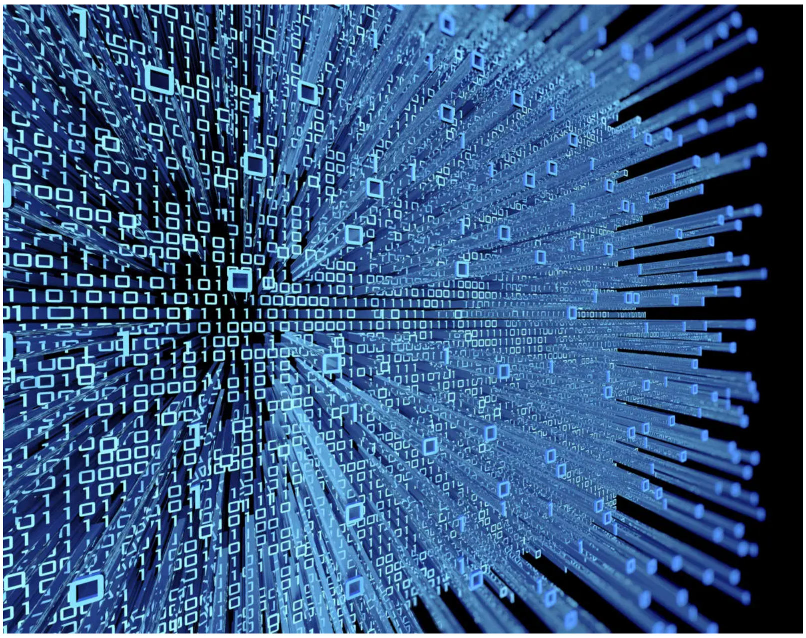

LINMA2171 Numerical Analysis

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Introduction

1.1 General Framework

— Data :

- $\chi \subseteq \mathbb{R}^d$ (here, $d = 1$ often).
- $f : \chi \to \mathbb{R}$, with $f \in \mathfrak{f}$.

— Design :

- $\hat{\mathfrak{f}} \subseteq \mathbb{R}^{\hat{\chi}}$ is the set of admissible function, and is a subset of all function from $\hat{\chi}$ to $\mathbb{R}$.
- $\mathcal{L} : \hat{\mathfrak{f}} \times \mathfrak{f} \to \mathbb{R}$ is the loss function.
- $\mathcal{R} : \hat{\mathfrak{f}} \to \mathbb{R}$ is the regularizer.

— Optimisation problem :

$$\arg_{\hat{f} \in \hat{\mathfrak{f}}} \min \quad \mathcal{L}(\hat{f}, f) + \lambda \mathcal{R}(\hat{f})$$

— Optimisation algorithm.

Polynomials

$\mathcal{P}_n$ is the set of all real polynomials of degree at most n.

— The Runge phenomenon is the explosion of the polynomial near the boundary of the domain when the interpolation points are chosen to be equidistant. A solution to that is to put more points near the boundary and less in the middle of the domain, e.g. Chebyshev points.

2.1 Lagrange interpolation

Let $x_0, \ldots, x_n$ be distinct real numbers. The Lagrange polynomial L_k of degree n is such that it is equal to 0 for all x_i, $i \neq k$ and 1 for x_k. This serves as a base for the next interpolations. The general formula for the Lagrange polynomial is

$$L_k(x) = \prod_{i=0 i\neq k}^{n} \frac{x - x_i}{x_k - x_i} \qquad k = 0, 1, \ldots, n \tag{2.1}$$

— N.B. : we usually denote $L_k(x; x_0, \ldots, x_n)$ or let $\chi = (x_0, \ldots, x_n)$ and $L_k(x; \chi)$.

2.2 Hermite interpolation

Let $x_0, \ldots, x_n$ be distinct real numbers. Then, given two sets of real numbers $(y_0, \ldots y_n)$ and $(z_0, \ldots, z_n)$, there is a unique polynomial $p_{2n+1} \in \mathcal{P}_{2n+1}$ such that

$$p_{2n+1}(x_i) = y_i \qquad p'_{2n+1}(x_i) = z_i \qquad i = 0, \ldots, n \tag{2.2}$$

The polynomial p_{2n+1} is termed the Hermite interpolation polynomial of degree at most $2n+1$ for the data points $(x_0, y_0, z_0), \ldots, (x_n, y_n, z_n)$. The expression is

$$p_{2n+1}(x) = \sum_{k=0}^{n} \left(H_k(x) y_k + K_k(x) z_k\right) \qquad \begin{cases} H_k(x) = (L_k(x))^2 (1 - 2L'_k(x_k)(x - x_k)) \\ K_k(x) = (L_k(x))^2 (x - x_k) \end{cases} \tag{2.3}$$

where $L_k(x)$ is the Lagrange polynomial.

— The $H_k(x)$ are such that their derivative is zero for all x_i, and their value is zero for all x_i except x_k, where it is 1.

$$H_k(x_i) = \delta_{ik} \qquad H'_k(x_i) = 0 \qquad \forall i$$

— The $K_k(x)$ are such that their derivative is zero for all x_i except x_k where it is one, and their value is zero for all x_i.

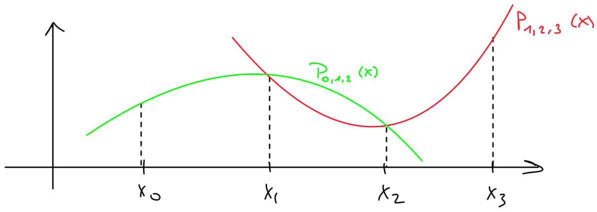
$$K_k(x_i) = 0 \qquad K'_k(x_i) = \delta_{ik} \qquad \forall i$$

2.3 Neville's algorithm

Let us assume we are given a set of support points (x_i, y_i), $i = 0, 1, \dots, n$, and p_n is their Lagrange interpolation polynomial. Let us now define the notation $P_{i_0 i_1 \dots i_k} \in \mathcal{P}_k$, the polynomial for which $P_{i_0 i_1 \dots i_k}(x_{i_j} = y_{i_j}$ for all $j = 0, 1, \dots, k$. We work by recursion, with the following formula :

$$\begin{cases} P_i(x) = y_i \\ P_{i_0 i_1 \dots i_k} = \frac{(x - x_{i_0}) P_{i_1 i_2 \dots i_k}(x) - (x - x_{i_k}) P_{i_0 i_1 \dots i_{k-1}}(x)}{x_{i_k} - x_{i_0}} \end{cases} \tag{2.4}$$

Example :
Let us have four points $(x_0, y_0), \dots (x_3, y_3)$. We want the polynomial interpolating all of them, using Neville's algorithm.



Here,

$$P_{0123}(x) = \frac{x - x_0}{x_3 - x_0} P_{123}(x) + \frac{x_3 - x}{x_3 - x_0} P_{012}(x) \tag{2.5}$$

2.4 Newton's interpolation formula

Newton's interpolation formula is used to evaluate polynomials with a computer, as it only needs to compute each operation $(x - x_i)$ one time. We write it like :

$$p_n(x) = ((\dots (y_{0 \dots n}(x - x_n) + y_{0 \dots n-1})(x - x_{n-1}) + y_{0 \dots n-2})(x - x_{n-2}) + \dots) + y_0 \tag{2.6}$$

And the recursive formula is

$$P_{i_0 i_1 \dots i_k} = P_{i_0 i_1 \dots i_{k-1}}(x) + y_{i_0 i_1 \dots i_k}(x - x_{i_0})(x - x_{i_1}) \dots (x - x_{i_{k-1}}) \tag{2.7}$$

for push only