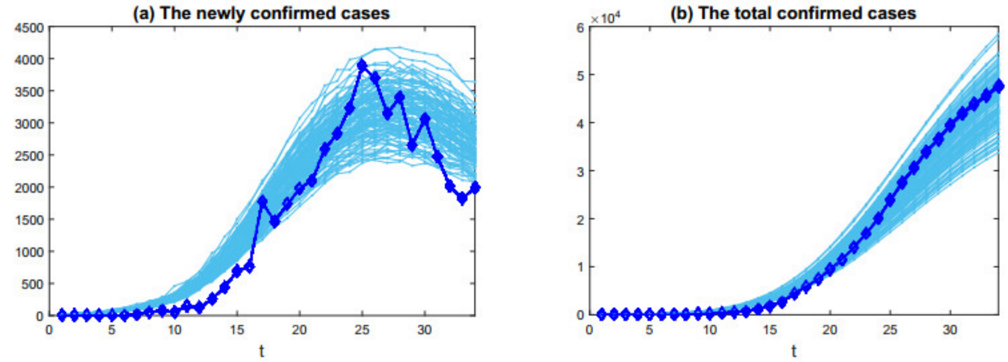

LINMA2470 Stochastic Modelling

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UCLouvain

Contents

Reminders

1.1 General properties of probability

- $P[A \cup B] = P[A] + P[B] - P[1 \cap B]$;
- $P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[AB]}{P[B]}$;
- A and B are independent iff $P[AB] = P[A]P[B] \Longrightarrow P[A|B] = P[A]$;
- $P[X \leq x] = F_X(x)$ is the distribution function, i.e. a monotone increasing function of x going from 0 to 1 when x goes from $-\infty$ to $+\infty$.
- Its derivative is the density function $f_X(x)$ such that $f_X(x)\delta \approx P[x \leq X \leq x + \delta]$ for an infinitesimal δ.
- A random variable X is said to be memoryless if $\forall t, x > 0$, $P[X > t + x | X > t] = P[X > x]$.
- Markov inequality (for a nonnegative random variable): $P[Y \geq y] \leq \frac{\mathbb{E}[Y]}{y}$;
- Chebyshev inequality: $P[|Z - \mathbb{E}[Z]| \geq \varepsilon] \leq \frac{\sigma_Z^2}{\varepsilon^2}$;

1.2 Expectation and variance

- For a discrete random variable, $\mathbb{E}[X] = \sum_{n=-\infty}^{\infty} nP[X = n]$;
- For a continuous random variable, $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$;
- $\mathbb{E}[X] = \int_0^{\infty} (1 - F_X(x)) dx$.
- $Var[X] = \sigma_X^2 = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$;

1.3 Law of large numbers

Let $X_1, \ldots, X_n$ be a series of independent and uniformly distributed (IID) random variables with expectation $\bar{X}$ and finite variance σ_X^2. Let $S_n = X_1 + \cdots + X_n$. Then,

- Weak version:

$$\lim_{n\to\infty} P\left[|\frac{S_n}{n} - \bar{X}| \geq \varepsilon\right] = 0 \tag{1.1}$$

- Strong version:

$$\lim_{n\to\infty} P\left[\sup_{m\geq n}\left(\frac{S_m}{m} - \bar{X}\right) > \varepsilon\right] = 0 \Longleftrightarrow \lim_{n\to\infty} \frac{S_n}{n} = X \qquad \text{with probability 1} \tag{1.2}$$

1.4 Central limit theorem

$$\lim_{n\to\infty} P\left[\frac{S_n - n\bar{X}}{\sqrt{n}\sigma} \leq y\right] = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{1.3}$$

1.5 Exponential distribution

- $f_X(x) = \lambda e^{-\lambda x}$, for $x \geq 0$;
- $F_X(x) = 1 - e^{-\lambda x}$, for $x \geq 0$;
- $\mathbb{E}[X] = 1/\lambda$.

$\rightarrow$ Note: the exponential distribution is memoryless.

