The Grad limit for a system of soft sphere

A short review of [C. Cercignani, Comm. Pure Appl. Math. 36 (1983), 479-484]

CHEN Zhe

Paris Dauphine University

September 2022

Table of Contents

Table of Contents

Basic assumptions

We describe the hard sphere model by the N-particle distribution functions $g_N = g_N(t, X_N, V_N) \in [0,1]$ for N hard spheres with diameter $d > 0$, where $X_N := (x_1, \ldots, x_N)$, and $V_N := (v_1, \ldots, v_N)$. For $i = 1, \ldots, N$, $x_i, v_i \in \mathbb{R}^3$ denotes the position and velocity of i^{th} particle respectively, with the Lebesgue measure $\mathrm{d}\mu_N := \prod_{i=1}^N \mathrm{d}x_i \, \mathrm{d}v_i$.
Naturally, we suppose $\int g_N \, \mathrm{d}\mu_N = 1$ for all $t > 0$. It is worth noting that $g_N = 0$ when $|x_i - x_j| < d$ for $1 \leq i \neq j \leq N$. That means the spheres can not penetrate into each other.
Here we can assume g_N is symmetric. That's to say

$$\begin{aligned} & g_N(t, x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N, v_1, \ldots, v_i, \ldots, v_j, \ldots, v_N) \\ = & g_N(t, x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N, v_1, \ldots, v_j, \ldots, v_i, \ldots, v_N) \end{aligned}$$

for all $1 \leq i < j \leq N$.

Hard sphere model

We can write the Liouville equation here:

$$\begin{cases} \frac{\partial}{\partial t} g_N + \sum_{i=1}^{N} v_i \cdot \frac{\partial}{\partial x_i} g_N = 0 \\ |x_i - x_j| > d, \quad i \neq j \end{cases} \tag{1}$$

firstly we define the s-marginal as follows

$$g_N^{(s)} := \int_{(\mathbb{R}^6)^s} g_N \, \mathrm{d}\mu_N^{(s)}$$

where $\mathrm{d}\mu_N^{(s)} := \mathrm{d}X^{(s)} \, \mathrm{d}V^{(s)}$, and $\mathrm{d}X^{(s)}$, $\mathrm{d}V^{(s)}$ denote $\prod_{i=1}^{N-s} \mathrm{d}x_{s+i}$, $\prod_{i=1}^{N-s} \mathrm{d}v_{s+i}$ respectively.
Then we integrate the Liouville equation (1) from the $s+1^{th}$ particle to the N^{th} particle. Thus it gives

$$\frac{\partial}{\partial t} g_N^{(s)} + \sum_{i=1}^{s} \int v_i \cdot \frac{\partial}{\partial x_i} g_N^{(s)} \, \mathrm{d}X^{(s)} \, \mathrm{d}V^{(s)} + \sum_{i=s+1}^{N} \int v_i \cdot \frac{\partial}{\partial x_i} g_N^{(s)} \, \mathrm{d}X^{(s)} \, \mathrm{d}V^{(s)}$$

$$=0$$

Assumptions on collision

By integration by parts and Gauss theorem, and Simplified by normal vector $n_i := \frac{x_i}{|x_i|}$, surface element of sphere $\mathrm{d}S_i$, and relative velocity $V_{ij} = v_i - v_j$. Let the collision be elastic:

$$\begin{cases} v_i' = & v_i - n_{ij}(n_{ij} \cdot V_{ij}) \\ v_j' = & v_j + n_{ij}(n_{ij} \cdot V_{ij}) \end{cases}$$

and function H denotes the Heaviside step function. Correspondingly,

$$V_{N[ij]} := (v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_{j-1}, v_j', v_{j+1}, \ldots, v_N)$$
$$g'_{N[ij]} := g_N(t, X_N, V_{N[ij]})$$

Boltzmann hierarchy

One has the Boltzmann hierarchy for the hard sphere

$$\frac{\partial}{\partial t} g_N^{(s)} + \sum_{i=1}^{s} v_i \cdot \frac{\partial}{\partial x_i} g_N^{(s)}$$
$$= (N-s) d^2 \sum_{i=1}^{s} \int \left\{ g_N^{\prime(s+1)} H(V_i \cdot n) \quad - g_N^{(s+1)} H(-V_i \cdot n) \right\} |V_i \cdot n_i| \, \mathrm{d}n \, \mathrm{d}v_* \tag{2}$$

and the equation that N-particle distribution satisfies

$$\frac{\partial}{\partial t} g_N + \sum_{i=1}^{N} v_i \cdot \frac{\partial}{\partial x_i} g_N \, \mathrm{d}\mu_N$$
$$= \frac{1}{2} \sum_{i \neq j}^{N} \left(g'_{N[ij]} H(V_{ij} \cdot n_{ij}) - g_N H(-V_{ij} \cdot n_{ij}) \right) \mathcal{D}(|x_i - x_j| - d) |V_{ij} \cdot n_{ij}| \tag{3}$$

