



LINMA2370 Modelling and Analysis of Dynamical Systems

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Introduction

The tools introduced in this course are a simplifying view of the reality, yet very useful to build simple and effective models in view of the control and optimization of the dynamical behaviour of the real systems.

1.1 Reminders

- A subset of \mathbb{R} is said to be negligible if its Lebesgue measure is equal to zero and that a property is said to be true almost everywhere if it is false only on a negligible set.
- Let $I \subseteq \mathbb{R}$ be an interval the interior of which is not empty. A function $x : I \rightarrow \mathbb{R}^N$ is said to be absolutely continuous if

$\forall \varepsilon \in (0, \infty), \exists \delta \in (0, \infty) :$

$$\begin{aligned} & \forall n \in \mathbb{N} \setminus \{0\}, \forall a_1, b_1, \dots, a_n, b_n \in I : \\ & a_i < b_i \forall i \in \{1, \dots, n\}, b_i \leq a_{i+1} \forall i \in \{1, \dots, n-1\}, \\ & \sum_{i=1}^n (b_i - a_i) \leq \delta \implies \sum_{i=1}^n \|x(b_i) - x(a_i)\| \leq \varepsilon \end{aligned}$$

- Let $a, b \in \mathbb{R}$ with $a < b$. A function $x : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous iff there exists an integrable function $\phi : [a, b] \rightarrow \mathbb{R}$ such that, for every $t \in [a, b]$,

$$x(t) = x(a) + \int_a^t \phi(s) ds$$

in which case x is almost everywhere differentiable with $\dot{x}(t) = \phi(t)$ for almost every $t \in [a, b]$.

- A function $f : \Omega \rightarrow \mathbb{R}^N$, where Ω is a nonempty subset of $\mathbb{R} \times \mathbb{R}^N$, is said to be Lipschitz continuous in the second argument, uniformly with respect to the first argument, if there exists $L \in [0, \infty)$ such that for all $t \in \mathbb{R}$ and all $x, y \in \mathbb{R}^N$ such that $(t, x), (t, y) \in \Omega$,

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\|$$

It is said to be locally Lipschitz continuous on an open ball for each argument.

- Let Ω be a nonempty open subset of $\mathbb{R} \times \mathbb{R}^N$ and $f : \Omega \rightarrow \mathbb{R}^N$ be such that

- for all $t \in \mathbb{R}$, $f(t, \cdot) : \Omega_t \rightarrow \mathbb{R}^N$
- $\partial_2 f : \Omega \rightarrow \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N) : (t, x) \rightarrow \partial_2 f(t, x)$ is locally bounded.

Then, f is locally Lipschitz continuous in the second argument, uniformly with respect to the first argument.

- If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two real normed spaces, and the real vector space $\mathcal{L}(X, Y)$ of all continuous linear mappings from X to Y ¹ is equipped with the norm defined by

$$\|L\| := \sup_{x \in X \setminus \{0\}} \frac{\|Lx\|_Y}{\|x\|_X}$$

1.2 State-space model

A state-space model for a continuous dynamical system consists of an ODE of the form

$$\dot{x}(t) = f(t, x(t)) \quad (1.1)$$

where the function $f : \Omega \rightarrow \mathbb{R}^N$, Ω being a nonempty subset of $\mathbb{R} \times \mathbb{R}^N$, is called the vector field associated with the ODE. A continuous dynamical system with input $u : \mathbb{R} \rightarrow \mathbb{R}^M$ described by the ODE

$$\dot{x}(t) = g(x(t), u(t)) \quad (1.2)$$

for some function $g : \mathbb{R}^N \times \mathbb{R}^M \rightarrow \mathbb{R}^N$, can be written in the form 1.1 by defining the vector field

$$f_u : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N : (t, x) \rightarrow g(x, u(t)) \quad (1.3)$$

→ N.B.: the norm of each $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ is defined as $|t| + \|x\|$.

1.3 Integral curve

Let Ω be a nonempty subset of $\mathbb{R} \times \mathbb{R}^N$. An integral curve of $f : \Omega \rightarrow \mathbb{R}^N$ is a function $x : I \rightarrow \mathbb{R}^N$ where $I \subseteq \mathbb{R}$ is an interval, for which the interior is not empty, called the interval of existence of x , i.e. differentiable and satisfies $(t, x(t)) \in \Omega$ and $\dot{x}(t) = f(t, x(t))$ for all $t \in I$. The graph $\{(t, x(t)) | t \in I\}$ and the image $\{x(t) | t \in I\}$ of x are respectively called the trajectory and the orbit of x . Given an initial condition $(t_0, x_0) \in \Omega$, a solution to the initial value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \quad (1.4)$$

is an integral curve $x : I \rightarrow \mathbb{R}^N$ of f such that $t_0 \in I$ and $x(t_0) = x_0$.