Poisson Processes

A Poisson process $N(t)$ counts the number of arrivals with exponentially distributed inter-arrival times.

$$S_n = \sum_{i=1}^{n} X_i \qquad X_i \sim \exp(\lambda) \tag{2.1}$$

$\forall n, t$, we have the relation $\{S_n \leq t\} = \{N(t) \geq n\}$, where S_n is a random variable telling at which time the n-th occurence appears.

$\rightarrow$ Note: a Poisson process is memoryless: $P[Z_1 > x] = e^{-\lambda x}$, with Z_1 be the duration of the time interval from t until the first arrival after t.

For a Poisson process of rate λ, and any given $t > 0$, the length of the interval from t until the first arrival after t is an exponentially distributed random variable. This random variable is idenpendent of both $N(t)$ and of the $N(t)$ arrival epochs before time t. It is also independent of $N(\tau), \forall \tau \leq t$.

Let us consider the process after Z_1, Z_m, the time until the m-th arrival after time t. It is independent of $N(t)$ and of the entier previous history of the process.
Let us denote $\tilde{N}(t, t') = N(t') - N(t)$.

- Stationary increments property: It has the same distribution as $N(t' - t), \forall t' \geq t$ (stationary increments property);
- Independent increments property: For any sequence of times $0 < t_1 < \cdots < t_k$, the set $\{N(t_1), \tilde{N}(t_1, t_2), \ldots, \tilde{N}(t_{k-1}, t_k)\}$ is a set of independent random variables.

From the memoryless property, here is another definition of a Poisson process:

- A Poisson process is a counting process that has the stationay and independent increment properties and such that

$$\begin{aligned} P[\tilde{N}(t, t+\delta) = 0] &= 1 - \lambda\delta + o(\delta) \\ P[\tilde{N}(t, t+\delta) = 1] &= \lambda\delta + o(\delta) \\ P[\tilde{N}(t, t+\delta) \geq 2] &= o(\delta) \end{aligned} \tag{2.2}$$

2.1 Distribution of $N(t)$

S_n is the sum n IID random variables and f_{S_n} is the convolution of n times f_X:

$$f_{S_n}(t) = \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!} \tag{2.3}$$

From this,

$$P[N(t) = n-1] = \frac{(\lambda t)^n e^{-\lambda t}}{(n)!} \tag{2.4}$$

and finally,

$$\mathbb{E}[N(t)] = \lambda t \qquad Var[N(t)] = \lambda t \tag{2.5}$$

From equation (2.4), the Poisson process verifies the following probability conditions:

- $P[\tilde{N}(t, t+\delta) = 0] = 1 - \lambda\delta + o(\delta)$;
- $P[\tilde{N}(t, t+\delta) = 1] = \lambda\delta + o(\delta)$;
- $P[\tilde{N}(t, t+\delta) \geq 2] = o(\delta)$;

where we use a first-order approximation of the exponential term, with $o(\delta)$ its residual. As $o(\delta)$ is negligible, we can approximate the Poisson process as a Bernoulli process.

2.1.1 Combining Poisson processes

Let $N_1(t)$ and $N_2(t)$ be tow independent Poisson processes. Let the process $N(t) = N_1(t) + N_2(t)$. We can show using the three properties above that $N(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

2.1.2 Subdividing a Poisson process

Let $N(t)$ be a Poisson process with rate λ. We split the arrivals in 2 subprocesses $N_1(t)$ and $N_2(t)$. Each arrival of $N(t)$ is sent to $N_1(t)$ with probability p and to $N_2(t)$ with probability $(1-p)$, each split being independent from all others.
Then, the resulting processes $N_1(t)$ and $N_2(t)$ are two independent Poisson processes with respective rate $p\lambda$ and $(1-p)\lambda$.

