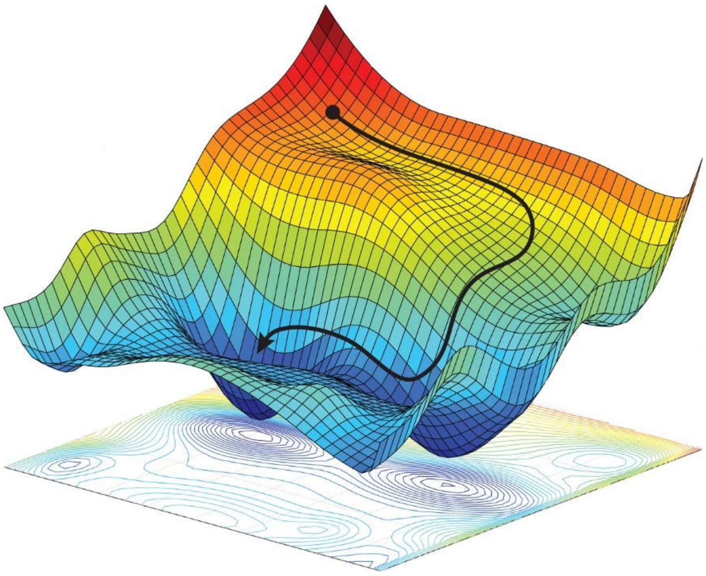

LINMA2460 Nonlinear Programming

Simon Desmidt
Issambre L'Hermite Dumont

Academic year 2024-2025 - Q2

UCLouvain

Contents

Definitions, notations and random properties

- The Taylor expansion of order p of the function f around x_k and evaluated at y is:
$$T_p(y; x_k) = f(x_k) + \sum_{i=1}^{p} \frac{1}{i!} D^i f(x_k)(y - x_k)^i \tag{1.1}$$

- We can thus define the gradient w.r.t. y of the Taylor expansion of order p of f around x_k and evaluated at x_{k+1}:
$$\nabla_y T_p(x_{k+1}; x_k) = \nabla_y T_p(y; x_k)\big|_{y=x_{k+1}} \tag{1.2}$$

- An oracle is a "black box" that gives information about the derivatives based on x. The general form of an oracle is:
$$\text{p-order oracle:} \quad x \mapsto \{D^i f(x)\}_{i=0}^{p} \tag{1.3}$$
And so we have the following simple oracles examples:
$$\begin{aligned} \text{Zero}^{th}\text{-order oracle:} &\quad x \mapsto \{f(x)\} \\ \text{First-order oracle:} &\quad x \mapsto \{f(x), \nabla f(x)\} \\ \text{Second-order oracle:} &\quad x \mapsto \{f(x), \nabla f(x), \nabla^2 f(x)\} \end{aligned} \tag{1.4}$$

- $\mathcal{C}_L^p(\mathbb{R}^n)$: Class of functions p-times continuously differentiable with L-Lipschitz continuous p-order derivative, i.e. $\|D^p f(x) - D^p f(y)\| \leq L\|x - y\|, \forall x, y \in \mathbb{R}^n$. And so we have the following simple classes of problems:
  - $\mathcal{C}_L^1(\mathbb{R}^n)$: Class of continuously differentiable functions with L-Lipschitz gradient;
  - $\mathcal{C}_L^2(\mathbb{R}^n)$: Class of continuously differentiable functions with L-Lipschitz hessian.

- pth-order method (generalization of GM):
$$x_{k+1} = \arg\min_{y \in \mathbb{R}^n} \Omega_{x_k, y, p}(y) \equiv T_{x_k, p}(y) + \frac{M}{(p+1)!}\|y - x_k\|^{p+1} \tag{1.5}$$

- Convergence rate:

– Linear:

$$\|x_{k+1} - x^*\| \leq \alpha \|x_k - x^*\| \quad \forall k \geq 0, \alpha \in (0,1) \tag{1.6}$$

– Super Linear:

$$\lim_{k \to +\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0 \tag{1.7}$$

– Quadratic:

$$\|x_{k+1} - x^*\| \leq \beta \|x_k - x^*\|^2 \quad \forall k \geq 0, \beta > 0 \tag{1.8}$$

1.1 Properties

- For a function $f \in C^1(\Omega)$ and Ω is bounded, the following holds: $\|\nabla f(x)\| \leq L$ for all $x \in \Omega$ for some $L \geq 0$.
- By the mean value theorem, for a continuously differentiable function f, $\forall x, y \in \Omega$, $\exists\, z \in \Omega : f(y) - f(x) = \langle \nabla f(z), y - x \rangle$.
- For a matrix A and a scalar b, $\|A\| \leq b \implies |\lambda(A)| \leq b \implies |A| \preceq bI_n$, where the absolute value of the matrix is taken component wise.

