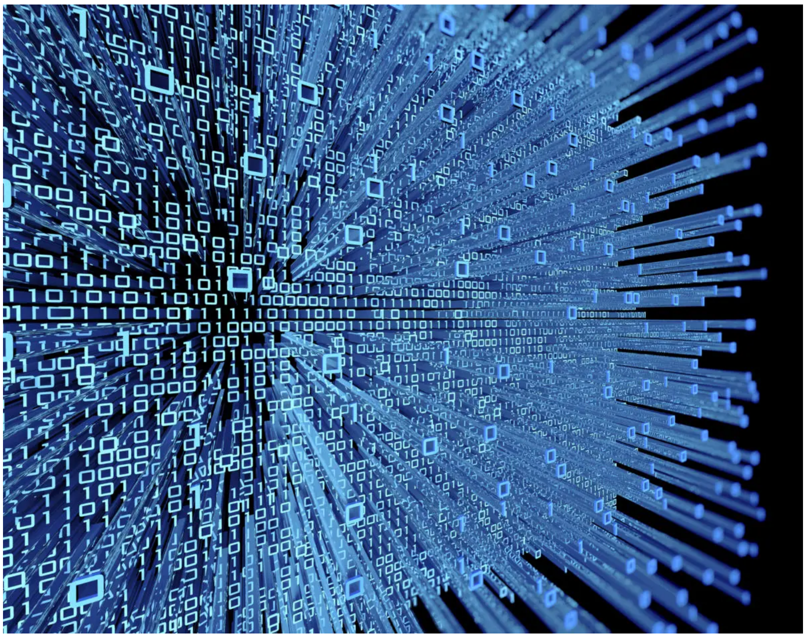

# LINMA2171 Numerical Analysis

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# Contents

# Polynomials

$\mathcal{P}_n$ is the set of all real polynomials of degree at most $n$.

- The Runge phenomenon is the explosion of the polynomial near the boundary of the domain when the interpolation points are chosen to be equidistant. A solution to that is to put more points near the boundary and less in the middle of the domain, e.g. Chebyshev points.

## 1.1 Lagrange interpolation

Let $x_0, \ldots, x_n$ be distinct real numbers. The Lagrange polynomial $L_k$ of degree $n$ is such that it is equal to 0 for all $x_i$, $i \neq k$ and 1 for $x_k$. This serves as a base for the next interpolations. The general formula for the Lagrange polynomial is

$$L_k(x) = \prod_{i=0 i\neq k}^{n} \frac{x - x_i}{x_k - x_i} \qquad k = 0, 1, \ldots, n \tag{1.1}$$

- N.B.: we usually denote $L_k(x; x_0, \ldots, x_n)$ or let $\chi = (x_0, \ldots, x_n)$ and $L_k(x; \chi)$.

## 1.2 Hermite interpolation

Let $x_0, \ldots, x_n$ be distinct real numbers. Then, given two sets of real numbers $(y_0, \ldots y_n)$ and $(z_0, \ldots, z_n)$, there is a unique polynomial $p_{2n+1} \in \mathcal{P}_{2n+1}$ such that

$$p_{2n+1}(x_i) = y_i \qquad p'_{2n+1}(x_i) = z_i \qquad i = 0, \ldots, n \tag{1.2}$$

The polynomial $p_{2n+1}$ is termed the Hermite interpolation polynomial of degree at most $2n+1$ for the data points $(x_0, y_0, z_0), \ldots, (x_n, y_n, z_n)$. The expression is

$$p_{2n+1}(x) = \sum_{k=0}^{n} \left(H_k(x) y_k + K_k(x) z_k\right) \qquad \begin{cases} H_k(x) = (L_k(x))^2 (1 - 2L'_k(x_k)(x - x_k)) \\ K_k(x) = (L_k(x))^2 (x - x_k) \end{cases} \tag{1.3}$$

where $L_k(x)$ is the Lagrange polynomial.

