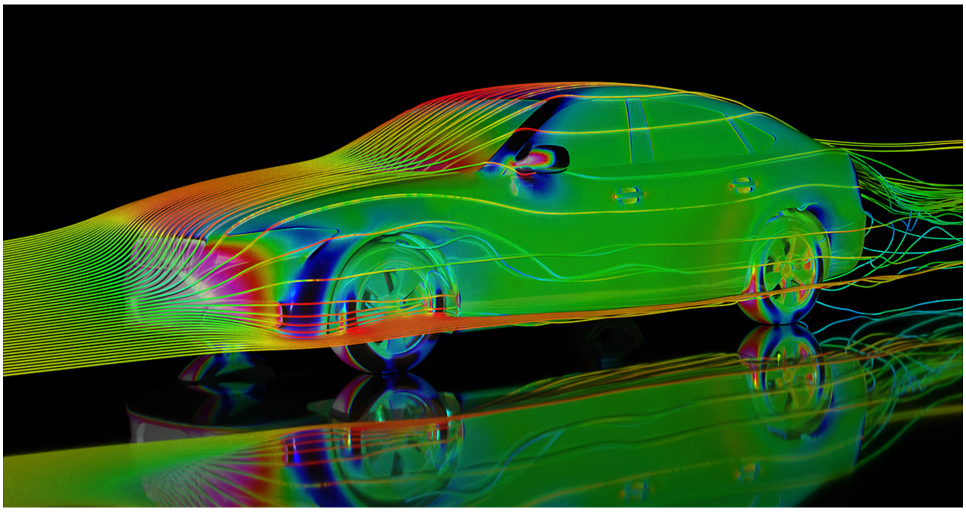

# LMECA2660 - Numerical Methods in Fluid Mechanics

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https://github.com/SimonDesmidt/Syntheses

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# Finite differences with uniform grid

## 1.1 Classical finite differences

Let us define a function $u(\cdot)$ that depends on a variable $x$. Suppose that in the dimension $x$, we discretize the function uniformly with a step $h$ and the values at the nodes are written $u_i$. Then, by a Taylor development series,

$$\begin{cases} u_{i+1} = u_i + h\left(\frac{\partial u}{\partial x}\right)_i + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{h^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \frac{h^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)_i + \dots \\ u_{i-1} = u_i - h\left(\frac{\partial u}{\partial x}\right)_i + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{h^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \frac{h^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)_i - \dots \end{cases} \tag{1.1}$$

This gives three possible finite-difference approximations:

$$\begin{aligned} \left(\frac{\partial u}{\partial x}\right)_i &= \frac{u_{i+1} - u_i}{h} + \mathcal{O}(h) && \text{(Forward differences)} \\ \left(\frac{\partial u}{\partial x}\right)_i &= \frac{u_i - u_{i-1}}{h} + \mathcal{O}(h) && \text{(Backward differences)} \\ \left(\frac{\partial u}{\partial x}\right)_i &= \frac{u_{i+1} - u_{i-1}}{2h} + \mathcal{O}(h^2) && \text{(Centered differences)} \end{aligned} \tag{1.2}$$

This also gives, for the second order,

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i + \dots \tag{1.3}$$

→ Note: in general, discentered differences are only used for stability reasons.

## 1.2 Richardson extrapolation

Richardson extrapolation combines centered finite differences at different scales to get a better error:

$$\begin{aligned} \frac{4}{3}\Bigg[\left(\frac{\partial u}{\partial x}\right)_i &= \frac{u_{i+1} - u_{i-1}}{2h} - \frac{h^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i - \frac{h^4}{120}\left(\frac{\partial^5 u}{\partial x^5}\right)_i - \dots\Bigg] \\ \frac{-1}{3}\Bigg[\left(\frac{\partial u}{\partial x}\right)_i &= \frac{u_{i+2} - u_{i-2}}{2(2h)} - \frac{(2h)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i - \frac{(2h)^4}{120}\left(\frac{\partial^5 u}{\partial x^5}\right)_i - \dots\Bigg] \\ \Longrightarrow \left(\frac{\partial u}{\partial x}\right)_i &= \frac{8(u_{i+1} - u_{i-1}) - (u_{i+2} - u_{i-2})}{12h} + \frac{h^4}{30}\left(\frac{\partial^5 u}{\partial x^5}\right)_i - \dots \end{aligned} \tag{1.4}$$

