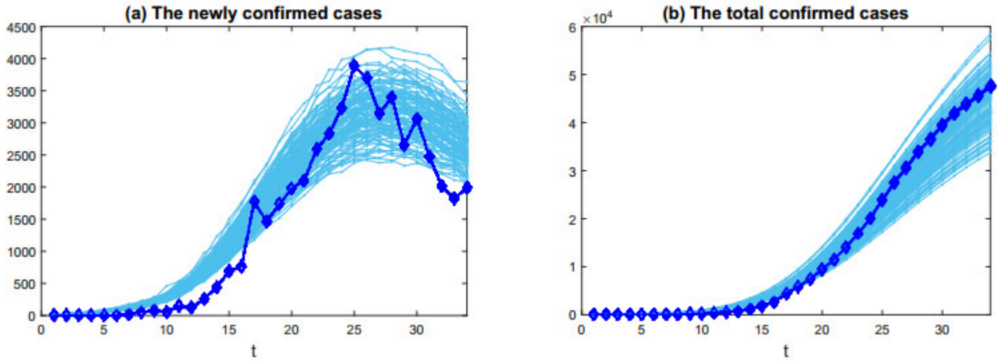

LINMA2470 Stochastic Modelling

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UCLouvain

Contents

Reminders

1.1 General properties of probability

- $P[A \cup B] = P[A] + P[B] - P[1 \cap B]$;
- $P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[AB]}{P[B]}$;
- A and B are independent iff $P[AB] = P[A]P[B] \implies P[A|B] = P[A]$;
- $P[X \leq x] = F_X(x)$ is the distribution function, i.e. a monotone increasing function of x going from 0 to 1 when x goes from $-\infty$ to $+\infty$.
- Its derivative is the density function $f_X(x)$ such that $f_X(x)\delta \approx P[x \leq X \leq x + \delta]$ for an infinitesimal δ.
- A random variable X is said to be memoryless if $\forall t, x > 0$, $P[X > t + x | X > t] = P[X > x]$.
- Markov inequality (for a nonnegative random variable): $P[Y \geq y] \leq \frac{\mathbb{E}[Y]}{y}$;
- Chebyshev inequality: $P[|Z - \mathbb{E}[Z]| \geq \varepsilon] \leq \frac{\sigma_Z^2}{\varepsilon^2}$;

1.2 Expectation and variance

- For a discrete random variable, $\mathbb{E}[X] = \sum_{n=-\infty}^{\infty} nP[X = n]$;
- For a continuous random variable, $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$;
- $\mathbb{E}[X] = \int_0^{\infty} (1 - F_X(x)) dx$.
- $Var[X] = \sigma_X^2 = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$;

1.3 Law of large numbers

Let $X_1, \ldots, X_n$ be a series of independent and uniformly distributed (IID) random variables with expectation $\bar{X}$ and finite variance σ_X^2. Let $S_n = X_1 + \cdots + X_n$. Then,

- Weak version:

$$\lim_{n\to\infty} P\left[|\frac{S_n}{n} - \bar{X}| \geq \varepsilon\right] = 0 \tag{1.1}$$

- Strong version:

$$\lim_{n\to\infty} P\left[\sup_{m\geq n}\left(\frac{S_m}{m} - \bar{X}\right) > \varepsilon\right] = 0 \Longleftrightarrow \lim_{n\to\infty} \frac{S_n}{n} = X \qquad \text{with probability 1} \tag{1.2}$$

1.4 Central limit theorem

$$\lim_{n\to\infty} P\left[\frac{S_n - n\bar{X}}{\sqrt{n}\sigma} \leq y\right] = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{1.3}$$

1.5 Exponential distribution

- $f_X(x) = \lambda e^{-\lambda x}$, for $x \geq 0$;
- $F_X(x) = 1 - e^{-\lambda x}$, for $x \geq 0$;
- $\mathbb{E}[X] = 1/\lambda$.

$\rightarrow$ Note: the exponential distribution is memoryless.