- The $H_k(x)$ are such that their derivative is zero for all $x_i$, and their value is zero for all $x_i$ except $x_k$, where it is 1.

$$H_k(x_i) = \delta_{ik} \qquad H'_k(x_i) = 0 \qquad \forall i$$

- The $K_k(x)$ are such that their derivative is zero for all $x_i$ except $x_k$ where it is one, and their value is zero for all $x_i$.

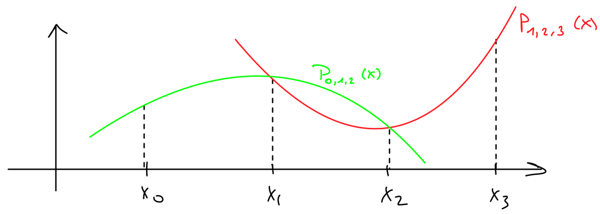
$$K_k(x_i) = 0 \qquad K'_k(x_i) = \delta_{ik} \qquad \forall i$$

## 1.3 Neville's algorithm

Let us assume we are given a set of support points $(x_i, y_i)$, $i = 0, 1, \ldots, n$, and $p_n$ is their Lagrange interpolation polynomial. Let us now define the notation $P_{i_0 i_1 \ldots i_k} \in \mathcal{P}_k$, the polynomial for which $P_{i_0 i_1 \ldots i_k}(x_{i_j}) = y_{i_j}$ for all $j = 0, 1, \ldots, k$. We work by recursion, with the following formula:

$$\begin{cases} P_i(x) = y_i \\ P_{i_0 i_1 \ldots i_k} = \frac{(x - x_{i_0}) P_{i_1 i_2 \ldots i_k}(x) - (x - x_{i_k}) P_{i_0 i_1 \ldots i_{k-1}}(x)}{x_{i_k} - x_{i_0}} \end{cases} \tag{1.4}$$

Example:
Let us have four points $(x_0, y_0), \ldots (x_3, y_3)$. We want the polynomial interpolating all of them, using Neville's algorithm.




Here,

$$P_{0123}(x) = \frac{x - x_0}{x_3 - x_0} \color{red}{P_{123}(x)} + \frac{x_3 - x}{x_3 - x_0} \color{green}{P_{012}(x)} \tag{1.5}$$

## 1.4 Newton's interpolation formula

Newton's interpolation formula is used to evaluate polynomials with a computer, as it only needs to compute each operation $(x - x_i)$ one time. We write it like:

$$p_n(x) = ((\ldots(y_{0\ldots n}(x - x_n) + y_{0\ldots n-1})(x - x_{n-1}) + y_{0\ldots n-2})(x - x_{n-2}) + \ldots) + y_0 \tag{1.6}$$

And the recursive formula is

$$P_{i_0 i_1 \ldots i_k} = P_{i_0 i_1 \ldots i_{k-1}}(x) + y_{i_0 i_1 \ldots i_k}(x - x_{i_0})(x - x_{i_1}) \ldots (x - x_{i_{k-1}}) \tag{1.7}$$

## 1.5 Linear algebra approach

Let $(\phi_0, \ldots, \phi_n)$ ba a basis of $\mathcal{P}_n$, which is known to be an $(n+1)$-dimensional linear space. The interpolation polynomial can thus be expressed in a unique way in the basis:

$$p_n(x) = \sum_{i=0}^{n} a_i \phi_i(x) \tag{1.8}$$

and the coefficient are obtained by solving the linear system

$$\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \ldots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \ldots & \phi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \ldots & \phi_n(x_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix} \tag{1.9}$$

This is called a Vandermonde matrix, and its determinant is

$$\det(V) = \prod_{0 \le i < j \le n} (x_j - x_i) \tag{1.10}$$

which is always non zero, as the $x_i$ are disinct, and the system has one unique solution.

→ N.B.: the condition number[1] of such a matrix grows exponentially with $n$.