with 1-dimensional Dirac function $\mathcal{D}$,

Setting of soft sphere model

As for soft sphere, we replace the Dirac function with $K_{d,\delta} = \frac{1}{\delta}\mathbb{1}_{[d,d+\delta]}$, and let

$$S_{\delta,d}(r) = C_{\delta,d} K_{\delta,d} \frac{d^2}{r^2+\delta^2}$$

with $C_{\delta,d} = \delta(1 - \arctan\frac{d+\delta}{\delta} + \arctan\frac{d}{\delta})$ to make sure $S_{\delta,d}$ is $O(d^2)$ and $\int_0^\infty S_{\delta,d}(r) r^2 \, \mathrm{d}r = d^2$ to approximate Dirac function $\delta(r-d)$
That means the N-particle distribution function $f_N = f_N(t, X_N, V_N)$ obeys

$$\begin{aligned} \frac{\partial}{\partial t} f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N = \quad & \frac{1}{2}\sum_{i,j=1}^N [f_N' H(V_{ij}\cdot n_{ij}) - f_N H(-V_{ij}\cdot n_{ij})] \\ & \cdot S_{\delta,d}(|x_i - x_j|)|V_{ij}\cdot n_{ij}| \end{aligned} \tag{4}$$

with $V_{ij} := v_i - v_j$, $n_{ij} := \frac{x_i - x_j}{|x_i - x_j|}$, and what's more, we assume f_N is same symmetry assumption as in hard sphere model.

Description of main equation

If $f_N \in L^1(\mathbb{R}^{6N};\ \mathrm{d}\mu_N)$ with $\mathrm{d}\mu_N = \prod_{i=1}^N \mathrm{d}x_i\, \mathrm{d}v_i$, then the original equation can be rewritten by Duhamel formula (in the sense of "mild solution")

$$f_N(t, X_N, V_N) = f_N(0, X_N - V_N t, V_N) + \int_0^t \mathcal{T} f_N(\theta, X_N - V_N(t-\theta), V_N)\, \mathrm{d}\theta \tag{5}$$

where

$$\mathcal{T} f_N := \frac{1}{2} \sum_{i,j} \left[f_N' H(V_{ij} \cdot n_{ij}) - f_N H(-V_{ij} \cdot n_{ij}) \right] S_{\delta,d}(|x_i - x_j|) |V_{ij} \cdot n_{ij}|$$

Modified H-theorem

Theorem

If we let $\Phi(r) := \int_0^r S_{\delta,d}(\rho)\ \mathrm{d}\rho \le \frac{\pi}{4-\pi}\frac{d^2}{\delta^2}$ and

$$H_N(t) := \frac{1}{N}\int \left[\ln f_N - \sum_{i,j=1}^{N} \Phi(|x_i - x_j|)\right] f_N\ \mathrm{d}\mu_N \tag{6}$$

Moreover, if we assume that $f_N \log f_N \in L^1(\mathbb{R}^{6N};\ \mathrm{d}\mu_N)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} H_N(t) \le 0 \tag{7}$$

Lemma

$$\int \ln f_N \mathcal{T} f_N\ \mathrm{d}\mu_N \le \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i,j}\int \Phi(|x_i - x_j|) f_N\ \mathrm{d}\mu_N \tag{8}$$

Proof of lemma

Firstly, we do a Post- to Pre-collision transformation

$$\begin{cases} v_i' = & v_i - n_{ij}(n_{ij} \cdot V_{ij}) \\ v_j' = & v_j + n_{ij}(n_{ij} \cdot V_{ij}) \end{cases}$$

It is easy to check that $V_{ij}' \cdot n_{ij}' = -V_{ij} \cdot n_{ij}$ and $v_i'' = v_i$, then one has

$$\int \ln f_N \mathcal{T} f_N \, \mathrm{d}\mu_N = \frac{1}{2} \sum_{i,j} \int f_N \ln \frac{f_N'}{f_N} |V_{ij} \cdot n_{ij}| H(-V_{ij} \cdot n_{ij}) S_{\delta,d}(|x_i - x_j|) \, \mathrm{d}\mu_N$$

Next, we note a basic inequality: $z \ln z - z + 1 \geq 0$:

$$\int \ln f_N \mathcal{T} f_N \, \mathrm{d}\mu_N \leq \frac{1}{2} \sum_{i,j} \int (f_N' - f_N) |V_{ij} \cdot n_{ij}| H(-V_{ij} \cdot n_{ij}) S_{\delta,d}(|x_i - x_j|) \, \mathrm{d}\mu_N$$