1.2 Complexity table

Method	Lipschitz	∇f	$\nabla^2 f$	$\dots$	$\nabla^p f$
Zero order		$O(n\varepsilon^{-2})$			
First order	$p = 1$	$O(\varepsilon^{-2})$			
Second order	$p = 2$	X	$O(\varepsilon^{-3/2})$		
$\vdots$		X	X	$\ddots$	
p order		X	X	X	$O(\varepsilon^{-\frac{p+1}{p}})$

1.3 GM VS Newton: table

	cost per iteration	cost of memory	Local rate
GM	$\mathcal{O}(n)$	$\mathcal{O}(n)$	Linear
Quasi-Newton	$\mathcal{O}(n^2)$	$\mathcal{O}(n^2)$	Super Linear
Newton	$\mathcal{O}(n^3)$	$\mathcal{O}(n^2)$	Quadratic

$\rightarrow$ For the GM, we assume that we don't need to compute the gradient at each iteration.

TODO

We can generalise the property of a L-Lipschitz function to $f \in \mathcal{C}_L^p(\mathbb{R}^n)$. For $p = 1$, we had

$$f(y) \leq f(x_k) + \langle \nabla f(x_k), y - x_k \rangle + \frac{L}{2}\|y - x_k\|^2 \qquad \forall y \in \mathbb{R}^n \tag{2.1}$$

For a general value of p, it becomes

$$f(y) \leq T_p(y; x_k) + \frac{L}{(p+1)!}\|y - x_k\|^{p+1} \qquad \forall y \in \mathbb{R}^n \tag{2.2}$$

Using this, we need a p-th order oracle for the method to work.
To solve $\min_{x \in \mathbb{R}^n} f(x)$, we can use the iteration

$$x_{k+1} = \arg\min_{y \in \mathbb{R}^n} T_p(y; x_k) + \frac{M}{(p+1)!}\|y - x_k\|^{p+1} \tag{2.3}$$

where the constant M is an approximation of the Lipschitz constant L. Assuming $f \in \mathcal{C}_L^p(\mathbb{R}^n)$, we have

$$\begin{aligned} f(x_{k+1}) &\leq T_p(x_{k+1}; x_k) + \frac{L}{(p+1)!}\|x_{k+1} - x_k\|^{p+1} \\ &= \underbrace{T_p(x_{k+1}; x_k) + \frac{M}{(p+1)!}\|x_{k+1} - x_k\|^{p+1}}_{\leq f(x_k)} + \frac{(L-M)}{(p+1)!}\|x_{k+1} - x_k\|^{p+1} \end{aligned} \tag{2.4}$$

where the inequality $\leq f(x_k)$ is due to the decrease of f and equation (2.3). Suppose that $M > 2L$. After some algebraic manipulations, we get

$$f(x_k) - f(x_{k+1}) \geq \frac{L}{(p+1)!}\|x_{k+1} - x_k\|^{p+1} \tag{2.5}$$

On the other hand, using the triangular inequality,

$$\begin{aligned} \|\nabla f(x_{k+1})\| &\leq \|\nabla f(x_{k+1}) - \nabla_y T_p(x_{k+1}; x_k)\| \\ &\quad + \underbrace{\left\| \nabla_y T_p(x_{k+1}; x_k) + \nabla \left(\frac{M}{(p+1)!}\|\cdot - x_k\|^{p+1} \right)\Big|_{y = x_{k+1}} \right\|}_{=0} \\ &\quad + \left\| \nabla \left(\frac{M}{(p+1)!}\|\cdot - x_k\|^{p+1} \right)\Big|_{y = x_{k+1}} \right\| \\ &\leq \frac{L}{p!}\|x_{k+1} - x_k\|^p + \frac{M}{p!}\|x_{k+1} - x_k\|^p \end{aligned} \tag{2.6}$$

$$\implies \|x_{k+1} - x_k\| \geq \left(\frac{p!}{L+M}\right)^{1/p} \|\nabla f(x_{k+1})\|^{1/p} \tag{2.7}$$

Combining equations (2.5) and (2.7),

$$f(x_k) - f(x_{k+1}) \geq \underbrace{\frac{L}{(p+1)!}\left(\frac{p!}{L+M}\right)^{\frac{p+1}{p}}}_{=:C(L)} \|\nabla f(x_{k+1})\|^{\frac{p+1}{p}} \tag{2.8}$$

Let $T(\varepsilon) = \inf\{k \in \mathbb{N} : \|\nabla f(x_k)\| \leq \varepsilon\}$. Assume that $T(\varepsilon) \geq 2$ and $f(x) \geq f_{low}$ $\forall x \in \mathbb{R}^n$. Summing up (2.8) for $k = 0, \ldots, T(\varepsilon) - 2$,

$$\begin{aligned} f(x_0) - f_{low} \geq f(x_0) - f(x_{T(\varepsilon)-1}) &= \sum_{k=0}^{T(\varepsilon)-2} f(x_k) - f(x_{k+1}) \\ &\geq (T(\varepsilon) - 1)C(L)\varepsilon^{\frac{p+1}{p}} \\ \implies T(\varepsilon) \leq 1 + \frac{f(x_0) - f_{low}}{C(L)}\varepsilon^{-\frac{p+1}{p}} &\equiv \mathcal{O}\left(\varepsilon^{-\frac{p+1}{p}}\right) \end{aligned} \tag{2.9}$$

Gradient descent without gradient

For this problem consider an adversarial attack on block-based image classifier. We have a machine learning model that given an image $a \in \mathbb{R}^p$ it returns $c(a) \in \mathbb{R}^m$, where $c_j(a) \in [0,1]$ is the probability of image a to be in class j. The classifier prediction is: $j(a) = \arg\max_{j\in[1,\dots,m]} c_j(a)$.
TODO - Add mise en situation ou pas?

