



LINMA2460 Nonlinear Programming

SIMON DESMIDT
ISSAMBRE L'HERMITE DUMONT

Academic year 2024-2025 - Q2



UCLouvain

Contents

1	Definitions, notations and random properties	2
1.1	Properties	3
1.2	Complexity table	3
1.3	GM VS Newton	3
2	TODO	4
3	Gradient descent without gradient	6
4	Local rates of convergence	8
4.1	Linear rate of GM	8
4.2	Local quadratic convergence of Newton's method	10
4.3	Quasi Newton methods	11
4.4	Cubically regularized Newton's method	16
5	Constrained nonlinear programming problems	17
6	Accelerated Gradient Method	19
6.1	Derivation of the algorithm	19
6.2	Accelerated Proximal Gradient Method	22
7	Path following Interior Point Method	23
7.1	Self concordant functions	23
7.2	Path-following Interior-point Method	24
7.3	Intermediate Newton method	25
7.4	Path-following Interior point Algorithm	27
8	Tips and Tricks	29
9	Final results and important theorems	31
9.1	TODO	31
9.2	Gradient descent without gradient	31
9.3	Local rates of convergence	32
9.4	Constrained nonlinear programming problems	33
9.5	Accelerated Gradient Method	33
9.6	Path following Interior Point Method	34

Definitions, notations and random properties

- The Taylor expansion of order p of the function f around x_k and evaluated at y is:

$$T_p(y; x_k) = f(x_k) + \sum_{i=1}^p \frac{1}{i!} D^i f(x_k) (y - x_k)^i \quad (1.1)$$

- We can thus define the gradient w.r.t. y of the Taylor expansion of order p of f around x_k and evaluated at x_{k+1} :

$$\nabla_y T_p(x_{k+1}; x_k) = \nabla_y T_p(y; x_k) \big|_{y=x_{k+1}} \quad (1.2)$$

- An oracle is a "black box" that gives information about the derivatives based on x . The general form of an oracle is:

$$\text{p-order oracle: } x \mapsto \{D^i f(x)\}_{i=0}^p \quad (1.3)$$

And so we have the following simple oracles examples:

$$\begin{aligned} \text{Zero}^{th}\text{-order oracle: } x &\mapsto \{f(x)\} \\ \text{First-order oracle: } x &\mapsto \{f(x), \nabla f(x)\} \\ \text{Second-order oracle: } x &\mapsto \{f(x), \nabla f(x), \nabla^2 f(x)\} \end{aligned} \quad (1.4)$$

- $\mathcal{C}_L^p(\mathbb{R}^n)$: Class of functions p -times continuously differentiable with L -Lipschitz continuous p -order derivative, i.e. $\|D^p f(x) - D^p f(y)\| \leq L\|x - y\|, \forall x, y \in \mathbb{R}^n$. And so we have the following simple classes of problems:

- $\mathcal{C}_L^1(\mathbb{R}^n)$: Class of continuously differentiable functions with L -Lipschitz gradient;
- $\mathcal{C}_L^2(\mathbb{R}^n)$: Class of continuously differentiable functions with L -Lipschitz hessian.

- p th-order method (generalization of GM):

$$x_{k+1} = \arg \min_{y \in \mathbb{R}^n} \Omega_{x_k, y, p}(y) \equiv T_{x_k, p}(y) + \frac{M}{(p+1)!} \|y - x_k\|^{p+1} \quad (1.5)$$

where M is an approximation of the Lipschitz constant L for the p th-order derivative of f .

- Convergence rate:

– Linear:

$$\|x_{k+1} - x^*\| \leq \alpha \|x_k - x^*\| \quad \forall k \geq 0, \alpha \in (0, 1) \quad (1.6)$$

– Super Linear:

$$\lim_{k \rightarrow +\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0 \quad (1.7)$$

– Quadratic:

$$\|x_{k+1} - x^*\| \leq \beta \|x_k - x^*\|^2 \quad \forall k \geq 0, \beta > 0 \quad (1.8)$$

1.1 Properties

- For a function $f \in \mathcal{C}^1(\Omega)$ and Ω is bounded, the following holds: $\|\nabla f(x)\| \leq L$ for all $x \in \Omega$ for some $L \geq 0$.
- By the mean value theorem, for a continuously differentiable function f , $\forall x, y \in \Omega$, $\exists z \in \Omega : f(y) - f(x) = \langle \nabla f(z), y - x \rangle$.
- For a matrix A and a scalar b , $\|A\| \leq b \implies |\lambda(A)| \leq b \implies |A| \preceq bI_n$, where the absolute value of the matrix is taken component wise.

1.2 Complexity table

Method	Lipschitz	∇f	$\nabla^2 f$...	$\nabla^p f$
Zero order		$O(n\varepsilon^{-2})$			
First order	$p = 1$	$O(\varepsilon^{-2})$			
Second order	$p = 2$	✗	$O(\varepsilon^{-3/2})$		
\vdots		✗	✗	\ddots	
p order		✗	✗	✗	$O(\varepsilon^{-\frac{p+1}{p}})$

1.3 GM VS Newton

	cost per iteration	cost of memory	Local rate
GM	$\mathcal{O}(n)$	$\mathcal{O}(n)$	Linear
Quasi-Newton	$\mathcal{O}(n^2)$	$\mathcal{O}(n^2)$	Super Linear
Newton	$\mathcal{O}(n^3)$	$\mathcal{O}(n^2)$	Quadratic

→ For the GM, we assume that we don't need to compute the gradient at each iteration.

TODO

We can generalise the property of a L-Lipschitz function to $f \in \mathcal{C}_L^p(\mathbb{R}^n)$. For $p = 1$, we had

$$f(y) \leq f(x_k) + \langle \nabla f(x_k), y - x_k \rangle + \frac{L}{2} \|y - x_k\|^2 \quad \forall y \in \mathbb{R}^n \quad (2.1)$$

For a general value of p , it becomes

$$f(y) \leq T_p(y; x_k) + \frac{L}{(p+1)!} \|y - x_k\|^{p+1} \quad \forall y \in \mathbb{R}^n \quad (2.2)$$

Using this, [we need a \$p\$ -th order oracle](#) for the method to work.

To solve $\min_{x \in \mathbb{R}^n} f(x)$, we can use the iteration

$$x_{k+1} = \arg \min_{y \in \mathbb{R}^n} T_p(y; x_k) + \frac{M}{(p+1)!} \|y - x_k\|^{p+1} \quad (2.3)$$

where the constant M is an approximation of the Lipschitz constant L . [Assuming \$f \in \mathcal{C}_L^p\(\mathbb{R}^n\)\$](#) , we have

$$\begin{aligned} f(x_{k+1}) &\leq T_p(x_{k+1}; x_k) + \frac{L}{(p+1)!} \|x_{k+1} - x_k\|^{p+1} \\ &= \underbrace{T_p(x_{k+1}; x_k) + \frac{M}{(p+1)!} \|x_{k+1} - x_k\|^{p+1}}_{\leq f(x_k)} + \frac{(L-M)}{(p+1)!} \|x_{k+1} - x_k\|^{p+1} \end{aligned} \quad (2.4)$$

where the inequality $\leq f(x_k)$ is due to the decrease of f and equation (2.3). [Suppose that \$M > 2L\$](#) . After some algebraic manipulations, we get

$$f(x_k) - f(x_{k+1}) \geq \frac{L}{(p+1)!} \|x_{k+1} - x_k\|^{p+1} \quad (2.5)$$

On the other hand, using the triangular inequality,

$$\begin{aligned} \|\nabla f(x_{k+1})\| &\leq \|\nabla f(x_{k+1}) - \nabla_y T_p(x_{k+1}; x_k)\| \\ &\quad + \underbrace{\left\| \nabla_y T_p(x_{k+1}; x_k) + \nabla \left(\frac{M}{(p+1)!} \|\cdot - x_k\|^{p+1} \right) \right\|_{y=x_{k+1}}}_{=0} \\ &\quad + \left\| \nabla \left(\frac{M}{(p+1)!} \|\cdot - x_k\|^{p+1} \right) \right\|_{y=x_{k+1}} \\ &\leq \frac{L}{p!} \|x_{k+1} - x_k\|^p + \frac{M}{p!} \|x_{k+1} - x_k\|^p \end{aligned} \quad (2.6)$$

$$\implies \|x_{k+1} - x_k\| \geq \left(\frac{p!}{L+M} \right)^{1/p} \|\nabla f(x_{k+1})\|^{1/p} \quad (2.7)$$

Combining equations (2.5) and (2.7),

$$f(x_k) - f(x_{k+1}) \geq \underbrace{\frac{L}{(p+1)!} \left(\frac{p!}{L+M} \right)^{\frac{p+1}{p}}}_{=:C(L)} \|\nabla f(x_{k+1})\|^{\frac{p+1}{p}} \quad (2.8)$$

Let $T(\varepsilon) = \inf\{k \in \mathbb{N} : \|\nabla f(x_k)\| \leq \varepsilon\}$. Assume that $T(\varepsilon) \geq 2$ and $f(x) \geq f_{low} \forall x \in \mathbb{R}^n$. Summing up (2.8) for $k = 0, \dots, T(\varepsilon) - 2$,

$$\begin{aligned} f(x_0) - f_{low} &\geq f(x_0) - f(x_{T(\varepsilon)-1}) = \sum_{k=0}^{T(\varepsilon)-2} f(x_k) - f(x_{k+1}) \\ &\geq (T(\varepsilon) - 1)C(L)\varepsilon^{\frac{p+1}{p}} \\ \implies T(\varepsilon) &\leq 1 + \frac{f(x_0) - f_{low}}{C(L)}\varepsilon^{-\frac{p+1}{p}} \equiv \mathcal{O}\left(\varepsilon^{-\frac{p+1}{p}}\right) \end{aligned} \quad (2.9)$$

Gradient descent without gradient

For this problem, consider an adversarial attack on block-based image classifier. We have a machine learning model that given an image $a \in \mathbb{R}^p$ it returns $c(a) \in \mathbb{R}^m$, where $c_j(a) \in [0, 1]$ is the probability of image a to be in class j . The classifier prediction is: $j(a) = \arg \max_{j \in [1, \dots, m]} c_j(a)$.

TODO - Add mise en situation ou pas?

