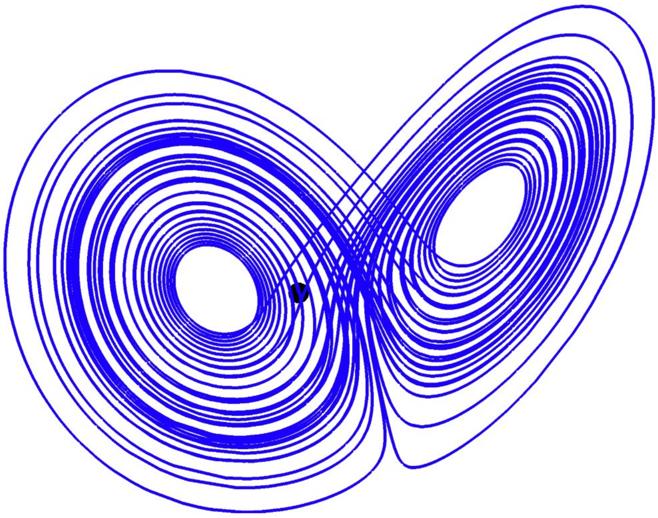

LINMA2370 Modelling and Analysis of Dynamical Systems

Simon Desmidt
Issambre L'Hermite Dumont

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UCLouvain

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Introduction

The tools introduced in this course are a simplifying view of the reality, yet very uselful to build simple and effective models in view of the control and optimization of the dynamical behaviour of the real systems.

1.1 Reminders

- A subset of $\mathbb{R}$ is said to be negligible if its Lebesgue measure is equal to zeroo and that a property is said to be true almost everywhere if it is false only on a negligible set.

- Let $I \subseteq \mathbb{R}$ be an interval the interior of which is not empty. A function $x : I \to \mathbb{R}^N$ is said to be absolutely continuous if

$$\forall \varepsilon \in (0, \infty),\ \exists \delta \in (0, \infty)\ : \\ \forall n \in \mathbb{N} \setminus \{0\},\ \forall a_1, b_1, \ldots, a_n, b_n \in I : \\ a_i < b_i\, \forall i \in \{1, \ldots, n\},\ b_i \leq a_{i+1}\, \forall i \in \{1, \ldots, n-1\}, \\ \sum_{i=1}^{n} (b_i - a_i) \leq \delta \Longrightarrow \sum_{i=1}^{n} \|x(b_i) - x(a_i)\| \leq \varepsilon$$

- Let $a, b \in \mathbb{R}$ with $a < b$. A function $x : [a, b] \to \mathbb{R}$ is absolutely continuous iff there exists an integrable fucntion $\varphi : [a, b] \to \mathbb{R}$ such that, for every $t \in [a, b]$,

$$x(t) = x(a0) + \int_a^t \phi(s)ds$$

  in which case x is almost everywhere differentiable with $\dot{x}(t) = \phi(t)$ for almost every $t \in [a, b]$.

- A function $f : \Omega \to \mathbb{R}^N$, where Ω is a nonempty subset of $\mathbb{R} \times \mathbb{R}^N$, is said to be Lipschitz continuous in the second argument, uniformly with respect to the first argument, if there exists $L \in [0, \infty)$ such that forall $t \in \mathbb{R}$ and all $x, y \in \mathbb{R}^N$ such that $(tx,), (t, y) \in \Omega$,

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

  It is said to be locally Lipschitz continuous on an open ball for each argument.

- Let Ω be a nonempty open subset of $\mathbb{R} \times \mathbb{R}^N$ and $f : \Omega \to \mathbb{R}^N$ be such that

– for all $t \in \mathbb{R}$, $f(t, \cdot) : \Omega_t \to \mathbb{R}^N$
– $\partial_2 f : \Omega \to \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N) : (t, x) \to \partial_2 f(t, x)$ is locally bounded.

Then, f is locally Lipschitz continuous in the second argument, uniformly with respect to the first argument.

- If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two real normed spaces, and the real vector space $\mathcal{L}(X, Y)$ of all continuous linear mappings from X to Y[1] is equipped with the norm defined by

$$\|L\| := \sup_{x \in X \setminus \{0\}} \frac{\|Lx\|_Y}{\|x\|_X}$$

1.2 State-space model

A state-space model for a continuous dynamical system consists of an ODE of the form

$$\dot{x}(t) = f(t, x(t)) \tag{1.1}$$

where the function $f : \Omega \to \mathbb{R}^N$, Ω being a nonempty subset of $\mathbb{R} \times$ *mathbbR*^{N}, is called the vector field associated with the ODE. A continuous dynamical system with input $u : \mathbb{R} \to \mathbb{R}^M$ described by the ODE

$$\dot{x}(t) = g(x(t), u(t)) \tag{1.2}$$

for some function $g : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$, can be written in the form 1.1 by defining the vector field

$$f_u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N : (t, x) \to g(x, u(t)) \tag{1.3}$$

$\to$ N.B.: the norm of each $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ is defined as $|t| + \|x\|$.

