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# LINMA2345 Game Theory

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# Basic models

In a model, we need to define the actions, the knowledge, the randomness and the outcomes.

A private information is any knowledge on the state of the world that is not a common knowledge. An information  $X$  is said to be common knowledge if the statement (everybody knows that) <sup>$K$</sup>  everybody knows  $X$  holds for any integer  $K$

## 1.1 Extensive form games

The extensive form is a tree. The nodes trigger something: they can be triggered by a player or by chance. The edges are the actual events of the game, from a player or from chance. The leaves are the payoffs, containing one item per player.

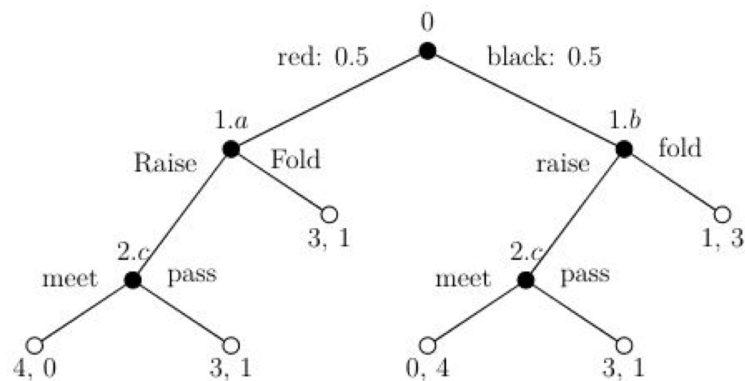


Figure 1.1: Tree form of a game.

The extensive form contains the following elements:

- $N$ : the set of players (e.g. by name);
- $S$ : the set of information states, i.e. the information to which each player has access;
- $D$ : the set of moves for a player that is at a certain information states (meet or pass, raise or stop, etc.);
- $V$ : the set of nodes (e.g. numbered);
- $V_0$ : the set of chance nodes;

- $Y$ : the set of action nodes, i.e. where someone does something;
- $X$ : the set of leaves;
- $t$ : the transitions, i.e. the outgoing edges from a node;
- $p$ : the probability distribution for the random transitions;
- $w$ : the payoffs for each player at each leaf.

### 1.1.1 Concepts

- Perfect recall: Players remember their past states and moves;
- Perfect information: Every player knows where they are in the game;
- Pure strategy: A function associates a move to each information state, i.e. if something happens, the strategy says what to do next;
- Payoff of a strategy: It is the expected final value, i.e. the probability of an result times its payoff;
- Randomized strategy: We assign, at each information state, a probability distribution on the moves.

## 1.2 Strategic form games

A strategic game is a tuple  $\Gamma = (N, C, u)$  where  $N$  is the set of players,  $C$  the set of pure strategies for all players, and  $u$  the payoff function. Using a randomized strategy, we have the following expected payoff formula, with the probability distribution  $\sigma_i$ :

$$E_{\sigma}(u_i(c)) = \sum_{c \in C} \sigma(c) u_i(c) = \sum_{c_i \in C_i} \sigma_i(c_i) \sum_{c_{-i} \in C_{-i}} \sigma_{-i}(c_{-i}) u_i(c_i, c_{-i}) \quad (1.1)$$

where  $-i$  denotes the set of all players except  $i$ , and  $\sigma_{-i}(c_{-i}) = \prod_{j \neq i} \sigma_j(c_j)$ .

Consider a player  $i \in N$ . If the players  $-i$  play following the strategy profile  $\sigma_{-i}$ , then the set of best response strategies for player  $i$  is given by

$$\arg \max_{c_i \in C_i} \sum_{c_{-i} \in C_{-i}} \sigma_{-i}(c_{-i}) u_i(c_i, c_{-i}) \quad (1.2)$$

### 1.2.1 Strong domination

A strategy is strongly dominated if it cannot be a best response.

Corentin vs François	A	B	C	D
a	8, 5	4, 6	1, 6	7, 8
b	5, 6	7, 7	4, 4	2, 9
c	6, 7	5, 5	2, 6	3, 6
d	5, 4	5, 2	6, 5	7, 5

It means that, for a given column, there exists another one where all the second numbers are strictly bigger than in the considered one. This works too for the rows, using the first numbers. In that case, the column or row can be completely removed, as the player never has an interest in using that strategy. In the example above, column B is strongly dominated by column D.