Given x_k , let us decide:

$$x_{k+1} = x_k - \frac{1}{\sigma} g_{h_k}(x_k) \quad h_k > 0, \sigma > 0 \quad (3.1)$$

where $g_{h_k}(x_k) \in \mathbb{R}^n$ is given by:

$$[g_{h_k}(x_k)]_j = \frac{f(x_k + h e_j) - f(x_k)}{h_k} \quad \forall j \in [1, \dots, m] \quad (3.2)$$

Suppose that $f \in \mathcal{C}_L^1(\mathbb{R}^n)$. Then,

$$\|\nabla f(x_k) - g_{h_k}(x_k)\| \leq \frac{L\sqrt{n}}{2} h_k \quad (3.3)$$

Thus

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - x_k\|^2 \\ &= f(x_k) + \langle g_{h_k}(x_k), x_{k+1} - x_k \rangle + \frac{\sigma}{2} \|x_{k+1} - x_k\|^2 \\ &\quad + \langle \nabla f(x_k) - g_{h_k}(x_k), x_{k+1} - x_k \rangle + \frac{(L - \sigma)}{2} \|x_{k+1} - x_k\|^2 \\ &\leq f(x_k) - \frac{1}{\sigma} \|g_{h_k}(x_k)\|^2 + \frac{1}{2\sigma} \|g_{h_k}(x_k)\|^2 \\ &\quad + \|\nabla f(x_k) - g_{h_k}(x_k)\| \frac{1}{\sigma} \|g_{h_k}(x_k)\| + \frac{(L - \sigma)}{2\sigma^2} \|g_{h_k}\|^2 \\ &\leq f(x_k) - \frac{1}{2\sigma} \|g_{h_k}(x_k)\|^2 + \frac{L\sqrt{n}}{2} h_k \frac{1}{\sigma} \|g_{h_k}\| + \frac{(L - \sigma)}{2\sigma^2} \|g_{h_k}\|^2 \\ &\leq f(x_k) - \frac{1}{2\sigma} \|g_{h_k}(x_k)\|^2 + \frac{L}{2} \left(\frac{nh_k^2}{2} + \frac{1}{2\sigma} \|g_{h_k}(x_k)\|^2 \right) + \frac{(L - \sigma)}{2\sigma^2} \|g_{h_k}\|^2 \\ &= f(x_k) - \left(\frac{2\sigma - L - 2(L - \sigma)}{4\sigma^2} \right) \|g_{h_k}(x_k)\|^2 + \frac{Ln}{4} h_k^2 \\ &= f(x_k) - \frac{(4\sigma - 3L)}{4\sigma} \|g_{h_k}(x_k)\|^2 + \frac{Ln}{4} h_k^2 \end{aligned} \quad (3.4)$$

$$\implies \frac{(4\sigma - 3L)}{4\sigma} \|g_{h_k}(x_k)\|^2 \leq f(x_k) - f(x_{k+1}) + \frac{Ln}{4} h_k^2 \quad (3.5)$$

If $\sigma \gg L$, then

$$\frac{1}{4\sigma} \|g_{h_k}(x_k)\|^2 \leq f(x_k) - f(x_{k+1}) + \frac{\sigma n}{4} h_k^2 \quad (3.6)$$

On the other hand, we have

$$\begin{aligned} \|\nabla f(x_k)\| &\leq \|\nabla f(x_k) - g_{h_k}(x_k)\| + \|g_{h_k}(x_k)\| \\ &\leq \frac{L\sqrt{n}}{2} h_k + \|g_{h_k}(x_k)\| \end{aligned} \quad (3.7)$$

Using trick (8.4) in chapter 8,

$$\implies \|\nabla f(x_k)\|^2 \leq \frac{L^2 n}{2} h_k^2 + 2\|g_{h_k}(x_k)\|^2 \quad (3.8)$$

$$\implies \frac{1}{8\sigma} \|\nabla f(x_k)\|^2 \leq \frac{L^2 n}{16\sigma} h_k^2 + \frac{1}{4\sigma} \|g_{h_k}(x_k)\|^2 \quad (3.9)$$

$$\implies \frac{1}{8\sigma} \|\nabla f(x_k)\|^2 \leq f(x_k) - f(x_{k+1}) + \frac{\sigma n}{4} h_k^2 + \frac{\sigma n}{16} h_k^2 \quad (3.10)$$

Let $T(\varepsilon) = \inf\{k \in \mathbb{N} : \|\nabla f(x_k)\| \leq \varepsilon\}$, with $f(x)$ bounded from below by f_{low} . Summing up (3.10) for $k = 0, \dots, T(\varepsilon) - 1$:

$$\frac{T(\varepsilon)}{8\sigma} \varepsilon^2 \leq f(x_0) - f_{low} + \frac{5\sigma n}{16} \sum_{k=0}^{T(\varepsilon)-1} h_k^2 \quad (3.11)$$

If $\{h_k^2\}_{k \geq 0}$ is summable

$$\implies T(\varepsilon) \leq 8\sigma \left(f(x_0) - f_{low} + \frac{5\sigma n}{16} \sum_{k=0}^{T(\varepsilon)-1} h_k^2 \right) \varepsilon^2 = \mathcal{O}(\varepsilon^2) \quad (3.12)$$

In terms of call to the oracle, we have a complexity bound of $\mathcal{O}(n\varepsilon^2)$.

Local rates of convergence

4.1 Linear rate of GM

Let $f \in \mathcal{C}_M^{2,2}(\mathbb{R}^n)$. Assume f has a local minimizer x^* such that

$$\mu I_n \preceq \nabla^2 f(x^*) \preceq M I_n \quad (4.1)$$

Let $x_{k+1} = x_k - \frac{1}{L} \nabla f(x_k)$ for a given $x_0 \in \mathbb{R}^n$.

Notice that

$$\begin{aligned} \nabla f(x_k) &= \nabla f(x_k) - \nabla f(x^*) \\ &= \int_0^1 \nabla^2 f(x^* + \tau(x_k - x^*)) (x_k - x^*) d\tau \\ &= \int_0^1 \nabla^2 f(x^* + \tau(x_k - x^*)) d\tau (x_k - x^*) \\ &= G_k(x_k - x^*) \end{aligned} \quad (4.2)$$

Then,

$$\begin{aligned} \|x_{k+1} - x^*\| &= \|x_k - \frac{1}{L} \nabla f(x_k) - x^*\| \\ &= \|(I_n - \frac{1}{L} G_k)(x_k - x^*)\| \\ &\leq \|I_n - \frac{1}{L} G_k\| \|x_k - x^*\| \end{aligned} \quad (4.3)$$

Since $f \in \mathcal{C}_M^{2,2}(\mathbb{R}^n)$, we have $\|\nabla^2 f(x^* + \tau(x_k - x^*)) - \nabla^2 f(x^*)\| \leq \tau M \|x_k - x^*\|$ and using this we get:

$$|\langle \nabla^2 f(x^* + \tau(x_k - x^*)) - \nabla^2 f(x^*) v, v \rangle| \leq \tau M \|x_k - x^*\| \|v\|^2 \quad \forall v \in \mathbb{R}^n \quad (4.4)$$

Using the bound (4.1) and the previous inequality, we get:

$$\begin{aligned} -\tau M \|x_k - x^*\| \|v\|^2 &\leq \left\langle \left(\nabla^2 f(x^* + \tau(x_k - x^*)) - \nabla^2 f(x^*) \right) v, v \right\rangle \leq \tau M \|x_k - x^*\| \|v\|^2 \\ \nabla^2 f(x^*) - \tau M \|x_k - x^*\| I_n &\preceq \nabla^2 f(x^* + \tau(x_k - x^*)) \preceq \nabla^2 f(x^*) + \tau M \|x_k - x^*\| I_n \\ (\mu - \tau M \|x_k - x^*\|) I_n &\preceq \nabla^2 f(x^* + \tau(x_k - x^*)) \preceq (L + \tau M \|x_k - x^*\|) I_n \end{aligned}$$

By the properties of the semi-definite matrices, and the trick (8.5), we have:

$$\begin{aligned} \int_0^1 (\mu - \tau M \|x_k - x^*\|) \|v\|^2 d\tau &\leq \int_0^1 \langle \nabla^2 f(x^* + \tau(x_k - x^*)) v, v \rangle d\tau \\ &\leq \int_0^1 (L + \tau M \|x_k - x^*\|) \|v\|^2 d\tau \quad \forall v \in \mathbb{R}^n \end{aligned} \quad (4.5)$$

By using G_k and some constants, we get:

$$-\frac{1}{L}(L + \frac{M}{2}\|x_k - x^*\|)I_n \preceq -\frac{1}{L}G_k \preceq -\frac{1}{L}(\mu - \frac{M}{2}\|x_k - x^*\|)I_n \quad (4.6)$$

$$\left(1 - \frac{1}{L}(L + \frac{M}{2}\|x_k - x^*\|)\right) I_n \preceq I_n - \frac{1}{L}G_k \preceq \left(1 - \frac{1}{L}(\mu - \frac{M}{2}\|x_k - x^*\|)\right) I_n \quad (4.7)$$

And finally,

$$\begin{aligned} \|I_n - \frac{1}{L}G_k\| &\leq \max \left\{ \left|1 - \frac{1}{L}(L + \frac{M}{2}\|x_k - x^*\|)\right|, \left|1 - \frac{1}{L}(\mu - \frac{M}{2}\|x_k - x^*\|)\right| \right\} \\ &= \max \left\{ \frac{M}{2L}\|x_k - x^*\|, 1 - \frac{\mu}{L} + \frac{M}{2L}\|x_k - x^*\| \right\} \\ &= 1 - \frac{\mu}{L} + \frac{M}{2L}\|x_k - x^*\| \end{aligned} \quad (4.8)$$

Suppose that $\frac{M}{2L}\|x_k - x^*\| \leq \frac{\mu}{2L} \iff \|x_k - x^*\| \leq \frac{\mu}{M}$

Then, in (4.8), we get:

$$\|I_n - \frac{1}{L}G_k\| \leq 1 - \frac{\mu}{2L} < 1 \quad (4.9)$$

And so, by (4.2)

$$\|x_{k+1} - x^*\| \leq \|I_n - \frac{1}{L}G_k\| \|x_k - x^*\| < \|x_k - x^*\| \quad (4.10)$$

If $\|x_0 - x^*\| < \frac{\mu}{M}$, it follows from the previous reasoning that:

$$\|x_2 - x^*\| \leq (1 - \frac{\mu}{2L})\|x_1 - x^*\| \leq (1 - \frac{\mu}{2L})^2\|x_0 - x^*\| \leq \frac{\mu}{M} \quad (4.11)$$

And so by induction, we can conclude that:

$$\|x_k - x^*\| \leq \left(1 - \frac{\mu}{2L}\right)^k \|x_0 - x^*\| \quad \forall k \geq 0 \quad (4.12)$$

\Rightarrow Linear rate of convergence

Given $\varepsilon > 0$, let $T(\varepsilon) = \inf\{k \in \mathbb{N} : \|x_k - x^*\| \leq \varepsilon\}$. Then, if $T(\varepsilon) \geq 1$ and using (4.12), we get:

$$\begin{aligned} \varepsilon &< \|x_{T(\varepsilon)-1} - x^*\| \leq \left(1 - \frac{\mu}{2L}\right)^{T(\varepsilon)-1} \|x_0 - x^*\| \\ \log \left(\frac{\varepsilon}{\|x_0 - x^*\|} \right) &\leq (T(\varepsilon) - 1) \log \left(1 - \frac{\mu}{2L}\right) \\ T(\varepsilon) - 1 &\leq \frac{\log \left(\frac{\varepsilon}{\|x_0 - x^*\|} \right)}{\log \left(1 - \frac{\mu}{2L}\right)} = \frac{\log (\|x_0 - x^*\| \varepsilon^{-1})}{|\log (1 - \frac{\mu}{2L})|} \end{aligned} \quad (4.13)$$

$$T(\varepsilon) \leq \mathcal{O}(\log(\varepsilon^{-1}))$$

\rightarrow Note: convexity was never assumed!

