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# LINMA2491 Operational Research

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# Definition and notation

- Given  $\Omega$ , a sigma-algebra  $\mathcal{A}$  is a set of subsets of  $\Omega$ , with the elements called events, such that:
  - $\Omega \in \mathcal{A}$
  - if  $A \in \mathcal{A}$  then also  $\Omega - A \in \mathcal{A}$
  - if  $A_i \in \mathcal{A}$  for  $i = 1, 2, \dots$  then also  $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$
  - if  $A_i \in \mathcal{A}$  for  $i = 1, 2, \dots$  then also  $\cap_{i=1}^{\infty} A_i \in \mathcal{A}$
- Consider:



- The state space is the set of all values of the system at each stage.

$$S_0 = \{C\}, \quad S_1 = \{C_u, C_d\}, \quad S_2 = \{C_{uu}, C_{ud}, C_{dd}\} \quad (1.1)$$

- The sample space is the set of all possible combination of the system.

$$\Omega = S_0 \times S_1 \times S_2 = \{(C, C_u, C_{uu}), (C, C_u, C_{ud}), (C, C_u, C_{dd}), \dots\} \quad (1.2)$$

- The power set of  $\Omega$  is the set of all of the subsets, denoted  $\mathcal{B}(\Omega)$ .
- The probability space is the triplet  $(\Omega, \mathcal{A}, P)$  where  $P$  is a probability measure.
  - $P(\emptyset) = 0$
  - $P(\Omega) = 1$
  - $P(\cup_{i=1}^{\infty} A_i) = \sum_i P(A_i)$  if  $A_i$  are disjoint
- $\forall t, A_t$  is the set of events on which we have information at stage  $t$ . For example,  $A_0 = \{C\}$ ,  $A_1 = \{C, C_u, C_d\}$ . Thus is it evident that  $t_1 \leq t_2 \Rightarrow \mathcal{A}_{t_1} \subseteq \mathcal{A}_{t_2}$

- Consider the following problem with  $x \in \mathbb{R}^n$  and domain  $\mathcal{D}$ :

$$\begin{aligned} \min f_0(x), \quad & \text{s.t.} \\ f_i(x) &\leq 0, i = 1, \dots, m \\ h_j(x) &= 0, j = 1, \dots, p \end{aligned} \quad (1.3)$$

Then the Lagrangian function is defined as  $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$ :

$$L(x, \lambda, v) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{j=1}^p v_j h_j(x) \quad (1.4)$$

- The Lagrange dual function is defined as  $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$ :

$$g(\lambda, v) = \inf_{x \in \mathcal{D}} L(x, \lambda, v) \quad (1.5)$$

- The Lagrange dual problem is a lower bound on the optimal value of the primal problem
- Lagrange relaxation of Stochastic Programs, consider the two problems:

$$\begin{aligned} \min f_1(x) + \mathbb{E}_\omega[f_2(y(\omega), \omega)] & \quad \min f_1(x) + \mathbb{E}_\omega[f_2(y(\omega), \omega)] \\ \text{s.t. } h_{1i}(x) \leq 0, i = 1, \dots, m_1 & \quad \text{s.t. } h_{1i}(x) \leq 0, i = 1, \dots, m_1 \\ h_{2i}(x, y(\omega), \omega) \leq 0, i = 1, \dots, m_2 & \quad h_{2i}(x(\omega), y(\omega), \omega) \leq 0, i = 1, \dots, m_2 \\ & \quad \textcolor{red}{x(\omega) = x} \end{aligned} \quad (1.6)$$

The red constraint is the non-anticipativity constraint, it transforms the deterministic variable into a stochastic variable. **A VERIFIER**

- The dual of a stochastic program is:

$$\begin{aligned} g(v) &= g_1(v) + \mathbb{E}_\omega(g_2(v, \omega)) \\ \text{where} \\ g_1(v) &= \inf f_1(x) + \left( \sum_{\omega \in \Omega} v(\omega) \right)^T x \\ \text{s.t. } h_{1i}(x) &\leq 0, i = 1, \dots, m_1 \\ \text{and} \\ g_2(v, \omega) &= \inf f_2(y(\omega), \omega) - vx(\omega) \\ \text{s.t. } h_{2i}(x(\omega), y(\omega), \omega) &\leq 0, i = 1, \dots, m_2 \end{aligned} \quad (1.7)$$

- With  $p^*$  the solution of the primal problem and  $d^*$  the solution of the dual problem, we have:

- Weak duality:  $d^* \leq p^*$
- Strong duality:  $d^* = p^*$

- The KKT conditions are necessary and sufficient for optimality in convex optimization, there aren't unique. They are:

- Primal constraint:  $f_i(x) \leq 0, i = 1, \dots, m, h_j(x) = 0, j = 1, \dots, p$
- Dual constraint:  $\lambda \geq 0$
- Complementarity slackness:  $\lambda_i f_i(x) = 0, i = 1, \dots, m$
- Gradient of the Lagrangian:  $\nabla_x L(x, \lambda, \nu) = 0$

