



LMECA2300 Advanced Numerical Methods

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2-D acoustic and electromagnetic waves

1.1 Physical laws

The expression of the force of the source applied to its surrounding is derived from Newton's law $F = ma$:

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p = r_v \quad (1.1)$$

where ρ_0 is the average density of the surrounding, \vec{u} is the velocity field, p is the variation of pressure and r_v is the "pressure force".

The conservation law of energy is

$$\nabla \cdot \vec{u} + \chi \frac{\partial p}{\partial t} = s_v \quad (1.2)$$

where χ is the compressibility [$kg^{-1}ms^2$] and is given by the equation $\frac{\rho}{\rho_0} = \chi p$. s_v is the "velocity source".

1.2 Wave equation

The wave equation is

$$\nabla^2 p - \rho_0 \chi \frac{\partial^2 p}{\partial t^2} = -\rho_0 \frac{\partial s_v}{\partial t} \quad (1.3)$$

In 1D, with $s_v = 0$, the solution is any function $f(t - \frac{x}{v})$ or $g(x - tv)$ with $v = 1/\sqrt{\rho_0 \chi}$.

1.3 Plane wave

In the plane, the standard wave is given by

$$p(x, t) = p_0 \cos(\omega t - kx) \quad (1.4)$$

where the phase velocity is $v_{ph} = \frac{\omega}{k} = \frac{1}{\sqrt{\chi \rho_0}}$.

The wave impedance is $\eta = \frac{p}{u_x} = \sqrt{\frac{\rho_0}{\chi}}$ [$kgm^{-2}s^{-1}$].

1.4 Harmonic case

In the case of a harmonic soundwave, we can use phasors:

$$p(\vec{r}, t) = \text{Re}\{P(\vec{r})e^{j\omega_0 t}\} \quad (1.5)$$

→ Note: there is a relationship between the phasors and the Fourier transform:

$$F\{p\} = \text{Re}(P)\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + j\text{Im}(P)\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] \quad (1.6)$$

with ω the frequency variable.

In the frequency domain, the PDE becomes

$$\nabla^2 P + k^2 P = -j\omega\rho_0 S_V = Q \quad (1.7)$$

where k is the wavenumber $k = 2\pi/\lambda$. This is useful to make the time dependence disappear.

1.4.1 Green function

The Green function is the solution G to the following Fourier PDE;

$$\nabla^2 G + k^2 G = \delta(x)\delta(y) \quad (1.8)$$

The general solution is

$$G(\rho) = -\frac{j}{4}H_0^{(2)} - \frac{j}{4}(J_0(k\rho) - jY_0(k\rho)) \quad (1.9)$$

where J_0 and Y_0 are respectively the Bessel functions of order 1 and 2, and ρ is the distance between the source and the observer.

The Bessel functions $J_n(x)$ are solutions of the equation

$$x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0 \quad (1.10)$$

and there exists an approximation of the Hankel function $H_0^{(2)}$ for $x \gg 1$:

$$H_0^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{j\pi/4} e^{-jx} \quad (1.11)$$

1.4.2 From velocity source to pressure

The pressure due to a velocity source spread over infinitesimal segment Γ' :

$$p(r) = G(\vec{r} - \vec{r}')(j\omega\rho_0)v_n(\vec{r}')d\Gamma' \quad (1.12)$$

To get the total pressure, we have to do an integral:

$$p = p_{inc} + \frac{\omega\rho_0}{4} \int_{\Gamma} H_0^{(2)}(k|r - r'|)v_n(\vec{r}')d\Gamma' \quad (1.13)$$

Which is obtained from the velocity source $-j\omega\rho_0 S_v$ from equation (1.7), and doing a convolution with the Green function.

This integral can be approximated with a pointwise approach:

$$I \approx C \sum_i v_i G(\rho_i) dl_i \quad (1.14)$$

where $C = -j\omega\rho_0$, v_i is the normal component of the velocity vector, and dl_i the length of the segment.