.1$



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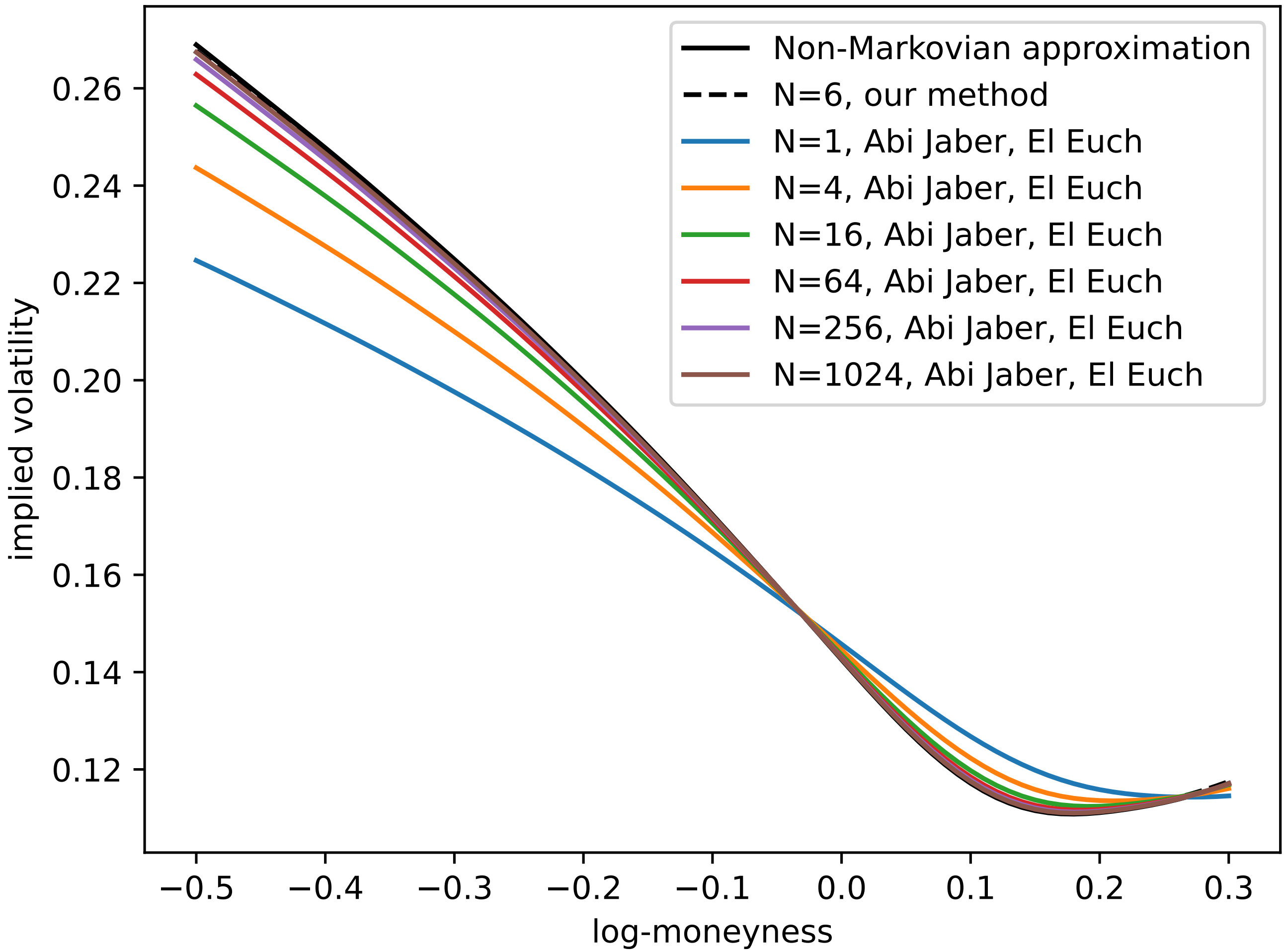

Thank you for your attention!