1.3 Integral curve

Let Ω be a nonempty subset of $\mathbb{R} \times \mathbb{R}^N$. An integral curve of $f : \Omega \to \mathbb{R}^N$ is a function $x : I \to \mathbb{R}^N$ where $I \subseteq \mathbb{R}$ is an interval, for which the interior is not empty, called the interval of existence of x, i.e. differentiable and satisfies $(t, x(t)) \in \Omega$ and $\dot{x}(t) = f(t, x(t))$ for all $t \in I$. The graph $\{(t, x(t)) | t \in I\}$ and the image $\{x(t) | t \in I\}$ of x are respectively called the trajectory and the orbit of x. Given an initial condition $(t_0, x_0) \in \Omega$, a solution to the initial value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \tag{1.4}$$

is an integral curve $x : I \to \mathbb{R}^N$ of f such that $t_0 \in I$ and $x(t_0) = x_0$.

[1] Meaning matrix from X to Y

If, for the IVP described hereabove, f is continuous, then a continuous function $x : I \to \mathbb{R}^N$ where $I \subseteq \mathbb{R}$ is an interval containing t_0 and the interior of which is not empty, is a solution iff its graph is contained in Ω and it satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

for all $t \in I$. In that case, $\dot{x}$ is continuous.

Let Ω be a nonempty subset of $\mathbb{R} \times \mathbb{R}^N$. An integral curve in the extended sense of $f : \Omega \to \mathbb{R}^N$ is a function $x : I \to \mathbb{R}^N$, where $I \subseteq \mathbb{R}$ is an interval the interior of which is not empty called the interval of existence of x, that is absolutely continuous and satisfies $(t, x(t)) \in \Omega$ for every $t \in I$ and $\dot{x}(t) = f(t(x(t)))$ for almost every $t \in I$.

$\to$ N.B.: If f is continuous, then the two definitions of integral curves are equivalent.

1.4 Existence of a solution

Consider the IVP defined hereabove with an integral curve in the extended sense, under the following assumptions:

- there exists $\tau, r \in (0, \infty)$, such that $[t_0 - \tau, t_0 + \tau] \times B(x_0, r) \subseteq \Omega$;
- for every $x \in B(x_0, r)$, the function $[t_0 - \tau, t_0 + \tau] \to \mathbb{R}^N : t \to f(t, x)$ is measurable;
- for every $t \in [t_0 - \tau, t_0 + \tau]$, the function $B(x_0, r) \to \mathbb{R}^N : x \to f(t, x)$ is continuous;
- there exists an integrable function $m : [t_0 - \tau, t_0 + \tau] \to [0, \infty)$ such that

$$\|f(t, x)\| \leq m(t) \text{ for all } (t, x) \in [t_0 - \tau, t_0\tau] \times B[x_0, r]$$

Then, there exists a solution defined on a compact interval the interior of which contains t_0.
In particular, for the IVP with an integral curve in the general sense, if (t_0, x_0) is an interior point of Ω and f is continuous, then there exists a solution defined on a compact interval the interior of which contains t_0.

Dynamical systems and state-space models

We will study first-order dynamical systems of the form

$$\dot{x} = f(x, u) \tag{2.1}$$

where f is a mapping from $\mathbb{R}^{n+m}$ to $\mathbb{R}^n$, while x and u are vector functions of time, respectively the state and the input.

2.1 Terminology and notation

- We assume that the input is a piecewise continuous and bounded function: $u \in \mathcal{U}$, where $\mathcal{U}$ is a set of piecewise continuous and bounded functions from $\mathbb{R}$ to $\mathbb{R}^m$.

- For a given value of the initial state $x(t_0) = x_0$ a,d a given input u, the solution $t \to x(t)$ for $t \geq t_0$, of the system of ODE 2.1 is called the trajectory of the system. It is denoted $x(t_0, x_0, u)$.

- When the input u can be freely chosen in $\mathcal{U}$, the system $\dot{x} = f(x, u)$ is said to be a forced/controlled system.

$\rightarrow$ N.B.: in this course, we will study the solution of the equation 2.1 when the input is actually an a priori set constant: $u(t) = \overline{u}\ \forall t \geq t_0$. The state-space model is then written as $\dot{x} = f(x, \overline{u}) = f_{\overline{u}}(x)$.

2.1.1 System with affine input

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i = f(x) + G(x)u \tag{2.2}$$

where f and g_i are mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$.

2.1.2 System with affine state

$$\dot{x} = \sum_{i=1}^{n} x_i a_i(u) + b(u) = A(u)x + b(u) \tag{2.3}$$

where b and a_i are mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$.

2.1.3 Bilinear systems

A bilinear system is affine both in the state and in the input:

$$\dot{x} = \left(A_0 + \sum_{i=1}^{m} u_i A_i \right) x + B_0 u \tag{2.4}$$

where A_i and B_i are matrices of dimensions $n \times n$ and $n \times m$ respectively.

2.1.4 Linear system

$$\dot{x} = Ax + Bu \tag{2.5}$$

where A and B are matrices of dimensions $n \times n$ and $n \times m$ respectively.