



LINMA2491 Operational Research

SIMON DESMIDT
ISSAMBRE L'HERMITE DUMONT

Academic year 2024-2025 - Q2



UCLouvain

Contents

1	Definition and notation	2
2	Modelling	5
2.1	Introduction	5
2.2	Representations	5
2.3	Multi Stage Stochastic Linear Program	6
3	Performance	9
3.1	Notation	9
3.2	The expected value of perfect information	9
3.3	The value of the stochastic solution	10
3.4	Basic inequalities	10
3.5	Bounds on EVPI and VSS	11

Definition and notation

- Given Ω , a sigma-algebra \mathcal{A} is a set of subsets of Ω , with the elements called events, such that:
 - $\Omega \in \mathcal{A}$
 - if $A \in \mathcal{A}$ then also $\Omega - A \in \mathcal{A}$
 - if $A_i \in \mathcal{A}$ for $i = 1, 2, \dots$ then also $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$
 - if $A_i \in \mathcal{A}$ for $i = 1, 2, \dots$ then also $\cap_{i=1}^{\infty} A_i \in \mathcal{A}$
- Consider:



- The state space is the set of all values of the system at each stage.

$$S_0 = \{C\}, \quad S_1 = \{C_u, C_d\}, \quad S_2 = \{C_{uu}, C_{ud}, C_{dd}\} \quad (1.1)$$

- The sample space is the set of all possible combination of the system.

$$\Omega = S_0 \times S_1 \times S_2 = \{(C, C_u, C_{uu}), (C, C_u, C_{ud}), (C, C_u, C_{dd}), \dots\} \quad (1.2)$$

- The power set of Ω is the set of all of the subsets, denoted $\mathcal{B}(\Omega)$.
- The probability space is the triplet (Ω, \mathcal{A}, P) where P is a probability measure.
 - $P(\emptyset) = 0$
 - $P(\Omega) = 1$
 - $P(\cup_{i=1}^{\infty} A_i) = \sum_i P(A_i)$ if A_i are disjoint
- $\forall t, A_t$ is the set of events on which we have information at stage t . For example, $A_0 = \{C\}$, $A_1 = \{C, C_u, C_d\}$. Thus is it evident that $t_1 \leq t_2 \Rightarrow \mathcal{A}_{t_1} \subseteq \mathcal{A}_{t_2}$

- Consider the following problem with $x \in \mathbb{R}^n$ and domain \mathcal{D} :

$$\begin{aligned} \min f_0(x), \quad & \text{s.t.} \\ f_i(x) &\leq 0, i = 1, \dots, m \\ h_j(x) &= 0, j = 1, \dots, p \end{aligned} \quad (1.3)$$

Then the Lagrangian function is defined as $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$:

$$L(x, \lambda, v) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{j=1}^p v_j h_j(x) \quad (1.4)$$

- The Lagrange dual function is defined as $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$:

$$g(\lambda, v) = \inf_{x \in \mathcal{D}} L(x, \lambda, v) \quad (1.5)$$

- The Lagrange dual problem is a lower bound on the optimal value of the primal problem
- Lagrange relaxation of Stochastic Programs, consider the two problems:

$$\begin{aligned} \min f_1(x) + \mathbb{E}_\omega[f_2(y(\omega), \omega)] & \quad \min f_1(x) + \mathbb{E}_\omega[f_2(y(\omega), \omega)] \\ \text{s.t. } h_{1i}(x) \leq 0, i = 1, \dots, m_1 & \quad \text{s.t. } h_{1i}(x) \leq 0, i = 1, \dots, m_1 \\ h_{2i}(x, y(\omega), \omega) \leq 0, i = 1, \dots, m_2 & \quad h_{2i}(x(\omega), y(\omega), \omega) \leq 0, i = 1, \dots, m_2 \\ & \quad \textcolor{red}{x(\omega) = x} \end{aligned} \quad (1.6)$$

The red constraint is the non-anticipativity constraint, it transforms the deterministic variable into a stochastic variable. **A VERIFIER**

