Assignment #3. Advanced numerical solvers based on spectral methods.

In this assignment, the spectral methods will be put into practice by assembling pieces from the fundamental spectral methods toolbox we have developed so far.

2D Lid-Driven Cavity flow for a newtonian fluid

In this part of the assignment, we will consider developing a numerical tool based on a Spectral Method. The tool will be useful for determining the solutions of incompressible steady Navier-Stokes equations which may be written in dimensionless form over some domain $\Omega \subset \mathbb{R}^2$ as

$$\nabla \cdot \mathbf{u} = 0 \tag{1a}$$

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2 \mathbf{u} \tag{1b}$$

where $\mathbf{u} = (u, v)$ denotes the velocity field, p the pressure and Re is the Reynolds number. The goal will be to solve this coupled equation system and validate the results against known standard benchmarks for steady lid driven cavity flow.

Similar to the procedure described by Zhang & Xi (2010) the governing equations are to be discretised using an explicit pseudospectral method. For the spatial discretisation we introduce a grid which includes the end points, e.g., a Legendre-Gauss-Lobatto or a Chebyshev-Gauss-Lobatto grid. By constructing a tensor product grid between the horizontal and vertical coordinates, it is possible to store and update the primary variables (p, u, v) on this set of collocation nodes.

A $P_N - P_{N-2}$ method is adopted to interpolate the variables. The velocities (u, v) are approximated by (nodal) Lagrange polynomials of order $N_x N_y$ on the grid consisting of $(N_x+1)(N_y+1)$ total nodes. The pressure p is approximated by a (nodal) Lagrange polynomial of order $(N_x - 2)(N_y - 2)$ on the inner $(N_x - 1)(N_y - 1)$ nodes. Thus, we seek to represent the variables as

$$\mathbf{u}(\mathbf{x}) = \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} \hat{\mathbf{u}}_{ij} h_i^{Nx+1}\left(\frac{x}{\xi}\right) h_j^{N_y+1}\left(\frac{y}{\eta}\right), \tag{2a}$$

$$p(\mathbf{x}) = \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} \hat{p}_{ij} h_i^{N_x-1}\left(\frac{x}{\xi}\right) h_j^{N_y-1}\left(\frac{y}{\eta}\right), \tag{2b}$$

where $\hat{\mathbf{u}}_{ij} = \mathbf{u}(\mathbf{x}_{ij})$ and h_i^P and h_j^Q are Lagrange polynomials of order P and order Q, respectively. The advantage of only calculating the pressure variable on the interior grid points is that it possible to avoid prescribing pressure boundary conditions and non-physical spurious modes.

Approximation of spatial derivatives is done using spectral differentiation matrices, which are defined on the collocation nodes used to represent the variable in question. Thus, the differentiation matrices will be different for use with the velocity variables on the full tensor grid and the pressure variable existing on the reduced tensor grid. This implies, that we can compute approximation using matrix-vector products of the form

$$\begin{aligned}
\partial_x u &\approx D_x \mathbf{u}, \quad \mathbf{u} \in \mathbb{R}^{(N_x+1)(N_y+1)\times 1}, \quad D_x \in \mathbb{R}^{(N_x+1)(N_y+1)\times(N_x+1)(N_y+1)} \\
\partial_y u &\approx D_y \mathbf{u}, \quad \mathbf{u} \in \mathbb{R}^{(N_x+1)(N_y+1)\times 1}, \quad D_y \in \mathbb{R}^{(N_x+1)(N_y+1)\times(N_x+1)(N_y+1)}
\end{aligned}$$

and similar for the vertical velocity variable. For the pressure variable, we will need to do approximations of the form

$$\begin{aligned}
\partial_x p &\approx D_x^p \mathbf{p}, \quad \mathbf{u} \in \mathbb{R}^{(N_x-1)(N_y-1)\times 1}, \quad D_x \in \mathbb{R}^{(N_x-1)(N_y-1)\times(N_x-1)(N_y-1)} \\
\partial_y p &\approx D_y^p \mathbf{p}, \quad \mathbf{u} \in \mathbb{R}^{(N_x-1)(N_y-1)\times 1}, \quad D_y \in \mathbb{R}^{(N_x-1)(N_y-1)\times(N_x-1)(N_y-1)}
\end{aligned}$$

The values of the pressure or its gradients can be interpolated/extrapolated to the full tensor grid.

No-slip boundary conditions are imposed along all boundaries and are specified as

$$u = v = 0, \quad \partial\Omega_q, \quad q = w, s, e \tag{3a}$$

$$u = U_{\text{wall}}, \quad v = 0, \quad \partial\Omega_n. \tag{3b}$$

To solve the discretised equations, we will use a pseudo time-stepping method, based on adding pseudo time derivative terms to each of the steady NS equation. We will solve

$$\frac{1}{\beta^2}\frac{\partial p}{\partial \tau} + \nabla \cdot \mathbf{u} = 0 \tag{4a}$$

$$\frac{\partial \mathbf{u}}{\partial \tau} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{\text{Re}}\nabla^2 \mathbf{u} \tag{4b}$$

where τ denotes the pseudo time. The additional term in the continuity equation is an artificial compressibility term which has the role of coupling velocities and pressure for enabling simultaneous updates. The value of the artificial compressibility coefficient β^2 can be set to 5.0 and will not alter the steady state solution in any way.

