



---

# **LINMA2370 Modelling and Analysis of Dynamical Systems**

---

SIMON DESMIDT

Academic year 2024-2025 - Q1



UCLouvain

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Reminders . . . . .	3
1.2	State-space model . . . . .	4
1.3	Integral curve . . . . .	4
1.4	Existence of a solution . . . . .	5
<b>2</b>	<b>Dynamical systems and state-space models</b>	<b>6</b>
2.1	Terminology and notation . . . . .	6
<b>3</b>	<b>State transformations</b>	<b>8</b>
3.1	Definition . . . . .	8
3.2	Linear state transformation . . . . .	9
3.3	Nonlinear state transformation . . . . .	9
3.4	Triangular system . . . . .	10
3.5	Brunovsky canonical form . . . . .	10
<b>4</b>	<b>Equilibria and invariant sets</b>	<b>12</b>
4.1	Equilibria of linear systems . . . . .	12
4.2	Invariant sets . . . . .	12
4.3	Periodic orbits . . . . .	13
<b>5</b>	<b>Local analysis of autonomous dynamical systems</b>	<b>14</b>
5.1	Linear planar systems . . . . .	14
5.2	Linearisation of nonlinear systems . . . . .	18
<b>6</b>	<b>Bifurcations</b>	<b>20</b>
6.1	Hopf bifurcation . . . . .	20
6.2	Transcritical bifurcation . . . . .	21
6.3	Saddle-node/Fold bifurcation . . . . .	21
6.4	Pitchfork bifurcation . . . . .	21
<b>7</b>	<b>Stability of equilibria</b>	<b>23</b>
7.1	Definitions . . . . .	23
7.2	Lyapunov's first method . . . . .	24
7.3	Lyapunov's second method . . . . .	24
7.4	The energy as a Lyapunov function . . . . .	25
7.5	Linear systems . . . . .	26
7.6	Bounded-input, bounded-state stability . . . . .	26

<b>8</b>	<b>Controllability and trajectory planning</b>	<b>28</b>
8.1	Controllability of LTI systems . . . . .	28
8.2	Controllability of nonlinear systems . . . . .	29
8.3	Drawbacks of linearisation . . . . .	29
8.4	Trajectory planning . . . . .	30
<b>9</b>	<b>Exercises</b>	<b>31</b>
9.1	Electrical systems . . . . .	31
9.2	Mechanical systems . . . . .	31
9.3	Compartmental systems . . . . .	34
9.4	Reaction systems . . . . .	35

# Introduction

The tools introduced in this course are a simplifying view of the reality, yet very useful to build simple and effective models in view of the control and optimization of the dynamical behaviour of the real systems.

## 1.1 Reminders

- A subset of  $\mathbb{R}$  is said to be negligible if its Lebesgue measure is equal to zero and that a property is said to be true almost everywhere if it is false only on a negligible set.
- Let  $I \subseteq \mathbb{R}$  be an interval the interior of which is not empty. A function  $x : I \rightarrow \mathbb{R}^N$  is said to be absolutely continuous if

$\forall \varepsilon \in (0, \infty), \exists \delta \in (0, \infty) :$

$$\begin{aligned} & \forall n \in \mathbb{N} \setminus \{0\}, \forall a_1, b_1, \dots, a_n, b_n \in I : \\ & a_i < b_i \forall i \in \{1, \dots, n\}, b_i \leq a_{i+1} \forall i \in \{1, \dots, n-1\}, \\ & \sum_{i=1}^n (b_i - a_i) \leq \delta \implies \sum_{i=1}^n \|x(b_i) - x(a_i)\| \leq \varepsilon \end{aligned}$$

- Let  $a, b \in \mathbb{R}$  with  $a < b$ . A function  $x : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous iff there exists an integrable function  $\phi : [a, b] \rightarrow \mathbb{R}$  such that, for every  $t \in [a, b]$ ,

$$x(t) = x(a) + \int_a^t \phi(s) ds$$

in which case  $x$  is almost everywhere differentiable with  $\dot{x}(t) = \phi(t)$  for almost every  $t \in [a, b]$ .

- A function  $f : \Omega \rightarrow \mathbb{R}^N$ , where  $\Omega$  is a nonempty subset of  $\mathbb{R} \times \mathbb{R}^N$ , is said to be Lipschitz continuous in the second argument, uniformly with respect to the first argument, if there exists  $L \in [0, \infty)$  such that for all  $t \in \mathbb{R}$  and all  $x, y \in \mathbb{R}^N$  such that  $(t, x), (t, y) \in \Omega$ ,

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

It is said to be locally Lipschitz continuous on an open ball for each argument.

- Let  $\Omega$  be a nonempty open subset of  $\mathbb{R} \times \mathbb{R}^N$  and  $f : \Omega \rightarrow \mathbb{R}^N$  be such that

- for all  $t \in \mathbb{R}$ ,  $f(t, \cdot) : \Omega_t \rightarrow \mathbb{R}^N$
- $\partial_2 f : \Omega \rightarrow \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N) : (t, x) \rightarrow \partial_2 f(t, x)$  is locally bounded.

Then,  $f$  is locally Lipschitz continuous in the second argument, uniformly with respect to the first argument.

- If  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  are two real normed spaces, and the real vector space  $\mathcal{L}(X, Y)$  of all continuous linear mappings from  $X$  to  $Y$ <sup>1</sup> is equipped with the norm defined by

$$\|L\| := \sup_{x \in X \setminus \{0\}} \frac{\|Lx\|_Y}{\|x\|_X}$$

## 1.2 State-space model

A state-space model for a continuous dynamical system consists of an ODE of the form

$$\dot{x}(t) = f(t, x(t)) \quad (1.1)$$

where the function  $f : \Omega \rightarrow \mathbb{R}^N$ ,  $\Omega$  being a nonempty subset of  $\mathbb{R} \times \mathbb{R}^N$ , is called the vector field associated with the ODE. A continuous dynamical system with input  $u : \mathbb{R} \rightarrow \mathbb{R}^M$  described by the ODE

$$\dot{x}(t) = g(x(t), u(t)) \quad (1.2)$$

for some function  $g : \mathbb{R}^N \times \mathbb{R}^M \rightarrow \mathbb{R}^N$ , can be written in the form (1.1) by defining the vector field

$$f_u : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N : (t, x) \rightarrow g(x, u(t)) \quad (1.3)$$

→ Note: the norm of each  $(t, x) \in \mathbb{R} \times \mathbb{R}^N$  is defined as  $|t| + \|x\|$ .

## 1.3 Integral curve

Let  $\Omega$  be a nonempty subset of  $\mathbb{R} \times \mathbb{R}^N$ . An integral curve of  $f : \Omega \rightarrow \mathbb{R}^N$  is a function  $x : I \rightarrow \mathbb{R}^N$  where  $I \subseteq \mathbb{R}$  is an interval, for which the interior is not empty, called the interval of existence of  $x$ , i.e. differentiable and satisfies  $(t, x(t)) \in \Omega$  and  $\dot{x}(t) = f(t, x(t))$  for all  $t \in I$ . The graph  $\{(t, x(t)) | t \in I\}$  and the image  $\{x(t) | t \in I\}$  of  $x$  are respectively called the trajectory and the orbit of  $x$ . Given an initial condition  $(t_0, x_0) \in \Omega$ , a solution to the initial value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \quad (1.4)$$

is an integral curve  $x : I \rightarrow \mathbb{R}^N$  of  $f$  such that  $t_0 \in I$  and  $x(t_0) = x_0$ .

---

<sup>1</sup>Meaning matrix from  $X$  to  $Y$

If, for the IVP described hereabove,  $f$  is continuous, then a continuous function  $x : I \rightarrow \mathbb{R}^N$  where  $I \subseteq \mathbb{R}$  is an interval containing  $t_0$  and the interior of which is not empty, is a solution iff its graph is contained in  $\Omega$  and it satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds$$

for all  $t \in I$ . In that case,  $\dot{x}$  is continuous.

Let  $\Omega$  be a nonempty subset of  $\mathbb{R} \times \mathbb{R}^N$ . An integral curve in the extended sense of  $f : \Omega \rightarrow \mathbb{R}^N$  is a function  $x : I \rightarrow \mathbb{R}^N$ , where  $I \subseteq \mathbb{R}$  is an interval the interior of which is not empty called the interval of existence of  $x$ , that is absolutely continuous and satisfies  $(t, x(t)) \in \Omega$  for every  $t \in I$  and  $\dot{x}(t) = f(t, x(t))$  for almost every  $t \in I$ .

→ Note: If  $f$  is continuous, then the two definitions of integral curves are equivalent.

## 1.4 Existence of a solution

Consider the IVP defined hereabove with an integral curve in the extended sense, under the following assumptions:

- there exists  $\tau, r \in (0, \infty)$ , such that  $[t_0 - \tau, t_0 + \tau] \times B(x_0, r) \subseteq \Omega$ ;
- for every  $x \in B(x_0, r)$ , the function  $[t_0 - \tau, t_0 + \tau] \rightarrow \mathbb{R}^N : t \rightarrow f(t, x)$  is measurable;
- for every  $t \in [t_0 - \tau, t_0 + \tau]$ , the function  $B(x_0, r) \rightarrow \mathbb{R}^N : x \rightarrow f(t, x)$  is continuous;
- there exists an integrable function  $m : [t_0 - \tau, t_0 + \tau] \rightarrow [0, \infty)$  such that

$$\|f(t, x)\| \leq m(t) \text{ for all } (t, x) \in [t_0 - \tau, t_0 + \tau] \times B(x_0, r)$$

Then, there exists a solution defined on a compact interval the interior of which contains  $t_0$ .