¹Meaning matrix from X to Y

If, for the IVP described hereabove, f is continuous, then a continuous function $x : I \rightarrow \mathbb{R}^N$ where $I \subseteq \mathbb{R}$ is an interval containing t_0 and the interior of which is not empty, is a solution iff its graph is contained in Ω and it satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds$$

for all $t \in I$. In that case, \dot{x} is continuous.

Let Ω be a nonempty subset of $\mathbb{R} \times \mathbb{R}^N$. An integral curve in the extended sense of $f : \Omega \rightarrow \mathbb{R}^N$ is a function $x : I \rightarrow \mathbb{R}^N$, where $I \subseteq \mathbb{R}$ is an interval the interior of which is not empty called the interval of existence of x , that is absolutely continuous and satisfies $(t, x(t)) \in \Omega$ for every $t \in I$ and $\dot{x}(t) = f(t, x(t))$ for almost every $t \in I$.

→ N.B.: If f is continuous, then the two definitions of integral curves are equivalent.

1.4 Existence of a solution

Consider the IVP defined hereabove with an integral curve in the extended sense, under the following assumptions:

- there exists $\tau, r \in (0, \infty)$, such that $[t_0 - \tau, t_0 + \tau] \times B(x_0, r) \subseteq \Omega$;
- for every $x \in B(x_0, r)$, the function $[t_0 - \tau, t_0 + \tau] \rightarrow \mathbb{R}^N : t \rightarrow f(t, x)$ is measurable;
- for every $t \in [t_0 - \tau, t_0 + \tau]$, the function $B(x_0, r) \rightarrow \mathbb{R}^N : x \rightarrow f(t, x)$ is continuous;
- there exists an integrable function $m : [t_0 - \tau, t_0 + \tau] \rightarrow [0, \infty)$ such that

$$\|f(t, x)\| \leq m(t) \text{ for all } (t, x) \in [t_0 - \tau, t_0 + \tau] \times B(x_0, r)$$

Then, there exists a solution defined on a compact interval the interior of which contains t_0 .

In particular, for the IVP with an integral curve in the general sense, if (t_0, x_0) is an interior point of Ω and f is continuous, then there exists a solution defined on a compact interval the interior of which contains t_0 .

Dynamical systems and state-space models

We will study first-order dynamical systems of the form

$$\dot{x} = f(x, u) \quad (2.1)$$

where f is a mapping from \mathbb{R}^{n+m} to \mathbb{R}^n , while x and u are vector functions of time, respectively the state and the input.

2.1 Terminology and notation

- We assume that the input is a piecewise continuous and bounded function: $u \in \mathcal{U}$, where \mathcal{U} is a set of piecewise continuous and bounded functions from \mathbb{R} to \mathbb{R}^m .
- For a given value of the initial state $x(t_0) = x_0$ and a given input u , the solution $t \rightarrow x(t)$ for $t \geq t_0$, of the system of ODE 2.1 is called the trajectory of the system. It is denoted $x(t_0, x_0, u)$.
- When the input u can be freely chosen in \mathcal{U} , the system $\dot{x} = f(x, u)$ is said to be a forced/controlled system.

→ N.B.: in this course, we will study the solution of the equation 2.1 when the input is actually an a priori set constant: $u(t) = \bar{u} \forall t \geq t_0$. The state-space model is then written as $\dot{x} = f(x, \bar{u}) = f_{\bar{u}}(x)$.

2.1.1 System with affine input

$$\dot{x} = f(x) + \sum_{i=1}^m u_i g_i = f(x) + G(x)u \quad (2.2)$$

where f and g_i are mappings from \mathbb{R}^n to \mathbb{R}^n .

2.1.2 System with affine state

$$\dot{x} = \sum_{i=1}^n x_i a_i(u) + b(u) = A(u)x + b(u) \quad (2.3)$$

where b and a_i are mappings from \mathbb{R}^m to \mathbb{R}^n .

