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# LINMA2491 Operational Research

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SIMON DESMIDT  
ISSAMBRE L'HERMITE DUMONT

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UCLouvain

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# Definition and notation

- Given  $\Omega$ , a sigma-algebra  $\mathcal{A}$  is a set of subsets of  $\Omega$ , with the elements called events, such that:
  - $\Omega \in \mathcal{A}$
  - if  $A \in \mathcal{A}$  then also  $\Omega - A \in \mathcal{A}$
  - if  $A_i \in \mathcal{A}$  for  $i = 1, 2, \dots$  then also  $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$
  - if  $A_i \in \mathcal{A}$  for  $i = 1, 2, \dots$  then also  $\cap_{i=1}^{\infty} A_i \in \mathcal{A}$
- Consider:



- The state space is the set of all values of the system at each stage.

$$S_0 = \{C\}, \quad S_1 = \{C_u, C_d\}, \quad S_2 = \{C_{uu}, C_{ud}, C_{dd}\} \quad (1.1)$$

- The sample space is the set of all possible combination of the system.

$$\Omega = S_0 \times S_1 \times S_2 = \{(C, C_u, C_{uu}), (C, C_u, C_{ud}), (C, C_u, C_{dd}), \dots\} \quad (1.2)$$

- The power set of  $\Omega$  is the set of all of the subsets, denoted  $\mathcal{B}(\Omega)$ .
- The probability space is the triplet  $(\Omega, \mathcal{A}, P)$  where  $P$  is a probability measure.
  - $P(\emptyset) = 0$
  - $P(\Omega) = 1$
  - $P(\cup_{i=1}^{\infty} A_i) = \sum_i P(A_i)$  if  $A_i$  are disjoint
- $\forall t, A_t$  is the set of events on which we have information at stage  $t$ . For example,  $A_0 = \{C\}$ ,  $A_1 = \{C, C_u, C_d\}$ . Thus is it evident that  $t_1 \leq t_2 \Rightarrow \mathcal{A}_{t_1} \subseteq \mathcal{A}_{t_2}$

- Consider the following problem with  $x \in \mathbb{R}^n$  and domain  $\mathcal{D}$ :

$$\begin{aligned} \min f_0(x), \quad & \text{s.t.} \\ f_i(x) &\leq 0, i = 1, \dots, m \\ h_j(x) &= 0, j = 1, \dots, p \end{aligned} \quad (1.3)$$

Then the Lagrangian function is defined as  $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$ :

$$L(x, \lambda, v) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{j=1}^p v_j h_j(x) \quad (1.4)$$

- The Lagrange dual function is defined as  $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$ :

$$g(\lambda, v) = \inf_{x \in \mathcal{D}} L(x, \lambda, v) \quad (1.5)$$

- The Lagrange dual problem is a lower bound on the optimal value of the primal problem
- Lagrange relaxation of Stochastic Programs, consider the two problems:

$$\begin{aligned} \min f_1(x) + \mathbb{E}_\omega[f_2(y(\omega), \omega)] & \quad \min f_1(x) + \mathbb{E}_\omega[f_2(y(\omega), \omega)] \\ \text{s.t. } h_{1i}(x) \leq 0, i = 1, \dots, m_1 & \quad \text{s.t. } h_{1i}(x) \leq 0, i = 1, \dots, m_1 \\ h_{2i}(x, y(\omega), \omega) \leq 0, i = 1, \dots, m_2 & \quad h_{2i}(x(\omega), y(\omega), \omega) \leq 0, i = 1, \dots, m_2 \\ & \quad \textcolor{red}{x(\omega) = x} \end{aligned} \quad (1.6)$$

The red constraint is the non-anticipativity constraint, it transforms the deterministic variable into a stochastic variable. **A VERIFIER**

- The dual of a stochastic program is:

$$\begin{aligned} g(v) &= g_1(v) + \mathbb{E}_\omega(g_2(v, \omega)) \\ \text{where} \\ g_1(v) &= \inf f_1(x) + \left( \sum_{\omega \in \Omega} v(\omega) \right)^T x \\ \text{s.t. } h_{1i}(x) &\leq 0, i = 1, \dots, m_1 \\ \text{and} \\ g_2(v, \omega) &= \inf f_2(y(\omega), \omega) - vx(\omega) \\ \text{s.t. } h_{2i}(x(\omega), y(\omega), \omega) &\leq 0, i = 1, \dots, m_2 \end{aligned} \quad (1.7)$$

- With  $p^*$  the solution of the primal problem and  $d^*$  the solution of the dual problem, we have:

- Weak duality:  $d^* \leq p^*$
- Strong duality:  $d^* = p^*$

- The KKT conditions are necessary and sufficient for optimality in convex optimization, there aren't unique. They are:

