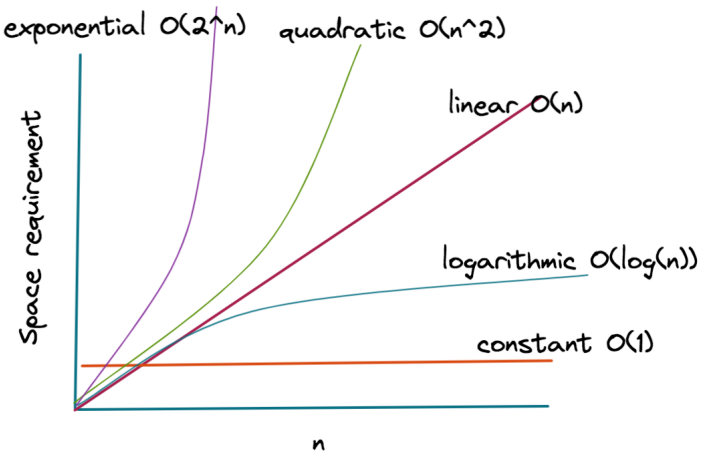

LINMA2111 - Discrete mathematics II

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Contents

Introduction

1.1 Sorting problems

A sorting problem is a problem that consists of taking a sequence of n objects and putting them in order. This kind of problem is made of three main elements:

- Context: set S with a partial order $<$;
- Input: n elements of S;
- Output: permutation of the input elements respecting the order.

To prove the correctness of an algorithm, we generally use the Hoare triple, i.e. a tuple for any input array x_0:

$$\begin{aligned} &\{\text{Algorithm to be used; Precondition; Postcondition}\} \\ &\{IS;\ x = x_0;\ x \text{ is sorted and is a permutation of } x_0\} \end{aligned} \tag{1.1}$$

where IS is the insertion sort algorithm, and x is the sorted array.
In practice, to prove the correctness of an algorithm, we define the invariant and the base case, and do an induction step show that the invariant is preserved. The proof is done is my notes at page 1.

1.2 Definition of complexity

For some functions $f, g : \mathbb{N} \to \mathbb{R}^+$,

$$\begin{aligned} &f \in \mathcal{O}(g) \iff \exists c > 0, \exists n_0, \forall n > n_0,\ f(n) \leq cg(n) \\ &f \in \Omega(g) \iff g \in \mathcal{O}(f) \\ &f \in \Theta(g) \iff f \in \mathcal{O}(g) \text{ and } f \in \Theta(g) \\ &f \in o(g) \iff \exists c > 0, \exists n_0, \forall n > n_0,\ f(n) < cg(n) \\ &f \in \omega(g) \iff g \in o(f) \end{aligned} \tag{1.2}$$

- The time complexity is the number of operations as a function of the input size;
- The space complexity is the amount of memory used (in addition to the input) as a function of the input size.

We define the average case complexity as an expectation of the time taken by the algorithm on each input possible, with a dependence in the size of the input.

$$t = \mathbb{E}_{x \sim D}[T(x)] \tag{1.3}$$

where D is the distribution of the input.

1.3 Complexity of sorting algorithms

No sorting algorithm can be faster than $\Omega(n \log(n))$. This can be proven through the complexity of comparison-based algorithms.

1.4 Sorting algorithms

1.4.1 Selection Sort

This algorithm is the naive approach in complexity $\mathcal{O}(n^2)$. It goes as follows:

Algorithm 1 Selection sort algorithm

1: **for** i from 1 to $n-1$ **do**
2: Select the smallest element in the subarray from index i to n;
3: Swap it with element at index i;
4: **end for**

The invariant of this algorithm is that the elements 1 to $i-1$ are sorted.

1.4.2 Insertion Sort

The idea of this algorithm is to shift the elements to the left until they are well placed. This also has a complexity $\mathcal{O}(n^2)$, although the best case is in $\mathcal{O}(n)$.

Algorithm 2 Insertion sort algorithm

1: **for** i from 2 to n **do**
2: shift element i to the left by successive swaps until it is well placed;
3: **end for**

1.4.3 Quick Sort

The quick sort algorithm is based on divide and conquer methods (see chapter 2): it splits and solves smaller sorting problems, and recombines the outputs at the end into a sorted instance.