2.1.3 Bilinear systems

A bilinear system is affine both in the state and in the input:

$$\dot{x} = \left(A_0 + \sum_{i=1}^m u_i A_i \right) x + B_0 u \quad (2.4)$$

where A_i and B_i are matrices of dimensions $n \times n$ and $n \times m$ respectively.

2.1.4 Linear system

$$\dot{x} = Ax + Bu \quad (2.5)$$

where A and B are matrices of dimensions $n \times n$ and $n \times m$ respectively.

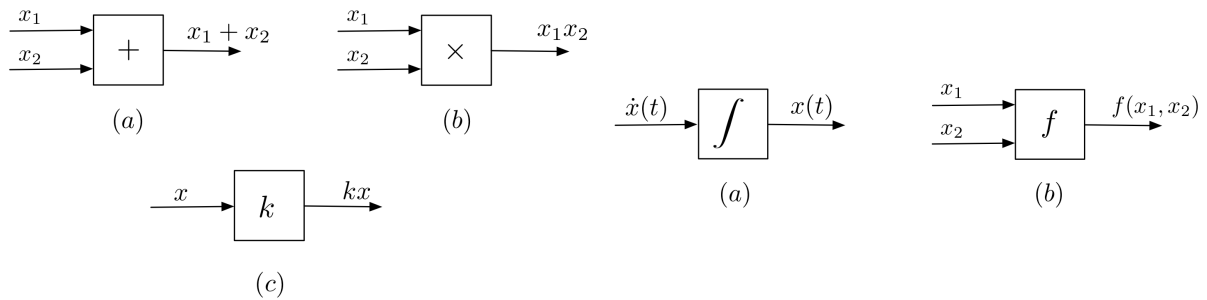
Articulated mechanical systems

Electrical and electromechanical systems

State transformations

5.1 Definition

The block diagram of a dynamical system is a visual representation of that system, necessarily containing n integrators whose outputs are the n state variables.



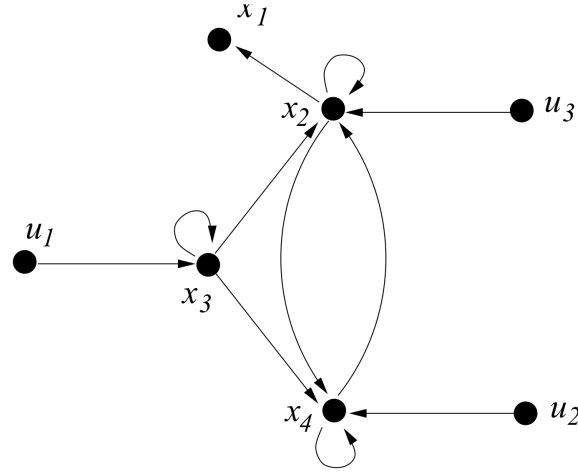
The graph of a dynamical system contains as nodes the inputs and states of the system, and its edges are the relations between those quantities. The construction rules of the graph of a dynamical system are the following:

- The $n + m$ nodes are the n state variables x_i and the m inputs u_j ;
- there is an oriented edge from x_i (or u_k) to x_j if the variable x_i (or u_k) appears explicitly in the equation of the derivative \dot{x}_j .

Example for a DC electric machine: the state space model is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = J^{-1}(-h(x_2) + K_m x_3 x_4 + u_3) \\ \dot{x}_3 = L_s^{-1}(-R_s x_3 + u_1) \\ \dot{x}_4 = L_r^{-1}(-R_r x_4 - K_e x_2 x_3 + u_2) \end{cases} \quad (5.1)$$

and its graph representation is



5.2 Linear state transformation

For a dynamical system $\dot{x} = f(x, u)$, a linear state transformation is a linear mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that is bijective and transforms the state of the system $x \in \mathbb{R}^n$ into a new state $z \in \mathbb{R}^n$ following the rule $z = Tx$, where $T \in \mathbb{R}^{n \times n}$ is an invertible matrix. The relation between the two systems is

$$\begin{cases} \dot{x} = f(x, u) \\ \dot{z} = g(z, u) \end{cases} \implies \begin{cases} z \triangleq T^{-1}x \\ g(z, u) \triangleq Tf(T^{-1}z, u) \end{cases} \quad (5.2)$$

For a linear system, we have

$$\dot{z} = Fz + Gu \quad F \triangleq TAT^{-1} \quad G \triangleq TB \quad (5.3)$$

5.3 Nonlinear state transformation

Let U, V be two open subsets of \mathbb{R}^n . A nonlinear state transformation is a mapping $T : U \rightarrow V$ that transforms the state of the system $x \in U$ into a new state $z \in V$: $z = T(x)$ and that has the following properties:

- T is bijective and has an inverse function $T^{-1} : V \rightarrow U$ such that $x = T^{-1}(z)$;
- T and T^{-1} are of class \mathcal{C}^1 , i.e. continuously differentiable.

