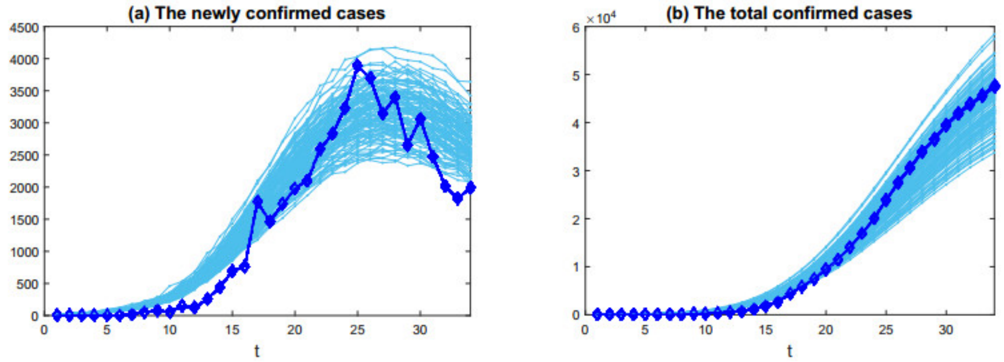

LINMA2470 Stochastic Modelling

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Reminders

1.1 General properties of probability

- $P[A \cup B] = P[A] + P[B] - P[1 \cap B]$;
- $P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[AB]}{P[B]}$;
- A and B are independent iff $P[AB] = P[A]P[B] \Longrightarrow P[A|B] = P[A]$;
- $P[X \leq x] = F_X(x)$ is the distribution function, i.e. a monotone increasing function of x going from 0 to 1 when x goes from $-\infty$ to $+\infty$.
- Its derivative is the density function $f_X(x)$ such that $f_X(x)\delta \approx P[x \leq X \leq x + \delta]$ for an infinitesimal δ.
- A random variable X is said to be memoryless if $\forall t, x > 0$, $P[X > t + x | X > t] = P[X > x]$.
- Markov inequality (for a nonnegative random variable): $P[Y \geq y] \leq \frac{\mathbb{E}[Y]}{y}$;
- Chebyshev inequality: $P[|Z - \mathbb{E}[Z]| \geq \varepsilon] \leq \frac{\sigma_Z^2}{\varepsilon^2}$;

1.2 Expectation and variance

- For a discrete random variable, $\mathbb{E}[X] = \sum_{n=-\infty}^{\infty} nP[X = n]$;
- For a continuous random variable, $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$;
- $\mathbb{E}[X] = \int_0^{\infty} (1 - F_X(x)) dx$.
- $Var[X] = \sigma_X^2 = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$;

1.3 Law of large numbers

Let $X_1, \ldots, X_n$ be a series of independent and uniformly distributed (IID) random variables with expectation $\bar{X}$ and finite variance σ_X^2. Let $S_n = X_1 + \cdots + X_n$. Then,

- Weak version:

$$\lim_{n\to\infty} P\left[|\frac{S_n}{n} - \bar{X}| \geq \varepsilon\right] = 0 \tag{1.1}$$

- Strong version:

$$\lim_{n\to\infty} P\left[\sup_{m\geq n}\left(\frac{S_m}{m} - \bar{X}\right) > \varepsilon\right] = 0 \Longleftrightarrow \lim_{n\to\infty}\frac{S_n}{n} = X \qquad \text{with probability 1} \tag{1.2}$$

1.4 Central limit theorem

$$\lim_{n\to\infty} P\left[\frac{S_n - n\bar{X}}{\sqrt{n}\sigma} \leq y\right] = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \tag{1.3}$$

1.5 Exponential distribution

- $f_X(x) = \lambda e^{-\lambda x}$, for $x \geq 0$;
- $F_X(x) = 1 - e^{-\lambda x}$, for $x \geq 0$;
- $\mathbb{E}[X] = 1/\lambda$.

$\rightarrow$ Note: the exponential distribution is memoryless.

Poisson Processes

A Poisson process $N(t)$ counts the number of arrivals with exponentially distributed inter-arrival times.

$$S_n = \sum_{i=1}^{n} X_i \qquad X_i \sim \exp(\lambda) \tag{2.1}$$

$\forall n, t$, we have the relation $\{S_n \leq t\} = \{N(t) \geq n\}$, where S_n is a random variable telling at which time the n-th occurence appears.

→ Note: a Poisson process is memoryless: $P[Z_1 > x] = e^{-\lambda x}$, with Z_1 be the duration of the time interval from t until the first arrival after t.

For a Poisson process of rate λ, and any given $t > 0$, the length of the interval from t until the first arrival after t is an exponentially distributed random variable. This random variable is idenpendent of both $N(t)$ and of the $N(t)$ arrival epochs before time t. It is also independent of $N(\tau)$, $\forall \tau \leq t$.

Let us consider the process after Z_1, Z_m, the time until the m-th arrival after time t. It is independent of $N(t)$ and of the entier previous history of the process.
Let us denote $\tilde{N}(t, t') = N(t') - N(t)$.

- Stationary increments property: It has the same distribution as $N(t' - t)$, $\forall t' \geq t$ (stationary increments property);
- Independent increments property: For any sequence of times $0 < t_1 < \cdots < t_k$, the set $\{N(t_1), \tilde{N}(t_1, t_2), \ldots, \tilde{N}(t_{k-1}, t_k)\}$ is a set of independent random variables.

