Project A1+A2 - Model Selection and Notation

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A1

0. We will use the classroom.csv data for this project.

a. math1st will be the outcome of interest for this first part

b. Recall that `math1st = mathkind + mathgain`
c. Read in the data (R: store as `dat`)
d. Fit all models using REML
e. It's best if you use `lmerTest::lmer` rather than `lme4::lmer` to call the MLM function. The former provides p-values for fixed effects in the summary.
f. There are 2 common error messages one can get from lmer calls: failed to converge (problem with hessian: negative eigenvalue; max|grad| = ...); and singularity. They may both be problematic in a real problem, but the latter suggests that a variance component is on the boundary of the parameter space.

1. In your discussion/writeup, consider the latter to be a "convergence problem" and ignore the former.

```
dat <- read.csv("~/Documents/GitHub/mlm_final_project/data/classroom.csv")
dat <- dat %>%
  mutate(math1st = mathkind + mathgain)
```

1. Estimate an Unconditional Means Model (UMM) with random intercepts for both schools and classrooms (nested in schools).

```
fit1 <- lmer( math1st ~ (1 | schoolid/classid), dat)
summary(fit1)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ (1 | schoolid/classid)
##    Data: dat
##
## REML criterion at convergence: 11944.6
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -5.1872 -0.6174 -0.0204  0.5821  3.8339
```

```
##
## Random effects:
##  Groups           Name        Variance Std.Dev.
##  classid:schoolid (Intercept)   85.46   9.244
##  schoolid         (Intercept)  280.68  16.754
##  Residual                      1146.80  33.864
## Number of obs: 1190, groups:  classid:schoolid, 312; schoolid, 107
##
## Fixed effects:
##             Estimate Std. Error      df t value Pr(>|t|)
## (Intercept)  522.540      2.037 104.407   256.6   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

a. Report the ICC for schools and the ICC for classrooms **Answer:** The ICC for schools is 0.1966269 and the ICC for classrooms is 0.0693518.

b. **WRITE OUT THIS MODEL** using your preferred notation, but use the same choice of notation for the remainder of your project

c. Be mindful and explicit about any assumptions made. $MATH1ST_{ijk} = b_0 + \zeta_{0k} + \eta_{0jk} + \varepsilon_{ijk}$, with $\zeta_{0k} \sim N(0, \sigma^2_{\zeta_0})$, $\eta_{0jk} \sim N(0, \sigma^2_{\eta_0})$ and $\varepsilon_{ijk} \sim N(0, \sigma^2_{\varepsilon})$, independently of one another, j represents classrooms and k represents *schools*.

2. ADD ALL School level predictors

```
fit2 <- lmer( math1st ~ housepov + (1 | schoolid/classid), dat)
summary(fit2)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ housepov + (1 | schoolid/classid)
##    Data: dat
##
## REML criterion at convergence: 11927.4
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -5.1142 -0.6011 -0.0350  0.5600  3.8154
##
## Random effects:
##  Groups           Name        Variance Std.Dev.
##  classid:schoolid (Intercept)   82.36   9.075
##  schoolid         (Intercept)  250.93  15.841
##  Residual                      1146.95  33.867
## Number of obs: 1190, groups:  classid:schoolid, 312; schoolid, 107
##
## Fixed effects:
##             Estimate Std. Error      df t value Pr(>|t|)
## (Intercept)  531.294      3.341 102.809 159.024   <2e-16 ***
## housepov     -45.783     14.236 111.063  -3.216   0.0017 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

```
## Correlation of Fixed Effects:
##          (Intr)
## housepov -0.810
```

```
anova(fit1,fit2)
```

```
## refitting model(s) with ML (instead of REML)
```

```
## Data: dat
## Models:
## fit1: math1st ~ (1 | schoolid/classid)
## fit2: math1st ~ housepov + (1 | schoolid/classid)
##      npar   AIC   BIC  logLik deviance  Chisq Df Pr(>Chisq)
## fit1    4 11956 11976 -5973.9    11948
## fit2    5 11948 11973 -5968.8    11938 10.125  1   0.001463 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

a. Report if adding the predictors as a block is justified There is only one school-level predictor which is housepov, its p-value is $0.0017029 < 0.05$, and I do a LRT on model with and without the school-level predictor, the p-value is $0.0014627 < 0.05$. So it is reasonable to add school-level predictor.
b. Report change in σ^2_ζ. The change in σ^2_ζ is 280.6812733-250.9258585 = 29.7554148.

3. ADD ALL Classroom level predictors

```
save.options = options()
options(na.action = "na.pass")
mm <- model.matrix(~math1st + ses + mathknow, data = dat)
in_sample <- apply(is.na(mm), 1, sum) == 0 # these rows aren't missing anything
options(save.options)
# remove those na
fit3 <- lmer( math1st ~ yearstea + mathknow + mathprep + housepov + (1 | schoolid/classid),
              dat, subset = in_sample)
summary(fit3)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula:
## math1st ~ yearstea + mathknow + mathprep + housepov + (1 | schoolid/classid)
##    Data: dat
##  Subset: in_sample
##
## REML criterion at convergence: 10821
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.5552 -0.6118 -0.0311  0.5863  3.8315
##
## Random effects:
##  Groups           Name        Variance Std.Dev.
##  classid:schoolid (Intercept)   94.36   9.714
##  schoolid         (Intercept)  223.31  14.943
```

