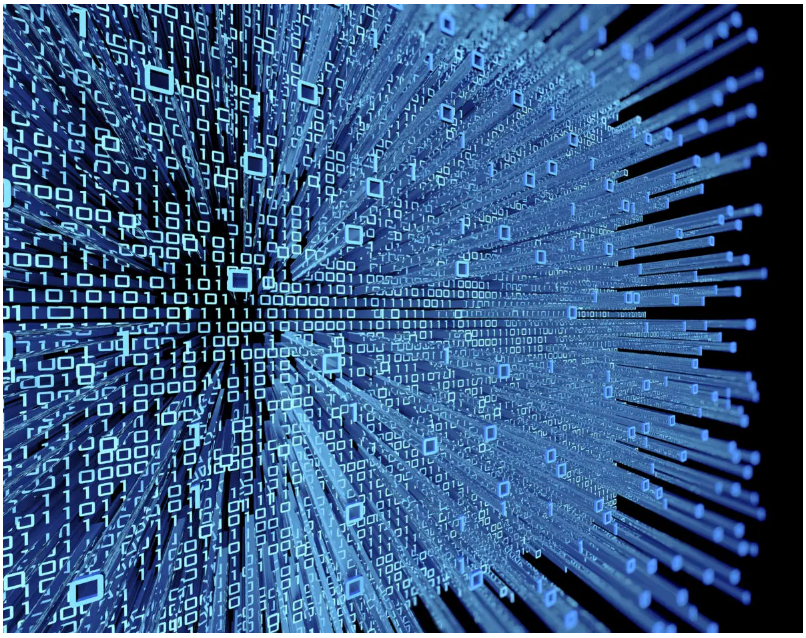

LINMA2491 Operational Research

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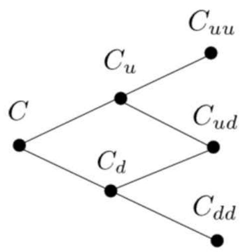
Contents

Definition and notation

- Given Ω, a sigma-algebra $\mathcal{A}$ is a set of subsets of Ω, with the elements called events, such that:
  - $\Omega \in \mathcal{A}$
  - if $A \in \mathcal{A}$ then also $\Omega - A \in \mathcal{A}$
  - if $A_i \in \mathcal{A}$ for $i = 1, 2, \ldots$ then also $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$
  - if $A_i \in \mathcal{A}$ for $i = 1, 2, \ldots$ then also $\cap_{i=1}^{\infty} A_i \in \mathcal{A}$
- Consider:



  - The state space is the set of all values of the system at each stage.

$$S_0 = \{C\}, \qquad S_1 = \{C_u, C_d\}, \qquad S_2 = \{C_{uu}, C_{ud}, C_{dd}\} \tag{1.1}$$

  - The sample space is the set of all possible combination of the system.

$$\Omega = S_0 \times S_1 \times S_2 = \{(C, C_u, C_{uu}), (C, C_u, C_{ud}), (C, C_u, C_{dd}), \ldots\} \tag{1.2}$$

- The power set of Ω is the set of all of the subsets, denoted $\mathcal{B}(\Omega)$.
- The probability space is the triplet $(\Omega, \mathcal{A}, P)$ where P is a probability measure.
  - $P(\emptyset) = 0$
  - $P(\Omega) = 1$
  - $P(\cup_{i=1}^{\infty} A_i) = \sum_i P(A_i)$ if A_i are disjoint
- $\forall t$, A_t is the set of events on which we have information at stage t. For example, $A_0 = \{C\}$, $A_1 = \{C, C_u, C_d\}$. Thus is it evident that $t_1 \leq t_2 \Rightarrow \mathcal{A}_{t_1} \subseteq \mathcal{A}_{t_2}$

- Consider the following problem with $x \in \mathbb{R}^n$ and domain $\mathcal{D}$:

$$\begin{aligned} &\min f_0(x), \qquad s.t. \\ &f_i(x) \leq 0, i = 1, \dots, m \\ &h_j(x) = 0, j = 1, \dots, p \end{aligned} \tag{1.3}$$

  Then the Lagrangian function is defined as $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$:

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x) \tag{1.4}$$

- The Lagrange dual function is defined as $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$:

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \tag{1.5}$$

- The Lagrange dual problem is a lower bound on the optimal value of the primal problem
- Lagrange relaxation of Stochastic Programs, consider the two problems:

$$\begin{aligned} &\min f_1(x) + \mathbb{E}_\omega[f_2(y(\omega), \omega)] \\ s.t \quad &h_{1i}(x) \leq 0, i = 1, \dots, m_1 \\ &h_{2i}(x, y(\omega), \omega) \leq 0, i = 1, \dots, m_2 \end{aligned} \qquad \begin{aligned} &\min f_1(x) + \mathbb{E}_\omega[f_2(y(\omega), \omega)] \\ s.t. \quad &h_{1i}(x) \leq 0, i = 1, \dots, m_1 \\ &h_{2i}(x(\omega), y(\omega), \omega) \leq 0, i = 1, \dots, m_2 \\ &\color{red}{x(\omega) = x} \end{aligned} \tag{1.6}$$