Algorithm 3 Quick Sort algorithm

```
pivot = T[1];
T_low = [T[i]  :  T[i] < pivot and i >1];
T_high = [T[i]  :  T[i] > pivot and i >1];
quicksort(T_low);
quicksort(T_high);
T = [T_low; pivot; T_high];
```

The worst-case complexity is still $\mathcal{O}(n^2)$. However, here, the average-case complexity is lower: $\mathcal{O}(n\log(n))$.
The randomized version of the algorithm consists in shuffling before applying the classical quick sort, in order to change the pivot value. As this is a random algorithm, the worst-case complexity corresponds to the expected complexity, i.e. $\mathcal{O}(n\log(n))$.

1.4.4 Merge Sort

The merge sort algorithm is also based on divide-and-conquer, and its complexity is also $\mathcal{O}(n^2)$.

Algorithm 4 Merge Sort algorithm

```
T_left = [T[i] such that i <= n/2]
T_right = [T[i] such that i > n/2]
mergesort(T_left)
mergesort(T_right)
T = merge(T_left, T_right)
```

The recurrence equation for the complexity is

$$t_n = 2t_{\lceil n/2 \rceil} + \Theta(n) \Longrightarrow t(n) = \Theta(n\log(n)) \tag{1.4}$$

The disadvatange of quick sort that this algorithm does not have is the randomness, as it is complex to simulate. However, the constant in the complexity order is higher than for the quick sort, and the memory usage can be higher too.

1.5 Decision tree

A decision tree focuses on strategies and outcomes. It captures the main operations and the leaves are the possible outcomes ($\mathcal{O}(n!)$ in sorting problems, for n the size of the data). This helps us compute the complexity of a sorting algorithm: the number of steps is the depth S of the tree. Then, we have

$$2^S \geq n! \Longrightarrow S = \mathcal{O}(log(n!)) \tag{1.5}$$

This is equivalent to $\mathcal{O}(n\log(n))$:

$$S = \log(n!) = \sum_{i=1}^{n} \log(i) \leq \int_1^n log(x)dx \tag{1.6}$$

and we know that $S \geq \int_1^{n+1} \log(x)dx$[1]. Then,

$$\begin{aligned} [x\log(x) - x]_1^n \leq S \leq [x\log(x) - x]_1^{n+1} \\ S = \mathcal{O}(n\log(n)) \end{aligned} \tag{1.7}$$

1.5.1 Shannon coding theorem

Let $\mathcal{S} = \{1, \ldots, N\}$ and P a random variable over $\mathcal{S}$ such that $Pr[P = i] = p_i$. Let $f : \mathcal{S} \to \{0,1\}^*$ maps a number to its binary representation, under the constraint that no number is a prefix of another one. Then,

$$\sum_{i=1}^{N} p_i |f(i)| \geq H(P) = -\sum_{i=1}^{N} p_i \log_2(p_i) \tag{1.8}$$

where $H(P)$ is called the entropy of the probability distribution P. This theorem states that the average length of the prefix of the encoding of N elements must be at least $\log(N)$ characters long.

1.5.2 Yao's minimax principle

Consider a probability distribution over instances of a given size of a given problem. There, there exists a deterministic algorithm solving the problem for thoses instances, whose average-case complexity for the given distribution is lower than the worst-case expected complexity of any random algorithm solving the same problem.
This means that a random algorithm cannot be better than a deterministic algorithm in every case of a problem.

1.5.3 What about the part on thermodynamics in CM2?

[1] Why?

Divide-and-conquer algorithms

The divide-and-conquer method consists in three steps:

1. Divide: create smaller subproblems;
2. Recurse: solve them;
3. Combine: merge the solutions.

In sorting problems, a divide-and-conquer algorithm is the merge sort algorithm. It consists in dividing the array in two, sorting each half recursively, and merging the two sorted halves. The merge operation is done in linear time.

2.1 Complexity of an integer multiplication

Given two n-digit numbers, we want to compute their product:

$$\begin{cases} a = a_{n-1}a_{n-2}...a_1a_0 \\ b = b_{n-1}b_{n-2}...b_1b_0 \end{cases} \implies c = a \cdot b = c_{2n-1}...c_0 \tag{2.1}$$

Let us define B as the basis (e.g. 10) and decompose a and b in two parts:

$$\begin{cases} a = \alpha_0 + B\alpha_1 \\ b = \beta_0 + B\beta_1 \end{cases} \implies a \cdot b = \alpha_0\beta_0 + B(\alpha_1\beta_0 + \alpha_0\beta_1) + B^2\alpha_1\beta_1 \tag{2.2}$$

In that case, we find a recurrence relation for the computation time:

$$T(n) = 4T(n/2) + \Theta(n) \tag{2.3}$$

The factor 4 comes from the fact that we need 4 products, and the $\Theta(n)$ is the complexity of the sum of the products.

