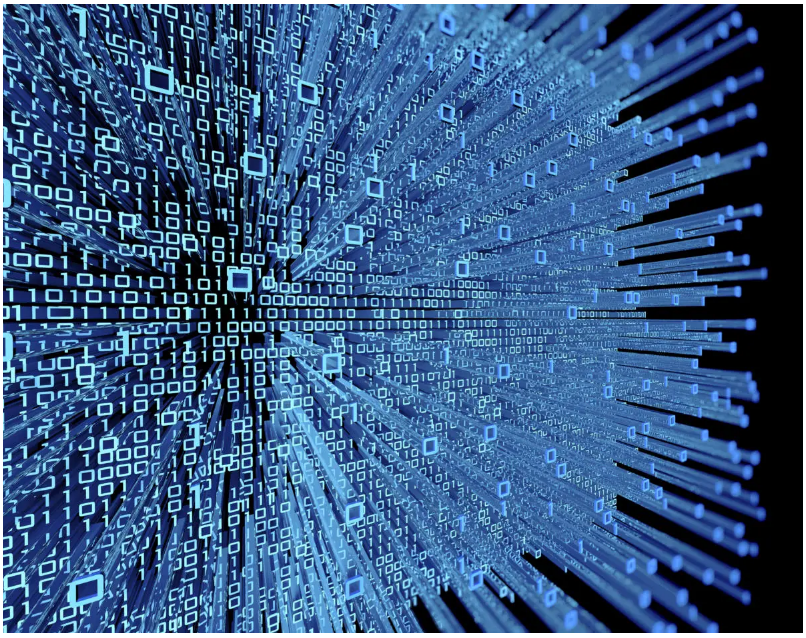

LINMA2171 Numerical Analysis

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Introduction

1.1 General Framework

— Data :
  - $\chi \subseteq \mathbb{R}^d$ (here, $d = 1$ often).
  - $f : \chi \to \mathbb{R}$, with $f \in \mathfrak{f}$.

— Design :
  - $\hat{\mathfrak{f}} \subseteq \mathbb{R}^{\hat{\chi}}$ is the set of admissible function, and is a subset of all function from $\hat{\chi}$ to $\mathbb{R}$.
  - $\mathcal{L} : \hat{\mathfrak{f}} \times \mathfrak{f} \to \mathbb{R}$ is the loss function.
  - $\mathcal{R} : \hat{\mathfrak{f}} \to \mathbb{R}$ is the regularizer.

— Optimisation problem :

$$\arg_{\hat{f} \in \hat{\mathfrak{f}}} \min \quad \mathcal{L}(\hat{f}, f) + \lambda \mathcal{R}(\hat{f})$$

— Optimisation algorithm.

Polynomials

$\mathcal{P}_n$ is the set of all real polynomials of degree at most n.

— The Runge phenomenon is the explosion of the polynomial near the boundary of the domain when the interpolation points are chosen to be equidistant. A solution to that is to put more points near the boundary and less in the middle of the domain, e.g. Chebyshev points.

2.1 Lagrange interpolation

Let $x_0, \ldots, x_n$ be distinct real numbers. The Lagrange polynomial L_k of degree n is such that it is equal to 0 for all x_i, $i \neq k$ and 1 for x_k. This serves as a base for the next interpolations. The general formula for the Lagrange polynomial is

$$L_k(x) = \prod_{i=0 i\neq k}^{n} \frac{x - x_i}{x_k - x_i} \qquad k = 0, 1, \ldots, n \tag{2.1}$$

— N.B. : we usually denote $L_k(x; x_0, \ldots, x_n)$ or let $\chi = (x_0, \ldots, x_n)$ and $L_k(x; \chi)$.

2.2 Hermite interpolation

Let $x_0, \ldots, x_n$ be distinct real numbers. Then, given two sets of real numbers $(y_0, \ldots y_n)$ and $(z_0, \ldots, z_n)$, there is a unique polynomial $p_{2n+1} \in \mathcal{P}_{2n+1}$ such that

$$p_{2n+1}(x_i) = y_i \qquad p'_{2n+1}(x_i) = z_i \qquad i = 0, \ldots, n \tag{2.2}$$

The polynomial p_{2n+1} is termed the Hermite interpolation polynomial of degree at most $2n+1$ for the data points $(x_0, y_0, z_0), \ldots, (x_n, y_n, z_n)$. The expression is

$$p_{2n+1}(x) = \sum_{k=0}^{n} \left(H_k(x) y_k + K_k(x) z_k\right) \qquad \begin{cases} H_k(x) = (L_k(x))^2 (1 - 2L'_k(x_k)(x - x_k)) \\ K_k(x) = (L_k(x))^2 (x - x_k) \end{cases} \tag{2.3}$$

where $L_k(x)$ is the Lagrange polynomial.