With this method, the truncation error is of order $\mathcal{O}(h^4)$. In the same way, for second order,

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{4}{3}\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{1}{3}\frac{u_{i+2} - 2u_i + u_{i-2}}{(2h)^2} + \mathcal{O}(h^4) \tag{1.5}$$

## 1.3 Operators

Let us define the following operators:

- Forward difference: $\Delta u_i = u_{i+1} - u_i$;
- Backward difference: $\nabla u_i = u_i - u_{i-1}$;
- Centered difference: $\delta u_i = u_{i+1/2} - u_{i-1/2}$;
- Mean: $\mu u_i = \frac{1}{2}(u_{i+1/2} + u_{i-1/2})$;

→ Note: $u_{i+1/2}$ and $u_{i-1/2}$ are not computable because they are not grid values, but can be used for derivations of other formulae.

- Identity operator: $I u_i = u_i$;
- Forward operator: $E u_i = u_{i+1}$;
- Backward operator: $E^{-1} u_i = u_{i-1}$;

→ Note: $E^{-1}E = I$.

Those operators have the following properties:

- $\mu\delta = \frac{1}{2}(E - E^{-1})$;
- $\mu^2 = I + \delta^2/4$;

The forward operator can be re-expressed using a Taylor development series:

$$\begin{aligned} E u_i = u_{i+1} &= u_i + h\frac{\partial}{\partial x}u_i + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}u_i + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3}{}_i + \dots \\ &= \left(I + hD + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \dots\right)u_i = \exp(hD)u_i \end{aligned} \tag{1.6}$$

From this, using a second Taylor development series,

$$hD = \log(I + \Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \tag{1.7}$$

And, in the same way,

$$hD = -\log(I - \nabla) = \nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots \tag{1.8}$$

We can do the same for another operator:

$$\mu\delta = \frac{1}{2}(E - E^{-1}) = \frac{1}{2}\left(\exp(hD) - \exp(-hD)\right) = \sinh(hD) \tag{1.9}$$

and we can use the Taylor series for $arc\,\sinh(x)$ but it is not very useful. By the property that $\mu^2 = I + \delta^2/4$, we get another form:

$$hD = \mu\delta\left(I - \frac{1}{6}\delta^2 + \frac{1}{30}\delta^4 - \frac{140}{\delta^6} + \dots\right) \tag{1.10}$$

If we keep only the first order term, we find the centered-difference scheme, and the terms up to second order give the Richardson extrapolation.

→ Note: in any scheme, using more information (more values, e.g. $u_{i+2}, u_{i+3}, \dots$) gives a more accurate solution and the order of the truncation error increases (e.g. to $\mathcal{O}(h^3)$).

## 1.4 2D Laplacian

For finite differences in 2D, we can define several types of stencils. For a second-order error, there is the cross operator, which is simply the sum of classical centered finite differences on both axes, and the box operator. This operator is a linear combination of the cross operator using the medians of the square, and the one that uses the diagonals of the square (see 1.1).

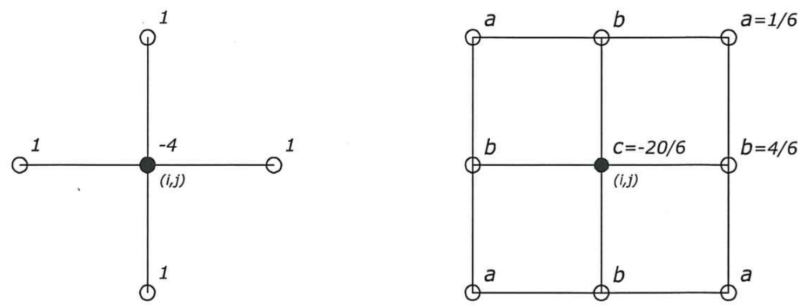


Figure 1.1: Cross operator and box operator.