By the same Post- to Pre-collision transformation, one has

$$\int \ln f_N \mathcal{T} f_N \, \mathrm{d}\mu_N \leq \frac{1}{2} \sum_{i,j} \int f_N V_{ij} \cdot n_{ij} S_{\delta,d}(|x_i - x_j|) \, \mathrm{d}\mu_N$$

Thanks to the symmetry assumption, we conclude that

$$f_N v_i \cdot n_{ij} S_{\delta,d}(|x_i - x_j|) = f_N v_j \cdot (-n_{ij}) S_{\delta,d}(|x_i - x_j|)$$

Finally, since the definition of Φ and using the integration by part,

$$\begin{aligned} \int \ln f_N \mathcal{T} f_N \, \mathrm{d}\mu_N &\leq \sum_{i,j} \int f_N v_i \cdot \frac{\partial \Phi(|x_i - x_j|)}{\partial x_i} \, \mathrm{d}\mu_N \\ &= -\sum_{i,j} \int v_i \cdot \frac{\partial f_N}{\partial x_i} \Phi(|x_i - x_j|) \, \mathrm{d}\mu_N \end{aligned}$$

Formal proof of H-theorem

replace $v_i \cdot \frac{\partial f_N}{\partial x_i}$ by $\mathcal{T} f_N - \frac{\partial}{\partial t} f_N$, one has

$$\int \mathcal{T} f_N \ln f \,\mathrm{d}\mu_N \leq \sum_{i,j} \left(\int \Phi(|x_i - x_j|)(\frac{\partial}{\partial t} f_N - \mathcal{T} f_N) \,\mathrm{d}\mu_N \right)$$

Go back to the theorem, we derivative for $H_N(t)$ along the flow:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} H_N(t) &= \frac{1}{N} \frac{\mathrm{d}}{\mathrm{d}t} \int f_N \big(\ln f_N - \sum_{i,j} \Phi(|x_i - x_j|)\big) \,\mathrm{d}\mu_N \\ &\leq \frac{1}{N} \left(\frac{\mathrm{d}}{\mathrm{d}t} \int f_N \,\mathrm{d}\mu_N + \int \mathcal{T} f_N \ln f_N \,\mathrm{d}\mu_N - \int \mathcal{T} f_N \ln f_N \,\mathrm{d}\mu_N \right) = 0\end{aligned}$$

Table of Contents

Cut-off assumption

In order to get the global existence of the equation, we should consider add a cut-off on it.

$$
\begin{aligned}
F_N(t, X_N, V_N) =& F_N(0, X_N - V_N t, X_N) \\
&+ \int_0^t \mathcal{T}^C F_N(\theta, X_N - (t-\theta) V_N, V_N)\, \mathrm{d}\theta
\end{aligned}
\tag{9}
$$

with $\mathcal{T}^C F_N := \mathbb{1}_{|V_{ij} \leq C|} \mathcal{T} F_N$. And initial data is restricted to have total energy not larger than a fixed value E, i.e. $F_N = 0$ when $\frac{1}{2} m \sum_{k=1}^{N} v_k^2 \geq E$ where the m denote the mass of molecule. Hence the equation (9) holds with $C = 2(\frac{2E}{m})^{1/2}$

Existence

Theorem (Existence)

The cutoff model

$$
\begin{aligned}
F_N(t, X_N, V_N) = & F_N(0, X_N - V_N t, X_N) \\
& + \int_0^t \mathcal{T}^C F_N(\theta, X_N - (t-\theta) V_N, V_N) \, \mathrm{d}\theta
\end{aligned} \tag{10}
$$

has one and only one solution $F_N(t, X_N, V_N) \in L^1(\mathbb{R}^{6N};\ \mathrm{d}\mu_N), t > 0$, for any initial datum $F_N(0, X_N, V_N) \in L^1(\mathbb{R}^{6N};\ \mathrm{d}\mu_N)$.
The same applies to the non-cutoff equation provide the initial data satisfying $f_N = 0$ when $\frac{1}{2} m \sum_{k=1}^N |v_k|^2 \geq E$.

Proof of existence

This theorem is given by the following result (refer to Chapter 3, theorem 1.1 in book [3])

Theorem (Pazy)

Let X be a Banach space, and let A be the infinitesimal of a C_0 semigroup $T(t)$ on X, such that $\|T(t)\| \leq Me^{\omega t}$. If B is bounded linear operator on X then $A + B$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on X, satisfying $\|S(t)\| \leq M\mathrm{e}^{(\omega + M\|B\|)t}$

because $F_N(0, X_N - V_N t, V_N) \in L^1$ and $\mathcal{T}^C$ is bounded and linear, hence Lipschitz continous.
Here the Banach space X is $L^1(\mathbb{R}^{6N};\ \mathrm{d}\mu_N)$, the semigroup $S(t)$ is generated by $\mathcal{T}^c$.

