do all of these questions by hand, i.e. without a computer. of course, check your answers with Julia.

matrix times vector

1. compute $A\mathbf{x}$ by dot products of the rows with the column vector. repeat, but now as a combination of the columns of A.

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

(the two methods should give consistent results.)

2. multiply A times $\mathbf{x}$ to find the three components of $A\mathbf{x}$, for the two systems below.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3. (a) what 2 by 2 matrix R rotates every vector by 90°?

$$R \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

(sketch out the two vectors to see this transformation does this rotation.)

(b) what 2 by 2 matrix R^2 rotates every vector by 180°?

4. what 3 by 3 matrix E multiplies $[x, y, z]$ to give $[x, y, z + x]$? what matrix E^{-1} multiplies $[x, y, z]$ to give $[x, y, z - x]$?

5. what 2 by 2 matrix P_1 projects the vector $[x, y]$ onto the x-axis to produce $[x, 0]$?

6. what 2 by 2 matrix R rotates every vector through 45° clockwise (without stretching or compressing the vector)?

$$R \begin{bmatrix} 1 \\ 0 \end{bmatrix} = ?$$

$$R \begin{bmatrix} 0 \\ 1 \end{bmatrix} = ?$$

with knowledge of how R acts on these two vectors, we can directly write R.

7. draw the row and column pictures for the equations $x - 2y = 0$ and $x + y = 6$.

8. a 9 by 9 Sudoku matrix S has the numbers 1, ..., 9 in every row and every column (in different order). for the all ones vector $\mathbf{x} = [1, ..., 1]$, what is $S\mathbf{x}$?

elimination [of unknowns in a linear system of equations]

9. apply elimination and back substitution to solve:

$$\begin{aligned} 2x - 3y &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

list the three row operations in the form "subtract ____times row ____from row ____".

10. what number d forces a row exchange, and what is the triangular system (not singular) for that d? which d makes this system singular?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

11. what number q makes this system singular and which right side t gives it infinitely many solutions?

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

12. what three matrices E_1, E_2, E_3 put A into upper-triangular form U i.e. $U = E_3E_2E_1A$?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

13. include $\mathbf{b} = [1, 0, 0]$ as a fourth column in the previous problem to produce $[A \quad \mathbf{b}]$. carry out the elimination steps on this augmented matrix to solve $A\mathbf{x} = \mathbf{b}$.

14. apply elimination to the 3 by 4 augmented matrix $[A \quad \mathbf{b}]$. how do you know this system has no solution? change the last number 6 so there *is* a solution.

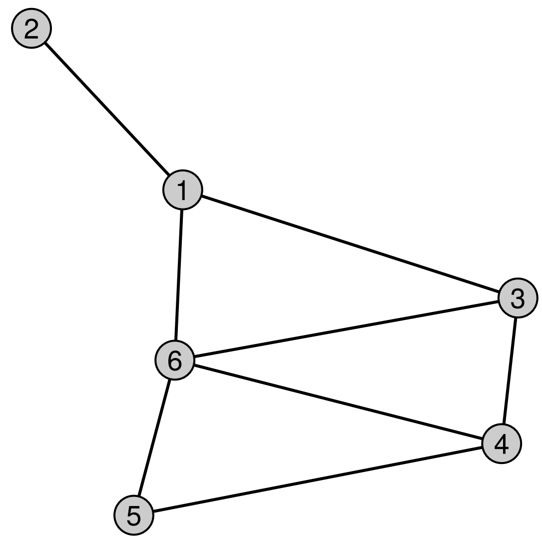
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

matrix times matrix

15. multiply AB using columns times rows:

$$\begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

16. consider the undirected graph below. write its adjacency matrix S. compute S^4 via Julia. using S^4, determine how many walks of length 4 exist from node 2 to node 4. sketch all of these walks.



matrix inverse

17. compute A^{-1} by setting up two systems of equations and solving both.

$$A = \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix}$$

18. compute A^{-1} by Gauss-Jordan elimination on $[A \quad \mathbf{b}]$.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

$A = LU$ factorization

19. what matrix E puts A into upper-triangular form $U = EA$? multiply by $E^{-1} = L$ to factor A into LU.

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

20. what matrix $E = E_3E_2E_1$ puts A into upper-triangular form $U = EA$? multiply by $E^{-1} = L$ to factor A into LU.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

21. (spend the most time understanding this one...) solve the following system $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ via (1) factoring $A = LU$ then (2) solving (a) the lower triangular system $L\mathbf{c} = \mathbf{b}$ followed by the upper triangular system $U\mathbf{x} = \mathbf{b}$. this is how computers solve linear systems of equations!

$$A = \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$$

(*source* for linear algebra problems: "Introduction to Linear Algebra" by Strang, 5th Ed., 2016)