→ Note: once a column or row has been removed, we can continue to iterate on the smaller table until no row or column can be removed again. The convergence to a certain table is guaranteed, whatever the order of removal.

### 1.2.2 Weak domination

Weak domination changes the strictly bigger into a bigger or equal. In that case, it is not always a good move to remove the strategy, particularly if the game includes randomness.

→ Note: Removing the weakly dominated strategies does not conserve convergence.

## 1.3 Equivalent models

Two games are fully equivalent if

$$\forall i \in N, \exists A_i > 0, B_i : \hat{u}_i(c) = A_i u_i(c) + B_i \quad \forall c \in C \quad (1.3)$$

This means that the scale of the payoffs changes nothing.

### 1.3.1 Best-response equivalence

Two games are best-response equivalent iff they have the best response sets, i.e.

$$c'_i \in \arg \max_{c_i \in C_i} \sum_{c_{-i} \in C_{-i}} \sigma_{-i}(c_{-i}) u_i(c_i, c_{-i}) \iff c'_i \in \arg \max_{c_i \in C_i} \sum_{c_{-i} \in C_{-i}} \sigma_{-i}(c_{-i}) \hat{u}_i(c_i, c_{-i}) \quad (1.4)$$

## 1.4 Bayesian game

In a Bayesian game, we add the set of player types  $T$  and the belief functions. What was before a set of players is now a set of agents, each of which is a player in a certain state depending on the randomness of some piece of information.

# Nash Equilibrium

## 2.1 Definitions

- Given a game in extensive form  $\Gamma_e$ , a pure strategy for a player  $i \in N$  is an element  $c \in C_i = \times_{s \in S_i} D_s$ , acting as a function that associates to each information state of the player a move to be played at that state. A pure strategy is such that, for any move of the opponent(s) and at any time in the game, it prescribes a move to be played.
- A randomized strategy  $(\sigma_1, \dots, \sigma_N)$  consists in choosing one of the pure strategies with a certain probability. For example, in rock paper scissors, a strategy is to do one of the three possibilities with probability  $1/3$  for each. They verify the following conditions:

$$\begin{aligned} \sum_{c_i \in D_i} \sigma_i(c_i) &= 1 \\ \forall d_i \in D_i : \quad \sigma_i(d_i) &\geq 0 \\ \forall e_i \in C_i \setminus D_i : \quad \sigma_i(e_i) &= 0 \end{aligned} \tag{2.1}$$

- A Nash equilibrium is such that no player alone has an interest to deviate from his current situation. For example, in the prisoner's dilemma, the Nash equilibrium is that both players defect, because none of them can decide to cooperate and have a higher payoff. In a Nash equilibrium, all strategies are best responses to  $\sigma_{-i}$ , and thus they pay-offs are equal:

$$\begin{aligned} \forall d_i \in D_i : \quad \sum_{c_{-i} \in D_{-i}} \left( \prod_{j \in N-i} \sigma_j(c_j) \right) u_i(c_{-i}, d_i) &= w_i \\ \forall e_i \in C_i \setminus D_i : \quad \sum_{c_{-i} \in D_{-i}} \left( \prod_{j \in N-i} \sigma_j(c_j) \right) u_i(c_{-i}, e_i) &\leq w_i \end{aligned} \tag{2.2}$$

→ Note: the product in those equations corresponds to the probabilities of the other players strategies, and the  $u_i$  next to it correspond to the rewards considering each player's strategy. The first equation means that all selected strategies have the same payoff, and the second one means that all strategies are the best response.

- The support of a game consists in the set of all possible combinations of strategies for all players. It can contain pure strategies as well as randomized strategies.

- A zero-sum game is such that the sum of payoffs is equal to zero in all situations, i.e. some players win what the other lose.

## 2.2 Properties

- There always exists at least one Nash equilibrium.
- If there exists more than one Nash equilibrium in pure strategies, then there exists an infinity of them in randomized strategies, as any linear combination of the Nash equilibria is also a Nash equilibrium.
- Given a two-person zero-sum game  $\Gamma(1, 2, (C_1, C_2), (u_1, -u_1))$ , the strategy profile  $\sigma = (\sigma_1, \sigma_2)$  is a Nash equilibrium iff

$$\sigma_1 \in \arg \max_{\tau_1 \in \Delta(C_1)} \min_{\tau_2 \in \Delta(C_2)} u_1(\tau_1, \tau_2) \quad (2.3)$$

and

$$\sigma_2 \in \arg \min_{\tau_2 \in \Delta(C_2)} \max_{\tau_1 \in \Delta(C_1)} u_1(\tau_1, \tau_2) \quad (2.4)$$