— The $H_k(x)$ are such that their derivative is zero for all x_i, and their value is zero for all x_i except x_k, where it is 1.

$$H_k(x_i) = \delta_{ik} \qquad H'_k(x_i) = 0 \qquad \forall i$$

— The $K_k(x)$ are such that their derivative is zero for all x_i except x_k where it is one, and their value is zero for all x_i.

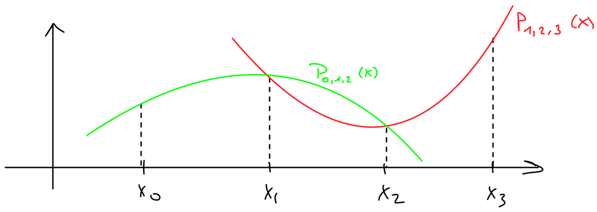
$$K_k(x_i) = 0 \qquad K'_k(x_i) = \delta_{ik} \qquad \forall i$$

2.3 Neville's algorithm

Let us assume we are given a set of support points (x_i, y_i), $i = 0, 1, \dots, n$, and p_n is their Lagrange interpolation polynomial. Let us now define the notation $P_{i_0 i_1 \dots i_k} \in \mathcal{P}_k$, the polynomial for which $P_{i_0 i_1 \dots i_k}(x_{i_j} = y_{i_j}$ for all $j = 0, 1, \dots, k$. We work by recursion, with the following formula :

$$\begin{cases} P_i(x) = y_i \\ P_{i_0 i_1 \dots i_k} = \frac{(x - x_{i_0}) P_{i_1 i_2 \dots i_k}(x) - (x - x_{i_k}) P_{i_0 i_1 \dots i_{k-1}}(x)}{x_{i_k} - x_{i_0}} \end{cases} \tag{2.4}$$

Example :
Let us have four points $(x_0, y_0), \dots (x_3, y_3)$. We want the polynomial interpolating all of them, using Neville's algorithm.



Here,

$$P_{0123}(x) = \frac{x - x_0}{x_3 - x_0} P_{123}(x) + \frac{x_3 - x}{x_3 - x_0} P_{012}(x) \tag{2.5}$$

2.4 Newton's interpolation formula

Newton's interpolation formula is used to evaluate polynomials with a computer, as it only needs to compute each operation $(x - x_i)$ one time. We write it like :

$$p_n(x) = ((\dots(y_{0\dots n}(x - x_n) + y_{0\dots n-1})(x - x_{n-1}) + y_{0\dots n-2})(x - x_{n-2}) + \dots) + y_0 \tag{2.6}$$

And the recursive formula is

$$P_{i_0 i_1 \dots i_k} = P_{i_0 i_1 \dots i_{k-1}}(x) + y_{i_0 i_1 \dots i_k}(x - x_{i_0})(x - x_{i_1}) \dots (x - x_{i_{k-1}}) \tag{2.7}$$

2.5 Linear algebra approach

Let $(\phi_0, \dots, \phi_n)$ ba a basis of $\mathcal{P}_n$, which is known to be an $(n+1)$-dimensional linear space. The interpolation polynomial can thus be expressed in a unique way in the basis :

$$p_n(x) = \sum_{i=0}^{n} a_i \phi_i(x) \tag{2.8}$$

and the coefficient are obtained by solving the linear system

$$\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \dots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \dots & \phi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \dots & \phi_n(x_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix} \tag{2.9}$$

This is called a Vandermonde matrix, and its determinant is

$$\det(V) = \prod_{0 \le i < j \le n} (x_j - x_i) \tag{2.10}$$

which is always non zero, as the x_i are disinct, and the system has one unique solution.

$\rightarrow$ N.B. : the condition number[1] of such a matrix grows exponentially with n.

