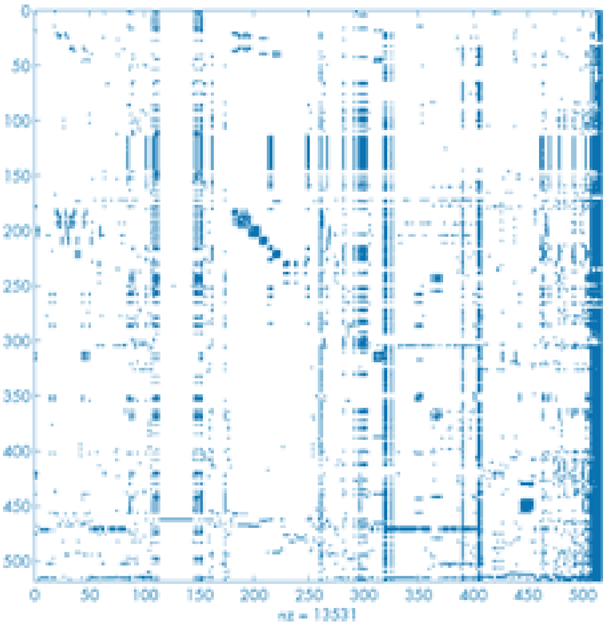

LINMA2380 Matrix Computations

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Contents

Reminders

1.1 Algebraic structures

- A semigroupe is a set together with an associative binary operation $(E, +)$.
- A monoid is a semigroup with a neutral element.
- A group is a monoid in which every element has an inverse.
- A commutative group is a group whose binary operation is commutative.
- A ring is a triple $(E, +, \cdot)$ such that
  - $(E, +)$ is a commutative group;
  - $(E, \cdot)$ is a monoid;
  - $\cdot$ is distirbutive with respect to $+$.
- An integral domain is a commutative ring in which the product of any two nonzero elements in nonzero :

  $$\forall x, y \in E, x, y \neq 0 \qquad xy \neq 0$$

  . This implies that the equation $ax = b$ with $a \neq 0$ has at most one solution.
- An Euclidean domain is an integral domain such that for every two elements in the domain, we can perform the Euclidean division:

  $$\forall (a_1, a_2), \quad \exists (q, r) : \quad a_1 = a_2 q + r \text{ with } r < a_2$$

- A field is a commutative ring $(E, +, \cdot)$ such that every $a \in E \setminus \{0\}$ has a multiplicative inverse.
- $(K, E, +)$ is a module over the ring $(K, +, \cdot)$ if
  - $(E, +)$ is a commutative group;
  - the external composition operation $\cdot : K \times E \to E$ satisfies
    - $(a + b) \cdot x = a \cdot x + b \cdot x \qquad a \cdot (x + y) = a \cdot x + a \cdot y$
    - $a \cdot (b \cdot x) = (a \cdot b) \cdot x$
    - $1 \cdot x = x$

- If, in addition to that, $(K, \cdot, +)$ is a field, then $(K, E, +)$ is a vector space over $(K, +, \cdot)$.
- $(K, E, +, \cdot)$ is an algebra if
  - $(K, E, +)$ is a module or a vector space;
  - the internal composition operation $\cdot : E \times E \to E$ is bilinear.

1.2 Matrix algebras

1.2.1 Product

Apart from the usual sum and product of two matrices, we can define the Hadamard and Kronecker products :

- Hadamard :
$$A_{m\times n} \odot B_{m\times n} := [a_{ij} \cdot b_{ij}]_{i,j=1}^{m,n}$$
- Kronecker :
$$A \otimes B := \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}$$

A square matrix $A \in \mathbb{C}^{n\times n}$ is said normal if $AA^* = A^*$. In the real case, it is said to be orthogonal and $*$ is equivalent to the transpose. Furthermore, it is said to be unitary if it satisfies the relations $AA^* = I_n = A^*A$.

1.2.2 Determinant

We define the quasi-diagonals of a matrix as the n-tuples of elements of a matrix A, $a_{1j_1,2j_2,\dots,nj_n}$ where the indices $\mathbf{j} = (j_1, \dots, j_n)$ constitute a permutation of the set $\{1, 2, \dots, n\}$. Thus a quasi-diagonal consists of n elements of the matrix A in such a way that no two of them lie in the same row or column of A. For each quasi-diagonal, we define the parity $t(\mathbf{j})$. It is the number of inversions $j_k > j_p$ for $k < p$ in $\mathbf{j}$.

- With the notation above, we define the determinant of a square matrix $A_{n\times n}$ as
$$\det(A) = \sum_{\mathbf{j}} (-1)^{t(\mathbf{j})} a_{1j_1} \cdot a_{2j_2} \cdot \dots \cdot a_{nj_n}$$

The determinant has the following properties :

- The determinant is multilinear in the rows of A :
$$\det \begin{bmatrix} a_{1:} \\ \vdots \\ b_{k:} + \lambda c_{k:} \\ \vdots \\ a_{n:} \end{bmatrix} = \det \begin{bmatrix} a_{1:} \\ \vdots \\ b_{k:} \\ \vdots \\ a_{n:} \end{bmatrix} + \det \begin{bmatrix} a_{1:} \\ \vdots \\ \lambda c_{k:} \\ \vdots \\ a_{n:} \end{bmatrix}$$

- The determinant is alternating in the rows of A : for $i \neq j$, $a_{i:} = a_{j:} \implies \det(A) = 0$

- $\det(I_n) = 1$, where I_n is the identity matrix.