- The dual of a stochastic program is:

$$\begin{aligned} g(v) &= g_1(v) + \mathbb{E}_\omega(g_2(v, \omega)) \\ \text{where} \\ g_1(v) &= \inf f_1(x) + \left(\sum_{\omega \in \Omega} v(\omega) \right)^T x \\ \text{s.t. } h_{1i}(x) &\leq 0, i = 1, \dots, m_1 \\ \text{and} \\ g_2(v, \omega) &= \inf f_2(y(\omega), \omega) - vx(\omega) \\ \text{s.t. } h_{2i}(x(\omega), y(\omega), \omega) &\leq 0, i = 1, \dots, m_2 \end{aligned} \quad (1.7)$$

- With p^* the solution of the primal problem and d^* the solution of the dual problem, we have:

- Weak duality: $d^* \leq p^*$
- Strong duality: $d^* = p^*$

- The KKT conditions are necessary and sufficient for optimality in convex optimization, there aren't unique. They are:

- Primal constraint: $f_i(x) \leq 0, i = 1, \dots, m, h_j(x) = 0, j = 1, \dots, p$
- Dual constraint: $\lambda \geq 0$
- Complementarity slackness: $\lambda_i f_i(x) = 0, i = 1, \dots, m$
- Gradient of the Lagrangian: $\nabla_x L(x, \lambda, \nu) = 0$

Modelling

2.1 Introduction

- For a certain sequence of events $x \rightarrow \omega \rightarrow y(\omega)$, where ω is the uncertainty,
 - A first-stage decision is a decision that is made before the uncertainty is revealed (i.e. in x);
 - A second-stage decision is a decision that is made after the uncertainty is revealed (i.e. in $y(\omega)$).
- We can have the following mathematical formulation:

$$\begin{aligned}
 \min \quad & c^T x + \mathbb{E}[\min q(\omega)^T y(\omega)] \\
 \text{s.t.} \quad & Ax = b \\
 & T(\omega)x + W(\omega)y(\omega) = h(\omega) \\
 & x \geq 0, y(\omega) \geq 0
 \end{aligned} \tag{2.1}$$

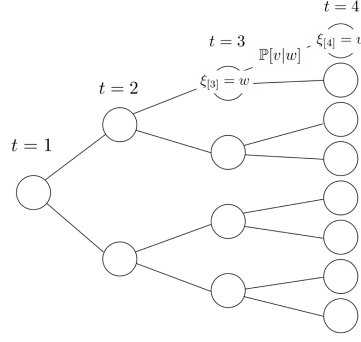
- First-stage decision variable: $x \in \mathbb{R}^{n_1}$
- First-stage parameter: $c \in \mathbb{R}^{n_1}$, $b \in \mathbb{R}^{m_1}$ and $A \in \mathbb{R}^{m_1 \times n_1}$
- Second-stage decision: $y(\omega) \in \mathbb{R}^{n_2}$
- Second-stage data: $q(\omega) \in \mathbb{R}^{n_2}$, $h(\omega) \in \mathbb{R}^{m_2}$ and $T(\omega) \in \mathbb{R}^{m_2 \times n_1}$, $W(\omega) \in \mathbb{R}^{m_2 \times n_2}$

2.2 Representations

2.2.1 Scenario Trees

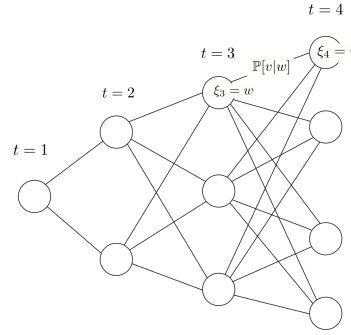
A scenario tree is a graphical representation of a Markov process $\{\xi_t\}_{t \in \mathbb{Z}}$, where the nodes are the history of realizations ($\xi_{[t]} = (\xi_1, \dots, \xi_t)$), and the edges are the transitions from $\xi_{[t]}$ to $\xi_{[t+1]}$.

- We denote the root as $t = 1$;
- An ancestor of a node $\xi_{[t]}$, $A(\xi_{[t]})$ is a unique adjacent node which precedes ξ_t ;
- The children of a node, $C(\xi_{[t]})$ are the nodes that are adjacent to $\xi_{[t]}$ and occur at stage $t + 1$.



2.2.2 Lattice

A lattice is a graphical representation of a Markov process $\{\xi_t\}_{t \in \mathbb{Z}}$, where the nodes are the realizations ξ_t and the edges correspond to the transitions from ξ_t to ξ_{t+1} .



