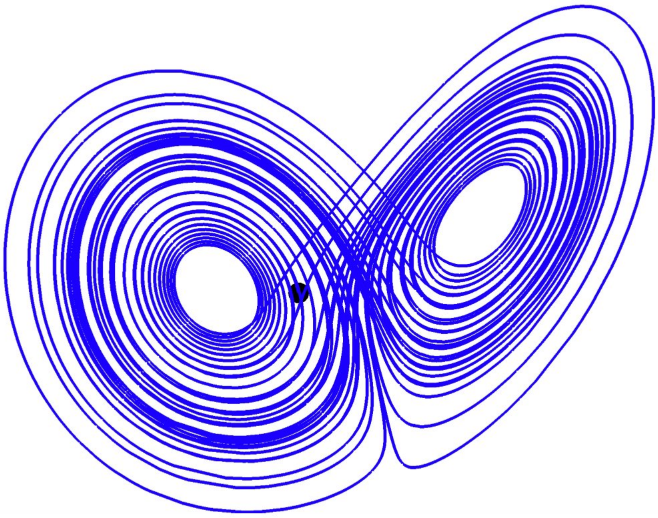

LINMA2370 Modelling and Analysis of Dynamical Systems

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UCLouvain

Contents

Introduction

The tools introduced in this course are a simplifying view of the reality, yet very uselful to build simple and effective models in view of the control and optimization of the dynamical behaviour of the real systems.

1.1 Reminders

- A subset of $\mathbb{R}$ is said to be negligible if its Lebesgue measure is equal to zeroo and that a property is said to be true almost everywhere if it is false only on a negligible set.
- Let $I \subseteq \mathbb{R}$ be an interval the interior of which is not empty. A function $x : I \to \mathbb{R}^N$ is said to be absolutely continuous if

$$\begin{aligned}
&\forall \varepsilon \in (0, \infty),\ \exists \delta \in (0, \infty)\ :\\
&\qquad \forall n \in \mathbb{N} \setminus \{0\},\ \forall a_1, b_1, \ldots, a_n, b_n \in I :\\
&\qquad a_i < b_i\, \forall i \in \{1, \ldots, n\},\ b_i \leq a_{i+1}\, \forall i \in \{1, \ldots, n-1\},\\
&\qquad\qquad \sum_{i=1}^{n} (b_i - a_i) \leq \delta \Longrightarrow \sum_{i=1}^{n} \|x(b_i) - x(a_i)\| \leq \varepsilon
\end{aligned}$$

- Let $a, b \in \mathbb{R}$ with $a < b$. A function $x : [a, b] \to \mathbb{R}$ is absolutely continuous iff there exists an integrable fucntion $\varphi : [a, b] \to \mathbb{R}$ such that, for every $t \in [a, b]$,

$$x(t) = x(a0) + \int_a^t \phi(s) ds$$

  in which case x is almost everywhere differentiable with $\dot{x}(t) = \phi(t)$ for almost every $t \in [a, b]$.

- A function $f : \Omega \to \mathbb{R}^N$, where Ω is a nonempty subset of $\mathbb{R} \times \mathbb{R}^N$, is said to be Lipschitz continuous in the second argument, uniformly with respect to the first argument, if there exists $L \in [0, \infty)$ such that forall $t \in \mathbb{R}$ and all $x, y \in \mathbb{R}^N$ such that $(tx,), (t, y) \in \Omega$,

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

  It is said to be locally Lipschitz continuous on an open ball for each argument.

- Let Ω be a nonempty open subset of $\mathbb{R} \times \mathbb{R}^N$ and $f : \Omega \to \mathbb{R}^N$ be such that

– for all $t \in \mathbb{R}$, $f(t,\cdot) : \Omega_t \to \mathbb{R}^N$
– $\partial_2 f : \Omega \to \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N) : (t,x) \to \partial_2 f(t,x)$ is locally bounded.

Then, f is locally Lipschitz continuous in the second argument, uniformly with respect to the first argument.

- If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two real normed spaces, and the real vector space $\mathcal{L}(X,Y)$ of all continuous linear mappings from X to Y[1] is equipped with the norm defined by
$$\|L\| := \sup_{x \in X \setminus \{0\}} \frac{\|Lx\|_Y}{\|x\|_X}$$

1.2 State-space model

A state-space model for a continuous dynamical system consists of an ODE of the form

$$\dot{x}(t) = f(t, x(t)) \tag{1.1}$$

where the function $f : \Omega \to \mathbb{R}^N$, Ω being a nonempty subset of $\mathbb{R} \times R^N$, is called the vector field associated with the ODE. A continuous dynamical system with input $u : \mathbb{R} \to \mathbb{R}^M$ described by the ODE

$$\dot{x}(t) = g(x(t), u(t)) \tag{1.2}$$

for some function $g : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$, can be written in the form (1.1) by defining the vector field

$$f_u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N : (t,x) \to g(x, u(t)) \tag{1.3}$$

→ N.B.: the norm of each $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ is defined as $|t| + \|x\|$.

1.3 Integral curve

Let Ω be a nonempty subset of $\mathbb{R} \times \mathbb{R}^N$. An integral curve of $f : \Omega \to \mathbb{R}^N$ is a function $x : I \to \mathbb{R}^N$ where $I \subseteq \mathbb{R}$ is an interval, for which the interior is not empty, called the interval of existence of x, i.e. differentiable and satisfies $(t, x(t)) \in \Omega$ and $\dot{x}(t) = f(t, x(t))$ for all $t \in I$. The graph $\{(t, x(t)) | t \in I\}$ and the image $\{x(t) | t \in I\}$ of x are respectively called the trajectory and the orbit of x. Given an initial condition $(t_0, x_0) \in \Omega$, a solution to the initial value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \tag{1.4}$$

is an integral curve $x : I \to \mathbb{R}^N$ of f such that $t_0 \in I$ and $x(t_0) = x_0$.

[1] Meaning matrix from X to Y

If, for the IVP described hereabove, f is continuous, then a continuous function $x : I \to \mathbb{R}^N$ where $I \subseteq \mathbb{R}$ is an interval containing t_0 and the interior of which is not empty, is a solution iff its graph is contained in Ω and it satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

for all $t \in I$. In that case, $\dot{x}$ is continuous.

Let Ω be a nonempty subset of $\mathbb{R} \times \mathbb{R}^N$. An integral curve in the extended sense of $f : \Omega \to \mathbb{R}^N$ is a function $x : I \to \mathbb{R}^N$, where $I \subseteq \mathbb{R}$ is an interval the interior of which is not empty called the interval of existence of x, that is absolutely continuous and satisfies $(t, x(t)) \in \Omega$ for every $t \in I$ and $\dot{x}(t) = f(t(x(t)))$ for almost every $t \in I$.

$\to$ N.B.: If f is continuous, then the two definitions of integral curves are equivalent.