# Modelling

## 2.1 Introduction

- For a certain sequence of events:  $x \rightarrow \omega \rightarrow y(\omega)$ , where  $\omega$  is the uncertainty:
  - A first-stage decision is a decision that is made before the uncertainty is revealed (in  $x$ )
  - A second-stage decision is a decision that is made after the uncertainty is revealed (in  $y(\omega)$ )
- We can have a mathematic formulation like this:

$$\begin{aligned}
 & \min c^T x + \mathbb{E}[\min q(\omega)^T y(\omega)] \\
 & Ax = b \\
 & T(\omega)x + W(\omega)y(\omega) = h(\omega) \\
 & x \geq 0, y(\omega) \geq 0
 \end{aligned} \tag{2.1}$$

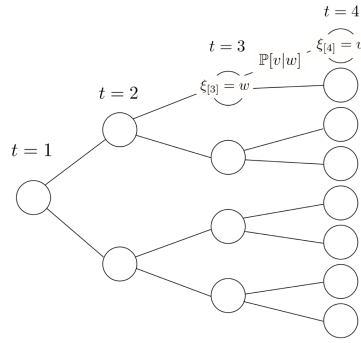
- First-stage decision:  $x \in \mathbb{R}^{n_1}$
- First-stage parameter:  $c \in \mathbb{R}^{n_1}$ ,  $b \in \mathbb{R}^{m_1}$  and  $A \in \mathbb{R}^{m_1 \times n_1}$
- Second-stage decision:  $y(\omega) \in \mathbb{R}^{n_2}$
- Second-stage data:  $q(\omega) \in \mathbb{R}^{n_2}$ ,  $h(\omega) \in \mathbb{R}^{m_2}$  and  $T(\omega) \in \mathbb{R}^{m_2 \times n_1}$ ,  $W(\omega) \in \mathbb{R}^{m_2 \times n_2}$

## 2.2 Representations

### 2.2.1 Scenario Trees

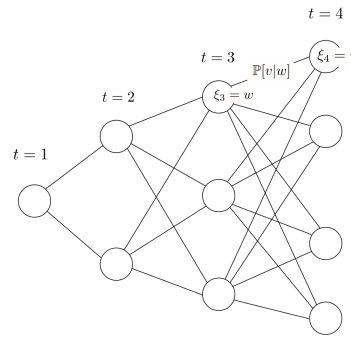
A scenario tree is a graphical representation of a Markov process  $\{\xi_t\}_{t \in \mathbb{Z}}$ , where the nodes are the histories of realizations ( $\xi_{[t]} = (\xi_1, \dots, \xi_t)$ ), and the edges are the transitions from  $\xi_{[t]}$  to  $\xi_{[t+1]}$ .

- We denote the root as  $t = 1$ ;
- An ancestor of a node  $\xi_{[t]}$ ,  $A(\xi_{[t]})$  is a unique adjacent node which precedes  $\xi_{[t]}$ ;
- The children of a node,  $C(\xi_{[t]})$  are the nodes that are adjacent to  $\xi_{[t]}$  and occur at stage  $t + 1$ .



### 2.2.2 Lattice

A lattice is a graphical representation of a Markov process  $\{\xi_t\}_{t \in \mathbb{Z}}$ , where the nodes are the realizations  $\xi_t$  and the edges correspond to the transitions from  $\xi_t$  to  $\xi_{t+1}$ .



### 2.2.3 Serial Independence

A process satisfies serial independence if, for every stage  $t$ ,  $\xi_t$  has a probability distribution that does not depend on the history of the process. Thus, the probability measure is

$$\mathbb{P}[\xi_t(\omega) = i | \xi_{[t-1]}(\omega)] = p_t(i) \quad \forall \xi_{[t-1]} \in \Xi_{[t-1]}, i \in \Xi_t \quad (2.2)$$

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## 2.3 Multi Stage Stochastic Linear Program

### 2.3.1 Notation

- Probability space:  $(\Omega, 2^\Omega, \mathbb{P})$  with filtration  $\{\mathcal{A}\}_{t \in \{1, \dots, H\}}$
- $c_t(\omega) \in \mathbb{R}^{n_t}$ : cost coefficients
- $h_t(\omega) \in \mathbb{R}^{m_t}$ : right-hand side parameters
- $W_t(\omega) \in \mathbb{R}^{m_t \times n_t}$ : coefficients of  $x_t(\omega)$
- $T_{t-1}(\omega) \in \mathbb{R}^{m_t \times n_{t-1}}$ : coefficients of  $x_{t-1}(\omega)$
- $x_t(\omega)$ : set of state and action variables in period  $t$

- We implicitly enforce non-anticipativity by requiring that  $x_t$  and  $\zeta_t$  are adapted to filtration  $\{\mathcal{A}\}_{t \in \{1, \dots, H\}}$
- $\forall A \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}, x_t(\omega_1) = x_t(\omega_2) \forall \omega_1, \omega_2 \in A$

### 2.3.2 General formulation