```
##  Residual                     1136.43  33.711
## Number of obs: 1081, groups:  classid:schoolid, 285; schoolid, 105
##
## Fixed effects:
##              Estimate Std. Error        df t value Pr(>|t|)
## (Intercept) 532.29852    5.20495 228.85767 102.268  < 2e-16 ***
## yearstea      0.06193    0.14717 223.76570   0.421  0.67432
## mathknow      2.55143    1.44530 231.06560   1.765  0.07883 .
## mathprep     -0.75440    1.42809 203.20755  -0.528  0.59790
## housepov    -41.62117   14.08834 109.83230  -2.954  0.00383 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
##          (Intr) yearst mthknw mthprp
## yearstea -0.264
## mathknow -0.052  0.030
## mathprep -0.666 -0.175  0.004
## housepov -0.568  0.077  0.082  0.032
```

```
wald.test(b = fixef(fit3), Sigma = summary(fit3)$vcov, Terms = 2:4)
```

```
## Wald test:
## ----------
##
## Chi-squared test:
## X2 = 3.5, df = 3, P(> X2) = 0.32
```

a. Report if adding the predictors as a block is justified [must use WALD test, not LRT] **Answer:** The Wald test generates a p-value = 0.22, which shows that we have no reason to add classroom-level predictors as a block. But it might be reasonable to include `mathknow` since it is significant according to the t-test.
b. Report change in σ^2_η and change in σ^2_ϵ. The change in σ^2_η is 94.3625825-85.4593745 = 8.903208 and change in σ^2_ϵ is 1136.4309806-1146.8001472 = -10.3691666.
c. Give a potential reason as to why σ^2_ϵ is reduced, but not σ^2_η?

A potential might be, the same teacher (same classroom)might have different effect on each student, therefore when we introduce a classroom-level fix effect, the model does not reduces classroom-level random effect, instead, reduces individual-level effect.

4. ADD (nearly) ALL student level predictors (but not `mathgain` or `mathkind`, as these are outcomes in this context).

```
fit4 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                housepov + (1 | schoolid/classid), dat, subset = in_sample)
summary(fit4)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
##     housepov + (1 | schoolid/classid)
##    Data: dat
```

```
##  Subset: in_sample
##
## REML criterion at convergence: 10729.5
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.8581 -0.6134 -0.0321  0.5971  3.6598
##
## Random effects:
##  Groups           Name        Variance Std.Dev.
##  classid:schoolid (Intercept)   93.89   9.689
##  schoolid         (Intercept)  169.45  13.017
##  Residual                      1064.96  32.634
## Number of obs: 1081, groups:  classid:schoolid, 285; schoolid, 105
##
## Fixed effects:
##               Estimate Std. Error         df t value Pr(>|t|)
## (Intercept)  539.63041    5.31209  275.39010 101.585  < 2e-16 ***
## ses           10.05076    1.54485 1066.56211   6.506 1.18e-10 ***
## minority     -16.18676    3.02605  704.47787  -5.349 1.20e-07 ***
## sex           -1.21419    2.09483 1022.42110  -0.580    0.562
## yearstea       0.01129    0.14141  226.80861   0.080    0.936
## mathknow       1.35004    1.39168  234.49768   0.970    0.333
## mathprep      -0.27705    1.37583  205.27111  -0.201    0.841
## housepov     -17.64850   13.21755  113.87814  -1.335    0.184
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
##          (Intr) ses    minrty sex    yearst mthknw mthprp
## ses      -0.121
## minority -0.320  0.162
## sex      -0.190  0.020 -0.011
## yearstea -0.259 -0.028  0.024  0.016
## mathknow -0.083 -0.007  0.115  0.007  0.029
## mathprep -0.631  0.053  0.001 -0.006 -0.172  0.004
## housepov -0.451  0.082 -0.178 -0.007  0.071  0.058  0.038
```

```
wald.test(b = fixef(fit4), Sigma = summary(fit4)$vcov, Terms = 2:4)
```

```
## Wald test:
## ----------
##
## Chi-squared test:
## X2 = 85.1, df = 3, P(> X2) = 0.0
```

a. Report if justified statistically as a block of predictors [must use WALD test, not LRT] The wald test gives a p-value less than 0.05, which justifies the significance of adding a block of individual predictors.
b. Report change in variance components for all levels The change in σ^2_η is 93.8853485-85.4593745 = 8.425974, the change in σ^2_ζ is 169.4480999-280.6812733 = -111.2331734 and change in σ^2_ϵ is 1064.9564422-1146.8001472 = -81.8437049.
c. Give a potential reason as to why the school level variance component drops from prior model Individual predictors are correlated with school-level effect.

d. WRITE OUT THIS MODEL using your chosen notation (include assumptions). $MATH1ST_{ijk} = b_0 + b_1 SES_{ijk} + b_2 MINORITY_{ijk} + b_3 SEX_{ijk} + b_4 YEARSTEA_{jk} + b_5 MATHKNOW_{jk} + b_6 MATHPREP_{jk} + b_7 HOUSEPOV_k + \zeta_{0k} + \eta_{0jk} + \varepsilon_{ijk}$, with $\zeta_{0k} \sim N(0, \sigma^2_{\zeta_0})$, $\eta_{0jk} \sim N(0, \sigma^2_{\eta_0})$ and $\varepsilon_{ijk} \sim N(0, \sigma^2_{\varepsilon})$, independently of one another, j represents classrooms and k represents *schools*. 5.a. Try to add a random slope for each teacher level predictor (varying at the school level; one by one separately- not all together)

e. Report the model fit or lack of fit