If  $(\sigma_1, \sigma_2)$  is a Nash equilibrium, then

$$u_1(\sigma_1, \sigma_2) = \max_{\tau_1 \in \Delta(C_1)} \min_{\tau_2 \in \Delta(C_2)} u_1(\tau_1, \tau_2) = \min_{\tau_2 \in \Delta(C_2)} \max_{\tau_1 \in \Delta(C_1)} u_1(\tau_1, \tau_2) \quad (2.5)$$

- In a game containing  $N$  players and  $M$  moves, the total number of pairs of strategies is  $(2^M - 1)^N$ .

## 2.3 How to find them

- Remove all strongly dominated strategies;
- Check if some of the remaining strategies will never be played, whatever the opponents choose.

## 2.4 Drawbacks

- There is no way to choose between the several equilibria, when there are more than one.
- No communication between agents can lead to a non optimal solution (see for example the prisoner's dilemma).
- The table form when looking for equilibria does not display all the information of the game.
- Finding the equilibria can be computationnally intensive.

# Decision theory

Decision theory consists in studying a rational and intelligent agent, taking decisions in an uncertain context to achieve a specific goal.

- Intelligence: to take in sylla
- Rationality: same

## 3.1 Lottery function

Uncertainty can be objective (e.g. dice) or subjective (e.g. a horse race).

A lottery function is  $f : \Omega \rightarrow \Delta(X)$ , where  $\Omega$  is the set of all possible states of the world, i.e. the possible realizations of the uncertainty (subejective part).  $\Delta(X)$  is the distribution of probabilities  $\Delta$  over the set of payoffs  $X$ .

- We denote  $P(\Omega)$  the set of all non-empty subsets of the set of states of the world  $\Omega$ .
- An event  $S \in P(\Omega)$  is a non-empty subset of  $\Omega$ .

## 3.2 Preferences

Given two lotteries  $f$  and  $g$ , and an event  $S \in P(\Omega)$ ,

- $f \succsim_S g$  if the player considers  $f$  to be at least as good as  $g$  when the tru state of the world belongs to  $S$ ;
- $f \sim_S g$  if both  $f \succsim_S g$  and  $g \succsim_S f$ , i.e. the player is indifferent between the two lotteries given  $S$ ;
- $f \succ_S g$  if  $f \succsim_S g$ , but  $g \not\succsim_S f$ , i.e. the player stricly prefers  $g$  over  $f$  given  $S$ .

## 3.3 Axioms

1. **Completeness:** either  $f \succsim_S g$  or  $g \succsim_S f$ ;
2. **Transitivity:** if  $f \succsim_S g$  and  $g \succsim_S h$ , then  $f \succsim_S h$ ;
3. **Relevance:** if  $f(\cdot|t) = g(\cdot|t)$  for all  $t \in S$ , then  $f \sim_S g$ ;



4. **Monotonicity:** if  $f \succ_S h$  and  $0 \leq \beta \leq \alpha \leq 1$ , then  $\alpha f + (1 - \alpha)h \succ_S \beta f + (1 - \beta)h$ ;
5. **Continuity:** if  $f \succ_S g \succ_S h$ , then  $\exists \alpha \in [0, 1]$  such that  $g \sim_S \alpha f + (1 - \alpha)h$ ;
6. **Objective substitution:** if  $e \succ_S f$  and  $g \succ_S h$  and  $\alpha \in (0, 1]$ , then  $\alpha e + (1 - \alpha)g \succ_S \alpha f + (1 - \alpha)h$ ;
7. **Strict objective substitution:** Identical with strict conditions;
8. **Subjective substitution:** if  $f \succ_S g$  and  $f \succ_T g$  and  $S \cap T = \emptyset$ , then  $f \succ_{S \cup T} g$ ;
9. **Strict subjective substitution:** Identical with strict conditions;
10. **Interest:** for every state  $t \in \Omega$ , there exists  $x, y \in X$  such that  $x \succ_t [y]$ ;
11. **State neutrality (optional):** For any states  $t, r \in \Omega$ , if  $f(\cdot, t) = f(\cdot, r)$ ,  $g(\cdot, t) = g(\cdot, r)$  and  $f \succ_r g$ , then  $f \succ_t g$ .

