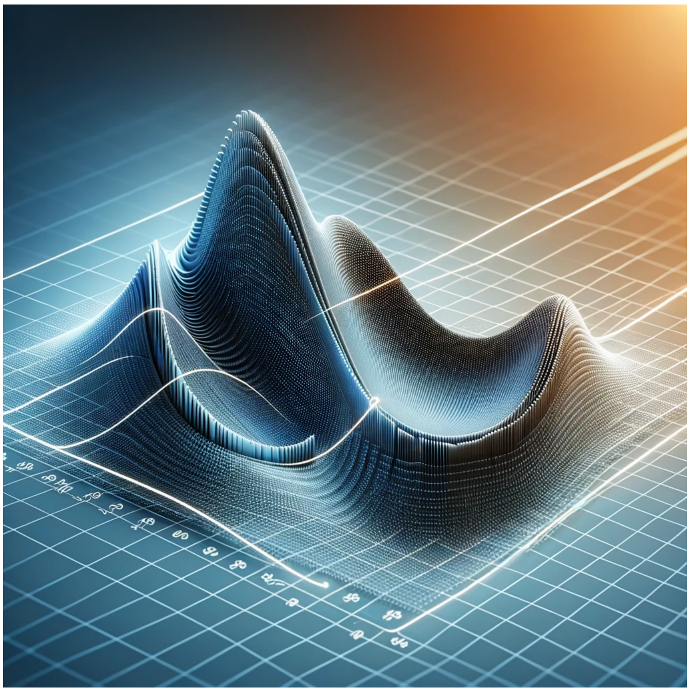

LINMA2491 Operational Research

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Academic year 2024-2025 - Q2

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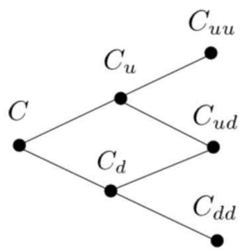
Contents

Definition and notation

- Given Ω, a sigma-algebra $\mathcal{A}$ is a set of subsets of Ω, with the elements called events, such that:
  - $\Omega \in \mathcal{A}$
  - if $A \in \mathcal{A}$ then also $\Omega - A \in \mathcal{A}$
  - if $A_i \in \mathcal{A}$ for $i = 1, 2, \ldots$ then also $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$
  - if $A_i \in \mathcal{A}$ for $i = 1, 2, \ldots$ then also $\cap_{i=1}^{\infty} A_i \in \mathcal{A}$
- Consider:



  - The state space is the set of all values of the system at each stage.

$$S_0 = \{C\}, \qquad S_1 = \{C_u, C_d\}, \qquad S_2 = \{C_{uu}, C_{ud}, C_{dd}\} \tag{1.1}$$

  - The sample space is the set of all possible combination of the system.

$$\Omega = S_0 \times S_1 \times S_2 = \{(C, C_u, C_{uu}), (C, C_u, C_{ud}), (C, C_u, C_{dd}), \ldots\} \tag{1.2}$$

- The power set of Ω is the set of all of the subsets, denoted $\mathcal{B}(\Omega)$.
- The probability space is the triplet $(\Omega, \mathcal{A}, P)$ where P is a probability measure.
  - $P(\emptyset) = 0$
  - $P(\Omega) = 1$
  - $P(\cup_{i=1}^{\infty} A_i) = \sum_i P(A_i)$ if A_i are disjoint
- $\forall t$, A_t is the set of events on which we have information at stage t. For example, $A_0 = \{C\}$, $A_1 = \{C, C_u, C_d\}$. Thus is it evident that $t_1 \leq t_2 \Rightarrow \mathcal{A}_{t_1} \subseteq \mathcal{A}_{t_2}$

- Consider the following problem with $x \in \mathbb{R}^n$ and domain $\mathcal{D}$:

$$\begin{aligned} &\min f_0(x), \qquad s.t. \\ &f_i(x) \leq 0, i = 1, \dots, m \\ &h_j(x) = 0, j = 1, \dots, p \end{aligned} \tag{1.3}$$

Then the Lagrangian function is defined as $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$:

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x) \tag{1.4}$$

- The Lagrange dual function is defined as $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$:

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \tag{1.5}$$

- The Lagrange dual problem is a lower bound on the optimal value of the primal problem