Given x_k let us decide:

$$x_{k+1} = x_k - \frac{1}{\sigma} g_{h_k}(x_k) \qquad h_k > 0,\ \sigma > 0 \tag{3.1}$$

where $g_{h_k}(x_k) \in \mathbb{R}^n$ is given by:

$$[g_{h_k}(x_k)]_j = \frac{f(x_k + he_j) - f(x_k)}{h_k} \qquad \forall j \in [1,\dots,m] \tag{3.2}$$

Suppose that $f \in \mathcal{C}^1_L(\mathbb{R}^n)$. Then,

$$\|\nabla f(x_k) - g_{h_k}(x_k)\| \le \frac{L\sqrt{n}}{2} h_k \tag{3.3}$$

Thus

$$\begin{aligned}
f(x_{k+1}) \le f(x_k) \quad & + \langle \nabla f(x_k), x_{k+1} - x_k\rangle + \frac{L}{2}\|x_{k+1} - x_k\|^2 \\
= f(x_k) \quad & + \langle g_{h_k}(x_k), x_{k+1} - x_k\rangle + \frac{\sigma}{2}\|x_{k+1} - x_k\| \\
& + \langle \nabla f(x_k) - g_{h_k}(x_k), x_{k+1} - x_k\rangle + \frac{(L-\sigma)}{2}\|x_{k+1} - x_k\|^2 \\
\le f(x_k) \quad & - \frac{1}{\sigma}\|g_{h_k}(x_k)\|^2 + \frac{1}{2\sigma}\|g_{h_k}(x_k)\|^2 \\
& + \|\nabla f(x_k) - g_{h_k}(x_k)\| \frac{1}{\sigma}\|g_{h_k}(x_k)\| + \frac{(L-\sigma)}{2\sigma^2}\|g_{h_k}\|^2 \\
\le f(x_k) \quad & - \frac{1}{2\sigma}\|g_{h_k}(x_k)\|^2 + \frac{L\sqrt{n}}{2} h_k \frac{1}{\sigma}\|g_{h_k}\| + \frac{(L-\sigma)}{2\sigma^2}\|g_{h_k}\|^2 \\
\le f(x_k) \quad & - \frac{1}{2\sigma}\|g_{h_k}(x_k)\|^2 + \frac{L}{2}\left(\frac{nh_k^2}{2} + \frac{1}{2\sigma}\|g_{h_k}(x_k)\|^2\right) + \frac{(L-\sigma)}{2\sigma^2}\|g_{h_k}\|^2 \\
= f(x_k) \quad & - \left(\frac{2\sigma - L - 2(L-\sigma)}{4\sigma^2}\right)\|g_{h_k}(x_k)\|^2 + \frac{Ln}{4} h_k^2 \\
= f(x_k) \quad & - \frac{(4\sigma - 3L)}{4\sigma}\|g_{h_k}(x_k)\|^2 + \frac{Ln}{4} h_k^2
\end{aligned} \tag{3.4}$$

$$\implies \frac{(4\sigma - 3L)}{4\sigma}\|g_{h_k}(x_k)\|^2 \leq f(x_k) - f(x_{k+1}) + \frac{Ln}{4}h_k^2 \tag{3.5}$$

If $\sigma \gg L$, then

$$\frac{1}{4\sigma}\|g_{h_k}(x_k)\|^2 \leq f(x_k) - f(x_{k+1}) + \frac{\sigma n}{4}h_k^2 \tag{3.6}$$

On the other hand, we have

$$\begin{aligned}\|\nabla f(x_k)\| &\leq \|\nabla f(x_k) - g_{h_k}(x_k)\| + \|g_{h_k}(x_k)\| \\ &\leq \frac{L\sqrt{n}}{2}h_k + \|g_{h_k}(x_k)\|\end{aligned} \tag{3.7}$$

Using trick (5.3) in chapter 5,

$$\begin{aligned}\implies \|\nabla f(x_k)\|^2 &\leq 2\left(\frac{L\sqrt{n}}{2}h_k\right)^2 + 2\|g_{h_k}(x_k)\|^2 \\ &\leq \frac{L^2 n}{2}h_k^2 + 2\|g_{h_k}(x_k)\|^2\end{aligned} \tag{3.8}$$

$$\implies \frac{1}{8\sigma}\|\nabla f(x_k)\|^2 \leq \frac{L^2 n}{16\sigma}h_k^2 + \frac{1}{4\sigma}\|g_{h_k}(x_k)\|^2 \tag{3.9}$$

$$\implies \frac{1}{8\sigma}\|\nabla f(x_k)\|^2 \leq f(x_k) - f(x_{k+1}) + \frac{\sigma n}{4}h_k^2 + \frac{\sigma n}{16}h_k^2 \tag{3.10}$$

