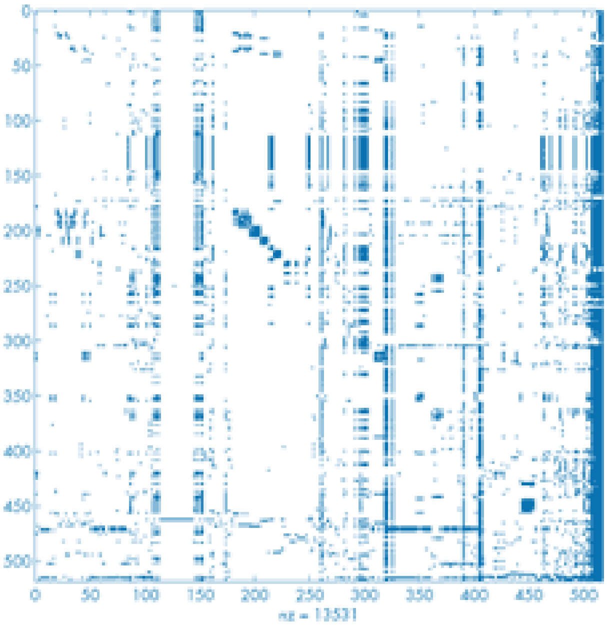

LINMA2380 Matrix Computations

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Contents

Reminders

1.1 Algebraic structures

- A semigroupe is a set together with an associative binary operation $(E, +)$.
- A monoid is a semigroup with a neutral element.
- A group is a monoid in which every element has an inverse.
- A commutative group is a group whose binary operation is commutative.
- A ring is a triple $(E, +, \cdot)$ such that
  - $(E, +)$ is a commutative group;
  - $(E, \cdot)$ is a monoid;
  - $\cdot$ is distirbutive with respect to $+$.
- An integral domain is a commutative ring in which the product of any two nonzero elements in nonzero :

  $$\forall x, y \in E, x, y \neq 0 \qquad xy \neq 0$$

  . This implies that the equation $ax = b$ with $a \neq 0$ has at most one solution.
- An Euclidean domain is an integral domain such that for every two elements in the domain, we can perform the Euclidean division:

  $$\forall (a_1, a_2), \quad \exists (q, r) : \quad a_1 = a_2 q + r \text{ with } r < a_2$$

- A field is a commutative ring $(E, +, \cdot)$ such that every $a \in E \setminus \{0\}$ has a multiplicative inverse.
- $(K, E, +)$ is a module over the ring $(K, +, \cdot)$ if
  - $(E, +)$ is a commutative group;
  - the external composition operation $\cdot : K \times E \to E$ satisfies
    * $(a + b) \cdot x = a \cdot x + b \cdot x \qquad a \cdot (x + y) = a \cdot x + a \cdot y$
    * $a \cdot (b \cdot x) = (a \cdot b) \cdot x$
    * $1 \cdot x = x$

- If, in addition to that, $(K, \cdot, +)$ is a field, then $(K, E, +)$ is a vector space over $(K, +, \cdot)$.
- $(K, E, +, \cdot)$ is an algebra if
  - $(K, E, +)$ is a module or a vector space;
  - the internal composition operation $\cdot : E \times E \to E$ is bilinear.

1.2 Matrix algebras

1.2.1 Product

Apart from the usual sum and product of two matrices, we can define the Hadamard and Kronecker products :

- Hadamard :

$$A_{m\times n} \odot B_{m\times n} := [a_{ij} \cdot b_{ij}]_{i,j=1}^{m,n}$$

- Kronecker :

$$A \otimes B := \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}$$

A square matrix $A \in \mathbb{C}^{n\times n}$ is said normal if $AA^* = A^*$. In the real case, it is said to be orthogonal and $*$ is equivalent to the transpose. Furthermore, it is said to be unitary if it satisfies the relations $AA^* = I_n = A^*A$.