1.4 Existence of a solution

Consider the IVP defined hereabove with an integral curve in the extended sense, under the following assumptions:

- there exists $\tau, r \in (0, \infty)$, such that $[t_0 - \tau, t_0 + \tau] \times B(x_0, r) \subseteq \Omega$;
- for every $x \in B(x_0, r)$, the function $[t_0 - \tau, t_0 + \tau] \to \mathbb{R}^N : t \to f(t, x)$ is measurable;
- for every $t \in [t_0 - \tau, t_0 + \tau]$, the function $B(x_0, r) \to \mathbb{R}^N : x \to f(t, x)$ is continuous;
- there exists an integrable function $m : [t_0 - \tau, t_0 + \tau] \to [0, \infty)$ such that

$$\|f(t, x)\| \leq m(t) \text{ for all } (t, x) \in [t_0 - \tau, t_0\tau] \times B[x_0, r]$$

Then, there exists a solution defined on a compact interval the interior of which contains t_0.
In particular, for the IVP with an integral curve in the general sense, if (t_0, x_0) is an interior point of Ω and f is continuous, then there exists a solution defined on a compact interval the interior of which contains t_0.

Dynamical systems and state-space models

We will study first-order dynamical systems of the form

$$\dot{x} = f(x, u) \tag{2.1}$$

where f is a mapping from $\mathbb{R}^{n+m}$ to $\mathbb{R}^n$, while x and u are vector functions of time, respectively the state and the input.

2.1 Terminology and notation

- We assume that the input is a piecewise continuous and bounded function: $u \in \mathcal{U}$, where $\mathcal{U}$ is a set of piecewise continuous and bounded functions from $\mathbb{R}$ to $\mathbb{R}^m$.
- For a given value of the initial state $x(t_0) = x_0$ a,d a given input u, the solution $t \to x(t)$ for $t \geq t_0$, of the system of ODE 2.1 is called the trajectory of the system. It is denoted $x(t_0, x_0, u)$.
- When the input u can be freely chosen in $\mathcal{U}$, the system $\dot{x} = f(x, u)$ is said to be a forced/controlled system.

→ N.B.: in this course, we will study the solution of the equation 2.1 when the input is actually an a priori set constant: $u(t) = \overline{u}\ \forall t \geq t_0$. The state-space model is then written as $\dot{x} = f(x, \overline{u}) = f_{\overline{u}}(x)$.

2.1.1 System with affine input

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i = f(x) + G(x)u \tag{2.2}$$

where f and g_i are mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$.

2.1.2 System with affine state

$$\dot{x} = \sum_{i=1}^{n} x_i a_i(u) + b(u) = A(u)x + b(u) \tag{2.3}$$

where b and a_i are mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$.

2.1.3 Bilinear systems

A bilinear system is affine both in the state and in the input:

$$\dot{x} = \left(A_0 + \sum_{i=1}^{m} u_i A_i \right) x + B_0 u \tag{2.4}$$

where A_i and B_i are matrices of dimensions $n \times n$ and $n \times m$ respectively.

2.1.4 Linear system

$$\dot{x} = Ax + Bu \tag{2.5}$$

where A and B are matrices of dimensions $n \times n$ and $n \times m$ respectively.

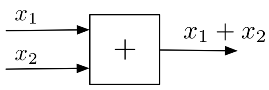
Articulated mechanical systems

Electrical and electromechanical systems

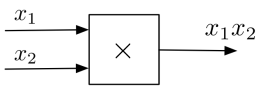
State transformations

5.1 Definition

The block diagram of a dynamical system is a visual representation of that system, necessarily containing n integrators whose outputs are the n state variables.



(a)



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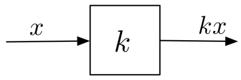

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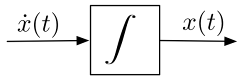
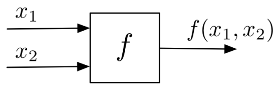


(a)



(b)

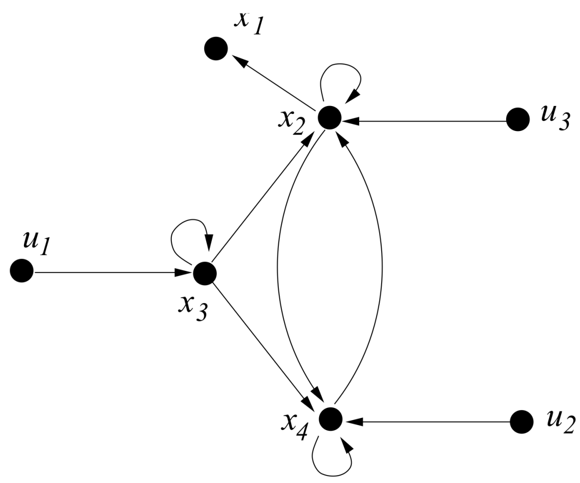
The graph of a dynamical system contains as nodes the inputs and states of the system, and its edges are the relations between those quantities. The construction rules of the graph of a dynamical system are the following:

- The $n + m$ nodes are the n state variables x_i and the m inputs u_j;
- there is an oriented edge from x_i (or u_k) to x_j if the varibale x_i (or u_k) appears explicitly in the equation of the derivative $\dot{x}_j$.

Example for a DC electric machine: the state space model is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = J^{-1}(-h(x_2) + K_m x_3 x_4 + u_3) \\ \dot{x}_3 = L_s^{-1}(-R_s x_3 + u_1) \\ \dot{x}_4 = L_r^{-1}(-R_r x_4 - K_e x_2 x_3 + u_2) \end{cases} \tag{5.1}$$

and its graph representation is



5.2 Linear state transformation

For a dynamical system $\dot{x} = f(x, u)$, a linear state transformation is a linear mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ that is bijective and transforms the state of the system $x \in \mathbb{R}^n$ into a new state $z \in \mathbb{R}^n$ following the rule $z = Tx$, where $T \in \mathbb{R}^{n\times n}$ is an invertible matrix.
The relation between the two systems is

$$\begin{cases} \dot{x} = f(x, u) \\ \dot{z} = g(z, u) \end{cases} \implies \begin{cases} z \triangleq T^{-1}x \\ g(z, u) \triangleq Tf(T^{-1}z, u) \end{cases} \tag{5.2}$$

For a linear system, we have

$$\dot{z} = Fz + Gu \qquad F \triangleq TAT^{-1} \qquad G \triangleq TB \tag{5.3}$$

5.3 Nonlinear state transformation

Let U, V be two open subsets of $\mathbb{R}^n$. A nonlinear state transformation is a mapping $T : U \to V$ that transforms the state of the system $x \in U$ into a new state $z \in V$: $z = T(x)$ and that has the following properties:

- T is bijective and has an inverse function $T^{-1} : V \to U$ such that $x = T^{-1}(z)$;
- T and T^{-1} are of class $\mathcal{C}^1$, i.e. continuously differentiable.

$\to$ N.B.: The state transformation is said to be global if $U = V = \mathbb{R}^n$.

Such a transformation T is called a diffeomorphism, and the new state space is

$$\dot{z} = \frac{\partial T}{\partial x}\dot{x} = \frac{\partial T}{\partial x} f(x, u) \iff f(x, u) \triangleq \left[\frac{\partial T^{-1}}{\partial z} g(z, u)\right]_{z=T(x)} \tag{5.4}$$

Lemma 5.1.
- If the jacobien matrix $\partial T/\partial x$ is nonsingular at x_0, then, by the inverse function theorem, there is a neighbourhood U of x_0 such that the mapping T restricted to U is a diffeomorphism on U.