Let $T(\varepsilon) = \inf\{k \in \mathbb{N} : \|\nabla f(x_k)\| \leq \varepsilon\}$, with $f(x)$ bounded below by f_{low}, summing up (3.10) for $k = 0, \ldots, T(\varepsilon) - 1$:

$$\frac{T(\varepsilon)}{8\sigma}\varepsilon^2 \leq f(x_0) - f_{low} + \frac{5\sigma n}{4}\sum_{k=0}^{T(\varepsilon)-1} h_k^2 \tag{3.11}$$

If $\{h_k^2\}$ is summable

$$\implies T(\varepsilon) \leq 8\sigma\left(f(x_0) - f_{low} + \frac{5\sigma n}{4}\sum_{k=0}^{T(\varepsilon)-1} h_k^2\right)\varepsilon^2 = \mathcal{O}(\varepsilon^2) \tag{3.12}$$

In terms of call to the oracle, we have a complexity bound of $\mathcal{O}(n\varepsilon^2)$.

Local rates of convergence

4.1 Linear rate of GM

Let $f \in \mathcal{C}_M^{2,2}(\mathbb{R}^n)$. Assume f has a local minimizer x^* such that

$$\mu I_n \preceq \nabla^2 f(x^*) \preceq M I_n \tag{4.1}$$

Let $x_{k+1} = x_k - \frac{1}{L}\nabla f(x_k)$ for a given $x_0 \in \mathbb{R}^n$.
Notice that

$$\begin{aligned}
\nabla f(x_k) &= \nabla f(x_k) - \nabla f(x^*) \\
&= \int_0^1 \nabla^2 f(x^* + \tau(x_k - x^*))(x_k - x^*)d\tau \\
&= \int_0^1 \nabla^2 f(x^* + \tau(x_k - x^*))d\tau(x_k - x^*) \\
&= G_k(x_k - x^*)
\end{aligned} \tag{4.2}$$

Then,

$$\begin{aligned}
\|x_{k+1} - x^*\| &= \|x_k - \frac{1}{L}\nabla f(x_k) - x^*\| \\
&= \|(I_n - \frac{1}{L}G_k)(x_k - x^*)\| \\
&\leq \|I_n - \frac{1}{L}G_k\|\|x_k - x^*\|
\end{aligned} \tag{4.3}$$

Since $f \in C_M^{2,2}(\mathbb{R}^n)$, we have $\|\nabla^2 f(x^* + \tau(x_k - x^*)) - \nabla^2 f(x^*)\| \leq \tau M\|x_k - x^*\|$ and using this we get:

$$|\langle \nabla^2 f(x^* + \tau(x_k - x^*)) - \nabla^2 f(x^*)v, v\rangle| \leq \tau M\|x_k - x^*\|\|v\|^2 \quad \forall v \in \mathbb{R}^n \tag{4.4}$$

Using the bound (4.1) and the previous inequality, we get:

$$\tau M\|x_k - x^*\|\|v\|^2 \leq |\langle \nabla^2 f(x^* + \tau(x_k - x^*)) - \nabla^2 f(x^*)v, v\rangle| \leq \tau M\|x_k - x^*\|\|v\|^2$$

$$\nabla^2 f(x^*) - \tau M\|x_k - x^*\|I_n \preceq \nabla^2 f(x^* + \tau(x_k - x^*)) \preceq \nabla^2 f(x^*) + \tau M\|x_k - x^*\|I_n$$

$$(\mu - \tau M\|x_k - x^*\|)I_n \preceq \nabla^2 f(x^* + \tau(x_k - x^*)) \preceq (L + \tau M\|x_k - x^*\|)I_n$$

By the properties of the semi-definite matrices, and the trick (5.4), we have:

$$\begin{aligned}
\int_0^1 (\mu - \tau M\|x_k - x^*\|)\|v\|^2 d\tau &\leq \int_0^1 \langle \nabla^2 f(x^* + \tau(x_k - x^*))v, v\rangle d\tau \\
&\leq \int_0^1 (L + \tau M\|x_k - x^*\|)\|v\|^2 d\tau \quad \forall v \in \mathbb{R}^n
\end{aligned} \tag{4.5}$$

By using G_k and some constants, we get:

$$-\frac{1}{L}(L+\frac{M}{2}\|x_k-x^*\|)I_n \preceq -\frac{1}{L}G_k \preceq -\frac{1}{L}(\mu-\frac{M}{2}\|x_k-x^*\|)I_n \tag{4.6}$$

$$\left(1-\frac{1}{L}(L+\frac{M}{2}\|x_k-x^*\|)\right)I_n \preceq I_n-\frac{1}{L}G_k \preceq \left(1-\frac{1}{L}(\mu-\frac{M}{2}\|x_k-x^*\|)\right)I_n \tag{4.7}$$

And finally, we get:

$$\begin{aligned}\|I_n-\frac{1}{L}G_k\| &\leq \max\left\{\left|1-\frac{1}{L}(L+\frac{M}{2}\|x_k-x^*\|)\right|,\left|1-\frac{1}{L}(\mu-\frac{M}{2}\|x_k-x^*\|)\right|\right\}\\ &= \max\left\{\frac{M}{2L}\|x_k-x^*\|, 1-\frac{\mu}{L}+\frac{M}{2L}\|x_k-x^*\|\right\}\\ &= 1-\frac{\mu}{L}+\frac{M}{2L}\|x_k-x^*\|\end{aligned} \tag{4.8}$$

