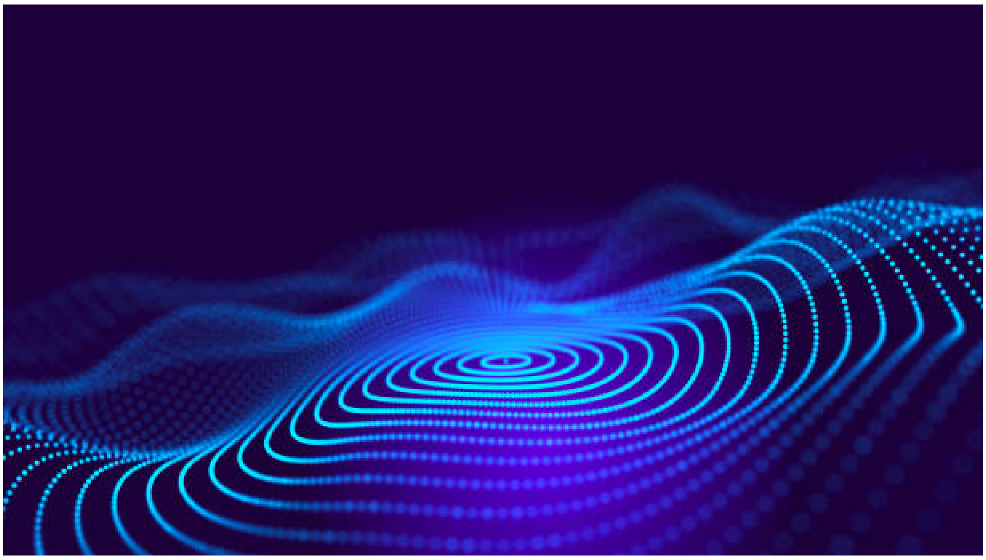

LMECA2300 Advanced Numerical Methods

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Contents

2-D acoustic and electromagnetic waves

1.1 Physical laws

The expression of the force of the source applied to its surrounding is derived from Newton's law $F = ma$:

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p = r_v \tag{1.1}$$

where ρ_0 is the average density of the surrounding, $\vec{u}$ is the velocity field, p is the variation of pressure and r_v is the "pressure force".
The conservation law of energy is

$$\nabla \cdot \vec{u} + \chi \frac{\partial p}{\partial t} = s_v \tag{1.2}$$

where χ is the compressibility $[kg^{-1}ms^2]$ and is given by the equation $\frac{\rho}{\rho_0} = \chi p$. s_v is the "velocity source".

1.2 Wave equation

The wave equation is

$$\nabla^2 p - \rho_0 \chi \frac{\partial^2 p}{\partial t^2} - -\rho_0 \frac{\partial s_V}{\partial t} \tag{1.3}$$

In 1D, with $s_V = 0$, the solution is any function $f\left(t - \frac{x}{v}\right)$ or $g\left(x - tv\right)$ with $v = 1/\sqrt{\rho_0 \chi}$.

1.3 Plane wave

In the plane, the standard wave is given by

$$p(x,t) = p_0 \cos(\omega t - kx) \tag{1.4}$$

where the phase velocity is $v_{ph} = \frac{\omega}{k} = \frac{1}{\sqrt{\chi \rho_0}}$.
The wave impedance is $\eta = \frac{p}{u_x} = \sqrt{\frac{\rho_0}{\chi}}$ $[kgm^{-2}s^{-1}]$.

1.4 Harmonic case

In the case of a harmonic soundwave, we can use phasors:

$$p(\vec{r}, t) = Re\{P(\vec{r})e^{j\omega_0 t}\} \tag{1.5}$$

$\rightarrow$ Note: there is a relationship between the phasors and the Fourier transform:

$$F\{p\} = Re(P)\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + jIm(P)\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] \tag{1.6}$$

with ω the frequency variable.
In the frequency domain, the PDE becomes

$$\nabla^2 P + k^2 P = -j\omega\rho_0 S_V = Q \tag{1.7}$$

where k is the wavenumber $k = 2\pi/\lambda$. This is useful to make the time dependence disappear.

1.4.1 Green function

The Green function is the solution G to the following Fourier PDE;

$$\nabla^2 G + k^2 G = \delta(x)\delta(y) \tag{1.8}$$

The general solution is

$$G(\rho) = -\frac{j}{4}H_0^{(2)} - \frac{j}{4}(J_0(k\rho) - jY_0(k\rho)) \tag{1.9}$$

where J_0 and Y_0 are respectively the Bessel functions of order 1 and 2, and ρ is the distance between the source and the observer.
The Bessel functions $J_n(x)$ are solutions of the equation

$$x^2 J_n''(x) + xJ_n'(x) + (x^2 - n^2)J_n(x) = 0 \tag{1.10}$$

and there exists an approximation of the Hankel function $H_0^{(2)}$ for $x \gg 1$:

$$H_0^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}}e^{j\pi/4}e^{-jx} \tag{1.11}$$

1.4.2 From velocity source to pressure

The pressure due to a velocity source spread over infinitesimal segment Γ':

$$p(r) = G(\vec{r} - \vec{r}')(j\omega\rho_0)v_n(\vec{r}')d\Gamma' \tag{1.12}$$

To get the total pressure, we have to do an integral:

$$p = p_{inc} + \frac{\omega\rho_0}{4}\int_\Gamma H_0^{(2)}(k|r - r'|)v_n(\vec{r}')d\Gamma' \tag{1.13}$$

Which is obtained from the velocity source $-j\omega\rho_0 S_v$ from equation (1.7), and doing a convolution with the Green function.

This integral can be approximated with a pointwise approach:

$$I \approx C\sum_i v_i G(\rho_i)dl_i \tag{1.14}$$

where $C = -j\omega\rho_0$, v_i is the normal component of the velocity vector, and dl_i the length of the segment.