- T is a global diffeomorphism iff
    1. $\partial T/\partial x$ is a nonsingular for every $x \in \mathbb{R}^n$;
    2. $\lim_{\|x\|\to\infty}\|T(x)\| = \infty$.

5.4 Triangular system

Definition 5.2. A single input dynamical system is triangular if there is a state variable x_i such that the shortest path from u t x_i in the graph of the system is of length n.

We can thus renumber the state variables such that the system is expressed as

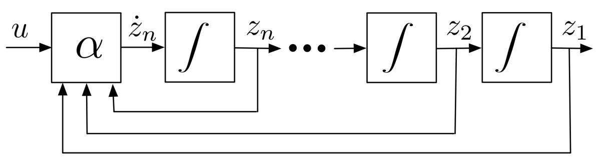
$$\begin{aligned}
\dot{x}_1 &= g_1(x_1, x_2)\\
&\vdots\\
\dot{x}_i &= g_i(x_1, \dots, x_{i+1})\\
&\vdots\\
\dot{x}_{n-1} &= g_{n-1}(x_1, \dots, x_n)\\
\dot{x}_n &= g_n(x_1, \dots, x_n, u)
\end{aligned} \tag{5.5}$$

5.5 Brunovsky canonical form

Definition 5.3. A single input dynamical system can be written in Brunovsky canonical form if there exists a state transformation $T : U \to V$ and an open interval $W \subseteq \mathbb{R}$ such that, in the new state variables $z = T(x)$, the system takes on the following particular triangular form:

$$\begin{aligned}
\dot{z}_1 &= z_2\\
\dot{z}_2 &= z_3\\
&\vdots\\
\dot{z}_n &= \alpha(z_1, \dots, z_n, u)
\end{aligned} \tag{5.6}$$

where the function α is continuous and invertible according to u over W for all $z \in V$. The block diagram of the Brunovsky canonical form is



Lemma 5.4. A triangular dynamical system described by the state-space model (5.5) can be put under Brunovsky canonical form around (x_0, u_0) if the inequalities

$$\begin{cases}
\frac{\partial g_i}{\partial x_{i+1}} \neq 0 & i = 1; \dots, n-1\\
\frac{\partial g_n}{\partial u} \neq 0
\end{cases} \tag{5.7}$$

Lemma 5.5. A control-affine system $\dot{x} = f(x) + g(x)u$ with $x \in \mathbb{R}^n, u \in mathbbR$ can be written in Brunovsky form in a domain $U \subseteq \mathbb{R}^n$ if there exists a state transformation $z = T(x)$ that fulfills the following conditions:

- $T_{i+1}(x) = \frac{\partial T_i}{\partial x} f(x)$, for $i = 1, \ldots, n-1$;
- $\frac{\partial T_i}{\partial x} g(x) = 0$, for $i = 1, \ldots, n-1$;
- $\frac{\partial T_n}{\partial x} g(x) \neq 0$

for every $x \in U$.

Equilibria and invariant sets

In this chapter, we assume that f is locally Lipschitz continuous on an open set $\Omega \subseteq \mathbb{R}^n$.

Definition 6.1. The pair $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an equilibrium of the system $\dot{x} = f(x, u)$ if $f(\bar{x}, \bar{u}) = 0$.

Definition 6.2. The equilibrium $(\bar{x}, \bar{u})$ is said to be isolated if there exists a neighbourhood of $\bar{x}$ that contains no other vector x such that $f(x, \bar{u}) = 0$.

6.1 Equilibria of linear systems

$$\dot{x} = Ax + Bu \tag{6.1}$$

Theorem 6.3. If the matrix A is regular, then for each $\bar{u}$, the pair $(-A^{-1}B\bar{u}, \bar{u})$ is an isolated equilibrium.
If the matrix A is singular, the system (6.1) has a continuum of non-isolated equilibria provided that $B\bar{u} \in Im(A)$. Those equilibria are the solutions of the system $A\bar{x} = -B\bar{u}$, forming an affine space. On the other side, for each $\bar{u}$ such that $B\bar{u} \notin Im(A)$, the system does not have any equilibrium.

6.2 Invariant sets

Definition 6.4. A set $\mathcal{X} \times U \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is said to be (positively) invariant for the dynamical system $\dot{x} = f(x, u)$ if, for all $x_0 \in \mathcal{X}$ and for all input signal $t \to u(t) \in U$, the trajectory $t \to x(t, x_0, u(t))$ remains in $\mathcal{X}$ for all $t \geq t_0$ whenever it is defined.

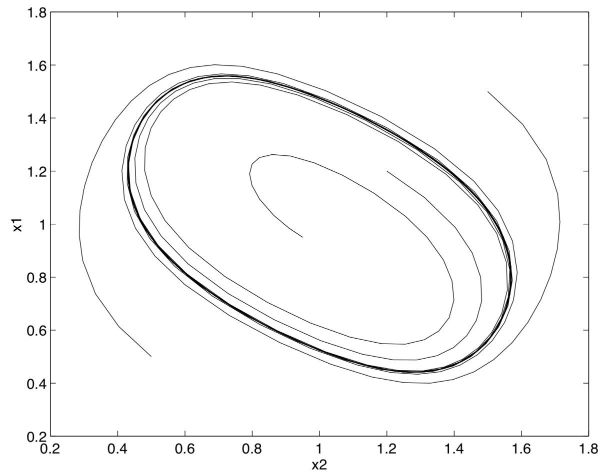
Definition 6.5. An outward normal vector to $\mathcal{X} \subseteq \mathbb{R}^n$ at $x \in \partial\mathcal{X}$ is a vector $n \in \mathbb{R}^n$ such that $n = \lambda(y - x)$, where $\lambda > 0$ and y is the center of an open ball $B \subseteq \mathbb{R}^n$ such that $x \in \partial B$ and $B \cap \mathcal{X} = \emptyset$; if no such open ball exists, $\mathcal{X}$ has no outward normal vector at x.

Theorem 6.6 (Bony's theorem)**.** Let f be a locally Lipschitz continuous vector field defined on an open set $\Omega \subseteq \mathbb{R}^n$, and $\mathcal{X}$ a closed set of Ω. If $\langle f(x), n(x) \rangle \leq 0$ for every $x \in \partial\mathcal{X}$, and every vector $n(x)$ is outward normal to $\mathcal{X}$ at x, then $\mathcal{X}$ is (positively) invariant for f.

- $\to$ N.B.: no condition has to be verified at a poitn where $\mathcal{X}$ does not have an outward normal vector.

6.3 Periodic orbits

A periodic orbit is such that it is arising from a trajectory of the dynamical system, verifying $x(t) = x(t+T)$ for all t and for some $T > 0$[1]. The infimum of possible values for T is called the period of the trajectory.
We denote $x(t, x_0, \bar{u})$ as the solution at time t with $x(t_0) = x_0$ and a constant input $u(t) = \bar{u}$.



Definition 6.7. The point z is called a limit point of y for the dynamical system subject to a constant input $\bar{u}$ if there exists a real sequence (t_n) such that $t_n \to \infty$ when $n \to \infty$ and $\lim_{n\to\infty} x(t_n, y, \bar{u}) = z$.