- Primal constraint:  $f_i(x) \leq 0, i = 1, \dots, m, h_j(x) = 0, j = 1, \dots, p$
- Dual constraint:  $\lambda \geq 0$
- Complementarity slackness:  $\lambda_i f_i(x) = 0, i = 1, \dots, m$
- Gradient of the Lagrangian:  $\nabla_x L(x, \lambda, \nu) = 0$
- An extreme point of a polyhedron  $P$  is a point  $x \in P$  such that it cannot be expressed as a linear combination of two distinct points in  $P$ , i.e. an extreme point is a vertex of the polyhedron.
- An extreme ray of a polyhedron  $P$  is  $\sigma \in \mathbb{R}^n$  such that for all  $x \in P$ , for all  $\lambda \in [0, 1]$ ,

$$(x + \lambda\sigma) \in P \quad (1.8)$$

i.e. it is a direction in which we can travel infinitely without leaving the polyhedron.

## 1.1 Reminders on subgradients

$\pi$  is a subgradient of the function  $g$  at  $u$  if

$$g(w) \geq g(u) + \pi^T(w - u) \quad \forall w \quad (1.9)$$

If  $g = \max\{g_1, g_2\}$  with  $g_{1,2}$  convex and differentiable, the subgradient of  $g$  at  $u_0$  is

- $\pi = \nabla g_1(u_0)$  if  $g_1(u_0) > g_2(u_0)$
- $\pi = \nabla g_2(u_0)$  if  $g_2(u_0) > g_1(u_0)$
- The line segment  $[\nabla g_1(u_0), \nabla g_2(u_0)]$  if  $g_1(u_0) = g_2(u_0)$

The subdifferential of  $g$  at  $u$  is the set of all subgradients of  $g$  at  $u$ , denoted  $\partial g(u)$ . If  $g$  is convex, then its subdifferential is nonempty on its domain, and  $g$  is differentiable at  $u$  if its  $\partial g(u) = \{\pi\}$ .

### 1.1.1 Use in duality

Define  $c(u)$  as the optimal value of

$$\begin{aligned} c(u) &= \min f_0(x) \\ f_i(x) &\leq u_i \quad i = 1, \dots, m \end{aligned} \quad (1.10)$$

where  $x \in \text{dom} f_0$  and  $f_0, f_i$  are convex functions.

- $c(u)$  is convex;
- If strong duality holds, denote  $\lambda^*$  as the maximizer of the dual function

$$\inf_{x \in \text{dom} f_0} (f_0(x) - \lambda^T(f(x) - u)) \quad (1.11)$$

for  $\lambda \leq 0$ . Then,  $\lambda^* \in \partial c(u)$ .  $\lambda_i$  represents the sensitivity of  $c(u)$  to a marginal change in the right-hand side of the  $i$ -th constraint.

# Modelling

## 2.1 Introduction

- For a certain sequence of events  $x \rightarrow \omega \rightarrow y(\omega)$ , where  $\omega$  is the uncertainty,
  - A first-stage decision is a decision that is made before the uncertainty is revealed (i.e. in  $x$ );
  - A second-stage decision is a decision that is made after the uncertainty is revealed (i.e. in  $y(\omega)$ ).
- We can have the following mathematical formulation:

$$\begin{aligned}
 \min \quad & c^T x + \mathbb{E}[\min q(\omega)^T y(\omega)] \\
 \text{s.t.} \quad & Ax = b \\
 & T(\omega)x + W(\omega)y(\omega) = h(\omega) \\
 & x \geq 0, y(\omega) \geq 0
 \end{aligned} \tag{2.1}$$

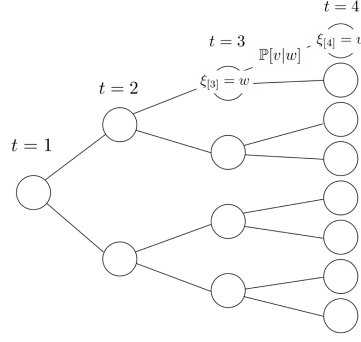
- First-stage decision variable:  $x \in \mathbb{R}^{n_1}$
- First-stage parameter:  $c \in \mathbb{R}^{n_1}$ ,  $b \in \mathbb{R}^{m_1}$  and  $A \in \mathbb{R}^{m_1 \times n_1}$
- Second-stage decision:  $y(\omega) \in \mathbb{R}^{n_2}$
- Second-stage data:  $q(\omega) \in \mathbb{R}^{n_2}$ ,  $h(\omega) \in \mathbb{R}^{m_2}$  and  $T(\omega) \in \mathbb{R}^{m_2 \times n_1}$ ,  $W(\omega) \in \mathbb{R}^{m_2 \times n_2}$