In particular, for the IVP with an integral curve in the general sense, if  $(t_0, x_0)$  is an interior point of  $\Omega$  and  $f$  is continuous, then there exists a solution defined on a compact interval the interior of which contains  $t_0$ .

# Dynamical systems and state-space models

We will study first-order dynamical systems of the form

$$\dot{x} = f(x, u) \quad (2.1)$$

where  $f$  is a mapping from  $\mathbb{R}^{n+m}$  to  $\mathbb{R}^n$ , while  $x$  and  $u$  are vector functions of time, respectively the state and the input.

## 2.1 Terminology and notation

- We assume that the input is a piecewise continuous and bounded function:  $u \in \mathcal{U}$ , where  $\mathcal{U}$  is a set of piecewise continuous and bounded functions from  $\mathbb{R}$  to  $\mathbb{R}^m$ .
- For a given value of the initial state  $x(t_0) = x_0$  and a given input  $u$ , the solution  $t \rightarrow x(t)$  for  $t \geq t_0$ , of the system of ODE 2.1 is called the trajectory of the system. It is denoted  $x(t_0, x_0, u)$ .
- When the input  $u$  can be freely chosen in  $\mathcal{U}$ , the system  $\dot{x} = f(x, u)$  is said to be a forced/controlled system.

→ Note: in this course, we will study the solution of the equation 2.1 when the input is actually an a priori set constant:  $u(t) = \bar{u} \forall t \geq t_0$ . The state-space model is then written as  $\dot{x} = f(x, \bar{u}) = f_{\bar{u}}(x)$ .

### 2.1.1 System with affine input

$$\dot{x} = f(x) + \sum_{i=1}^m u_i g_i = f(x) + G(x)u \quad (2.2)$$

where  $f$  and  $g_i$  are mappings from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ .

### 2.1.2 System with affine state

$$\dot{x} = \sum_{i=1}^n x_i a_i(u) + b(u) = A(u)x + b(u) \quad (2.3)$$

where  $b$  and  $a_i$  are mappings from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ .

### 2.1.3 Bilinear systems

A bilinear system is affine both in the state and in the input:

$$\dot{x} = \left( A_0 + \sum_{i=1}^m u_i A_i \right) x + B_0 u \quad (2.4)$$

where  $A_i$  and  $B_i$  are matrices of dimensions  $n \times n$  and  $n \times m$  respectively.

### 2.1.4 Linear system

$$\dot{x} = Ax + Bu \quad (2.5)$$

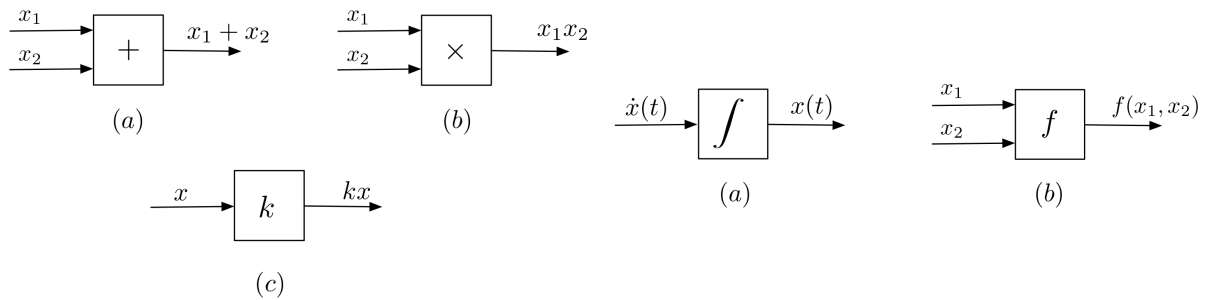
where  $A$  and  $B$  are matrices of dimensions  $n \times n$  and  $n \times m$  respectively.



# State transformations

## 3.1 Definition

The block diagram of a dynamical system is a visual representation of that system, necessarily containing  $n$  integrators whose outputs are the  $n$  state variables.



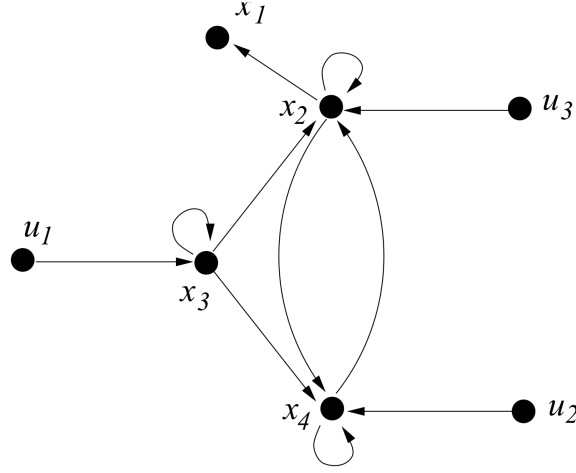
The graph of a dynamical system contains as nodes the inputs and states of the system, and its edges are the relations between those quantities. The construction rules of the graph of a dynamical system are the following:

- The  $n + m$  nodes are the  $n$  state variables  $x_i$  and the  $m$  inputs  $u_j$ ;
- there is an oriented edge from  $x_i$  (or  $u_k$ ) to  $x_j$  if the variable  $x_i$  (or  $u_k$ ) appears explicitly in the equation of the derivative  $\dot{x}_j$ .

Example for a DC electric machine: the state space model is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = J^{-1}(-h(x_2) + K_m x_3 x_4 + u_3) \\ \dot{x}_3 = L_s^{-1}(-R_s x_3 + u_1) \\ \dot{x}_4 = L_r^{-1}(-R_r x_4 - K_e x_2 x_3 + u_2) \end{cases} \quad (3.1)$$

and its graph representation is



## 3.2 Linear state transformation

For a dynamical system  $\dot{x} = f(x, u)$ , a linear state transformation is a linear mapping  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  that is bijective and transforms the state of the system  $x \in \mathbb{R}^n$  into a new state  $z \in \mathbb{R}^n$  following the rule  $z = Tx$ , where  $T \in \mathbb{R}^{n \times n}$  is an invertible matrix. The relation between the two systems is

$$\begin{cases} \dot{x} = f(x, u) \\ \dot{z} = g(z, u) \end{cases} \implies \begin{cases} z \triangleq T^{-1}x \\ g(z, u) \triangleq Tf(T^{-1}z, u) \end{cases} \quad (3.2)$$

For a linear system, we have

$$\dot{z} = Fz + Gu \quad F \triangleq TAT^{-1} \quad G \triangleq TB \quad (3.3)$$

## 3.3 Nonlinear state transformation

Let  $U, V$  be two open subsets of  $\mathbb{R}^n$ . A nonlinear state transformation is a mapping  $T : U \rightarrow V$  that transforms the state of the system  $x \in U$  into a new state  $z \in V$ :  $z = T(x)$  and that has the following properties:

- $T$  is bijective and has an inverse function  $T^{-1} : V \rightarrow U$  such that  $x = T^{-1}(z)$ ;
- $T$  and  $T^{-1}$  are of class  $\mathcal{C}^1$ , i.e. continuously differentiable.

→ Note: The state transformation is said to be global if  $U = V = \mathbb{R}^n$ .

Such a transformation  $T$  is called a diffeomorphism, and the new state space is

$$\dot{z} = \frac{\partial T}{\partial x} \dot{x} = \frac{\partial T}{\partial x} f(x, u) \iff f(x, u) \triangleq \left[ \frac{\partial T^{-1}}{\partial z} g(z, u) \right]_{z=T(x)} \quad (3.4)$$

**Lemma 3.1.** • If the jacobian matrix  $\partial T / \partial x$  is nonsingular at  $x_0$ , then, by the inverse function theorem, there is a neighbourhood  $U$  of  $x_0$  such that the mapping  $T$  restricted to  $U$  is a diffeomorphism on  $U$ .

- $T$  is a global diffeomorphism iff
  1.  $\frac{\partial T}{\partial x}$  is a nonsingular for every  $x \in \mathbb{R}^n$ ;
  2.  $\lim_{\|x\| \rightarrow \infty} \|T(x)\| = \infty$ .

### 3.4 Triangular system

**Definition 3.2.** A single input dynamical system is triangular if there is a state variable  $x_i$  such that the shortest path from  $u$  to  $x_i$  in the graph of the system is of length  $n$ .