Propagation of positivity

Theorem

If initial datum $F_N(0, X_N, V_N)$ belongs to $L^1_+(\mathbb{R}^{6N};\ \mathrm{d}\mu_N)$, then so does $F_N(t, X_N, V_N)$, which means $F_N(t, X_N, V_N) \geq 0$ hold for all $t > 0$.

It can be obtained by

$$\frac{\mathrm{d}}{\mathrm{d}t} F_N(t, \cdot) = \mathcal{T}^C F_N = \frac{1}{2} \sum_{i,j} F_N(V_{ij} \cdot n_{ij}) S_{\delta,d} \geq -\frac{N(N-1)}{4} \frac{d^2}{\delta^3} C \cdot F_N(t, \cdot)$$

hence $\frac{\mathrm{d}}{\mathrm{d}t}(\exp(\frac{N(N-1)}{4}\frac{d^2}{\delta^3}C) F_N(t, \cdot)) \geq 0$

Conservation law

Theorem

Under the assumption above, if the initial datum is such that $(1+\sum_{k=1}^{N}\frac{|v_k|^2}{N})F_N(t,\cdot)\in L^1(\mathbb{R}^{6N};\ \mathrm{d}\mu_N)$ for all $t=0$, then it also belongs to L^1 for $t>0$.
Moreover,

$$M_0 := \int F_N(t,X_N,V_N)\ \mathrm{d}\mu_N$$

$$M_1 := \int v_k F_N(t,X_N,V_N)\ \mathrm{d}\mu_N$$

$$M_2 := \int |v_k|^2 F_N(t,X_N,V_N)\ \mathrm{d}\mu_N$$

all are constant in time.

Proof of conservation law

This theorem is nothing but multiplying both side of original equation by $1, \frac{1}{N}\sum_{k=1}^{N} v_k, \frac{1}{N}\sum_{k=1}^{N} v_k^2$, and note that

$$\int \mathcal{T} F_N(t, X_N, V_N) \, \mathrm{d}\mu_N = 0$$

$$\int \mathcal{T} F_N(t, X_N, V_N) \sum_{k=1}^{N} v_k \, \mathrm{d}\mu_N = 0$$

$$\int \mathcal{T} F_N(t, X_N, V_N) \sum_{k=1}^{N} |v_k|^2 \, \mathrm{d}\mu_N = 0$$

thanks to the symmetry of F_N, we can replace $\frac{1}{N}\sum_{k=1}^{N} v_k$ and $\frac{1}{N}\sum_{k=1}^{N} |v_k|^2$ by v_k and $|v_k|^2$ respectively.

Modified H-theorem

Theorem(Formal)

If $F_N \in L^1(\mathbb{R}^{6N};\ \mathrm{d}\mu_N)$ and $F_N \ln F_N \in L^1$ for $t = 0$, and the assumptions above hold, then $F_N \ln F_N \in L^1(\mathbb{R}^{6N};\ \mathrm{d}\mu_N)$ for any $t > 0$ and $\frac{\mathrm{d}}{\mathrm{d}t} H_N(t) \leq 0$.

The formal proof is same as before.

Table of Contents

Assumptions on Grad limit

In this section, we consider the Grad limit of the system, i.e. as $N \to \infty, d \to 0$ and Nd^2 tends to a finite limit. Moreover, we assume that the initial datum satisfies the cutoff assumption for any fixed unit energy $\frac{E}{Nm}$.
For a fixed s, we are going to find a subsequence $f_{N_j}^s$ that converges weakly.

$$h_N := \prod_{r=0}^{[N/s]-1} f_N^{(s)}(x_{rs+1}, \ldots, x_{rs+s}, v_{rs+1}, \ldots, v_{rs+s}, t) \prod_{r=[N/s]s+1}^{N} f_N^{(1)}(x_r, v_r, t)$$

note $\frac{f_N}{h_N}\ln\frac{f_N}{h_N} + 1 - \frac{f_N}{h_N} \geq 0$, multiply it by h_N, and integrating with respect to $\mathrm{d}\mu_N$

$$\frac{1}{s}\int f_N^{(s)}\ln f_N^{(s)} = \frac{N/s}{N[N/s]}\int f_N \ln f_N - \left\{\frac{N/s}{s[N/s]} - 1\right\}\int f_N^{(1)}\ln f_N^{(1)}$$

Theorem (Boundedness of energy and entropy)

