LINMA2171 Numerical Analysis

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Contents

Linear continuous-time 2D dynamical systems

1.1 Introduction

Consider the 2D dynamical system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. Let $\Omega \subseteq \mathbb{R}^n$ that is compact and positively invariant. Assume that $f \in \mathcal{C}^1$ on Ω, i.e. it is continuously differentiable on Ω. Let $x(0) \in \Omega$. Then the system has one and only one solution for all positive times.

Ω is positively invariant if $x(0) \in \Omega \implies x(t) \in \Omega \; \forall t \geq 0$.

1.2 General form

The general form of a linear dynamical system is

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n \tag{1.1}$$

and its solution is of the form $x(t) = \exp(At)x(0)$. We will now study the different possibilities of stability, based on the matrix A:

1. A is diagonalizable:

    (a) $\lambda_1 > \lambda_2 > 0$: unstable (repelling) node (see LINMA2370).
    (b) $\lambda_1 > \lambda_2 = 0$:
    (c) $\lambda_1 > 0 > \lambda_2$: saddle point.
    (d) $0 = \lambda_1 > \lambda_2$:
    (e) $0 > \lambda_1 > \lambda_2$: attracting node.
    (f) $\lambda_1 = \lambda_2 > 0$: unstable star, the eigenvectors are perpendicular and the direction is repelling from the origin.
    (g) $\lambda_1 = \lambda_2 = 0$: every point is an equilibrium.
    (h) $\lambda_1 = \lambda_2 < 0$: stable star, the eigenvectors are perpendicular and the direction is going to the origin.

2. A has two equal eigenvalues with only one eigenvector, i.e. is not diagonalizable:

    (a) $\lambda < 0$: convergence to the equilibrium point.
    (b) $\lambda = 0$:

(c) $\lambda > 0$:

3. A has complex conjugate eigenvalues:

   (a) $Re(\lambda) > 0$: diverges from the equilibrium in a spiral form.
   (b) $Re(\lambda) = 0$: the trajectory is periodic, does not converge nor diverge.
   (c) $Re(\lambda) < 0$: converges to the equilibrium in a spiral form.

Nonlinear CT, 2D systems

2.1 General form

The general form of a nonlinear CT 2D dynamical system is

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^2, f : \mathbb{R}^2 \to \mathbb{R}^2 \tag{2.1}$$

A nullcline is a curve such that for an element x_i of x, $\dot{x}_i = 0$. There are two in a 2D system and their intersections are the equilibrium points. The vector field has a horizontal or vertical direction on those curves., and their sense is given by the sign of $\dot{x}$ or $\dot{y}$.

→ N.B.: The nullclines cross trajectories, as they are not trajectories themselves.

An equilibrium point is stable if all eigenvalues of the jacobian matrix evaluated at that point have nonpositive real part. It is said to be hyperbolic if its real part is nonzero.

Theorem 2.1. Consider the initial value problem $\dot{x} = f(x), x(0) = x_0$. Suppose that f is continuous and all its partial derivaties are continuous for x in some open connected set $D \subseteq \mathbb{R}^n$. Then for $x_0 \in D$, the initial value problem has a solution $x(t)$ and the solution is unique.
This means that existence and uniqueness of solutions are guaranteed if f is continuously differentiable.

2.2 Methodology

1. Nullclines: Find the graph of the functions such that $\dot{x} = 0, \dot{y} = 0$.

2. Equilibrium points: Those are the intersections of both nullclines.

3. Stability: Analyze the eigenvalues of the Jacobian matrix at each equilibrium point.

2.3 Linearization

The system here is

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y) \tag{2.2}$$

If (x^*, y^*) is one of its equilibrium points, then the Jacobian matrix for that point is

$$A := \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)} \tag{2.3}$$

and the linearized system is

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} \tag{2.4}$$

with $u = x - x^*$ and $v = y - y^*$.