We can thus renumber the state variables such that the system is expressed as

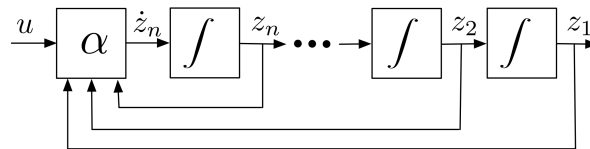
$$\begin{aligned}
 \dot{x}_1 &= g_1(x_1, x_2) \\
 &\vdots \\
 \dot{x}_i &= g_i(x_1, \dots, x_{i+1}) \\
 &\vdots \\
 \dot{x}_{n-1} &= g_{n-1}(x_1, \dots, x_n) \\
 \dot{x}_n &= g_n(x_1, \dots, x_n, u)
 \end{aligned} \tag{3.5}$$

### 3.5 Brunovsky canonical form

**Definition 3.3.** A single input dynamical system can be written in Brunovsky canonical form if there exists a state transformation  $T : U \rightarrow V$  and an open interval  $W \subseteq \mathbb{R}$  such that, in the new state variables  $z = T(x)$ , the system takes on the following particular triangular form:

$$\begin{aligned}
 \dot{z}_1 &= z_2 \\
 \dot{z}_2 &= z_3 \\
 &\vdots \\
 \dot{z}_n &= \alpha(z_1, \dots, z_n, u)
 \end{aligned} \tag{3.6}$$

where the function  $\alpha$  is continuous and invertible according to  $u$  over  $W$  for all  $z \in V$ . The block diagram of the Brunovsky canonical form is



**Lemma 3.4.** A triangular dynamical system described by the state-space model (3.5) can be put under Brunovsky canonical form around  $(x_0, u_0)$  if the inequalities

$$\begin{cases} \frac{\partial g_i}{\partial x_{i+1}} \neq 0 & i = 1, \dots, n-1 \\ \frac{\partial g_n}{\partial u} \neq 0 \end{cases} \tag{3.7}$$

**Lemma 3.5.** A control-affine system  $\dot{x} = f(x) + g(x)u$  with  $x \in \mathbb{R}^n, u \in \mathbb{R}$  can be written in Brunovsky form in a domain  $U \subseteq \mathbb{R}^n$  if there exists a state transformation  $z = T(x)$  that fulfills the following conditions:

- $T_{i+1}(x) = \frac{\partial T_i}{\partial x} f(x)$ , for  $i = 1, \dots, n-1$ ;
- $\frac{\partial T_i}{\partial x} g(x) = 0$ , for  $i = 1, \dots, n-1$ ;
- $\frac{\partial T_n}{\partial x} g(x) \neq 0$

for every  $x \in U$ .

# Equilibria and invariant sets

In this chapter, we assume that  $f$  is locally Lipschitz continuous on an open set  $\Omega \subseteq \mathbb{R}^n$ .

**Definition 4.1.** The pair  $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$  is called an equilibrium of the system  $\dot{x} = f(x, u)$  if  $f(\bar{x}, \bar{u}) = 0$ .

**Definition 4.2.** The equilibrium  $(\bar{x}, \bar{u})$  is said to be isolated if there exists a neighbourhood of  $\bar{x}$  that contains no other vector  $x$  such that  $f(x, \bar{u}) = 0$ .

## 4.1 Equilibria of linear systems

$$\dot{x} = Ax + Bu \quad (4.1)$$

**Theorem 4.3.** If the matrix  $A$  is regular, then for each  $\bar{u}$ , the pair  $(-A^{-1}B\bar{u}, \bar{u})$  is an isolated equilibrium.

If the matrix  $A$  is singular, the system (4.1) has a continuum of non-isolated equilibria provided that  $B\bar{u} \in \text{Im}(A)$ . Those equilibria are the solutions of the system  $A\bar{x} = -B\bar{u}$ , forming an affine space. On the other side, for each  $\bar{u}$  such that  $B\bar{u} \notin \text{Im}(A)$ , the system does not have any equilibrium.

## 4.2 Invariant sets

**Definition 4.4.** A set  $\mathcal{X} \times U \subseteq \mathbb{R}^n \times \mathbb{R}^m$  is said to be (positively) invariant for the dynamical system  $\dot{x} = f(x, u)$  if, for all  $x_0 \in \mathcal{X}$  and for all input signal  $t \rightarrow u(t) \in U$ , the trajectory  $t \rightarrow x(t, x_0, u(t))$  remains in  $\mathcal{X}$  for all  $t \geq t_0$  whenever it is defined.

**Definition 4.5.** An outward normal vector to  $\mathcal{X} \subseteq \mathbb{R}^n$  at  $x \in \partial\mathcal{X}$  is a vector  $n \in \mathbb{R}^n$  such that  $n = \lambda(y - x)$ , where  $\lambda > 0$  and  $y$  is the center of an open ball  $B \subseteq \mathbb{R}^n$  such that  $x \in \partial B$  and  $B \cap \mathcal{X} = \emptyset$ ; if no such open ball exists,  $\mathcal{X}$  has no outward normal vector at  $x$ .

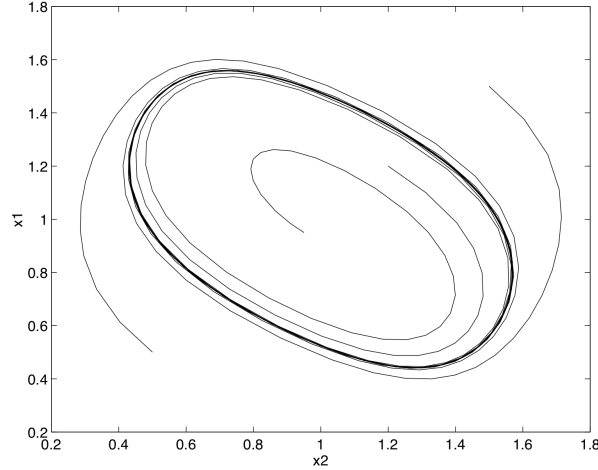
**Theorem 4.6** (Bony's theorem). Let  $f$  be a locally Lipschitz continuous vector field defined on an open set  $\Omega \subseteq \mathbb{R}^n$ , and  $\mathcal{X}$  a closed set of  $\Omega$ . If  $\langle f(x), n(x) \rangle \leq 0$  for every  $x \in \partial\mathcal{X}$ , and every vector  $n(x)$  is outward normal to  $\mathcal{X}$  at  $x$ , then  $\mathcal{X}$  is (positively) invariant for  $f$ .

→ Note: no condition has to be verified at a point where  $\mathcal{X}$  does not have an outward normal vector.

### 4.3 Periodic orbits

A periodic orbit is such that it is arising from a trajectory of the dynamical system, verifying  $x(t) = x(t + T)$  for all  $t$  and for some  $T > 0$ <sup>1</sup>. The infimum of possible values for  $T$  is called the period of the trajectory.

We denote  $x(t, x_0, \bar{u})$  as the solution at time  $t$  with  $x(t_0) = x_0$  and a constant input  $u(t) = \bar{u}$ .



**Definition 4.7.** The point  $z$  is called a limit point of  $y$  for the dynamical system subject to a constant input  $\bar{u}$  if there exists a real sequence  $\{t_n\}_n$  such that  $t_n \rightarrow \infty$  when  $n \rightarrow \infty$  and  $\lim_{n \rightarrow \infty} x(t_n, y, \bar{u}) = z$ .

**Definition 4.8.** A limit cycle is a closed orbit  $\gamma$  such that at least one point of  $\gamma^2$  is a limit point of at least another point of the phase plane not in  $\gamma$ .

→ Note: These definitions are only valid in  $\mathbb{R}^2$ .

**Theorem 4.9** (Bendixson-Dulac). Let  $D$  be a simply connected domain in  $\mathbb{R}^2$ . If the divergence of  $f^3$  is not identically zero and does not change sign in  $D$ , then  $D$  does not contain any closed orbit.

**Theorem 4.10** (Poincaré-Bendixson). If  $E$  is a closed and bounded subset of  $\mathbb{R}^2$ , invariant for the system  $\dot{x} = f(x, u)$ , and if  $\gamma$  is an orbit starting in  $E$ , then:

- either  $\gamma$  converges to an equilibrium (which is the unique limit point of  $\gamma$ );
- or  $\gamma$  converges to a periodic orbit (which is the set of all limit points of  $\gamma$ ).

This theorem can be used to prove the existence of a limit cycle:

1. Find a compact invariant set (proved by showing that on the border of this set, the vector field points inwards);
2. If there is no equilibrium in this set, it must contain a limit cycle or only periodic trajectories.

<sup>1</sup>Equilibria are trivial periodic orbits.

<sup>2</sup>Implying that they all are.

<sup>3</sup> $\text{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$

# Local analysis of autonomous dynamical systems

A dynamical system is said to be autonomous if the input is constant:

$$\dot{x} = f(x, \bar{u}) \quad (5.1)$$

## 5.1 Linear planar systems

Let us consider the linear planar system such as (4.1) with constant input  $u = \bar{u}$ . Let  $\bar{x}$  be an equilibrium point corresponding to  $\bar{u}$ . We will use the state transformation  $z = M^{-1}(x - \bar{x})$ . We obtain the linear system

$$\dot{z} = A'z \quad A' = M^{-1}AM \quad (5.2)$$

As  $A$  and  $A'$  have the same eigenvalues, we can choose  $A'$  to have a canonical form:

- Two distinct real eigenvalues or double real eigenvalue ( $\lambda_1 = \lambda_2$ ) with a geometric multiplicity equal to 2.