4.2 Local quadratic convergence of Newton's method

Let $f \in \mathcal{C}_M^{2,2}(\mathbb{R}^n)$. Assume f has a local minimizer x^* such that

$$\mu I_n \preceq \nabla^2 f(x^*) \quad \mu > 0 \quad (4.14)$$

Given $x_0 \in \mathbb{R}^n$, let:

$$x_{k+1} = x_k - \nabla^{-2} f(x_k) \nabla f(x_k) \quad (4.15)$$

We have, by the previous equation and the definition of G_k (4.2):

$$\begin{aligned} \|x_{k+1} - x^*\| &= \|x_k - \nabla^{-2} f(x_k) \nabla f(x_k) - x^*\| \\ &= \|(x_k - x^*) - \nabla^{-2} f(x_k) G_k(x_k - x^*)\| \\ &= \|\nabla^{-2} f(x_k) \left(\nabla^2 f(x_k) - \int_0^1 \nabla^2 f(x^* + \tau(x_k - x^*)) d\tau \right) (x_k - x^*)\| \\ &= \|\nabla^{-2} f(x_k) \left(\int_0^1 \nabla^2 f(x_k) - \nabla^2 f(x^* + \tau(x_k - x^*)) d\tau \right) (x_k - x^*)\| \\ &\leq \|\nabla^{-2} f(x_k)\| \left(\int_0^1 \|\nabla^2 f(x_k) - \nabla^2 f(x^* + \tau(x_k - x^*))\| d\tau \right) \|x_k - x^*\| \\ &\leq \|\nabla^{-2} f(x_k)\| \left(\int_0^1 M(1 - \tau) \|x_k - x^*\| d\tau \right) \|x_k - x^*\| \\ &\leq \|\nabla^{-2} f(x_k)\| \|x_k - x^*\|^2 \frac{M}{2} \end{aligned} \quad (4.16)$$

Since $f \in \mathcal{C}_M^{2,2}(\mathbb{R}^n)$, we have

$$\nabla^2 f(x^* + \tau(x_k - x^*)) - \nabla^2 f(x^*) \succeq \tau M \|x_k - x^*\| I_n \quad (4.17)$$

$$\begin{aligned} \nabla^2 f(x_k) &\succeq \nabla^2 f(x^*) - M \|x_k - x^*\| I_n \\ &\succeq (\mu - M \|x_k - x^*\|) I_n \\ \lambda_{\min}(\nabla^2 f(x_k)) &\geq \mu - M \|x_k - x^*\| \end{aligned} \quad (4.18)$$

Suppose that $-M \|x_k - x^*\| \geq -\frac{\mu}{2} \Leftrightarrow \|x_k - x^*\| \leq \frac{\mu}{2M}$. Then,

$$\begin{aligned} \lambda_{\min}(\nabla^2 f(x_k)) &\geq \frac{\mu}{2} \\ \lambda_{\max}(\nabla^{-2} f(x_k)) &\leq \frac{2}{\mu} \\ \Rightarrow \|\nabla^{-2} f(x_k)\| &\leq \frac{2}{\mu} \end{aligned} \quad (4.19)$$

Therefore, by (4.16), we conclude that:

$$\begin{aligned} \|x_{k+1} - x^*\| &\leq \frac{M}{2} \|\nabla^{-2} f(x_k)\| \|x_k - x^*\|^2 \\ &\leq \frac{M}{\mu} \|x_k - x^*\|^2 \end{aligned} \quad (4.20)$$

If $\|x_k - x^*\| \leq \frac{\mu}{2M}$ then,

$$\|x_{k+1} - x^*\| \leq \frac{M}{\mu} \|x_k - x^*\|^2 = \frac{1}{2} \|x_k - x^*\| \quad (4.21)$$

If $\|x_0 - x^*\| \leq \frac{\mu}{2M}$ then $\{x_k\}_{k \geq 0} \subset B[x^*, \frac{\mu}{2M}]$.

Denote $\delta_k = \frac{M}{\mu} \|x_k - x^*\|$, then we have $\delta_0 = \frac{M}{\mu} \|x_0 - x^*\| \leq \frac{1}{2}$, and if we combine this with (4.21), we get:

$$\delta_{k+1} \leq \delta_k^2 \quad \forall k \geq 0 \quad (4.22)$$

And if we proceed by recurrence, we get:

$$\begin{aligned} \delta_1 &\leq \delta_0^2 \leq \left(\frac{1}{2}\right)^2 \\ \delta_2 &\leq \delta_1^2 \leq \left(\frac{1}{2}\right)^4 \\ &\vdots \\ \delta_k &\leq \left(\frac{1}{2}\right)^{2^k} \quad \forall k \geq 0 \end{aligned} \quad (4.23)$$

$$\Rightarrow \|x_k - x^*\| \leq \frac{\mu}{M} \left(\frac{1}{2}\right)^{2^k} \quad (4.24)$$

Let $T(\varepsilon) = \inf\{k \in \mathbb{N} : \|x_k - x^*\| \leq \varepsilon\}$ and [suppose that \$T\(\varepsilon\) \geq 1\$](#) . Then using the convergence rate (4.24), we can state the maximal number of iterations:

$$\varepsilon \leq \|x_{T(\varepsilon)-1} - x^*\| \leq \frac{\mu}{M} \left(\frac{1}{2}\right)^{2^{T(\varepsilon)-1}} \quad (4.25)$$

$$2^{2^{T(\varepsilon)-1}} \leq \frac{\mu}{M} \varepsilon^{-1} \quad (4.26)$$

$$\Rightarrow T(\varepsilon) \leq \log_2(\log_2(\frac{\mu}{M} \varepsilon^{-1}))$$

4.3 Quasi Newton methods

4.3.1 SR1 Update

One step of a Quasi-Newton method is given by:

$$x_{k+1} = x_k - B_k \nabla f(x_k) \quad (4.27)$$

With $B_k \in \mathbb{R}^{n \times n}$, [symmetric and non-singular](#).

[Suppose that \$x_k \rightarrow x^*\$ when \$k \rightarrow \infty\$, and that \$\nabla^2 f\(x_k\) \succeq \mu I_n\$ with \$\mu \geq 0\$.](#)

We want the condition on B_k to have a Super Linear convergence (1.7) of the Quasi-Newton method. So let us [assume that \$f \in \mathcal{C}_M^{2,2}\(\mathbb{R}^n\)\$](#) .

Then,

$$\|\nabla^2 f(x_{k+1}) - \nabla^2 f(x_k)\| \leq M \|x_{k+1} - x_k\| \quad (4.28)$$

GOOD LABEL ?

$$\|\nabla f(x_{k+1}) - \nabla f(x_k) - \nabla^2 f(x_k)(x_{k+1} - x_k)\| \leq \frac{M}{2} \|x_{k+1} - x_k\|^2 \quad (4.29)$$

Therefore

$$\begin{aligned} \nabla f(x_{k+1}) &= \nabla f(x_{k+1}) - \nabla f(x_k) - \nabla^2 f(x_k)(x_{k+1} - x_k) \\ &\quad + \nabla f(x_k) + \nabla^2 f(x_k)(x_{k+1} - x_k) \end{aligned} \quad (4.30)$$

Using the relation (4.27) we get:

$$\begin{aligned} \nabla f(x_{k+1}) &= \nabla f(x_{k+1}) - \nabla f(x_k) - \nabla^2 f(x_k)(x_{k+1} - x_k) \\ &\quad - B_k^{-1}(x_{k+1} - x_k) \\ &\quad + \nabla^2 f(x_k)(x_{k+1} - x_k) \\ &= \nabla f(x_{k+1}) - \nabla f(x_k) - \nabla^2 f(x_k)(x_{k+1} - x_k) \\ &\quad - \left(B_k^{-1} - \nabla^2 f(x^*) \right) (x_{k+1} - x_k) \\ &\quad + \left(\nabla^2 f(x_k) - \nabla^2 f(x^*) \right) (x_{k+1} - x_k) \end{aligned} \quad (4.31)$$

$$\begin{aligned} \|\nabla f(x_{k+1})\| &\leq \|\nabla f(x_{k+1}) - \nabla f(x_k) - \nabla^2 f(x_k)(x_{k+1} - x_k)\| \\ &\quad + \left\| \left(B_k^{-1} - \nabla^2 f(x^*) \right) (x_{k+1} - x_k) \right\| \\ &\quad + \left\| \left(\nabla^2 f(x_k) - \nabla^2 f(x^*) \right) (x_{k+1} - x_k) \right\| \\ &\leq \frac{M}{2} \|x_{k+1} - x_k\|^2 + M \|x_k - x^*\| \|x_{k+1} - x_k\| \\ &\quad + \left\| \left(B_k^{-1} - \nabla^2 f(x_k) \right) (x_{k+1} - x_k) \right\| \end{aligned}$$

On the line before we used (4.28) and (4.29). And so we can write:

$$\frac{\|\nabla f(x_{k+1})\|}{\|x_{k+1} - x_k\|} \leq \frac{M}{2} \|x_{k+1} - x_k\| + M \|x_k - x^*\| + \frac{\left\| \left(B_k^{-1} - \nabla^2 f(x_k) \right) (x_{k+1} - x_k) \right\|}{\|x_{k+1} - x_k\|} \quad (4.32)$$

From now on , suppose that this condition (Dimis-Mori condition) is true:

$$\lim_{k \rightarrow \infty} \frac{\left\| \left(B_k^{-1} - \nabla^2 f(x_k) \right) (x_{k+1} - x_k) \right\|}{\|x_{k+1} - x_k\|} = 0 \quad (4.33)$$

Under this condition and by (4.32), we have:

$$\lim_{k \rightarrow \infty} \frac{\|\nabla f(x_{k+1})\|}{\|x_{k+1} - x_k\|} = 0 \quad (4.34)$$

As $\|x_{k+1} - x_k\| \rightarrow 0$, we conclude that $\lim_{x \rightarrow \infty} \|\nabla f(x_{k+1})\| = 0$ and so $\|\nabla f(x^*)\| = 0 \Rightarrow \nabla f(x^*) = 0$, meaning that x^* is a stationary point of $f(\cdot)$.

We have $\nabla^2 f(x^*) \succeq \mu I_n$ and given $y \in \mathbb{R}^n$, we have:

$$\begin{aligned} \nabla^2 f(y) - \nabla^2 f(x^*) &\succeq -M \|y - x^*\| I_n \\ \nabla^2 f(y) &\succeq (\mu - M \|y - x^*\|) I_n \end{aligned} \quad (4.35)$$

Thus, if $-M\|y - x^*\| \geq -\frac{\mu}{2}$ then $\nabla^2 f(y) \succeq \frac{\mu}{2} I_n$.

Since $x_k \rightarrow x^*$, there exists $k_0 \in \mathbb{N}$ such that $\|x_{k+1} - x^*\| \leq \frac{\mu}{2M} \forall k \geq k_0$. Thus for any $\tau \in [0, 1]$:

$$\|x^* + \tau(x_{k+1} - x^*) - x^*\| \leq \frac{\mu}{2M}, \quad \forall k \geq k_0 \quad (4.36)$$

and so $\nabla^2 f(x^* + \tau(x_{k+1} - x^*)) \succeq \frac{\mu}{2} I_n \forall k \geq k_0$.

$$\begin{aligned} \|x_{k+1} - x^*\| \|\nabla f(x_{k+1})\| &\geq (x_{k+1} - x^*)^T \nabla f(x_{k+1}) \\ &= (x_{k+1} - x^*)^T (\nabla f(x_{k+1}) - \nabla f(x^*)) \\ &= (x_{k+1} - x^*)^T \int_0^1 \nabla^2 f(x^* + \tau(x_{k+1} - x^*)) (x_{k+1} - x^*) d\tau \\ &\geq \int_0^1 (x_{k+1} - x^*)^T \frac{\mu}{2} I_n (x_{k+1} - x^*) d\tau \\ &= \frac{\mu}{2} \|x_{k+1} - x^*\|^2 \end{aligned} \quad (4.37)$$

$$\|\nabla f(x_{k+1})\| \geq \frac{\mu}{2} \|x_{k+1} - x^*\| \quad (4.38)$$

Let $\rho_k = \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|}$ then, using (8.6), we obtain:

$$\begin{aligned} \frac{\|\nabla f(x_{k+1})\|}{\|x_{k+1} - x_k\|} &\geq \frac{(\frac{\mu}{2})\|x_{k+1} - x^*\|}{\|x_{k+1} - x_k\|} \\ &\geq \frac{(\frac{\mu}{2})\|x_{k+1} - x^*\|}{\|x_{k+1} - x^*\| + \|x_k - x^*\|} \\ &= \frac{(\frac{\mu}{2})\rho_k}{\rho_k + 1} \end{aligned} \quad (4.39)$$