### 3.4 Utility maximization theorem

- A utility function is a function  $u : X \times \Omega \rightarrow \mathbb{R}$  that assigns a value to each prize in each state of the world.
- A conditional probability function on  $\Omega$  is a function  $p : P(\Omega) \rightarrow \Delta(\Omega)$  that specifies for all  $S \in P(\Omega)$  a probability distribution on  $\Omega$  such that, for all  $t \in \Omega$ ,

$$p(t|S) = 0 \text{ if } t \notin S \text{ and } \sum_{r \in S} p(r|S) = 1.$$

- The expected utility value of a lottery  $f$  given an event  $S$  and a conditional probability function  $p$  is given by

$$\mathbb{E}_p(u(f)|S) = \sum_{t \in S} \left[ p(t|S) \sum_{x \in X} u(x, t) f(x, t) \right].$$

$p(t|S)$  is the probability of each outcome occurring,  $u(x, t)$  the utility for said outcome and prize, and  $f(x, t)$  the probability of said outcome and prize.

Consider a decision problem. The axioms 1 to 10 are satisfied iff there exists a utility function  $u$  and a conditional probability function  $p : P(\Omega) \rightarrow \Delta(\Omega)$  such that

1. **Normative assumption:** the utility function satisfies  $\max_{x \in X} u(x, t) = 1$  and  $\min_{x \in X} u(x, t) = 0 \forall t \in \Omega$ ;
2. **Bayes:** the conditional probability function satisfies, for all sets  $R \subseteq S \subseteq T \subseteq \Omega$  with  $S \neq \emptyset$ ,

$$p(R|T) = p(R|S)p(S|T),$$

where, for two sets  $A, B \subseteq \Omega$ ,  $p(A|B) = \sum_{a \in A} p(a|B)$ .

3. for any two lotteries  $f$  and  $g$  and for any event  $S$ , the preference relation  $f \succ_S g$  holds iff

$$\mathbb{E}_p(u(f)|S) \geq \mathbb{E}_p(u(g)|S).$$

Axiom 11 is also satisfied on top of the others iff the above holds with a state-independent utility function, i.e. there exists a function  $U : X \rightarrow \mathbb{R}$  such that  $u(x, t) = U(x)$  for all  $t \in \Omega$ .

### 3.4.1 Equivalence of the representations

Let us consider two decision making problems, the first with a utility function  $u$  and a conditional probability function  $p$ , and the second with a utility function  $v$  and a conditional probability function  $q$ . The two problems are equivalent iff,  $\forall S \subseteq \Omega, \forall t \in S, \forall x \in X$ , there exists  $A > 0$  and  $B : S \rightarrow \mathbb{R}$  such that  $q(t|S)v(x, t) = Ap(t|S)u(x, t) + B(t)$ .

## 3.5 Bayesian probability system

A conditional probability function  $p : P(\Omega) \rightarrow \Delta(\Omega)$  is a Bayesian probability system on the set  $\Omega$  if  $\forall S \subseteq \Omega, S \neq \emptyset, p(\cdot|S)$  is a probability distribution on  $\Omega$  such that  $\forall R \subseteq S, \forall T \supseteq S : p(S|S) = 1$  and  $p(R|T) = p(R|S)p(S|T)$ .

## 3.6 General method for decision making

1. Compute the probability distribution;
2. Express the loss function for each decision;
3. Compute the expected values of the loss function for each decision;
4. Select the decision that minimizes the loss.

Add the syllabus information in all previous chapters.

# Sequential Equilibria

## 4.1 Rational and irrational equilibrium

In this chapter, we only work in extensive form, to be able to be rational at each state. Nash equilibria are not necessarily rational, and this chapter aims to find tools to distinguish this characterization.

**Theorem 4.1.** If  $\Gamma^e$  is an extensive form game with perfect information, then there exists at least one sequential equilibrium of  $\Gamma^e$  in pure strategies. Furthermore, for all generic games with perfect information, there is exactly one sequential equilibrium.