$$A' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (5.3)$$

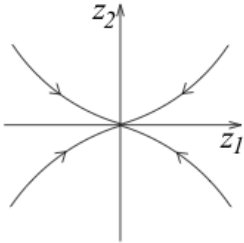
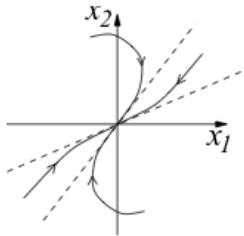
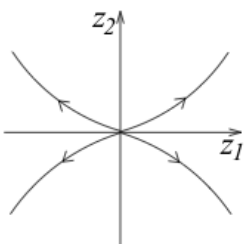
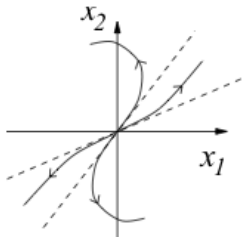
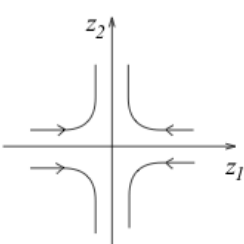
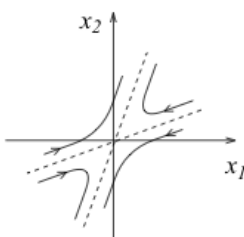
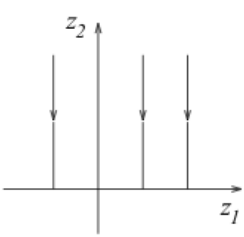
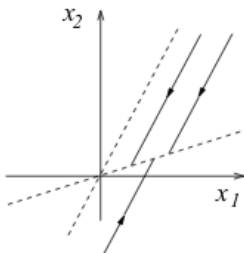
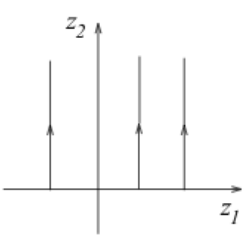
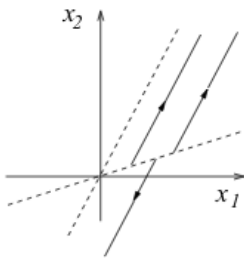
- Double real eigenvalue of geometric multiplicity equal to 1.

$$A' = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad (5.4)$$

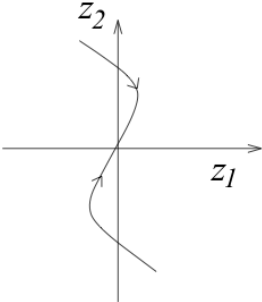
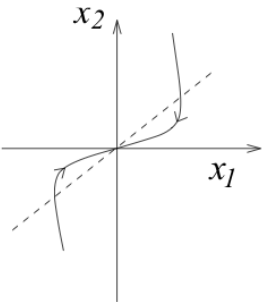
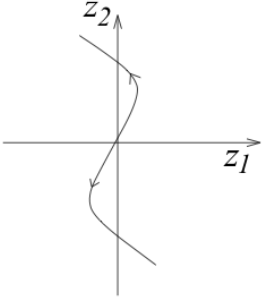
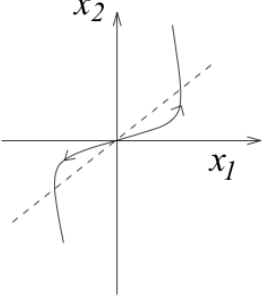
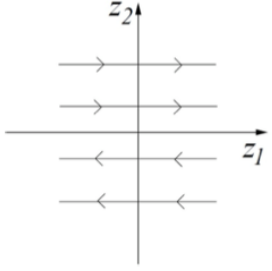
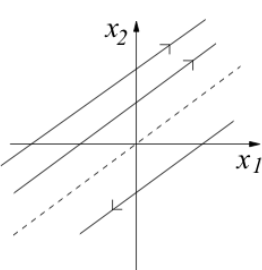
- Two complex conjugate eigenvalues  $\alpha \pm \omega i$ .

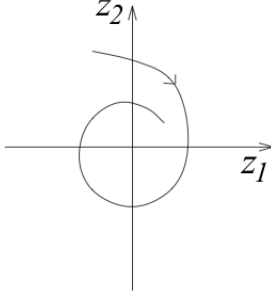
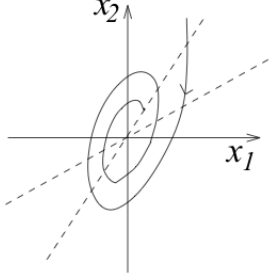
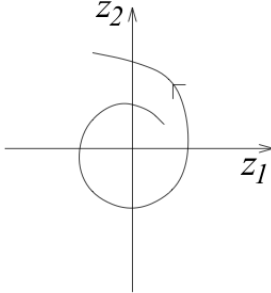
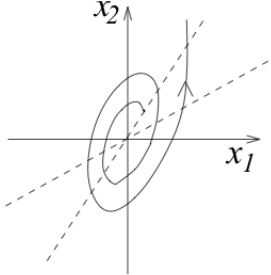
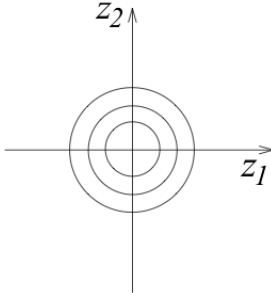
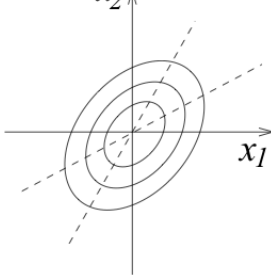
$$A' = \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix} \quad \omega > 0 \quad (5.5)$$

And the types of equilibrium for a linear planar system are:

Type of equilibrium	Behaviour of orbits $(z_1, z_2)$	Behaviour of orbits $(x_1, x_2)$	Conditions on the eigenvalues
Attractive node			$\lambda_2 \leq \lambda_1 < 0$
Repulsive node			$0 < \lambda_1 \leq \lambda_2$
Saddle point			$\lambda_1 < 0 < \lambda_2$
Non-isolated attractive equilibrium			$\lambda_1 = 0,$ $\lambda_2 < 0$
Non-isolated repulsive equilibrium			$\lambda_1 = 0,$ $\lambda_2 > 0$

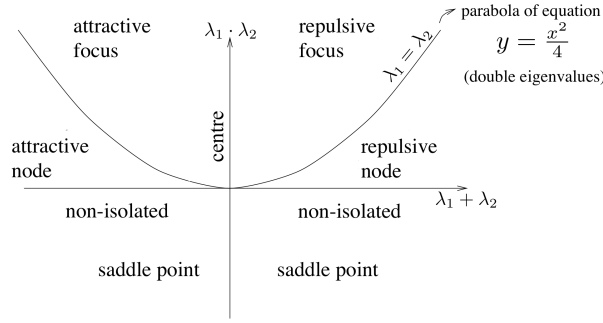


Type of equilibrium	Behaviour of orbits $(z_1, z_2)$	Behaviour of orbits $(x_1, x_2)$	Conditions on the eigenvalues
Degenerate attractive node			$\lambda < 0$ (defective)
Degenerate repulsive node			$\lambda > 0$ (defective)
Non-isolated equilibrium			$\lambda = 0$ (defective)

Type of equilibrium	Behaviour of orbits $(z_1, z_2)$	Behaviour of orbits $(x_1, x_2)$	Conditions on the eigenvalues
Attractive focus			$\lambda_{1,2} = \alpha \pm \omega i$ $\alpha < 0, \omega \neq 0$
Repulsive focus			$\lambda_{1,2} = \alpha \pm \omega i$ $\alpha > 0, \omega \neq 0$
Centre			$\lambda_{1,2} = \pm \omega i$ $\omega \neq 0$

- Note: if one of the eigenvalues is zero, the equilibrium is not isolated.

**Definition 5.1.** If all trajectories of a linear system converge to an equilibrium, we say that it is an attractive equilibrium. It is a repulsive equilibrium if they all diverge to infinity (save for the equilibrium itself.)



- An attractive (resp. repulsive) equilibrium will remain attractive (resp. repulsive) after a perturbation and a saddle point will remain a saddle point. Such equilibria are called structurally stable. However, a center equilibrium (zero real part) is never structurally stable: even a small perturbation of the matrix  $A$  can shift eigenvalues away from the imaginary axis, and the corresponding trajectories then converge to the equilibrium or diverge from it.

**Definition 5.2.** If all eigenvalues of  $A$  have nonzero real part, the equilibrium of  $\dot{x} = Ax$  is said to be hyperbolic.

## 5.2 Linearisation of nonlinear systems

We assume the existence of an equilibrium  $(\bar{x}, \bar{u})$  for the nonlinear system  $\dot{x} = f(x, \bar{u})$ . The Taylor expansion is

$$\dot{x} = f(\bar{x}, \bar{u}) + \left( \frac{\partial f(x, \bar{u})}{\partial x} \right)_{\bar{x}} (x - \bar{x}) + \mathcal{O}(\|x - \bar{x}\|^2) \quad (5.6)$$

Thus, the linear approximation of the system is, for  $\tilde{x} = x - \bar{x}$ ,

$$\dot{\tilde{x}} = \left( \frac{\partial f(x, \bar{u})}{\partial x} \right)_{\bar{x}} \tilde{x} \quad (5.7)$$

We define  $A \triangleq \left( \frac{\partial f(x, \bar{u})}{\partial x} \right)_{\bar{x}}$  as the Jacobian matrix of  $f$  at the equilibrium.

**Definition 5.3.** The equilibrium  $(\bar{x}, \bar{u})$  of the nonlinear system is said to be hyperbolic if all the eigenvalues of the jacobian matrix  $A$  have a nonzero real part.

**Definition 5.4.** Two dynamical systems are topologically conjugate if there exists a homeomorphism, i.e. a continuous bijection whose inverse is also continuous, that maps the trajectories of the first system to the trajectories of the second one in a time respecting way. That means that the trajectories of  $\dot{x} = f(x)$  on a domain  $D$  and  $\dot{y} = g(y)$  on a domain  $E$  are topologically conjugate through the homeomorphism  $\phi : D \rightarrow E$  if every curve  $[0, t_0] \rightarrow D : t \rightarrow x(t)$  is a trajectory of the system  $f$  iff the corresponding curve  $[0, t_0] \rightarrow E : t \rightarrow \phi(x(t))$  is a trajectory of the system  $g$ .