2.4 Population dynamics

The Lotka-Volterra model of competition is used to model the population growth of two species fighting for the same finite resources, but not eating each other, e.g. sheeps and rabbits. The state-space model is

$$\begin{aligned} \dot{x} &= x(3 - x - 2y) \\ \dot{y} &= y(2 - y - x) \end{aligned} \tag{2.5}$$

Its equilibrium points are $(0,0), (0,2), (3,0), (1,1)$. The first is an unstable node, the next two are stable and the last is a saddle point.
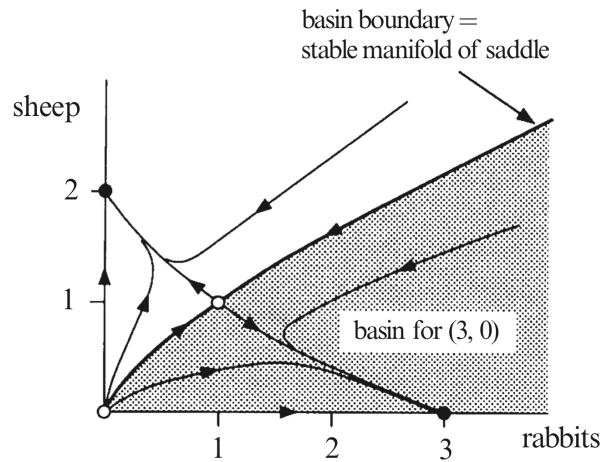
→ N.B.: the stable manifold is the boundary between the regions of the two stable equilibria. On one side of the stable manifold, evey trajectory converges to one equilibrium, and to the other one on the other side. Due to this characteristic, it is also called the basin boundary.

→ N.B.: if the real part of the eigenvalues is zero, additionnal information is needed to conclude on the stability.

2.5 Conservative systems

A conservative system is such that there exists a quantity $V : \mathbb{R} \to \mathbb{R}$ such that it is constant, according to the time, along every trajectory. It often has the dimensions of an energy.

Theorem 2.2. Consider the system $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ and $f \in \mathcal{C}^1$. Suppose that there exists a conserved quantity $E(x)$ and suppose that x^* is an isolated fixed point. If x^* is a local minimum of E, then all trajectories sufficiently close to x^* are closed.

2.6 Reversible systems

Definition 2.3. If the system $\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$ has the same equations under the transformation of coordinates

$$\begin{cases} \tau := -t \\ X := x \\ Y := -y \end{cases} \tag{2.6}$$

then the system is reversible. In 2D, this is saying that the system is reversible if f is odd and g is even.

Theorem 2.4. If (x^*, y^*) is an equilibrium point, center for the linearized system, and the system is reversible, then, when close enough to the equilibrium point, all trajectories are closed curves.
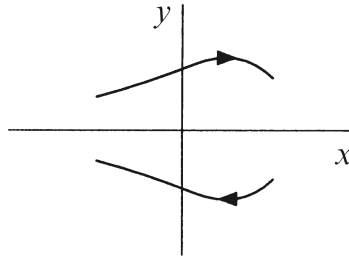
$\rightarrow$ N.B.: Trajectories that start and end at the same fixed point are called homoclinic orbits.

2.7 Invariance principle

Definition 2.5. x^* is a $(\omega-)$limit point of $x(t)$ if there exists $\lim_{t \to \infty} t_n = \infty$ such that $\lim_{n \to \infty} x(t_n) = x^*$.

Definition 2.6. A limit set of a trajectory $x(t)$ is the set of all limit points of that trajectory. It is denoted $\lim_\omega (x(\cdot))$.

Theorem 2.7. Let the trajectory $\{x(t); t > 0\}$ be bounded. Then, $\lim_\omega (x(\cdot))$ is nonempty, bounded, closed, invariant and $\lim_{t \to \infty} x(t) = \lim_\omega (x(\cdot))$, meaning that the distance between $x(t)$ and the limit set tends to 0.