1.2.2 Determinant

We define the quasi-diagonals of a matrix as the n-tuples of elements of a matrix A, $a_{1j_1,2j_2,\dots,nj_n}$ where the indices $\mathbf{j} = (j_1, \dots, j_n)$ constitute a permutation of the set $\{1, 2, \dots, n\}$. Thus a quasi-diagonal consists of n elements of the matrix A in such a way that no two of them lie in the same row or column of A. For each quasi-diagonal, we define the parity $t(\mathbf{j})$. It is the number of inversions $j_k > j_p$ for $k < p$ in $\mathbf{j}$.

- With the notation above, we define the determinant of a square matrix $A_{n\times n}$ as

$$\det(A) = \sum_{\mathbf{j}} (-1)^{t(\mathbf{j})} a_{1j_1} \cdot a_{2j_2} \cdot \dots \cdot a_{nj_n}$$

The determinant has the following properties :

- The determinant is multilinear in the rows of A :

$$\det \begin{bmatrix} a_{1:} \\ \vdots \\ b_{k:} + \lambda c_{k:} \\ \vdots \\ a_{n:} \end{bmatrix} = \det \begin{bmatrix} a_{1:} \\ \vdots \\ b_{k:} \\ \vdots \\ a_{n:} \end{bmatrix} + \det \begin{bmatrix} a_{1:} \\ \vdots \\ \lambda c_{k:} \\ \vdots \\ a_{n:} \end{bmatrix}$$

- The determinant is alternating in the rows of A : for $i \neq j$, $a_{i:} = a_{j:} \implies \det(A) = 0$
- $\det(I_n) = 1$, where I_n is the identity matrix.
- $\det \begin{bmatrix} a_{1:} \\ \vdots \\ a_{i:} \\ \vdots \\ a_{j:} \\ \vdots \\ a_{n:} \end{bmatrix} = -\det \begin{bmatrix} a_{1:} \\ \vdots \\ a_{j:} \\ \vdots \\ a_{i:} \\ \vdots \\ a_{n:} \end{bmatrix}$
- $\det \begin{bmatrix} a_{1:} \\ \vdots \\ a_{i:} + \lambda a_{j:} \\ \vdots \\ a_{n:} \end{bmatrix} = \det \begin{bmatrix} a_{1:} \\ \vdots \\ a_{i:} \\ \vdots \\ a_{n:} \end{bmatrix}$
- $\det(\lambda A) = \lambda^n \det(A)$
- for $i \neq j$, $a_{i:} = \lambda a_{j:} \implies \det(A) = 0$
- $\det(A^T) = \det(A)$
- $\det(A^*) = \overline{\det(A)}$, if $A \in \mathbb{C}^{n \times n}$

→ N.B.: any property of the determinant established for the rows of matrices also holds for the columns.

- The minor $A_{(kl)}$ of dimension $n-1$ of a matrix $A_{n\times n}$ is the determinant of the submatrix obtained by removing the $k$the row and the lth column. From this, we can note the determinant as a linear combination of the elements of a row or column :

  $$\det(A) = a_{1j}A^c_{1j} + a_{2j}A^c_{2j} + \cdots + a_{nj}A^c_{nj} \det(A) = a_{i1}A^c_{i1} + a_{i2}A^c_{i2} + \cdots + a_{in}A^c_{in}$$

  where the coefficient A^c_{kl} is called the cofactors of the corresponding element a_{kl}[1]

Laplace and Binet-Cauchy relations

For the pairs of p-tuples

$$\mathbf{i}_p := (i_1, \ldots, i_p) \text{ and } \mathbf{j}_p := (j_1, \ldots j_p)$$

satisfying

$$1 \leq i_1 < \cdots < i_p \leq n \text{ and } 1 \leq j_1 < \cdots < j_p \leq n$$

we define the minors of order p of A as

$$A \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} := \det[a_{i_k, j_l}]^p_{k,l=1} \tag{1.1}$$

We also define the complementary cofactors of A as

$$A^c \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} := (-1)^s A \begin{pmatrix} \mathbf{i}^c_p \\ \mathbf{j}^c_p \end{pmatrix} \tag{1.2}$$