→ N.B.: The state transformation is said to be global if $U = V = \mathbb{R}^n$.

Such a transformation T is called a diffeomorphism, and the new state space is

$$\dot{z} = \frac{\partial T}{\partial x} \dot{x} = \frac{\partial T}{\partial x} f(x, u) \iff f(x, u) \triangleq \left[\frac{\partial T^{-1}}{\partial z} g(z, u) \right]_{z=T(x)} \quad (5.4)$$

Lemma 5.1. • If the jacobian matrix $\partial T / \partial x$ is nonsingular at x_0 , then, by the inverse function theorem, there is a neighbourhood U of x_0 such that the mapping T restricted to U is a diffeomorphism on U .

- T is a global diffeomorphism iff
 1. $\partial T / \partial x$ is a nonsingular for every $x \in \mathbb{R}^n$;
 2. $\lim_{\|x\| \rightarrow \infty} \|T(x)\| = \infty$.

5.4 Triangular system

Definition 5.2. A single input dynamical system is triangular if there is a state variable x_i such that the shortest path from u to x_i in the graph of the system is of length n .

We can thus renumber the state variables such that the system is expressed as

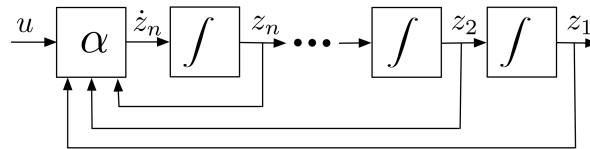
$$\begin{aligned}
 \dot{x}_1 &= g_1(x_1, x_2) \\
 &\vdots \\
 \dot{x}_i &= g_i(x_1, \dots, x_{i+1}) \\
 &\vdots \\
 \dot{x}_{n-1} &= g_{n-1}(x_1, \dots, x_n) \\
 \dot{x}_n &= g_n(x_1, \dots, x_n, u)
 \end{aligned} \tag{5.5}$$

5.5 Brunovsky canonical form

Definition 5.3. A single input dynamical system can be written in brunovsky canonical form if there exists a state transformation $T : U \rightarrow V$ and an open interval $W \subseteq \mathbb{R}$ such that, in the new state variables $z = T(x)$, the system takes on the following particular triangular form:

$$\begin{aligned}
 \dot{z}_1 &= z_2 \\
 \dot{z}_2 &= z_3 \\
 &\vdots \\
 \dot{z}_n &= \alpha(z_1, \dots, z_n, u)
 \end{aligned} \tag{5.6}$$

where the function α is continuous and invertible according to u over W for all $z \in V$. The block diagram of the Brunovsky canonical form is



Lemma 5.4. A triangular dynamical system described by the state-space model (5.5) can be put under Brunovsky canonical form around (x_0, u_0) if the inequalities

$$\begin{cases} \frac{\partial g_i}{\partial x_{i+1}} \neq 0 & i = 1, \dots, n-1 \\ \frac{\partial g_n}{\partial u} \neq 0 \end{cases} \tag{5.7}$$

Lemma 5.5. A control-affine system $\dot{x} = f(x) + g(x)u$ with $x \in \mathbb{R}^n, u \in \mathbb{R}$ can be written in Brunovsky form in a domain $U \subseteq \mathbb{R}^n$ if there exists a state transformation $z = T(x)$ that fulfills the following conditions:

- $T_{i+1}(x) = \frac{\partial T_i}{\partial x} f(x)$, for $i = 1, \dots, n-1$;
- $\frac{\partial T_i}{\partial x} g(x) = 0$, for $i = 1, \dots, n-1$;
- $\frac{\partial T_n}{\partial x} g(x) \neq 0$

for every $x \in U$.

Equilibria and invariant sets

In this chapter, we assume that f is locally Lipschitz continuous on an open set $\Omega \subseteq \mathbb{R}^n$.