We introduce the Master theorem to solve this equation, see section 2.1.1. It gives a complexity of $T(n) = \Theta(n^{\log_2(4)}) = \Theta(n^2)$.
To reduce the complexity, we can change the value of the parameter a from 4 to 3 by calculating only 3 products:

$$\begin{cases} \gamma_0 = \alpha_0\beta_0 \\ \gamma_2 = \alpha_1\beta_1 \\ \gamma_1 = (\alpha_0 + \alpha_1)(\beta_0 + \beta_1) - \gamma_0 - \gamma_2 \end{cases} \tag{2.4}$$

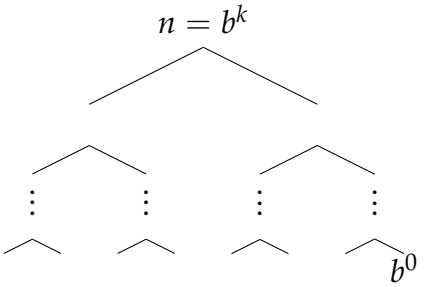
This reduces the complexity to $\Theta(n^{1.58})$.
Following a similar reasoning, we can instead divide a and b into 3 sums instead of 2, and get 5 multiplications. This gives a complexity of $\Theta(n^{\log_3(5)}) = \Theta(n^{1.46})$. Dividing in 4, 5, etc, we converge to a complexity of $\Theta(n)$ and this is the optimal complexity for the multiplication of two numbers of n digits. The problem is that the constant in front of the n starts to grow as we divide into more and more limbs, and so a bigger exponent can be enough for most arrays[1].

2.1.1 Master Theorem

Let $a \geq 1$ and $b > 1$ be constants and $f(n)$ a positive function, and let $T(n)$ be defined by $T(0) > 0$ and $T(n) = aT(\lfloor n/b \rfloor) + f(n)$. Then,

- If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b(a)})$;
- If $f(n) = \Theta(n^{\log_b(a)})$, then, $T(n) = \Theta(n^{\log_b(a)} \log(n))$;
- If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ for some $\epsilon > 0$ and if, for some $c > 1$ and n_0 such that $af(\lfloor n/b \rfloor) \leq cf(n)$ for all $n \geq n_0$, then $T(n) = \Theta(f(n))$;

2.1.2 Proof of the master theorem



In that tree, for a level i, the size of the problem is b^i and the number of subproblems is a^{k-i}. By summing up,

$$T(b^k) = \sum_{i=0}^{k} a^{k-i} f(b^i) \tag{2.5}$$

For a function $f(n) = n^\alpha$,

$$T(n) = a^k \sum_{i=0}^{k} \left(\frac{b^\alpha}{a}\right)^i = a^k \frac{1 - (\frac{b^\alpha}{a})^{k+1}}{1 - b^\alpha/a} \tag{2.6}$$

Under the assumption that $\alpha \neq \log_b(a)$. As $k = \log_b(n)$, we can replace above and simplify using the formula $a^{\log_b(n)} = n^{\log_b(a)}$. In the case where $a = b^\alpha$, then

$$T(n) = a^k(k+1) = a^{\log_b(n)}(\log_b(n) + 1) = \Theta(n^{\log_b(a)} \log(n)) \tag{2.7}$$

[1]We call galactical algorithm an algorithm that is asymptotically good but the constant grows so large that it is only useful for huge arrays (e.g. $\sim 10^{80}$).