[1] $A^c_{kl} = (-1)^{k+l} A_{(kl)}$.

where $s = \sum_{k=1}^{p}(i_k + j_k)$ and $\mathbf{i}_p^c$ is the set complement of $\mathbf{i}_p$ (same for $\mathbf{j}_p$).
Laplace Theorem:
Let A ne a matrix of dimensions $n \times n$ and $\mathbf{i}_p$ be a p-tuple of rows (and $\mathbf{j}_p$ for the columns). Then, $\det(A)$ is equal to the sum of the products of all possibles minors located in these rows/columns with their complementary cofactors:

$$\begin{cases} \det(A) = \sum_{\mathbf{j}_p} A \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} A^c \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} \\ \det(A) = \sum_{\mathbf{i}_p} A \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} A^c \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} \end{cases} \tag{1.3}$$

Binet-Cauchy Theorem:
Let $\mathbf{m}$ be the m-tuple $(1, \ldots, m)$. Let A and B be matrices of dimensions $m \times n$ and $n \times m$ respectively. If $m \leq n$, then

$$\det(AB) = \sum_{\mathbf{j}_m} = A \begin{pmatrix} \mathbf{m} \\ \mathbf{j}_m \end{pmatrix} B \begin{pmatrix} \mathbf{j}_m \\ \mathbf{m} \end{pmatrix} \tag{1.4}$$

1.2.3 Inverse and rank

- The adjugate matrix of a square matrix $A_{n\times n}$ is defined as

$$adj(A) := [A^c_{ji}]_{i,j=1}^n$$

Then, for every square matrix $A_{n\times n}$, we have

$$A \cdot adj(A) = \det(A) I_n = adj(A) \cdot A \tag{1.5}$$

Every matrix $A_{m\times n}$ whose elements belong to a field $\mathcal{F}$ can be brought to the following form by means of invertible (or elementary) transformations of rows and columns:

$$RAQ = \left(\begin{array}{c|c} I_r & 0_{r\times(n-r)} \\ \hline 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{array}\right) \tag{1.6}$$

The rank of a matrix $A_{m\times n}$ whose elements belong to a field $\mathcal{F}$ is equal to the largest size of its nonzero minors. As a corollary, any non-singular matrix whose elements belong to a field $\mathcal{F}$ can be written as a product of elementary transformations.

Schur complement:
Let $A_{n\times n}$ be an invertible submatrix of the matrix

$$M_{(n+p)\times(n+m)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Then the rank of M satisfies

$$rank(M) = n + rank(D - CA^{-1}B) \tag{1.7}$$

And the matrix $D - CA^{-1}B$ is called the Schur complement of M.

QR form

TODO

Unitary transformations and SVD

3.1 Introduction and definitions

- A unitary matrix is a matrix $U \in \mathbb{C}^{n\times n}$ such that $U^*U = I$, i.e. its column are orthogonal.
- An isometry is a matrix $U \in \mathbb{C}^{m\times n}$, $m \neq n$, such that $U^*U = I$. We have $\|Ux\| = \|x\|$.

3.2 Diagonalization by unitary transformations

The goal here is to have a matrix decomposition of the form

$$A = R \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q^{-1} \tag{3.1}$$

for any arbitrary matrix $A_{m\times n}$, and with R, Q being unitary (if A is complex) or orthogonal (if A is real). We limit ourselves here to transformation matrices that are isometries[1]. This means that the invariants that we obtain characterize the way the matrix act on the norm of vectors.

Theorem 3.1. Every Hermitian[2] matrix $A \in \mathbb{C}^{n\times n}$ can be diagonalized by a unitary transformation $U \in \mathbb{C}^{n\times n}$:

$$U^*AU = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots \vdots & \\ \dots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} \tag{3.2}$$

with $\lambda_i \in \mathbb{R}$.

Theorem 3.2. The eigenvalues of a Hermitian matrix $A \in \mathbb{C}^{n\times n}$ are invariant under unitary similarity transformations:

$$B = U^*AU \tag{3.3}$$

Every class of equivalence defined by this transformation group has a unique canonical representative which is the diagonal matrix Λ with the eigenvalues of A decreasing along the diagonal.