Combining (4.39) and (4.32), we get:

$$\frac{\mu}{2} \frac{\rho_k}{\rho_k + 1} \leq \frac{M}{2} \|x_{k+1} - x_k\| + M \|x_k - x^*\| + \frac{\| (B_k^{-1} - \nabla^2 f(x^*)) (x_{k+1} - x_k) \|}{\|x_{k+1} - x_k\|} \quad (4.40)$$

Since the right hand side goes to zero when $k \rightarrow +\infty$, then we have: **IDK how to write that**

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\rho_k}{1 + \rho_k} &= 0 \\ \lim_{k \rightarrow \infty} \frac{1}{\frac{1}{\rho_k} + 1} &= 0 \\ \Rightarrow \lim_{k \rightarrow \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} &\Rightarrow \lim_{k \rightarrow \infty} \rho_k = 0 \end{aligned} \quad (4.41)$$

For $n = 1$, the Quasi-Newton update is written:

$$x_{k+1} = x_k - b_k f'(x_k), \quad k \geq 0 \quad (4.42)$$

with $b_k \in \mathbb{R}$. We want $b_k \approx f''(x_k)^{-1}$ and by finite difference we can express it like that $b_k^{-1} \approx \frac{f'(x_{k-1}+h) - f'(x_{k-1})}{h}$. And with $h = x_k - x_{k-1}$, we can define:

$$b_k^{-1} = \frac{f'(x_k) - f'(x_{k-1})}{x_k - x_{k-1}} \quad (4.43)$$

Thus if $x_k \rightarrow x^*$ then:

$$\lim_{k \rightarrow \infty} \frac{|(b_k^{-1} - f''(x^*))(x_k - x_{k-1})|}{|x_k - x_{k-1}|} = 0 \quad (4.44)$$

Because we can notice that:

$$\frac{|(b_k^{-1} - f''(x^*))(x_k - x_{k-1})|}{|x_k - x_{k-1}|} = |b_k^{-1} - f''(x_{k-1})| + |f''(x_{k-1}) - f''(x^*)| \quad (4.45)$$

Since $x_k \rightarrow x^*$, we have $h = x_k - x_{k-1}$ and so:

$$b_k^{-1} = \frac{f'(x_k) - f'(x_{k-1}))}{x_k - x_{k-1}} \rightarrow f''(x_{k-1}) \quad (4.46)$$

Thus, $\lim_{k \rightarrow \infty} |b_k^{-1} - f''(x_k)| = 0$.

Assuming that f'' is continuous, we have $\lim_{k \rightarrow \infty} |f''(x_k) - f''(x^*)| = 0$.

If we define $s_{k-1} = x_k - x_{k-1}$ and $y_{k-1} = f'(x_k) - f'(x_{k-1})$ and knowing (4.43), we can write:

$$\begin{aligned} b_k(f'(x_k) - f'(x_{k-1})) &= x_k - x_{k-1} \\ b_k y_{k-1} &= s_{k-1} \end{aligned} \quad (4.47)$$

This suggests that for $n > 1$, we should define the secant condition, B_k such that:

$$B_k y_{k-1} = s_{k-1} \quad (4.48)$$

Let us define $f(x) = \frac{1}{2} \|Ax - b\|^2 = \frac{1}{2} x^T A^T A x - (A^T b)^T x + \frac{1}{2} b^T b$. If A is full rank then f is a strongly convex quadratic function. And we have $\nabla f(x_k) = A^T A x_k - A^T b$. Then,

$$y_{k-1} = \nabla f(x_k) - \nabla f(x_{k-1}) = A^T A (x_k - x_{k-1}) = \nabla^2 f(x_k) s_{k-1} \quad (4.49)$$

And so

$$\nabla^2 f(x_k) y_{k-1} = s_{k-1} \quad (4.50)$$

Therefore, $\nabla^2 f$ satisfies the secant condition (4.48), when f is a strongly convex quadratic function. Thus it is reasonable to require the secant for any approximation to $\nabla^2 f(x_k)$.

Now, how can we compute B_k such that it satisfies the secant condition (4.48)?

Given a matrix B_{k-1} , our goal is to find a perturbation matrix $P_{k-1} \in \mathbb{R}^{n \times n}$ such that:

$$(B_{k-1} + P_{k-1}) y_{k-1} = s_{k-1} \quad (4.51)$$

If we get such P_{k-1} , we can define $B_k = B_{k-1} + P_{k-1}$, which would satisfy the secant condition (4.48).

For that we need at least n degrees of freedom and a symmetric matrix, so it is natural to try:

$$P_{k-1} = v_{k-1} v_{k-1}^T, \quad v_{k-1} \in \mathbb{R}^n \quad (4.52)$$

So we get:

$$(B_{k-1} + v_{k-1} v_{k-1}^T) y_{k-1} = s_{k-1} \quad (4.53)$$

By algebraic manipulations, we get:

$$\begin{aligned} (v_{k-1}^T y_{k-1}) v_{k-1} &= s_{k-1} - B_{k-1} y_{k-1} \\ v_{k-1} &= \frac{s_{k-1} - B_{k-1} y_{k-1}}{\beta} \quad \text{for } \beta = v_{k-1}^T y_{k-1} \end{aligned} \quad (4.54)$$

Combining the two previous equations, we get:

$$\begin{aligned} \left(\frac{1}{\beta} (s_{k-1} - B_{k-1} y_{k-1})^T y_{k-1} \right) \frac{1}{\beta} (s_{k-1} - B_{k-1} y_{k-1}) &= s_{k-1} - B_{k-1} y_{k-1} \\ \frac{1}{\beta^2} (s_{k-1} - B_{k-1} y_{k-1})^T y_{k-1} &= 1 \end{aligned} \quad (4.55)$$

We can isolate β :

$$\beta = \sqrt{(s_{k-1} - B_{k-1} y_{k-1})^T y_{k-1}} \quad (4.56)$$

Combining (4.54) and (4.56), we get:

$$v_{k-1} = \frac{s_{k-1} - B_{k-1} y_{k-1}}{\sqrt{(s_{k-1} - B_{k-1} y_{k-1})^T y_{k-1}}} \quad (4.57)$$

This leads us to the following update for B_k :

$$\begin{aligned} B_k &= B_{k-1} + v_{k-1} v_{k-1}^T \\ &= B_{k-1} + \frac{(s_{k-1} - B_{k-1} y_{k-1}) (s_{k-1} - B_{k-1} y_{k-1})^T}{(s_{k-1} - B_{k-1} y_{k-1})^T y_{k-1}} \end{aligned} \quad (4.58)$$

This is called the **SR1 update** (symmetric rank 1 update).

4.3.2 BFGS Update

Let's take back $B_{k+1} y_k = s_k$ and defining $H_{k+1} = B_{k+1}^{-1} \approx \nabla^2 f(x_{k+1})$, we get $H_{k+1} s_k = y_k$.

The idea is to find a rank 2 update that consists in finding $a, b \in \mathbb{R}$ and $v, u \in \mathbb{R}^n$ such that:

$$(H_k + a u u^T + b v v^T) s_k = y_k \quad (4.59)$$

Noticing that $u^T s_k$ and $v^T s_k$ are scalars, we can impose that:

$$\begin{cases} a(u^T s_k) u = -H_k s_k \\ b(v^T s_k) v = y_k \end{cases} \quad (4.60)$$

It suggests that we should take $a = \frac{1}{u^T s_k}$ and $b = \frac{1}{v^T s_k}$. Which gives us:

$$\begin{cases} u = -H_k s_k \\ v = y_k \end{cases} \quad (4.61)$$

Combining the two equations, we get:

$$H_{k+1} = H_k - \frac{H_k s_k s_k^T H_k}{s_k^T H_k s_k} + \frac{y_k y_k^T}{y_k^T s_k} \quad (4.62)$$

Using linear algebra, we can compute:

$$\begin{aligned} B_{k+1} &= H_{k+1}^{-1} \\ &= \left(I - \rho_k s_k y_k^T \right) B_k \left(I - \rho_k y_k s_k^T \right) + \rho_k s_k s_k^T \text{ with } \rho_k = \frac{1}{y_k^T s_k} \end{aligned} \quad (4.63)$$

Remarks:

- If $B_k \succ 0$ and $s_k^T y_k > 0$ then $B_{k+1} \succ 0$.
- If $B_k \succ 0$ and $d_k = -B_k \nabla f(x_k)$, then

$$\langle \nabla f(x_k), d_k \rangle = -\langle \nabla f(x_k), B_k \nabla f(x_k) \rangle < 0 \quad (4.64)$$

and so d_k is a descent direction for f at x_k .

- The LBFGS is a low memory of BFGS, that does not require the storage of the matrices B_k . Given a vector $v \in \mathbb{R}^n$, it computes $B_k v$, which is all that we need to implement QN method.

4.4 Cubically regularized Newton's method

The cubically regularized Newton's method is defined as follows:

$$x_{k+1} = \arg \min_{y \in \mathbb{R}^n} f(x_k) + \nabla f(x_k)^T (y - x_k) + \frac{1}{2} (y - x_k)^T \nabla^2 f(x_k) (y - x_k) + \frac{M}{6} \|y - x_k\|^3 \quad (4.65)$$

Based on the following equation, seen in the first lecture:

$$f(x_k) - f(x_{k+1}) \geq \frac{L_p}{(p+1)!} \left(\frac{p!}{(M + L_p)} \right)^{\frac{p+1}{p}} \|\nabla f(x_{k+1})\|^{\frac{p+1}{p}} \quad (4.66)$$

We can get the following expression a cubically regularized Newton's method (assuming $M = 0$):

$$f(x_k) - f(x_{k+1}) \geq \frac{1}{\sqrt{L_p}} \frac{\sqrt{2}}{3} \|\nabla f(x_{k+1})\|^{\frac{3}{2}} \quad (4.67)$$

NOT SURE ABOUT THAT

Constrained nonlinear programming problems

Consider the constrained problem

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad c_i(x) = 0, \quad i \in \{1, \dots, m\} \quad (5.1)$$

where $f, c_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are \mathcal{C}^1 and there exists at least a \hat{x} such that $c_i(\hat{x}) = 0$.

A natural approach to solve this problem is to consider the related unconstrained problem in which we try to minimize $f(x)$ plus a term that penalizes the violation of the constraints (quadratic penalty function).

$$\min_{x \in \mathbb{R}^n} Q_\sigma(x) \equiv f(x) + \frac{\sigma}{2} \|c(x)\|_2^2 \quad (5.2)$$

For the problem (5.1), we would like to find a KKT point x^* for which there exists $\lambda^* \in \mathbb{R}^m$ such that:

$$\begin{cases} \nabla f(x^*) - \sum_{i=1}^m \lambda_i^* \nabla c_i(x^*) = 0 & \text{(stationarity)} \\ c(x^*) = 0 & \text{(feasibility)} \end{cases} \quad (5.3)$$

In practice, we are happy if we can find an $(\varepsilon_1, \varepsilon_2)$ -KKT point for (5.1), i.e. a point x^+ such that there exists λ^+ with:

$$\begin{cases} \|\nabla f(x^+) - \sum_{i=1}^m \lambda_i^+ \nabla c_i(x^+)\| \leq \varepsilon_1 \\ \|c(x^+)\| \leq \varepsilon_2 \end{cases} \quad (5.4)$$

Let us relate (5.2) and (5.1). Notice that¹:

$$\begin{aligned} \|\nabla Q_\sigma(x)\| &= \|\nabla f(x) + \sigma \mathbf{J}_c(x)^T c(x)\| \\ &= \|\nabla f(x) + \sigma \sum_{i=1}^m c_i(x) \nabla c_i(x)\| \\ &= \|\nabla f(x) - \sum_{i=1}^m \lambda_i^+ \nabla c_i(x)\| \quad \text{with} \quad \lambda_i^+ = -\sigma c_i(x^+) \end{aligned} \quad (5.5)$$

¹ $J_c(\cdot)$ is the Jacobian of $c(\cdot)$.