- Lagrange relaxation of Stochastic Programs, consider the two problems:

$$\begin{aligned} &\min f_1(x) + \mathbb{E}_\omega[f_2(y(\omega), \omega)] & &\min f_1(x) + \mathbb{E}_\omega[f_2(y(\omega), \omega)] \\ s.t \quad &h_{1i}(x) \leq 0, i = 1, \dots, m_1 & s.t. \quad &h_{1i}(x) \leq 0, i = 1, \dots, m_1 \\ &h_{2i}(x, y(\omega), \omega) \leq 0, i = 1, \dots, m_2 & &h_{2i}(x(\omega), y(\omega), \omega) \leq 0, i = 1, \dots, m_2 \\ & & &\color{red}{x(\omega) = x} \end{aligned} \tag{1.6}$$

The red constraint is the non-anticipativity constraint, it transforms the deterministic variable into a stochastic variable. A VERIFIER

- The dual of a stochastic program is:

$$\begin{aligned} &g(\nu) = g1(\nu) + \mathbb{E}_\omega(g2(\nu, \omega)) \\ \textit{where} \\ &g_1(\nu) = \inf f_1(x) + \left(\sum_{\omega \in \Omega} \nu(\omega) \right)^T x \\ &s.t. \quad h_{1i}(x) \leq 0, i = 1, \dots, m_1 \\ \textit{and} \\ &g_2(\nu, \omega) = \inf f_2(y(\omega)\omega) - \nu x(\omega) \\ &s.t. \quad h_{2i}(x(\omega), y(\omega), \omega) \leq 0, i = 1, \dots, m_2 \end{aligned} \tag{1.7}$$

- With p^* the solution of the primal problem and d^* the solution of the dual problem, we have:
  - Weak duality: $d^* \leq p^*$
  - Strong duality: $d^* = p^*$

- The KKT conditions are necessary and sufficient for optimality in convex optimization, there aren't unique. They are:

- Primal constraint: $f_i(x) \leq 0, i = 1, \ldots, m, h_j(x) = 0, j = 1, \ldots, p$
- Dual constraint: $\lambda \geq 0$
- Complemetarity slackness: $\lambda_i f_i(x) = 0, i = 1, \ldots, m$
- Gradient of the Lagrangian: $\nabla_x L(x, \lambda, \nu) = 0$

Modelling

2.1 Introduction

- For a certain sequence of events $x \to \omega \to y(\omega)$, where ω is the uncertainty,
  - A first-stage decision is a decision that is made before the uncertainty is revealed (i.e. in x);
  - A second-stage decision is a decision that is made after the uncertainty is revealed (i.e. in $y(\omega)$).
- We can have the following mathematical formulation:

$$
\begin{aligned}
\min c^T x + \mathbb{E}[\min q(\omega)^T y(\omega)] \\
Ax = b \\
T(\omega)x + W(\omega)y(\omega) = h(\omega) \\
x \geq 0, y(\omega) \geq 0
\end{aligned}
\tag{2.1}
$$

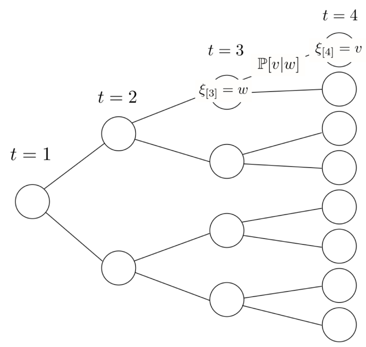
  - First-stage decision variable: $x \in \mathbb{R}^{n_1}$
  - First-stage parameter: $c \in \mathbb{R}^{n_1}$, $b \in \mathbb{R}^{m_1}$ and $A \in \mathbb{R}^{m_1 \times n_1}$
  - Second-stage decision: $y(\omega) \in \mathbb{R}^{n_2}$
  - Second-stage data: $q(\omega) \in \mathbb{R}^{n_2}$, $h(\omega) \in \mathbb{R}^{m_2}$ and $T(\omega) \in \mathbb{R}^{m_2 \times n_1}$, $W(\omega) \in \mathbb{R}^{m_2 \times n_2}$