## 1.6 Barycentric interpolation formula

This formula is interesting, because it is numerically stable, contrary to the linear algebra method described before. We use the following notation, called the nodal polynomial:

$$\pi_{n+1}(x) = \prod_{i=0}^{n} (x - x_i) \tag{1.11}$$

We now define

$$\lambda_j = \frac{1}{\prod_{k \neq j} (x_j - x_k)} \tag{1.12}$$

The modified Lagrange formula is then

$$p_n(x) = \pi_{n+1}(x) \sum_{j=0}^{n} \frac{\lambda_j}{x - x_j} y_i \tag{1.13}$$

For the polynomial $p_n(x) = 1$, we have the following expression:

$$1 = \pi_{n+1}(x) \sum_{j=0}^{n} \frac{\lambda_j}{x - x_j}$$

and thus we generally prefer to use the equivalent formula for equation (1.13):

$$p_n(x) = \sum_{j=0}^{n} \frac{\lambda_j y_j}{x - x_j} / \sum_{j=0}^{n} \frac{\lambda_j}{x - x_j} \tag{1.14}$$

for all $x \notin \{x_n, \ldots, x_n\}$.

## 1.7 Trigonometric interpolation

Let us consider the evenly spaced points $x_j = \frac{2\pi j}{N}$, $j = 0, \ldots, N$, on the interval $[0, 2\pi]$, and the interpolation values $f_0, \ldots, f_N \in \mathbb{C}$, with $f_0 = f_N$. The trigonometric interpolation problem consists of finding $\beta_k$ such that

$$p(x) = \sum_{k=0}^{N-1} \beta_k e^{ikx} \text{ such that } p(x_j) = f_j \qquad j = 0, \ldots, N-1 \tag{1.15}$$

[1]It is a measure of the reaction of the system to a small perturbation

→ N.B.: the bound is $N-1$ because the last condition $p(x_N) = f_N$ is satisfied when the others are (periodicity).

This is equivalent to the generalization to $\mathbb{C}$ of the polynomial interpolation problem: if we denote $\omega := e^{ix}$, the complex polynomial is

$$P(\omega) = \sum_{k=0}^{N-1} \beta_k \omega^k \tag{1.16}$$

The Vandermonde matrix in the complex case is defined as in the real case. We denote it $W$.

Theorem: $W^*W = NI$ for a complex Vandermonde matrix in an interpolation problem.

From this, the solution to the interpolation problem is solved by multiplying both sides by $W^*$. We get

$$\beta = \frac{1}{N} W^* f \Longrightarrow \beta_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-i2\pi kj/N} \qquad k = 0, \ldots, N-1 \tag{1.17}$$

And that is the discrete Fourier transform (DFT)

## 1.8 Rational interpolation

Let the interpolation points be $x_0 < x_1 < \cdots < x_\sigma$, with the values $y_0, \ldots, y_\sigma \in \mathbb{R}$. We define the polynomial

$$\Phi(x) = \frac{p_\mu(x)}{q_\nu(x)} \qquad p_\mu \in \mathcal{P}_\mu, q_\nu \in \mathcal{P}_\nu \tag{1.18}$$

$$\text{such that } \Phi(x_i) = y_i \qquad i = 0, \ldots, \sigma \tag{1.19}$$

The interpolation polynomial can be written

$$\Phi(x) = \frac{\sum_{k=0}^{\mu} a_k x^k}{\sum_{k=0}^{\nu} b_k x^k} = \frac{\lambda p_\mu(x)}{q\nu(x)} \tag{1.20}$$

The number of constraints, i.e. points needed for the interpolation is then $\sigma = \mu + \nu$. This implies that

If $\Phi$ is a solution to the equation (1.18), then $p_\mu, q_\nu$ are solutions of

$$p_\mu(x_i) - y_i q_\nu(x_i) = 0 \qquad i = 0, \ldots, \mu + \nu \tag{1.21}$$

$$\left( \sum_{k=0}^{\mu} a_k x_i^k \right) - y_i \left( \sum_{k=0}^{\nu} b_k x_i^k \right) = 0 \tag{1.22}$$

The theorem of existence states that the equation (1.21) always has a non trivial solution, i.e. $(p_\mu, q_\nu) \neq (0, 0)$.

The theorem of uniqueness states that if $\Phi_1$ and $\Phi_2$ are non trivial solutions of (1.21), then they are equivalent, i.e. they differ only by a common polynomial factor in the numerator and denominator.