```
fit5.1 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                  housepov + (1 | schoolid/classid) + (0 + yearstea | schoolid),
                dat, subset = in_sample)
```

```
## Warning in checkConv(attr(opt, "derivs"), opt$par, ctrl = control$checkConv, :
## Model failed to converge with max|grad| = 0.00805459 (tol = 0.002, component 1)
```

```
summary(fit5.1)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
##     housepov + (1 | schoolid/classid) + (0 + yearstea | schoolid)
##    Data: dat
##  Subset: in_sample
##
## REML criterion at convergence: 10729.5
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.8482 -0.6147 -0.0322  0.5979  3.6603
##
## Random effects:
##  Groups           Name        Variance  Std.Dev.
##  classid.schoolid (Intercept) 9.247e+01  9.6159
##  schoolid         (Intercept) 1.684e+02 12.9758
##  schoolid.1       yearstea    1.008e-02  0.1004
##  Residual                     1.065e+03 32.6361
## Number of obs: 1081, groups:  classid:schoolid, 285; schoolid, 105
##
## Fixed effects:
##               Estimate Std. Error         df t value Pr(>|t|)
## (Intercept)  539.59885    5.30780  266.47954 101.662  < 2e-16 ***
## ses           10.04528    1.54492 1066.09816   6.502 1.21e-10 ***
## minority     -16.16715    3.02635  702.61831  -5.342 1.24e-07 ***
## sex           -1.21060    2.09480 1022.21558  -0.578    0.563
## yearstea       0.01128    0.14192  122.87743   0.079    0.937
## mathknow       1.33106    1.39155  234.33195   0.957    0.340
## mathprep      -0.26584    1.37588  204.90504  -0.193    0.847
## housepov     -17.72082   13.21686  113.58577  -1.341    0.183
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
##          (Intr) ses    minrty sex    yearst mthknw mthprp
```

```
## ses      -0.121
## minority -0.320  0.162
## sex      -0.191  0.020 -0.010
## yearstea -0.258 -0.027  0.023  0.015
## mathknow -0.082 -0.007  0.115  0.006  0.028
## mathprep -0.632  0.053  0.001 -0.006 -0.172  0.003
## housepov -0.450  0.082 -0.179 -0.007  0.070  0.057  0.037
## optimizer (nloptwrap) convergence code: 0 (OK)
## Model failed to converge with max|grad| = 0.00805459 (tol = 0.002, component 1)
```

```
fit5.2 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                  housepov + (1 | schoolid/classid) + (0 + mathknow | schoolid),
                dat, subset = in_sample)
```

```
## boundary (singular) fit: see ?isSingular
```

```
summary(fit5.2)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
##     housepov + (1 | schoolid/classid) + (0 + mathknow | schoolid)
##    Data: dat
##  Subset: in_sample
##
## REML criterion at convergence: 10729.5
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.8580 -0.6134 -0.0321  0.5971  3.6598
##
## Random effects:
##  Groups           Name        Variance  Std.Dev.
##  classid.schoolid (Intercept) 9.389e+01  9.689654
##  schoolid         (Intercept) 1.694e+02 13.017245
##  schoolid.1       mathknow    2.323e-07  0.000482
##  Residual                     1.065e+03 32.633630
## Number of obs: 1081, groups:  classid:schoolid, 285; schoolid, 105
##
## Fixed effects:
##               Estimate Std. Error         df t value Pr(>|t|)
## (Intercept)  539.63042    5.31210  275.38873 101.585  < 2e-16 ***
## ses           10.05075    1.54484 1066.56223   6.506 1.18e-10 ***
## minority     -16.18678    3.02605  704.47917  -5.349 1.20e-07 ***
## sex           -1.21419    2.09483 1022.42143  -0.580    0.562
## yearstea       0.01129    0.14141  226.80898   0.080    0.936
## mathknow       1.35004    1.39169  234.49763   0.970    0.333
## mathprep      -0.27705    1.37583  205.27161  -0.201    0.841
## housepov     -17.64848   13.21759  113.87742  -1.335    0.184
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
```

```
##          (Intr) ses    minrty sex    yearst mthknw mthprp
## ses      -0.121
## minority -0.320  0.162
## sex      -0.190  0.020 -0.011
## yearstea -0.259 -0.028  0.024  0.016
## mathknow -0.083 -0.007  0.115  0.007  0.029
## mathprep -0.631  0.053  0.001 -0.006 -0.172  0.004
## housepov -0.451  0.082 -0.178 -0.007  0.071  0.058  0.038
## optimizer (nloptwrap) convergence code: 0 (OK)
## boundary (singular) fit: see ?isSingular
```

```
fit5.3 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                  housepov + (1 | schoolid/classid) + (0 + mathprep | schoolid),
                dat, subset = in_sample)
```

