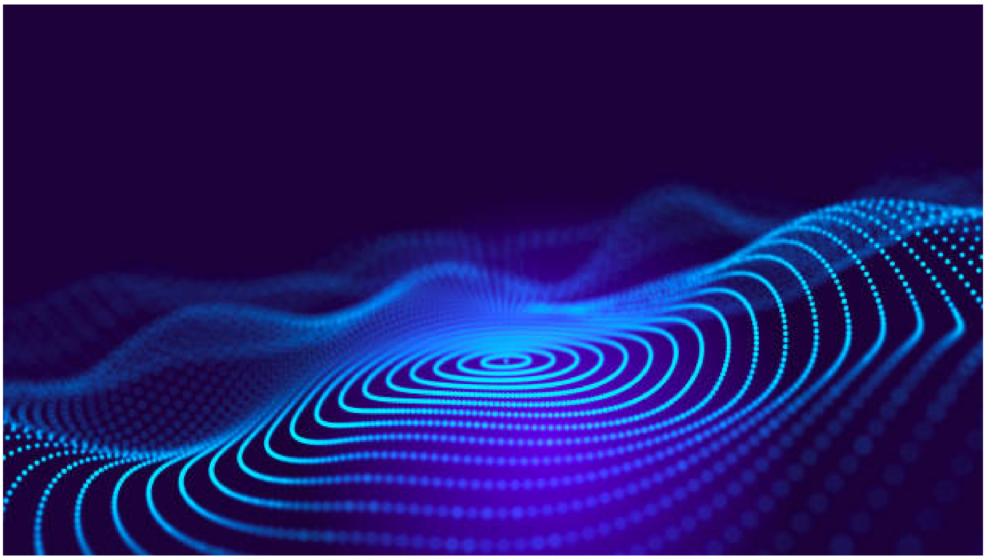

LMECA2300 Advanced Numerical Methods

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Academic year 2024-2025 - Q2

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Contents

2-D acoustic and electromagnetic waves

1.1 Physical laws

The expression of the force of the source applied to its surrounding is derived from Newton's law $F = ma$:

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p = r_v \tag{1.1}$$

where ρ_0 is the average density of the surrounding, $\vec{u}$ is the velocity field, p is the variation of pressure and r_v is the "pressure force".
The conservation law of energy is

$$\nabla \cdot \vec{u} + \chi \frac{\partial p}{\partial t} = s_v \tag{1.2}$$

where χ is the compressibility $[kg^{-1}ms^2]$ and is given by the equation $\frac{\rho}{\rho_0} = \chi p$. s_v is the "velocity source".

1.2 Wave equation

The wave equation is

$$\nabla^2 p - \rho_0 \chi \frac{\partial^2 p}{\partial t^2} - -\rho_0 \frac{\partial s_V}{\partial t} \tag{1.3}$$

In 1D, with $s_V = 0$, the solution is any function $f\left(t - \frac{x}{v}\right)$ or $g\left(x - tv\right)$ with $v = 1/\sqrt{\rho_0 \chi}$.

1.3 Plane wave

In the plane, the standard wave is given by

$$p(x, t) = p_0 \cos(\omega t - kx) \tag{1.4}$$

where the phase velocity is $v_{ph} = \frac{\omega}{k} = \frac{1}{\sqrt{\chi \rho_0}}$.
The wave impedance is $\eta = \frac{p}{u_x} = \sqrt{\frac{\rho_0}{\chi}}$ $[kgm^{-2}s^{-1}]$.

1.4 Harmonic case

In the case of a harmonic soundwave, we can use phasors:

$$p(\vec{r},t) = Re\{P(\vec{r})e^{j\omega_0 t}\} \tag{1.5}$$

→ Note: there is a relationship between the phasors and the Fourier transform:

$$F\{p\} = Re(P)\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] + jIm(P)\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)] \tag{1.6}$$

with ω the frequency variable.
In the frequency domain, the PDE becomes

$$\nabla^2 P + k^2 P = -j\omega\rho_0 S_V = Q \tag{1.7}$$

where k is the wavenumber $k = 2\pi/\lambda$. This is useful to make the time dependence disappear.