- $p_\mu, q_\nu$ are relatively prime if they do not have zeros in common.

Given $\Phi = \frac{p_\mu}{q_\nu}$, let $\tilde{\Phi} = \frac{\tilde{p}_\mu}{\tilde{q}_\nu}$ be the equivalent expression for which $\tilde{p}_\mu$ and $\tilde{q}_\nu$ are relatively prime. $\Phi$ is the solution of (1.18) $\Longleftrightarrow$ $\tilde{p}_\mu(x_i) - y_i \tilde{q}_\nu(x_i) = 0, i = 0, \ldots, \mu + \nu$.

# Splines

## 2.1 Definition

Let $\mathcal{S} = \mathcal{S}(k) = \mathcal{S}(k; x_0, \ldots, x_m) = \{s \in \mathcal{C}^{k-1}[a,b] : s|_{[x_{i-1},x_i]} \in \mathcal{P}_k, i = 1, \ldots, m\}$ denote the linear space of splines of degree $k \geq 1$, with knots $a = x_0 < x_1 < \cdots < x_m = b$. The conditions at the knots are the following:

$$s^{(j)}(x_i^-) = s^{(j)}(x_i^+) \quad j = 0, \ldots, k-1 \tag{2.1}$$

$s^{(j)}$ denoting the $j$th derivative of the spline $s$. A basis of that set $\mathcal{S}$ is

$$\{x^0, \ldots, x^k, (x - x_1)_+^k, \ldots, (x - x_{m-1})_+^k\} \tag{2.2}$$

where $(x - x_j)_+^k = (\max\{0, x - x_j\})^k$.

**Theorem 2.1.** The dimension of the linear space $\mathcal{S}(k; x_0, \ldots, x_m)$ is $m + k$.

## 2.2 B-splines

The basis defined above is not well suited for computation, we will instead use the B-splines. These are functions $\phi$ such that

$$\phi(x) = 0 \qquad \forall x \in [x_0, x_p] \cup [x_q, x_m] \tag{2.3}$$

with $0 < p < q < m$ and $q - p$ as small as possible. We will use $\phi$ of the form

$$\phi(x) = \sum_{j=p}^{q} d_j (x - x_j)_+^k, \quad a \leq x \leq b \tag{2.4}$$

where the parameters $d_j$ satisfy

$$r_k(x) := \sum_{j=p}^{q} d_j (x - x_j)^k = 0, \quad x_q \leq x \leq b \tag{2.5}$$

Playing with arithmetics and algebra, we finally get the general formula for a B-spline:

$$B_p(x) = \sum_{j=p}^{p+k+1} \left( \prod_{l=pl\neq j}^{p+k+1} \frac{1}{x_\ell - x_j} \right) (x - x_j)_+^k, \qquad x \in \mathbb{R} \tag{2.6}$$

It belongs to $\mathcal{S}$ and verifies the condition (2.3). It is well-defined for $(p = 0, \ldots, m-k-1)$ and thus gives $m-k$ B-splines. To define a basis of $\mathcal{S}$, we need $2k$ more functions. We are going to add $k$ knots on the left of $x_0$ and on the right of $x_m$:

$$x_{-k} < x_{-k+1} < \ldots, x_{-1} < x_0 = a < x_1 < \cdots < x_m = b < x_{m+1} < \cdots < x_{m+k} \tag{2.7}$$

and we will now define $B_{-k}, \ldots, B_{m-1}$ on these dots. We now have $m+k$ linearly independent functions, and thus a basis of $\mathcal{S}$.

**Theorem 2.2.** Let $x_{-k}, \ldots x_{m+k}$ satisfy (2.7). Then, the $m+k$ functions $B_p$, $p = -k, \ldots, m-1$, given by (2.6) form a basis of the space $\mathcal{S}(k; x_0, x_m)$, with small support, meaning that $B_p$ is null outside the interval $(x_p, x_{p+k+1})$.