## 2.2 Representations

### 2.2.1 Scenario Trees

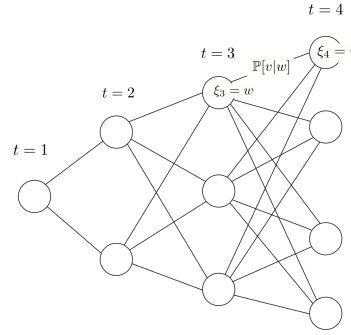
A scenario tree is a graphical representation of a Markov process  $\{\xi_t\}_{t \in \mathbb{Z}}$ , where the nodes are the history of realizations ( $\xi_{[t]} = (\xi_1, \dots, \xi_t)$ ), and the edges are the transitions from  $\xi_{[t]}$  to  $\xi_{[t+1]}$ .

- We denote the root as  $t = 1$ ;
- An ancestor of a node  $\xi_{[t]}$ ,  $A(\xi_{[t]})$  is a unique adjacent node which precedes  $\xi_t$ ;
- The children of a node,  $C(\xi_{[t]})$  are the nodes that are adjacent to  $\xi_{[t]}$  and occur at stage  $t + 1$ .



### 2.2.2 Lattice

A lattice is a graphical representation of a Markov process  $\{\xi_t\}_{t \in \mathbb{Z}}$ , where the nodes are the realizations  $\xi_t$  and the edges correspond to the transitions from  $\xi_t$  to  $\xi_{t+1}$ .



### 2.2.3 Serial Independence

A process satisfies serial independence if, for every stage  $t$ ,  $\xi_t$  has a probability distribution that does not depend on the history of the process. Thus, the probability measure is

$$\mathbb{P} \left[ \xi_t(\omega) = i \mid \xi_{[t-1]}(\omega) \right] = p_t(i) \quad \forall \xi_{[t-1]} \in \Xi_{[t-1]}, i \in \Xi_t \quad (2.2)$$

## 2.3 Multi Stage Stochastic Linear Program

### 2.3.1 Notation

- Probability space:  $(\Omega, 2^\Omega, \mathbb{P})$  with filtration  $\{\mathcal{A}\}_{t \in \{1, \dots, H\}}$
- $c_t(\omega) \in \mathbb{R}^{n_t}$ : cost coefficients
- $h_t(\omega) \in \mathbb{R}^{m_t}$ : right-hand side parameters
- $W_t(\omega) \in \mathbb{R}^{m_t \times n_t}$ : coefficients of  $x_t(\omega)$
- $T_{t-1}(\omega) \in \mathbb{R}^{m_t \times n_{t-1}}$ : coefficients of  $x_{t-1}(\omega)$
- $x_t(\omega)$ : set of state and action variables in period  $t$

- We implicitly enforce non-anticipativity by requiring that  $x_t$  and  $\xi_t$  are adapted to filtration  $\{\mathcal{A}\}_{t \in \{1, \dots, H\}}$
- $\forall A \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}, x_t(\omega_1) = x_t(\omega_2) \forall \omega_1, \omega_2 \in A$

### 2.3.2 General formulation of the MSLP

The extended formulation of the MSLP is:

$$\begin{aligned}
& \min c_1^T x_1 + \mathbb{E}[c_2(\omega)^T x_2(\omega) + \dots + c_H(\omega)^T x_H(\omega)] \\
& \text{s.t. } W_1 x_1 = h_1 \\
& T_1(\omega) x_1 + W_2(\omega) x_2(\omega) = h_2(\omega), \omega \in \Omega \\
& \quad \vdots \\
& T_{t-1}(\omega) x_{t-1}(\omega) + W_t(\omega) x_t(\omega) = h_t(\omega), \omega \in \Omega \\
& \quad \vdots \\
& T_{H-1}(\omega) x_{H-1}(\omega) + W_H(\omega) x_H(\omega) = h_H(\omega), \omega \in \Omega \\
& x_1 \geq 0, x_t(\omega) \geq 0, t = 2, \dots, H
\end{aligned} \tag{2.3}$$

We can now consider two specific instantiations of the MSLP: the scenario tree (MSLP-ST) and the lattice (MSLP-L). Using these notations:

- $\omega_t \in S_t$  : index in the support  $\Xi_t$  of random input  $\xi_t$
- $\omega_{[t]} \in S_1 \times \dots \times S_t$  (interpretation: index in  $\Xi_{[t]} = \Xi_1 \times \dots \times \Xi_t$ , which is the history of realizations, up to period  $t$ )