[1] To define.

[2] $A = A^*$

Theorem 3.3 (**Singular Value Decomposition**)**.** For every matrix $A \in \mathbb{C}^{m\times n}$, there exist unitary transformations $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that

$$A = U\Sigma V^* \qquad \Sigma = \left(\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0_{r\times(n-r)} \\ 0 & & \sigma_r & \\ \hline & 0_{(m-r)\times r} & & 0_{(m-r)\times(n-r)} \end{array}\right) \tag{3.4}$$

with real positive singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$. The value r and the r-tuple $(\sigma_1, \ldots, \sigma_r)$ are uniquely defined and, as a consequence, the matrix Σ constitutes a canonical form under unitary transformations, i.e. under transformations of the forme $B = \tilde{U}^* A \tilde{V}$. Where $\tilde{U}, \tilde{V}$ are two unitary matrices.

Properties:

- If the matrix A is real, U, V are orthogonal matrices;
- The transformations U, V diagonalize the matrices AA^* and A^*A respectively, since $U^*AA^*A = \Sigma\Sigma^T$, $V^*A^*AV = \Sigma^T\Sigma$, and the columns of U, V are the eigenvectors of AA^* and A^*A respectively.
- The transformations U, V are not uniquely defined.

3.3 Linear operator point of view

We define the compact SVD form: $A = U_1\Sigma_r V_1^*$, to have Σ_r invertible. In this form, Σ_r is $r \times r$, r being the number of nonzero singular values. U_1 contains the r first columns of U and V_1^* the r first lines of V^*. The other columns (resp. rows) of U (resp. V) are denoted by the matrix U_2 (resp. V_2^*).

Definition 3.4. If $\mathcal{X}_1, \mathcal{X}_2$ are subspaces of $\mathbb{R}^n$ such that their intersection is the origin, then we note $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{X}_1 + \mathcal{X}_2$ the direct sum of the two spaces.

Any vector $x \in \mathcal{X}_1 \oplus \mathcal{X}_2$ has a unique decomposition $x = x_1 + x_2$, $x_i \in \mathcal{X}_i$.
For the SVD, we have

$$\mathcal{X}_1 = Im(V_1) \qquad \mathcal{X}_2 = Im(V_2) = Ker(A) \tag{3.5}$$

$$\mathcal{Y}_1 = Im(U_1) = Im(A) \qquad \mathcal{Y}_2 = Im(U_2) \tag{3.6}$$

3.4 Polar decomposition - formal point of view

Any matrix $A_{n\times n}$ can be expressed in the following form:

$$A = \underbrace{U\Sigma U^*}_{=:H_1} UV^* = H_1 Q = H_1 \exp(iH_2) \tag{3.7}$$

with H_1 a positive definite Hermitian matrix, Q unitary and H_2 also Hermitian.

3.5 Projectors and generalized inverses - algebraic point of view

Definition 3.5. A projector is a matrix $P \in \mathbb{C}^{n\times n}$ such that $P^2 = P$.
It is said to be orthogonal if $\forall x,\ (Px)^*(x - Px) = 0$.

Theorem 3.6. Any projector P can be written $P = XY^*$ with $Y^*X = I_r$, r being the rank of P. If P is orthogonal, then $X = Y$.

- $Im(P) = Ker(P^{\perp})$
- $P = P^*$

3.6 Least squares

Theorem 3.7. Given a linear system $Ax = y$, the generalized inverse $A^I = V_1\Sigma_r^{-1}U_1^*$ gives $x^* = A^I y$ the solution minimizing the norm of $Ax - y$. If there are more than one such solution, it returns the one of smallest norm.