The box operator expresses the following quantity:

$$h^2\nabla^2\left(u + \frac{h^2}{12}\nabla^2 u + \dots\right) = h^2\left(\nabla^2 u + \frac{h^2}{12}\nabla^2(\nabla^2 u)\right) \tag{1.11}$$

Those stencils can be generalized to higher orders using more points (bigger cross and bigger square).

→ Note: the coefficients are found using the constraint that the truncation error is independent of orientation of the stencil.

## 1.5 Convection equation

The convection equation is

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{1.12}$$

The analytic solution of this equation on an infinite domain, for a speed $c$ constant, is

$$u(x,t) = A(t)e^{ikx} = A(0)e^{ik(x-ct)} \tag{1.13}$$

On a periodic domain of length $L$, the solution is

$$\begin{aligned} u(x,t) &= \sum_{k=-\infty}^{\infty} A_k(t)e^{ikx} \qquad k = \frac{2\pi}{L}p \qquad p \in \mathbb{Z} \\ &\implies u(x,t) = \sum_{p=-\infty}^{\infty} A_p(t)e^{i\frac{2\pi x}{L}} \end{aligned} \tag{1.14}$$

where the coefficients verify the condition $A_k = A^*_{-k}$ for all $k$, since the solution must be real. In the exact solution, all the modes have the same speed $c$. However, it is not the case when we use explicit finite differences. Let us show it in the case of an infinite domain (same thing happens for a periodic domain, adding the sum on $p$):
Let $u_i(t) = A(t)e^{jkx_i}$. Then,

$$\begin{aligned} \left.\frac{\partial u}{\partial x}\right|_i &= Ajk^* \exp(jkx_i) \implies \frac{dA}{dt} + jk^*cA = 0 \\ \implies u_i(t) &= A(0)e^{j(kx_i - k^*ct)} = A(0)e^{ik\left(x_i - \frac{k^*h}{kh}ct\right)} \end{aligned} \tag{1.15}$$

we call $k^*$ the modified wave number. It is different from $k$ because all the modes do not move at the same speed. For example, for the E2 stencil, its expression is derived in the following way:

$$\begin{aligned} \left.\frac{\partial u}{\partial x}\right|_i = \frac{u_{i+1} - u_i}{2h} = \frac{A}{h}\frac{1}{2}(e^{jkh} - e^{-jkh})e^{jkx_i} = \frac{A}{h}j\sin(kh)e^{ikx_i} = Ajk^*e^{jkx_i} \\ \implies k^*h = \sin(kh) \end{aligned} \tag{1.16}$$

This is a **phase** error, as the modes do not **move** with the right **velocity**.
By a Taylor development,

$$\frac{k^*h}{kh} = 1 - \frac{(kh)^2}{6} + \mathcal{O}((kh)^4) \tag{1.17}$$

and although $k$ is not constant for all modes, the error of the stencil is still of the same order: $\mathcal{O}((kh)^2)$. Moreover, the speed of the modes is $c^* = \frac{k^*h}{kh}c$.

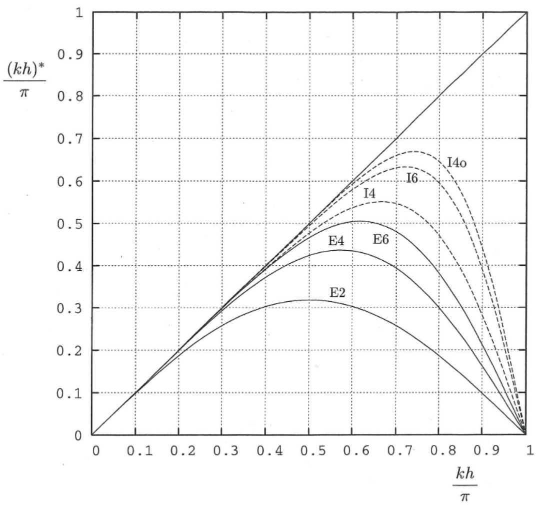


Figure 1.2: Evolution of the exact wave number and the modified wave number.