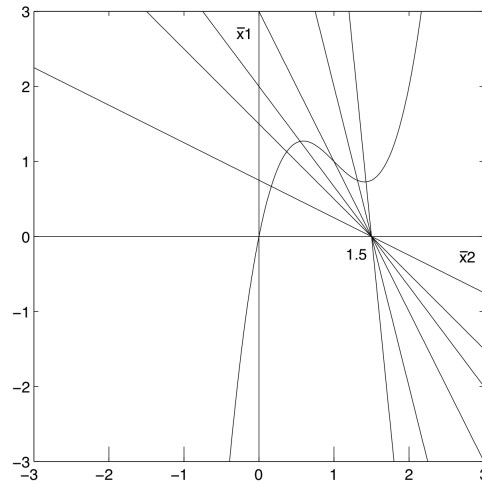
**Theorem 5.5.** If the equilibrium  $(\bar{x}, \bar{u})$  is hyperbolic, then the trajectories of the nonlinear system in a neighbourhood of the equilibrium  $(\bar{x}, \bar{u})$  are topologically conjugate to those of the linear approximation (5.7). Specifically, there exists a neighbourhood  $X$  of  $\bar{x}$ , a neighbourhood  $\tilde{X}$  of 0, and a homeomorphism  $\phi : X \rightarrow \tilde{X}$  with  $\phi(\bar{x}) = 0$  such that if  $t \rightarrow x(t)$  is a trajectory of the nonlinear system contained in  $X$ , then  $t \rightarrow \phi(x(t))$  is a trajectory of the linear system.

That means that if the equilibrium is a node/focus (attractive or repulsive) or a saddle point (but not a centre) in the linearised system, then the linearised system is a good representation for the local behaviour of the nonlinear trajectories around the equilibrium as well. However, this theorem is local, and the higher-order terms are needed to conclude in the case of a non-hyperbolic equilibrium.

# Bifurcations

Bifurcation theory looks at the impact of the value  $\bar{u}$  on the nature and number of equilibria.

## 6.1 Hopf bifurcation



Depending on the slope of the straight line, the characterization of the equilibrium changes: it is an attractive focus, then changes to repulsive focus and goes back to being an attractive focus.

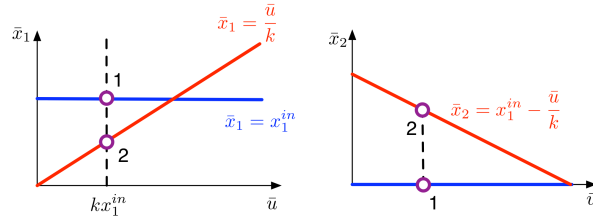
**Theorem 6.1.** Suppose that a system has a family of isolated equilibria  $(\bar{x}, \bar{u})$  parametrized by  $\bar{u}$ . Suppose that there exists a value  $\bar{u}^*$  such that a pair of eigenvalues of the Jacobian matrix evaluated in this equilibrium have a zero real part and a nonzero imaginary part. These values depend continuously on  $\bar{u}$ , at least in the neighbourhood of  $\bar{u}^*$ , and are denoted by

$$\lambda_i(\bar{u}) = \alpha(\bar{u}) \pm i\omega(\bar{u}) \quad (6.1)$$

Suppose also that  $\frac{d\alpha(\bar{u}^*)}{d\bar{u}} > 0$ . Thus, for  $\bar{u}$  close enough to  $\bar{u}^*$ , the equilibrium is attractive for  $\bar{u} < \bar{u}^*$  and repulsive for  $\bar{u} > \bar{u}^*$ .

Then, there generically exists either an attractive closed orbit (i.e. limit cycle) for all  $\bar{u}^* < \bar{u} < \bar{u}^* + \varepsilon$  or a repulsive closed cycle for  $\bar{u}^* - \varepsilon < \bar{u} < \bar{u}^*$  (for some  $\varepsilon > 0$ ) unique in the neighbourhood of the equilibrium.

## 6.2 Transcritical bifurcation

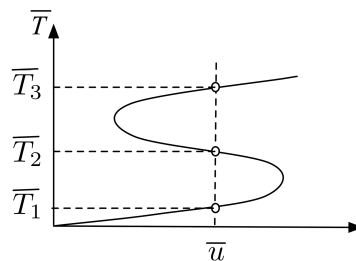


In this example, the first equilibrium is attractive if  $\bar{u} > kx_1^{in}$  and is a saddle point otherwise. The second however is a saddle point when the above condition is met and attractive when it is not.

A transcritical bifurcation is thus such that the characterization of the two equilibria switch when passing a certain threshold value of  $\bar{u}$ .

## 6.3 Saddle-node/Fold bifurcation

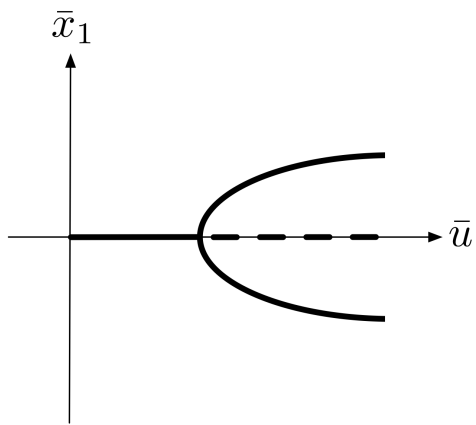
For small values of  $\bar{u}$ , the system has a single equilibrium. Then, for a critical value of  $\bar{u}$ , the system exhibits two more equilibrium values (one a saddle point and the other attractive). By further increasing  $\bar{u}$ , we cross a new critical value beyond which the system has only a single equilibrium that is also attractive.



As the input  $\bar{u}$  is slowly modified from low to high values, the state of the system, initially following the bottom line of equilibria, goes through a brutal change at the rightmost bifurcation, where it jumps to a different equilibrium. It is called a catastrophe. As the input decreases again to low values, the catastrophe happens the other way. This is a hysteresis.

## 6.4 Pitchfork bifurcation

A pitchfork bifurcation is the split, for some value  $\bar{u}^*$  of the bifurcation parameter, of a single attractive (resp. repulsive) equilibrium into three equilibria, one being repulsive (resp. attractive) and the other two being attractive (resp. repulsive).



# Stability of equilibria

In this chapter we assume that the locally Lipschitz continuous vector field  $f(\cdot, \bar{u})$  is defined on an open set  $\Omega \subseteq \mathbb{R}^n$  for a fixed value of  $\bar{u}$ , where the trajectories are defined and stay in  $\Omega$  for all positive times.

## 7.1 Definitions

**Definition 7.1.** The equilibrium  $(\bar{x}, \bar{u})$  is said to be stable if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x_0 \in \Omega : \|x(t_0) - \bar{x}\| < \delta \implies \|x(t, x(t_0), \bar{u}) - \bar{x}\| < \varepsilon \quad \forall t \geq t_0 \quad (7.1)$$

That means that an equilibrium is stable if the trajectories remain arbitrarily close to it, provided that they start close enough from this equilibrium.

**Definition 7.2.** The equilibrium  $(\bar{x}, \bar{u})$  is said to be attractive if

$$\exists \delta > 0 \text{ such that } \|x(t_0) - \bar{x}\| < \delta \implies \lim_{t \rightarrow \infty} \|x(t, x(t_0), \bar{u}) - \bar{x}\| = 0 \quad (7.2)$$

An attractive equilibrium  $\bar{x}$  is thus a point to which each solution  $x$  converges provided that it starts close enough to  $\bar{x}$ .

→ Note: stability and attractiveness do not imply each other.

**Definition 7.3.** The equilibrium  $(\bar{x}, \bar{u})$  is said to be asymptotically stable if it is both stable and attractive. The set of points  $x_0$  for which the trajectory  $x(t, x_0, \bar{u})$  converges to  $\bar{x}$  is called the basin of attraction of the asymptotically stable equilibrium.

**Definition 7.4.** The equilibrium  $(\bar{x}, \bar{u})$  is said to be exponentially stable if

$$\exists a, b, \delta > 0 \text{ such that } \|x(t_0) - \bar{x}\| < \delta \implies \|x(t, x(t_0), \bar{u}) - \bar{x}\| \leq a \|x(t_0) - \bar{x}\| e^{-bt} \quad \forall t \geq t_0 \quad (7.3)$$

→ Note: exponential stability implies asymptotic stability.

→ Note: for a linear system, an attractive equilibrium and center are both stable, while a saddle or repulsive equilibrium is unstable. Attractive equilibrium is also exponentially stable, thus asymptotically stable: these three notions coincide for linear systems.



## 7.2 Lyapunov's first method

**Theorem 7.5.** • If the equilibrium is attractive in the linearised system, i.e. all eigenvalues of the Jacobian matrix have a negative real part, then the equilibrium  $(\bar{x}, \bar{u})$  is exponentially stable.

- If the equilibrium is repulsive or a saddle point in the linearised system, i.e. the Jacobian matrix has at least one eigenvalue with a positive real part, then the equilibrium  $(\bar{x}, \bar{u})$  is unstable.

This theorem does not conclude on the stability of a non-hyperbolic equilibrium.

