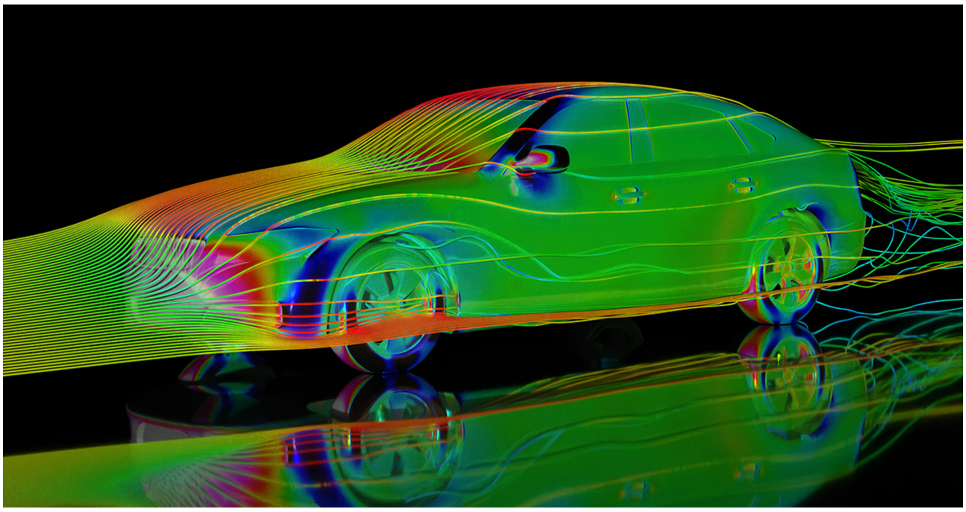

LMECA2660 - Numerical Methods in Fluid Mechanics

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This summary may not be up-to-date, the newer version is available at this address:
https://github.com/SimonDesmidt/Syntheses

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UCLouvain

Contents

Finite differences with uniform grid

1.1 Classical finite differences

Let us define a function $u(\cdot)$ that depends on a variable x. Suppose that in the dimension x, we discretize the function uniformly with a step h and the values at the nodes are written u_i. Then, by a Taylor development series,

$$\begin{cases} u_{i+1} = u_i + h\left(\frac{\partial u}{\partial x}\right)_i + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{h^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \frac{h^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)_i + \dots \\ u_{i-1} = u_i - h\left(\frac{\partial u}{\partial x}\right)_i + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{h^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \frac{h^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)_i - \dots \end{cases} \tag{1.1}$$

This gives three possible finite-difference approximations:

$$\begin{aligned} \left(\frac{\partial u}{\partial x}\right)_i &= \frac{u_{i+1} - u_i}{h} + \mathcal{O}(h) && \text{(Forward differences)} \\ \left(\frac{\partial u}{\partial x}\right)_i &= \frac{u_i - u_{i-1}}{h} + \mathcal{O}(h) && \text{(Backward differences)} \\ \left(\frac{\partial u}{\partial x}\right)_i &= \frac{u_{i+1} - u_{i-1}}{2h} + \mathcal{O}(h^2) && \text{(Centered differences)} \end{aligned} \tag{1.2}$$

This also gives, for the second order,

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i + \dots \tag{1.3}$$

$\rightarrow$ Note: in general, discentered differences are only used for stability reasons.

1.2 Richardson extrapolation

Richardson extrapolation combines centered finite differences at different scales to get a better error:

$$\begin{aligned} \frac{4}{3}\Bigg[\left(\frac{\partial u}{\partial x}\right)_i &= \frac{u_{i+1} - u_{i-1}}{2h} - \frac{h^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i - \frac{h^4}{120}\left(\frac{\partial^5 u}{\partial x^5}\right)_i - \dots\Bigg] \\ \frac{-1}{3}\Bigg[\left(\frac{\partial u}{\partial x}\right)_i &= \frac{u_{i+2} - u_{i-2}}{2(2h)} - \frac{(2h)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i - \frac{(2h)^4}{120}\left(\frac{\partial^5 u}{\partial x^5}\right)_i - \dots\Bigg] \\ \implies \left(\frac{\partial u}{\partial x}\right)_i &= \frac{8(u_{i+1} - u_{i-1}) - (u_{i+2} - u_{i-2})}{12h} + \frac{h^4}{30}\left(\frac{\partial^5 u}{\partial x^5}\right)_i - \dots \end{aligned} \tag{1.4}$$

With this method, the truncation error is of order $\mathcal{O}(h^4)$. In the same way, for second order,

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{4}{3}\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{1}{3}\frac{u_{i+2} - 2u_i + u_{i-2}}{(2h)^2} + \mathcal{O}(h^4) \tag{1.5}$$

1.3 Operators

Let us define the following operators:

- Forward difference: $\Delta u_i = u_{i+1} - u_i$;
- Backward difference: $\nabla u_i = u_i - u_{i-1}$;
- Centered difference: $\delta u_i = u_{i+1/2} - u_{i-1/2}$;
- Mean: $\mu u_i = \frac{1}{2}(u_{i+1/2} + u_{i-1/2})$;

→ Note: $u_{i+1/2}$ and $u_{i-1/2}$ are not computable because they are not grid values, but can be used for derivations of other formulae.

- Identity operator: $I u_i = u_i$;
- Forward operator: $E u_i = u_{i+1}$;
- Backward operator: $E^{-1} u_i = u_{i-1}$;

→ Note: $E^{-1}E = I$.

Those operators have the following properties:

- $\mu\delta = \frac{1}{2}(E - E^{-1})$;
- $\mu^2 = I + \delta^2/4$;

The forward operator can be re-expressed using a Taylor development series:

$$\begin{aligned} E u_i = u_{i+1} &= u_i + h\frac{\partial}{\partial x}u_i + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}u_i + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3}{}_i + \dots \\ &= \left(I + hD + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \dots\right) u_i = \exp(hD)u_i \end{aligned} \tag{1.6}$$

From this, using a second Taylor development series,

$$hD = \log(I + \Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \tag{1.7}$$

And, in the same way,

$$hD = -\log(I - \nabla) = \nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots \tag{1.8}$$

We can do the same for another operator:

$$\mu\delta = \frac{1}{2}(E - E^{-1}) = \frac{1}{2}\left(\exp(hD) - \exp(-hD)\right) = \sinh(hD) \tag{1.9}$$

and we can use the Taylor series for $arc\,\sinh(x)$ but it is not very useful. By the property that $\mu^2 = I + \delta^2/4$, we get another form:

$$hD = \mu\delta\left(I - \frac{1}{6}\delta^2 + \frac{1}{30}\delta^4 - \frac{140}{\delta^6} + \dots\right) \tag{1.10}$$

If we keep only the first order term, we find the centered-difference scheme, and the terms up to second order give the Richardson extrapolation.

→ Note: in any scheme, using more information (more values, e.g. $u_{i+2}, u_{i+3}, \dots$) gives a more accurate solution and the order of the truncation error increases (e.g. to $\mathcal{O}(h^3)$).

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