2.6 Barycentric interpolation formula

This formula is interesting, because it is numerically stable, contrary to the linear algebra method described before. We use the following notation, called the nodal polynomial :

$$\pi_{n+1}(x) = \prod_{i=0}^{n} (x - x_i) \tag{2.11}$$

We now define

$$\lambda_j = \frac{1}{\prod_{k \neq j} (x_j - x_k)} \tag{2.12}$$

The modified Lagrange formula is then

$$p_n(x) = \pi_{n+1}(x) \sum_{j=0}^{n} \frac{\lambda_j}{x - x_j} y_i \tag{2.13}$$

For the polynomial $p_n(x) = 1$, we have the following expression :

$$1 = \pi_{n+1}(x) \sum_{j=0}^{n} \frac{\lambda_j}{x - x_j}$$

and thus we generally prefer to use the equivalent formula for equation (2.13) :

$$p_n(x) = \sum_{j=0}^{n} \frac{\lambda_j y_j}{x - x_j} \Big/ \sum_{j=0}^{n} \frac{\lambda_j}{x - x_j} \tag{2.14}$$

for all $x \notin \{x_n, \dots, x_n\}$.

2.7 Trigonometric interpolation

Let us consider the evenly spaced points $x_j = \frac{2\pi j}{N}$, $j = 0, \dots, N$, on the interval $[0, 2\pi]$, and the interpolation values $f_0, \dots, f_N \in \mathbb{C}$, with $f_0 = f_N$. The trigonometric interpolation problem consists of finding β_k such that

$$p(x) = \sum_{k=0}^{N-1} \beta_k e^{ikx} \text{ such that } p(x_j) = f_j \qquad j = 0, \dots, N-1 \tag{2.15}$$

1. It is a measure of the reaction of the system to a small perturbation

→ N.B. : the bound is $N-1$ because the last condition $p(x_N) = f_N$ is satisfied when the others are (periodicity).

This is equivalent to the generalization to $\mathbb{C}$ of the polynomial interpolation problem : if we denote $\omega := e^{ix}$, the complex polynomial is

$$P(\omega) = \sum_{k=0}^{N-1} \beta_k \omega^k \tag{2.16}$$

The Vandermonde matrix in the complex case is defined as in the real case. We denote it W.
Theorem : $W^*W = NI$ for a complex Vandermonde matrix in an interpolation problem.

From this, the solution to the interpolation problem is solved by multiplying both sides by W^*. We get

$$\beta = \frac{1}{N} W^* f \Longrightarrow \beta_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-i2\pi kj/N} \qquad k = 0, \dots, N-1 \tag{2.17}$$

And that is the discrete Fourier transform (DFT)

2.8 Rational interpolation

Let the interpolation points be $x_0 < x_1 < \cdots < x_\sigma$, with the values $y_0, \dots, y_\sigma \in \mathbb{R}$. We define the polynomial

$$\Phi(x) = \frac{p_\mu(x)}{q_\nu(x)} \qquad p_\mu \in \mathcal{P}_\mu, q_\nu \in \mathcal{P}_\nu \tag{2.18}$$

$$\text{such that } \Phi(x_i) = y_i \qquad i = 0, \dots, \sigma \tag{2.19}$$

The interpolation polynomial can be written

$$\Phi(x) = \frac{\sum_{k=0}^{\mu} a_k x^k}{\sum_{k=0}^{\nu} b_k x^k} = \frac{\lambda p_\mu(x)}{q\nu(x)} \tag{2.20}$$

The number of constraints, i.e. points needed for the interpolation is then $\sigma = \mu + \nu$. This implies that

If Φ is a solution to the equation (2.18), then p_μ, q_ν are solutions of

$$p_\mu(x_i) - y_i q_\nu(x_i) = 0 \qquad i = 0, \dots, \mu + \nu \tag{2.21}$$

$$\left(\sum_{k=0}^{\mu} a_k x_i^k \right) - y_i \left(\sum_{k=0}^{\nu} b_k x_i^k \right) = 0 \tag{2.22}$$

The theorem of existence states that the equation (2.21) always has a non trivial solution, i.e. $(p_\mu, q_\nu) \neq (0, 0)$.
The theorem of uniqueness states that if Φ_1 and Φ_2 are non trivial solutions of (2.21), then they are equivalent, i.e. they differ only by a common polynomial factor in the numerator and denominator.

— p_μ, q_ν are relatively prime if they do not have zeros in common.

Given $\Phi = \frac{p_\mu}{q_\nu}$, let $\tilde{\Phi} = \frac{\tilde{p}\mu}{\tilde{q}_\nu}$ be the equivalent expression for which $\tilde{p}_\mu$ and $\tilde{q}_\nu$ are relatively prime. Φ is the solution of (2.18) $\Longleftrightarrow \tilde{p}_\mu(x_i) - y_i\tilde{q}_\nu(x_i) = 0$, $i = 0, \ldots, \mu + \nu$.