```
## boundary (singular) fit: see ?isSingular
```

```
summary(fit5.3)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
##     housepov + (1 | schoolid/classid) + (0 + mathprep | schoolid)
##    Data: dat
##  Subset: in_sample
##
## REML criterion at convergence: 10729.5
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.8581 -0.6134 -0.0321  0.5971  3.6598
##
## Random effects:
##  Groups           Name        Variance  Std.Dev.
##  classid.schoolid (Intercept) 9.388e+01 9.689e+00
##  schoolid         (Intercept) 1.694e+02 1.302e+01
##  schoolid.1       mathprep    2.171e-07 4.659e-04
##  Residual                     1.065e+03 3.263e+01
## Number of obs: 1081, groups:  classid:schoolid, 285; schoolid, 105
##
## Fixed effects:
##               Estimate Std. Error         df t value Pr(>|t|)
## (Intercept)  539.63039    5.31207  275.39223 101.586  < 2e-16 ***
## ses           10.05076    1.54485 1066.56201   6.506 1.18e-10 ***
## minority     -16.18676    3.02605  704.47629  -5.349 1.20e-07 ***
## sex           -1.21419    2.09483 1022.42070  -0.580    0.562
## yearstea       0.01129    0.14141  226.80838   0.080    0.936
## mathknow       1.35003    1.39167  234.49786   0.970    0.333
## mathprep      -0.27705    1.37582  205.27063  -0.201    0.841
## housepov     -17.64851   13.21749  113.87941  -1.335    0.184
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

```
## Correlation of Fixed Effects:
##          (Intr) ses    minrty sex    yearst mthknw mthprp
## ses      -0.121
## minority -0.320  0.162
## sex      -0.190  0.020 -0.011
## yearstea -0.259 -0.028  0.024  0.016
## mathknow -0.083 -0.007  0.115  0.007  0.029
## mathprep -0.631  0.053  0.001 -0.006 -0.172  0.004
## housepov -0.451  0.082 -0.178 -0.007  0.071  0.058  0.038
## optimizer (nloptwrap) convergence code: 0 (OK)
## boundary (singular) fit: see ?isSingular
```

Answer 5b: The model with random slope on `mathknow` and the model with random slope on `mathprep` have convergent problem.

c. Retry the above, allowing the slopes to be correlated with the random intercepts (still one by one)

```
fit5.c.1 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                    housepov + (yearstea | schoolid) + (1 | classid),
                  dat, subset = in_sample)
```

```
## Warning in checkConv(attr(opt, "derivs"), opt$par, ctrl = control$checkConv, :
## Model failed to converge with max|grad| = 0.00352934 (tol = 0.002, component 1)
```

```
fit5.c.2 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                    housepov + (mathknow| schoolid) + (1 | classid),
                  dat, subset = in_sample)
fit5.c.2 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                    housepov + (mathknow| classid) + (1 | schoolid/classid),
                  dat, subset = in_sample)
```

```
## boundary (singular) fit: see ?isSingular
```

```
summary(fit5.c.2)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
##     housepov + (mathknow | classid) + (1 | schoolid/classid)
##    Data: dat
##  Subset: in_sample
##
## REML criterion at convergence: 10728.9
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.8153 -0.6180 -0.0277  0.5980  3.6480
##
## Random effects:
##  Groups           Name        Variance  Std.Dev.  Corr
##  classid          (Intercept) 9.325e+01 9.657e+00
```

```
##                   mathknow    2.875e+00 1.696e+00 -1.00
##  classid:schoolid (Intercept) 7.566e-07 8.698e-04
##  schoolid         (Intercept) 1.703e+02 1.305e+01
##  Residual                     1.063e+03 3.260e+01
## Number of obs: 1081, groups:
## classid, 285; classid:schoolid, 285; schoolid, 105
##
## Fixed effects:
##               Estimate Std. Error         df t value Pr(>|t|)
## (Intercept)  538.76572    5.31426  276.86273 101.381  < 2e-16 ***
## ses           10.10871    1.54203 1056.01776   6.555 8.66e-11 ***
## minority     -15.93572    3.01904  692.79393  -5.278 1.75e-07 ***
## sex           -1.14293    2.09059 1009.92162  -0.547    0.585
## yearstea       0.04044    0.14127  228.73831   0.286    0.775
## mathknow       1.29405    1.37532  238.60451   0.941    0.348
## mathprep      -0.17654    1.37089  207.77481  -0.129    0.898
## housepov     -17.59393   13.28370  114.68680  -1.324    0.188
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
##          (Intr) ses    minrty sex    yearst mthknw mthprp
## ses      -0.123
## minority -0.316  0.163
## sex      -0.193  0.020 -0.010
## yearstea -0.270 -0.027  0.023  0.016
## mathknow -0.098 -0.009  0.114  0.004  0.015
## mathprep -0.630  0.054 -0.003 -0.005 -0.157  0.003
## housepov -0.451  0.083 -0.178 -0.004  0.072  0.063  0.034
## optimizer (nloptwrap) convergence code: 0 (OK)
## boundary (singular) fit: see ?isSingular
```

```
fit5.c.3 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                    housepov + (mathprep | schoolid) + (1|classid),
                  dat, subset = in_sample)
```

```
## boundary (singular) fit: see ?isSingular
```

```
summary(fit5.c.3)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
##     housepov + (mathprep | schoolid) + (1 | classid)
##    Data: dat
##  Subset: in_sample
##
## REML criterion at convergence: 10724.7
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.8542 -0.6034 -0.0221  0.5914  3.6475
##
```

```
## Random effects:
##  Groups   Name        Variance Std.Dev. Corr
##  classid  (Intercept)   78.46   8.858
##  schoolid (Intercept)  552.78  23.511
##           mathprep      15.89   3.986   -1.00
##  Residual             1064.26  32.623
## Number of obs: 1081, groups:  classid, 285; schoolid, 105
##
## Fixed effects:
##               Estimate Std. Error         df t value Pr(>|t|)
## (Intercept)  538.60853    5.60817  159.88504  96.040  < 2e-16 ***
## ses           10.14166    1.53961 1060.93429   6.587 7.04e-11 ***
## minority     -16.46420    2.99525  663.67458  -5.497 5.52e-08 ***
## sex           -1.16760    2.08697 1023.15165  -0.559    0.576
## yearstea      -0.02587    0.13949  223.50105  -0.185    0.853
## mathknow       1.29890    1.37194  229.68059   0.947    0.345
## mathprep       0.04076    1.34846  139.04922   0.030    0.976
## housepov     -14.01322   12.88712  116.05270  -1.087    0.279
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
##          (Intr) ses    minrty sex    yearst mthknw mthprp
## ses      -0.121
## minority -0.275  0.161
## sex      -0.183  0.024 -0.013
## yearstea -0.260 -0.033  0.025  0.023
## mathknow -0.071 -0.001  0.107  0.002  0.049
## mathprep -0.692  0.061 -0.035 -0.008 -0.155  0.012
## housepov -0.461  0.095 -0.187  0.003  0.089  0.027  0.107
## optimizer (nloptwrap) convergence code: 0 (OK)
## boundary (singular) fit: see ?isSingular
```