Therefore, if $\|\nabla Q_\sigma(x^+)\| \leq \varepsilon_1$, then there exists $\lambda^+ \in \mathbb{R}^m, \lambda^+ = -\sigma c(x^+)$ such that $\|\nabla f(x^+) - \sum_{i=1}^m \lambda_i^+ \nabla c_i(x^+)\| \leq \varepsilon_1$.

Given $\bar{x} \in \mathbb{R}^n$, suppose that we compute x^+ such that

$$\begin{aligned} Q_\sigma(x^+) &\leq Q_\sigma(\bar{x}) \\ f(x^+) + \frac{\sigma}{2} \|c(x^+)\|^2 &\leq f(\bar{x}) + \frac{\sigma}{2} \|c(\bar{x})\|^2 \\ \frac{\sigma}{2} \|c(x^+)\|^2 &\leq f(\bar{x}) - f(x^+) + \frac{\sigma}{2} \|c(\bar{x})\|^2 \\ \|c(x^+)\|^2 &\leq \frac{2}{\sigma} (f(\bar{x}) - f(x^+)) + \|c(\bar{x})\|^2 \end{aligned} \quad (5.6)$$

If $f(x) \geq f_{low} \quad \forall x \in \mathbb{R}^n$, we get $\|c(x^+)\|^2 \leq \frac{2}{\sigma} (f(\bar{x}) - f_{low}) + \|c(\bar{x})\|^2$.

If $\|c(\bar{x})\| \leq \frac{\varepsilon_2}{\sqrt{2}}$ and $\sigma \geq \frac{4}{\varepsilon_2^2} (f(\bar{x}) - f_{low})$, then $\|c(x^+)\|^2 \leq \varepsilon_2^2$ and so $\|c(x^+)\| \leq \varepsilon_2$.

In summary, if we have $\bar{x} \in \mathbb{R}^n$ such that $\|c(\bar{x})\| \leq \frac{\varepsilon_2}{\sqrt{2}}$, and using a method for unconstrained optimization (e.g. GM), we compute x^+ with

$$Q_\sigma(x^+) \leq Q_\sigma(\bar{x}) \quad \text{and} \quad \|\nabla Q_\sigma(x^+)\| \leq \varepsilon_1 \quad (5.7)$$

for $\sigma \geq \frac{4}{\varepsilon_2^2} (f(\bar{x}) - f_{low})$, then x^+ is a $(\varepsilon_1, \varepsilon_2)$ -KKT point for the unconstrained problem (5.1).

Algorithm 1 Quadratic Penalty Method

- 1: **Input:** $\varepsilon_1, \varepsilon_2 \in (0, 1)$, $x_0 \in \mathbb{R}^n$ such that $\|c(x_0)\|_2 \leq \frac{\varepsilon_2}{\sqrt{2}}$, $\sigma_0 > 0$
- 2: $k = 0$
- 3: **while** $\|c(x_{k+1})\| > \varepsilon_1$ **do**
- 4: Compute $x_{k+1} \in \mathbb{R}^n$ as an approximate solution to

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} Q_{\sigma_k}(x) \\ \text{such that} \quad &Q_{\sigma_k}(x_{k+1}) \leq Q_{\sigma_k}(x_0) \\ \text{and} \quad &\|\nabla Q_{\sigma_k}(x_{k+1})\| \leq \varepsilon_2 \end{aligned} \quad (5.8)$$

- 5: $\sigma_{k+1} \leftarrow 2\sigma_k$
 - 6: $k \leftarrow k + 1$
 - 7: **end while**
-

→ Note: We can compute x_{k+1} satisfying (5.8) by using any monotone optimization method starting from:

$$x_k^* = \arg \min \{Q_{\sigma_k}(x_0), Q_{\sigma_k}(x_k)\} \quad (5.9)$$

- For a constrained problem of the form $\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t. } c_i \leq 0 \quad i = 0, \dots, m$, we can add slack variables to obtain an equivalent equality constrained problem:

$$\begin{aligned} &\min_{x \in \mathbb{R}^n, s \in \mathbb{R}^m} f(x) \\ \text{s.t. } &c_i(x) + s_i^2 = 0 \quad i = 1, \dots, m \end{aligned} \quad (5.10)$$

Accelerated Gradient Method

6.1 Derivation of the algorithm

$$\min_{x \in \mathbb{R}^n} f(x) \quad (6.1)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, ∇f is L -Lipschitz and has a minimizer x^* . The Accelerated Gradient Method combines present and past information to obtain a point y_k (prediction) and then perform a gradient step using this point as reference point.

$$\begin{cases} y_k = (1 - \gamma_k)x_k + \gamma_k v_k, & \gamma_k \in (0, 1) \\ x_{k+1} = x_k - \frac{1}{L} \nabla f(y_k) \end{cases} \quad (6.2)$$

We will identify ways to define v_k and γ_k based on the following guiding inequalities:

$$\begin{aligned} v_k &= \arg \min_{x \in \mathbb{R}^n} \Psi_k(x) \\ \Psi_k(x) &\leq A_k f(x) + \frac{1}{2} \|x - x_0\|^2 \\ A_k f(x_k) &\leq \min_{x \in \mathbb{R}^n} \Psi_k(x) \equiv \Psi_k^*, \quad A_k \geq 0 \\ A_k &\geq c(k-1)^2 \quad \forall k \geq 2 \end{aligned} \quad (6.3)$$

Assuming the 3 last guiding inequalities (6.3) hold, we have:

$$\begin{aligned} A_k f(x_k) &\leq \min_{x \in \mathbb{R}^n} \Psi_k(x) \\ &\leq \Psi_k(x^*) \\ &\leq A_k f(x^*) + \frac{1}{2} \|x^* - x_0\|^2 \\ (f(x_k) - f(x^*)) &\leq \frac{\|x_k - x^*\|^2}{2A_k} \quad \forall k \geq 2 \\ &\leq \frac{\|x_k - x^*\|^2}{2C(k-1)^2} = \mathcal{O}(k^{-2}) = \mathcal{O}(\varepsilon^{-1/2}) \quad \forall k \geq 2 \end{aligned} \quad (6.4)$$

If we take $A_0 = 0$ and $\Psi_0(x) = \frac{1}{2} \|x - x_0\|^2$, then the second inequality from (6.3) is true for $k = 0$. Let us assume the inequality is true for some $k \geq 0$. Looking at the case $k = 1$, it appears that we can define:

$$\Psi_{k+1}(x) = \Psi_k(x) + b_k (f(y_k) + \langle \nabla f(y_k), x - y_k \rangle) \quad (6.5)$$

with $b_k > 0$ (to be determined).

Suppose that the inequality holds for $k \geq 0$. Then, by the convexity of f and doing an induction assumption:

$$\begin{aligned}\Psi_{k+1}(x) &\leq \Psi_k(x) + b_k f(x) \\ &\leq A_k f(x) + \frac{1}{2} \|x - x_0\|^2 + b_k f(x) \\ &= (A_k + b_k) f(x) + \frac{1}{2} \|x - x_0\|^2\end{aligned}\tag{6.6}$$

Therefore, if we define $A_{k+1} = A_k + b_k$, then the second inequality of (6.3) will also hold for $k + 1$. Regarding of the third inequality of (6.3), notice that:

$$\begin{aligned}A_0 f(x_0) = 0 &= \min_{x \in \mathbb{R}^n} \frac{1}{2} \|x - x_0\|^2 \\ &= \min_{x \in \mathbb{R}^n} \Psi_0(x)\end{aligned}\tag{6.7}$$

It holds for $k = 0$, suppose that it still holds for $k \geq 0$. We want to show that it is also true for $k + 1$. Notice that:

$$\begin{aligned}\Psi_1 &= \frac{1}{2} \|x - x_0\|^2 + b_0 (f(y_0) + \langle \nabla f(y_0), x - y_0 \rangle) \\ \Psi_2 &= \frac{1}{2} \|x - x_0\|^2 + \sum_{i=0}^1 b_0 (f(y_i) + \langle \nabla f(y_i), x - y_i \rangle) \\ &\vdots \\ \Psi_k &= \frac{1}{2} \|x - x_0\|^2 + \sum_{i=0}^{k-1} b_0 (f(y_i) + \langle \nabla f(y_i), x - y_i \rangle)\end{aligned}\tag{6.8}$$

Thus, $\Psi_k(x)$ is a μ -strongly convex function with $\mu = 1$. Therefore:

$$\begin{aligned}\Psi_k(x) &\geq \Psi_k(v_k) + \frac{1}{2} \|v_k - x_0\|^2 \\ &= \min_{x \in \mathbb{R}^n} \Psi_k(x) + \frac{1}{2} \|v_k - x_0\|^2 \\ &\geq A_k f(x_k) + \frac{1}{2} \|v_k - x_0\|^2\end{aligned}\tag{6.9}$$

And so:

$$\begin{aligned}\min_x \Psi_{k+1}(x) &= \min_x \Psi_k + b_k (f(y_k) + \langle \nabla, x - y_k \rangle) \\ &\geq \min_x A_k f(x_k) + \frac{1}{2} \|v_k - x_0\|^2 + b_k (f(y_k) + \langle \nabla, x - y_k \rangle) \\ &\geq \min_x A_k (f(x_k) + \langle \nabla, x_k - y_k \rangle) + b_k (f(y_k) + \langle \nabla, x - y_k \rangle) \\ &\geq (A_k + b_k) f(y_k) + \langle \nabla f(y_k), A_k x_k + b_k x - A_{k+1} y_k \rangle + \frac{1}{2} \|v_k - x_0\|^2 \\ &\geq (A_{k+1}) f(y_k) + \langle \nabla f(y_k), A_k x_k + b_k x - A_{k+1} y_k \rangle + \frac{1}{2} \|v_k - x_0\|^2\end{aligned}\tag{6.10}$$

To make things consistent, let us impose

$$A_k x_k - A_{k+1} y_k + b_k x = b_k (x - v_k) \iff y_k = \frac{A_k}{A_{k+1}} x_k + \frac{b_k}{A_{k+1}} v_k \quad (6.11)$$

And so we can continue equation (6.10):

$$\min_{x \in \mathbb{R}^n} \Psi_{k+1}(x) A_{k+1} \min_{x \in \mathbb{R}^n} \geq f(y_k) + \langle \nabla f(y_k), \gamma_k (x - v_k) \rangle + \frac{1}{2A_{k+1}\gamma_k^2} \|\gamma_k (v_k - x)\|^2 \quad (6.12)$$

To verify the Lipschitz condition, we impose

$$\frac{1}{2A_{k+1}\gamma_k^2} = \frac{L}{2} \iff b_k^2 - \frac{1}{L} b_k - \frac{A_k}{L} = 0 \implies b_k = \frac{1 + \sqrt{1 + 4A_k L}}{2L} \quad (6.13)$$

From all that have been computed previously, we can find a bound in terms of iterations needed. If $x^* = \arg \min f(x)$, we have

$$\begin{aligned} A_k f(x_k) &\leq \min_{x \in \mathbb{R}^n} \Psi_k(x) \leq \Psi_k(x^*) \leq A_k f(x^*) + \frac{1}{2} \|x^* - x_k\|^2 \\ \Rightarrow A_k (f(x_k) - f(x^*)) &\leq \frac{1}{2} \|x^* - x_k\|^2 \\ \Rightarrow f(x_k) - f(x^*) &\leq \frac{1}{2A_k} \|x^* - x_k\|^2 \end{aligned} \quad (6.14)$$

From the relation $A_{k+1} = A_k + b_k$ and the definition of b_k , we can show that $A_k \geq C(k-1)^2$ with $C > 0$ and $k \geq 2$. Thus, we get

$$f(x_k) - f(x^*) \leq \frac{\|x_0 - x^*\|^2}{2C(k-1)^2} = \mathcal{O}(1/k^2) \quad \forall k \geq 1 \quad (6.15)$$

A recap is given in algorithm 2.