### 2.3.3 Scenario Tree formulation

$$\begin{aligned}
& \min c_1^T x_1 + \mathbb{E} \left[ c_2(\omega_{[2]})^T x_2(\omega_{[2]}) + \dots + c_H(\omega_{[H]})^T x_H(\omega_{[H]}) \right] \\
& \text{s.t. } W_1 x_1 = h_1 \\
& T_1(\omega_{[2]}) x_1 + W_2(\omega_{[2]}) x_2(\omega_{[2]}) = h_2(\omega_{[2]}), \omega_{[2]} \in S_1 \times S_2 \\
& \quad \vdots \\
& T_{t-1}(\omega_{[t]}) x_{t-1}(\omega_{[t-1]}) + W_t(\omega_{[t]}) x_t(\omega_{[t]}) = h_t(\omega_{[t]}), \omega_{[t]} \in S_1 \times \dots \times S_t \\
& \quad \vdots \\
& T_{H-1}(\omega_{[H]}) x_{H-1}(\omega_{[H-1]}) + W_H(\omega_{[H]}) x_H(\omega_{[H]}) = h_H(\omega_{[H]}), \omega_{[H]} \in S_1 \times \dots \times S_H \\
& x_1 \geq 0, x_t(\omega_{[t]}) \geq 0, t = 2, \dots, H
\end{aligned} \tag{2.4}$$



### 2.3.4 Lattice formulation

$$\begin{aligned}
& \min c_1^T x_1 + \mathbb{E} \left[ c_2(\omega_2)^T x_2(\omega_{[2]}) + \cdots + c_H(\omega_H)^T x_H(\omega_{[H]}) \right] \\
& s.t. \quad W_1 x_1 = h_1 \\
& \quad T_1(\omega_2) x_1 + W_2(\omega_2) x_2(\omega_{[2]}) = h_2(\omega_2), \omega_{[2]} \in S_1 \times S_2 \\
& \quad \vdots \\
& \quad T_{t-1}(\omega_t) x_{t-1}(\omega_{[t-1]}) + W_t(\omega_t) x_t(\omega_{[t]}) = h_t(\omega_t), \omega_{[t]} \in S_1 \times \cdots \times S_t \\
& \quad \vdots \\
& \quad T_{H-1}(\omega_H) x_{H-1}(\omega_{[H-1]}) + W_H(\omega_H) x_H(\omega_{[H]}) = h_H(\omega_H), \omega_{[H]} \in S_1 \times \cdots \times S_H \\
& \quad x_1 \geq 0, x_t(\omega_{[t]}) \geq 0, t = 2, \dots, H
\end{aligned} \tag{2.5}$$

→ Note: There exists some relations to other decision making problems such as statistical decision theory, dynamic programming, online optimization and stochastic control.

# Performance

## 3.1 Notations

Using (2.1), let's define the following:

- $z(x, \xi) = c^T x + Q(x, \xi) + \delta(x|K_1)$
- $Q(x, \xi) = \min_y \{q(\omega)^T y \mid W(\omega)y = h(\omega) - T(\omega)x\}$
- $K_1 = \{x \mid Ax = b, x \geq 0\}$  is the set of feasible first-stage decisions
- $K_2(\omega) = \{x \mid \exists y \geq 0 : W(\omega)y = h(\omega) - T(\omega)x\}$  is the set of first-stage decisions that have a feasible reaction in the second stage for  $\omega \in \Omega$
- It is possible that  $z(x, \xi) = +\infty$  (if  $x \notin K_1 \cap K_2(\omega)$ )
- It is possible that  $z(x, \xi) = -\infty$  (unbounded below)

## 3.2 Expected value of perfect information

There are 2 tactics:

- **wait-and-see** value is the expected value of reacting with perfect foresight (we know everything that will happen)  $x^*(\xi)$  to  $\xi$ :

$$WS = \mathbb{E}[\min_x z(x, \xi)] = \mathbb{E}[z(x^*(\xi), \xi)] \quad (3.1)$$

- **here-and-now** value is the expected value of the recourse problem (remove non-anticipativity constraint):

$$SP = \min_x \mathbb{E}[z(x, \xi)] \quad (3.2)$$

The **expected value of perfect information** is like the value we give to getting a perfect forecast for the future and is thus defined like this:

$$EVPI = SP - WS \quad (3.3)$$

### 3.3 The value of the stochastic solution

Here too there are 2 tactics:

- **expected value problem**

$$EV = \min_x z(x, \bar{\zeta}) = \mathbb{E}[\bar{\zeta}] \quad (3.4)$$

and its **expected value solution** is noted  $x^*(\bar{\zeta})$ .