d. Report anything unusual about the variance components (changes that are in a direction you didn't expect) and any potential explanation for why those changes occurred (hint: what did you add to the model?).

```
VarCorr(fit5.1)
```

```
##  Groups           Name        Std.Dev.
##  classid.schoolid (Intercept)  9.61590
##  schoolid         (Intercept) 12.97582
##  schoolid.1       yearstea     0.10039
##  Residual                     32.63608
```

```
VarCorr(fit5.c.1)
```

```
##  Groups   Name        Std.Dev. Corr
##  classid  (Intercept)  6.15207
##  schoolid (Intercept) 19.13695
##           yearstea     0.74341 -0.780
##  Residual             32.65709
```

```
VarCorr(fit5.2)
```

```
##  Groups           Name        Std.Dev.
##  classid.schoolid (Intercept) 9.6897e+00
##  schoolid         (Intercept) 1.3017e+01
##  schoolid.1       mathknow    4.8198e-04
##  Residual                     3.2634e+01
```

```
VarCorr(fit5.c.2)
```

```
##  Groups           Name        Std.Dev.   Corr
##  classid          (Intercept) 9.6569e+00
##                   mathknow    1.6956e+00 -1.000
##  classid:schoolid (Intercept) 8.6984e-04
##  schoolid         (Intercept) 1.3049e+01
##  Residual                     3.2604e+01
```

```
VarCorr(fit5.3)
```

```
##  Groups           Name        Std.Dev.
##  classid.schoolid (Intercept) 9.6893e+00
##  schoolid         (Intercept) 1.3017e+01
##  schoolid.1       mathprep    4.6594e-04
##  Residual                     3.2634e+01
```

```
VarCorr(fit5.c.3)
```

```
##  Groups   Name        Std.Dev. Corr
##  classid  (Intercept)  8.8579
##  schoolid (Intercept) 23.5112
##           mathprep     3.9857  -1.000
##  Residual             32.6230
```

It seems that in all these models, the random slope and random intercepts are highly correlated. After introducing correlation between random slope and random intercept, we found that school-level random intercept increase a lot in the model with `mathprep` or `yearstea`, while it decrease in the model with `mathknow`.

6. Question:

a. Why is it a bad idea to include a classroom-level variable with random slopes at the classroom level?
Answer: The coefficient on classroom-level variable and the classroom-level variable accounts for the same variation in the population, as a result, adding a classroom-level random slope on classroom-level variable would be redundant.
Alternative: Classroom-level variables only vary at school-level, so the classroom-level random slope would capture limited variation.

A2

7. Question:

a. For UMM, write down: V_S, V_C, V_E for the three variance components (simply the estimates) **Answer:** We have that V_S = 280.68, V_C = 85.46, and V_E = 1146.8
b. For the most complicated (all fixed effects) random INTERCEPTS ONLY model, what are: V_C, V_S, V_E? **Answer:** We have in this model that V_S = 169.45, V_C = 93.89, V_E = 1064.96
c. By what fraction did these each decrease with the new predictors in the model? **Answer:** The fraction decrease for V_S, and V_E are `round((280.68-169.45)/280.68, 2)`, and `round((1146.8-1064.96)/1146.8, 2)`, respectively. But for V_C it actually increased `round((93.89-85.46)/85.46, 2)` fraction-wise.