Let f_N denote the solution of

$$
\begin{aligned}
F_N(t, X_N, V_N) =& F_N(0, X_N - V_N t, X_N) \\
&+ \int_0^t \mathcal{T} F_N(\theta, X_N - (t-\theta)V_N, V_N)\, \mathrm{d}\theta
\end{aligned}
$$

and $F_N = 0$ if $\frac{1}{2} m \sum_{k=1}^N |v_k|^2 \geq E$. Moreover, $F_N(1 + \frac{1}{N}\sum_{k=1}^N |v_k|^2)$ is in L^1_+ at time $t = 0$, and their mass, momentum, energy all are uniformly bounded with respect to N. Then

$$
\begin{aligned}
H_N^s &:= \int F_N^{(s)} \ln F_N^{(s)}\, \mathrm{d}\mu_s \\
E_N^s &:= \int F_N^{(s)} (1 + \frac{1}{s}\sum_{k=1}^s |v_k|^2)\, \mathrm{d}\mu_s = \int f_N^{(1)} (1 + |v_1|^2)\, \mathrm{d}\mu_1
\end{aligned}
$$

are uniformly bounded with respect to N for any $t > 0$.

Tend N to infinity

Lemma

Let $\{h_N(z)\}$ be a sequence of non-negative $L^1(\mu)$ functions of $z \in \mathbb{R}^m$ with respect to finite measure μ, such that for constant A, B

$$\|h_N\| = A \quad \forall N = 1, 2, \ldots$$

$$\int h_N \ln h_N \, \mathrm{d}\mu \leq B \quad \forall N = 1, 2, \ldots$$

Then for all $\varepsilon > 0$, $h_N(1 + |z|^2)^{-\varepsilon}$, contains a subsequence $\{h_{N_j}(1 + |z|^2)^{-\varepsilon}\}$ converging weakly to a function $h(1 + |z|^2)^{-\varepsilon} \in L^1(\mu)$ and $\lim_{j\to\infty} h_{N_j}\phi \, \mathrm{d}\mu = \int h\phi \, \mathrm{d}\mu$ provided $(1 + |z|^2)^{\varepsilon}\phi(z) \in L^\infty(\mu)$

We apply the lemma to $h_N = \frac{F_N^{(s)}}{F_0^{(s)}}$ with $F_0^{(s)} := \exp\left(-\frac{1}{s}\sum_{k=1}^{s} |v_k|^2 \right)$, and $\mathrm{d}\mu = F_0^{(s)} \, \mathrm{d}\mu_s$.

Proof of lemma

We divide the domain $\mathbb{R}^m$ into two parts: $S_M := \{z | h(z) > \mathrm{e}^M\}$ and $\mathbb{R}^m \backslash S_M$, for any measurable subset $S \subset \mathbb{R}^m$, and we use the notation $\ln^+ h(x) := \mathbb{1}_{\{x | h(x) \geq 1\}} \ln h(x)$, $\ln^- h(x) := -\mathbb{1}_{\{x | 0 < h(x) < 1\}} \ln h(x)$

$$
\begin{aligned}
\int_S h_N \, \mathrm{d}\mu_N \leq & \mathrm{e}^M \int_{S \backslash S_M} \mathrm{d}\mu + \int_{S_M} h_N \, \mathrm{d}\mu \\
& \leq \mathrm{e}^M \int_{S \backslash S_M} \mathrm{d}\mu + \int_{S_M} h_N \frac{\ln^+ h_N}{M} \, \mathrm{d}\mu \\
& \leq \mathrm{e}^M \int_{S \backslash S_M} \mathrm{d}\mu + \int_{S_M} h_N \frac{\ln h_N}{M} \, \mathrm{d}\mu + \int_{S_M} g_N \frac{\ln^- h_N}{M} \, \mathrm{d}\mu \\
& \leq \mathrm{e}^M \int_S \mathrm{d}\mu + \frac{B + \mathrm{e}^{-1}}{M}
\end{aligned}
$$

which gives the equi-integrability, then the tightness are given by the following estimate

$$
\int_{\mathbb{R}^m \backslash B_r} h_N (1 + |z|^2)^{-\varepsilon} \, \mathrm{d}\mu \leq r^{-2\varepsilon} \int_{\mathbb{R}^m \backslash B_r} h_N \, \mathrm{d}\mu \to 0 \quad \text{as } r \to \infty
$$

Subsequence weakly convergence

Theorem

Let f_N denote the solution of

$$\begin{aligned} f_N(t, X_N, V_N) =& f_N(0, X_N - V_N t, X_N) \\ &+ \int_0^t \mathcal{T} f_N(\theta, X_N - (t-\theta)V_N, V_N)\, \mathrm{d}\theta \end{aligned}$$

and $f_N = 0$ if $\frac{1}{2} m \sum_{k=1}^N |v_k|^2 \geq E$. Moreover, $f_N(t,\cdot)(1 + \frac{1}{N}\sum_{k=1}^N |v_k|^2)$ is in L^1_+ at time $t = 0$, and their mass, momentum, energy all are uniformly bounded with respect to N. Then for each $t > 0$ and for each s, there is a subsequence $f^s_{N_j}$ converging weakly in L^1 to a function f^s in such a way that

$$\lim_{j\to\infty} \int \phi f^s_{N_j}\, \mathrm{d}\mu_s = \int \phi f^s_N\, \mathrm{d}\mu_s$$

for any ϕ such that $\phi(1 + \frac{1}{s}\sum_{k=1}^s |v_k|^2)^{\varepsilon - 2} \in L^\infty$ for all $\varepsilon > 0$.