From the memoryless property, here is another definition of a Poisson process:

- A Poisson process is a counting process that has the stationay and independent increment properties and such that

$$\begin{aligned}
&P[\tilde{N}(t, t+\delta) = 0] = 1 - \lambda\delta + o(\delta) \\
&P[\tilde{N}(t, t+\delta) = 1] = \lambda\delta + o(\delta) \\
&P[\tilde{N}(t, t+\delta) \geq 2] = o(\delta)
\end{aligned} \tag{2.2}$$

2.1 Distribution of $N(t)$

S_n is the sum n IID random variables and f_{S_n} is the convolution of n times f_X:

$$f_{S_n}(t) = \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!} \tag{2.3}$$

From this,

$$P[N(t) = n-1] = \frac{(\lambda t)^n e^{-\lambda t}}{(n)!} \tag{2.4}$$

and finally,

$$\mathbb{E}[N(t)] = \lambda t \qquad Var[N(t)] = \lambda t \tag{2.5}$$

From equation (2.4), the Poisson process verifies the following probability conditions:

- $P[\tilde{N}(t, t+\delta) = 0] = 1 - \lambda\delta + o(\delta)$;
- $P[\tilde{N}(t, t+\delta) = 1] = \lambda\delta + o(\delta)$;
- $P[\tilde{N}(t, t+\delta) \geq 2] = o(\delta)$;

where we use a first-order approximation of the exponential term, with $o(\delta)$ its residual. As $o(\delta)$ is negligible, we can approximate the Poisson process as a Bernoulli process.

2.1.1 Combining Poisson processes

Let $N_1(t)$ and $N_2(t)$ be tow independent Poisson processes. Let the process $N(t) = N_1(t) + N_2(t)$. We can show using the three properties above that $N(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

2.1.2 Subdividing a Poisson process

Let $N(t)$ be a Poisson process with rate λ. We split the arrivals in 2 subprocesses $N_1(t)$ and $N_2(t)$. Each arrival of $N(t)$ is sent to $N_1(t)$ with probability p and to $N_2(t)$ with probability $(1-p)$, each split being independent from all others.
Then, the resulting processes $N_1(t)$ and $N_2(t)$ are two independent Poisson processes with respective rate $p\lambda$ and $(1-p)\lambda$.

2.1.3 Conditional arrival distribution

The density probability function when we have n Poisson processes, under the condition that $N(t) = n$, is

$$f(s_1, \ldots, s_n | N(t) = n) = \frac{n!}{t^n} \tag{2.6}$$

From the previous results, we can compute that

$$P[S_1 > \tau | N(t) = n] = \left(\frac{t-\tau}{t}\right)^n \tag{2.7}$$

and the expectation is

$$E[S_1|N(t) = n] = \frac{t}{n+1} \tag{2.8}$$

And from this, we derive that

$$P[X_i > \tau|N(t) = n] = \left(\frac{t-\tau}{t}\right)^n \tag{2.9}$$

with expectation

$$E[X_i] = \frac{t}{n+1} \tag{2.10}$$

And thus the density function is

$$f_{S_i}(x|N(t) = n) = \frac{x^{i-1}(t-x)^{n-i}n!}{t^n(n-i)!(i-1)!} \tag{2.11}$$

2.2 Non-homogenous Poisson processes

A non-homogenous Poisson rocess $N(t)$ is a counting process with increments that are independent but not stationary, with

- $P[\tilde{N}(t, t+\delta) = 0] = 1 - \lambda(t)\delta + o(\delta)$;
- $P[\tilde{N}(t, t+\delta) = 1] = \lambda(t)\delta + o(\delta)$;
- $P[\tilde{N}(t, t+\delta) \geq 2] = o(\delta)$;

where $\tilde{N}(t, t+\delta) = N(t+\delta) - N(t)$. The time-varying arrival rate $\lambda(t)$ is assumed to be continuous and stricly positive.

2.3 Bernoulli process approximation

We can approximate the non-homogenous Poisson process with a Bernoulli process where the time is partitioned into increments of lengths inversely proportional to $\lambda(t)$ (i.e. using a nonlinear time scale).

- $P[\tilde{N}(t, t+\epsilon/\lambda(t)) = 0] = 1 - \epsilon + o(\epsilon)$;
- $P[\tilde{N}(t, t+\epsilon/\lambda(t)) = 1] = \epsilon + o(\epsilon)$;
- $P[\tilde{N}(t, t+\epsilon/\lambda(t)) \geq 2] = o(\epsilon)$;

Letting ϵ tend to zero, we obtain

$$P[N(t) = n] = \frac{(m(t))^n e^{-m(t)}}{n!} \qquad P[\tilde{N}(t, t') = n] = \frac{(m(t,t'))^n e^{-m(t,t')}}{n!} \tag{2.12}$$

with

$$m(t) = \int_0^t \lambda(\tau)d\tau \qquad m(t, t') = \int_t^{t'} \lambda(\tau)d\tau \tag{2.13}$$

2.4 Classification of queueing systems

- We note $A/B/k$ where A is the type of distribution for the arrival process, B for the service time and k the number of servers.

We suppose that the arrivals wait in a single queue. Commonly used letters are

- M: exponential distribution (for A) or Poisson process (for B);
- D: deterministic time intervals;
- E: Erlang distribution;
- G: general distribution.