The bridge survival game in *Squid Game*

E. Adrian Henle[1], Nick Gantzler[2], and Cory M. Simon[1*]

[1]School of Chemical, Biological, and Environmental Engineering. Oregon State University. Corvallis, OR. USA.
[2]Department of Physics. Oregon State University. Corvallis, OR. USA.

[*]Cory.Simon@oregonstate.edu

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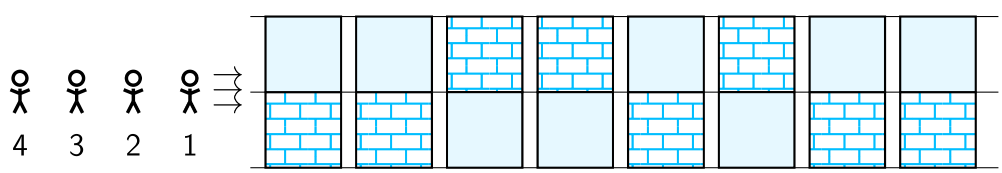
Abstract

In episode seven ("VIPs") of the television series *Squid Game*, the characters play a game of chance where they must sequentially traverse a bridge composed of glass panels—some sufficiently strong to hold the weight of a player and some not. Inspired by *Squid Game*, we propose, analyze, and simulate a stochastic process, "The Bridge Survival Game".

The Bridge Survival Game

In the Bridge Survival Game, N players line up to sequentially attempt to cross a bridge.

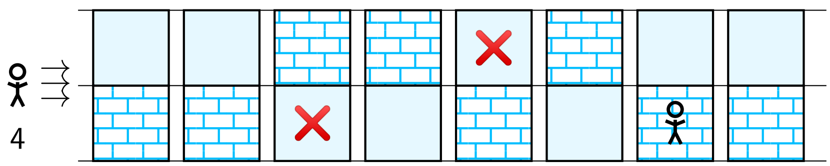
The bridge is comprised of a $2 \times L$ grid of glass panels. Each column contains (1) a tempered glass panel that can support the weight of a player and (2) a normal glass panel that shatters under the weight of a player. Under this constraint, the tempered glass panels are randomly distributed on the bridge. Owing to spacing between the columns, each player must traverse the bridge column-by-column, in L hops.



An initial condition for a Bridge Survival Game with $N = 4$ players attempting to traverse a bridge with $L = 8$ columns of glass panels (tempered: brick pattern, normal: solid).

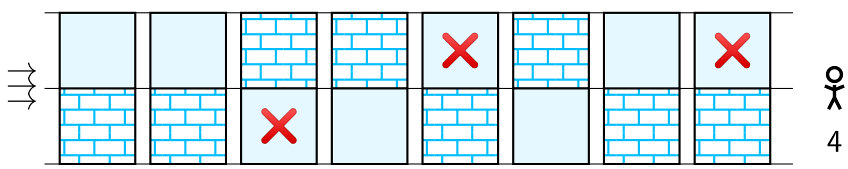
To the players, the glass panels are visually indistinguishable. Therefore, for each hop onto an unvisited (by any player) column, the active player (the player at the front, currently attempting bridge traversal) chooses a glass panel at random. If the player hops onto the tempered glass panel, he/she proceeds to hop onto the next column. On the other hand, if the player hops onto the normal glass panel, the glass shatters, he/she is eliminated, and the player behind then becomes the active player to attempt traversal of the bridge.

The players behind observe the outcomes of the players in front of them. Owing to perfect memory and survival instincts, if a player in front has successfully traversed a column by hopping onto a tempered glass panel, the remaining players will hop onto the same sturdy panel in their [attempted] traversal of the bridge.



Player 3 is traversing the bridge and, from observing the outcomes of the hops of the (eliminated) two players in front of him/her, has taken only two *risky* hops to arrive at his/her current position.

The game proceeds until (i) all players are eliminated or (ii) all columns of the bridge have been traversed and a subset of the players safely cross the bridge.



Continuing with the scenario above, in this outcome (end game), only player four successfully traversed the bridge.

❓ What is the probability that player $i \in \{1, 2, \ldots, N\}$ (where player 1 is the first to attempt bridge traversal) survives by successfully crossing the length-L bridge?

Analysis

💡 Regardless of its outcome, each hop onto an unvisited (by any player) column by an active player constitutes an observation uncovering which panel in that column is composed of sturdy, tempered glass. From the perspective of the group of N players (observers), this observation incurs a cost (elimination of a player) only if the active player chooses a normal glass panel.

Probability that exactly n players are eliminated

Let E_n be the event that *exactly* $n \in \{0, 1, 2, \ldots, N\}$ players are eliminated in the Bridge Survival Game. We wish to find the probability of this event, $P(E_n)$.

Case $N > L$

Suppose the number of players is greater than the bridge length ($N > L$).

At most, L players can be eliminated, by all of the first L players choosing a normal glass panel in their first hop. This outcome would elucidate the path for safe traversal over the bridge for the remaining $N - L$ players behind them. Therefore, $P(E_n) = 0$ for $n > L$.