Suppose that $\frac{M}{2L}\|x_k-x^*\| \leq \frac{\mu}{2L} \iff \|x_k-x^*\| \leq \frac{\mu}{M}$
Then, in (4.8), we get:

$$\|I_n-\frac{1}{L}G_k\| \leq 1-\frac{\mu}{2L} < 1 \tag{4.9}$$

And so, by (4.2)

$$\|x_{k+1}-x^*\| \leq \|I_n-\frac{1}{L}G_k\|\|x_k-x^*\| < \|x_k-x^*\| \tag{4.10}$$

If $\|x_0-x^*\| < \frac{\mu}{M}$, it follows from the previous reasoning that:

$$\|x_2-x^*\| \leq (1-\frac{\mu}{2L})\|x_1-x^*\| \leq (1-\frac{\mu}{2L})^2\|x_0-x^*\| \leq \frac{\mu}{M} \tag{4.11}$$

And so by induction, we can conclude that:

$$\|x_k-x^*\| \leq \left(1-\frac{\mu}{2L}\right)^k\|x_0-x^*\| \quad \forall k \geq 0 \tag{4.12}$$

$$\boxed{\Rightarrow \text{Linear rate of convergence}}$$

Given $\varepsilon > 0$, let $T(\varepsilon) = \inf\{k \in \mathbb{N} : \|x_k-x^*\| \leq \varepsilon\}$. Then, if $T(\varepsilon) \geq 1$ and using (4.12), we get:

$$\begin{aligned}\varepsilon < \|x_{T(\varepsilon)-1}-x^*\| &\leq \left(1-\frac{\mu}{2L}\right)^{T(\varepsilon)-1}\|x_0-x^*\|\\ \log\left(\frac{\varepsilon}{\|x_0-x^*\|}\right) &\leq (T(\varepsilon)-1)\log\left(1-\frac{\mu}{2L}\right)\\ T(\varepsilon)-1 &\leq \frac{\log\left(\frac{\varepsilon}{\|x_0-x^*\|}\right)}{\log\left(1-\frac{\mu}{2L}\right)} = \frac{\log\left(\|x_0-x^*\|\varepsilon^{-1}\right)}{|\log\left(1-\frac{\mu}{2L}\right)|}\end{aligned} \tag{4.13}$$

$$\boxed{T(\varepsilon) \leq \mathcal{O}(\log(\varepsilon^{-1}))}$$

$\rightarrow$ Note: convexity was never assumed!

4.2 Local quadratic convergence of Newton's method

Let $f \in \mathcal{C}^{2,2}_M(\mathbb{R}^n)$. Assume f has a local minimizer x^* such that

$$\mu I_n \preceq \nabla^2 f(x^*) \quad \mu > 0 \tag{4.14}$$

Given $x_0 \in \mathbb{R}^n$, let:

$$x_{k+1} = x_k - \nabla^{-2} f(x_k) \nabla f(x_k) \tag{4.15}$$

We have, by the previous equation and the definition of G_k (4.2):

$$\begin{aligned}
\|x_{k+1} - x^*\| &= \|x_k - \nabla^{-2} f(x_k)\nabla f(x_k) - x^*\| \\
&= \|(x_k - x^*) - \nabla^{-2} f(x_k) G_k (x_k - x^*)\| \\
&= \|\nabla^{-2} f(x_k) \left(\nabla^2 f(x_k) - \int_0^1 \nabla^2 f(x^* + \tau(x_k - x^*)) d\tau\right)(x_k - x^*)\| \\
&= \|\nabla^{-2} f(x_k) \left(\int_0^1 \nabla^2 f(x_k) - \nabla^2 f(x^* + \tau(x_k - x^*)) d\tau\right)(x_k - x^*)\| \\
&\leq \|\nabla^{-2} f(x_k)\| \left(\int_0^1 \left\|\nabla^2 f(x_k) - \nabla^2 f(x^* + \tau(x_k - x^*))\right\| d\tau\right)\|x_k - x^*\| \\
&\leq \|\nabla^{-2} f(x_k)\| \left(\int_0^1 M(1-\tau)\|x_k - x^*\| d\tau\right)\|x_k - x^*\| \\
&\leq \|\nabla^{-2} f(x_k)\| \|x_k - x^*\|^2 \frac{M}{2}
\end{aligned} \tag{4.16}$$

Since $f \in \mathcal{C}^{2,2}_M(\mathbb{R}^n)$, we have

$$\nabla^2 f(x^* + \tau(x_k - x^*)) - \nabla^2 f(x^*) \succeq \tau M \|x_k - x^*\| I_n \tag{4.17}$$

$$\begin{aligned}
\nabla^2 f(x_k) &\succeq \nabla^2 f(x^*) - M\|x_k - x^*\| I_n \\
&\succeq (\mu - M\|x_k - x^*\|) I_n
\end{aligned} \tag{4.18}$$

$$\lambda_{\min}(\nabla^2 f(x_k)) \geq \mu - M\|x_k - x^*\|$$

Suppose that $-M\|x_k - x^*\| \geq -\frac{\mu}{2} \Leftrightarrow \|x_k - x^*\| \leq \frac{\mu}{2M}$
Then,

$$\begin{aligned}
\lambda_{\min}(\nabla^2 f(x_k)) &\geq \frac{\mu}{2} \\
\lambda_{\max}(\nabla^{-2} f(x_k)) &\leq \frac{2}{\mu} \\
\Rightarrow \|\nabla^{-2} f(x_k)\| &\leq \frac{2}{\mu}
\end{aligned} \tag{4.19}$$