2.2.3 Serial Independence

A process satisfies serial independence if, for every stage t , ξ_t has a probability distribution that does not depend on the history of the process. Thus, the probability measure is

$$\mathbb{P} \left[\xi_t(\omega) = i \mid \xi_{[t-1]}(\omega) \right] = p_t(i) \quad \forall \xi_{[t-1]} \in \Xi_{[t-1]}, i \in \Xi_t \quad (2.2)$$

2.3 Multi Stage Stochastic Linear Program

2.3.1 Notation

- Probability space: $(\Omega, 2^\Omega, \mathbb{P})$ with filtration $\{\mathcal{A}\}_{t \in \{1, \dots, H\}}$
- $c_t(\omega) \in \mathbb{R}^{n_t}$: cost coefficients
- $h_t(\omega) \in \mathbb{R}^{m_t}$: right-hand side parameters
- $W_t(\omega) \in \mathbb{R}^{m_t \times n_t}$: coefficients of $x_t(\omega)$
- $T_{t-1}(\omega) \in \mathbb{R}^{m_t \times n_{t-1}}$: coefficients of $x_{t-1}(\omega)$
- $x_t(\omega)$: set of state and action variables in period t

- We implicitly enforce non-anticipativity by requiring that x_t and ξ_t are adapted to filtration $\{\mathcal{A}\}_{t \in \{1, \dots, H\}}$
- $\forall A \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}, x_t(\omega_1) = x_t(\omega_2) \forall \omega_1, \omega_2 \in A$

2.3.2 General formulation of the MSLP

The extended formulation of the MSLP is:

$$\begin{aligned}
& \min c_1^T x_1 + \mathbb{E}[c_2(\omega)^T x_2(\omega) + \dots + c_H(\omega)^T x_H(\omega)] \\
& \text{s.t. } W_1 x_1 = h_1 \\
& T_1(\omega) x_1 + W_2(\omega) x_2(\omega) = h_2(\omega), \omega \in \Omega \\
& \quad \vdots \\
& T_{t-1}(\omega) x_{t-1}(\omega) + W_t(\omega) x_t(\omega) = h_t(\omega), \omega \in \Omega \\
& \quad \vdots \\
& T_{H-1}(\omega) x_{H-1}(\omega) + W_H(\omega) x_H(\omega) = h_H(\omega), \omega \in \Omega \\
& x_1 \geq 0, x_t(\omega) \geq 0, t = 2, \dots, H
\end{aligned} \tag{2.3}$$

We can now consider two specific instantiations of the MSLP: the scenario tree (MSLP-ST) and the lattice (MSLP-L). Using these notations:

- $\omega_t \in S_t$: index in the support Ξ_t of random input ξ_t
- $\omega_{[t]} \in S_1 \times \dots \times S_t$ (interpretation: index in $\Xi_{[t]} = \Xi_1 \times \dots \times \Xi_t$, which is the history of realizations, up to period t)

2.3.3 Scenario Tree formulation

$$\begin{aligned}
& \min c_1^T x_1 + \mathbb{E} \left[c_2(\omega_{[2]})^T x_2(\omega_{[2]}) + \dots + c_H(\omega_{[H]})^T x_H(\omega_{[H]}) \right] \\
& \text{s.t. } W_1 x_1 = h_1 \\
& T_1(\omega_{[2]}) x_1 + W_2(\omega_{[2]}) x_2(\omega_{[2]}) = h_2(\omega_{[2]}), \omega_{[2]} \in S_1 \times S_2 \\
& \quad \vdots \\
& T_{t-1}(\omega_{[t]}) x_{t-1}(\omega_{[t-1]}) + W_t(\omega_{[t]}) x_t(\omega_{[t]}) = h_t(\omega_{[t]}), \omega_{[t]} \in S_1 \times \dots \times S_t \\
& \quad \vdots \\
& T_{H-1}(\omega_{[H]}) x_{H-1}(\omega_{[H-1]}) + W_H(\omega_{[H]}) x_H(\omega_{[H]}) = h_H(\omega_{[H]}), \omega_{[H]} \in S_1 \times \dots \times S_H \\
& x_1 \geq 0, x_t(\omega_{[t]}) \geq 0, t = 2, \dots, H
\end{aligned} \tag{2.4}$$