8. a.

```
fit8.a.1 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                housepov + (1 | schoolid/classid) + (0 + ses || schoolid), dat, subset = in_sample)
summary(fit8.a.1)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
##     housepov + (1 | schoolid/classid) + (0 + ses || schoolid)
##    Data: dat
##  Subset: in_sample
##
## REML criterion at convergence: 10724.8
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.6138 -0.6185 -0.0290  0.5798  3.7130
##
## Random effects:
##  Groups           Name        Variance Std.Dev.
##  classid.schoolid (Intercept)   88.56   9.411
##  schoolid         (Intercept)  167.98  12.961
##  schoolid.1       ses           72.50   8.515
##  Residual                     1035.12  32.173
## Number of obs: 1081, groups:  classid:schoolid, 285; schoolid, 105
##
## Fixed effects:
##               Estimate Std. Error         df t value Pr(>|t|)
## (Intercept)  539.13754    5.27918  270.54292 102.125  < 2e-16 ***
## ses            9.78982    1.82217   79.01642   5.373 7.62e-07 ***
## minority     -16.52526    3.02189  700.06722  -5.469 6.32e-08 ***
## sex           -1.40185    2.08170 1011.28952  -0.673    0.501
## yearstea       0.03079    0.14052  223.94368   0.219    0.827
## mathknow       1.35576    1.38459  232.20020   0.979    0.329
## mathprep      -0.19801    1.35994  198.59489  -0.146    0.884
## housepov     -16.94561   13.21117  112.82498  -1.283    0.202
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

```
## Correlation of Fixed Effects:
##          (Intr) ses    minrty sex    yearst mthknw mthprp
## ses      -0.091
## minority -0.323  0.124
## sex      -0.190  0.017 -0.010
## yearstea -0.260 -0.019  0.024  0.018
## mathknow -0.079  0.006  0.110  0.006  0.028
## mathprep -0.628  0.042  0.001 -0.007 -0.172  0.002
## housepov -0.451  0.076 -0.180 -0.007  0.070  0.056  0.041
```

```
fit8.a.2 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                housepov + (1 | schoolid/classid) + (0 + sex || schoolid), dat, subset = in_sample)
summary(fit8.a.2)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
##     housepov + (1 | schoolid/classid) + (0 + sex || schoolid)
##    Data: dat
##  Subset: in_sample
##
## REML criterion at convergence: 10728.9
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.8578 -0.6110 -0.0259  0.5922  3.5557
##
## Random effects:
##  Groups           Name        Variance Std.Dev.
##  classid.schoolid (Intercept)   96.08   9.802
##  schoolid         (Intercept)  161.63  12.713
##  schoolid.1       sex           35.84   5.986
##  Residual                     1054.36  32.471
## Number of obs: 1081, groups:  classid:schoolid, 285; schoolid, 105
##
## Fixed effects:
##               Estimate Std. Error         df t value Pr(>|t|)
## (Intercept)  539.43517    5.30740  272.54993 101.638  < 2e-16 ***
## ses            9.98477    1.54243 1058.27916   6.473 1.46e-10 ***
## minority     -16.16537    3.02861  704.25756  -5.338 1.27e-07 ***
## sex           -1.33535    2.18747  138.09087  -0.610    0.543
## yearstea       0.01448    0.14163  226.44539   0.102    0.919
## mathknow       1.40067    1.39464  234.45909   1.004    0.316
## mathprep      -0.27193    1.38011  205.78530  -0.197    0.844
## housepov     -16.77652   13.22879  112.39634  -1.268    0.207
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
##          (Intr) ses    minrty sex    yearst mthknw mthprp
## ses      -0.120
## minority -0.320  0.161
## sex      -0.179  0.020 -0.015
## yearstea -0.259 -0.029  0.024  0.013
```

```
## mathknow -0.081 -0.007  0.114  0.007  0.028
## mathprep -0.633  0.052  0.001 -0.004 -0.172  0.004
## housepov -0.449  0.081 -0.178 -0.010  0.070  0.055  0.036
```

```
fit8.a.3 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                housepov + (1 | schoolid/classid) + (0 + minority || schoolid), dat, subset = in_sample)
```

```
## boundary (singular) fit: see ?isSingular
```

```
summary(fit8.a.3)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
##     housepov + (1 | schoolid/classid) + (0 + minority || schoolid)
##    Data: dat
##  Subset: in_sample
##
## REML criterion at convergence: 10729.5
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.8580 -0.6134 -0.0321  0.5971  3.6598
##
## Random effects:
##  Groups           Name        Variance Std.Dev.
##  classid.schoolid (Intercept)   93.89   9.69
##  schoolid         (Intercept)  169.45  13.02
##  schoolid.1       minority       0.00   0.00
##  Residual                     1064.96  32.63
## Number of obs: 1081, groups:  classid:schoolid, 285; schoolid, 105
##
## Fixed effects:
##               Estimate Std. Error         df t value Pr(>|t|)
## (Intercept)  539.63041    5.31209  275.39107 101.585  < 2e-16 ***
## ses           10.05075    1.54484 1066.56217   6.506 1.18e-10 ***
## minority     -16.18677    3.02605  704.47765  -5.349 1.20e-07 ***
## sex           -1.21419    2.09483 1022.42106  -0.580    0.562
## yearstea       0.01129    0.14141  226.80889   0.080    0.936
## mathknow       1.35003    1.39168  234.49798   0.970    0.333
## mathprep      -0.27705    1.37583  205.27126  -0.201    0.841
## housepov     -17.64847   13.21752  113.87889  -1.335    0.184
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
##          (Intr) ses    minrty sex    yearst mthknw mthprp
## ses      -0.121
## minority -0.320  0.162
## sex      -0.190  0.020 -0.011
## yearstea -0.259 -0.028  0.024  0.016
## mathknow -0.083 -0.007  0.115  0.007  0.029
## mathprep -0.631  0.053  0.001 -0.006 -0.172  0.004
```

```
## housepov -0.451  0.082 -0.178 -0.007  0.071  0.058  0.038
## optimizer (nloptwrap) convergence code: 0 (OK)
## boundary (singular) fit: see ?isSingular
```

b. Retry part (a), allowing the slopes to be correlated with the random intercepts.