An explicit four-stage Runge-Kutta method can be used for iteratively updating the solution of the pseudo time-dependent system of equations as

$$\begin{aligned}
\phi^{(1)} &= \phi^n + \frac{1}{4}\Delta\tau R(\phi^n) \\
\phi^{(2)} &= \phi^n + \frac{1}{3}\Delta\tau R(\phi^{(1)}) \\
\phi^{(3)} &= \phi^n + \frac{1}{2}\Delta\tau R(\phi^{(2)}) \\
\phi^{n+1} &= \phi^n + \Delta\tau R(\phi^{(3)})
\end{aligned} \tag{5}$$

where the variables are $\phi = (p, u, v)$ and $\Delta\tau$ denotes the time step. It may be convenient to update all variables (p, u, v) on the full tensor grid, despite that pressure gradients are computed on the reduced tensor grid.

The function R represents the residuals

$$R = -\beta^2(\nabla \cdot \mathbf{u}) \tag{6}$$

for the continuity equation and

$$R = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \mathrm{Re}^{-1}\nabla^2\mathbf{u} \tag{7}$$

for the momentum equations.

The explicit pseudo time-stepping is subject to a conditional CFL stability condition. The pseudo-time steps can be taken adaptively as

$$\Delta\tau = \frac{\mathrm{CFL}}{\lambda_x + \lambda_y} \tag{8}$$

where

$$\lambda_x = \frac{|u_{max}| + \sqrt{u_{max}^2 + \beta^2}}{\Delta x} + \frac{1}{\mathrm{Re} \cdot \Delta x^2}, \quad \lambda_y = \frac{|v_{max}| + \sqrt{v_{max}^2 + \beta^2}}{\Delta y} + \frac{1}{\mathrm{Re} \cdot \Delta y^2} \tag{9}$$

where CFL= $\mathcal{O}(1)$ and Δx and Δy are the minimum collocation grid spacing in $x-$ and $y-$directions, respectively. The velocity variables u_{max} and v_{max} are the maximum velocities in a given pseudo time step.

We will assume that the flow starts from rest such that the initial velocity field is everywhere zero. Thus, as initial conditions you can take all variables to be zero initially except for the lid speed defined for the north boundary. For computational purposes it may be necessary to smooth the velocity field smoothly from 0 to 1 near the corners of the cavity to avoid non-physical spurious modes.

Calculate the residual between two time steps as

$$R(u^{n+1}) = \frac{||u^{n+1} - u^n||_2}{||u^n||_2}, \tag{10}$$

and likewise for v^{n+1} at each time step and observe the convergence to steady state. Plot the residual for u^n and v^n as a function of n (i.e. time) for both resolutions.

In the study it may be interesting to determine the global quantities such as total kinetic energy E, the enstrophy Z and the palinstrophy P for different Reynolds Numbers defined by

$$E = \frac{1}{2}\int_\Omega ||U||^2 dx, \quad Z = \frac{1}{2}\int_\Omega ||\omega||^2 dx, \quad P = \frac{1}{2}\int_\Omega ||\nabla\omega||^2 dx,$$

where the vorticity is $\omega = \partial_x v - \partial_y u$. The results can be compared to those of Brunau & Saad (2006).

1. Formulate mathematically a pseudospectral formulation to be used as a basis for solving the incompressible Navier-Stokes equations approximately on $(x, y) \in \Omega = \{(x, y)|x \in [0, 1],\ y \in [0, 1\}]$. The description can be based on the details given, but the entire scheme needs to be formulated in the report.

2. Implement the model based on the formulated pseudospectral scheme.

3. To validate the code compare computed results for the lid-driven cavity in two horizontal dimensions with those provided by Ghaia et al. (1984) for the centreline in the cavity and see if you can reproduce some of these. Investigate how much resolution is needed, CPU timing, size of Reynolds number that can be used, etc. Reproduce results for Re$\in \{1, 100, 400, 1000\}$.

4. Plot relevant snap-shots and figures of the computed solutions of numerical experiments. This includes velocity profiles, streamlines, global quantities of the steady state solution, vortices, global quantities and their convergence or something else of interest.

5. If time permits, it can be of interest to try and speedup the code by using one of the Multigrid strategies discussed in Zhang & Xi (2010).

Communicate your results in a written report. Make sure to describe and comment on important details and findings. Include sufficiently details for the reader to both reproduce and understand the reported results and conclusions.

Enjoy!

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Deadline for this assignment is Monday, 7 Dec 2021, 14:00.
Hand in at Secretary's office Building 303, room 108.