Poisson Processes

A Poisson process $N(t)$ counts the number of arrivals with exponentially distributed inter-arrival times.

$$S_n = \sum_{i=1}^{n} X_i \qquad X_i \sim \exp(\lambda) \tag{2.1}$$

$\forall n, t$, we have the relation $\{S_n \leq t\} = \{N(t) \geq n\}$, where S_n is a random variable telling at which time the n-th occurence appears.

$\rightarrow$ Note: a Poisson process is memoryless: $P[Z_1 > x] = e^{-\lambda x}$, with Z_1 be the duration of the time interval from t until the first arrival after t.

For a Poisson process of rate λ, and any given $t > 0$, the length of the interval from t until the first arrival after t is an exponentially distributed random variable. This random variable is idenpendent of both $N(t)$ and of the $N(t)$ arrival epochs before time t. It is also independent of $N(\tau), \forall \tau \leq t$.

Let us consider the process after Z_1, Z_m, the time until the m-th arrival after time t. It is independent of $N(t)$ and of the entier previous history of the process.
Let us denote $\tilde{N}(t, t') = N(t') - N(t)$.

- Stationary increments property: It has the same distribution as $N(t' - t), \forall t' \geq t$ (stationary increments property);
- Independent increments property: For any sequence of times $0 < t_1 < \cdots < t_k$, the set $\{N(t_1), \tilde{N}(t_1, t_2), \ldots, \tilde{N}(t_{k-1}, t_k)\}$ is a set of independent random variables.

From the memoryless property, here is another definition of a Poisson process:

- A Poisson process is a counting process that has the stationay and independent increment properties and such that

$$\begin{aligned} P[\tilde{N}(t, t+\delta) = 0] &= 1 - \lambda\delta + o(\delta) \\ P[\tilde{N}(t, t+\delta) = 1] &= \lambda\delta + o(\delta) \\ P[\tilde{N}(t, t+\delta) \geq 2] &= o(\delta) \end{aligned} \tag{2.2}$$

2.1 Distribution of $N(t)$

S_n is the sum n IID random variables and f_{S_n} is the convolution of n times f_X:

$$f_{S_n}(t) = \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!} \tag{2.3}$$

From this,

$$P[N(t) = n - 1] = \frac{(\lambda t)^n e^{-\lambda t}}{(n)!} \tag{2.4}$$

and finally,

$$\mathbb{E}[N(t)] = \lambda t \qquad Var[N(t)] = \lambda t \tag{2.5}$$

From equation (2.4), the Poisson process verifies the following probability conditions:

- $P[\tilde{N}(t, t+\delta) = 0] = 1 - \lambda\delta + o(\delta)$;
- $P[\tilde{N}(t, t+\delta) = 1] = \lambda\delta + o(\delta)$;
- $P[\tilde{N}(t, t+\delta) \geq 2] = o(\delta)$;

where we use a first-order approximation of the exponential term, with $o(\delta)$ its residual. As $o(\delta)$ is negligible, we can approximate the Poisson process as a Bernoulli process.

2.1.1 Combining Poisson processes

Let $N_1(t)$ and $N_2(t)$ be tow independent Poisson processes. Let the process $N(t) = N_1(t) + N_2(t)$. We can show using the three properties above that $N(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

2.1.2 Subdividing a Poisson process

Let $N(t)$ be a Poisson process with rate λ. We split the arrivals in 2 subprocesses $N_1(t)$ and $N_2(t)$. Each arrival of $N(t)$ is sent to $N_1(t)$ with probability p and to $N_2(t)$ with probability $(1-p)$, each split being independent from all others.
Then, the resulting processes $N_1(t)$ and $N_2(t)$ are two independent Poisson processes with respective rate $p\lambda$ and $(1-p)\lambda$.