2.2 Representations

2.2.1 Scenario Trees

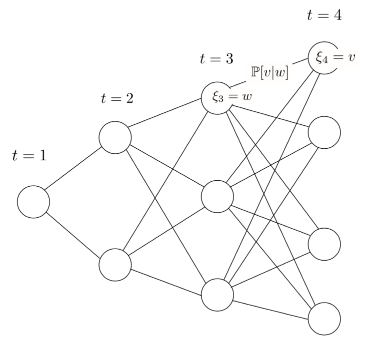
A scenario tree is a graphical representation of a Markov process $\{\xi_t\}_{t \in \mathbb{Z}}$, where the nodes are the history of realizations ($\xi_{[t]} = (\xi_1, \ldots, \xi_t)$), and the edges are the transitions from $\xi_{[t]}$ to $\xi_{[t+1]}$.

- We denote the root as $t = 1$;
- An ancestor of a node $\xi_{[t]}$, $A(\xi_{[t]})$ is a unique adjacent node which precedes ξ_t;
- The children of a node, $C(\xi_{[t]})$ are the nodes that are adjacent to $\xi_{[t]}$ and occur at stage $t + 1$.



2.2.2 Lattice

A lattice is a graphical representation of a Markov process $\{\xi_t\}_{t\in\mathbb{Z}}$, where the nodes are the realizations ξ_t and the edges correspond to the transitions from ξ_t to ξ_{t+1}.



2.2.3 Serial Independence

A process satisfies serial independence if, for every stage t, ξ_t has a probability distribution that does not depend on the history of the process. Thus, the probability measure is

$$\mathbb{P}\left[\xi_t(\omega)=i\,\middle|\,\xi_{[t-1]}(\omega)\right]=p_t(i)\qquad\forall\,\xi_{[t-1]}\in\Xi_{[t-1]},i\in\Xi_t \tag{2.2}$$

2.3 Multi Stage Stochastic Linear Program

2.3.1 Notation

- Probability space: $(\Omega,2^\Omega,\mathbb{P})$ with filtration $\{\mathcal{A}\}_{t\in\{1,\ldots,H\}}$
- $c_t(\omega)\in\mathbb{R}^{n_t}$: cost coefficients
- $h_t(\omega)\in\mathbb{R}^{m_t}$: right-hand side parameters
- $W_t(\omega)\in\mathbb{R}^{m_t\times n_t}$: coefficients of $x_t(\omega)$
- $T_{t-1}(\omega)\in\mathbb{R}^{m_t\times n_{t-1}}$: coefficients of $x_{t-1}(\omega)$
- $x_t(\omega)$: set of state and action variables in period t

- We implicitly enforce non-anticipativity by requiring that x_t and ξ_t are adapted to filtration $\{\mathcal{A}\}_{t\in\{1,\dots,H\}}$
- $\forall A \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}$, $x_t(\omega_1) = x_t(\omega_2)\ \forall \omega_1, \omega_2 \in A$

2.3.2 General formulation of the MSLP

The extended formulation of the MSLP is:

$$\begin{aligned}
&\min c_1^T x_1 + \mathbb{E}[c_2(\omega)^T x_2(\omega) + \cdots + c_H(\omega)^T x_H(\omega)]\\
&s.t.\quad W_1 x_1 = h_1\\
&T_1(\omega)x_1 + W_2(\omega)x_2(\omega) = h_2(\omega), \omega \in \Omega\\
&\qquad\qquad\vdots\\
&T_{t-1}(\omega)x_{t-1}(\omega) + W_t(\omega)x_t(\omega) = h_t(\omega), \omega \in \Omega\\
&\qquad\qquad\vdots\\
&T_{H-1}(\omega)x_{H-1}(\omega) + W_H(\omega)x_H(\omega) = h_H(\omega), \omega \in \Omega\\
&x_1 \geq 0, x_t(\omega) \geq 0, t = 2, \dots, H
\end{aligned} \tag{2.3}$$

We can now consider two specific instantiations of the MSLP: the scenario tree (MSLP-ST) and the lattice (MSLP-L). Using these notations:

- $\omega_t \in S_t$: index in the support Ξ_t of random input ξ_t
- $\omega_{[t]} \in S_1 \times \cdots \times S_t$ (interpretation: index in $\Xi_{[t]} = \Xi_1 \times \cdots \times \Xi_t$, which is the history of realizations, up to period t)