Algorithm 2 Accelerated Gradient Method

- 1: **Input:** Given $x_0 \in \mathbb{R}^n$, define $\Psi_0(x) = \frac{1}{2} \|x - x_0\|^2$, $A_0 = 0$, $b_0 = 0$, $k = 0$;
- 2: Compute

$$b_k = \frac{1 + \sqrt{1 + 4A_k L}}{2L} > 0; \quad (6.16)$$

- 3: Set $\gamma_k = \frac{b_k}{A_{k+1}} \in (0, 1]$ and compute $y_k = (1 - \gamma_k)x_k + \gamma_k v_k$;

- 4: Set

$$x_{k+1} = \arg \min_{x \in \mathbb{R}^n} f(y_k) + \langle \nabla f(y_k), x - y_k \rangle + \frac{L}{2} \|x - y_k\|^2 \quad (6.17)$$

and $A_{k+1} = A_k + b_k$;

- 5: Define

$$\Psi_{k+1}(x) = \Psi_k(x) + b_k (f(y_k) + \langle \nabla f(y_k), x - y_k \rangle) \quad \forall x \in \mathbb{R}^n \quad (6.18)$$

and set

$$v_{k+1} = \arg \min_{x \in \mathbb{R}^n} \Psi_{k+1}(x) \quad (6.19)$$

- 6: $k \leftarrow k + 1$ and go back to Step 1;
-

6.2 Accelerated Proximal Gradient Method

In this section, we consider the minimisation of a function over a nonempty, closed and convex set Ω . We decompose the objective function F into a smooth and a possibly non smooth part:

$$\min_{x \in \Omega \subseteq \mathbb{R}^n} F(x) \equiv f(x) + \varphi(x) \quad (6.20)$$

The accelerated proximal gradient method consists in using the proximal operator of the non smooth part φ to define x_{k+1} :

Algorithm 3 Accelerated Proximal Gradient Method

- 1: **Input:** Given $x_0 \in \text{dom}F$, define $\Psi_0(x) = \frac{1}{2}\|x - x_0\|^2$, $A_0 = 0$, $b_0 = 0$, $k = 0$;
- 2: Compute

$$b_k = \frac{1 + \sqrt{1 + 4A_k L}}{2L} > 0; \quad (6.21)$$

- 3: Set $\gamma_k = \frac{b_k}{A_{k+1}} \in (0, 1]$ and compute $y_k = (1 - \gamma_k)x_k + \gamma_k v_k$;
- 4: Set

$$x_{k+1} = \text{Prox}_{\frac{1}{L}\varphi}(y_k - \frac{1}{L}\nabla f(y_k)) \quad (6.22)$$

and $A_{k+1} = A_k + b_k$;

- 5: Define

$$\Psi_{k+1}(x) = \Psi_k(x) + b_k(f(y_k) + \langle \nabla f(y_k), x - y_k \rangle) \quad \forall x \in \mathbb{R}^n \quad (6.23)$$

and set

$$v_{k+1} = \arg \min_{x \in \mathbb{R}^n} \Psi_{k+1}(x) \quad (6.24)$$

- 6: $k \leftarrow k + 1$ and go back to Step 1;
-

Theorem 6.1. If $\{x_k\}_{k \geq 0}$ is generated by the accelerated proximal gradient method, then

$$F(x_k) - F(x^*) \leq \frac{8L\|x_0 - x^*\|^2}{(k-1)^2} \quad \forall k \geq 2 \quad (6.25)$$

Path following Interior Point Method

7.1 Self concordant functions

7.1.1 Definition

Definition 7.1. Given a convex function $f \in \mathcal{C}^3(\text{dom} f)$, with $\text{dom} f \subseteq \mathbb{R}^n$ open and convex, $f(\cdot)$ is said to be self-concordant with constant M_f when

$$\left| D^3 f(x)[u, u, u] \right| \leq 2M_f \|u\|_x^3 \quad \forall x \in \text{dom} f \quad \forall u \in \mathbb{R}^n \quad (7.1)$$

where $\|u\|_x := \sqrt{\langle \nabla^2 f(x) u, u \rangle}$.

From this definition, we can derive two lemmas:

- Let f_1, f_2 be self-concordant functions with constants M_1 and M_2 respectively. Then, given constants $\alpha, \beta > 0$, the function $f = \alpha f_1 + \beta f_2$ is self-concordant with constant $M_f = \max \left\{ \frac{M_1}{\sqrt{\alpha}}, \frac{M_2}{\sqrt{\beta}} \right\}$.
- Let $f(\cdot)$ be a self-concordant function with constant $M_f \geq 0$. Given $x, y \in \text{dom} f$, we have

$$\|y - x\|_y \geq \frac{\|y - x\|_x}{1 + M_f \|y - x\|_x} \quad (7.2)$$

7.1.2 With μ -strongly convex

As a reminder, a function f is said to be μ -strongly convex if

$$\begin{aligned} f(y) &\geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2 \quad \forall x, y \in \text{dom} f \\ &\Rightarrow \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2 \end{aligned} \quad (7.3)$$

Taking $y = x^* = \arg \min f(x)$, we find

$$\|\nabla f(x)\| \geq \mu \|x - x^*\| \quad \forall x \in \text{dom} f \quad (7.4)$$

after using the Cauchy-Schwarz inequality. This implies that the norm of the gradient tends to 0 as x approaches the minimizer x^* .

We can show that, for a self concordant function f with constant M_f , given $x, y \in \text{dom} f$, we have

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \frac{\|y - x\|_x^2}{1 + M_f \|y - x\|_x} \quad (7.5)$$

Theorem 7.2. Let $f(\cdot)$ be a self-concordant function with constant M_f . Consider $x_f^* = \arg \min_{x \in \text{dom} f} f(x)$. Given $x \in \text{dom} f$, with $\nabla^2 f(x)$ is nonsingular, we have

$$\|x - x^*\|_x \leq \frac{\|\nabla f(x)\|_x^*}{1 - M_f \|\nabla f(x)\|_x^*} \quad (7.6)$$

whenever $M_f \|\nabla f(x)\|_x^* < 1$, with $\|\nabla f(x)\|_x^* = \sqrt{\langle h, \nabla^{-2} f(x) h \rangle}$.

→ Note: $|\langle h, u \rangle| \leq \|h\|_x^* \|u\|_x$ if $\nabla^2 f(x)$ is nonsingular.

7.1.3 Self-concordant barrier

Definition 7.3. Let $F(\cdot)$ be a self-concordant function with constant $M_f = 1$. We say that $F(\cdot)$ is a ν -self-concordant barrier for the set $\overline{\text{dom} F}$ when

$$\langle \nabla F(x), u \rangle^2 \leq \nu \langle \nabla^2 F(x) u, u \rangle \quad x \in \text{dom} F \quad \forall u \in \mathbb{R}^n \quad (7.7)$$

The typical example is $F(x) = -\log(x)$.

→ Note: If $F(\cdot)$ is a ν -self-concordant barrier for the set $\overline{\text{dom} F}$, then $\langle \nabla F(x), y - x \rangle < \nu \forall x, y \in \text{dom} F$.

→ If, in addition, $\nabla^2 F(x)$ is nonsingular, then $\|\nabla F(x)\|_x^* \leq \sqrt{\nu}$.

7.2 Path-following Interior-point Method

Consider the optimization problem

$$\min_{x \in \mathbb{R}^n} f_0(x) \equiv \langle c, x \rangle \quad x \in \Omega \quad (7.8)$$

where $\Omega = \overline{\text{dom} F}$ for some ν -self-concordant barrier F and it is bounded. From these assumptions, it follows from the Weierstraß theorem that it has a solution x^* .

The barrier strategy consists in solving the problem iteratively by solving unconstrained optimization problems of the form

$$\min_{x \in \text{dom} F} t f_0(x) + F(x) \quad t > 0 \quad (7.9)$$

Let us denote $f(t; x) \equiv t \langle c, x \rangle + F(x)$, and $x^*(t) = \arg \min_{x \in \text{dom} F} f(t; x)$, which we call the central path function. Then,

$$\nabla_x f(t; x^*(t)) = t c + \nabla F(x^*(t)) = 0 \implies c = -\frac{1}{t} \nabla F(x^*(t)) \quad (7.10)$$

Consequently,

$$f_0(x^*(t)) - f_0(x) = \langle c, x^*(t) - x \rangle = \frac{1}{t} \langle \nabla F(x^*(t)), x^* - x^*(t) \rangle < \frac{\nu}{t} \quad (7.11)$$

The last inequality following equation (7.5). This means that

$$\lim_{t \rightarrow \infty} f_0(x^*(t)) = f_0(x^*) \quad (7.12)$$

And in particular, for $\epsilon > 0$, if $t \geq \nu\epsilon^{-1}$, then

$$f_0(x^*(t)) - f_0(x^*) < \epsilon \quad (7.13)$$

But, since $x^*(t)$ is not computable, one way to get an implementable method is to compute $\bar{x}(t)$ such that

$$\|\nabla_x f(t; \bar{x}(t))\|_x^* \leq \beta \quad \beta \in (0, 1) \quad (7.14)$$

This implies

$$\begin{aligned} f_0(\bar{x}(t)) - f_0(x^*) &= f_0(\bar{x}(t)) - f_0(x^*(t)) - (f_0(x^*) - f_0(x^*(t))) \\ &< \frac{\nu}{t} + f_0(\bar{x}(t)) - f_0(x^*(t)) \\ &= \frac{\nu}{t} + \frac{1}{t} \langle t\mathbf{c}, \bar{x}(t) - x^*(t) \rangle \\ &= \frac{\nu}{t} + \frac{1}{t} \langle \nabla_x f(t; \bar{x}(t)) - \nabla F(\bar{x}(t)), \bar{x}(t) - x^*(t) \rangle \end{aligned} \quad (7.15)$$

To get to the next line, we use the Cauchy-Schwarz and triangular inequalities:

$$\leq \frac{\nu}{t} + \frac{1}{t} [\|\nabla_x f(t; \bar{x}(t))\|_x^* + \|\nabla F(\bar{x}(t))\|_x^*] \|\bar{x}(t) - x^*(t)\|_x \quad (7.16)$$

From equations (7.14) and (7.6), and a property of self-concordant barriers, this means that

$$f_0(\bar{x}(t)) - f_0(x^*) < \frac{\nu}{t} + \frac{1}{t} (\beta + \sqrt{\nu}) \underbrace{\frac{\|\nabla_x f(t; \bar{x}(t))\|_x^*}{1 - \|\nabla_x f(t; \bar{x}(t))\|_x^*}}_{=:\omega(\|\nabla_x f(t; \bar{x}(t))\|_x^*)} \quad (7.17)$$

where $\omega(x) = \frac{x}{1-x}$ is a monotone increasing function, meaning that

$$\omega(\beta) > \omega(\|\nabla_x f(t; \bar{x}(t))\|_x^*) \quad (7.18)$$

and thus

$$f_0(\bar{x}(t)) - f_0(x^*) < \frac{1}{t} \left(\nu + (\beta + \sqrt{\nu}) \frac{\beta}{1 - \beta} \right) \quad (7.19)$$