2.1.3 Conditional arrival distribution

The density probability function when we have n Poisson processes, under the condition that $N(t) = n$, is

$$f(s_1, \ldots, s_n | N(t) = n) = \frac{n!}{t^n} \tag{2.6}$$

From the previous results, we can compute that

$$P[S_1 > \tau | N(t) = n] = \left(\frac{t - \tau}{t}\right)^n \tag{2.7}$$

and the expectation is
$$E[S_1|N(t) = n] = \frac{t}{n+1} \tag{2.8}$$
And from this, we derive that
$$P[X_i > \tau|N(t) = n] = \left(\frac{t-\tau}{t}\right)^n \tag{2.9}$$
with expectation
$$E[X_i] = \frac{t}{n+1} \tag{2.10}$$
And thus the density function is
$$f_{S_i}(x|N(t) = n) = \frac{x^{i-1}(t-x)^{n-i}n!}{t^n(n-i)!(i-1)!} \tag{2.11}$$

2.2 Non-homogenous Poisson processes

A non-homogenous Poisson rocess $N(t)$ is a counting process with increments that are independent but not stationary, with

- $P[\tilde{N}(t, t+\delta) = 0] = 1 - \lambda(t)\delta + o(\delta)$;
- $P[\tilde{N}(t, t+\delta) = 1] = \lambda(t)\delta + o(\delta)$;
- $P[\tilde{N}(t, t+\delta) \geq 2] = o(\delta)$;

where $\tilde{N}(t, t+\delta) = N(t+\delta) - N(t)$. The time-varying arrival rate $\lambda(t)$ is assumed to be continuous and stricly positive.

2.3 Bernoulli process approximation

We can approximate the non-homogenous Poisson process with a Bernoulli process where the time is partitioned into increments of lengths inversely proportional to $\lambda(t)$ (i.e. using a nonlinear time scale).

- $P[\tilde{N}(t, t+\epsilon/\lambda(t)) = 0] = 1 - \epsilon + o(\epsilon)$;
- $P[\tilde{N}(t, t+\epsilon/\lambda(t)) = 1] = \epsilon + o(\epsilon)$;
- $P[\tilde{N}(t, t+\epsilon/\lambda(t)) \geq 2] = o(\epsilon)$;

Letting ϵ tend to zero, we obtain
$$P[N(t) = n] = \frac{(m(t))^n e^{-m(t)}}{n!} \qquad P[\tilde{N}(t, t') = n] = \frac{(m(t, t'))^n e^{-m(t,t')}}{n!} \tag{2.12}$$
with
$$m(t) = \int_0^t \lambda(\tau) d\tau \qquad m(t, t') = \int_t^{t'} \lambda(\tau) d\tau \tag{2.13}$$

2.4 Classification of queueing systems

- We note $A/B/k$ where A is the type of distribution for the arrival process, B for the service time and k the number of servers.

We suppose that the arrivals wait in a single queue. Commonly used letters are

- M: exponential distribution (for A) or Poisson process (for B);
- D: deterministic time intervals;
- E: Erlang distribution;
- G: general distribution.

Renewal Processes

A renewal process is a counting process with IID interarrival intervals. We note X_i the interval between arrivals, $\bar{X} = \mathbb{E}[X]$ is supposed to be finite with probability $P[X_i] > 0 = 1$[1], σ can be finite, and we denote $S_n = \sum_{i=1} X_i$ the time of the n-th arrival.

3.1 Strong law of large numbers

Let $\{N(t); t \geq 0\}$ be a renewal process, then

$$\lim_{t\to\infty} N(t) = \infty \qquad \lim_{t\to\infty} \mathbb{E}[N(t)] = \infty \tag{3.1}$$

This implies that

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\bar{X}} \text{ with probability } 1 \tag{3.2}$$

3.2 Central limit theorem

If the interarrival intervals of the renewal process $N(t)$ have a finite standard deviation, then from the CLT for IID random variables, we have

$$\lim_{t\to\infty} P\left[\frac{S_n - n\bar{X}}{\sqrt{n}\sigma} \leq \alpha\right] = \Phi(\alpha) \tag{3.3}$$

What is $\Phi(\alpha)$?
and

$$\lim_{t\to\infty} P\left[\frac{N(t) - t/\bar{X}}{\sigma\bar{X}^{-3/2}\sqrt{t}} < \alpha\right] = \Phi(\alpha) \tag{3.4}$$

$\to$ Note: The reliability of the observed mean of successive results that are supposed to be IID depends a lot on the rule used to decide when we stop repeating the experiment.