```
fit8.b.1 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                housepov + (1 | schoolid/classid) + (ses || schoolid), dat, subset = in_sample)
```

```
## Warning in checkConv(attr(opt, "derivs"), opt$par, ctrl = control$checkConv, :
## Model failed to converge with max|grad| = 0.00340779 (tol = 0.002, component 1)
```

```
summary(fit8.b.1)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
##     housepov + (1 | schoolid/classid) + (ses || schoolid)
##    Data: dat
##  Subset: in_sample
##
## REML criterion at convergence: 10724.8
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.6138 -0.6185 -0.0290  0.5798  3.7130
##
## Random effects:
##  Groups           Name        Variance Std.Dev.
##  classid.schoolid (Intercept)   88.56   9.411
##  schoolid         (Intercept)   24.35   4.934
##  schoolid.1       (Intercept)  143.63  11.985
##  schoolid.2       ses           72.50   8.515
##  Residual                      1035.12  32.173
## Number of obs: 1081, groups:  classid:schoolid, 285; schoolid, 105
##
## Fixed effects:
##               Estimate Std. Error         df t value Pr(>|t|)
## (Intercept)  539.13752    5.27918  270.54021 102.125  < 2e-16 ***
## ses            9.78982    1.82216   79.01642   5.373 7.61e-07 ***
## minority     -16.52525    3.02189  700.06821  -5.469 6.32e-08 ***
## sex           -1.40185    2.08170 1011.28946  -0.673    0.501
## yearstea       0.03079    0.14052  223.94290   0.219    0.827
## mathknow       1.35578    1.38459  232.19932   0.979    0.329
## mathprep      -0.19800    1.35994  198.59443  -0.146    0.884
## housepov     -16.94568   13.21125  112.82290  -1.283    0.202
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
##          (Intr) ses    minrty sex    yearst mthknw mthprp
## ses      -0.091
```

```
## minority -0.323  0.124
## sex      -0.190  0.017 -0.010
## yearstea -0.260 -0.019  0.024  0.018
## mathknow -0.079  0.006  0.110  0.006  0.028
## mathprep -0.628  0.042  0.001 -0.007 -0.172  0.002
## housepov -0.451  0.076 -0.180 -0.007  0.070  0.056  0.041
## optimizer (nloptwrap) convergence code: 0 (OK)
## Model failed to converge with max|grad| = 0.00340779 (tol = 0.002, component 1)
```

```
fit8.b.2 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                housepov + (1 | schoolid/classid) + (ses || schoolid), dat, subset = in_sample)
```

```
## Warning in checkConv(attr(opt, "derivs"), opt$par, ctrl = control$checkConv, :
## Model failed to converge with max|grad| = 0.00340779 (tol = 0.002, component 1)
```

```
summary(fit8.b.2)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
##     housepov + (1 | schoolid/classid) + (ses || schoolid)
##    Data: dat
##  Subset: in_sample
##
## REML criterion at convergence: 10724.8
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.6138 -0.6185 -0.0290  0.5798  3.7130
##
## Random effects:
##  Groups           Name        Variance Std.Dev.
##  classid.schoolid (Intercept)   88.56   9.411
##  schoolid         (Intercept)   24.35   4.934
##  schoolid.1       (Intercept)  143.63  11.985
##  schoolid.2       ses           72.50   8.515
##  Residual                      1035.12  32.173
## Number of obs: 1081, groups:  classid:schoolid, 285; schoolid, 105
##
## Fixed effects:
##               Estimate Std. Error         df t value Pr(>|t|)
## (Intercept)  539.13752    5.27918  270.54021 102.125  < 2e-16 ***
## ses            9.78982    1.82216   79.01642   5.373 7.61e-07 ***
## minority     -16.52525    3.02189  700.06821  -5.469 6.32e-08 ***
## sex           -1.40185    2.08170 1011.28946  -0.673    0.501
## yearstea       0.03079    0.14052  223.94290   0.219    0.827
## mathknow       1.35578    1.38459  232.19932   0.979    0.329
## mathprep      -0.19800    1.35994  198.59443  -0.146    0.884
## housepov     -16.94568   13.21125  112.82290  -1.283    0.202
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Correlation of Fixed Effects:
```

```
##          (Intr) ses    minrty sex    yearst mthknw mthprp
## ses      -0.091
## minority -0.323  0.124
## sex      -0.190  0.017 -0.010
## yearstea -0.260 -0.019  0.024  0.018
## mathknow -0.079  0.006  0.110  0.006  0.028
## mathprep -0.628  0.042  0.001 -0.007 -0.172  0.002
## housepov -0.451  0.076 -0.180 -0.007  0.070  0.056  0.041
## optimizer (nloptwrap) convergence code: 0 (OK)
## Model failed to converge with max|grad| = 0.00340779 (tol = 0.002, component 1)
```

```
fit8.b.3 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
               housepov + (1 | schoolid/classid) + (ses || schoolid), dat, subset = in_sample)
```

```
## Warning in checkConv(attr(opt, "derivs"), opt$par, ctrl = control$checkConv, :
## Model failed to converge with max|grad| = 0.00340779 (tol = 0.002, component 1)
```