- **expected value of using the EV solution** measures the performance of  $x^*(\bar{\zeta})$ :

$$EEV = \mathbb{E}[z(x^*(\bar{\zeta}), \zeta)] \quad (3.5)$$

The **value of the stochastic solution** is noted like this:

$$VSS = EEV - SP \quad (3.6)$$

### 3.4 Basic inequalities

#### 3.4.1 Crystal Ball

For every  $\zeta$ , we have  $z(x^*(\bar{\zeta}), \zeta) \leq z(x^*, \zeta)$  where  $x^*$  is the optimal solution to the stochastic program. And if we take the expectation of this inequality, we have  $WS \leq SP$ , because  $WS$  is a relaxation. It explains that we can do better with a crystal ball.

#### 3.4.2 Lazy solution

Knowing that  $x^*$  is the optimal solution of  $\min_x \mathbb{E}[z(x, \zeta)]$  and  $x^*(\bar{\zeta})$  is a solution but not necessarily optimal then we have  $SP \leq EEV$ , because:

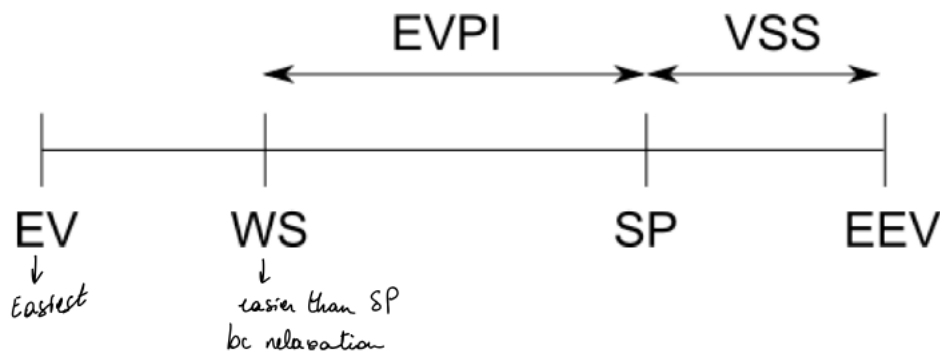
$$\min_x \mathbb{E}[z(x, \zeta)] = SP \leq EEV = \mathbb{E}[z(x^*(\bar{\zeta}), \zeta)] \quad (3.7)$$

#### 3.4.3 Link between all the values

We know that:

- $VSS \geq 0$
- $EVPI \leq EEV - EV$
- $EVPI \geq 0$
- If  $EEV - EV = 0$  then  $VSS = EVPI = 0$
- $VSS \leq EEV - EV$

and the inequalities can be summarized in the following diagram:



### 3.5 Bounds on EVPI and VSS

First let's introduce the pairs subproblem of  $\xi^r$  and  $\xi^k$ :

$$\begin{aligned} \min z^P(x, \xi^r, \xi^k) &= c^T x + p^r q^T y(\xi^r) + (1 - p^r) q^T y(\xi^k) \\ \text{s.t. } Ax &= b \\ Wy(\xi^r) &= \xi^r - Tx \\ Wy(\xi^k) &= \xi^k - Tx \\ x, y &\geq 0 \end{aligned} \tag{3.8}$$

- $(\bar{x}^k, \bar{y}^k, y(\xi^r))$  denotes an optimal solution to the problem and  $z^k$  is the optimal objective function value  $z^P(\bar{x}^k, \bar{y}^k, y(\xi^r))$
- $z^P(x, \xi^r, \xi^r)$  corresponds to the deterministic optimization against the reference scenario
- if  $\xi^r \notin \Xi$ ,  $p^r = 0$  and  $z^P(x, \xi^r, \xi^k) = z(x, \xi^k)$

The **sum of pairs expected value (SPEV)**:

$$SPEV = \frac{1}{1 - p^r} \sum_{k=1, k \neq r}^K p^k \min z^P(x, \xi^r, \xi^k) \tag{3.9}$$

When  $\xi^r \notin \Xi$  then  $SPEV = WS$ : When  $p^r = 0$ ,  $z^P(x, \xi^r, \xi^k)$  coincides with  $z(x, \xi^k)$ . Therefore  $SPEV = \sum_{k=1}^K p^k \min_x z(x, \xi^k) = WS$ . We then know  $WS \leq SPEV \leq SP$ .

#### 3.5.1 Upper bound on SP: EVRS and EPEV

- The **expected value of the reference scenario** is  $EVRS = \mathbb{E}_{\xi}(\bar{x}^r, \xi)$ , where  $\bar{x}^r$  is the optimal solution to  $z(x, \xi^r)$ .
- The **expectation of pairs of expected value** is defined as

$$EPEV = \min_{k=1, \dots, K \cup \{r\}} \mathbb{E}_{\xi}(\bar{x}^r, \xi)$$

where  $(\bar{x}^k, \bar{y}^k, y(\xi^k))$  is the optimal solution to the pairs subproblem of  $\xi^r$  and  $\xi^k$ .