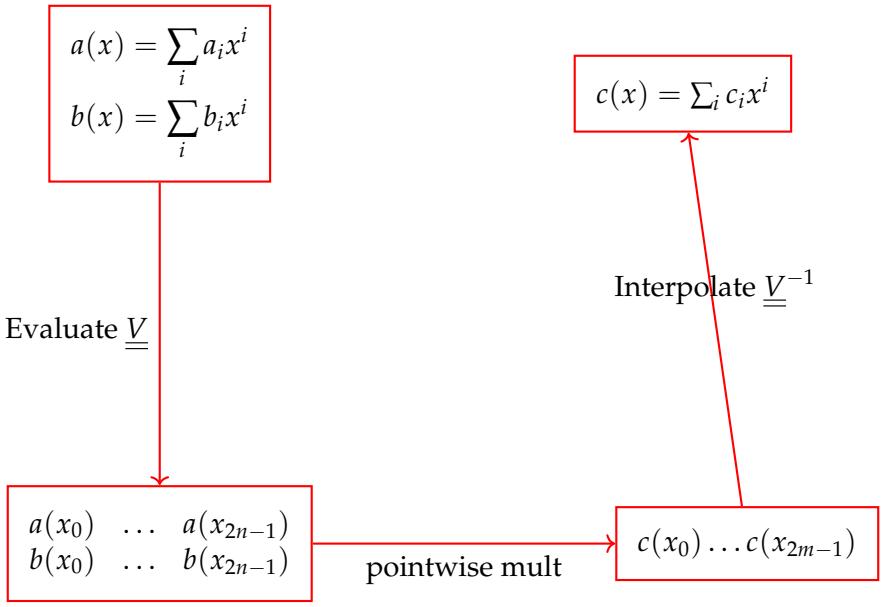
2.2 Multiplying polynomials

Consider $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$. It can either be represented by its coefficients, or some values $(a(x_0), \ldots, a(x_{n-1}))$ for n distinct values. To compute those values, we can use a matrix-vector product:

$$\begin{bmatrix} 1 & x_0 & \ldots & x_0^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & \ldots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix} = \underline{\underline{V}}\mathbf{a} \tag{2.8}$$

where the matrix $\underline{\underline{V}}$ is called the Vandermonde matrix.

2.2.1 Convolution theorem



This method gives bad conditioning, but this is not a problem here as we consider integer matrices.

2.3 Discrete Fourier Transform

From the previous section, if we take the points as the n complex roots of 1, i.e. $x_0 = e^{2\pi i/n}$ and $x_j = e^{2\pi ij/n}$, then we get a Discrete Fourier Transform matrix. Defining $\omega_n = x_0$,

$$\underline{\underline{V}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \vdots & \omega_n & \ldots & \omega_n^{n-1} \\ \vdots & \ddots & \omega_n^{(i-1)(j-1)} & \vdots \\ 1 & \omega_n^{n-1} & \ldots & \omega_n^{(n-1)^2} \end{bmatrix} \tag{2.9}$$

This gives us

$$\underline{\underline{V}}^{-1} = \frac{1}{n}\left[\omega_n^{-(i-1)(j-1)}\right] \tag{2.10}$$

and so $\underline{\underline{V}} = \frac{1}{n}\underline{\underline{V}}^*$. We conclude on

$$DFT(\mathbf{x}) = \underline{\underline{V}}\mathbf{x} \tag{2.11}$$

2.3.1 Fast Fourier Transform

The Fast Fourier Transform (FFT) algorithm uses the divide-and-conquer approach from above to compute the Fourier transform of a vector x. The exact algorithm is the following.
Assume that $n = 2^k$, $k \geq 1$, $\mathbf{x} = (x_0, \ldots, x_{n-1})$ and $(y_0, \ldots, y_{n-1}) = \underline{\underline{V}}_n\mathbf{x}$. The explicit formula to calculate y_i, $i = 0, \ldots, n/2 - 1$ is

$$\begin{aligned} y_i &= \sum_{j=0}^{n-1} x_j \omega_n^{ji} = \sum_{j=0}^{n/2-1} x_{2j}\omega_n^{2ji} + \sum_{j=0}^{n/2-1} x_{2j+1}\omega_n^{(2j+1)i} \\ &= \sum_{j=0}^{n/2-1} x_{2j}\omega_{n/2}^{ji} + \omega_n^i \sum_{j=0}^{n/2-1} x_{2j+1}\omega_{n/2}^{ji} \end{aligned} \tag{2.12}$$

And in matrix form:

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n/2-1} \end{pmatrix} = \underline{\underline{V}}_{n/2} \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_{n-2} \end{pmatrix} + \begin{pmatrix} \omega_n^0 & & \\ & \ddots & \\ & & \omega_n^{n/2-1} \end{pmatrix} \underline{\underline{V}}_{n/2} \begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix} \tag{2.13}$$

And for the next half of y, we have a similar expression:

$$\begin{aligned} y_{i+n/2} &= \sum_{j=0}^{n/2-1} x_{2j}\omega_n^{2ji}\left(\omega_{n/2}^{n/2}\right)^j + \omega_n^{i+n/2}\sum_{j=0}^{n/2-1} x_{2j+1}\omega_n^{ji}\left(\omega_{n/2}^{n/2}\right)^j \\ &= \sum_{j=0}^{n/2-1} x_{2j}\omega_{n/2}^{ji} + (-1)\cdot\omega_n^i \sum_{j=0}^{n/2-1} x_{2j+1}\omega_{n/2}^{ji} \end{aligned} \tag{2.14}$$