Definition 6.1. The pair $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an equilibrium of the system $\dot{x} = f(x, u)$ if $f(\bar{x}, \bar{u}) = 0$.

Definition 6.2. The equilibrium (\bar{x}, \bar{u}) is said to be isolated if there exists a neighbourhood of \bar{x} that contains no other vector x such that $f(x, \bar{u}) = 0$.

6.1 Equilibria of linear systems

$$\dot{x} = Ax + Bu \quad (6.1)$$

Theorem 6.3. If the matrix A is regular, then for each \bar{u} , the pair $(-A^{-1}B\bar{u}, \bar{u})$ is an isolated equilibrium.

If the matrix A is singular, the system (6.1) has a continuum of non-isolated equilibria provided that $B\bar{u} \in \text{Im}(A)$. Those equilibria are the solutions of the system $A\bar{x} = -B\bar{u}$, forming an affine space. On the other side, for each \bar{u} such that $B\bar{u} \notin \text{Im}(A)$, the system does not have any equilibrium.

6.2 Invariant sets

Definition 6.4. A set $\mathcal{X} \times U \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is said to be (positively) invariant for the dynamical system $\dot{x} = f(x, u)$ if, for all $x_0 \in \mathcal{X}$ and for all input signal $t \rightarrow u(t) \in U$, the trajectory $t \rightarrow x(t, x_0, u(t))$ remains in \mathcal{X} for all $t \geq t_0$ whenever it is defined.

Definition 6.5. An outward normal vector to $\mathcal{X} \subseteq \mathbb{R}^n$ at $x \in \partial\mathcal{X}$ is a vector $n \in \mathbb{R}^n$ such that $n = \lambda(y - x)$, where $\lambda > 0$ and y is the center of an open ball $B \subseteq \mathbb{R}^n$ such that $x \in \partial B$ and $B \cap \mathcal{X} = \emptyset$; if no such open ball exists, \mathcal{X} has no outward normal vector at x .

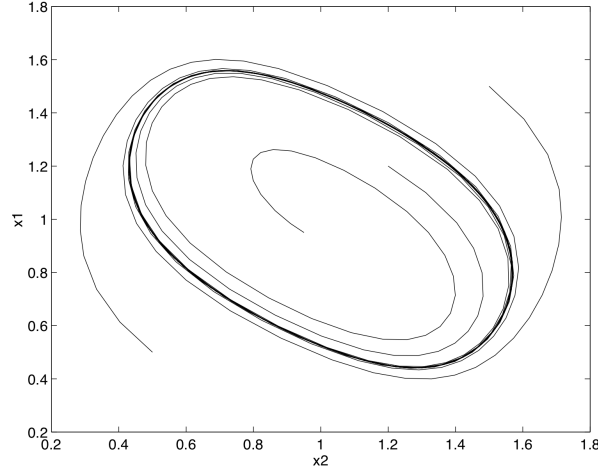
Theorem 6.6 (Bony's theorem). Let f be a locally Lipschitz continuous vector field defined on an open set $\Omega \subseteq \mathbb{R}^n$, and \mathcal{X} a closed set of Ω . If $\langle f(x), n(x) \rangle \leq 0$ for every $x \in \partial\mathcal{X}$, and every vector $n(x)$ is outward normal to \mathcal{X} at x , then \mathcal{X} is (positively) invariant for f .

→ N.B.: no condition has to be verified at a point where \mathcal{X} does not have an outward normal vector.

6.3 Periodic orbits

A periodic orbit is such that it is arising from a trajectory of the dynamical system, verifying $x(t) = x(t + T)$ for all t and for some $T > 0$ ¹. The infimum of possible values for T is called the period of the trajectory.

We denote $x(t, x_0, \bar{u})$ as the solution at time t with $x(t_0) = x_0$ and a constant input $u(t) = \bar{u}$.



Definition 6.7. The point z is called a limit point of y for the dynamical system subject to a constant input \bar{u} if there exists a real sequence (t_n) such that $t_n \rightarrow \infty$ when $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} x(t_n, y, \bar{u}) = z$.

Definition 6.8. A limit cycle is a closed orbit γ such that at least one point of γ^2 is a limit point of at least another point of the phase plane not in γ .

→ N.B.: These definitions are only valid in \mathbb{R}^2 .