7.3 Intermediate Newton method

Let us consider the problem (7.8), and let $\hat{f}(\cdot)$ be a self-concordant function with constant $M_{\hat{f}} = 1$. Consider $x \in \text{dom} \hat{f}$ with $\nabla^2 \hat{f}(x)$ nonsingular. Assume that $\|\nabla \hat{f}(x)\|_x^* \leq \tau$ with $\tau + \tau^2 + \tau^3 \leq 1$. The iterate of the intermediate Newton method is given by

$$x^+ = x - \frac{1}{1 + \xi} \nabla^{-2} \hat{f}(x) \nabla \hat{f}(x) \quad \xi = \frac{(\|\nabla \hat{f}(x)\|_x^*)^2}{1 + \|\nabla \hat{f}(x)\|_x^*} \quad (7.20)$$

Then, $x^+ \in \text{dom} \hat{f}$ and

$$\|\nabla \hat{f}(x^+)\|_{x^+}^* \leq \tau^2 \left(1 + \tau + \frac{\tau}{1 + \tau + \tau^2}\right) \quad (7.21)$$

Consider now the function $f(t; x) \equiv t\langle c, x \rangle + F(x)$, a self-concordant function with constant $M_f = 1$. The gradient and hessian are

$$\nabla_x f(t; x) = tc + \nabla F(x) \quad \nabla_x^2 f(t; x) = \nabla^2 F(x) \quad (7.22)$$

Let us define the iterate $t^+ = t + \frac{\gamma}{\|c\|_x^*}$ with $\gamma > 0$. The iterate of the intermediate Newton method becomes

$$x^+ = x - \frac{1}{1 - \xi} \nabla_x^{-2} f(t^+; x) \nabla_x f(t^+; x) = x - \frac{1}{1 + \xi} \nabla^{-2} F(x) (t^+ c + \nabla F(x)) \quad (7.23)$$

As previously, suppose that $\|\nabla_x f(t; x)\|_x^* \leq \beta$. Then,

$$\begin{aligned} \|\nabla_x f(t^+; x)\|_x^* &= \|t^+ c + \nabla F(x)\|_x^* = \|t^+ c - tc + tc + \nabla F(x)\|_x^* \\ &\leq (t^+ - t) \|c\|_x^* + \|\nabla_x f(t; x)\|_x^* = \gamma + \beta \end{aligned} \quad (7.24)$$

This inequality is derived using the hypothesis and the definition of t^+ . This means that, choosing $\gamma \leq \tau - \beta$ for $\tau + \tau^2 + \tau^3 \leq 1$, we get

$$\|\nabla_x f(t^+; x)\|_x^* \leq \tau \quad (7.25)$$

By equation (7.21), we have

$$\|\nabla_x f(t^+; x^+)\|_{x^+}^* \leq \tau^2 \left(1 + \tau + \frac{\tau}{1 + \tau + \tau^2}\right) = \frac{\tau^2(1 + \tau)}{1 - \tau^3} \quad (7.26)$$

And so taking $\beta = \tau^2 \left(1 + \tau + \frac{\tau}{1 + \tau + \tau^2}\right)$ seems reasonable.

→ Note: notice that $\tau > \beta$ for every $\tau \in (0, 1/2]$ and verifies $\tau + \tau^2 + \tau^3 \leq 1$.

From all those inequalities and properties, we can derive an algorithm.

7.4 Path-following Interior point Algorithm

7.4.1 Algorithm

Algorithm 4 Path-following Interior Point Algorithm

- 1: **Input:** Given $\tau \in (0, 1/2]$, define $\beta = \tau^2 \left(1 + \tau + \frac{\tau}{1+\tau+\tau^2}\right)$. Choose $0 < \gamma \leq \tau - \beta$. Find $x_0 \in \text{dom}F$ such that $\|\nabla F(x_0)\|_{x_0}^* \leq \beta$ and set $t_0 = 0$ and $k := 0$;
- 2: **Step 1:** Compute

$$\begin{aligned} t_{k+1} &= t_k + \frac{\gamma}{\|c\|_x^*} \\ x_{k+1} &= x_k - \frac{1}{1 + \tilde{\xi}_k} \nabla^{-2} F(x_k) (t_{k+1} c + \nabla F(x_k)) \\ \tilde{\xi}_k &= \frac{(\|\nabla f(t_k; x_k)\|_{x_k}^*)^2}{1 + \|\nabla f(t_k; x_k)\|_{x_k}^*} \end{aligned} \quad (7.27)$$

- 3: **Step 2:** $k \leftarrow k + 1$ and go back to Step 1.
-

7.4.2 Complexity bound

Notice that, by construction, $\|\nabla_x f(t_k; x_k)\|_{x_k}^* \leq \beta$, $\forall k \geq 0$, and so

$$t_k \|c\|_{x_k}^* = \|\nabla_x f(t_k; x_k) - \nabla F(x_k)\|_{x_k}^* \leq \beta + \sqrt{\nu} \quad (7.28)$$

This can be used to bound t_{k+1} :

$$t_{k+1} - t_k = \frac{\gamma}{\|c\|_{x_k}^*} \geq \frac{\gamma t_k}{\beta + \sqrt{\nu}} \iff \left(1 + \frac{\gamma}{\beta + \sqrt{\nu}}\right) t_k \quad \forall k \geq 0 \quad (7.29)$$

Thus,

$$t_k \geq \left(1 + \frac{\gamma}{\beta + \sqrt{\nu}}\right)^{k-1} t_1 = \left(1 + \frac{\gamma}{\beta + \sqrt{\nu}}\right)^{k-1} \frac{\gamma}{\|c\|_{x_0}^*} \quad (7.30)$$

Combining this to (7.19), it follows that

$$\begin{aligned} f_0(x_k) - f_0^* &\leq \frac{1}{t_k} \left(\nu + \frac{(\beta + \sqrt{\nu})\beta}{1 - \beta} \right) \\ &\leq \frac{\|c\|_{x_0}^* \left(\nu + \frac{(\beta + \sqrt{\nu})\beta}{1 - \beta} \right)}{\gamma \left(1 + \frac{\gamma}{\beta + \sqrt{\nu}}\right)^{k-1}} \end{aligned} \quad (7.31)$$

Thus, to obtain a point x_k with $f_0(x_k) - f_0^* \leq \epsilon$, it is sufficient to have

$$\begin{aligned} \frac{\|c\|_{x_0}^* \left(\nu + \frac{(\beta + \sqrt{\nu})\beta}{1 - \beta} \right)}{\gamma \left(1 + \frac{\gamma}{\beta + \sqrt{\nu}} \right)^{k-1}} &\leq \epsilon \\ (k-1) \ln \left(1 + \frac{\gamma}{\beta + \sqrt{\nu}} \right) &\geq \ln \left(\frac{\|c\|_{x_0}^* \left(\nu + \frac{(\beta + \sqrt{\nu})\beta}{1 - \beta} \right)}{\gamma} \epsilon^{-1} \right) \\ &\Rightarrow k \geq \mathcal{O}(\epsilon^{-1}) \end{aligned} \quad (7.32)$$

Notice that $\ln(1+x) \geq cx$ for $x > 0$ and c a constant **TO BE CHECKED**. We can apply it to $x = \frac{\gamma}{\beta + \sqrt{\nu}}$ to find a bound on the number of iterations: we will have $f_0(x_k) - f_0^* \leq \epsilon$ whenever

$$(k-1)c \left(\frac{\gamma}{\beta + \sqrt{\nu}} \right) \geq \ln \left(\|c\|_{x_0}^* \left(\nu + \frac{(\beta + \sqrt{\nu})\beta}{1 - \beta} \right) \gamma^{-1} \epsilon^{-1} \right) \quad (7.33)$$

Therefore, to find a ϵ -approximate solution of problem (7.8), the algorithm 4 takes no more than $\mathcal{O}(\sqrt{\nu} \ln(\epsilon^{-1}))$ iterations.

7.4.3 Example

Consider the following problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} q_0(x) &\equiv c_0 + \langle b_0, x \rangle + \frac{1}{2} \langle A_0 x, x \rangle \\ \text{s.t. } q_i(x) &\equiv c_i + \langle b_i, x \rangle + \frac{1}{2} \langle A_i x, x \rangle \leq \beta_i \quad i = 1, \dots, m \end{aligned} \quad (7.34)$$

where $A_i = A_i^T \succeq 0$ for $i = 0, \dots, m$. To be able to use the algorithm derived previously, we need to change the objective function:

$$\min_{(x, \beta) \in \mathbb{R}^n \times \mathbb{R}} \beta_0 \equiv f_0(x, \beta) \quad \text{s.t. } q_i(x) \leq \beta_i \quad i = 0, \dots, m \quad (7.35)$$

The feasible set of this problem is the closure of the domain of the following self-concordant barrier, with constant $\nu = m + 1$:

$$F(x, \beta_0) = - \sum_{i=0}^m \ln(\beta_i - q_i(x)) \quad (7.36)$$

From the complexity of algorithm 4, it takes at most $\mathcal{O}(\sqrt{m+1} \ln(\epsilon^{-1}))$ iterations to find x_k such that

$$f_0(x_k, \beta_{0,k}) - f_0^* \leq \epsilon \quad (7.37)$$

and the operation complexity multiplies it by $\mathcal{O}(m^3)$ because it solves a linear system at each iteration.

Tips and Tricks

1. μ -strongly convex function:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2 \quad \forall x, y \in \mathbb{R}^n \quad (8.1)$$

2. Approximation of the max:

$$\max\{z, 0\} = \frac{z + |z|}{2} = \frac{z + \sqrt{z^2}}{2} \approx \frac{z + \sqrt{z^2 + \delta}}{2} \quad (8.2)$$

3.

$$ab \leq \frac{a^2 + b^2}{2} \quad (8.3)$$

4.

$$(a + b)^2 \leq 2a^2 + 2b^2 \quad (8.4)$$

5. V-trick:

$$\langle xv, v \rangle \leq \|x\| \|v\|^2 \quad (8.5)$$

6. Triangular inequality by the minimizer:

$$\|x_{k+1} - x_k\| \leq \|x_{k+1} - x^*\| + \|x_k - x^*\| \quad (8.6)$$

7. Mean Value Theorem $\forall x, y \in \Omega, \exists z \in \Omega$ s.t.:

$$f(y) - f(x) = \langle \nabla f(z), y - x \rangle \quad z \in [x, y] \quad (8.7)$$

8. By definition if a function is C_M^p , then

$$|f(y) - T_p(y; x)| \leq \frac{M}{(p+1)!} \|y - x\|^{p+1} \quad (8.8)$$

9. Fundamental theorem of calculus:

$$\nabla f(y) - \nabla f(x) = \int_0^1 \nabla^2 f(x + \tau(y - x))(y - x) d\tau \quad (8.9)$$

10. Cauchy-Schwarz inequality;

11. Triangular inequality;

12. Dimis-Mori condition for Quasi Newton SR1:

$$\lim_{k \rightarrow \infty} \frac{\| (B_k^{-1} - \nabla^2 f(x_k)) (x_{k+1} - x_k) \|}{\|x_{k+1} - x_k\|} = 0 \quad (8.10)$$

13. KKT conditions:

$$\begin{cases} \nabla f(x^*) - \sum_{i=1}^m \lambda_i^* \nabla c_i(x^*) = 0 & \text{(stationarity)} \\ c(x^*) = 0 & \text{(feasibility)} \end{cases} \quad (8.11)$$

14. For a function $f \in \mathcal{C}_M^{2,2}$,

$$\left| f(y) - f(x) - \nabla f(x)^T (y - x) - \frac{1}{2} (y - x)^T \nabla^2 f(x) (y - x) \right| \leq \frac{M}{6} \|y - x\|^3 \quad (8.12)$$

Final results and important theorems

9.1 TODO

- Generalisation the property of a L-Lipschitz function to $f \in \mathcal{C}_L^p(\mathbb{R}^n)$. For $p = 1$, we had

$$f(y) \leq f(x_k) + \langle \nabla f(x_k), y - x_k \rangle + \frac{L}{2} \|y - x_k\|^2 \quad \forall y \in \mathbb{R}^n \quad (9.1)$$

For a general value of p , it becomes

$$f(y) \leq T_p(y; x_k) + \frac{L}{(p+1)!} \|y - x_k\|^{p+1} \quad \forall y \in \mathbb{R}^n \quad (9.2)$$

- Gradient method of order p : To solve $\min_{x \in \mathbb{R}^n} f(x)$, we can use the iteration

$$x_{k+1} = \arg \min_{y \in \mathbb{R}^n} T_p(y; x_k) + \frac{M}{(p+1)!} \|y - x_k\|^{p+1} \quad (9.3)$$

where the constant M is an approximation of the Lipschitz constant L .