Definition 6.8. A limit cycle is a closed orbit γ such that at least one point of γ[2] is a limit point of at least another point of the phase plane not in γ.

$\rightarrow$ N.B.: These definitions are only valid in $\mathbb{R}^2$.

Theorem 6.9 (Bendixson-Dulac)**.** Let D be a simply connected domain in $\mathbb{R}^2$. If the divergence of f[3] is not identically zero and does not change sign in D, then D does not contain any closed orbit.

Theorem 6.10 (PoincarÃľ-Bendixson)**.** If E is a closed and bounded subset of $\mathbb{R}^2$, invariant for the system $\dot{x} = f(x, u)$, and if γ is an orbit starting in E, then:

- either γ converges to an equilibrium (which is the unique limit point of γ);
- or γ converges to a periodic orbit (which is the set of all limit points of γ).

This theorem can be used to prove the existence of a limit cycle:

1. Find a compact invariant set (proved by showing that on the border of this set, the vector field points inwards);
2. If there is no equilibrium in this set, it must contain a limit cycle or only periodic trajectories.

[1]Equilibria are trivial periodic orbits.
[2]Implying that they all are.
[3]$\div(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$

Local analysis of autonomous dynamical systems

A dynamical system is said to be autonomous if the input is constant:

$$\dot{x} = f(x\bar{u}) \tag{7.1}$$

7.1 Linear planar systems

Let us consider the linear planar system such as (6.1) with constant input $u = \bar{u}$. Let $\bar{x}$ be an equilibrium point corresponding to $\bar{u}$. We will use the state transformation $z = M^{-1}(x - x\bar{x})$. We obtain the linear system

$$\dot{z} = A'z \qquad A' = M^{-1}AM \tag{7.2}$$

As A and A' have the same eigenvalues, we can choose A' to have a canonical form:

- Two distinct real eigenvalues or double real eigenvalue ($\lambda_1 = \lambda_2$) with a geometric multiplicity equal to 2.

$$A' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{7.3}$$

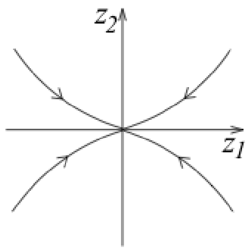
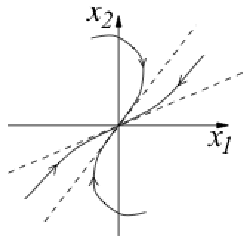
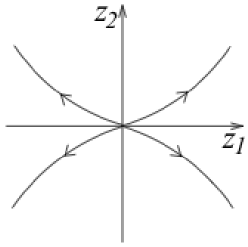
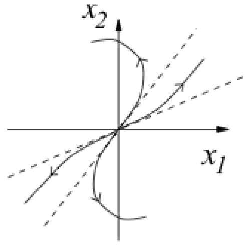
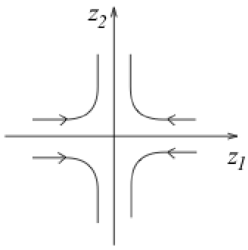
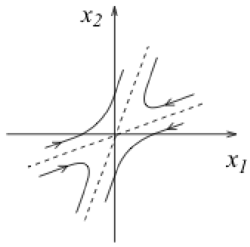
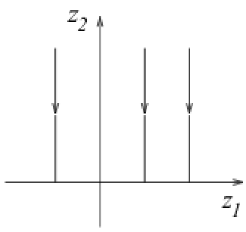
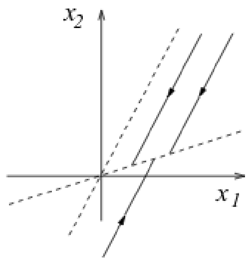
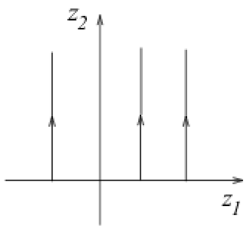
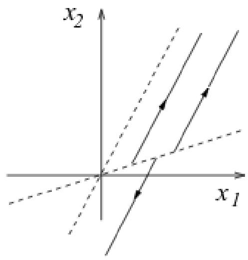
- Double real eigenvalue of geometric multiplicity equal to 1.

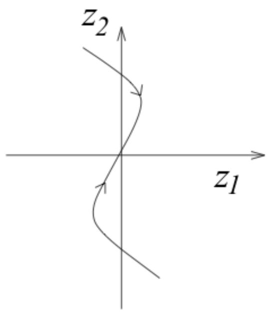
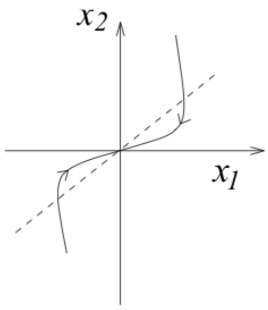
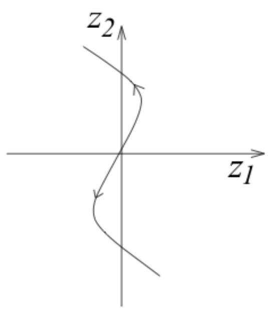
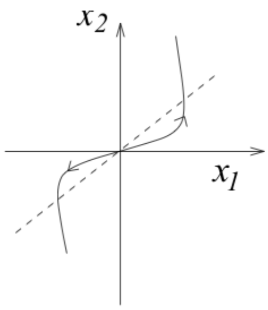
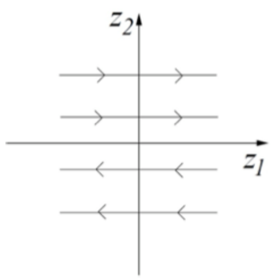
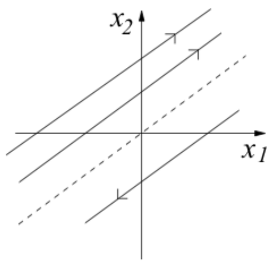
$$A' = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \tag{7.4}$$

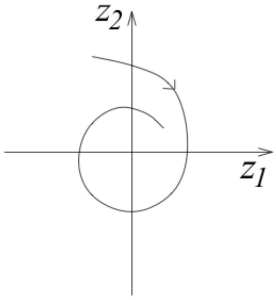
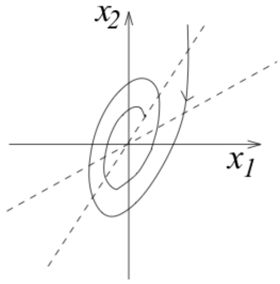
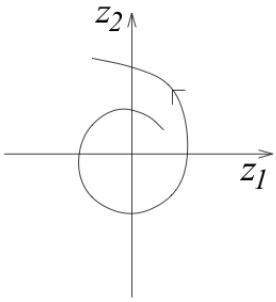
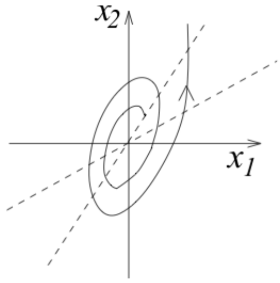
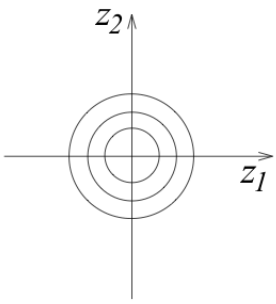
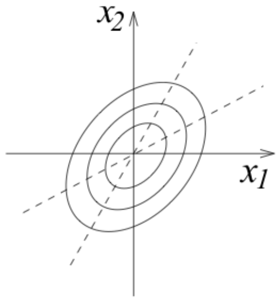
- Two complex conjugate eigenvalues $\alpha \pm \omega i$.