2.1.3 Conditional arrival distribution

The density probability function when we have n Poisson processes, under the condition that $N(t) = n$, is

$$f(s_1, \ldots, s_n | N(t) = n) = \frac{n!}{t^n} \tag{2.6}$$

From the previous results, we can compute that

$$P[S_1 > \tau | N(t) = n] = \left(\frac{t-\tau}{t}\right)^n \tag{2.7}$$

and the expectation is

$$E[S_1|N(t) = n] = \frac{t}{n+1} \tag{2.8}$$

And from this, we derive that

$$P[X_i > \tau|N(t) = n] = \left(\frac{t-\tau}{t}\right)^n \tag{2.9}$$

with expectation

$$E[X_i] = \frac{t}{n+1} \tag{2.10}$$

And thus the density function is

$$f_{S_i}(x|N(t) = n) = \frac{x^{i-1}(t-x)^{n-i}n!}{t^n(n-i)!(i-1)!} \tag{2.11}$$

2.2 Non-homogenous Poisson processes

A non-homogenous Poisson rocess $N(t)$ is a counting process with increments that are independent but not stationary, with

- $P[\tilde{N}(t, t+\delta) = 0] = 1 - \lambda(t)\delta + o(\delta)$;
- $P[\tilde{N}(t, t+\delta) = 1] = \lambda(t)\delta + o(\delta)$;
- $P[\tilde{N}(t, t+\delta) \geq 2] = o(\delta)$;

where $\tilde{N}(t, t+\delta) = N(t+\delta) - N(t)$. The time-varying arrival rate $\lambda(t)$ is assumed to be continuous and stricly positive.

2.3 Bernoulli process approximation

We can approximate the non-homogenous Poisson process with a Bernoulli process where the time is partitioned into increments of lengths inversely proportional to $\lambda(t)$ (i.e. using a nonlinear time scale).

- $P[\tilde{N}(t, t+\epsilon/\lambda(t)) = 0] = 1 - \epsilon + o(\epsilon)$;
- $P[\tilde{N}(t, t+\epsilon/\lambda(t)) = 1] = \epsilon + o(\epsilon)$;
- $P[\tilde{N}(t, t+\epsilon/\lambda(t)) \geq 2] = o(\epsilon)$;

Letting ϵ tend to zero, we obtain

$$P[N(t) = n] = \frac{(m(t))^n e^{-m(t)}}{n!} \qquad P[\tilde{N}(t, t') = n] = \frac{(m(t,t'))^n e^{-m(t,t')}}{n!} \tag{2.12}$$

with

$$m(t) = \int_0^t \lambda(\tau)d\tau \qquad m(t,t') = \int_t^{t'} \lambda(\tau)d\tau \tag{2.13}$$

2.4 Classification of queueing systems

- We note $A/B/k$ where A is the type of distribution for the arrival process, B for the service time and k the number of servers.

We suppose that the arrivals wait in a single queue. Commonly used letters are

- M: exponential distribution (for A) or Poisson process (for B);
- D: deterministic time intervals;
- E: Erlang distribution;
- G: general distribution.

Renewal Processes

A renewal process is a counting process with IID interarrival intervals. We note X_i the interval between arrivals, $\bar{X} = \mathbb{E}[X]$ is supposed to be finite with probability $P[X_i] > 0 = 1$[1], σ can be finite, and we denote $S_n = \sum_{i=1} X_i$ the time of the n-th arrival.

3.1 Strong law of large numbers

Let $\{N(t); t \geq 0\}$ be a renewal process, then

$$\lim_{t\to\infty} N(t) = \infty \qquad \lim_{t\to\infty} \mathbb{E}[N(t)] = \infty \tag{3.1}$$

This implies that

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\bar{X}} \text{ with probability 1} \tag{3.2}$$

3.2 Central limit theorem

If the interarrival intervals of the renewal process $N(t)$ have a finite standard deviation, then from the CLT for IID random variables, we have

$$\lim_{t\to\infty} P\left[\frac{S_n - n\bar{X}}{\sqrt{n}\sigma} \leq \alpha\right] = \Phi(\alpha) \tag{3.3}$$

What is $\Phi(\alpha)$?
and

$$\lim_{t\to\infty} P\left[\frac{N(t) - t/\bar{X}}{\sigma\bar{X}^{-3/2}\sqrt{t}} < \alpha\right] = \Phi(\alpha) \tag{3.4}$$

$\rightarrow$ Note: The reliability of the observed mean of successive results that are supposed to be IID depends a lot on the rule used to decide when we stop repeating the experiment.