- Bound on the number of iterations of the p -th order gradient method:

$$T(\varepsilon) \leq 1 + \frac{f(x_0) - f_{low}}{C(L)} \varepsilon^{-\frac{p+1}{p}} \equiv \mathcal{O} \left(\varepsilon^{-\frac{p+1}{p}} \right) \quad C(L) = \frac{L}{(p+1)!} \left(\frac{p!}{L+M} \right)^{\frac{p+1}{p}} \quad (9.4)$$

9.2 Gradient descent without gradient

We want to minimize a function f without computing its gradient.

$$x_{k+1} = x_k - \frac{1}{\sigma} g_{h_k}(x_k) \quad h_k > 0, \sigma > 0 \quad (9.5)$$

where $g_{h_k}(x_k) \in \mathbb{R}^n$ is given by:

$$[g_{h_k}(x_k)]_j = \frac{f(x_k + h_k e_j) - f(x_k)}{h_k} \quad \forall j \in [1, \dots, m] \quad (9.6)$$

Suppose that $f \in \mathcal{C}_L^1(\mathbb{R}^n)$. Then,

$$\|\nabla f(x_k) - g_{h_k}(x_k)\| \leq \frac{L\sqrt{n}}{2} h_k \quad (9.7)$$

And the convergence rate is

$$\implies T(\varepsilon) \leq 8\sigma \left(f(x_0) - f_{low} + \frac{5\sigma n}{16} \sum_{k=0}^{T(\varepsilon)-1} h_k^2 \right) \varepsilon^2 = \mathcal{O}(\varepsilon^2) \quad (9.8)$$

9.3 Local rates of convergence

9.3.1 Linear rate of GM

As a reminder, the gradient method follows the iterate

$$x_{k+1} = x_k - \frac{1}{L} \nabla f(x_k) \quad (9.9)$$

We define in some proves the quantity G_k as

$$G_k = \int_0^1 \nabla^2 f(x^* + \tau(x_k - x^*)) d\tau \quad (9.10)$$

The local convergence rate, i.e. for iterates such that $\|x_k - x^*\| \leq \frac{\mu}{M}$, of the gradient method is linear:

$$\begin{aligned} \|x_k - x^*\| &\leq \left(1 - \frac{\mu}{2L}\right)^k \|x_0 - x^*\| \quad \forall k \geq 0 \\ T(\varepsilon) &\leq 1 + \frac{\log(\|x_0 - x^*\| \varepsilon^{-1})}{|\log(1 - \frac{\mu}{2L})|} \equiv \mathcal{O}(\log(\varepsilon^{-1})) \end{aligned} \quad (9.11)$$

9.3.2 Local quadratic convergence of Newton's method

As a reminder, the Newton's method follows the iterate

$$x_{k+1} = x_k - \nabla^{-2} f(x_k) \nabla f(x_k) \quad (9.12)$$

The local convergence rate, i.e. for iterates such that $\|x_k - x^*\| \leq \frac{\mu}{2M}$, of the Newton's method is quadratic:

$$\begin{aligned} \|x_{k+1} - x^*\| &\leq \frac{M}{\mu} \|x_k - x^*\|^2 \\ T(\varepsilon) &\leq \log_2(\log_2(\frac{\mu}{M} \varepsilon^{-1})) \end{aligned} \quad (9.13)$$

9.3.3 Quasi Newton methods

SR1 Update

As a reminder, the SR1 update is given by

$$x_{k+1} = x_k - B_k \nabla f(x_k) \quad B_k = B_{k-1} + \frac{(s_{k-1} - B_{k-1} y_{k-1})(s_{k-1} - B_{k-1} y_{k-1})^T}{(s_{k-1} - B_{k-1} y_{k-1})^T y_{k-1}} \quad (9.14)$$

where B_k is found using Dimis-Mori and the secant condition:

$$\lim_{k \rightarrow \infty} \frac{\| (B_k^{-1} - \nabla^2 f(x_k)) (x_{k+1} - x_k) \|}{\|x_{k+1} - x_k\|} = 0 \quad (9.15)$$

$B_k y_{k-1} = s_{k-1}$

BFGS Update

BFGS uses an approximation of the hessian instead of its inverse:

$$H_k = B_k^{-1} = \nabla^2 f(x_k) \quad (9.16)$$

The secant condition becomes

$$H_{k+1}s_k = y_k \quad (9.17)$$

The idea is to use a rank 2 update:

$$H_{k+1} = H_k + auu^T + bvv^T \quad a, b \in \mathbb{R} \quad u, v \in \mathbb{R}^n \quad (9.18)$$

The update is equation (4.63).

9.4 Constrained nonlinear programming problems

Consider the constrained problem

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad c_i(x) = 0, \quad i \in \{1, \dots, m\} \quad (9.19)$$

where $f, c_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are \mathcal{C}^1 and there exists at least a \hat{x} such that $c_i(\hat{x}) = 0$.

To do that, we use a quadratic penalty term:

$$\min_{x \in \mathbb{R}^n} Q_\sigma(x) \equiv f(x) + \frac{\sigma}{2} \|c(x)\|_2^2 \quad (9.20)$$

The initial problem is constrained. We suppress those constraints by adding their norm in the objective function, with a parameter σ . We define ε_1 as the tolerance on the norm of the constraints, and ε_2 as the tolerance on the objective function. The concept of the algorithm is to solve the unconstrained problem, and increase σ until the constraints are satisfied with tolerance ε_1 .

9.5 Accelerated Gradient Method

9.5.1 Derivation of the algorithm

This algorithm minimizes a convex function f with L -Lipschitz gradient. The method combines past and present information for the step of the gradient method.

$$\begin{cases} y_k = (1 - \gamma_k)x_k + \gamma_k v_k, & \gamma_k \in (0, 1) \\ x_{k+1} = x_k - \frac{1}{L} \nabla f(y_k) \end{cases} \quad (9.21)$$

where v_k and γ_k are defined in the algorithm 2. The method is based on the following inequalities, which allow to have the convergence rate that we want:

$$\begin{aligned} v_k &= \arg \min_{x \in \mathbb{R}^n} \Psi_k(x) \\ \Psi_k(x) &\leq A_k f(x) + \frac{1}{2} \|x - x_0\|^2 \\ A_k f(x_k) &\leq \min_{x \in \mathbb{R}^n} \Psi_k(x) \equiv \Psi_k^*, \quad A_k \geq 0 \\ A_k &\geq c(k-1)^2 \quad \forall k \geq 2 \end{aligned} \quad (9.22)$$

This convergence rate is

$$(f(x_k) - f(x^*)) \leq \frac{\|x_k - x^*\|^2}{2C(k-1)^2} = \mathcal{O}(k^{-2}) = \mathcal{O}(\varepsilon^{-1/2}) \quad \forall k \geq 2 \quad (9.23)$$

9.5.2 Accelerated Proximal Gradient Method

In this algorithm, the function to minimise is defined over a nonempty, closed and convex set Ω . The function has a smooth part $f(\cdot)$ and a possibly nonsmooth part $\phi(\cdot)$.

$$\min_{x \in \Omega \subseteq \mathbb{R}^n} F(x) \equiv f(x) + \phi(x) \quad (9.24)$$

The APGM consists in changing the iterate x_{k+1} to

$$x_{k+1} = \text{Prox}_{\frac{1}{L}\phi} \left(y_k - \frac{1}{L} \nabla f(y_k) \right) \quad (9.25)$$

The convergence rate is

$$F(x_k) - F(x^*) \leq \frac{8L\|x_0 - x^*\|^2}{(k-1)^2} \quad \forall k \geq 2 \quad (9.26)$$

9.6 Path following Interior Point Method

9.6.1 Self-concordant functions

We have 2 lemmas:

- Let f_1, f_2 be self-concordant functions with constants M_1 and M_2 respectively. Then, given constants $\alpha, \beta > 0$, the function $f = \alpha f_1 + \beta f_2$ is self-concordant with constant $M_f = \max \left\{ \frac{M_1}{\sqrt{\alpha}}, \frac{M_2}{\sqrt{\beta}} \right\}$.
- Let $f(\cdot)$ be a self-concordant function with constant $M_f \geq 0$. Given $x, y \in \text{dom} f$, we have

$$\|y - x\|_y \geq \frac{\|y - x\|_x}{1 + M_f \|y - x\|_x} \quad (9.27)$$

For a self concordant function f with constant M_f , given $x, y \in \text{dom} f$, we have

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \frac{\|y - x\|_x^2}{1 + M_f \|y - x\|_x} \quad (9.28)$$

Theorem 9.1. Let $f(\cdot)$ be a self-concordant function with constant M_f . Consider $x_f^* = \arg \min_{x \in \text{dom} f} f(x)$. Given $x \in \text{dom} f$, with $\nabla^2 f(x)$ is nonsingular, we have

$$\|x - x^*\|_x \leq \frac{\|\nabla f(x)\|_x^*}{1 - M_f \|\nabla f(x)\|_x^*} \quad (9.29)$$

whenever $M_f \|\nabla f(x)\|_x^* < 1$, with $\|\nabla f(x)\|_x^* = \sqrt{\langle h, \nabla^2 f(x) h \rangle}$.

9.6.2 Path-following Interior Point Method

We consider the optimization problem

$$\min_{x \in \mathbb{R}^n} f_0(x) \equiv \langle c, x \rangle \quad x \in \Omega \quad (9.30)$$

where $\Omega = \overline{\text{dom}F}$ for some self-concordant barrier F . The method consists in solving

$$\min_{x \in \text{dom}F} t f_0(x) + F(x) \quad t > 0 \quad (9.31)$$

With this method,

$$f_0(\bar{x}(t)) - f_0(x^*) < \frac{1}{t} \left(\nu + (\beta + \sqrt{\nu}) \frac{\beta}{1 - \beta} \right) \quad (9.32)$$

9.6.3 Intermediate Newton method

The iterate of the intermediate Newton method for a self-concordant function \hat{f} with $M_{\hat{f}} = 1$ is

$$x^+ = x - \frac{1}{1 + \xi} \nabla^{-2} \hat{f}(x) \nabla \hat{f}(x) \quad \xi = \frac{(\|\nabla \hat{f}(x)\|_x^*)^2}{1 + \|\nabla \hat{f}(x)\|_x^*} \quad (9.33)$$

This is used in algorithm 4 to solve the problem. The complexity bound is

$$k \geq \mathcal{O}(\sqrt{\nu} \ln(\epsilon^{-1})) \quad (9.34)$$