2.3.4 Lattice formulation

$$\begin{aligned}
& \min c_1^T x_1 + \mathbb{E} \left[c_2(\omega_2)^T x_2(\omega_{[2]}) + \cdots + c_H(\omega_H)^T x_H(\omega_{[H]}) \right] \\
& s.t. \quad W_1 x_1 = h_1 \\
& \quad T_1(\omega_2) x_1 + W_2(\omega_2) x_2(\omega_{[2]}) = h_2(\omega_2), \omega_{[2]} \in S_1 \times S_2 \\
& \quad \quad \quad \vdots \\
& \quad T_{t-1}(\omega_t) x_{t-1}(\omega_{[t-1]}) + W_t(\omega_t) x_t(\omega_{[t]}) = h_t(\omega_t), \omega_{[t]} \in S_1 \times \cdots \times S_t \\
& \quad \quad \quad \vdots \\
& \quad T_{H-1}(\omega_H) x_{H-1}(\omega_{[H-1]}) + W_H(\omega_H) x_H(\omega_{[H]}) = h_H(\omega_H), \omega_{[H]} \in S_1 \times \cdots \times S_H \\
& \quad x_1 \geq 0, x_t(\omega_{[t]}) \geq 0, t = 2, \dots, H
\end{aligned} \tag{2.5}$$

→ Note: There exists some relations to other decision making problems such as statistical decision theory, dynamic programming, online optimization and stochastic control.

Performance

3.1 Notation

Using (2.1), let's define the following:

- $z(x, \xi) = c^T x + Q(x, \xi) + \delta(x|K_1)$
- $Q(x, \xi) = \min_y \{q(\omega)^T y | W(\omega)y = h(\omega) - T(\omega)x\}$
- $K_1 = \{x | Ax = b, x \geq 0\}$ is the set of feasible first-stage decisions
- $K_2(\omega) = \{x | \exists y \geq 0 : W(\omega)y = h(\omega) - T(\omega)x\}$ is the set of first-stage decisions that have a feasible reaction in the second stage for $\omega \in \Omega$
- It can be that $z(x, \xi) = +\infty$ (if $x \notin K_1 \cap K_2(\omega)$)
- It can be that $z(x, \xi) = -\infty$ (unbounded below)

3.2 The expected value of perfect information

There is 2 tactics:

- **wait-and-see** value is the expected value of reacting with perfect foresight $x^*(\xi)$ to ξ :

$$WS = \mathbb{E}[\min_x z(x, \xi)] = \mathbb{E}[z(x^*(\xi), \xi)] \quad (3.1)$$

- **here-and-now** value is the expected value of the recourse problem (remove non-anticipativity):

$$SP = \min_x \mathbb{E}[z(x, \xi)] \quad (3.2)$$

The **expected value of perfect information** is like the value of a perfect forecast for the future and is thus defined like this:

$$EVPI = SP - WS \quad (3.3)$$

3.3 The value of the stochastic solution

Here too there is 2 tactics:

- **Expected value problem**

$$EV = \min_x z(x, \bar{\xi}) = \mathbb{E}[\tilde{z}] \quad (3.4)$$

and it's **expected value solution** is noted $x^*(\bar{\xi})$

- **expected value of using the EV solution** measures the performance of $x^*(\bar{\xi})$:

$$EEV = \mathbb{E}[z(x^*(\bar{\xi}), \tilde{\xi})] \quad (3.5)$$

The **value of the stochastic solution** is noted like this:

$$VSS = EEV - SP \quad (3.6)$$

3.4 Basic inequalities

3.4.1 Crystal Ball

For every ξ , we have $z(x^*(\xi), \xi) \leq z(x^*, \xi)$ where x^* is the optimal solution to the stochastic program. And if we take the expectation of this inequality, we have $WS \leq SP$, because WS is a relaxation. It explains that we can do better with a Crystal Ball.