[1] A probability of 1 means that the opposite can happen, but is so rare that the probability is 0.

3.3 Stopping time

Let N be the rv corresponding to the total number of experiments observed. Let I_n be a series of rv being the indicator fucnction of $\{N \geq n\}$:

$$I_n = \begin{cases} 1 & \text{if the } n\text{-th experiment is observed} \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

N is a stopping time if I_n depends only on $X_1, \ldots, X_{n-1}$. This means that stopping at 3pm, for example, is not a stopping time, because it can depend on X_n, depending if the n-th arrival is before or after 3pm.

3.3.1 Wald's inequality

Let N be a stopping time for $\{X_n; n \geq 1\}$. Then, $\mathbb{E}[S_N] = \mathbb{E}[N]\bar{X}$.

3.4 Blackwell's renewal theorem

3.4.1 Arithmetic distribution

If interarrival intervals can only have a length that is a multiple of some real number d, the interarrival distribution will be called an arithmetic distribution, and d the span of the distribution.

3.4.2 Blackwell's inequality

If the interarrival distribution of a renewal process $N(t)$ is not arithmetic, then

$$\lim_{t \to \infty} (m(t+\delta) - m(t)) = \frac{\delta}{\bar{X}} \qquad \forall \delta \tag{3.6}$$

If the interarrival distribution is arithmetic with span d, then

$$\lim_{t \to \infty} (m(t+nd) - m(t)) = \frac{nd}{\bar{X}} \qquad \forall n \geq 1 \tag{3.7}$$

3.4.3 Relationship with a Poisson process

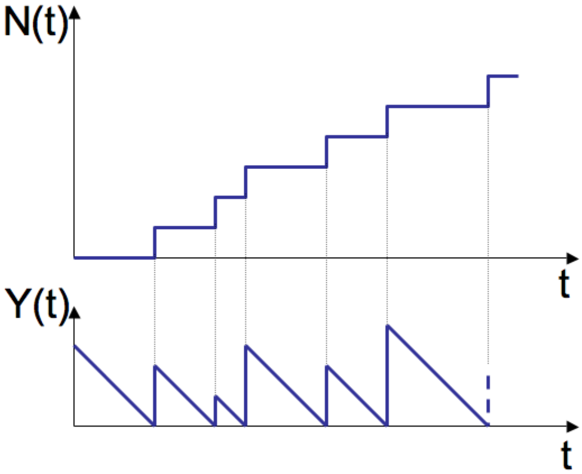
The sum of many renewal processes tends to a Poisson process: for a non-arithmetic renewal process with $P[X_i = 0] = 0$, we have

$$\begin{aligned} &\lim_{t \to \infty} P[N(t+\delta) - N(t) = 0] = 1 - \delta/\bar{X} + o(\delta) \\ &\lim_{t \to \infty} P[N(t+\delta) - N(t) = 1] = \delta/\bar{X} + o(\delta) \\ &\lim_{t \to \infty} P[N(t+\delta) - N(t) \geq 2] = o(\delta) \end{aligned} \tag{3.8}$$

The increments are asymptotically stationary, but not independent. |sectionRenewal reward process Along to the renewal process $N(t)$, we can add a reward function $R(t)$.

It models the rate at which the process is accumulating a reward or cost. It can however only depend on the current renewal but not the previous ones.
Let $Y(y)$ be the residual life at time t for the current renewal:

$$R(t) = Y(t) = S_{N(t)+1} - t \tag{3.9}$$

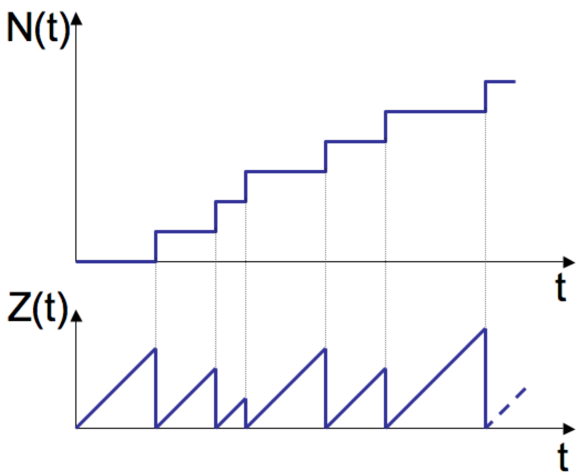


The time average residual life is $\frac{1}{t}\int_0^t Y(\tau)d\tau$.
From the definition of $Y(t)$, we can calculate that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Y(\tau)d\tau = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} = \frac{1}{2}\mathbb{E}[X] + \frac{Var(X)}{\mathbb{E}[X]} > \frac{1}{2}\mathbb{E}[X] \text{ with probability 1} \tag{3.10}$$

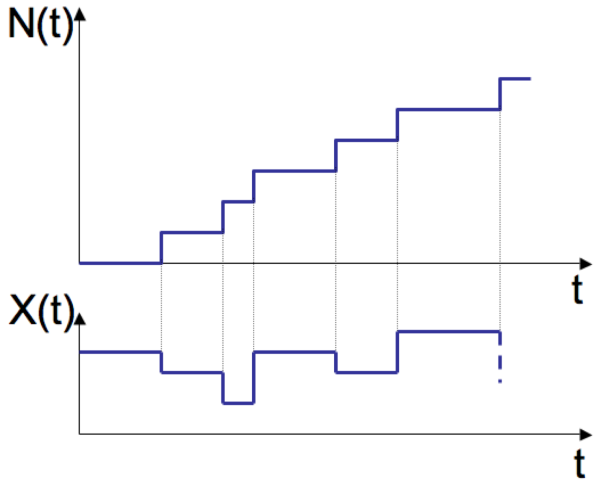
3.4.4 Time average age

Let $Z(t)$ be the age of the current renewal at time t: $R(t) = Z(t) = t - S_{N(t)}$. The time average age is $\lim_{t\to\infty} \frac{1}{t}\int_0^t Z(\tau)d\tau = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]}$.



3.4.5 Time average duration

Let $X(t)$ be the duration of the renewal containing time t: $R(t) = X(t) = S_{N(t)+1} - S_{N(t)}$. The time average duration is $\lim_{t\to\infty} \frac{1}{t}\int_0^t X(\tau)d\tau = \frac{\mathbb{E}[X^2]}{\mathbb{E}[X]}$.



| subsectionGeneral renewal reward functions Let $R(t)$ be a reward function for a renewal process with expected inter-renewal times $\bar{X} < \infty$, then with probability 1,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t R(\tau)d\tau = \frac{\mathbb{E}[R_n]}{\mathbb{E}[X]} \tag{3.11}$$

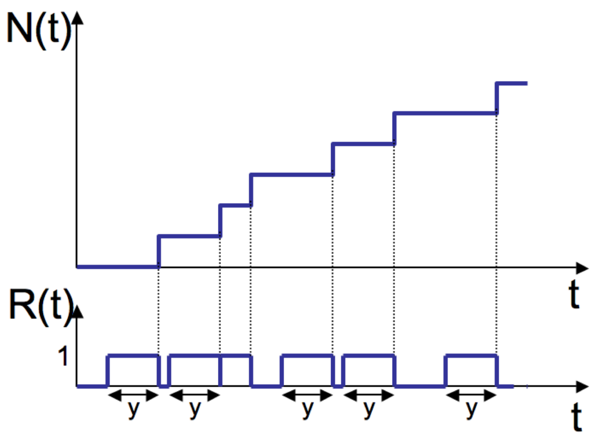
where R_n is defined as

$$R_n = \int_{S_n}^{S_{n+1}} R(\tau)d\tau \tag{3.12}$$

3.4.6 Distribution of residual life

We are interested in the fraction of time that $Y(t) \leq y$:

$$R(t) = I\{Y(t) \leq y\} \qquad\qquad R_n = \min\{y, X_n\} \tag{3.13}$$



And we can calculate that

$$\begin{aligned} \mathbb{E}[R_n] &= \int_0^y P[X > x]dx \\ F_Y(y) &= \frac{1}{\mathbb{E}[X]} \int_0^y P[X > x]dx \end{aligned} \tag{3.14}$$