- $\det \begin{bmatrix} a_{1:} \\ \vdots \\ a_{i:} \\ \vdots \\ a_{j:} \\ \vdots \\ a_{n:} \end{bmatrix} = -\det \begin{bmatrix} a_{1:} \\ \vdots \\ a_{j:} \\ \vdots \\ a_{i:} \\ \vdots \\ a_{n:} \end{bmatrix}$

- $\det \begin{bmatrix} a_{1:} \\ \vdots \\ a_{i:} + \lambda a_{j:} \\ \vdots \\ a_{n:} \end{bmatrix} = \det \begin{bmatrix} a_{1:} \\ \vdots \\ a_{i:} \\ \vdots \\ a_{n:} \end{bmatrix}$

- $\det(\lambda A) = \lambda^n \det(A)$

- for $i \neq j$, $a_{i:} = \lambda a_{j:} \implies \det(A) = 0$

- $\det(A^T) = \det(A)$

- $\det(A^*) = \overline{\det(A)}$, if $A \in \mathbb{C}^{n \times n}$

→ N.B.: any property of the determinant established for the rows of matrices also holds for the columns.

- The minor $A_{(kl)}$ of dimension $n-1$ of a matrix $A_{n \times n}$ is the determinant of the submatrix obtained by removing the $k$the row and the lth column. From this, we can note the determinant as a linear combination of the elements of a row or column :

  $$\det(A) = a_{1j}A^c_{1j} + a_{2j}A^c_{2j} + \cdots + a_{nj}A^c_{nj} \det(A) = a_{i1}A^c_{i1} + a_{i2}A^c_{i2} + \cdots + a_{in}A^c_{in}$$

  where the coefficient A^c_{kl} is called the cofactors of the corresponding element a_{kl}[1]

Laplace and Binet-Cauchy relations

For the pairs of p-tuples

$$\mathbf{i}_p := (i_1, \ldots, i_p) \text{ and } \mathbf{j}_p := (j_1, \ldots j_p)$$

satisfying

$$1 \leq i_1 < \cdots < i_p \leq n \text{ and } 1 \leq j_1 < \cdots < j_p \leq n$$

we define the minors of order p of A as

$$A \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} := \det[a_{i_k, j_l}]_{k,l=1}^{p} \tag{1.1}$$

We also define the complementary cofactors of A as

$$A^c \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} := (-1)^s A \begin{pmatrix} \mathbf{i}^c_p \\ \mathbf{j}^c_p \end{pmatrix} \tag{1.2}$$

[1] $A^c_{kl} = (-1)^{k+l} A_{(kl)}$.

where $s = \sum_{k=1}^{p}(i_k + j_k)$ and $\mathbf{i}_p^c$ is the set complement of $\mathbf{i}_p$ (same for $\mathbf{j}_p$).
Laplace Theorem:
Let A ne a matrix of dimensions $n \times n$ and $\mathbf{i}_p$ be a p-tuple of rows (and $\mathbf{j}_p$ for the columns). Then, $\det(A)$ is equal to the sum of the products of all possibles minors located in these rows/columns with their complementary cofactors:

$$\begin{cases} \det(A) = \sum_{\mathbf{j}_p} A \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} A^c \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} \\ \det(A) = \sum_{\mathbf{i}_p} A \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} A^c \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} \end{cases} \tag{1.3}$$

Binet-Cauchy Theorem:
Let $\mathbf{m}$ be the m-tuple $(1, \ldots, m)$. Let A and B be matrices of dimensions $m \times n$ and $n \times m$ respectively. If $m \leq n$, then

$$\det(AB) = \sum_{\mathbf{j}_m} = A \begin{pmatrix} \mathbf{m} \\ \mathbf{j}_m \end{pmatrix} B \begin{pmatrix} \mathbf{j}_m \\ \mathbf{m} \end{pmatrix} \tag{1.4}$$

1.2.3 Inverse and rank

- The adjugate matrix of a square matrix $A_{n\times n}$ is defined as

$$adj(A) := [A_{ji}^c]_{i,j=1}^n$$

Then, for every square matrix $A_{n\times n}$, we have

$$A \cdot adj(A) = \det(A) I_n = adj(A) \cdot A \tag{1.5}$$

Every matrix $A_{m\times n}$ whose elements belong to a field $\mathcal{F}$ can be brought to the following form by means of invertible (or elementary) transformations of rows and columns:

$$RAQ = \left(\begin{array}{c|c} I_r & 0_{r\times(n-r)} \\ \hline 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{array}\right) \tag{1.6}$$

The rank of a matrix $A_{m\times n}$ whose elements belong to a field $\mathcal{F}$ is equal to the largest size of its nonzero minors. As a corollary, any non-singular matrix whose elements belong to a field $\mathcal{F}$ can be written as a product of elementary transformations.

Schur complement:
Let $A_{n\times n}$ be an invertible submatrix of the matrix

$$M_{(n+p)\times(n+m)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Then the rank of M satisfies

$$rank(M) = n + rank(D - CA^{-1}B) \tag{1.7}$$

And the matrix $D - CA^{-1}B$ is called the Schur complement of M.