1.4.1 Green function

The Green function is the solution G to the following Fourier PDE;

$$\nabla^2 G + k^2 G = \delta(x)\delta(y) \tag{1.8}$$

The general solution is

$$G(\rho) = -\frac{j}{4}H_0^{(2)} - \frac{j}{4}(J_0(k\rho) - jY_0(k\rho)) \tag{1.9}$$

where J_0 and Y_0 are respectively the Bessel functions of order 1 and 2, and ρ is the distance between the source and the observer.
The Bessel functions $J_n(x)$ are solutions of the equation

$$x^2 J_n''(x) + xJ_n'(x) + (x^2 - n^2)J_n(x) = 0 \tag{1.10}$$

and there exists an approximation of the Hankel function $H_0^{(2)}$ for $x \gg 1$:

$$H_0^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}}e^{j\pi/4}e^{-jx} \tag{1.11}$$

1.4.2 From velocity source to pressure

The pressure due to a velocity source spread over infinitesimal segment Γ':

$$p(r) = G(\vec{r}-\vec{r}')(j\omega\rho_0)v_n(\vec{r}')d\Gamma' \tag{1.12}$$

To get the total pressure, we have to do an integral:

$$p = p_{inc} + \frac{\omega\rho_0}{4}\int_\Gamma H_0^{(2)}(k|r-r'|)v_n(\vec{r}')d\Gamma' \tag{1.13}$$

Which is obtained from the velocity source $-j\omega\rho_0 S_v$ from equation (1.7), and doing a convolution with the Green function.

This integral can be approximated with a pointwise approach:

$$I \approx C\sum_i v_i G(\rho_i)dl_i \tag{1.14}$$

where $C = -j\omega\rho_0$, v_i is the normal component of the velocity vector, and dl_i the length of the segment.

1.5 Singularity

In the case of a singularity, the Green function is infinite. We thus need to find a workaround. For this, we use the following trick:

$$H_0^{(2)}(x) = \left(H_0^{(2)}(x) + \frac{2j}{\pi}\ln(x)\right) - \frac{2j}{\pi}\ln(x) \tag{1.15}$$

The first term will be integrate numerically, while the second can be done analytically. When $x \approx 0$, we use a series development of the Green function for the integral of the first term:

$$H_0^{(2)}(x) \approx 1 - \frac{2j}{\pi}\left(\ln(x/2) + \gamma\right) \tag{1.16}$$

where γ is the Euler constant. For the second term near zero, the integral is finite and no approximation is required.

1.6 Radiated wave from an incident wave

Let us now imagine the situation where a planar wave is coming from a certain direction. The object on which we analyse the radiation is discretized into m segments. On each segment, we define a basis function $v_{n,i}$ for the normal speed. It is piecewise constant, i.e. it has a certain value on the segment, and 0 on the rest of the object. To determine the overall normal speed on each segment, we write it as a linear combination of the $v_{n,i}$:

$$v_n = \sum_i x_i v_{n,i} \tag{1.17}$$

where the x_i are unknown. We impose on the middle of each segment that the radiated pressure is 0, so that we get a system $Ax = b$ to solve, where

$$\begin{aligned} A_{ij} &= \frac{\omega\rho_0}{4}\int_{\Gamma_i'} H_0^{(2)}(k|r_{0,j} - r'|)v_{n,i}d\Gamma_i' \\ b_j &= -p_{incident}(r_{0,j}) \end{aligned} \tag{1.18}$$

from equation (1.13). Once the coefficients x_i are computed, the pressure field on the domain can be computed as it was previously in section 1.4.2.

If we define $\hat{u}$ as the unit vector in the direction of propagation of the incident wave, the final pressure in the domain is

$$p(\vec{r}) = P_0 e^{-jk\hat{u}\cdot\vec{r}} \tag{1.19}$$