As the error increases when $k$ increases, it is important to use a very refined grid so that all points whose amplitude is non negligible have $k^*h \approx kh$.

## 1.6 Implicit finite differences

The general scheme of implicit finite differences is the following:

$$\begin{aligned} \frac{\partial u}{\partial x}_i + \alpha\frac{1}{2}\left(\left.\frac{\partial u}{\partial x}\right|_{i+1} + \left.\frac{\partial u}{\partial x}\right|_{i-1}\right) + \beta\frac{1}{2}\left(\left.\frac{\partial u}{\partial x}\right|_{i+2} + \left.\frac{\partial u}{\partial x}\right|_{i-2}\right) \\ = a\frac{u_{i+1}-u_{i-1}}{2h} + b\frac{u_{i+2}+u_{i-2}}{4h} + c\frac{u_{i+3}-u_{i-3}}{6h} \end{aligned} \tag{1.18}$$

Usually, we use $\beta = c = 0$ to keep only the nearest grid points to $i$. Those schemes are called compact.
We can use $\left.\frac{\partial u}{\partial x}\right|_i = Ajk^*e^{jkx_i}$ and $\frac{u_{i+1}-u_{i-1}}{2h} = \sin(kh)$ to show that

$$k^*h = \frac{a\sin(kh) + \frac{b}{2}\sin(2kh) + \frac{c}{3}\sin(3kh)}{1+\alpha\cos(kh)+\beta\cos(2kh)} \tag{1.19}$$

We can do a Taylor development of this quantity to get different schemes:

$$\begin{aligned} &\text{Taylor of order 1:} && 1+\alpha+\beta = a+b+c \Longrightarrow \text{Error of order 2} \\ &\text{Taylor of order 2:} && 3(\alpha+2^2\beta) = a+2^2b+3^2c \Longrightarrow \text{Error of order 4} \\ &\text{Taylor of order 3:} && 5(\alpha+2^4\beta) = a+2^4b+3^4c \Longrightarrow \text{Error of order 6} \\ &\text{Taylor of order 4:} && 7(\alpha+2^6\beta) = a+2^6b+3^6c \Longrightarrow \text{Error of order 8} \\ &\text{Taylor of order 5:} && 9(\alpha+2^8\beta) = a+2^8b+3^8c \Longrightarrow \text{Error of order 10} \end{aligned} \tag{1.20}$$

For example, in the case where $\beta = 0$ and $\alpha \neq 0$, the system is tridiagonal and the solver has a time complexity of $\mathcal{O}(N)$. For $\beta = 0$ and $\alpha = 0$, we find the explicit scheme.

→ Note: The I4o scheme is the scheme with the smallest constant before the $(kh)^4$ using all parameters. For this one, the sign of that constant is positive, meaning that it goes above the correct velocity. This behaviour is not present on the other scheme.

## 1.7 Diffusion equation

The diffusion equation is

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} \tag{1.21}$$

This equation is not reversible because it represents a loss of information: the high-wave-number terms decay fast and have no impact after a small time.
The exact solution is $u(x,t) = A_k(t)e^{jkx}$ with $A_k(t) = A_k(0)e^{\alpha k^2 t}$.

### 1.7.1 Explicit scheme

The explicit finite differences scheme is

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i = a\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + b\frac{u_{i+2} - 2u_i + u_{i-2}}{4h^2} + c\frac{u_{i+3} - 2u_i + u_{i-3}}{9h^2} \tag{1.22}$$

Let $u_i(t) = A(t)e^{jkx_i}$. Then,

$$\frac{dA}{dt} = -\alpha(k^2)^* A \implies A(t) = A(0)e^{-\alpha(k^2)^* t}e^{jkx_i} \implies u_i(t) = A(0)e^{-\alpha k^2\left(\frac{(k^2)^* h^2}{k^2h^2}\right)}e^{jkx_i} \tag{1.23}$$