## 4.2 Behavioural strategies

**Definition 4.2. Game in extensive form.** For a game in extensive form  $\Gamma^e$ , we define the following:

- $N$ : set of players;
- $S = \cup_{i \in N} S_i$ : set of all information states, where  $S_i$  are the information states of  $i \in N$ , and  $S_i \cap S_j = \emptyset$ ;
- $D = \cup_{s \in S} D_s$ : set of all possible moves, where  $D_s$  is the set of moves available at state  $s \in S$ ;
- $Y_s$ : set of nodes having information state  $s \in S$ ;
- $X = \cup_{s \in S} Y_s$ : set of nodes in the tree;
- $\Omega \subset X$ : set of leafs in the tree, with  $X^0$  the root.

See figure 1.1 to distinguish states (1.a, 1.b, etc.) and nodes ( $X_0, X_1, \dots$ ).

Let us call  $C_i$  the set of pure strategies available to player  $i$ , and  $\Delta C_i$  the set of mixed strategies available to player  $i$ .

**Definition 4.3.** A behavioural strategy for player  $i \in N$  specifies, at each information state  $s \in S_i$ , a probability distribution on the moves  $D_s$  of the player. For  $s \in S_i$ ,  $d \in D_s$ , and  $\tau_i \in \times_{s \in S_i} \Delta D_s$ ,  $\tau_{i,s}(d)$  is the probability of occurrence of the move  $d$  at information state  $s$  when implementing  $\tau_i$ .

Mixed strategy	Behavioural strategy	↓
$(0.5[a_1, y_1] + 0.3[b_1, z_1] + 0.2[b_1, y_1])$	$(0.5[a_1] + 0.5[b_1], 0.6[z_1] + 0.4[y_1])$	

Figure 4.1: Those strategies represent the same thing, they just display information in different form.

**Definition 4.4. Expected utility at node  $x$ .** Consider a game in extensive form  $\Gamma^e$ . Given a player  $i \in N$ , a node  $x \in X$  and a behavioural strategy profile  $\tau$ , we let

$$U_i(\tau, x) = \sum_{y \in \Omega} P(y|\tau, x) w_i(y) \quad (4.1)$$

where  $w_i(y)$  is the payoff of  $i$  at the terminal node  $y \in \Omega$ .  $U_i(\tau, x)$  denotes the expected utility of  $i$  at  $x$  following  $\tau$ .

The difference between a Nash equilibrium and a sequential equilibrium is that the Nash equilibrium is rational at the start of the game, while a sequential equilibrium is rational at each node. Thus, every sequential equilibrium is Nash, but not the other way around.

**Definition 4.5. Belief vector.** A belief vector for player  $i$  is such that each of the component is the probability assigned by player  $i$  to be at node  $x$  given the fact that  $i$  knows his information state  $s$ . This supposes that there is some incomplete information for player  $i$ .

$$\pi_s(y) = \frac{P(y|\tau, x^0)}{\sum_{x \in Y_s} P(x|\tau, x^0)} \quad (4.2)$$

**Definition 4.6. Sequential value.** The sequential value of a move  $d$  for player  $i$  is the sum on the states  $x$  of the utilities weighted by the belief vector:

$$U_i(d|s, (\tau, \pi)) = \sum_{x \in Y_s} \pi_s(x) U_i((\tau_{-i}, d), x) \quad (4.3)$$

### 4.2.1 Weak consistency

**Definition 4.7.** A belief vector  $\pi$  is weakly consistent with a behavioural strategy profile  $\tau$  if it satisfies

$$\forall s \in S \forall y \in Y_s \quad \left( \sum_{x \in Y_s} P(x|\tau, x^0) \right) \pi_s(y) = P(y|\tau, x^0) \quad (4.4)$$

This simply means that it must satisfy Bayes' theorem.

### 4.2.2 Strong consistency

**Definition 4.8.** A belief vector  $\pi$  is strongly consistent with a behavioural strategy profile  $\tau$  in the game  $\Gamma^e$  if there exists a sequence of profiles  $(\tau^k)_{k=1}^\infty$  such that

- there is a non-zero probabilities on every move;

- $\tau(d) = \lim_{k \rightarrow \infty} \tau^k(d)$ ,  $\forall i \in N$ ,  $\forall s \in S$ ,  $\forall x \in Y_s$ . This means that the sequence converges to the strategy;
- $\pi_s(x) = \lim_{k \rightarrow \infty} \frac{P(x|\tau^k, x^0)}{\sum_{y \in Y_s} P(y|\tau^k, x^0)}$ ,  $\forall s \in S$ ,  $\forall x \in Y_s$ . The belief vector is the limit of the sequence of beliefs given by the Bayes rule.