$$A' = \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix} \qquad \omega > 0 \tag{7.5}$$

And the types of equilibrium for a linear planar system are:

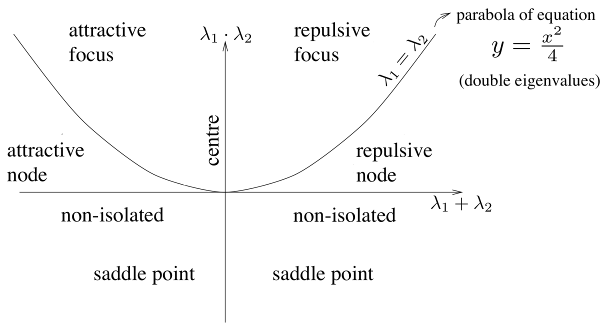
Type of equilibrium	Behaviour of orbits (z_1, z_2)	Behaviour of orbits (x_1, x_2)	Conditions on the eigenvalues
Attractive node			$\lambda_2 \leq \lambda_1 < 0$
Repulsive node			$0 < \lambda_1 \leq \lambda_2$
Saddle point			$\lambda_1 < 0 < \lambda_2$
Non-isolated attractive equilibrium			$\lambda_1 = 0$, $\lambda_2 < 0$
Non-isolated repulsive equilibrium			$\lambda_1 = 0$, $\lambda_2 > 0$

Type of equilibrium	Behaviour of orbits (z_1, z_2)	Behaviour of orbits (x_1, x_2)	Conditions on the eigenvalues
Degenerate attractive node			$\lambda < 0$ (defective)
Degenerate repulsive node			$\lambda > 0$ (defective)
Non-isolated equilibrium			$\lambda = 0$ (defective)

Type of equilibrium	Behaviour of orbits (z_1, z_2)	Behaviour of orbits (x_1, x_2)	Conditions on the eigenvalues
Attractive focus			$\lambda_{1,2} = \alpha \pm \omega i$ $\alpha < 0, \quad \omega \neq 0$
Repulsive focus			$\lambda_{1,2} = \alpha \pm \omega i$ $\alpha > 0, \quad \omega \neq 0$
Centre			$\lambda_{1,2} = \pm\omega i$ $\omega \neq 0$

- N.B.: if one of the eigenvalues is zero, the equilibrium is not isolated.

Definition 7.1. If all trajectories of a linear system converge to an equilibrium, we say that it is an attractive equilibrium. It is a repulsive equilibrium if they all diverge to infinity (save for the equilibrium itself.)



- An attractive (resp. repulsive) equilibrium will remain attractive (resp.repulsive) after a perturbation and a saddle point will remain a saddle point. Such equilibria are called structurally stable. However, a center equilibrium (zero real part) is never structurally stable: even a small perturbation of the matrix A can shift eigenvalues away from the imaginary axis, and the corresponding trajectories then converge to the equilibrium or diverge from it.

Definition 7.2. If all eigenvalues of A have nonzero real part, the equilibrium of $\dot{x} = Ax$ is said to be hyperbolic.

7.2 Linearisation of nonlinear systems

We assume the existence of an equilibrium $(\bar{x}, \bar{u})$ for the nonlinear system $\dot{x} = f(x, \bar{u})$. The Taylor expansion is

$$\dot{x} = f(\bar{x}, \bar{u}) + \left(\frac{\partial f(x, \bar{u})}{\partial x}\right)_{\bar{x}} (x - \bar{x}) + \mathcal{O}(\|x - \bar{x}\|^2) \tag{7.6}$$

Thus, the linear approximation of the system is, for $\tilde{x} = x - \bar{x}$,

$$\dot{\tilde{x}} = \left(\frac{\partial f(x, \bar{u})}{\partial x}\right)_{\bar{x}} \tilde{x} \tag{7.7}$$

We define $A \triangleq \left(\frac{\partial f(x,\bar{u})}{\partial x}\right)_{\bar{x}}$ as the Jacobian matrix of f at the equilibrium.

Definition 7.3. The equilibrium $(\bar{x}, \bar{u})$ of the nonlinear system is said to be hyperbolic if all the eigenvalues of the jacobian matrix A have a nonzero real part.

Definition 7.4. Two dynamical systems are topologically conjugate if there exists a homeomorphism, i.e. a continuous bijection whose inverse is also continuous, that maps the trajectories of the first system to the trajectories of the second one in a time respecting way. That means that the trajectories of $\dot{x} = f(x)$ on a domain D and $\dot{y} = g(y)$ on a domain E are topologically conjnugate through the homeomorphism $\phi : D \to E$ if every curve $[0, t_0] \to D : t \to x(t)$ is a trajectory of the system f iff the corresponding curve $[0, t_0] \to E : t \to \phi(x(t))$ is a trajectory of the system g.

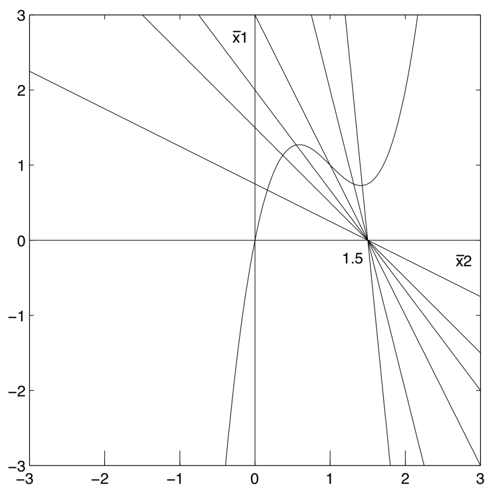
Theorem 7.5. If the equilibrium $(\bar{x}, \bar{u})$ is hyperbolic, then the trajectories of the nonlinear system in a neighbourhood of the equilibrium $(\bar{x}, \bar{u})$ are topologically conjugate to those of the linear approximation (7.7). Specifically, there exists a neighbourhood X of $\bar{x}$, a neighbourhood $\tilde{X}$ of 0, and a homeomorphism $\phi : X \to \tilde{X}$ with $\phi(\bar{x}) = 0$ such that if $t \to x(t)$ is a trajectory of the nonlinear system contained in X, then $t \to \phi(x(t))$ is a trajectory of the linear system.

That means that if the equilibrium is a node/focus (attractive or repulsive) or a saddle point (but not a centre) in the linearised system, then the linearised system is a good representation for the local behaviour of the nonlinear trajectories around the equilibrium as well. However, this theorem is local, and the higher-order terms are needed to conclude in the case of a non-hyperbolic equilibrium.

Bifurcations

Bifurcation theory looks at the impact of the value $\bar{u}$ on the nature and number of equilibria.