3.4.7 Key theorem

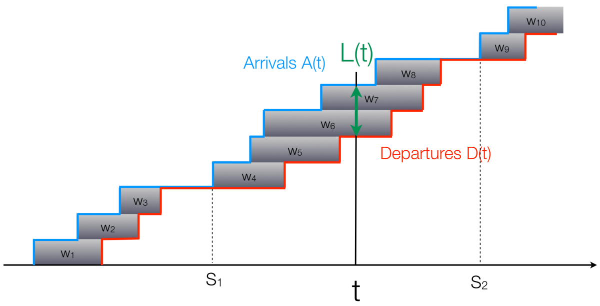
Let $N(t)$ be a non-arithmetic renewal process, let $R(z,x) \geq 0$ be such that $r(z) = \int_{x=z}^{\infty} R(z,x)dF_X(x)$ is directly Rieman integrable. Then,

$$\lim_{t\to\infty} \mathbb{E}[R(t)] = \frac{\mathbb{E}[R_n]}{\bar{X}} \tag{3.15}$$

3.5 Little's Law

Let a queueing system be such that

- $A(t)$ is the number of arrivals between 0 and t;
- $D(t)$ is the number of departures between 0 and t;
- $L(t) = A(t) - D(t)$ is the number of customers in the system at time t;
- w_i the time the i^{th} customer spends in the system;
- $N(t)$ is the renewal process counting the number of busy periods of the system (each time a customer arrives when the system is empty).



A general queueing system

Let us use $L(t)$ as a reward function for the renewal process $N(t)$. This implies

$$\begin{aligned} \sum_{n=1}^{N(t)} R_n \leq \int_0^t L(\tau)d\tau \leq \sum_{i=1}^{A(t)} w_i \leq \sum_{n=1}^{N(t)+1} R_n \\ \lim_{t\to\infty} \frac{1}{t} \int_0^t L(\tau)d\tau = \frac{\mathbb{E}[R_n]}{\mathbb{E}[X]} \end{aligned} \tag{3.16}$$

Putting all this together, we can show that $\bar{L} = \lim_{t\to\infty} \frac{1}{t} \int_0^t L(\tau)d\tau = \bar{W}\lambda$.

3.5.1 M/G/1 queue

Let $R(t)$ be the remaining time for the customer being served. Let $U(t)$ be the time an arrival at time t would have to wait before being served. Let $L_q(t)$ be the number of customers in queue at time t, independent of the Z_i. We define

$$U(t) = \sum_{i=1}^{L_q(t)} Z_i + R(t) \Longrightarrow \mathbb{E}[U(t)] = \mathbb{E}[L_q(t)]\mathbb{E}[Z] + \mathbb{E}[R(t)] \tag{3.17}$$

We can show that

$$\int_0^{S_N(t)} R(\tau)d\tau \leq \int_0^{S_N(t)+1} R(\tau)d\tau \tag{3.18}$$

And from Little's Law,

$$\lim_{t\to\infty} \mathbb{E}[L_q(t)] = \lambda \bar{W}_q \Longrightarrow \lim_{t\to\infty} \mathbb{E}[U(t)] = \lambda \bar{W}_q \mathbb{E}[Z] + \lambda \frac{\mathbb{E}[Z^2]}{2} \tag{3.19}$$

Poisson arrival process implies that arrivals occur with identical probability at any moment, this implies independence with $U(t)$. Hence $\mathbb{E}[W_q(t)] = \mathbb{E}[U(t)]$. Hence $\bar{W}_q = \lambda \bar{W}_q \mathbb{E}[Z] + \lambda \frac{\mathbb{E}[Z^2]}{2}$. And we can isolate $\bar{W}_q$:

$$\bar{W}_q = \frac{\lambda(\mathbb{E}[Z]^2 + \sigma^2)}{2(1 - \lambda \mathbb{E}[Z])} \tag{3.20}$$

And we remember $\bar{W} = \bar{W}_q + \mathbb{E}[Z]$.