The recurrence formula for B-splines is the following, for $k > 1$:

$$\begin{cases} B_p^k(x) = \frac{(x-x_p)B_p^{k-1}(x) + (x_{p+k+1}-x)B_{p+1}^{k-1}(x)}{x_{p+k+1}-x_p} \\ B_p^0(x) = 1_{[x_p, x_{p+1})} \end{cases} \tag{2.8}$$

## 2.3 Regression with splines

Let $B_{-k}, \ldots, B_{m-1}$ be a basis of the linear space of splines $\mathcal{S}(k; u_0, \ldots, u_m)$. We have a function $f \in \mathcal{C}[a,b]$ and sampling points $w_0, \ldots, w_q$, assuming $q+1 \geq k+m$. The goal of this section is to find a spline function $s \in \mathcal{S}$ that is the closest to the data points $(w_i, f(w_i)), i = 0, \ldots, q$, i.e.

$$\arg\min_{s \in \mathcal{S}} \sum_{i=0}^{q} |f(w_i) - s(w_i)|^2 \tag{2.9}$$

Using $s = \sum_{j=-k}^{m-1} c_j B_j$, we must solve the system

$$\underbrace{\begin{bmatrix} B_{-k}(w_0) & \ldots & B_{m-1}(w_0) \\ \vdots & \ddots & \vdots \\ B_{-k}(w_q) & \ldots & B_{m-1}(w_q) \end{bmatrix}}_{=:A} \underbrace{\begin{bmatrix} c_{-k} \\ \vdots \\ c_{m-1} \end{bmatrix}}_{=:c} = \underbrace{\begin{bmatrix} f(w_0) \\ \vdots \\ f(w_q) \end{bmatrix}}_{=:F} \tag{2.10}$$

This is solved using the normal equations: $A^T A c = A^T F$.

**Theorem 2.3.** Under the above assumptions, the columns of $A$ are linearly independent iff there exists a subset of $m+k$ sampling times $w_{i_{-k}} < \cdots < w_{i_{m-1}}$ such that

$$u_p < w_{i_p} < u_{p+k+1} \quad p = -k, \ldots, m-1 \tag{2.11}$$

meaning that $w_{i_p}$ must be in the support of $B_p$.

## 2.4 Interpolation by natural cubic splines

Let us define the set of cubic splines $\mathcal{S}(k = 3; \xi_0, \dots, \xi_m)$. The set of natural cubic splines with those knots is the set

$$\mathcal{S}_N(k = 3; \xi_0, \dots, \xi_m) = \{s \in \mathcal{C}^2[\xi_0, \xi_m] : s|_{[\xi_{i-1},\xi_i]} \in \mathcal{P}_3,\ i = 1, \dots, m \text{ and } s''(\xi_0) = s''(\xi_m) = 0\} \tag{2.12}$$

For an arbitrary piece $[\xi_{i-1}, \xi_i]$, we have 4 conditions:

- $s|_{[\xi_{i-1},\xi_i]}(\xi_{i-1}) = s_{i-1}$
- $s|_{[\xi_{i-1},\xi_i]}(\xi_i) = s_i$
- $s''|_{[\xi_{i-1},\xi_i]}(\xi_{i-1}) = \sigma_{i-1}$
- $s''|_{[\xi_i,\xi_i]}(\xi_{i-1}) = \sigma_i$

And we thus write

$$s|_{[\xi_{i-1},\xi_i]} = s_{i-1}A(x) + s_iB(x) + \sigma_{i-1}C(x) + \sigma_iD(x) \tag{2.13}$$

where all functions are $\in \mathcal{P}_3$ and satisfy 4 conditions themselves:

| | | | |
|---|---|---|---|
| $A(\xi_{i-1}) = 1$ | $B(\xi_{i-1}) = 0$ | $C(\xi_{i-1}) = 0$ | $D(\xi_{i-1}) = 0$ |
| $A(\xi_i) = 0$ | $B(\xi_i) = 1$ | $C(\xi_i) = 0$ | $D(\xi_i) = 0$ |
| $A''(\xi_{i-1}) = 0$ | $B''(\xi_{i-1}) = 0$ | $C''(\xi_{i-1}) = 1$ | $D''(\xi_{i-1}) = 0$ |
| $A''(\xi_i) = 0$ | $B''(\xi_i) = 0$ | $C''(\xi_i) = 0$ | $D''(\xi_i) = 1$ |