## 7.3 Lyapunov's second method

**Theorem 7.6.** The equilibrium  $(\bar{x}, \bar{u})$  of the system  $\dot{x} = f(x, \bar{u})$ , where  $f$  is locally Lipschitz continuous on an open set  $\Omega \subseteq \mathbb{R}^n$  is stable if there exists a continuously differentiable function  $V : \Omega \rightarrow \mathbb{R}$  with the following properties:

- $\Omega \subseteq \mathbb{R}^n$  is a neighbourhood of  $\bar{x}$ ;
- $V(x) > V(\bar{x}) \forall x \in \Omega \setminus \{\bar{x}\}$ , i.e.  $V$  has a strict minimum point at  $\bar{x}$ ;
- $\dot{V}(x) \leq 0 \forall x \in \Omega \setminus \{\bar{x}\}$ .

That means that a sufficient condition for the equilibrium  $(\bar{x}, \bar{u})$  to be stable is to have a positive-definite function  $V - V(\bar{x})$  whose temporal derivative  $\dot{V}$  along trajectories is negative-semidefinite in a neighbourhood of  $\bar{x}$ , the temporal derivative being defined as

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x} \dot{x} = \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(x, \bar{u}) = \langle \nabla V(x(t)), f(x(t), \bar{u}) \rangle \quad (7.4)$$

**Theorem 7.7.** The equilibrium  $(\bar{x}, \bar{u})$  of the system  $\dot{x} = f(x, \bar{u})$  is asymptotically stable if there exists a continuously differentiable function  $V : \Omega \rightarrow \mathbb{R}$  with the following properties:

- $\Omega \subseteq \mathbb{R}^n$  is a neighbourhood of  $\bar{x}$ ;
- $V(x) > V(\bar{x}) \forall x \in \Omega \setminus \{\bar{x}\}$ , i.e.  $V$  has a strict global minimum point at  $\bar{x}$ ;
- $\dot{V}(x) < 0 \forall x \in \Omega \setminus \{\bar{x}\}$ .

**Theorem 7.8.** The basin of attraction of an asymptotically stable equilibrium is an open, connected, invariant set; and its boundary is formed by trajectories.

**Definition 7.9.** Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$  be locally Lipschitz continuous and  $\bar{x} \in \mathbb{R}^N$  be an equilibrium of  $f$ . The equilibrium  $\bar{x}$  is said to be globally asymptotically stable if it is asymptotically stable and its basin of attraction is the whole state space  $\mathbb{R}^N$ .

**Definition 7.10.** A function  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  is said to be radially unbounded if  $\|x\| \rightarrow \infty$  implies  $V(x) \rightarrow \infty$ , i.e. for every  $M \in \mathbb{R}$ , there exists  $R \geq 0$  such that, for every  $x \in \mathbb{R}^N$ ,  $\|x\| \geq R$  implies  $V(x) \geq M$ .

**Theorem 7.11.** Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$  be locally Lipschitz continuous and  $\bar{x} \in \mathbb{R}^N$  be an equilibrium of  $f$ . If there exists a radially unbounded continuously differentiable function  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  such that, for every  $x \in \mathbb{R}^N \setminus \{\bar{x}\}$ ,  $V(x) > V(\bar{x})$  and  $\dot{V}_f(x) < 0$ , then  $\bar{x}$  is globally asymptotically stable.

### 7.3.1 LaSalle's invariance principle

**Definition 7.12.** A function  $x : \mathbb{R}^+ \rightarrow \mathbb{R}^N$  is said to asymptotically approach a nonempty set  $M \subseteq \mathbb{R}^N$  if

$$\lim_{t \rightarrow \infty} \underbrace{\inf_{y \in M} \|x(t) - y\|}_{=: \text{dist}(x(t), M)} = 0 \quad (7.5)$$

**Theorem 7.13.** Let  $F \subseteq \Omega$  be a compact (positively) invariant set for  $f$ . Let  $V : \Omega \rightarrow \mathbb{R}$  be a continuously differentiable and such that  $\dot{V}_f \leq 0$  for every  $x \in F$ . Let  $E := \{x \in F \mid \dot{V}_f(x) = 0\}$ . Let  $M$  be the largest (positively and negatively) invariant set in  $E$ . Then, every trajectory of  $f$  starting in  $F$  asymptotically approaches  $M$ .

**Corollary 7.14.** Let  $\bar{x}$  be an equilibrium of  $f$ . Let  $V : \Omega \rightarrow \mathbb{R}$  be continuously differentiable and such that, for every  $x \in \Omega \setminus \{\bar{x}\}$ ,  $V(x) > V(\bar{x})$  and  $\dot{V}(x) \leq 0$ . Let  $S := \{x \in \Omega \mid \dot{V}(x) = 0\}$ . If the equilibrium is the only trajectory fully contained (past and future) in  $S$ , then  $\bar{x}$  is asymptotically stable.

## 7.4 The energy as a Lyapunov function

→ Note: the Lyapunov function  $V(x)$  often has the dimensions of an energy.

### 7.4.1 In mechanical systems

The general equation in a mechanical system<sup>1</sup> is

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) + k(q) + h(\dot{q}) = G\bar{u} \quad (7.6)$$

Here we assume the kinematic matrix  $G$  to be constant. The Lyapunov function taken for this general system is

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + E_p(q) - q^T G\bar{u} \quad (7.7)$$

$$\dot{V}(q, \dot{q}) = -\dot{q}^T h(\dot{q}) \quad (7.8)$$

The first term is the kinetic energy, the second is the potential energy and the third is the work realized by the applied forces and torques.

### 7.4.2 In electrical systems

For electrical systems, there isn't a general formula of the state-space model. The Lyapunov function will however still have the dimensions of an energy, with different terms:

- Inductance:  $E = \frac{1}{2}Li^2$
- Capacitance:  $E = \frac{1}{2}Cv^2$
- Resistance:  $E = Ri^2$

---

<sup>1</sup>See chapter 9.2

## 7.5 Linear systems

Let us study once again the system (4.1) with an equilibrium  $(\bar{x}, \bar{u})$ . We define the Lyapunov function

$$V(x) = (x - \bar{x})^T P (x - \bar{x})$$

where  $P$  is a symmetric positive-definite matrix. Its derivative is  $\dot{V}(x) = -(x - \bar{x})^T Q (x - \bar{x})$ , with  $-Q = A^T P + PA$ .

$$V(x) = (x - \bar{x})^T P (x - \bar{x}) \quad (7.9)$$

$$\dot{V}(x) = -(x - \bar{x})^T Q (x - \bar{x}) \quad -Q = A^T P + PA \quad (7.10)$$

**Theorem 7.15.** Let  $A$  be a real matrix of order  $n$ . For every positive-definite matrix  $Q$ , equation (7.10) owns a unique positive-definite solution  $P$  iff  $A$  is a Hurwitz matrix, i.e. all its eigenvalues have a negative real part.

## 7.6 Bounded-input, bounded-state stability

We are here interested in an input signal  $u(t)$  that is bounded and close to  $\bar{u}$ . We need to analyse this case because a constant signal is not feasible in reality. We study the linear system

$$\dot{x} = Ax + Bu \quad x(t_0) = x_0 \quad (7.11)$$

The trajectory of the system is

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^t e^{A(t-\tau)} B u(\tau) d\tau \quad (7.12)$$

The equilibrium is  $(\bar{x}, \bar{u}) = (0, 0)$ . It is asymptotically stable iff the matrix  $A$  is a Hurwitz matrix. This would mean that  $\|e^{At}\|$  is bounded for all  $t \geq t_0$  and there are non-negative constants  $k$  and  $\lambda$  such that

$$\|e^{A(t-t_0)}\| \leq k e^{-\lambda(t-t_0)} \quad (7.13)$$

and thus

$$\|x(t)\| \leq k e^{-\lambda(t-t_0)} \|x_0\| + \frac{k\|B\|}{\lambda} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\| \quad (7.14)$$

This means that a bounded input  $u(t)$ , however big its magnitude is, generates a bounded state  $x(t)$ , and the effect of the initial condition  $x_0$  fades away with time.

**Theorem 7.16.** If the equilibrium  $(\bar{x}, \bar{u})$  of the linear system is asymptotically stable,

- there are three nonnegative constants  $c_1, c_2, c_3$  such that, for each initial state  $x_0$  with  $\|x_0 - \bar{x}\| < c_1$  and each input signal  $u$  with  $\|u(t) - \bar{u}\| < c_2 \forall t \geq t_0$ , the solution  $x$  is bounded:  $\|x(t) - \bar{x}\| < c_3 \forall t \geq t_0$ ;
- there is a nonnegative constant  $c_0$  and a continuous function  $\alpha : [0, a) \rightarrow [0, \infty)$  passing through the origine, i.e.  $\alpha(0) = 0$ , and increasing such that, for each input signal  $u$  with  $\|u(t) - \bar{u}\| < c_0 \forall t \geq t_0$ , the ultimate bound on  $x$  is an increasing function of the bound on  $u$  :

$$\limsup_{t \rightarrow \infty} \|x(t)\| \leq \alpha(\|u\|_{\mathcal{L}_\infty}) \quad (7.15)$$

**Theorem 7.17.** If  $f$  is globally continuously differentiable and globally Lipschitz continuous, and if the equilibrium  $(\bar{x}, \bar{u})$  is globally exponentially stable, then for each initial condition  $x_0$ , and each input signal  $u$ , the solution  $x$  is bounded.