8.1 Hopf bifurcation



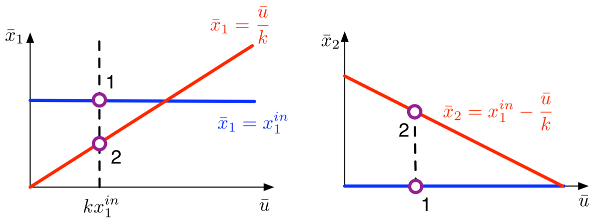
Depending on the slope of the straight line, the characterization of the equilibrium changes: it is an attractive focus, then changes to repulsive focus and goes back to being an attractive focus.

Theorem 8.1. Suppose that a system has a family of isolated equilibria $(\bar{x}, \bar{u})$ parametrized by $\bar{u}$. Suppose that there exists a value $\bar{u}^*$ such that a pair of eigenvalues of the Jacobian matrix evaluated in this equilibrium have a zero real part and a nonzero imaginary part. These values depend continuously on $\bar{u}$, at least in the neighbourhood of $\bar{u}^*$, and are denoted by

$$\lambda_i(\bar{u}) = \alpha(\bar{u}) \pm i\omega(\bar{u}) \tag{8.1}$$

Suppose also that $\frac{d\alpha(\bar{u}^*)}{d\bar{u}} > 0$. Thus, for $\bar{u}$ close enough to $\bar{u}^*$, the equilibrium is attractive for $\bar{u} < \bar{u}^*$ and repulsive for $\bar{u} > \bar{u}^*$.
Then, there generically exists either an attractive closed orbit (i.e. limit cycle) for all $\bar{u}^* < \bar{u} < \bar{u}^* + \varepsilon$ or a repulsive closed cycle for $\bar{u}^* - \varepsilon < \bar{u} < \bar{u}^*$ (for some $\varepsilon > 0$) unique in the neighbourhood of the equilibrium.

8.2 Transcritical bifurcation

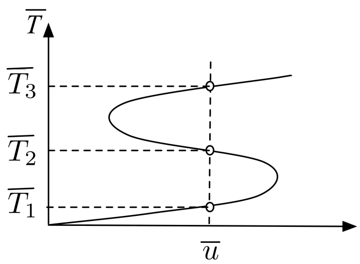


In this example, the first equilibrium is attractive if $\bar{u} > kx_1^{in}$ and is a saddle point otherwise. The second however is a saddle point when the above condition is met and attractive when it is not.

A transcritical bifurcation is thus such that the characterization of the two equilibria switch when passing a certain threshold value of $\bar{u}$.

8.3 Saddle-node/Fold bifurcation

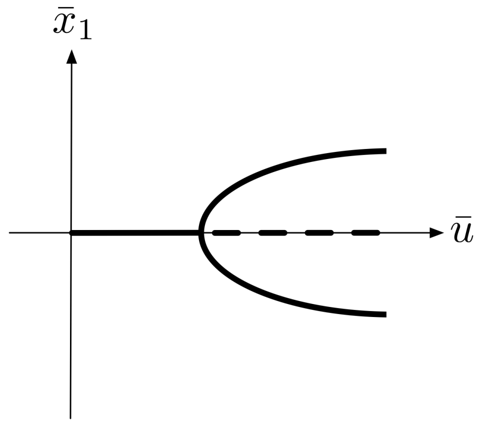
For small values of $\bar{u}$, the system has a single equilibrium. Then, for a critical value of $\bar{u}$, the system exhibits two more equilibrium values (one a saddle point and the other attractive). By further increasing $\bar{u}$, we cross a new critical value beyond which the system has only a single equilibrium that is also attractive.



As the input $\bar{u}$ is slowly modified from low to high values, the state of the system, initially following the bottom line of equilibria, goes through a brutal change at the rightmost bifurcation, where it jumps to a different equilibrium. It is called a catastrophe. As the input decreases again to low values, the catastrophe happens the other way. This is a hysteresis.

8.4 Pitchfork bifurcation

A pitchfork bifurcation is the split, for some value $\bar{u}^*$ of the birufcation parameter, of a single attractive (resp. repulsive) equilibrium into three equilibria, one being repulsive (resp. attractive) and the other two being attractive (resp. repulsive).



$\bar{x}_1$
$\bar{u}$

Stability of equilibria

In this chapter we assume that the locally Lipschitz continuous vector field $f(\cdot, \bar{u})$ is defined on an open set $\Omega \subseteq \mathbb{R}^n$ for a fixed value of $\bar{u}$, where the trajectories are defined and stay in Ω for all positive times.

9.1 Definitions

Definition 9.1. The equilibrium $(\bar{x}, \bar{u})$ is said to be stable if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x_0 \in \Omega : \ \|x(t_0) - \bar{x}\| < \delta \Longrightarrow \|x(t, x(t_0), \bar{u}) - \bar{x}\| < \varepsilon \quad \forall t \geq t_0 \tag{9.1}$$

That means that an equilibrium is stable if the trajectories remain arbitrarily close to it, provided that they start close enough from this equilibrium.

Definition 9.2. The equilibrium $(\bar{x}, \bar{u})$ is said to be attractive if

$$\exists \delta > 0 \text{ such that } \|x(T_0) - \bar{x}\| < \delta \Longrightarrow \lim_{t \to \infty} \|x(t, x(t_0), \bar{u}) - \bar{u}\| = 0 \tag{9.2}$$

An attractive equilibrium $\bar{x}$ is thus a point to which each solution x converges provided that it starts close enough to $\bar{x}$.

→ N.B.: stability and attractiveness do not imply each other.

Definition 9.3. The equilibrium $(\bar{x}, \bar{u})$ is said to be asymptotically stable if it is both stable and attractive. The set of points x_0 for which the trajectory $x(t, x_0, \bar{u})$ converges to $\bar{x}$ is called the basin of attraction of the saymptotically stable equilibrium.

Definition 9.4. The equilibrium $(\bar{x}, \bar{u})$ is said to be exponentially stable if

$$\exists a, b, \delta > 0 \text{ such that } \|x(t_0) - \bar{x}\| < \delta \Longrightarrow \|x(t, x(t_0), \bar{u}) - \bar{x}\| \leq a\|x(t_0) - \bar{x}\| e^{-bt} \qquad \forall t \geq t_0 \tag{9.3}$$

→ N.B.: exponential stablity implies asymptotic stability.

→ N.B.: for a linear system, an attractive equilibrium and center are both stable, while a saddle or repulsive equilibrium is unstable. Attractive equilibrium is also exponentially stable, thus asymptotically stable: these three notions coincide for linear systems.

9.2 Lyapunov's first method

Theorem 9.5.
- If the equilibrium is attractive in the linearised system, i.e. all eigenvalues of the Jacobian matrix have a negative real part, then the equilibrium $(\bar{x}, \bar{u})$ is exponentially stable.
- If the equilibrium is repulsive or a saddle point in the linearised syste, i.e. the Jacobian matrix has at least one eigenvalue with a positive real part, then the equilibrium $(\bar{x}, \bar{u})$ is unstable.

This theorem does not conclude on the stability of a non-hyperbolic equilibrium.

