



LINMA2460 Nonlinear Programming

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Definitions and Notations

- The Taylor expansion of order p of the function f around x_k and evaluated at y is:

$$T_p(y; x_k) = f(x_k) + \sum_{i=1}^p \frac{1}{i!} D^i f(x_k) (y - x_k)^i \quad (1.1)$$

- We can thus define the gradient w.r.t. y of the Taylor expansion of order p of f around x_k and evaluated at x_{k+1} :

$$\nabla_y T_p(x_{k+1}; x_k) = \nabla_y T_p(y; x_k) \big|_{y=x_{k+1}} \quad (1.2)$$

- An oracle is a "black box" that gives information about the derivatives based on x . The general form of an oracle is:

$$\text{p-order oracle: } x \mapsto \{D^i f(x)\}_{i=0}^p \quad (1.3)$$

And so we have the following simple oracles examples:

$$\begin{aligned} \text{Zero}^{th}\text{-order oracle: } & x \mapsto \{f(x)\} \\ \text{First-order oracle: } & x \mapsto \{f(x), \nabla f(x)\} \\ \text{Second-order oracle: } & x \mapsto \{f(x), \nabla f(x), \nabla^2 f(x)\} \end{aligned} \quad (1.4)$$

- $\mathcal{C}_L^p(\mathbb{R}^n)$: Class of functions p -times continuously differentiable with L -Lipschitz continuous p -order derivative, i.e. $\|D^p f(x) - D^p f(y)\| \leq L\|x - y\|, \forall x, y \in \mathbb{R}^n$. And so we have the following simple classes of problems:
 - $\mathcal{C}_L^1(\mathbb{R}^n)$: Class of continuously differentiable functions with L -Lipschitz gradient;
 - $\mathcal{C}_L^2(\mathbb{R}^n)$: Class of continuously differentiable functions with L -Lipschitz hessian.

TODO

We can generalise the property of a L -Lipschitz function to $f \in \mathcal{C}_L^p(\mathbb{R}^n)$. For $p = 1$, we had

$$f(y) \leq f(x_k) + \langle \nabla f(x_k), y - x_k \rangle + \frac{L}{2} \|y - x_k\|^2 \quad \forall y \in \mathbb{R}^n \quad (2.1)$$

For a general value of p , it becomes

$$f(y) \leq T_p(y; x_k) + \frac{L}{(p+1)!} \|y - x_k\|^{p+1} \quad \forall y \in \mathbb{R}^n \quad (2.2)$$

Using this, [we need a \$p\$ -th order oracle](#) for the method to work.

To solve $\min_{x \in \mathbb{R}^n} f(x)$, we can use the iteration

$$x_{k+1} = \arg \min_{y \in \mathbb{R}^n} T_p(y; x_k) + \frac{M}{(p+1)!} \|y - x_k\|^{p+1} \quad (2.3)$$

where the constant M is an approximation of the Lipschitz constant L . [Assuming \$f \in \mathcal{C}_L^p\(\mathbb{R}^n\)\$](#) , we have

$$\begin{aligned} f(x_{k+1}) &\leq T_p(x_{k+1}; x_k) + \frac{L}{(p+1)!} \|x_{k+1} - x_k\|^{p+1} \\ &= \underbrace{T_p(x_{k+1}; x_k) + \frac{M}{(p+1)!} \|x_{k+1} - x_k\|^{p+1}}_{\leq f(x_k)} + \frac{(L-M)}{(p+1)!} \|x_{k+1} - x_k\|^{p+1} \end{aligned} \quad (2.4)$$

where the inequality $\leq f(x_k)$ is due to the decrease of f and equation (2.3). [Suppose that \$M > 2L\$](#) . After some algebraic manipulations, we get

$$f(x_k) - f(x_{k+1}) \geq \frac{L}{(p+1)!} \|x_{k+1} - x_k\|^{p+1} \quad (2.5)$$

On the other hand, using the triangular inequality,

$$\begin{aligned} \|\nabla f(x_{k+1})\| &\leq \|\nabla f(x_{k+1}) - \nabla_y T_p(x_{k+1}; x_k)\| \\ &\quad + \underbrace{\left\| \nabla_y T_p(x_{k+1}; x_k) + \nabla \left(\frac{M}{(p+1)!} \|\cdot - x_k\|^{p+1} \right) \right\|_{y=x_{k+1}}}_{=0} \\ &\quad + \left\| \nabla \left(\frac{M}{(p+1)!} \|\cdot - x_k\|^{p+1} \right) \right\|_{y=x_{k+1}} \\ &\leq \frac{L}{p!} \|x_{k+1} - x_k\|^p + \frac{M}{p!} \|x_{k+1} - x_k\|^p \end{aligned} \quad (2.6)$$

Le + rouge doit être un -?

$$\implies \|x_{k+1} - x_k\| \geq \left(\frac{p!}{L+M} \right)^{1/p} \|\nabla f(x_{k+1})\|^{1/p} \quad (2.7)$$

Combining equations (2.5) and (2.7),

$$f(x_k) - f(x_{k+1}) \geq \underbrace{\frac{L}{(p+1)!} \left(\frac{p!}{L+M} \right)^{\frac{p+1}{p}}}_{=:C(L)} \|\nabla f(x_{k+1})\|^{\frac{p+1}{p}} \quad (2.8)$$

Let $T(\varepsilon) = \inf\{k \in \mathbb{N} : \|\nabla f(x_k)\| \leq \varepsilon\}$. Assume that $T(\varepsilon) \geq 2$ and $f(x) \geq f_{low} \forall x \in \mathbb{R}^n$. Summing up (2.8) for $k = 0, \dots, T(\varepsilon) - 2$,

$$\begin{aligned} f(x_0) - f_{low} &\geq f(x_0) - f(x_{T(\varepsilon)-1}) = \sum_{k=0}^{T(\varepsilon)-2} f(x_k) - f(x_{k+1}) \\ &\geq (T(\varepsilon) - 1)C(L)\varepsilon^{\frac{p+1}{p}} \\ \implies T(\varepsilon) &\leq 1 + \frac{f(x_0) - f_{low}}{C(L)}\varepsilon^{-\frac{p+1}{p}} \equiv \mathcal{O}\left(\varepsilon^{-\frac{p+1}{p}}\right) \end{aligned} \quad (2.9)$$