Theorem 2.8. [LaSalle theorem] Let $V : \mathbb{R}^n \to \mathbb{R}$, $V \in \mathcal{C}^1$, such that, $\dot{V}(x) := \frac{d}{dt}(V(\varphi_T(x)))_{t=0} \leq 0$, $\forall x \in \mathbb{R}^n$, φ being a mapping such that $\varphi_T(x(t)) = x(t + T)$. Let us define the sets $S := \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$ and I the largest invariant subset of S. Then $\forall x_0 \in \mathbb{R}^n$, $\lim_\omega (x(\cdot; x_0)) \subseteq I$. Moreover, if $\{x(t; x_0) : t > 0\}$ is bounded, then $\lim_{t \to \infty} x(t; x_0) = I$.

Corollary 2.9. Let x^* be an equilibrium point of $\dot{x} = f(x), x \in \mathbb{R}^n$. Let $V : \mathbb{R}^n \to \mathbb{R}, V \in \mathcal{C}^1$ be positive-definite[1]. Assume $\dot{V}(x) \leq 0 \ \forall x \in \mathbb{R}^n$ and S is defined as above. Assume that no solution trajectory stays in S except $x(t) = x^*$. Then, x^* is asymptotically stable.

Example 2.10. Let us consider the spring-damper system $\ddot{x} = -g(x) - h(\dot{x})$. where g, h are gloablly Lipschitz and verifying

- $g(0) = 0, yg(y) > 0 \ \forall y \neq 0, y \in (-a, a)$;

- $h(0) = 0, yh(y) > 0 \ \forall y \neq 0, y \in (-a, a)$;

Then, the origin is a stable equilibrium point.

2.8 Index theory

Let us consider the 2D system $\dot{x} = f(x), x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2, f \in \mathcal{C}^0$. Let C be a Jordan curve, i.e. a loop that is continuous and not self intersecting[2].

Definition 2.11. The index of C for f $I_C(f)$ is defined as the number of loops that $f(C)$ does around the origin, counting positive for CCW and negative for CW.

- If C_1, C_2 contain the same fixed points, i.e. we can continuously deform C_1 into C_2 without passing through a fixed point, there indices are equal.

- If a Jordan curve contains no fixed point, its index is zero.

- The change of variable $t \to -t$ does not change the index.

- If C is a trajectory or an orbit for the system, then $I_C(f) = +1$.

- For any Jordan curve C, $I_C(f) = I_C(-f)$.

The index of an isolated equilibrium point x^* is $I_{x^*}(f) := I_C(f)$, C being any Jordan curve taken close enought to x^* to have no other equilibrium point inside.

- For any C, $I_C(f) = \sum_i I_{x_i^*}(f)$, x_i^* being the equilibrium points of the system.

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[1]$V(x^*) = 0, V(x) > 0 \ \forall x \neq x^*$.
[2]Not necessarily a trajectory.

Bifurcations

We study here bifurcations in 1 dimension and continuous time. A bifurcation is the evolution of the equilibrium points of the system with a parameter r. The system is

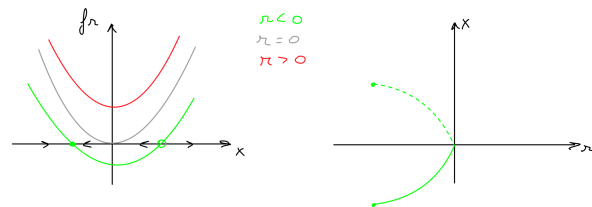$$\dot{x} = f_r(x), \; x \in \mathbb{R} \tag{3.1}$$

and we study the three cases $r > r_c$, $r = r_c$ and $r < r_c$, r_c being the critical value.

3.1 Saddle-node bifurcation

The normal form of a system with saddle-node bifurcation is

$$\dot{x} = r + x^2 =: f_r(x) \tag{3.2}$$

It is the Taylor-developped, shifted and scaled general form.