```
summary(fit8.b.3)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
##     housepov + (1 | schoolid/classid) + (ses || schoolid)
##    Data: dat
##  Subset: in_sample
##
## REML criterion at convergence: 10724.8
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.6138 -0.6185 -0.0290  0.5798  3.7130
##
## Random effects:
##  Groups           Name        Variance Std.Dev.
##  classid.schoolid (Intercept)   88.56   9.411
##  schoolid         (Intercept)   24.35   4.934
##  schoolid.1       (Intercept)  143.63  11.985
##  schoolid.2       ses           72.50   8.515
##  Residual                      1035.12  32.173
## Number of obs: 1081, groups:  classid:schoolid, 285; schoolid, 105
##
## Fixed effects:
##               Estimate Std. Error         df t value Pr(>|t|)
## (Intercept)  539.13752    5.27918  270.54021 102.125  < 2e-16 ***
## ses            9.78982    1.82216   79.01642   5.373 7.61e-07 ***
## minority     -16.52525    3.02189  700.06821  -5.469 6.32e-08 ***
## sex           -1.40185    2.08170 1011.28946  -0.673    0.501
## yearstea       0.03079    0.14052  223.94290   0.219    0.827
## mathknow       1.35578    1.38459  232.19932   0.979    0.329
## mathprep      -0.19800    1.35994  198.59443  -0.146    0.884
## housepov     -16.94568   13.21125  112.82290  -1.283    0.202
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
##
## Correlation of Fixed Effects:
##          (Intr) ses    minrty sex    yearst mthknw mthprp
## ses      -0.091
## minority -0.323  0.124
## sex      -0.190  0.017 -0.010
## yearstea -0.260 -0.019  0.024  0.018
## mathknow -0.079  0.006  0.110  0.006  0.028
## mathprep -0.628  0.042  0.001 -0.007 -0.172  0.002
## housepov -0.451  0.076 -0.180 -0.007  0.070  0.056  0.041
## optimizer (nloptwrap) convergence code: 0 (OK)
## Model failed to converge with max|grad| = 0.00340779 (tol = 0.002, component 1)
```

c. Report anything unusual about the variance components (changes that are unexpected) **Answer:** The last 3 models with slopes correlated with random intercepts failed to converge. One unusual thing is that the model with a random slope for minority has no change in the variance components.

9. a. Take the two predictors that had significant (at .05 level) random slopes, in the forms in which they worked (indep. or correlated) and add both to the model, and test for need of one conditional on already including the other.

```
fit9 <- lmer( math1st ~ ses + minority + sex + yearstea + mathknow + mathprep +
                housepov + (1 | schoolid/classid) + (0 + sex + ses || schoolid), dat, subset = in_sample

anova(fit8.a.1, fit9) #P = 0.4282
```

```
## refitting model(s) with ML (instead of REML)
```

```
## Data: dat
## Subset: in_sample
## Models:
## fit8.a.1: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep + housepov + (1 | schoolid/
## fit9: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep + housepov + (1 | schoolid/clas
##          npar   AIC   BIC  logLik deviance  Chisq Df Pr(>Chisq)
## fit8.a.1   12 10772 10832 -5373.9    10748
## fit9       13 10773 10838 -5373.6    10747 0.6276  1     0.4282
```

```
anova(fit8.a.2, fit9) #P = 0.0333
```

```
## refitting model(s) with ML (instead of REML)
```

```
## Data: dat
## Subset: in_sample
## Models:
## fit8.a.2: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep + housepov + (1 | schoolid/
## fit9: math1st ~ ses + minority + sex + yearstea + mathknow + mathprep + housepov + (1 | schoolid/clas
##          npar   AIC   BIC  logLik deviance  Chisq Df Pr(>Chisq)
## fit8.a.2   12 10776 10836 -5375.9    10752
## fit9       13 10773 10838 -5373.6    10747 4.5286  1    0.03333 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

b. Is the more complex model (with both random slopes in it) justified? **Answer:** It is only justified compared to the model that has just sex as a random slope. Based on the LRT having ses with a random slope as well is justified based on the 0.03 p-value.

c. WRITE OUT THIS MODEL in your preferred notation (include assumptions) $MATH1ST_{ijk} = b_0 + (b_1 + \zeta_{1k})SES_{ijk} + b_2 MINORITY_{ijk} + (b_3 + \zeta_{3k})SEX_{ijk} + b_4 YEARSTEA_{jk} + b_5 MATHKNOW_{jk} + b_6 MATHPREP_{jk} + b_7 HOUSEPOV_k + \zeta_{0k} + \eta_{0jk} + \varepsilon_{ijk}$, with $\zeta_{0k} \sim N(0, \sigma^2_{\zeta_0})$, $\zeta_{1k} \sim N(0, \sigma^2_{\zeta_1})$, $\zeta_{3k} \sim N(0, \sigma^2_{\zeta_3})$ $\eta_{0jk} \sim N(0, \sigma^2_{\eta_0})$ and $\varepsilon_{ijk} \sim N(0, \sigma^2_{\varepsilon})$

10. Now consider the model with a random slope only in minority. We will make predictions at levels of minority in the range 0 to 1 for illustrative purposes.

a. What are: V_C, V_S(minority=0), V_E? i. We need to list 'minority=0' here, or we don't know how to use the slope variance.

b. What are: V_S(minority=0.25), V_S(minority=+0.50), V_S(minority=+0.75)?

c. Is the variance between schools monotonically increasing in the value of minority?