Defining $h_i = \xi_i - \xi_{i-1}$, the final formula is

$$s(x) = \frac{(x - \xi_{i-1})s_i + (\xi_i - x)s_{i-1}}{h_i} \tag{2.14}$$

$$- \frac{1}{6}(x - \xi_{i-1})(\xi_i - x)\left[\left(1 + \frac{x - \xi_{i-1}}{h_i}\right)\sigma_i + \left(1 + \frac{\xi_i - x}{h_i}\right)\sigma_{i-1}\right] \qquad x \in [\xi_{i-1}, \xi_i] \tag{2.15}$$

Now, we have the additional conditions that $s'(\xi_j^-) = s'(\xi_j^+)$, $j = 1, \dots, m-1$, which we write in the following matrix form:

$$Q^T s = R\sigma \tag{2.16}$$

where the matrices are:

$$Q^T = \begin{pmatrix} h_1^{-1} & -h_1^{-1} - h_2^{-1} & h_2^{-1} & 0 & \dots & 0 \\ 0 & h_2^{-1} & -h_2^{-1} - h_3^{-1} & h_3^{-1} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & h_{m-1}^{-1} & -h_{m-1}^{-1} - h_m^{-1} & h_m^{-1} \end{pmatrix} \tag{2.17}$$

$$R = \begin{pmatrix} \frac{1}{3}(h_1 + h_2) & \frac{h_2}{6} & 0 & \dots & 0 \\ \frac{h_2}{6} & \frac{1}{3}(h_2 + h_3) & \frac{h_3}{6} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{h_{m-1}}{6} & \frac{1}{3}(h_{m-1} + h_m) \end{pmatrix} \tag{2.18}$$

**Theorem 2.4.** $s \in \mathcal{S}_N(k = 3; \xi_0, \ldots, \xi_m)$ iff $Q^T s = R\sigma$.

**Theorem 2.5.** Consider $\xi_0 \ldots, x_m$ distinct and $y_0, \ldots, y_m$. The interpolation at the knots

$$s \in \mathcal{S}_N(k = 3; \xi_0, \ldots, \xi_m) \text{ such that } s(\xi_i) = y_i \qquad i = 0, \ldots, m \tag{2.19}$$

exists and is unique.

**Theorem 2.6.** Let $s$ be a natural cubic spline. Then,

$$\int_{\xi_0}^{\xi_m} \left(s''(x)\right)^2 dx = s^T K s \qquad K = QR^{-1}Q^T \tag{2.20}$$

**Theorem 2.7.** Let $s$ be the function in $\mathcal{S}_N(k = 3; \xi_0, \ldots, \xi_m)$ such that $s(\xi_i) = y_i$, $i = 0, \ldots, m$. Let $v$ be any function in $H^2[a, b]$ that satisfiers the same interpolation conditions. Then

$$\int_{\xi_0}^{\xi_m} \left(v''(x)\right)^2 dx \geq \int_{\xi_0}^{\xi_m} \left(s''(x)\right)^2 dx \tag{2.21}$$

with equality iff $v = s$.

## 2.5 Smoothing splines

The problem studied in this section is

$$\arg \min_{s \in H^2[a,b]} F_\lambda(s) := \sum_{i=0}^{m} (y_i - s(x_i))^2 + \lambda \int_a^b (s''(x))^2 dx \tag{2.22}$$

where $a = x_0 < x_1 < \cdots < x_m = b$, $y_i \in \mathbb{R}$ are given and $\lambda > 0$ is a parameter. The first term is the data-attachment and the second is the roughness penalty.