# Controllability and trajectory planning

**Definition 8.1.** For the dynamical system  $\dot{x} = f(x, u)$ , the final state  $x_f \in \mathbb{R}^n$  is reachable from the initial state  $x_0 \in \mathbb{R}^n$  within time  $T$  if there exists an input function  $u : [t_0, t_0 + T] \rightarrow \mathbb{R}^m$  such that  $x(t_0) = x_0$  and  $x(t_0 + T) = x_f$ .

## 8.1 Controllability of LTI systems

**Definition 8.2.** The system  $\dot{x} = Ax + Bu$ , for  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ , is controllable if, for each initial state  $x_0$ , it is possible to reach any other final state  $x_f$  within any positive time  $T$ .

**Theorem 8.3.** The LTI system  $\dot{x} = Ax + Bu$  is completely controllable iff one of the two following criteria is satisfied:

- The matrix  $\mathcal{C} = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$  has full rank; The rank of the matrix  $[sI - A \ B]$  is equal to  $n$  for each  $s \in \mathbb{C}$

If the controllability matrix has rank  $d < n$ , we define a matrix  $T = (T_a \ T_b)$  such that  $T_a$  contains  $d$  linearly independent columns of  $\mathcal{C}$  and  $T_b$  completes the matrix by  $n - d$  vectors independent of the columns of  $T_a$ . Its inverse is  $T^{-1} := \begin{pmatrix} U_a \\ U_b \end{pmatrix}$ , where the matrices  $U_a, U_b$  are chosen such that

$$T^{-1}T = \begin{pmatrix} U_a T_a & U_a T_b \\ U_b T_a & U_b T_b \end{pmatrix} = \begin{pmatrix} I_d & 0 \\ 0 & I_{n-d} \end{pmatrix} \quad (8.1)$$

From that, we define a state transformation

$$z = \begin{pmatrix} z_a \\ z_b \end{pmatrix} = (U_a x \ U_b x) \quad (8.2)$$

The new state-space model is

$$\dot{z}_a = U_a A T_a z_a + U_a A T_b z_b + U_a B u \quad (8.3)$$

$$\dot{z}_b = U_b A T_b z_b \quad (8.4)$$

The part  $z_b$  is the non controllable part of the system, it is not influenced by the input  $u$ .

## 8.2 Controllability of nonlinear systems

**Definition 8.4.** The system  $\dot{x} = f(x, u)$  for  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$  is locally accessible from the initial state  $x_0$  if for any time  $T$ , the set of states reachable from  $x_0$  within  $T$  contains a nonempty open set.

**Definition 8.5.** The system  $\dot{x} = f(x, u)$  for  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$  is locally controllable from the initial state  $x_0$  if for any time  $T$ , the set of states reachable from  $x_0$  within  $T$  contains a neighbourhood of  $x_0$ .

We can also define the global controllability, which requires that the whole state space is reachable from any initial condition.

→ Note: local accessibility, local controllability and global controllability are equivalent for linear systems.

## 8.3 Drawbacks of linearisation

**Theorem 8.6.** Let us consider the linearisation of the system  $\dot{x} = f(x, u)$  around an equilibrium  $(\bar{x}, \bar{u})$ :

$$\dot{x} = Ax + Bu \quad A = \left( \frac{\partial f}{\partial x} \right)_{(\bar{x}, \bar{u})} \quad B = \left( \frac{\partial f}{\partial u} \right)_{(\bar{x}, \bar{u})} \quad (8.5)$$

If the linearised system is controllable, then for each  $\varepsilon > 0$ , the set of states reachable from  $\bar{x}$  within time  $T$  with inputs  $u(t)$  such that  $\|u(t) - \bar{u}\| < \varepsilon$ , contains a neighbourhood of  $\bar{x}$ .

We now define a new operator, the Lie bracket of two vector fields:

$$[g_1, g_2] := \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2 \quad (8.6)$$

**Theorem 8.7.** Let us consider the control-linear system<sup>1</sup> in some open set  $X \subseteq \mathbb{R}^n$ :

$$\dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m \quad (8.7)$$

for some analytic<sup>2</sup> vector fields  $g_1, \dots, g_m$ . Consider the set of all vector fields  $g_1, \dots, g_m$  and their repeated Lie brackets. Consider also the vector space generated by all these vector fields evaluated at a state  $x_0$ . This vector space is of dimension  $n$  iff the system is locally controllable from  $x_0$ . Moreover, if this condition is met everywhere in the state space  $X$ , then the system is globally controllable.

**Theorem 8.8.** Let us consider the system in  $X \subseteq \mathbb{R}^n$ :

$$\dot{x} = f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m \quad (8.8)$$

for some analytic vector fields  $f, g_1, \dots, g_m$ . Consider the set of all vector fields  $f, g_1, \dots, g_m$  and those obtained by repeated Lie brackets. Consider also the vector space generated by those vector fields evaluated at a state  $x_0$ . This vector space is of dimension  $n$  iff the system is locally accessible from  $x_0$ .

<sup>1</sup>Control-linear means linear in every input.

<sup>2</sup>Analytic means that all coordinates of all fields  $g_i$  have a Taylor series that converges in a neighbourhood, i.e. all derivatives of all order exist.

## 8.4 Trajectory planning

We work here with the following Brunovsky form of the nonlinear system:

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= \alpha(z) + \beta(z)u \quad \beta(z) \neq 0\end{aligned}\tag{8.9}$$

It is sufficient to define a polynomial trajectory for  $z_1$ :

$$z_1(t) = \sum_{i=0}^{2n-1} \lambda_i \left( \frac{t}{T} \right)^i\tag{8.10}$$

By calculating the successive derivatives of  $z_1(t)$ , we obtain the expressions of  $z_j(t)$ , for  $j = 2, \dots, n$ :

$$z_j(t) = \sum_{i=j-1}^{2n-1} \frac{i!}{(i-j+1)!} \frac{\lambda_i}{T^{j-1}} \left( \frac{t}{T} \right)^{1-j}\tag{8.11}$$

and we can find the values of coefficients  $\lambda_i$  by calculating those values for  $t = 0$  and  $t = T$ . Finally, the input we need to obtain the wanted results is

$$u(t) = \frac{\dot{z}_n(t) - \alpha(z(t))}{\beta(z(t))}\tag{8.12}$$

# Exercises

## 9.1 Electrical systems

The three main components are

- Resistance :  $v = Ri$ , does not induce an ODE.
- Capacitance :  $q = CV \rightarrow i_c = C \frac{dV_C}{dt}$
- Inductance :  $v_L = L \frac{di_L}{dt}$

The graph of the circuit is such that every component is an edge and every cable connecting them is a node. We have  $M$  edges and  $N$  nodes.

### 9.1.1 Resolution method

- Draw the graph of the circuit;
- Write the equations with Kirchhoff's laws, they give  $N - 1$  current equations and  $M + 1 - N$  voltage equations;
- Simplify equations;
- Write the final system.

A capacitance mesh is a mesh of more than 2 capacitances.

An inductance cut is a node with more than 2 inductances.

## 9.2 Mechanical systems

### 9.2.1 For a single body

For a single body, we first set an arbitrary orthonormal inertial basis. The position of the body is characterized by its coordinates and its orientation:

$$q := \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \quad (9.1)$$



The equations we use for the ODE system are

$$m\ddot{x} = F_x \quad (9.2)$$

$$m\ddot{y} = F_y \quad (9.3)$$

$$I\ddot{\theta} = T \quad (9.4)$$

Where  $m$  is the mass of the body,  $I$  its moment of inertia,  $F_x, F_y$  the forces applied to the body, and  $T$  the resultant of the torques.

The final equation to solve for that system is

$$J\ddot{q} + b(q) = B(q)u \quad (9.5)$$

where  $J$  is the diagonal and constant inertia matrix,  $b(q)$  represents the effect of the gravity,  $B(q)$  is a the kinematic matrix which depends nonlinearly on the state variables, and  $u$  contains the input variables, e.g. some forces applied to the body. The state-space model is thus

$$\begin{aligned} \dot{q} &= v \\ \dot{v} &= J^{-1}(-b(q) + B(q)u) \end{aligned} \quad (9.6)$$

## 9.2.2 Articulated mechanical systems

General procedure:

- Choose an inertial basis and  $N$  moving frames attached to the centers of mass of the  $N$  bodies:

We get a vector of dimension  $3N$ :

$$\xi = (x_1, y_1, \theta_1, \dots, x_N, y_N, \theta_N)^T \quad (9.7)$$

- Write the expressions of the constraints to which the motion of the system is subjected:

The algebraic relations are written such that

$$\Phi(\xi) = 0 \quad (9.8)$$

where  $\Phi : \Omega \rightarrow \mathbb{R}^p$  is a  $\mathcal{C}^1$  mapping, and  $p$  is the number of constraints. We choose a partition  $\xi = (q, \bar{q})$  such that the dimension  $\sigma$  of  $\bar{q}$  is equal to the rank of the Jacobian matrix of the mapping  $\Phi$ :

$$\sigma := \dim(\bar{q}) = \text{rank} \frac{\partial \Phi}{\partial \xi} \quad (9.9)$$

and such that we can express  $\bar{q}$  as a function of  $q$ :  $\bar{q} = \phi(q)$ . This removes the redundant coordinates  $\bar{q}$  of the system description. The number of degrees of freedom is thus  $\delta := 3N - \sigma$ .