9.3 Lyapunov's second method

Theorem 9.6. The equilibrium $(\bar{x}, \bar{u})$ of the system $\dot{x} = f(x, \bar{u})$, where f is locally Lipschitz continuous on an open set $\Omega \subseteq \mathbb{R}^n$ is stable if there exists a continuously differentiable function $V : \Omega \to \mathbb{R}$ with the following properties:

- $\Omega \subseteq \mathbb{R}^n$ is a neighbourhood of $\bar{x}$;
- $V(x) > V(\bar{x}) \, \forall x \in \Omega \setminus \{\bar{x}\}$, i.e. V has a strict minimum point at $\bar{x}$;
- $\dot{V}(x) \leq 0 \, \forall x \in \Omega \setminus \{\bar{x}\}$.

That means that a sufficient condition for the equilibrium $(\bar{x}, \bar{u})$ to be stable is to have a positive-definite function $V - V(\bar{x})$ whose temporal derivative $\dot{V}$ along trajectories is negative-semidefinite in a neighbourhood of $\bar{x}$, the temporal derivative being defined as

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x}\dot{x} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x, \bar{u}) = \langle \nabla V(x(t)), f(x(t), \bar{u}) \rangle \tag{9.4}$$

Theorem 9.7. The equilibrium $(\bar{x}, \bar{u})$ of the system $\dot{x} = f(x, \bar{u})$ is asymptotically stable if there exists a continuously differentiable function $V : \Omega \to \mathbb{R}$ with the following properties:

- $\Omega \subseteq \mathbb{R}^n$ is a neighbourhood of $\bar{x}$;
- $V(x) > V(\bar{x}) \, \forall x \in \Omega \setminus \{\bar{x}\}$, i.e. V has a strict global minimum point at $\bar{x}$;
- $\dot{V}(x) < 0 \, \forall x \in \Omega \setminus \{\bar{x}\}$.

Theorem 9.8. The basin of attraction of an asymptotically stable equilibrium is an open, connected, invariant set; and its boundary is formed by trajectories.

Definition 9.9. Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be locally Lipschitz continuous and $\bar{x} \in \mathbb{R}^N$ be an equilibrium of f. The equilibrium $\bar{x}$ is said to be globally asymptotically stable if it is asymptotically stable and its basin of attraction is the whole state spce $\mathbb{R}^N$.

Definition 9.10. A function $V : \mathbb{R}^N \to \mathbb{R}$ is said to be radially unbounded if $\|x\| \to \infty$ implies $V(x) \to \infty$, i.e. for every $M \in \mathbb{R}$, there exists $R \geq 0$ such that, for every $x \in \mathbb{R}^N$, $\|x\| \geq R$ implies $V(x) \geq M$.

Theorem 9.11. Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be locally Lipschitz continuous and $\bar{x} \in \mathbb{R}^N$ be an equilibrium of f. If there exists a radially unbounded continuously differentiable function $V : \mathbb{R}^N \to \mathbb{R}$ such that, for every $x \in \mathbb{R}^N \setminus \{\bar{x}\}$, $V(x) > V(\bar{x})$ and $\dot{V}_f(x) < 0$, then $\bar{x}$ is globally asymptotically stable.

9.3.1 LaSalle's invariance principle

Definition 9.12. A function $x : \mathbb{R}^+ \to \mathbb{R}^N$ is said to asymptotically approach a nonempty set $M \subseteq \mathbb{R}^N$ if

$$\lim_{t\to\infty} \underbrace{\inf_{y\in M} \|x(t) - y\|}_{=:\text{dist}(x(t),M)} = 0 \tag{9.5}$$

Theorem 9.13. Let $F \subseteq \Omega$ be a compact (positively) invariant set for f. Let $V : \Omega \to \mathbb{R}$ be a continuously differentiable and such that $\dot{V}_f \leq 0$ for every $x \in F$. Let $E := \{x \in F | \dot{V}_f(x) = 0\}$. Let M be the largest (positively and negatively) invariant set in E. Then, every trajectory of f starting in F asymptotically approaches M.

Corollary 9.14. Let $\bar{x}$ be an equilibrium of f. Let $V : \Omega \to \mathbb{R}$ be continuously differentiable and such that, for every $x \in \Omega \setminus \{\bar{x}\}$, $V(x) > V(\bar{x})$ and $\dot{V}(x) \leq 0$. Let $S := \{x \in \Omega | \dot{V}(x) = 0\}$. If the equilibrium is the only trajectory fully contained (past and future) in S, then $\bar{x}$ is asymptotically stable.

9.4 The energy as a Lyapunov function

→ N.B.: the Lyapunov function $V(x)$ often has the dimensions of an energy.

9.4.1 In mechanical systems

The general equation in a mechanical system[1] is

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + k(q) + h(\dot{q}) = G\bar{u} \tag{9.6}$$

Here we assume the kinematic matrix G to be constant. The Lyapunov function taken for this general system is

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + E_p(q) - q^T G\bar{u} \tag{9.7}$$

$$\dot{V}(q,\dot{q}) = -\dot{q}^T h(\dot{q}) \tag{9.8}$$

The first term is the kinetic energy, the second is the potential energy and the third is the work realized by the applied forces and torques.

9.4.2 In electrical systems

For electrical systems, there isn't a general formula of the state-space model. The Lyapunov function will however still have the dimensions of an energy, with different terms:

- Inductance: $E = \frac{1}{2}Li^2$
- Capacitance: $E = \frac{1}{2}Cv^2$
- Resistance: $E = Ri^2$

[1]See chapter 3

9.5 Linear systems

Let us study once again the system (6.1) with an equilibrium $(\bar{x}, \bar{u})$. We define the Lyapunov function

$$V(x) = (x - \bar{x})^T P(x - \bar{x})$$

where P is a symmetric positive-definite matrix. Its derivative is $\dot{V}(x) = -(x - \bar{x})^T Q(x - \bar{x})$, with $-Q = A^T P + PA$.

$$V(x) = (x - \bar{x})^T P(x - \bar{x}) \tag{9.9}$$

$$\dot{V}(x) = -(x - \bar{x})^T Q(x - \bar{x}) \qquad -Q = A^T P + PA \tag{9.10}$$

Theorem 9.15. Let A be a real matrix of order n. For every positive-definite matrix Q, equation (9.10) owns a unique positive-definite solution P iff A is a Hurwitz matrix, i.e. all its eigenvalues have a negative real part.

9.6 Bounded-input, bounded-state stability

We are here interested in an input signal $u(t)$ that is bounded and close to $\bar{u}$. We need to analyse this case because a constant signal is not feasible in reality. We study the linear system

$$\dot{x} = Ax + Bu \quad x(t_0) = x_0 \tag{9.11}$$

The trajectory of the system is

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau \tag{9.12}$$

The equilibrium is $(\bar{x}, \bar{u}) = (0, 0)$. IT is asymptotically stable iff the matrix A is a Hurwitz matrix. This would mean that $\|e^{At}\|$ is bounded for all $t \geq t_0$ and there are nonnegative constants k and λ such that

$$\|e^{A(t-t_0)}\| \leq ke^{-\lambda(t-t_0)} \tag{9.13}$$

and thus

$$\|x(t)\| \leq ke^{-\lambda(t-t_0)} \|x_0\| + \frac{k\|B\|}{\lambda} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\| \tag{9.14}$$

This means that a bounded input $u(t)$, however big its magnitude is, generates a bounded state $x(t)$, and the effect of the initial condition x_0 fades away with time.