[1] A probability of 1 means that the opposite can happen, but is so rare that the probability is 0.

3.3 Stopping time

Let N be the rv corresponding to the total number of experiments observed. Let I_n be a series of rv being the indicator fucnction of $\{N \geq n\}$:

$$I_n = \begin{cases} 1 & \text{if the } n\text{-th experiment is observed} \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

N is a stopping time if I_n depends only on $X_1, \ldots, X_{n-1}$. This means that stopping at 3pm, for example, is not a stopping time, because it can depend on X_n, depending if the n-th arrival is before or after 3pm.

3.3.1 Wald's inequality

Let N be a stopping time for $\{X_n; n \geq 1\}$. Then, $\mathbb{E}[S_N] = \mathbb{E}[N]\bar{X}$.

3.4 Blackwell's renewal theorem

3.4.1 Arithmetic distribution

If interarrival intervals can only have a length that is a multiple of some real number d, the interarrival distribution will be called an arithmetic distribution, and d the span of the distribution.

3.4.2 Blackwell's inequality

If the interarrival distribution of a renewal process $N(t)$ is not arithmetic, then

$$\lim_{t\to\infty} (m(t+\delta) - m(t)) = \frac{\delta}{\bar{X}} \qquad \forall \delta \tag{3.6}$$

If the interarrival distribution is arithmetic with span d, then

$$\lim_{t\to\infty} (m(t+nd) - m(t)) = \frac{nd}{\bar{X}} \qquad \forall n \geq 1 \tag{3.7}$$

3.4.3 Relationship with a Poisson process

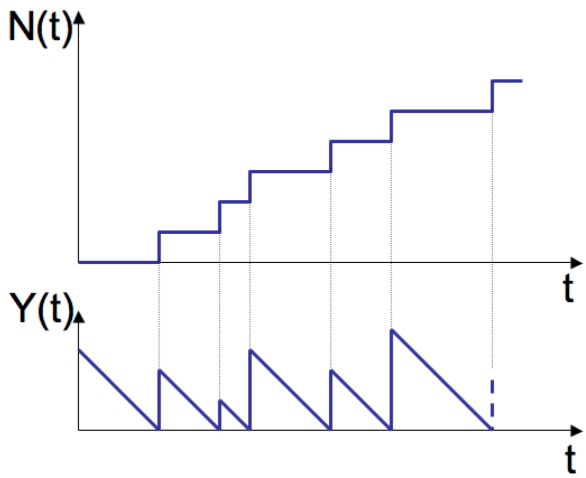
The sum of many renewal processes tends to a Poisson process: for a non-arithmetic renewal process with $P[X_i = 0] = 0$, we have

$$\begin{aligned} &\lim_{t\to\infty} P[N(t+\delta) - N(t) = 0] = 1 - \delta/\bar{X} + o(\delta) \\ &\lim_{t\to\infty} P[N(t+\delta) - N(t) = 1] = \delta/\bar{X} + o(\delta) \\ &\lim_{t\to\infty} P[N(t+\delta) - N(t) \geq 2] = o(\delta) \end{aligned} \tag{3.8}$$

The increments are asymptotically stationary, but not independent. \sectionRenewal reward process Along to the renewal process $N(t)$, we can add a reward function $R(t)$.

It models the rate at which the process is accumulating a reward or cost. It can however only depend on the current renewal but not the previous ones.
Let $Y(y)$ be the residual life at time t for the current renewal:

$$R(t) = Y(t) = S_{N(t)+1} - t \tag{3.9}$$

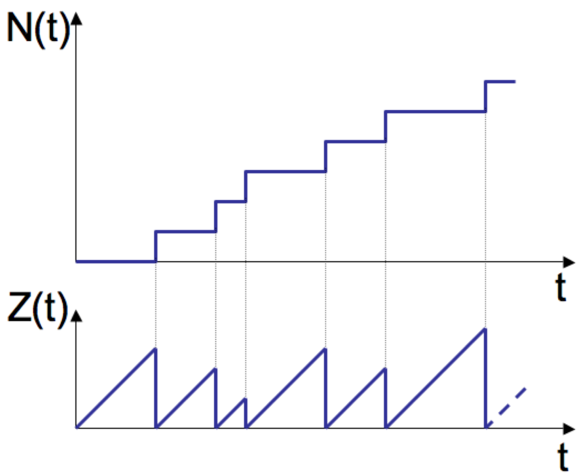


The time average residual life is $\frac{1}{t}\int_0^t Y(\tau)d\tau$.
From the definition of $Y(t)$, we can calculate that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t Y(\tau)d\tau = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} = \frac{1}{2}\mathbb{E}[X] + \frac{Var(X)}{\mathbb{E}[X]} > \frac{1}{2}\mathbb{E}[X] \text{ with probability 1} \tag{3.10}$$

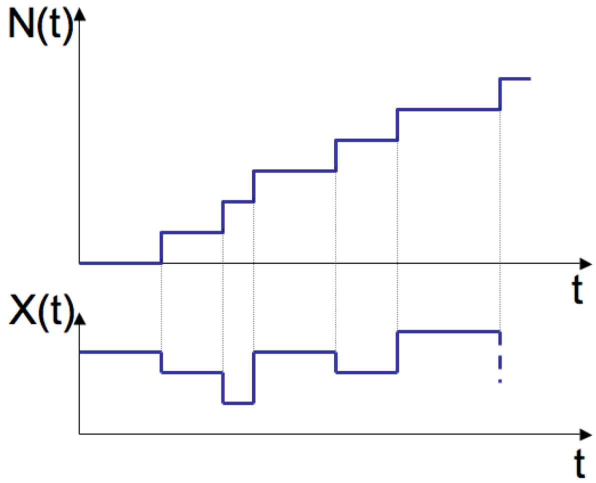
3.4.4 Time average age

Let $Z(t)$ be the age of the current renewal at time t: $R(t) = Z(t) = t - S_{N(t)}$. The time average age is $\lim_{t\to\infty}\frac{1}{t}\int_0^t Z(\tau)d\tau = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]}$.



3.4.5 Time average duration

Let $X(t)$ be the duration of the renewal containing time t: $R(t) = X(t) = S_{N(t)+1} - S_{N(t)}$. The time average duration is $\lim_{t\to\infty}\frac{1}{t}\int_0^t X(\tau)d\tau = \frac{\mathbb{E}[X^2]}{\mathbb{E}[X]}$.



| subsectionGeneral renewal reward functions Let $R(t)$ be a reward function for a renewal process with expected inter-renewal times $\bar{X} < \infty$, then with probability 1,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t R(\tau)d\tau = \frac{\mathbb{E}[R_n]}{\mathbb{E}[X]} \tag{3.11}$$

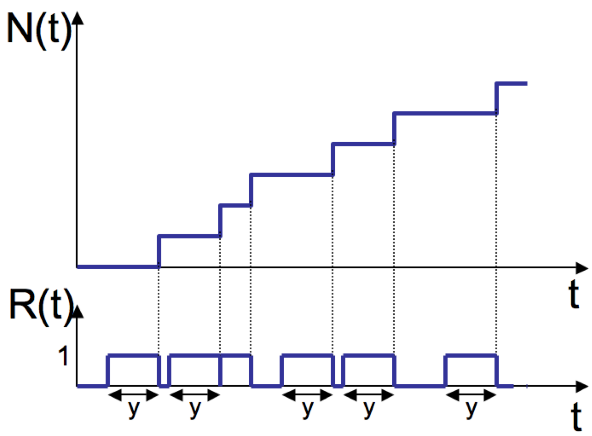
where R_n is defined as

$$R_n = \int_{S_n}^{S_{n+1}} R(\tau)d\tau \tag{3.12}$$

3.4.6 Distribution of residual life

We are interested in the fraction of time that $Y(t) \leq y$:

$$R(t) = I\{Y(t) \leq y\} \qquad R_n = \min\{y, X_n\} \tag{3.13}$$



And we can calculate that

$$\begin{aligned} \mathbb{E}[R_n] &= \int_0^y P[X > x]dx \\ F_Y(y) &= \frac{1}{\mathbb{E}[X]} \int_0^y P[X > x]dx \end{aligned} \tag{3.14}$$