And, once again, in matrix form:

$$\begin{pmatrix} y_{n/2} \\ y_{n/2+1} \\ \vdots \\ y_{n-1} \end{pmatrix} = \underline{\underline{V}}_{n/2} \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_{n-2} \end{pmatrix} - \begin{pmatrix} \omega_n^0 & & \\ & \ddots & \\ & & \omega_n^{n/2-1} \end{pmatrix} \underline{\underline{V}}_{n/2} \begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix} \tag{2.15}$$

Let us define three notations:

$$\begin{aligned} \mathbf{x}^{[0]} &= (x_0, \ldots, x_{n/2-1}) \\ \mathbf{x}^{[1]} &= (x_{n/2}, \ldots, x_n) \\ \underline{\underline{T}}_n &= \begin{pmatrix} \omega_n^0 & & \\ & \ddots & \\ & & \omega_n^{n/2-1} \end{pmatrix} \end{aligned} \tag{2.16}$$

Then,

$$\underline{\underline{DFT}}(\mathbf{x}) = \begin{pmatrix} \underline{\underline{DFT}}(\mathbf{x}^{[0]} + \underline{\underline{T}}_n \underline{\underline{DFT}}(\mathbf{x}^{[1]})) \\ \underline{\underline{DFT}}(\mathbf{x}^{[0]} - \underline{\underline{T}}_n \underline{\underline{DFT}}(\mathbf{x}^{[1]})) \end{pmatrix} \tag{2.17}$$

The time complexity of this algorithm is $\Theta(n \log(n))$.

→ Note: there exists other algorithm to compute the FFT, such as the non power-of-two n alorithm.

2.4 Matrix multiplication

Given $A, B \in \mathbb{Z}^{n \times n}$, we want to compute $C = A \cdot B$. The basic algorithm is in $\Theta(n^3)$, but we want a better one.

2.4.1 Straßen algorithm

Let us divide the matrices in blocks:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \qquad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \tag{2.18}$$

The Straßen algorithm uses the same idea as in section section 2.1: we define 7 block matrices to reduce the number of products done.

$$\begin{cases} M_1 = (A_{11} + A_{22})(B_{11} + B_{22}) \\ M_2 = (A_{21} + A_{22})B_{11} \\ M_3 = A_{11}(B_{12} - B_{22}) \\ M_4 = A_{22}(B_{21} - B_{12}) \\ M_5 = (A_{11} + A_{12})B_{22} \\ M_6 = (A_{21} - A_{11})(B_{11} + B_{12}) \\ M_7 = (A_{12} - A_{22})(B_{21} + B_{22}) \end{cases} \implies C = \begin{pmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 - M_2 + M_3 + M_6 \end{pmatrix} \tag{2.19}$$

Which has a complexity of $\Theta(n^{\log_2 7}) \approx \Theta(n^{2.81})$. The lower bound for the complexity is $\Omega(n^2)$, as we need to make at least one operation per element of C, and the current best algorithm (galactic) is in $\Theta(n^{371339})$.

2.4.2 Matrix inversion

As we can assume intuitively, matrix multiplication and inversion are closely linked. Therefore, the complexity to compute both should be linked too. Let us call $M(n)$ the time complexity of multiplication of matrices of size n, and $I(n)$ the time complexity of inversion of a matrix of size n. Then,

$$D = \begin{pmatrix} I & A & 0 \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix} \implies D^{-1} = \begin{pmatrix} I & -A & AB \\ 0 & I & -B \\ 0 & 0 & I \end{pmatrix} \tag{2.20}$$

This means that the complexity of inversion is an upper bound on the complexity of multiplication: $M(n) \leq I(3n) \leq 3^3 I(n)\ \forall n \geq n_0$.

This kind of inequality is called reduction. Given problems A and B, if A can be transformed in a problem B of size $f(n)$ in time $T_R(n)$, then

$$T_A(n) \leq T_R(n) + T_B(f(n)) \tag{2.21}$$

For example, the problem of finding the median of an array can be transformed into the problem of sorting the array.