  The red constraint is the non-anticipativity constraint, it transforms the deterministic variable into a stochastic variable. A VERIFIER

- The dual of a stochastic program is:

$$\begin{aligned} & g(\nu) = g1(\nu) + \mathbb{E}_\omega(g2(\nu, \omega)) \\ where \quad & \\ & g_1(\nu) = \inf f_1(x) + \left(\sum_{\omega \in \Omega} \nu(\omega) \right)^T x \\ & s.t. \quad h_{1i}(x) \leq 0, i = 1, \dots, m_1 \\ and \quad & \\ & g_2(\nu, \omega) = \inf f_2(y(\omega)\omega) - \nu x(\omega) \\ & s.t. \quad h_{2i}(x(\omega), y(\omega), \omega) \leq 0, i = 1, \dots, m_2 \end{aligned} \tag{1.7}$$

- With p^* the solution of the primal problem and d^* the solution of the dual problem, we have:
  - Weak duality: $d^* \leq p^*$
  - Strong duality: $d^* = p^*$
- The KKT conditions are necessary and sufficient for optimality in convex optimization, there aren't unique. They are:

- Primal constraint: $f_i(x) \leq 0, i = 1, \ldots, m, h_j(x) = 0, j = 1, \ldots, p$
- Dual constraint: $\lambda \geq 0$
- Complemetarity slackness: $\lambda_i f_i(x) = 0, i = 1, \ldots, m$
- Gradient of the Lagrangian: $\nabla_x L(x, \lambda, \nu) = 0$

Modelling

2.1 Introduction

- For a certains sequence of events: $x \to \omega \to y(\omega)$, where ω is the uncertainty:
  - A first-stage decision is a decision that is made before the uncertainty is revealed (in x)
  - A second-stage decision is a decision that is made after the uncertainty is revealed (in $y(\omega)$)
- We can have a mathematic formulation like this:

$$
\begin{aligned}
&\min c^T x + \mathbb{E}[\min q(\omega)^T y(\omega)] \\
&Ax = b \\
&T(\omega)x + W(\omega)y(\omega) = h(\omega) \\
&x \geq 0, y(\omega) \geq 0
\end{aligned}
\tag{2.1}
$$

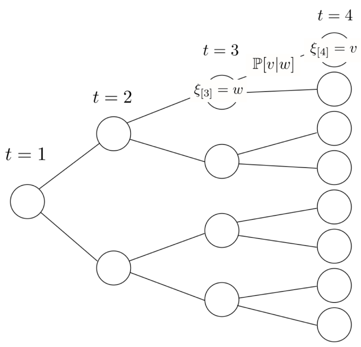
  - First-stage decision: $x \in \mathbb{R}^{n_1}$
  - First-stage parameter: $c \in \mathbb{R}^{n_1}$, $b \in \mathbb{R}^{m_1}$ and $A \in \mathbb{R}^{m_1 \times n_1}$
  - Second-stage decision: $y(\omega) \in \mathbb{R}^{n_2}$
  - Second-stage data: $q(\omega) \in \mathbb{R}^{n_2}$, $h(\omega) \in \mathbb{R}^{m_2}$ and $T(\omega) \in \mathbb{R}^{m_2 \times n_1}$, $W(\omega) \in \mathbb{R}^{m_2 \times n_2}$

2.2 Representations

2.2.1 Scenario Trees

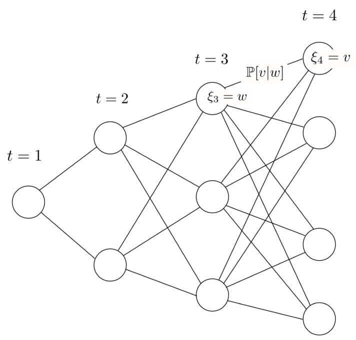
A scenario tree is a graphical representation of a Markov process $\{\xi_t\}_{t \in \mathbb{Z}}$, where the nodes are the histoires of realizations ($\xi_{[t]} = (\xi_1, \ldots, \xi_t)$), and the edges are the transitions from $\xi_{[t]}$ to $\xi_{[t+1]}$.

- We denote the root as $t = 1$;
- An ancestor of a node $\xi_{[t]}$, $A(\xi_{[t]})$ is a unique adjacent node which precedes ξ_t;
- The children of a node, $C(\xi_{[t]})$ are the nodes that are adjacent to $\xi_{[t]}$ and occur at stage $t + 1$.



2.2.2 Lattice

A lattice is a graphical representation of a Markov process $\{\xi_t\}_{t\in\mathbb{Z}}$, where the nodes are the realizations ξ_t and the edges correspond to the transitions from ξ_t to ξ_{t+1}.