Therefore, by (4.16), we conclude that:

$$\begin{aligned}
\|x_{k+1} - x^*\| &\leq \frac{M}{2} \|\nabla^{-2} f(x_k)\| \|x_k - x^*\| \\
&\leq \frac{M}{\mu} \|x_k - x^*\|^2
\end{aligned} \tag{4.20}$$

If $\|x_k - x^*\| \leq \frac{\mu}{2M}$ then,

$$\|x_{k+1} - x^*\| \leq \frac{M}{\mu}\|x_k - x^*\|^2 = \frac{1}{2}\|x_k - x^*\| \tag{4.21}$$

If $\|x_0 - x^*\| \leq \frac{\mu}{2M}$ then $\{x\}_{k\geq 0} \subset B[x^*, \frac{\mu}{2M}]$.
Denote $\delta_k = \frac{M}{\mu}\|x_k - x^*\|$, then we have $\delta_0 = \frac{M}{\mu}\|x_0 - x^*\| \leq \frac{1}{2}$, and if we combine this with (4.21), we get:

$$\delta_{k+1} \leq \delta_k^2 \quad \forall k \geq 0 \tag{4.22}$$

And if we proceed by recurcence, we get:

$$\begin{aligned} \delta_1 &\leq \delta_0^2 \leq \left(\frac{1}{2}\right)^2 \\ \delta_2 &\leq \delta_1^2 \leq \left(\frac{1}{2}\right)^4 \\ &\vdots \\ \delta_k &\leq \left(\frac{1}{2}\right)^{2^k} \quad \forall k \geq 0 \end{aligned} \tag{4.23}$$

$$\Rightarrow \|x_k - x^*\| \leq \frac{\mu}{M}\left(\frac{1}{2}\right)^{2^k} \tag{4.24}$$

Let $T(\varepsilon) = \inf\{k \in \mathbb{N} : \|x_k - x^*\| \leq \varepsilon\}$ and suppose that $T(\varepsilon) \geq 1$. Then using the convergence rate (4.24), we can state the maximal number of iterations:

$$\varepsilon \leq \|x_{T(\varepsilon)-1} - x^*\| \leq \frac{\mu}{M}\left(\frac{1}{2}\right)^{2^{T(\varepsilon)-1}} \tag{4.25}$$

$$2^{2^{T(\varepsilon)-1}} \leq \frac{\mu}{M}\varepsilon^{-1} \tag{4.26}$$

$$\boxed{\Rightarrow T(\varepsilon) \leq \log_2(\log_2(\frac{\mu}{M}\varepsilon^{-1}))}$$

4.3 Quasi Newton methods

One step of a Quasi-Newton method is given by:

$$x_{k+1} = x_k - B_k \nabla f(x_k) \tag{4.27}$$

With $B_k \in \mathbb{R}^{n\times n}$, symmetric and non-singular
Suppose that $x_k \to x^*$ when $k \to \infty$, and that $\nabla^2 f(x_k) \succeq \mu I_n$ with $\mu \geq 0$
We want the condition on B_k to have a Super Linear convergence (1.7) of the Quasi-Newton method. So lets assume that $f \in \mathcal{C}_M^{2,2}(\mathbb{R}^n)$.
Then,

$$\|\nabla^2 f(x_{k+1} - \nabla^2 f(x_k))\| \leq M\|x_{k+1} - x_k\| \tag{4.28}$$

GOOD LABEL ?

$$\|\nabla f(x_{k+1} - \nabla f(x_k) - \nabla^2 f(x_k)(x_{k+1} - x_k))\| \leq \frac{M}{2}\|x_{k+1} - x_k\|^2 \tag{4.29}$$

Therefore

$$\begin{aligned} \nabla f(x_{k+1}) = \nabla f(x_{k+1}) &- \nabla f(x_k) - \nabla^2 f(x_k)(x_{k+1} - x_k) \\ &+ \nabla f(x_k) + \nabla^2 f(x_k)(x_{k+1} - x_k) \end{aligned} \tag{4.30}$$