3.7 Unitarily invariant matrix norms - geometric point of view

A matrix norm is unitarily invariant if, for every $A \in \mathbb{C}^{m\times n}$, we have $\|A\| = \|U^*AV\|$ if U, V are unitary.
The 2-norm and the Frobenius norm of $A \in \mathbb{C}^{m\times n}$ are unitarily invariant.

$$\|A\|_2 := \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \qquad \|A\|_F := \left(\sum_{i,j} |a_{i,j}^2|\right)^{1/2} \tag{3.8}$$

3.8 Canonical angles

Theorem 3.8. Given two subspaces $\mathcal{S}_i \subseteq \mathbb{C}^n$ $(i = 1, 2)$, there exist orthonormal bases given by the columns of $\hat{S}_i$ respectively, and satisfying

$$\hat{S}_1^*\hat{S}_2 = \left(\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0_{r\times(r_2-r)} \\ 0 & & \sigma_r & \\ \hline & 0_{(r_1-r)\times r} & & 0_{(r_1-r)\times r_2-r} \end{array}\right) \qquad 1 \geq \sigma_1 \geq \cdots \geq \sigma_r > 0 \tag{3.9}$$

Add the paper sheet of notes.

3.9 Variational problems

Theorem 3.9. For a Hermitian matrix $H \in \mathbb{C}^{n\times n}$, the Rayleigh quotient is defined as

$$R(x) := \frac{\langle Hx, x\rangle}{\langle x, x\rangle} = \frac{x^*Hx}{x^*x} \qquad x \neq 0 \in \mathbb{C}^n \tag{3.10}$$

The Rayleigh quotient of a Hermitian matrix $H \in \mathbb{C}^{n\times n}$ is real and satisfies

$$\lambda_{\min}(H) \leq R(x) \leq \lambda_{\max}(H) \tag{3.11}$$

Furthermore, supposing that $\lambda_1 \geq \cdots \geq \lambda_n$, we have

$$\lambda_n = \min_{x\neq 0} R(x) \qquad \lambda_1 = \max_{x\neq 0} R(x) \tag{3.12}$$

Lemma 3.10. Let $\mathcal{S}_j \subseteq \mathbb{C}^n$ be a subspace of dimension j. Then, it holds that

$$\min_{x\neq 0\in\mathcal{S}_j} R(x) \leq \lambda_j \qquad \max_{x\neq 0\in\mathcal{S}_j} R(x) \geq \lambda_{n-j+1} \tag{3.13}$$

Theorem 3.11 (Courant-Fisher)**.** For any Hermitian matrix $H \in \mathbb{C}^{n\times n}$, the Rayleigh quotient $R(x)$ satisfies

$$\lambda_j = \max_{\mathcal{S}_j} \min_{x\neq 0\in\mathcal{S}_j} R(x) \qquad \lambda_{n-j+1} = \min_{\mathcal{S}_j} \max_{x\neq 0\in\mathcal{S}_j} R(x) \tag{3.14}$$

Theorem 3.12. The singular values of an arbitrary matrix $A \in \mathbb{C}^{m\times n}$ are given by

$$\sigma_j(A) = \max_{\mathcal{S}_j} \min_{x\neq 0\in\mathcal{S}_j} \frac{\|Ax\|_2}{\|x\|_2} \tag{3.15}$$

$$\sigma_{n-j+1}(A) = \min_{\mathcal{S}_j} \max_{x\neq 0\in\mathcal{S}_j} \frac{\|Ax\|_2}{\|x\|_2} \tag{3.16}$$

The following theorem is a major application of the SVD, as it allows to store a matrix with much less information that it contains.

Theorem 3.13. Let $A \in \mathbb{C}^{m\times n}$ be a matrix of rank r. The best approximation of A by a matrix $B \in \mathbb{C}^{m\times n}$ of rank $s < r$ satisfies

$$\min_{\operatorname{rank}(B)\leq s} \|A - B\|_2 = \sigma_{s+1}(A) \tag{3.17}$$

Theorem 3.14 (Eckart-Young)**.** Furthermore, the matrix A from the last theorem also satisfies

$$\min_{\operatorname{rank}(B)\leq s} \|A - B\|_F^2 = \sigma_{s+1}^2 + \cdots + \sigma_r^2 \tag{3.18}$$

Eigenvalues, eigenvectors and similarity transformations

The eigenvalues of a matrix are invariant under similarity transformations. The similiary transformations $A \to TAT^{-1}$ define an equivalence class of matrices and every matrix A_T belonging to the similarity class of A has the same eigenvalues.