As  $SP, EPEV, EVRS$  are the optimal values of  $\min_x \mathbb{E}_{\xi} z(x, \xi)$  over smaller feasible sets:

$$SP \leq EPEV \leq EVRS \tag{3.10}$$

Because

- $SP$ :  $x \in K_1 \cap K_2$
- $EPEV$ :  $x \in K_1 \cap K_2 \cap \{\bar{x}^k, k = 1, \dots, K \cup \{r\}\}$
- $EVRS$ :  $x \in \bar{x}^r \cap K_1 \cap K_2$

## 3.6 Estimations of WS and EEV

An estimation of WS and EEV can be done through a sample mean approximation: from samples  $\tilde{\xi}_i = \tilde{\xi}(\omega_i)$  for  $i = 1, \dots, K$ ,

1. Compute  $x^*(\tilde{\xi})$ ;
2. Compute  $WS_i = z(x^*(\tilde{\xi}_i), \tilde{\xi}_i)$  and  $EEV_i = c^T x^*(\tilde{\xi}_i) + Q(x^*(\tilde{\xi}), \tilde{\xi}_i)$ ;
3. Estimate  $\bar{WS} = \frac{1}{K} \sum_{i=1}^K WS_i$  and  $\bar{EEV} = \frac{1}{K} \sum_{i=1}^K EEV_i$ .

### 3.6.1 Central Limit Theorem

Suppose  $\{X_1, \dots, X_K\}$  is a sequence of iid rv with  $\mathbb{E}[X_i] = \mu$  and  $\text{Var}[X_i] = \sigma^2 < \infty$ . Then, as  $n$  approaches infinity,  $\sqrt{n}(S_n - \mu)$  converge in distribution to a normal  $\mathcal{N}(0, \sigma^2)$ :

$$\sqrt{n} \left( \left( \frac{1}{n} \sum_{i=1}^n X_i \right) - \mu \right) \xrightarrow{d} \mathcal{N}(0, \sigma^2) \quad (3.11)$$

The central limit theorem is useful to decrease the importance of rare but extreme events.

### 3.6.2 Importance sampling

Suppose we wish to estimate  $\mathbb{E}[C(\omega)]$ , where  $\omega$  is distributed according to  $f(\omega)$  and estimates  $\mathbb{E}[C(\omega)]$  with  $\sum_{i=1}^N \frac{1}{N} C(\omega_i)$ . A sample average pulls samples  $\omega_i$  according to the distribution function  $f(\omega)$ , while the importance sampling pulls the samples  $\omega_i$  according to the distribution  $g(\omega) = \frac{f(\omega)C(\omega)}{\mathbb{E}[C(\omega)]}$ , where the  $\mathbb{E}[C(\omega)]$  is an approximation of the real expectation. It then estimates  $\mathbb{E}[C(\omega)]$  with  $\sum_{i=1}^N \frac{1}{N} \frac{f(\omega_i)C(\omega_i)}{g(\omega_i)}$ .

# Benders Decomposition

## 4.1 Cutting plane methods

A cutting plane method is an optimisation method based on the idea of iteratively refining the objective function, or a set of feasible constraints of a problem through linear inequalities (see LINMA2450).

### 4.1.1 Nomenclature

- The benders decomposition is a specific method for obtaining the cutting planes when  $F(x)$  is the value function of a second-stage linear program.
- The L-shaped method is a specific instance of Benders decomposition when the second-stage linear program is decomposable into a set of scenarios.
- The multi-cut L-shaped method is an alternative to the L-shaped method which generates multiple cutting planes at step 1 of Kelley's method (see 4.1.2).

### 4.1.2 Kelley's Cutting Plane Algorithm

This algorithm is designed to solve convex but non-differentiable optimization problems of the form

$$\begin{aligned} z^* &= \min c^T x + F(x) \\ \text{s.t. } x &\in X \end{aligned} \tag{4.1}$$

where  $X \subseteq \mathbb{R}^n$  is convex and compact,  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex and  $c \in \mathbb{R}^n$  is a parameter vector.

Let us define

- $L_k : \mathbb{R}^n \rightarrow \mathbb{R}$  a lower bound function of  $F(x)$  at iteration  $k$ ;
- A lower bound  $L_k$  of  $z^*$  at iteration  $k$ ;
- An upper bound  $U_k$  of  $z^*$  at iteration  $k$ .