Using the relation (4.27) we get:

$$
\begin{aligned}
\nabla f(x_{k+1}) = \nabla f(x_{k+1}) \quad & - \nabla f(x_k) - \nabla^2 f(x_k)(x_{k+1} - x_k) \\
& - B_k^{-1}(x_{k+1} - x_k) \\
& + \nabla^2 f(x_k)(x_{k+1} - x_k) \\
= \nabla f(x_{k+1}) \quad & - \nabla f(x_k) - \nabla^2 f(x_k)(x_{k+1} - x_k) \\
& - \left(B_k^{-1} - \nabla^2 f(x^*)\right)(x_{k+1} - x_k) \\
& + \left(\nabla^2 f(x_k) - \nabla^2 f(x^*)\right)(x_{k+1} - x_k) \\
\|\nabla f(x_{k+1})\| \leq \|\nabla f(x_{k+1}) \quad & - \nabla f(x_k) - \nabla^2 f(x_k)(x_{k+1} - x_k)\| \\
& + \| \left(B_k^{-1} - \nabla^2 f(x^*)\right)(x_{k+1} - x_k)\| \\
& + \| \left(\nabla^2 f(x_k) - \nabla^2 f(x^*)\right) \| \|(x_{k+1} - x_k)\| \\
\leq \frac{M}{2}\|x_{k+1} - x_k\|^2 \quad & + M\|x_k - x^*\| \|x_{k+1} - x_k\| \\
& + \| \left(B_k^{-1} - \nabla^2 f(x_k)\right)(x_{k+1} - x_k)\|
\end{aligned}
\tag{4.31}
$$

On the line before we used (4.28) and (4.29). And so we can write:

$$
\frac{\|\nabla f(x_{k+1})\|}{\|x_{k+1} - x_k\|} \leq \frac{M}{2}\|x_{k+1} - x_k\| + M\|x_k - x^*\| + \frac{\| \left(B_k^{-1} - \nabla^2 f(x_k)\right)(x_{k+1} - x_k)\|}{\|x_{k+1} - x_k\|}
\tag{4.32}
$$

From now on , suppose that this condition (Dimis-Mori condition) is true:

$$
\lim_{k \to \infty} \frac{\| \left(B_k^{-1} - \nabla^2 f(x_k)\right)(x_{k+1} - x_k)\|}{\|x_{k+1} - x_k\|} = 0
\tag{4.33}
$$

Under this condition and by (4.32), we have:

$$
\lim_{k \to \infty} \frac{\|\nabla f(x_{k+1})\|}{\|x_{k+1} - x_k\|} = 0
\tag{4.34}
$$

As $\|x_{k+1} - x_k\| \to 0$, we conclude that $\lim_{x \to \infty} \|\nabla f(x_{k+1})\| = 0$ and so $\|\nabla f(x^*)\| = 0 \Rightarrow \nabla f(x^*) = 0$. ($x^*$ is a stationary point of f)
We have $\nabla^2 f(x^*) \succeq \mu I_n$ and given $y \in \mathbb{R}^n$, we have:

$$
\begin{aligned}
\nabla^2 f(y) - \nabla^2 f(x^*) &\succeq -M\|y - x^*\| I_n \\
\nabla^2 f(y) &\succeq (\mu - M\|y - x^*\|) I_n
\end{aligned}
\tag{4.35}
$$

Thus, if $-M\|y - x^*\| \geq -\frac{\mu}{2}$ then $\nabla^2 f(y) \succeq \frac{\mu}{2} I_n$.
Since $x_k \to x^*$, there exists $k_0 \in \mathbb{N}$ such that $\|x_{k+1} * x^*\| \leq \frac{\mu}{2M}$ $\forall k \geq k_0$. Thus for any $\tau \in [0,1]$:

$$
\|x^* + \tau(x_{k+1} - x^*) - x^*\| \leq \frac{\mu}{2M}, \quad \forall k \geq k_0
\tag{4.36}
$$

and so $\nabla^2 f(x^* + \tau(x_{k+1} - x^*)) \succeq \frac{\mu}{2} I_n \; \forall k \geq k_0$.

$$\begin{aligned}
\|x_{k+1} - x^*\| \|\nabla f(x_{k+1})\| &\geq (x_{k+1} - x^*)^T \nabla f(x_{k+1}) \\
&= (x_{k+1} - x^*)^T \left(\nabla f(x_{k+1}) - \nabla f(x^*) \right) \\
&= (x_{k+1} - x^*)^T \int_0^1 \nabla^2 f(x^* + \tau(x_{k+1} - x^*))(x_{k+1} - x^*) d\tau \\
&\geq \int_0^1 (x_{k+1} - x^*)^T \frac{\mu}{2} I_n (x_{k+1} - x^*) d\tau \\
&= \frac{\mu}{2} \|x_{k+1} - x^*\|^2
\end{aligned} \tag{4.37}$$

$$\|\nabla f(x_{k+1})\| \geq \frac{\mu}{2} \|x_{k+1} - x^*\| \tag{4.38}$$

Let $\rho_k = \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|}$ then, using (5.5), we obtain:

$$\begin{aligned}
\frac{\|\nabla f(x_{k+1})\|}{\|x_{k+1} - x_k\|} &\geq \frac{(\frac{\mu}{2})\|x_{k+1} - x^*\|}{\|x_{k+1} - x_k\|} \\
&\geq \frac{(\frac{\mu}{2})\|x_{k+1} - x^*\|}{\|x_{k+1} - x^*\| + \|x_k - x^*\|} \\
&= \frac{(\frac{\mu}{2})\rho_k}{\rho_k + 1}
\end{aligned} \tag{4.39}$$

Combining (4.39) and (4.32), we get:

$$\frac{\mu}{2} \frac{\rho_k}{\rho_k + 1} \leq \frac{M}{2} \|x_{k+1} - x_k\| + M\|x_k - x^*\| + \frac{\| \left(B_k^{-1} - \nabla^2 f(x^*) \right) (x_{k+1} - x_k) \|}{\|x_{k+1} - x_k\|} \tag{4.40}$$