3.4.2 Lazy solution

Knowing that x^* is the optimal solution of $\min_x \mathbb{E}[z(x, \tilde{\xi})]$ and $x^*(\bar{\xi})$ is a solution but not necessarily optimal then we have $SP \leq EEV$, because:

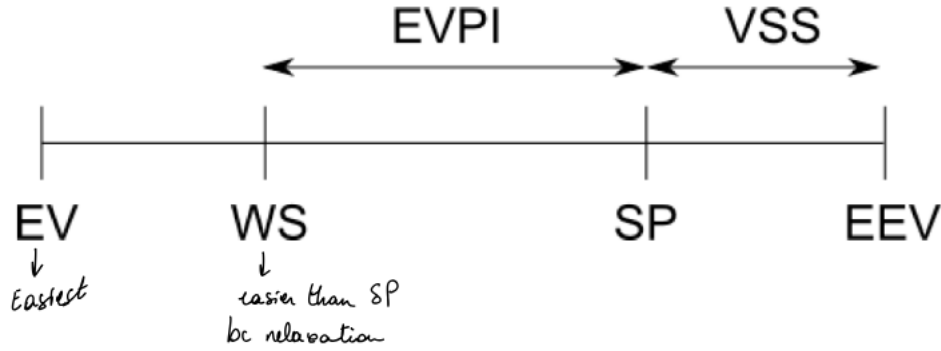
$$\min_x \mathbb{E}[z(x, \tilde{\xi})] = SP \leq EEV = \mathbb{E}[z(x^*(\bar{\xi}), \tilde{\xi})] \quad (3.7)$$

3.4.3 Link between all the values

We know that:

- $VSS \geq 0$
- $EVPI \geq 0$
- $VSS \leq EEV - EV$
- $EVPI \leq EEV - EV$
- If $EEV - EV = 0$ then $VSS = EVPI = 0$

and the inequalities can be summarized in the following diagram:



3.5 Bounds on EVPI and VSS

First let's introduce the pairs subproblem of ζ^r and ζ^k is

$$\begin{aligned}
 \min z^P(x, \zeta^r, \zeta^k) &= c^T x + p^r q^T y(\zeta^r) + (1 - p^r) q^T y(\zeta^k) \\
 \text{s.t. } Ax &= b \\
 Wy(\zeta^r) &= \zeta^r - Tx \\
 Wy(\zeta^k) &= \zeta^k - Tx \\
 x, y &\geq 0
 \end{aligned} \tag{3.8}$$

- $(\bar{x}^k, \bar{y}^k, y(\zeta^r))$ denotes an optimal solution to the problem and z^k is the optimal objective function value $z^P(\bar{x}^k, \bar{y}^k, y(\zeta^r))$
- $z^P(x, \zeta^r, \zeta^r)$ corresponds to the deterministic optimization against the reference scenario
- if $\zeta^r \notin \Xi$, $p^r = 0$ and $z^P(x, \zeta^r, \zeta^k) = z(x, \zeta^k)$

The **sum of pairs expected value (SPEV)**:

$$SPEV = \frac{1}{1 - p^r} \sum_{k=1, k \neq r}^K p^k \min z^P(x, \zeta^r, \zeta^k) \tag{3.9}$$

When $\zeta^r \notin \Xi$ then $SPEV = WS$: When $p^r = 0$, $z^P(x, \zeta^r, \zeta^k)$ coincides with $z(x, \zeta^k)$. Therefore $SPEV = \sum_{k=1}^K p^k \min_x z(x, \zeta^k) = WS$. We then know $WS \leq SPEV \leq SP$.

3.5.1 Upper bound on SP: EVRS and EPEV

- The **expected value of the reference scenario** is $EVRS = \mathbb{E}_{\zeta}(\bar{x}^r, \zeta)$, where \bar{x}^r is the optimal solution to $z(x, \zeta^r)$
- The **expectation of pairs of expected value** is defined as $EPEV = \min_{k=1, \dots, K \cup \{r\}} \mathbb{E}_{\zeta}(\bar{x}^r, \zeta)$ where $(\bar{x}^k, \bar{y}^k, y(\zeta^k))$ is the optimal solution to the pairs subproblem of ζ^r and ζ^k

As $SP, EPEV, EVRS$ are the optimal values of $\min_x \mathbb{E}_{\tilde{\zeta}} z(x, \tilde{\zeta})$ over smaller feasible sets:

$$SP \leq EPEV \leq EVRS \quad (3.10)$$

Because

- SP : $x \in K_1 \cap K_2$
- $EPEV$: $x \in K_1 \cap K_2 \cap \{\bar{x}^k, k = 1, \dots, K \cup \{r\}\}$
- $EVRS$: $x \in \bar{x}^r \cap K_1 \cap K_2$