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**Algorithm 1** Kelley's Cutting plane algorithm

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- 1: **Step 0:** Set  $k = 0$  and assume  $x_1 \in X$  is given. Set  $L_0(x) = -\infty$  for all  $x \in X$ ,  $U_0 = c^T x_1 + F(x_1)$ , and  $L_0 = -\infty$ .  
2: **Step 1:** Set  $k = k + 1$ . Find  $a_k \in \mathbb{R}$  and  $b_k \in \mathbb{R}^n$  such that

$$F(x_k) = a_k + b_k^T x_k$$

$$F(x) \geq a_k + b_k^T x \quad x \in X$$

- 3: **Step 2:** Set

$$U_k = \min(U_{k-1}, c^T x_k + F(x_k))$$

and

$$L_k(x) = \max(L_{k-1}(x), a_k + b_k^T x) \quad x \in X$$

- 4: **Step 3:** Compute

$$L_k = \min_{x \in X} L_k(x) + c^T x$$

and denote  $x_{k+1}$  as the optimal solution of this problem.

- 5: **Step 4:** If  $U_k - L_k = 0$ , stop. Otherwise, go to step 1.
- 

## 4.2 Context and description

Consider the following optimization problem:

$$\begin{aligned} z^* &= \min c^T x + q^T y \\ Ax &= b \\ Tx + Wy &= h \\ x, y &\geq 0 \end{aligned} \tag{4.2}$$

with  $x \in \mathbb{R}^{n_1}$ ,  $y \in \mathbb{R}^{n_2}$ ,  $c \in \mathbb{R}^{n_1}$ ,  $q \in \mathbb{R}^{n_2}$ ,  $A \in \mathbb{R}^{m_1 \times n_1}$ ,  $b \in \mathbb{R}^{m_1}$ ,  $T \in \mathbb{R}^{m_2 \times n_1}$ ,  $W \in \mathbb{R}^{m_2 \times n_2}$ ,  $h \in \mathbb{R}^{m_2}$ <sup>1</sup>.

We use Benders decomposition when the entire problem is difficult to solve, and if the constraint  $Tx + Wy = h$  is ignored, the problem becomes easy to solve, or if fixing  $x$  simplifies the computation of the solution.

### 4.2.1 Idea of Benders decomposition

Define the value function  $V : \mathbb{R}^{n_1} \rightarrow \mathbb{R}$ :

$$\begin{aligned} (S) : V(x) &= \min_y q^T y \\ Wy &= h - Tx \\ y &\geq 0 \end{aligned} \tag{4.3}$$

---

<sup>1</sup>It is not necessarily a stochastic problem

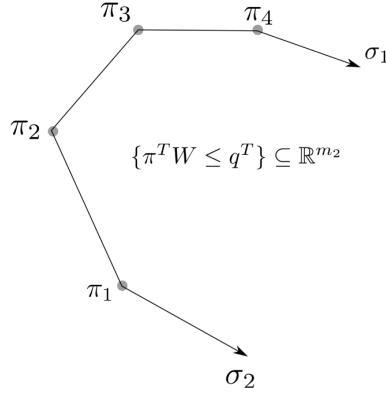
Or equivalently,

$$\begin{aligned}
\min \quad & c^T x + V(x) \\
\text{subject to} \quad & Ax = b \\
& x \in \text{dom}(V) \\
& x \geq 0
\end{aligned} \tag{4.4}$$

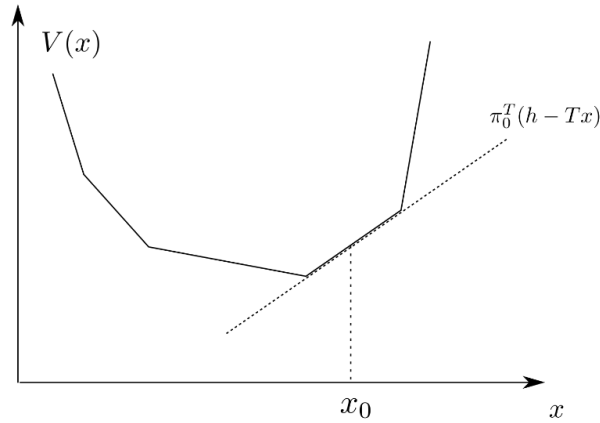
where  $\text{dom}(V) = \{x \in \mathbb{R}^{n_1} \mid \exists y \geq 0 : Wy = h - Tx\}$ .  
The dual of (4.3) is

$$\begin{aligned}
\max_{\pi} \quad & \pi^T (h - Tx) \\
\text{subject to} \quad & \pi^T W \leq q^T
\end{aligned} \tag{4.5}$$

Let us call  $E$  the set of extreme points of  $\pi^T W \leq q^T$  and  $R$  the set of extreme rays of  $\pi^T W \leq q^T$  (see (1.8) for definitions).