Theorem 6.9 (Bendixson-Dulac). Let D be a simply connected domain in \mathbb{R}^2 . If the divergence of f^3 is not identically zero and does not change sign in D , then D does not contain any closed orbit.

Theorem 6.10 (Poincaré-Bendixson). If E is a closed and bounded subset of \mathbb{R}^2 , invariant for the system $\dot{x} = f(x, u)$, and if γ is an orbit starting in E , then:

- either γ converges to an equilibrium (which is the unique limit point of γ);
- or γ converges to a periodic orbit (which is the set of all limit points of γ).

This theorem can be used to prove the existence of a limit cycle:

1. Find a compact invariant set (proved by showing that on the border of this set, the vector field points inwards);
2. If there is no equilibrium in this set, it must contain a limit cycle or only periodic trajectories.

¹Equilibria are trivial periodic orbits.

²Implying that they all are.

³ $\div(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$

Local analysis of autonomous dynamical systems

A dynamical system is said to be autonomous if the input is constant:

$$\dot{x} = f(x\bar{u}) \quad (7.1)$$

7.1 Linear planar systems

Let us consider the linear planar system such as (6.1) with constant input $u = \bar{u}$. Let \bar{x} be an equilibrium point corresponding to \bar{u} . We will use the state transformation $z = M^{-1}(x - x\bar{x})$. We obtain the linear system

$$\dot{z} = A'z \quad A' = M^{-1}AM \quad (7.2)$$

As A and A' have the same eigenvalues, we can choose A' to have a canonical form:

- Two distinct real eigenvalues or double real eigenvalue ($\lambda_1 = \lambda_2$) with a geometric multiplicity equal to 2.

$$A' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (7.3)$$

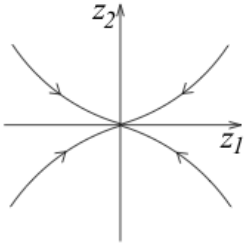
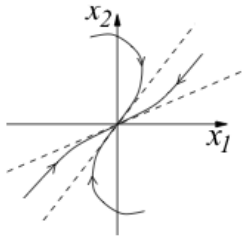
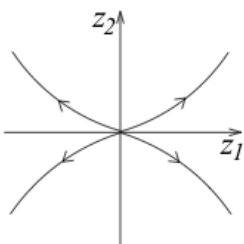
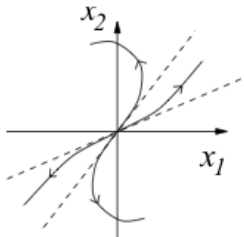
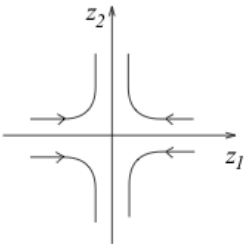
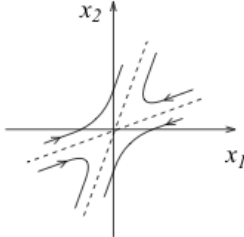
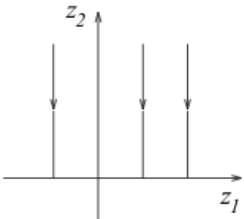
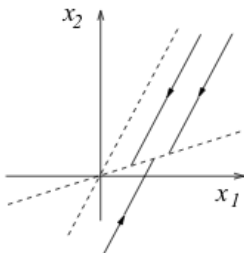
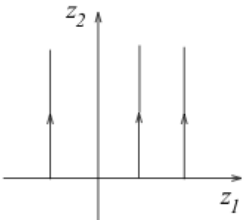
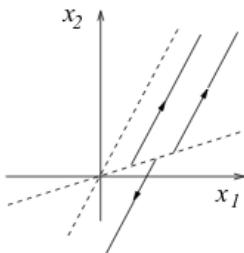
- Double real eigenvalue of geometric multiplicity equal to 1.