2.4.3 Matrix inversion upper bound

Let us assume that $A = A^T$ and $A \succ 0$.

$$A = \begin{pmatrix} B & C \\ C^T & D \end{pmatrix} \Longrightarrow A^{-1} = \begin{pmatrix} B^{-1} + B^{-1}CS^{-1}C^TB^{-1} & -B^{-1}CS^{-1} \\ -S^{-1}C^TB^{-1} & S^{-1} \end{pmatrix} \tag{2.22}$$

where $S = D - C^TB^{-1}C$ is the Schur complement of A. From Equation 2.21, $f(n) = \Theta(M(n))$ in our case, and so we have the following relation between inversion and multiplication:

$$I(n) = \mathcal{O}(M(n)) \tag{2.23}$$

$\rightarrow$ Note: the hypotheses that $A = A^T$ and $A \succ 0$ are not binding. If A is invertible but does not verify those conditions, we can work with AA^T that does, and we find the inverse of A with $A^{-1} = A^T(AA^T)^{-1}$.

Dynamic Programming

3.1 Two approaches

There are two approaches for a dynamic programming algorithm: bottom-up and top-down. In the bottom-up approach, we solve the subproblems first, and then use their solutions to solve bigger problems. In the top-down approach, we start from the main problem, and recursively solve the subproblems as needed.

- The main idea of bottom-up is to use memoization, i.e. storing the solutions of subproblems to avoid recomputing them.
- The main idea of top-down is recursion.

Dynamic programming is used to improve time complexity by trading time for space.

3.2 Examples

3.2.1 Rod cutting

The problem of rod cutting consists in wanting to cut a beam of length n, given commands of clients.Finding the optimal solution can be done by computing a solution on sub-beams and merging them.

Algorithm 5 Rod Cutting Algorithm

```
1:  function CutRod(p, n)
2:      if n = 0 then
3:          return 0
4:      end if
5:      q ← −∞
6:      for i = 1 to n do
7:          q ← max{q, p_i + CutRod(p, n − i)}
8:      end for
9:      return q
10: end function
```

The complexity of this algorithm is given by a recurrence relation:

$$T(n) = 1 + \sum_{j=0}^{n-1} T(j) = 1 + T(n-1) + \sum_{j=0}^{n-2} T(j) = 2T(n-1) \Longrightarrow T(n) = \mathcal{O}(e^n) \quad (3.1)$$

This basic algorithm has a very bad complexity, but memoization can improve it.

Algorithm 6 Memoized rod cutting algorithm

```
1: function CutRod_Memoized(p, n)
2:     r[0 : n] = −∞
3:     function CRM_Aux(p, n, r)
4:         if r[n] ≥ 0 then
5:             return r
6:         end if
7:         if n = 0 then
8:             return 0
9:         end if
10:        q = −∞
11:        for i=1 to n do
12:            q = max{q, p_i + CRM_Aux(p, n − i, r)}
13:        end for
14:        r[n] = q return r
15:    end function
16:     return CRM_Aux(p,n,r)
17: end function
```

Finally, another algorithm exists, using a bottom-up approach in dynamic programming. It has a complexity $\Theta(n^2)$.

Algorithm 7 Bottom-up efficient rod cutting algorithm

```
1: function Bottom-up-CR(p, n)
2:     r[0 : n]
3:     r[0] = 0
4:     for j=1 to n do
5:         q = −∞
6:         for i=1 to j do
7:             q = max{q, p_i + r_{j−1}}
8:         end for
9:         r[j] = q
10:    end for
        return r[n]
11: end function
```

3.2.2 Matrix chain multiplication

This problems consists in doing the multiplication of matrices $A_1 A_2 \dots A_n$ with dimensions $p_0, p_1, \dots, p_n$. The total number of operations depends heavily on the order of the multiplication. The idea for the algorithm is to split the chain into smaller chains until we get to a product of two matrices. The splitting happens at the index i that minimizes the total number of operations, knowing the cost of computing a subchain:

$$Algo(p) = \min_{1 \leq i \leq n-1} \{Algo(p[0:i]) + Algo(p[i+1:n]) + p_0 p_i p_n\} \tag{3.2}$$

Let us define $m[i,j]$ the optimal cost of the product $A_i \dots A_j$. Our goal is to compute $m[1,n]$, with the initial condition $m[i,i] = 0$ for all i. The recurrence relation is

$$m[i,j] = \min_{i \le k \le j} \{m[i,k] + m[k+1,j] + p_i p_k p_j\} \qquad \forall 1 \le i \le j < n \tag{3.3}$$

This gives a complexity $\Theta(n^2)$, as we need to compute the entries of a triangular matrix $n \times n$.