Theorem 9.16. If the equilibrium $(\bar{x}, \bar{u})$ of the linear system is asymptotically stable,

- there are three nonnegative constants c_1, c_2, c_3 such that, for each initial state x_0 with $\|x_0 - \bar{x}\| < 0$ and each input signal u with $\|u(t) - \bar{u}\| < c_2 \ \forall t \geq t_0$, the solution x is bounded: $\|x(t) - x_0\| < c_3 \ \forall t \geq t_0$;
- there is a nonnegative constant c_0 and a continuous function $\alpha : [0, a) \to [0, \infty)$ passing through the origine, i.e. $\alpha(0) = 0$, and increasing such that, for each input signal u with $\|u(t) - \bar{u}\| < c_0 \ \forall t \geq t_0$, the ultimate bound on x is an increasing function of the bound on u :

$$\lim_{t\to\infty} \sup \|x(t)\| \leq \alpha(\|u\|_{\mathcal{L}_\infty}) \tag{9.15}$$

Theorem 9.17. If f is globally continuously differentiable and globally Lipschitz continuous, and if the equilibrium $(\bar{x}, \bar{u})$ is globally exponentially stable, then for each initial condition x_0, and each input signal u, the solution x is bounded.

Controllability and trajectory planning

Definition 10.1. For the dyanmical system $\dot{x} = f(x, u)$, the final state $x_f \in \mathbb{R}^n$ is reachable from the initial state $x_0 \in \mathbb{R}^n$ within time T if there exists an input function $u : [t_0, t_0 + T] \to \mathbb{R}^m$ such that $x(t_0) = x_0$ and $x(t_0 + T) = x_f$.

10.1 Controllability of LTI systems

Definition 10.2. The system $\dot{x} = Ax + Bu$, for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, is controllable if, for each initial state x_0, it is possible to reach any other final state x_f within any positive time T.

Theorem 10.3. The LTI system $\dot{x} = Ax + Bu$ is completely controllable iff one of the two following criteria is satisfied:

- The matrix $\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1} \end{bmatrix}$ has full rank; The rank of the matrix $\begin{bmatrix} sI - A & B \end{bmatrix}$ is equal to n for each $s \in \mathbb{C}$

If the controllability matrix has rank $d < n$, we define a matrix $T = (T_a \; T_b)$ such that T_a contains d linearly independent columns of $\mathcal{C}$ and T_b completes the matrix by $n - d$ vectors independent of the columns of T_a. Its inverse is $T^{-1} := \begin{pmatrix} U_a \\ U_b \end{pmatrix}$, where the matrices U_a, U_b are chosen such that

$$T^{-1}T = \begin{pmatrix} U_a T_a & U_a T_b \\ U_b T_a & U_b T_b \end{pmatrix} = \begin{pmatrix} I_d & 0 \\ 0 & I_{n-d} \end{pmatrix} \tag{10.1}$$

From that, we define a state transformation

$$z = \begin{pmatrix} z_a \\ z_b \end{pmatrix} = \begin{pmatrix} U_a x & U_b x \end{pmatrix} \tag{10.2}$$

The new state-space model is

$$\dot{z}_a = U_a A T_a z_a + U_a A T_b z_b + U_a B u \tag{10.3}$$
$$\dot{z}_b = U_b A T_b z_b \tag{10.4}$$
$$\tag{10.5}$$

The part z_b is the non controllable part of the system, it is not influenced by the input u.

10.2 Controllability of nonlinear systems

Definition 10.4. The system $\dot{x} = f(x, u)$ for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is locally accessible from the initial state x_0 if for any time T, the set of states reachable from x_0 within T contains a nonempty open set.

Definition 10.5. The system $\dot{x} = f(x, u)$ for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is locally controllable from the initial state x_0 if for any time T, the set of states reachable from x_0 within T contains a neighbourhood of x_0.

We can also define the global controllability, which requires that the whole state space is reachable from any initial condition.

→ N.B.: local accessibility, local controllability and global controllability are equivalent for linear systems.

10.3 Drawbacks of linearisation

Theorem 10.6. Let us consider the linearisation of the system $\dot{x} = f(x, u)$ around an equilibrium$(\bar{x}, \bar{u})$:

$$\dot{x} = Ax + Bu \qquad A = \left(\frac{\partial f}{\partial x}\right)_{(\bar{x},\bar{u})} \qquad B = \left(\frac{\partial f}{\partial u}\right)_{(\bar{x},\bar{u})} \tag{10.6}$$

If the linearised system is controllable, then for each $\varepsilon > 0$, the set of states reachable from $\bar{x}$ within time T with inputs $u(t)$ such that $\|u(t) - \bar{u}\| < \varepsilon$, contains a neighbourhood of $\bar{x}$.

We now define a new operator, the Lie bracket of two vector fields:

$$[g_1, g_2] := \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2 \tag{10.7}$$

Theorem 10.7. Let us consider the control-linear system[1] in some open set $X \subseteq \mathbb{R}^n$:

$$\dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m \tag{10.8}$$

for some analytic[2] vector fields $g_1, \ldots, g_m$. Consider the set of all vector fields $g_1, \ldots, g_m$ and their repeated Lie brackets. Consider also the vector space generated by all these vector fields evaluated at a state x_0. This vector space is of dimension n iff the system is locally controllable from x_0. Moreover, if this condition is met everywhere in the state space X, then the system is globally controllable.

Theorem 10.8. Let us consider the system in $X \subseteq \mathbb{R}^n$:

$$\dot{x} = f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m \tag{10.9}$$

for some analytic vector fields $f, g_1, \ldots, g_m$. Consider the set of all vector fields $f, g_1, \ldots, g_m$ and those obtained by repeated Lie brackets. Consider also the vector space generated by those vector fields evaluated at a state x_0. This vector space if of dimension n iff the system is locally accessible from x_0.

[1] Control-linear means linear in every input.

[2] Analytic means that all coordinates of all fields g_i have a Taylor series that converges in a neighbourhood, i.e. all derivatives of all order exist.

10.4 Trajectory planning

We work here with the following Brunovsky form of the nonlinear system:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_n &= \alpha(z) + \beta(z)u \qquad \beta(z) \neq 0
\end{aligned} \tag{10.10}$$

It is sufficient to define a polynomial trajectory for z_1:

$$z_1(t) = \sum_{i=0}^{2n-1} \lambda_i \left(\frac{t}{T}\right)^i \tag{10.11}$$

By calculating the successive derivatives of $z_1(t)$, we obtain the expressions of $z_j(t)$, for $j = 2, \ldots, n$:

$$z_j(t) = \sum_{i=j-1}^{2n-1} \frac{i!}{(i-j+1)!} \frac{\lambda_i}{T^{j-1}} \left(\frac{t}{T}\right)^{1-j} \tag{10.12}$$

and we can find the values of coefficients λ_i by calculating those values for $t = 0$ and $t = T$. Finally, the input we need to obtain the wanted results is

$$u(t) = \frac{\dot{z}_n(t) - \alpha(z(t))}{\beta(z(t))} \tag{10.13}$$