Theorem 4.1 (Schur)**.** Every matrix $A \in \mathbb{C}^{n\times n}$ can be upper triangularized under unitary similarity transformations:

$$U^*AU = \begin{bmatrix} \lambda_1 & \times & \cdots & \times \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \times \\ 0 & \dots & 0 & \lambda_n \end{bmatrix} =: A_S \tag{4.1}$$

where the diagonal of A_S consists of the eigenvalues of A.

- If A is Hermitian, then A_S is Hermitian, and thus diagonal and real. This is its canonical form.
- The eigenvalues in the Schur form can be ordered.
- If $A \in \mathbb{R}^{n\times n}$, the eigenvalues and eigenvectors can be complex. Then, their complex conjugates are also eigenvalues and eigenvectors of A.

Definition 4.2. A normal matrix is a square matrix $A \in \mathbb{C}^{n\times n}$ satisfying $AA^* = A^*A$.

Theorem 4.3. A matrix $A \in \mathbb{C}^{n\times n}$ is normal if and only if it is diagonalizable under unitary similarity transformations: $A = U\Lambda U^*$.

4.1 Invariant subspaces

Definition 4.4. A subset $\mathcal{X} \subseteq \mathbb{C}^n$ is an invariant subspace under the operator $A \in \mathbb{C}^{n\times n}$ if $A\mathcal{X} \subseteq \mathcal{X}$.

Theorem 4.5. Let $\mathcal{X} \subseteq \mathbb{C}^n$ be a subspace of dimension k. Let $X \in \mathbb{C}^{n\times k}$ be such that the columns of X form a basis of $\mathcal{X}$, and let X_c be a completion of X such that $T := [X|X_c]$ is non-singular. Then, the following three propositions are equivalent:

- $A\mathcal{X} \subseteq \mathcal{X}$;

- $AX = XA_{11}$;
- $T^{-1}AT = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0_{(n-k)\times k} & A_{22} \end{array}\right]$

where $A_{11} \in \mathbb{C}^{k\times k}$, $A_{12} \in \mathbb{C}^{k\times(n-k)}$ and $A_{22} \in \mathbb{C}^{(n-k)\times(n-k)}$.

Theorem 4.6 (Real Schur form)**.** Every matrix $A \in \mathbb{R}^{n\times n}$ can be almost triangularized under real similarity transformations $U \in \mathbb{R}^{n\times n}$, and with blocks of dimension 1×1 or 2×2:

$$U^T A U = \begin{bmatrix} A_{11} & \times & \dots & \times \\ & A_{22} & \ddots & \vdots \\ & & \ddots & \times \\ & & & A_{kk} \end{bmatrix} \qquad A_{ii} \in \mathbb{R}^{1\times 1} \cup \mathbb{R}^{2\times 2} \tag{4.2}$$

4.2 Jordan canonical form

Theorem 4.7. Every matrix $A \in \mathbb{C}^{n\times n}$ admits a block-diagonal form under similarity transformations:

$$T^{-1}AT = A_S = diag\{A_{11}, \dots, A_{kk}\} \tag{4.3}$$

where each block A_{ii} has only one eigenvalue (with multiplicity possibly larger than 1).

Theorem 4.8. Every matrix $A \in \mathbb{C}^{n\times n}$ can be transformed by similarity transformations into a block-diagonal form:

$$T^{-1}AT = diag\{J_1(\lambda), \dots, J_k(\lambda)\} \tag{4.4}$$

where each $J_i(\lambda) \in \mathbb{C}^{n_i\times n_i}$ is a Jordan block:

$$J_i(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} \tag{4.5}$$

Corollary 4.9. Two matrices $A, B \in \mathbb{C}^{n\times n}$ are similar iff they have the same Jordan form.