**Theorem 2.8.** If $\hat{s}$ is a solution of (2.22), then $\hat{s} \in \mathcal{S}_N(k = 3; x_0, \ldots, x_m)$.

To find the solution of (2.22), we can rewrite the function to minimize:

$$F_\lambda(s) = (y - s)^T (y - s) + \lambda s^T K s \tag{2.23}$$

This function is strictly convex and quadratic and thus $s$ is the solution of the linear system

$$(I + \lambda K)s = y \tag{2.24}$$

Meaning that (2.22) has one and only one solution. The easiest way to compute $s$ is

$$s = y - \lambda Q \sigma \tag{2.25}$$

- When $\lambda \to 0$, we get a simple interpolation problem and there is an infinity of solutions.
- When $\lambda \to \infty$, we find the linear regression solution.

## 2.6 Interpolation by natural splines

We define the linear space of natural splines as follows:

$$\mathcal{S}_N(2\kappa+1) = \{s \in \mathcal{S}(2\kappa+1) : s^{(j)}(a) = s^{(j)}(b) = 0, j = \kappa+1, \ldots, 2\kappa\} \qquad \kappa \geq 1 \tag{2.26}$$

Provided that $m \geq \kappa$, the dimension of $\mathcal{S}_N(2\kappa+1)$ is $m+1$. Given a function $f \in \mathcal{C}[a,b]$, the set of interpolatory natural splines is

$$I_f\mathcal{S}_N(2\kappa+1) = \{s \in \mathcal{S}_N(2\kappa+1) : s(x_i) = f(x_i), i = 0, \ldots, m\} \tag{2.27}$$

**Theorem 2.9.** If $m \geq \kappa$, then $I_f\mathcal{S}_N(2\kappa+1)$ is a singleton, meaning that the interpolating natural spline exists and is unique.

**Theorem 2.10.** Let $m \geq \kappa$ and let $s$ ne the unique element of $I_f\mathcal{S}_N(2\kappa+1)$. Then, for all $v \in H^{\kappa+1}(a,b)$ that also interpolate $f$ at $x_0, \ldots, x_m$, it holds that

$$\|s^{(\kappa+1)}\|_2 \leq \|v^{(\kappa+1)}\|_2 \tag{2.28}$$

with equality iff $v = s$. In particular, the interpolating natural cubic spline, i.e. $\kappa = 1$, is the unique minimizer of the mean square acceleration under the interpolation conditions.

## 2.7 Error bounds of interpolation by natural splines

**Theorem 2.11.** Let $s$ be the natural cubic spline interpolant of $f \in \mathcal{C}^4[a,b]$, where the interpolation is at equally spaced knots. Then,

$$\|(f-s)^{(r)}\|_\infty \leq C_r \|f^{(4)}\|_\infty h^{4-r} \qquad r = 0,1,2,3 \tag{2.29}$$

with $C_0 = 5/384, C_1 = 1/24, c_2 = 3/8, C_3 = 1$ and $h$ the space between two knots. This means that the error between the interpolation and the function tends to 0 as the number of interpolation points goes to infinity.

## 2.8 Vector-valued splines

Vector-valued splines just work component-wise:

$$\mathbf{s}(x) = \sum_{j=-k}^{m-1} \mathbf{c}_j B_j(x) \tag{2.30}$$

where $x$ and $B_j(x)$ are real and scalar, but $\mathbf{s}$ and $\mathbf{c}_j$ are in $\mathbb{R}^n$ for all $j$.
We deifne the Bernstein polynomials of degree $n$ as

$$b_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i} \qquad i = 0, \ldots, n \tag{2.31}$$

They form a basis of $\mathcal{P}_n$ and a partition of unity, i.e. $\sum_{i=0}^n b_i^n(x) = 1$. We can thus write any polynomial piece $\mathbf{s}|_{[x_j, x_{j+1}]} \in \mathcal{P}_k$ in the Bernstein form:

$$\mathbf{s}|_{[x_j,x_{j+1}]} = \sum_{i=0}^{k} \mathbf{c}_i^j b_i^k \frac{x - x_j}{x_{j+1} - x_j} \tag{2.32}$$

aka the BÃl'zier curve on $[x_j, x_{j+1}]$ with control points $\mathbf{c}_0^j, \ldots, \mathbf{c}_k^j$.