3.4.7 Key theorem

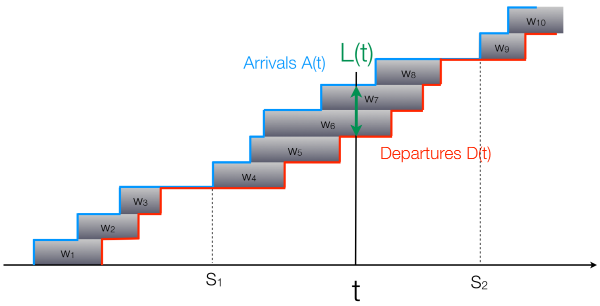
Let $N(t)$ be a non-arithmetic renewal process, let $R(z,x) \geq 0$ be such that $r(z) = \int_{x=z}^{\infty} R(z,x)dF_X(x)$ is directly Rieman integrable. Then,

$$\lim_{t\to\infty} \mathbb{E}[R(t)] = \frac{\mathbb{E}[R_n]}{\bar{X}} \tag{3.15}$$

3.5 Little's Law

Let a queueing system be such that

- $A(t)$ is the number of arrivals between 0 and t;
- $D(t)$ is the number of departures between 0 and t;
- $L(t) = A(t) - D(t)$ is the number of customers in the system at time t;
- w_i the time the i^{th} customer spends in the system;
- $N(t)$ is the renewal process counting the number of busy periods of the system (each time a customer arrives when the system is empty).



A general queueing system

Let us use $L(t)$ as a reward function for the renewal process $N(t)$. This implies

$$\begin{aligned} \sum_{n=1}^{N(t)} R_n \leq \int_0^t L(\tau)d\tau \leq \sum_{i=1}^{A(t)} w_i \leq \sum_{n=1}^{N(t)+1} R_n \\ \lim_{t\to\infty} \frac{1}{t}\int_0^t L(\tau)d\tau = \frac{\mathbb{E}[R_n]}{\mathbb{E}[X]} \end{aligned} \tag{3.16}$$

Putting all this together, we can show that $\bar{L} = \lim_{t\to\infty} \frac{1}{t}\int_0^t L(\tau)d\tau = \bar{W}\lambda$.

3.5.1 M/G/1 queue

Let $R(t)$ be the remaining time for the customer being served. Let $U(t)$ be the time an arrival at time t would have to wait before being served. Let $L_q(t)$ be the number of customers in queue at time t, independent of the Z_i. We define

$$U(t) = \sum_{i=1}^{L_q(t)} Z_i + R(t) \Longrightarrow \mathbb{E}[U(t)] = \mathbb{E}[L_q(t)]\mathbb{E}[Z] + \mathbb{E}[R(t)] \tag{3.17}$$

We can show that

$$\int_0^{S_N(t)} R(\tau)d\tau \leq \int_0^{S_N(t)+1} R(\tau)d\tau \tag{3.18}$$

And from Little's Law,

$$\lim_{t\to\infty} \mathbb{E}[L_q(t)] = \lambda\bar{W}_q \Longrightarrow \lim_{t\to\infty} \mathbb{E}[U(t)] = \lambda\bar{W}_q\mathbb{E}[Z] + \lambda\frac{\mathbb{E}[Z^2]}{2} \tag{3.19}$$

Poisson arrival process implies that arrivals occur with identical probability at any moment, this implies independence with $U(t)$. Hence $\mathbb{E}[W_q(t)] = \mathbb{E}[U(t)]$. Hence $\bar{W}_q = \lambda\bar{W}_q\mathbb{E}[Z] + \lambda\frac{\mathbb{E}[Z^2]}{2}$. And we can isolate $\bar{W}_q$:

$$\bar{W}_q = \frac{\lambda(\mathbb{E}[Z]^2 + \sigma^2)}{2(1 - \lambda\mathbb{E}[Z])} \tag{3.20}$$

And we remember $\bar{W} = \bar{W}_q + \mathbb{E}[Z]$.