2.2.3 Serial Independence

A process satisfies serial independence if, for every stage t, ξ_t has a probability distribution that does not depend on the history of the process. Thus, the probability measure is

$$\mathbb{P}[\xi_t(\omega) = i|\xi_{[t-1]}(\omega)] = p_t(i) \qquad \forall \xi_{[t-1]} \in \Xi_{[t-1]}, i \in \Xi_t \tag{2.2}$$

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2.3 Multi Stage Stochastic Linear Program

2.3.1 Notation

- Probability space: $(\Omega, 2^\Omega, \mathbb{P})$ with filtration $\{\mathcal{A}\}_{t\in\{1,\ldots,H\}}$
- $c_t(\omega) \in \mathbb{R}^{n_t}$: cost coefficients
- $h_t(\omega) \in \mathbb{R}^{m_t}$: right-hand side parameters
- $W_t(\omega) \in \mathbb{R}^{m_t\times n_t}$: coefficients of $x_t(\omega)$
- $T_{t-1}(\omega) \in \mathbb{R}^{m_t\times n_{t-1}}$: coefficients of $x_{t-1}(\omega)$
- $x_t(\omega)$: set of state and action variables in period t

- We implicitly enforce non-anticipativity by requiring that x_t and ξ_t are adapted to filtration $\{\mathcal{A}\}_{t\in\{1,\dots,H\}}$
- $\forall A \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}$, $x_t(\omega_1) = x_t(\omega_2)\ \forall \omega_1, \omega_2 \in A$

2.3.2 General formulation of the MSLP

The extended formulation of the MSLP is:

$$
\begin{aligned}
&\min c_1^T x_1 + \mathbb{E}[c_2(\omega)^T x_2(\omega) + \cdots + c_H(\omega)^T x_H(\omega)] \\
&s.t. \quad W_1 x_1 = h_1 \\
&T_1(\omega)x_1 + W_2(\omega)x_2(\omega) = h_2(\omega), \omega \in \Omega \\
&\vdots \\
&T_{t-1}(\omega)x_{t-1}(\omega) + W_t(\omega)x_t(\omega) = h_t(\omega), \omega \in \Omega \\
&\vdots \\
&T_{H-1}(\omega)x_{H-1}(\omega) + W_H(\omega)x_H(\omega) = h_H(\omega), \omega \in \Omega \\
&x_1 \geq 0, x_t(\omega) \geq 0, t = 2, \dots, H
\end{aligned}
\tag{2.3}
$$

We can now consider two specific instantiations of the MSLP: the scenario tree (MSLP-ST) and the lattice (MSLP-L). Using these notation:

- $\omega_t \in S_t$ (interpretation: index in the support Ξ_t of random input ξ_t)
- $\omega_{[t]} \in \S_1 \times \cdots \times S_T$ (interpretation: index in $\Xi_{[t]} = \Xi_1 \times \cdots \times \Xi_t$, which is the history of realizations, up to period t)

The scenario tree case can be formulated as:

$$
\begin{aligned}
&\min c_1^T x_1 + \mathbb{E}[c_2(\omega_{[2]})^T x_2(\omega_{[2]}) + \cdots + c_H(\omega_{[H]})^T x_H(\omega_{[H]})] \\
&s.t. \quad W_1 x_1 = h_1 \\
&T_1(\omega_{[2]})x_1 + W_2(\omega_{[2]})x_2(\omega_{[2]}) = h_2(\omega_{[2]}), \omega_{[2]} \in S_1 \times S_2 \\
&\vdots \\
&T_{t-1}(\omega_{[t]})x_{t-1}(\omega_{[t-1]}) + W_t(\omega_{[t]})x_t(\omega_{[t]}) = h_t(\omega_{[t]}), \omega_{[t]} \in S_1 \times \cdots \times S_t \\
&\vdots \\
&T_{H-1}(\omega_{[H]})x_{H-1}(\omega_{[H-1]}) + W_H(\omega_{[H]})x_H(\omega_{[H]}) = h_H(\omega_{[H]}), \omega_{[H]} \in S_1 \times \cdots \times S_H \\
&x_1 \geq 0, x_t(\omega_{[t]}) \geq 0, t = 2, \dots, H
\end{aligned}
\tag{2.4}
$$

The lattice case can be formulated as:

$$
\begin{aligned}
&\min c_1^T x_1 + \mathbb{E}[c_2(\omega_2)^T x_2(\omega_{[2]}) + \cdots + c_H(\omega_H)^T x_H(\omega_{[H]})] \\
&s.t. \quad W_1 x_1 = h_1 \\
&T_1(\omega_2) x_1 + W_2(\omega_2) x_2(\omega_{[2]}) = h_2(\omega_2), \omega_{[2]} \in S_1 \times S_2 \\
&\vdots \\
&T_{t-1}(\omega_t) x_{t-1}(\omega_{[t-1]}) + W_t(\omega_t) x_t(\omega_{[t]}) = h_t(\omega_t), \omega_{[t]} \in S_1 \times \cdots \times S_t \\
&\vdots \\
&T_{H-1}(\omega_H) x_{H-1}(\omega_{[H-1]}) + W_H(\omega_H) x_H(\omega_{[H]}) = h_H(\omega_H), \omega_{[H]} \in S_1 \times \cdots \times S_H \\
&x_1 \geq 0, x_t(\omega_{[t]}) \geq 0, t = 2, \ldots, H
\end{aligned}
\tag{2.5}
$$

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