For $r > r_c$, there is no equilibrium point. For $r = r_c$, we have one single equilibrium point, and for $r < r_c$, we have one stable and one unstable equilibrium point. The conditions for this bifurcation to appear are

$$\begin{aligned}
f_{r_c}(x^*) &= 0 \\
\partial_x f_{r_c}(x^*) &= 0 \\
\partial_r f_{(r_c)}(x^*) &\neq 0 \\
\partial_{xx}^2 f_{r_c}(x^*) &\neq 0
\end{aligned} \tag{3.3}$$

3.2 Transcritical bifurcation

The normal form for this case is

$$\dot{x} = rx - x^2 =: f_r(x) \tag{3.4}$$
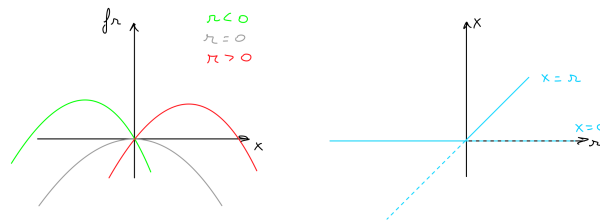
At the critical value, the equilibrium points merge and exchange stability.
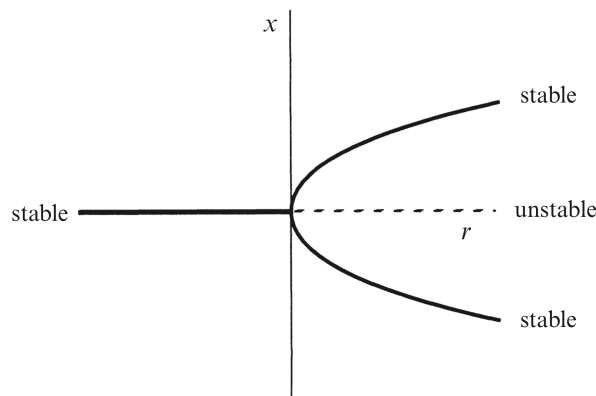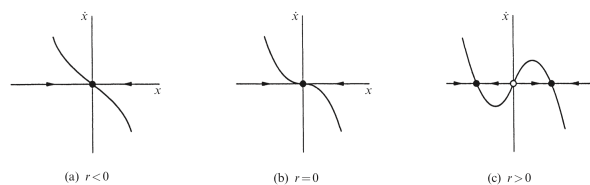
3.3 Pitchfork bifurcation

A pitchfork bifurcation often occurs in physical problems that have a symmetry.

### 3.3.1 Supercritical bifurcation

The normal form of this bifurcation is

$$\dot{x} = rx - x^3 =: f_r(x) \tag{3.5}$$

(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

There are three stable branches and one unstable.

3.3.2 Subcritical bifurcation

In the subcritical case, there are three unstable branches and one stable. This is however not physically possible and we thus add a x^5 term so that the system does not tend to infinity. The canonical form is
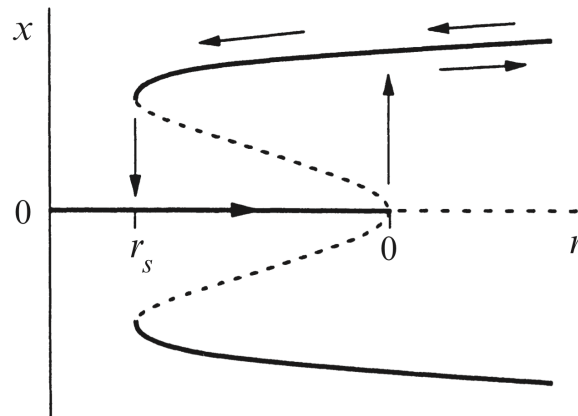
- In the range $r_s < r < 0$, two stable states coexist, and it is the inition condition that determines which fixed point is approach when $t \to \infty$. That means that the origin is stable under small perturbations, but not big ones (local stability vs global stability).

- The jump between stable curves is called a hysteresis.

- The bifurcation at $r = r_s$ is a saddle-node bifurcation.