- Write the equations of motion for each coordinate with the Lagrange method:

Including the bonding forces related to the constraints, we get

$$J\ddot{q} + b(q, \bar{q}) = B(q, \bar{q})u + w \quad (9.10)$$

$$\bar{J}\ddot{\bar{q}}(q, \bar{q}) = \bar{B}(q, \bar{q})u + \bar{w} \quad (9.11)$$

$$(9.12)$$

where  $w$  and  $\bar{w}$  represent the bonding forces that ensure that the constraints are satisfied at any time during the motion of the system. Their expressions are

$$w = -A(q)\lambda \quad \bar{w} = \lambda \quad (9.13)$$

where  $\lambda$  is the vector of Lagrange coefficients of dimension  $\sigma$ , and  $A(q)$  is the matrix of dimension  $\delta \times \sigma$  defined as

$$A(q) := \left( \frac{\partial \phi}{\partial q} \right)^T \quad (9.14)$$

- Remove the Lagrange coefficients and the redundant coordinates:

The final equation is

$$M(q)\ddot{q} + f(q, \dot{q}) + g(q) = G(q)u \quad (9.15)$$

The matrices in this formula are defined as follows:

$$M(q) := J + A(q)\bar{J}A^T(q) \quad (9.16)$$

$$f(q, \dot{q}) := A(q)\bar{J}\dot{A}^T(q)\dot{q} \quad (9.17)$$

$$g(q) := b(q, \phi(q)) + A(q)\bar{q}(q, \phi(q)) \quad (9.18)$$

$$G(q) := B(q, \phi(q)) + A(q)\bar{B}(q, \phi(q)) \quad (9.19)$$

Here we have the following properties:

- $q \in \mathbb{R}^\delta$  is the vector of coordinates necessary for the description of the system;
- $M(q) \in \mathbb{R}^{\delta \times \delta}$  is the symmetric and positive definite inertia matrix.
- $f(q, \dot{q}) \in \mathbb{R}^\delta$  represents the forces and torques resulting from the links related to the constraints:

$$f(q, \dot{q}) = C(q, \dot{q})\dot{q} \quad (9.20)$$

- $g(q) \in \mathbb{R}^\delta$  represents the forces and torques resulting from the gravity;
- $u \in \mathbb{R}^m$  represents the forces and torques applied to the system;
- $G(q) \in \mathbb{R}^{\delta \times m}$  is the kinematic matrix.

The state-space model is thus

$$\begin{aligned} \dot{q} &= v \\ \dot{v} &= M^{-1}(q) (-f(q, v) - g(q) + G(q)u) \end{aligned} \quad (9.21)$$

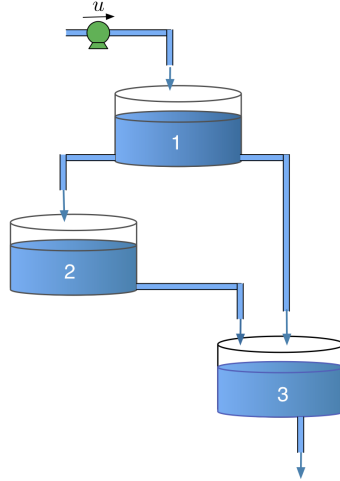
### 9.2.3 Properties

- $\dot{M}(q) = A(q)\bar{J}\dot{A}^T(q) + \dot{A}(q)\bar{J}A^T(q) = C(q, \dot{q}) + C^T(q, \dot{q})$
- $\frac{\partial}{\partial q}(\dot{q}^T M(q) \dot{q}) = \dot{q}^T C(q, \dot{q})$

### 9.3 Compartmental systems

We denote here  $x_i \geq 0$  the quantity of content in compartment  $i$  for  $i \in \{1, \dots, n\}$ , and  $q_{ij} \geq 0$  specifies the flow from compartment  $i$  to compartment  $j$ , with  $i, j \in \{1, \dots, n\}$ . We denote respectively  $q_{i0}$  and  $q_{0i}$  the flow from compartment  $i$  to the environment and vice-versa. The general form of the state-space equations is

$$\dot{x}_i = \sum_{j=0}^n q_{ji}(x, u) - \sum_{j=0}^n q_{ij}(x, u) \quad i = 1, \dots, n \quad (9.22)$$



Defining  $q(x, u)$  the vector of flows (in an arbitrary order), we get the matrix form:

$$\dot{x} = Lq(x, u) \quad (9.23)$$

where  $L$  is the incidence matrix of the oriented graph whose coefficients all belong to  $\{-1, 0, 1\}$ .

**Definition 9.1.** A vector is nonnegative if each of its components is a nonnegative real number. The nonnegative orthant is the set of all nonnegative vectors of dimension  $n$ .

**Definition 9.2.** A dynamical system  $\dot{x} = f(x, u)$  is nonnegative if, for every admissible input  $u$ , its state is confined in the nonnegative orthant when the initial state is nonnegative.

**Theorem 9.3.** A dynamical system  $\dot{x} = f(x, u)$  is nonnegative if  $f$  is continuously differentiable and if

$$x \in \mathbb{R}_+^n, x_i = 0 \implies \dot{x}_i \geq 0 \quad \forall i \quad (9.24)$$

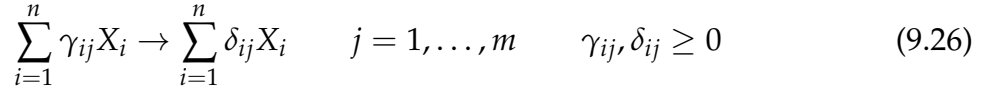
**Theorem 9.4.** Under the following conditions, a dynamical compartmental system  $\dot{x} = Lq(x, u)$  is nonnegative.

- The functions  $q_{ij}$  are nonnegative functions of their arguments on their domain of definition:  $q_{ij} : \mathbb{R}_+^n \times \mathbb{R}^m \rightarrow \mathbb{R}_+ : (x, u) \rightarrow q_{ij}(x, u)$ .
- The functions  $q_{ij}$  are continuous and differentiable functions of their arguments on their domain of definition.
- As there cannot be an outflow from an empty compartment, the functions  $q_{ij}$  verify the condition

$$x_i = 0 \implies q_{ij}(x, u) = 0 \quad \forall i \quad (9.25)$$

## 9.4 Reaction systems

The number  $n$  of species is finite and these species are denoted by  $X_i$ . A reaction is thus a set of  $m$  reactions of the following form:



The coefficients  $\gamma_{ij}$  and  $\delta_{ij}$  are positives real numbers called stoichiometric coefficients. We create 3 matrices:

- $\Gamma = [\gamma_{ij}]$
- $\Delta = [\delta_{ij}]$
- $C = \Delta - \Gamma$

The rank  $p$  of  $C$  is called the reaction network rank, it is the number of independent reactions.

### 9.4.1 State-space model

Let  $x_i(t)$  denote the quantity of the species  $X_i$  per unit of volume in the system at time  $t$ , and let  $r_j : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  be the reaction rate associated to each reaction. It verifies

- $r_j(x) \geq 0 \forall j \forall x \in \mathbb{R}_+^n$
- $r_j(x) = 0$  if  $x_i = 0$  for some  $i \in I^j$ ,  $I^j$  being the set of indices of all reactants involved in reaction  $j$ .

The equations of the system are thus

$$\dot{x}_i = \sum_{j=1}^m (\delta_{ij} - \gamma_{ij}) r_j(x(t)) + \frac{1}{V} (Q_{0i}(t) - Q_{i0}(t)) \quad (9.27)$$

with  $V$  the constant volume,  $Q_{i0}(t)$  the flux going from the domain to the outside and  $Q_{0i}(t)$  the flux going from the outside into the domain. In matrix form,

$$\dot{x} = Cr(t) + q_{in}(x, u) - q_{out}(x, u) \quad (9.28)$$

### 9.4.2 Modelling of reaction kinetics

$$r_j(x) = k_j \prod_{i \in I^j} x_i^{v_{ij}} \quad (9.29)$$

where  $k_j$  is the kinetic constant of the  $j$ th reaction, and  $v_{ij}$  the partial order of the  $i$ th reactant in the  $j$ th reaction. Generally,  $v_{ij} = \gamma_{ij}$ .

### 9.4.3 Continuous Stirred Tank Reactors (CSTR)

We study here a perfectly mixed chemical reactor of constant volume  $V^1$ . We call  $F_{in}$  the feed volumetric flowrate. The reactional equation of the system is, with  $u \triangleq F_{in}/V$ ,

$$\dot{x} = Cr(x) - ux + ux^{in} \quad (9.30)$$

If the volume is not constant,

$$\frac{d}{dt}(xV) = Cr(x)V - F_{out}x + F_{in}x^{in} \quad (9.31)$$

Balancing the volumes,

$$\dot{V} = F_{in} - F_{out} \quad (9.32)$$

and thus

$$\dot{x} = Cr(x) + \frac{u_1}{x_{n+1}}(x^{in} - x) \quad \dot{x}_{n+1} = u_1 - u_2 \quad \begin{cases} u_1 = F_{in} \\ u_2 = F_{out} \\ x_{n+1} = V \end{cases} \quad (9.33)$$

Note that in a closed system,  $q_{in} = q_{out} = 0$ . And a system is conservative if there exists  $\omega \in \mathbb{R}^n$  such that  $C^T \omega = 0$  and  $\omega_i > 0$  for all  $i$ .

---

<sup>1</sup>This is achieved by adjusting the in and out flowrates.