By Cantor diagonal argument, we can assume the sequence is the same for all rational value of $s, t \geq 0$. Furthermore, the sequence is equicontinous in t. Thus from equation of f_N

$$\|\mathcal{T} f_N\|_{L^1} \leq A\|(1+\frac{1}{N}\sum_{k=1}^{N}|v_k|^2)f_N\|_{L^1} = A\|(1+\frac{1}{s}\sum_{k=1}^{s}|v_k|^2)f_N\|_{L^1}$$

Theorem

The subsequence of the last theorem can be taken the same for all s and all $t > 0$.

Mollified Boltzmann hierarchy

Theorem

The same subsequence converges to a solution of the (mollified) Boltzmann hierarchy, given by equation

$$
\begin{aligned}
&f^{(s)}(t, X_N, V_N) \\
=&f^{(s)}(0, X_N - V_N t, V_N) + \sum_{k=1}^{s} \int_0^t \int S_\delta(|x_k - x_*|) \\
&\cdot \left[f^{(s+1)\prime} H(V_{k*} \cdot n_{i*}) - f^{(s+1)} H(-V_{i*} \cdot n_{i*}) \right] \mathrm{d}x_* \, \mathrm{d}v_* \, \mathrm{d}t
\end{aligned}
$$

in a integral form, equivalent to the following integro-differential system

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \sum_{i=1}^{s} v_i \cdot \frac{\partial}{\partial x_i}\right) f^{(s)} = &\sum_{i=1}^{s} \int S_\delta(|x - x_*|) |V_{i*} \cdot n_{i*}| \\
&\cdot \left[f^{(s+1)\prime} H(V_{i*} \cdot n_{i*}) - f^{(s+1)} H(-V_{i*} \cdot n_{i*}) \right] \mathrm{d}x_* \, \mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
f_N^{(s)}(x_j, v_j, t) = {} & f_N^{(s)}(x_j - v_j t, v_j, 0) \\
& + \frac{1}{2} \int_0^t [f^{(s)\prime} H(V_{ij} \cdot n_{ij}) - f^{(s)} H(-V_{ij} \cdot n_{ij})] S_{\delta,d}(|x_i - x_j|) |V_{ij} \\
& + (N - s) \int_0^t \int \sum_{i=1}^{s} [f^{(s+1)\prime} H(V_i \cdot n_i) - f^{(s+1)} H(-V_i \cdot n_i)] \\
& \cdot |V_i \cdot n_i| S_{\delta,d}(|x_i - x_*|) \, \mathrm{d}x_* \, \mathrm{d}v_* \, \mathrm{d}t
\end{aligned}
$$

Taking Grad limit in the both sides, and

$$(N - s) S_{\delta,d}(r) \to S_\delta(r)$$

we will get the conclusion. Consider the factored solution of this hierarchy, each factor being a solution of the Boltzmann equation with an initial datum $F^{(1)}(x_k, v_k, 0)$. Moreover, if the datum has a bounded 4^{th} order moment, the solution exists and is unique. We will prove it in the next section.

Factorization theorem

Theorem

If the solution of the hierarchy

$$(\frac{\partial}{\partial t} + \sum_{i=1}^{s} v_i \cdot \frac{\partial}{\partial x_i}) f^{(s)} = \sum_{i=1}^{s} \int S_\delta(|x - x_*|)|V_i \cdot n_i| \cdot [f^{(s+1)\prime} H(V_i \cdot n_i) - f^{(s+1)} H(-V_i \cdot n_i)] \,\mathrm{d}x_* \,\mathrm{d}v_*$$

with a factorized initial datum has a 4^{th} order moment, then the solution of

$$f_N(t, X_N, V_N) = f_N(t, X_N - V_N t, V_N) + \int_0^t \mathcal{T} f_N(\theta, X_N - V_N(t - \theta, V_N)) \,\mathrm{d}\theta$$

process function $f_N^{(s)}$ weakly converges to $f^{(s)}$ under the Grad limit, exists at least along a subsequence and are factored for any $t > 0$ while $f = f^{(1)}$ satisfies the mollified Boltzmann hiearchy

Cont'd

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla_x\right) f = \int S_\delta(|x - x_*|) |V \cdot n| \big[f' f'_* H(V \cdot n) - f f_* H(-V \cdot n) \big] \, \mathrm{d}x_* \, \mathrm{d}v_*$$

Remark

We will obtain the Boltzmann equation if we interchange the order of the two limit(the Grad limit and $\delta \to 0$). Thus the interchange seems very difficult.