Given $n \leq L$ players were eliminated, there are $\binom{L}{n}$ ways to distribute the n broken glass panels among the L columns of the bridge. Each *particular* distribution of broken panels occurs with probability $\left(\frac{1}{2}\right)^L$ because it corresponds to a distinct sequence of outcomes of L hops to previously unvisited (by all players) columns, each of which is an independent event with probability $\frac{1}{2}$. Note, these L risky hops ("observations") were taken by (i) n players, if a panel was broken in the last column L, or (ii) $n+1$ players, if a panel in the last column L was not broken. The outcomes corresponding to the set of distributions of n broken panels on the bridge are mutually exclusive. Therefore:

$$P(E_n) = \begin{cases} \binom{L}{n}\left(\frac{1}{2}\right)^L & 0 \leq n \leq L \\ 0 & L < n \leq N. \end{cases} \tag{1}$$

Another way to arrive at eqn. 1: of 2^L equally likely outcomes, $\binom{L}{n}$ of them result in n eliminated players.

Two sanity checks on eqn. 1: (1) The probability that zero players are eliminated is $P(E_0) = \left(\frac{1}{2}\right)^L$, since then the first player must choose the tempered glass panel in each column for each of their L hops to cross the bridge; by similar reasoning, $P(E_L) = \left(\frac{1}{2}\right)^L$. (2) Since the events $\{E_0, E_1, ..., E_N\}$ are mutually exclusive and their union comprises the sample space, we must have $\sum_{n=0}^{N} P(E_n) = 1$, which holds since $\sum_{n=0}^{L} \binom{L}{n} = 2^L$ via the binomial theorem.

Case $N \leq L$

Suppose the number of players is less than or equal to the bridge length ($N \leq L$). Unlike the case above, no player is certain to survive.

Given $n < N$ players were eliminated, the argument above and eqn. 1 for $P(E_n)$ holds, since all L columns of the bridge must have been visited. However, the case $n = N$ is special because, then, all L columns of the bridge were not necessarily visited if $N < L$. For $n = N$, we sum over the mutually exclusive events that player N was eliminated at column $c \in \{N, N+1, ..., L\}$ of the bridge. For each column c at which player N was eliminated, there are $\binom{c-1}{N-1}$ ways to distribute the broken glass panels of the remaining $N-1$ players onto the previous $c-1$ columns on the bridge; and, each distribution corresponds to a distinct sequence of particular outcomes of c independent risky hops to unvisited (by any player) columns. Therefore,

$$P(E_n) = \begin{cases} \binom{L}{n} \left(\frac{1}{2}\right)^L & 0 \leq n < N \\ \sum_{c=N}^{L} \binom{c-1}{N-1} \left(\frac{1}{2}\right)^c & n = N. \end{cases} \tag{2}$$

As a sanity check on eqn. 2, we confirmed $\sum_{n=0}^{N} P(E_n) = 1$ numerically.

How to prove this? $P(E_N)$ appears related to the Hockey-stick identity, except for the $(1/2)^c$ power. Maybe a generating function could help?

Probability that player i survives

Let S_i be the event that player i survives. This event is the union of the events that $n \in \{0, 1, ..., i-1\}$ players are eliminated:

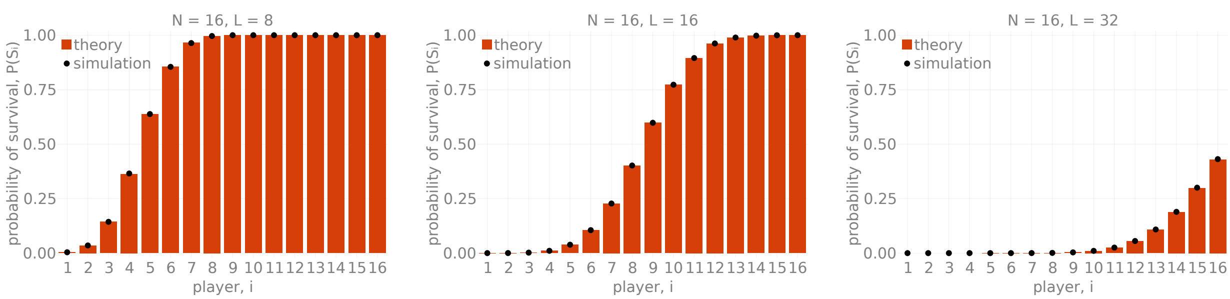
$$S_i = \cup_{n=0}^{i-1} E_n. \tag{3}$$

Since the events $\{E_0, E_1, ..., E_N\}$ are mutually exclusive:

$$P(S_i) = \sum_{n=0}^{i-1} P(E_n) = \begin{cases} \sum_{n=0}^{i-1} \binom{L}{n} \left(\frac{1}{2}\right)^L & 1 \leq i \leq \min(L, N) \\ 1 & i > L, N > L. \end{cases} \tag{4}$$

Eqn. 4 holds for both $N \geq L$ and $N < L$, and $P(E_N)$ is not involved.