This is a **amplitude** error, as the mode do not decay with the proper **rate**.
For example, for the explicit scheme of order 2 ($a = 1$, $b = c = 0$), the modified $k^2$ is given by

$$(k^1)^* h^2 = 4\sin^2\left(\frac{kh}{2}\right) \iff \frac{(k^2)^* h^2}{k^2h^2} = \frac{\sin^2\left(\frac{kh}{2}\right)}{\left(\frac{kh}{2}\right)^2} \tag{1.24}$$

### 1.7.2 Implicit schemes

The general form of the implicit scheme follows the same reasoning as for (1.18).

$$\begin{aligned}\left.\frac{\partial^2 u}{\partial x^2}\right|_i + \alpha\frac{1}{2}\left(\left.\frac{\partial^2 u}{\partial x^2}\right|_{i+1} + \left.\frac{\partial^2 u}{\partial x^2}\right|_{i-1}\right) + \beta\frac{1}{2}\left(\left.\frac{\partial^2 u}{\partial x^2}\right|_{i+2} + \left.\frac{\partial^2 u}{\partial x^2}\right|_{i-2}\right) \\ = a\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + b\frac{u_{i+2} - 2u_i + u_{i-2}}{4h^2} + c\frac{u_{i+3} - 2u_i + u_{i-3}}{9h^2}\end{aligned} \tag{1.25}$$

By using $\left.\frac{\partial^2 u}{\partial x^2}\right|_i = -A(t)(k^2)^* e^{jkx_i}$, we can show that

$$(k^2)^* h^2 = \frac{4\left[a\sin^2\left(\frac{kh}{2}\right) + \frac{b}{4}\sin^2\left(\frac{2kh}{2}\right) + \frac{c}{9}\sin^2\left(\frac{3kh}{2}\right)\right]}{1 + \alpha\cos(kh) + \beta\cos(2kh)} \tag{1.26}$$

As previously, we can do the Taylor development series of this expression to get different schemes:

$$\begin{aligned}\text{Taylor of order 1:} \quad & 1 + \alpha + \beta = a + b + c \implies \text{Error of order 2} \\ \text{Taylor of order 2:} \quad & 2\cdot 3(\alpha + 2^2\beta) = a + 2^2 b + 2^2 c \implies \text{Error of order 4} \\ \text{Taylor of order 3:} \quad & 3\cdot 5(\alpha + 2^4\beta) = a + 2^4 b + 2^4 c \implies \text{Error of order 6} \\ \text{Taylor of order 4:} \quad & 4\cdot 7(\alpha + 2^6\beta) = a + 2^6 b + 2^6 c \implies \text{Error of order 8} \\ \text{Taylor of order 5:} \quad & 5\cdot 9(\alpha + 2^8\beta) = a + 2^8 b + 2^8 c \implies \text{Error of order 10}\end{aligned} \tag{1.27}$$

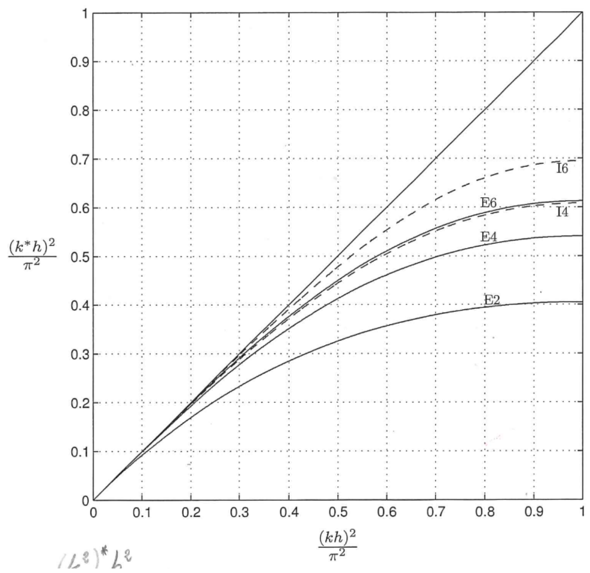


Figure 1.3: Evolution of the modified $k^2$ with $k^2$.

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