2.3.3 Scenario Tree formulation

$$\begin{aligned}
&\min c_1^T x_1 + \mathbb{E}\left[c_2(\omega_{[2]})^T x_2(\omega_{[2]}) + \cdots + c_H(\omega_{[H]})^T x_H(\omega_{[H]})\right]\\
&s.t.\quad W_1 x_1 = h_1\\
&T_1(\omega_{[2]})x_1 + W_2(\omega_{[2]})x_2(\omega_{[2]}) = h_2(\omega_{[2]}), \omega_{[2]} \in S_1 \times S_2\\
&\qquad\qquad\vdots\\
&T_{t-1}(\omega_{[t]})x_{t-1}(\omega_{[t-1]}) + W_t(\omega_{[t]})x_t(\omega_{[t]}) = h_t(\omega_{[t]}), \omega_{[t]} \in S_1 \times \cdots \times S_t\\
&\qquad\qquad\vdots\\
&T_{H-1}(\omega_{[H]})x_{H-1}(\omega_{[H-1]}) + W_H(\omega_{[H]})x_H(\omega_{[H]}) = h_H(\omega_{[H]}), \omega_{[H]} \in S_1 \times \cdots \times S_H\\
&x_1 \geq 0, x_t(\omega_{[t]}) \geq 0, t = 2, \dots, H
\end{aligned} \tag{2.4}$$

2.3.4 Lattice formulation

$$
\begin{aligned}
& \min c_1^T x_1 + \mathbb{E}\left[c_2(\omega_2)^T x_2(\omega_{[2]}) + \cdots + c_H(\omega_H)^T x_H(\omega_{[H]})\right] \\
& s.t. \quad W_1 x_1 = h_1 \\
& T_1(\omega_2)x_1 + W_2(\omega_2)x_2(\omega_{[2]}) = h_2(\omega_2), \omega_{[2]} \in S_1 \times S_2 \\
& \qquad \vdots \\
& T_{t-1}(\omega_t)x_{t-1}(\omega_{[t-1]}) + W_t(\omega_t)x_t(\omega_{[t]}) = h_t(\omega_t), \omega_{[t]} \in S_1 \times \cdots \times S_t \\
& \qquad \vdots \\
& T_{H-1}(\omega_H)x_{H-1}(\omega_{[H-1]}) + W_H(\omega_H)x_H(\omega_{[H]}) = h_H(\omega_H), \omega_{[H]} \in S_1 \times \cdots \times S_H \\
& x_1 \geq 0, x_t(\omega_{[t]}) \geq 0, t = 2, \ldots, H
\end{aligned}
\tag{2.5}
$$

→ Note: There exists some relations to other decision making problems such as statistical decision theory, dynamic programming, online optimization and stochastic control.

Performance

3.1 Notation

Using (2.1), let's define the following:

- $z(x, \xi) = c^T x + Q(x, \xi) + \delta(x|K_1)$
- $Q(x, \xi) = \min_y \{q(\omega)^T y | W(\omega) y = h(\omega) - T(\omega) x\}$
- $K_1 = \{x | Ax = b, x \geq 0\}$ is the set of feasible first-stage decisions
- $K_2(\omega) = \{x | \exists y \geq 0 : W(\omega) y = h(\omega) - T(\omega) x\}$ is the set of first-stage decisions that have a feasible reaction in the second stage for $\omega \in \Omega$
- It can be that $z(x, \xi) = +\infty$ (if $x \notin K_1 \cap K_2(\omega)$)
- It can be that $z(x, \xi) = -\infty$ (unbounded below)

3.2 The expected value of perfect information

There is 2 tactics:

- **wait-and-see** value is the expected value of reacting with perfect foresight $x^*(\xi)$ to ξ:
$$WS = \mathbb{E}[\min_x z(x, \xi)] = \mathbb{E}[z(x^*(\xi), \xi)] \tag{3.1}$$
- **here-and-now** value is the expected value of the recourse problem (remove non-anticipativity):
$$SP = \min_x \mathbb{E}[z(x, \xi)] \tag{3.2}$$