Fig. 1 shows $P(S_i)$ under three different scenarios: (a) $N > L$, (b) $N = L$, and (c) $N < L$. The players at the front of the line to cross the bridge have the lowest probability of survival.



(a) case $N > L$ (b) case $N = L$ (c) case $N < L$

Figure 1: We visualize $P(S_i)$, the probability that player i, of $N = 16$ total players, survives the bridge game for a length $L \in \{8, 16, 32\}$ bridge, according to (i) theory via eqn. 4 and (ii) Monte Carlo simulation.

Expected number of players that survive

Let $\theta \in \{0, 1, ..., N\}$ be the random variable denoting the number of players that successfully traverse the bridge. The expected value of θ is:

$$\mathbb{E}[\theta] = \sum_{n=0}^{N} (N-n) P(E_n), \tag{5}$$

with $P(E_n)$ given in eqn. 2 since (i) the [union of the] mutually exclusive events $\{E_0, E_1, ..., E_N\}$ comprise the sample space and (ii) exactly n players eliminated implies $N - n$ players survived. Fig. 2 shows $\mathbb{E}[\theta]$ as a function of the bridge length, L, for $N = 16$ total players. As the bridge lengthens, fewer players are expected to survive.

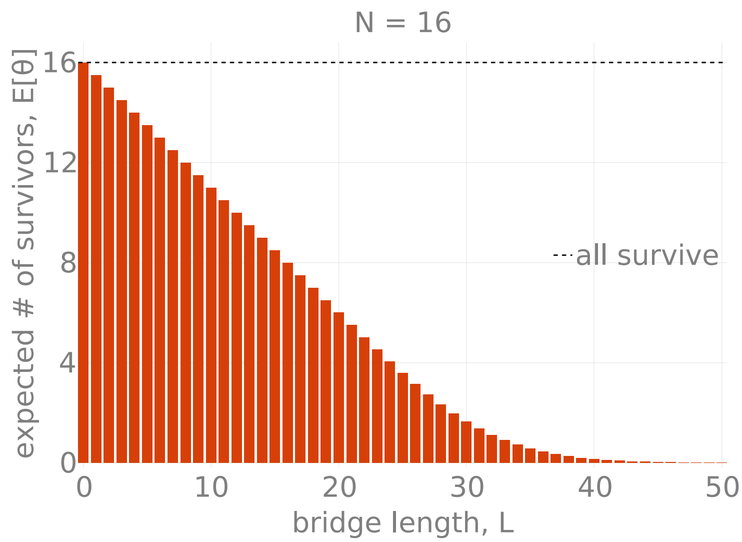


Figure 2: We visualize the expected number of players that survive, $\mathbb{E}[\theta]$ in eqn. 5, as a function of bridge length L, for $N = 16$ total players.

Monte Carlo simulation

We write Julia code to run a Monte Carlo simulation of the Bridge Survival Problem to estimate $P(E_n)$ and compare to our analytical solution.

Data structure for the bridge

First, we define a data structure for the bridge, which stores (i) its length L, (ii) which row of each column contains the safe, tempered glass panel, and (iii) which columns have been visited by a player.

```
struct Bridge
    L::Int
    safe_panels::Array{Int, 1}
    observed::Array{Bool, 1}
end
```

Second, we write a constructor of a bridge that (i) randomly chooses an arrangement of the tempered glass panels and (ii) initializes each column as unobserved.

```
function Bridge(L::Int)
    safe_panels = [sample(1:2) for col in 1:L]
```

```
    observed = [false for col in 1:L]
    return Bridge(L, safe_panels, observed)
end
```

Simulating hopping and bridge traversal

Third, we write a function that simulates a player hopping onto column c of the bridge, marks the column as observed, and returns true if the player survived and false otherwise.

```
function hop!(bridge::Bridge, c::Int)
    if bridge.observed[c]
        return true
    else
        panel_to_hop_on = sample(1:2)
        bridge.observed[c] = true
        if bridge.safe_panels[c] == panel_to_hop_on
            return true
        else
            return false
        end
    end
end
```

Next, we write a function to simulate bridge traversal by an active player. The function returns true if the player survives and false otherwise.

```
function traverse!(bridge::Bridge)
    for c in 1:bridge.L
        survived = hop!(bridge, c)
        if ! survived
            return false
        end
    end
    return true
end
```

Simulating bridge traversal by all players

Finally, we write a function to simulate sequential bridge traversal by all N players, which returns the number of players eliminated.

```
function simulate(L::Int, N::Int)
    bridge = Bridge(L)
    for p in 1:N
        survived = traverse!(bridge)
```

```
        if survived
            return p - 1
        end
    end
    return N
end
```

By running the Monte Carlo simulation 100000 times, we estimate $P(S_i)$ as the fraction of the simulations that player i survived, shown as the points in Fig. 1.