3.3 Generating functions

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence. We associate the function $\sum_{k=0}^{\infty} a_k x^k$ to the sequence, on which we have addition, multiplication, differentiation, etc.
Example:

$$a_k = 1 \; \forall k \in \mathbb{N} \Longrightarrow f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} = x \sum_{k=0}^{\infty} x^k + 1 = xf(x) + 1 \tag{3.4}$$

Another example is the Fibonacci sequence:

$$f_{n+2} = f_{n+1} + f_n \qquad f_0 = 0 \qquad f_1 = 1 \tag{3.5}$$

From the recurrence equation, we can get the generative function:

$$\begin{aligned} f(x) &= f_0 x^0 + f_1 x^1 + \sum_{k \ge 2} f_k x^k = f_0 + f_1 x + \sum_{k \ge 0} f_{k+2} x^{k+2} \\ &= f_0 + f_1 x + \sum_{k \ge 0} f_{k+1} x^k + \sum_{k \ge 0} f_k x^k \\ &= f_0 + f_1 x + x(f(x) - f_0) + x^2 f(x) \\ &= \frac{f_0 + (f_1 - f_0)x}{1 - x - x^2} \end{aligned} \tag{3.6}$$

Randomized Algorithms

4.1 Probabilistic analysis and randomized algorithm

In probabilistic computing, we need a distribution of the inputs to be able to sample. Based on prior knowledge or assumptions, we can determine the average-case complexity. The goal is for it to be lower than the deterministic algorithm, although it comes at the price of the accuracy of the solution.

Let us use as example the hiring problem. We interview n candidates, one at a time, and we need to decide wether we hire the candidate or not right after the interview. Our goal is of course to hire the best candidate.
One approach would be to interview the $n/2$ first candidates and only observe their skills. Then, hire the first candidate that is more skilled than the average (or all) of the first half of candidates.

4.1.1 Indicator Random Variable

Given a sample space S and an event A, we define the indicator random variable as

$$X_A = \begin{cases} 1 \text{ if } A \text{ occurs} \\ 0 \text{ otherwise} \end{cases} \tag{4.1}$$

where $Pr[X_A = 1] = Pr[A]$. This is useful because $\mathbb{E}[X_A] = Pr[A]$.

4.2 Random Sampling and Applications

4.2.1 Birthday paradox

The birthday paradox is not stricly speaking a paradox, but is called that way because it goes against the first intuition. What is the number of people that must be in a room so that the probability that two people have the same birthday is higher than $1/2$, assuming that the birthdays are uniformly distributed? Let $r = 1, \ldots, n$ and $Pr[b_i = r] = 1/n$. Then, for two people,

$$Pr[b_i = b_j] = \sum_{r=1}^{n} Pr[b_i = r \text{ and } b_j = r] = \sum_{r=1}^{n} Pr[b_i = r]Pr[b_j = r] = \sum_{r=1}^{n} \frac{1}{n^2} = \frac{1}{n} \tag{4.2}$$

And for k people, denoting B_k the event that k people have distinct birthdays, we have

$$\begin{aligned} Pr[B_k] &= 1 \cdot \frac{n-1}{n}\frac{n-2}{n}\dots\frac{n-k+1}{n} = 1 \cdot \left(1-\frac{1}{n}\right)\cdot(1-\frac{2}{n})\dots\left(1-\frac{k-1}{n}\right) \\ &\leq e^{-1/n}e^{-2/n}\dots e^{-(k-1)/n} = e^{-k(k-1)/2n} \end{aligned} \tag{4.3}$$

Then, for $Pr[B_k] \geq 1/2$, we get

$$k(k-1) \geq 2n\ln 2 \iff k \geq 23 \tag{4.4}$$

We can also compute the expectation: let X_{ij} be the 1 if the people i and j have the same birthday and 0 otherwise. Then $\mathbb{E}[X_{ij}] = 1/n$ and for $X = \sum_{i=1}^{k}\sum_{j=i+1}^{k} X_{ij}$,

$$\mathbb{E}[X] = \sum_{i=1}^{k}\sum_{j=i+1}^{k} X_{ij} = \frac{k(k-1)}{2n} \tag{4.5}$$

This means that for at least $\sqrt{2n}+1$ people in a room, we can EXPECT at least one collision. For $n = 365$, this is $k = 28$ people.

4.2.2 Hash table

A hash table is a data structure in which inserting, searching and deleting is done in $\mathcal{O}(1)$ in average. J'ai pas compris les slides 18-19/19 du CM7.