Proof of the existence

If the initial data are such that $(1+|v|^2)f \in L^1(\mathbb{R}^{6N};\ \mathrm{d}\mu_N)$ and $f\ln f \in L^1(\mathbb{R}^{6N};\ \mathrm{d}\mu_N)$, by first introducing a suitable cutoff for large velocities and then using the weak compactness lemma considered also in the main text to show that the solution still exists when the cutoff is removed.(c.f.[4])
In order to obtain uniqueness, however, the assumption $|v|^4 f \in L^1(\mathbb{R}^{6N};\ \mathrm{d}\mu_N)$ is needed.(c.f.[5])
In this way a uniquely determined nonlinear semigroup T^t is defined such that if f is an arbitrarily assigned initial datum, provided it satisfies the conditions that have been mentioned, then $T^t f$ is the solution of the Boltzmann equation taking f as initial value.

Table of Contents

Proof of the uniqueness

A distribution function f_N^s are associated with the measure $\mathrm{d}\nu_N^s = F_N^s \,\mathrm{d}\mu_N^{(s)}$ with the Lebesgue measure $\mathrm{d}\mu_N^{(s)}$.
In this section, we use the language of measure. The uniqueness of measures can conclude the uniqueness in $L^1(\mathbb{R}^{6N};\ \mathrm{d}\mu_N)$ directly. Let $\mathcal{N}$ be the measure space on $\mathbb{R}^6$ with total weight no more than one, and finite 4^{th} order moment, equipped with the weak*-topology is compact;
Let $\Omega = (\mathbb{R}^{6N})$ denote the N-particle phase space. The topology on Ω is generated by $\{w | w_j \in E_j, j = 1, 2, \ldots, s; E_j \subset \mathbb{R}^6$ is Borel set$\}$, with $w = (w_1, w_2, \ldots, w_N) \in \Omega$;
Let $\mathcal{M}$ be the set of all probability measure on Ω;
Let $\mathrm{d}\pi(\nu)$ denote the product measure formed from $\nu \in \mathcal{N}$;
We define the inner product $(\mathrm{d}\pi, \phi) := \int_\Omega \phi \,\mathrm{d}\pi$;

Let $C(\mathcal{N})$ be the space of bounded and continuous functions from $\mathcal{N}$ to $\mathbb{R}$;
We define the semigroup U^t to ensure the following table interchangeable:

$$\begin{tikzcd} & \mathcal{N} \arrow[dl, dashed, "(U^t G)"'] \arrow[d, "T^t"] \\ \mathcal{M} & \mathcal{N} \arrow[l, "G"'] \end{tikzcd}$$

for all $G \in C(\mathcal{N})$, where the T^t obtained from the existence of the solution of the mollifies Boltzmann equation. Let A with domain $D \subset C(\mathcal{N})$ be its generator. In particular, $\|U^t\| = 1$.

Define $\mathcal{L} := \mathrm{span}\{ \mathrm{d}\nu \to (\mathrm{d}\pi, \phi^s)|\phi^s \in C^1(\mathbb{R}^{6N}), s = 1, 2, \dots\}$. In other words, $\mathcal{L}$ is an algebra of polynomials of finite degree. The Stone-Weierstrass theorem tells us $\overline{\mathcal{L}} = C(\mathcal{N})$. On $\mathcal{L}$, we define the linear operator L as follows:

$$L : (\mathrm{d}\pi(\nu), \phi^s) \mapsto \left(\mathrm{d}\pi(\nu), \left(\sum_{j=1}^{s} v_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{s} \mathcal{T}_{j,s+1}^{\dagger} \right) \phi^s \right) = (\mathrm{d}\pi, A^{\dagger}\phi^s)$$

where

$$\begin{aligned} \mathcal{T}_{j,s+1}^{\dagger}\phi^s := & \Big(\phi^{s\prime} H(V_{j,s+1} \cdot n_{j,s+1}) - \phi^s H(-V_{j,s+1} \cdot n_{j,s+1}) \Big) \\ & \cdot S_{\delta}(|x_j - x_{s+1}|)|V_{j,s+1} \cdot n_{j,s+1}| \end{aligned}$$

Let $\mathcal{L}_1 := \cup_{t\in\mathbb{R}} U^t\mathcal{L}$. Functions in $U^t\mathcal{L}$ are of the form $(U^t \mathrm{d}\pi, \phi^s)$. Similarly, we define the semigroup U_N^t with respect to N-particle equation.