Since the right hand side goes to zero when $k \to +\infty$, then we have: IDK how to write that

$$\begin{aligned}
\lim_{k \to \infty} \frac{\rho_k}{1 + \rho_k} &= 0 \\
\lim_{k \to \infty} \frac{1}{\frac{1}{\rho_k} + 1} &= 0 \\
\Rightarrow \lim_{k \to \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} &\Rightarrow \lim_{k \to \infty} \rho_k = 0
\end{aligned} \tag{4.41}$$

Suppose that $n = 1$, then the quasi-newton update is writed:

$$x_{k+1} = x_k - b_k f'(x_k), \quad k \geq 0 \tag{4.42}$$

with $b_k \in \mathbb{R}$. We want $b_k \approx f''(x_k)^{-1}$ and by finite difference we can express it like that $b_k^{-1} \approx \frac{f'(x_{k-1} + h) - f'(x_{k-1})}{h}$. And with $h = x_h - x_{k-1}$, we can define:

$$b_k^{-1} = \frac{f'(x_k) - f'(x_{k-1})}{x_k - x_{k-1}} \tag{4.43}$$

Thus if $x_k \to x^*$ then:

$$\lim_{k \to \infty} \frac{|(b_k^{-1} - f''(x^*))(x_k - x_{k-1})|}{|x_k - x_{k-1}|} = 0 \tag{4.44}$$

Because we can notice that:

$$\frac{|(b_k^{-1} - f''(x^*))(x_k - x_{k-1})|}{|x_k - x_{k-1}|} = |b_k^{-1} - f''(x_{k-1})| + |f''(x_{k-1}) - f''(x^*)| \tag{4.45}$$

Since $x_k \to x^*$, we have $h = x_k - x_{k-1}$ and so:

$$b_k^{-1} = \frac{f'(x_k) - f'(x_{k-1})}{x_k - x_{k-1}} \to f''(x_{k-1}) \tag{4.46}$$

Thus, $\lim_{k\to\infty} |b_k^{-1} - f''(x_k)| = 0$.
Assuming that f'' is continuous, we have $\lim_{k\to\infty} |f''(x_k) - f''(x^*)| = 0$.
If we define $S_{k-1} = x_k - x_{k-1}$ and $y_{k-1} = f'(x_k) - f'(x_{k-1})$ and knowing (4.43), we can write:

$$\begin{aligned} b_k(f'(x_k) - f'(x_{k-1})) &= x_k - x_{k-1} \\ b_k y_{k-1} &= S_{k-1} \end{aligned} \tag{4.47}$$

This suggests that for $n > 1$, we should define the secant condition, B_k such that:

$$B_k y_{k-1} = S_{k-1} \tag{4.48}$$

Lets define $f(x) = \frac{1}{2}\|Ax - b\|^2 = \frac{1}{2}x^T A^T A x - (A^T b)^T x + \frac{1}{2}b^T b$. If A is full rank then f is a strongly convex quadratic function. And we have $\nabla f(x_k) = A^T A x_k - A^T b$. Then,

$$y_{k-1} = \nabla f(x_k) - \nabla f(x_{k-1}) = A^T A(x_k - x_{k-1}) = \nabla^2 f(x_k) S_{k-1} \tag{4.49}$$

And so

$$\nabla^2 f(x_k) y_{k-1} = S_{k-1} \tag{4.50}$$

Therefore, $\nabla^{-2} f$ satisfies the secant condition (4.48), when f is a strongly convex quadratic function. Thus it is reasonnable to require the secant for any approximation to $\nabla^{-2} f(x_k)$.

Now, how can we compute B_k such that it satisfies the secant condition (4.48)?
Given a matrix B_{k-1}, our goal is to find a perturbation matrix $P_{k-1} \in \mathbb{R}^{n\times n}$ such that:

$$(B_{k-1} + P_{k-1})\, y_{k-1} = S_{k-1} \tag{4.51}$$

If we get such P_{k-1}, we can define $B_k = B_{k-1} + P_{k-1}$, which would satisfy the secant condition (4.48).
For that we need at least n degrees of freedom and a symmetric matrix, so it is natural to try:

$$P_{k-1} = V_{k-1} V_{k-1}^T \quad V_{k-1} \in \mathbb{R}^n \tag{4.52}$$

Tips and Tricks

1. Approximation of the max:

$$\max\{z, 0\} = \frac{z + |z|}{2} = \frac{z + \sqrt{z^2}}{2} \approx \frac{z + \sqrt{z^2 + \delta}}{2} \tag{5.1}$$

2. $$ab \leq \frac{a^2 + b^2}{2} \tag{5.2}$$

3. $$(a + b)^2 \leq 2a^2 + 2b^2 \tag{5.3}$$

4. V-trick:

$$\langle xv, v \rangle \leq \|x\| \|v\|^2 \tag{5.4}$$

5. Triangular inequality by the minimizer:

$$\|x_{k+1} - x_k\| \leq \|x_{k+1} - x^*\| + \|x_k - x^*\| \tag{5.5}$$