4.2.3 Randomized algorithm for computing integrals

To approximate π , we can put n random points on a square of size 1. For each point, we define the associated random variable $X_i = 1$ if $x_i^2 + y_i^2 \leq 1$ and 0 otherwise. If we observe k points in the orthant, we have

$$\frac{k}{n} \approx \frac{\pi}{4} \tag{4.6}$$

And we can even compute the error using the variance of $X_n = \sum_{i=1}^{n} X_i$:

$$\mathbb{V}(X_n) = \frac{1}{n}\mathbb{V}(X_i) = \frac{1}{n}(\mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2) = \frac{1}{n}(\mathbb{E}[X_i] - \mathbb{E}[X_i]^2) = \frac{1}{n}\frac{\pi}{4}\left(1-\frac{\pi}{4}\right) \tag{4.7}$$

This is can be done for any curve, not just the quarter of the circle, and we can thus approximate any integral with this method, with an absolutr error (standard deviation) of order $\mathcal{O}(1/\sqrt{n})$. This bound is valid for any dimension of the integration.

4.2.4 Randomized algorithm for decision problems

Polynomials

An example of a decision problem is to determine wether a polynomial is identically 0 or not. In one dimension, it is easy: use $d+1$ points for a polynomial of degree d, and if all are 0, then $P \equiv 0$. In more than one dimension, the number of roots of the polynomial is infinite and we need something more. We can use the Schwartz-Zippel

lemma:

Given a finite set S of an integral domain, let P be a polynomial of n variables $(x_1, \ldots, x_n)$ and of degree at most $d \geq 0$. Let us pick a random point $(r_1, \ldots, r_n) \in S$ uniformly:

$$Pr[P(r_1, \ldots, r_n) = 0] \leq \frac{d}{|S|} \tag{4.8}$$

We can prove it by induction:

$$P(x_1, \ldots, x_n) = \sum_{i=0}^{d} x_1^i P_i(x_2, \ldots, x_n) \tag{4.9}$$

for some polynomials P_i of degree at most $d - i$ and $n - 1$ variables. We can go all the way to polynomials of one single variable and use the fundamental theorem of algebra.

From this lemma, the probability to predict correctly from a set of points if a polynomial is identically zero or not is

$$Pr[correct] = Pr[TRUE \ \& \ P \equiv 0] + Pr[FALSE \ \& \ P \not\equiv 0] \tag{4.10}$$

with $Pr[TRUE|P \equiv 0] = 1$ and $Pr[FALSE|P \not\equiv 0] \geq 1 - \left(\frac{d}{|S|}\right)^k$. Then,

$$Pr[correct] \geq 1 - (d/|S|)^k \tag{4.11}$$

Matrix product

Given $A, B, C \in \mathbb{R}^{n \times n}$, we want to check if $AB = C$. Doing the product is costly, so we can take random vectors $x \in \mathbb{R}^n$ and check the products:

$$A(Bx) - Cx \stackrel{?}{=} 0 \tag{4.12}$$

This is a n-variate polynomial of degree 1 and we can use the same method as in the previous section. Dunno what to do with slides 13->17 of CM8.

4.3 Las Vegas and Monte Carlo

An algorithm is said to be Las Vegas when the output is correct, but the time taken to get to this solution is random. On the other hand, an algorithm is said to be Monte Carlo when the output is correct with some probability, or we only get a bound on the real value.

4.4 Hash tables

TODO

4.5 Randomness generation

Let us take functions $f, g : X \to Y$ and an element $s \in X$. We define the sequence

$$\begin{cases} x_0 = s \\ x_{i+1} = f(x_i) \\ y_i = g(x_i) \end{cases} \tag{4.13}$$

The output of this pseudo-random generator is the sequence (usual bits) $y = (y_0, \ldots, y_n)$.
One famous example of pseudo-random generator is Blum-Blum-Shub (BBS):
select random κ-bit prime integers p, q such that $p = q = 3 \mod 4$, and let $N = pq$.
The PRG is

$$\begin{cases} x_0 = s \in \mathbb{Z}_N^* \\ f_N(x) = x^2 \mod N \\ g(x) = x \mod 2 \end{cases} \tag{4.14}$$

4.6 Derandomization

Derandomization transforms a randomized algorithm into a deterministic algorithm, or at least into an algorithm using less randomness, without increasing time and memory costs too much.

4.6.1 Techniques

- Use a PRG: cheap, the randomness goes in the time complexity, and some randomness still exists;
- Try all possible values: trades n random bits for 2^n time, can remove all randomness, but can have a polynomial time slowdown.

Computability