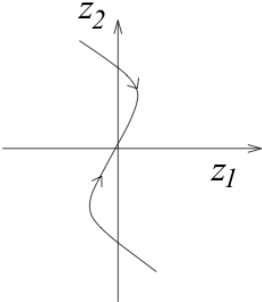
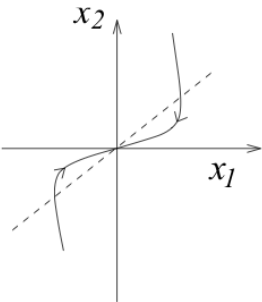
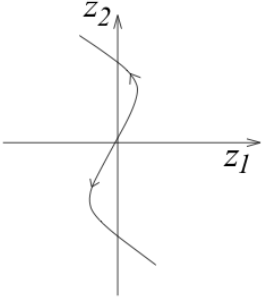
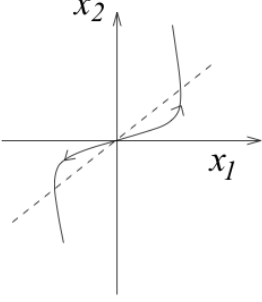
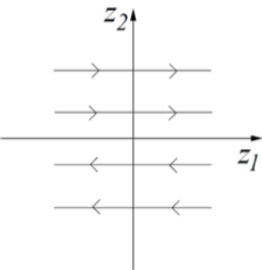
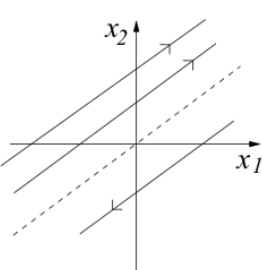
$$A' = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad (7.4)$$

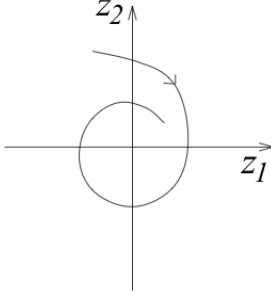
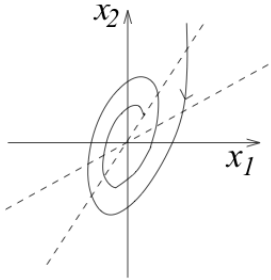
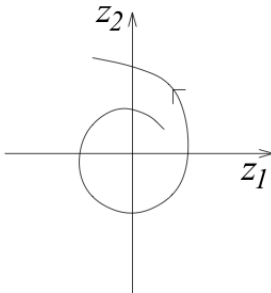
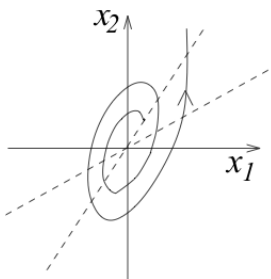
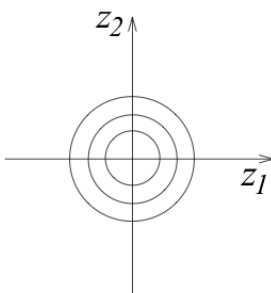
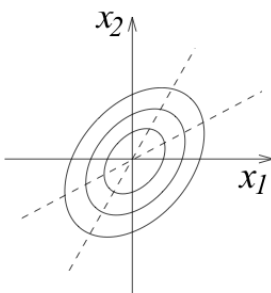
- Two complex conjugate eigenvalues $\alpha \pm \omega i$.

$$A' = \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix} \quad \omega > 0 \quad (7.5)$$

And the types of equilibrium for a linear planar system are:

Type of equilibrium	Behaviour of orbits (z_1, z_2)	Behaviour of orbits (x_1, x_2)	Conditions on the eigenvalues
Attractive node			$\lambda_2 \leq \lambda_1 < 0$
Repulsive node			$0 < \lambda_1 \leq \lambda_2$
Saddle point			$\lambda_1 < 0 < \lambda_2$
Non-isolated attractive equilibrium			$\lambda_1 = 0,$ $\lambda_2 < 0$
Non-isolated repulsive equilibrium			$\lambda_1 = 0,$ $\lambda_2 > 0$

Type of equilibrium	Behaviour of orbits (z_1, z_2)	Behaviour of orbits (x_1, x_2)	Conditions on the eigenvalues
Degenerate attractive node			$\lambda < 0$ (defective)
Degenerate repulsive node			$\lambda > 0$ (defective)
Non-isolated equilibrium			$\lambda = 0$ (defective)

Type of equilibrium	Behaviour of orbits (z_1, z_2)	Behaviour of orbits (x_1, x_2)	Conditions on the eigenvalues
Attractive focus			$\lambda_{1,2} = \alpha \pm \omega i$ $\alpha < 0, \omega \neq 0$
Repulsive focus			$\lambda_{1,2} = \alpha \pm \omega i$ $\alpha > 0, \omega \neq 0$
Centre			$\lambda_{1,2} = \pm \omega i$ $\omega \neq 0$