The **expected value of perfect information** is like the value of a perfect forecast for the future and is thus defined like this:

$$EVPI = SP - WS \tag{3.3}$$

3.3 The value of the stochastic solution

Here too there is 2 tactics:

- **Expected value problem**

$$EV = \min_x z(x, \bar{\xi}) = \mathbb{E}[\xi] \tag{3.4}$$

  and it's **expected value solution** is noted $x^*(\bar{\xi})$

- **expected value of using the EV solution** measures the performance of $x^*(\bar{\xi})$:

$$EEV = \mathbb{E}[z(x^*(\bar{\xi}), \xi)] \tag{3.5}$$

The **value of the stochastic solution** is noted like this:

$$VSS = EEV - SP \tag{3.6}$$

3.4 Basic inequalities

3.4.1 Crystal Ball

For every ξ, we have $z(x^*(\xi), \xi) \leq z(x^*, \xi)$ where x^* is the optimal solution to the stochastic program. And if we take the expectation of this inequality, we have $WS \leq SP$, because WS is a relaxation. It explains that we can do better with a Crystal Ball.

3.4.2 Lazy solution

Knowing that x^* is the optimal solution of $\min_x \mathbb{E}[z(x, \xi)]$ and $x^*(\bar{\xi})$ is a solution but not necessarily optimal then we have $SP \leq EEV$, because:

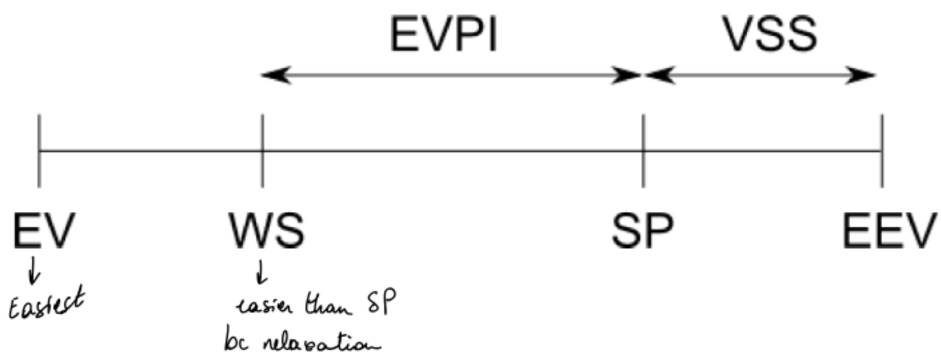
$$\min_x \mathbb{E}[z(x, \xi)] = SP \leq EEV = \mathbb{E}[z(x^*(\bar{\xi}), \xi)] \tag{3.7}$$

3.4.3 Link between all the values

We know that:

- $VSS \geq 0$
- $EVPI \geq 0$
- $VSS \leq EEV - EV$
- $EVPI \leq EEV - EV$
- If $EEV - EV = 0$ then $VSS = EVPI = 0$

and the inequalities can be summarized in the following diagram:



3.5 Bounds on EVPI and VSS

First let's introduce the **sum of pairs expected value (SPEV)**:

$$SPEV = \frac{1}{1-p^r} \sum_{k=1,k\neq r}^{K} p^k \min z^p(x, \xi^r, \xi^k) \tag{3.8}$$

When $\xi^r \notin \Xi$ then $SPEV = WS$: When $p^r = 0, z^p(x, \xi^r, \xi^k)$ coincides with $z(x, \xi^k)$. Therefore $SPEV = \sum_{k=1}^{K} p^k \min_x z(x, \xi^k) = WS$.
We then know $WS \leq SPEV \leq SP$.

3.5.1 Upper bound on SP: EVRS and EPEV

- The **expected value of the reference scenario** is $EVRS = \mathbb{E}_\xi(\bar{x}^r, \xi)$, where $\bar{x}^r$ is the optimal solution to $z(x, \xi^r)$
- The **expectation of pairs of expected value** is defined as $EPEV = \min_{k=1,\dots,K\cup\{r\}} \mathbb{E}_\xi(\bar{x}^r, \xi)$ where $(\bar{x}^k, \bar{y}^k, y(\xi^K))$ is the optimal solution to the pairs subproblem of ξ^r and ξ^k

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