Our idea is to approximate the measure by a series semigroup acting on a measure. The functions in $\mathcal{L}^1$ are bounded and continuous, but no longer polynomials. We can approximate it by a series of $(\,\mathrm{d}\pi, \psi^s_N)$, which means $(\,\mathrm{d}\pi, \psi^s_N) \to (U^t\,\mathrm{d}\pi, \phi^s)$. It shows that

$$(\,\mathrm{d}\pi, A^\dagger\psi^s_N) \to (U^t\,\mathrm{d}\pi, A^\dagger\phi^s)$$

Therefore, we extend the L on $\mathcal{L}_1$ by L_1. Put it clearly,

$$\begin{tikzcd}
 & (\,\mathrm{d}\pi, A^\dagger\psi^s_N) \arrow[d, "N\to\infty"] \\
U^t\mathcal{L} \ni (U^t\,\mathrm{d}\pi, \phi^s t) \arrow[ur, "L_1"] \arrow[r, "L"'] & (U^t\,\mathrm{d}\pi, A^+\phi^s)
\end{tikzcd}$$

Now we choose $\psi_N^s = (U_N^t)^\dagger \phi^s$, where $(U_N^t)^\dagger$ is the adjoint of U_N^t. By suitable combination, one has

$$
\begin{aligned}
&(\mathrm{d}\pi, A^\dagger \psi_N^s) \\
=&\Bigg((\mathrm{d}\nu)^{s+1}, \underbrace{\Bigg(\sum_{j=1}^{s} v_j \frac{\partial}{\partial x_j} + \frac{1}{N}\sum_{i,j}^{s} \mathcal{T}_{i,j}^\dagger + \frac{N-s}{N}\sum_{j=1}^{s} \mathcal{T}_{j,s+1}^\dagger\Bigg)}_{=:\mathcal{I}} (U_N^t)^\dagger \phi^s\Bigg) \\
&- \frac{1}{N}\Bigg((\mathrm{d}\nu)^{s+1}, \underbrace{\Bigg(\sum_{i,j}^{s} \mathcal{T}_{i,j}^\dagger - s\sum_{j=1}^{s} \mathcal{T}_{j,s+1}^\dagger\Bigg)}_{=:R_s} (U_N^t)^\dagger \phi^s\Bigg)
\end{aligned}
$$

Firstly, we focus on first term in the last equation. We can interchange the order of semigroup $(U_N^t)^\dagger$ and $\mathcal{I}$ which is the generator of $(U_N^t)^\dagger$.

$$
\begin{aligned}
(\mathrm{d}\pi, A^\dagger \psi_N^s) =&(U_N^t(\mathrm{d}\nu)^{s+1}, A^\dagger \phi^s) \\
&+ \frac{1}{N}\Bigg((\mathrm{d}\nu)^{s+1}, ((U_N^t)^\dagger R_s - R_s (U_N^t)^\dagger)\phi^s\Bigg)
\end{aligned}
$$

$(U_N^t)^\dagger$ is contracting, while R_s is bounded with bound independent of N. And

$$
\begin{aligned}
(\mathrm{d}\pi, A^\dagger \psi_N^s) =& (U_N^t(\mathrm{d}\nu)^{s+1}, A^\dagger \phi^s) \\
&+ \underbrace{\frac{1}{N}\Big((\mathrm{d}\nu)^{s+1}, ((U_N^t)^\dagger R_s - R_s (U_N^t)^\dagger)\phi^s \Big)}_{\to 0 \text{ as } N\to\infty}
\end{aligned}
$$

Since $U_N^t(\mathrm{d}\nu)^{s+1} \to U^t(\mathrm{d}\nu)^{s+1}$ weakly,
it shows that

$$(\mathrm{d}\pi, A^\dagger \psi_N^s) \to (U^t\,\mathrm{d}\pi, \phi^s)$$

One has $\overline{\mathcal{L}_1} = A$. Thus L has a unique closed extension $\overline{L} = A$, which implies that U^t is uniquely determined by L.

Part of reference

- Cercignani Carlo, The grad limit for a system of soft sphere, Comm. Pure Appl. Math. 07/1983, 36(4), 476-494.
- Haïm Brezis, Analyse fonctionnelle, 2e tirage, Masson Paris, 1983
- A.Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, 1983
- Arkeryd, L., Ark. Rat. Mech. Anal. 45, 1972, p.1.
- Povzner, A. Ya, On the Boltzmann equation in the kinetic theory of gases, Mat. Sb. (N.S.) 58, 1962, pp.65-86
- John B. Conway, A Course in Abstract Analysis, American Mathematical Society, 2012