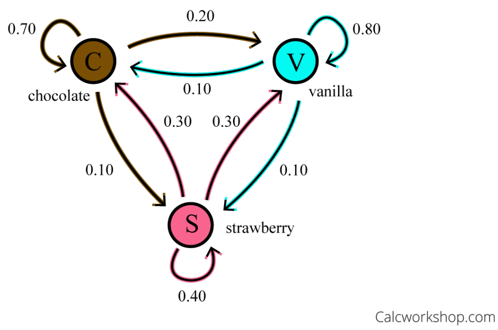
Finite State Markov Chains

4.1 Definitions

A Markov chain is a stochastic process with fixed intervals $\{X_n, n \geq 0\}$ such that each random variable $X_n, n \geq 1$ depends on the past only through the most recent random variable X_{n-1}:

$$P[X_n = j | X_{n-1} = i, X_{n-2} = k, \ldots, X_0 = m] = P[X_n = j | X_{n-1} = i] = P_{ij} \tag{4.1}$$

The rv X_n is called the state of the Markov chain, and the set of possible sample values for the states lie in a countable set. A Markov chain can be represented under a graph or matrix form:



$$P = \begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.10 & 0.80 & 0.10 \\ 0.30 & 0.30 & 0.40 \end{pmatrix} \tag{4.2}$$

- We say that a state j is accessible from i ($i \to j$) if there exists a xalk in the graph from i to j: $i \to j$ iff $P^n_{ij} = P[X_n = j | X_0 = i] > 0$ for some n.
- Two distinct states i, j communicate ($i \leftrightarrow j$) if i is accessible from j and vice versa.
- A class C of states is a non-empty set of states such that for each $i \in C$, each state $j \neq i$ satisfies $j \in C$ if $i \leftrightarrow j$ and $j \notin C$ if $i \not\leftrightarrow j$.
- A state i is recurrent if it is accessible from all states that are accessible from i. A transient state is a state that is not recurrent.

$\to$ Note: all states of a same class are of the same type.

- A finite-state Markov chain has at least one recurrent class.
- The period of a state i, denoted $d(i)$, is the greatest common divisor of all n such that $P^n_{ii} > 0$. A state is aperiodic if $d(i) = 1$.

$\to$ Note: All states of a class have the same periodicity.

- If a class has a period $d > 1$, then there existsa partition $\{C_i\}_{i=1}^d$ of the states of the class such that all the transitions from a state of C_n go to a state of class C_{n+1} and all transitions from C_d go to a state of C_1, i.e. we make a cycle of subclasses.
- A class is called ergodic if it is aperiodic and recurrent.
- A matrix is stochastic iff it is square, non negative ad each row sums to 1, i.e. $P\mathbb{1}_n = \mathbb{1}_n$.

4.2 Transition probabilities

We can calculate that

$$P[X_{n+2} = j | X_n = i] = P_{ij}^2 = \sum_{k=1}^{J} P_{ik}P_{kj} \Longrightarrow P^2 = P \cdot P \Longrightarrow P^n = P \cdot ... \cdot P \tag{4.3}$$

More generally, $P_{ij}^{n+m} = \sum_{k=1}^{J} P_{ik}^n P_{kj}^m$.
Because of ergodicity, all rows converge to the same value, and we store those values in a row vector called π. Then, $\pi = \pi P$ and the sum of all values of π is 1.

From this, we induce that π is a left eigenvector of P for the eigenvalue 1, and the number of linearly independent solutions corresponds to the multiplicity of the eigenvalue 1. There will be one independent solution for each recurrent class of P. Moreover, if P is ergodic, then $\lim_{n\to\infty} P^n = \mathbb{1}_n\pi$, else π will be the average over the different subclasses.

4.3 MArkov chains with rewards