We can see that  $V(x)$  is a piecewise linear convex function of  $x$  and, defining  $x_0$  as the dual optimal multiplier of (4.3) given  $x_0$ , then  $\pi_0^T (h - Tx_0)$  is a supporting hyperplane of  $V(x)$  at  $x_0$ , because it belongs to the subdifferential of  $V(x)$  at  $h - Tx_0$ .



From this, we can also express the domain of  $V$  as follows:

$$\text{dom}(V) = \{x \mid \sigma^T (h - Tx) \leq 0, \sigma \in R\} \tag{4.6}$$

where  $\sigma \in R$  is the set of extreme rays of  $\pi^T W \leq q^T$ .

→ Note: when a domain is unbounded in a direction that does not improve the objective value, it is not a problem to its resolution.



### 4.2.2 Reformulation

The objective of the reformulation is to find a general form for the algorithm. That way, each iteration simply adds constraints of the same form, involving the minimum number of changes to the problem.

$$\begin{aligned}
 & \min c^T x + \theta \\
 & Ax = b \\
 & \sigma_r^T (h - Tx) \leq 0 \quad \sigma_r \in R \\
 & \theta \geq \pi_e^T (h - Tx) \quad \pi_e \in E \\
 & x \geq 0
 \end{aligned} \tag{4.7}$$

where  $\theta$  is a free variable.

The idea is to relax some inequalities that define  $V(x)$  and  $\text{dom } V$ :

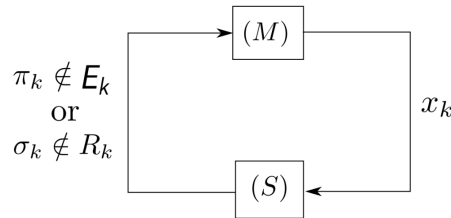
$$\begin{aligned}
 (M) : & \quad z_k = \min c^T x + \theta \\
 & \quad Ax = b \\
 & \quad \sigma_r^T (h - Tx) \leq 0 \quad \sigma_r \in R_k \subseteq R \\
 & \quad \theta \geq \pi_e^T (h - Tx) \quad \pi_e \in E_k \subseteq E \\
 & \quad x \geq 0 \\
 (S) : & \quad V(\bar{x}) = \min_{x,y} q^T y \\
 & \quad Wy = h - Tx \\
 & \quad x = \bar{x} \\
 & \quad y \geq 0
 \end{aligned} \tag{4.8}$$

The solution of the main problem (M) above provides:

- A lower bound  $z_k \leq z^*$ ;
- A candidate solution  $x_k$ ;
- An under-estimator of  $V(x_k)$ ,  $\theta_k \leq V(x_k)$ .

The solution of the subproblem (S) with input  $x_k$  provides:

- An upper bound  $c^T x_k + q^T y_{k+1} \geq z^*$ ;
- A new vertex  $\pi_{k+1}$  or a new extreme ray  $\sigma_{k+1}$ .



In addition to the two problems defined above, we have the dual of (S):

$$\begin{aligned}
 (D) : & \quad \max_{\pi, \lambda} \lambda^T x + \pi^T h \quad \pi^T W \leq g^T \\
 & \quad \pi^T T + \lambda = 0
 \end{aligned} \tag{4.10}$$

With this, the formulation of the optimality cut becomes

$$\theta \geq \lambda^T (x - \bar{x}) + V(\bar{x}) \tag{4.11}$$

## 4.3 Benders Decomposition Algorithm

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**Algorithm 2** Benders Decomposition Algorithm

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```
1: Step 0: Set  $k = 0$ ,  $E_0 = R_0 = \emptyset$ ;  
2: Step 1: Solve (M);  
3: if (M) is feasible then  
4:   Store  $x_k$ ;  
5: else  
6:   break;  
7: end if  
8: Step 2: Solve (S) (or (D)) with  $x_k$  as input;  
9: if (S) is infeasible then  
10:   Let  $R_{k+1} = R_k \cup \{\sigma_{k+1}\}$ ;  
11:    $k \leftarrow k + 1$ ;  
12: else  
13:   Let  $E_{k+1} = E_k \cup \{\pi_{k+1}\}$ ;  
14:   if  $E_{k+1} = E_k$  then  
15:     terminate with  $(x_k, y_{k+1})$  as optimal solution;  
16:   else  
17:     Let  $k \leftarrow k + 1$  and back to step 1.  
18:   end if  
19: end if
```

---

→ Note: The algorithm takes finite time, as  $E$  and  $R$  are finite.