do all of these questions by hand, i.e. without a computer. of course, check your answers with Julia.

transposes and permutations

1. with

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

   (a) compute $\mathbf{x}^\intercal A \mathbf{y}$.

   (b) this is the row $\mathbf{x}^\intercal A =?$ times the column $\mathbf{y}$.

   (c) this is the row $\mathbf{x}^\intercal$ times the column $A\mathbf{y} =?$

2. what permutation matrix P makes the matrix A upper triangular via PA?

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

3. factor the symmetric matrix into $S = LDL^\intercal$ with D the diagonal pivot matrix and L the lower-triangular matrix.

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

4. find the $PA = LU$ factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

5. which of the following are subspaces of $\mathbb{R}^3$?

   (a) the plane of vectors $[b_1, b_2, b_3]$ with $b_1 = b_2$

   (b) the plane of vectors with $b_1 = 1$

   (c) the vectors with $b_1 b_2 b_3 = 0$

   (d) all vectors that satisfy $b_1 + b_2 + b_3 = 0$

   (e) all vectors with $b_1 \leq b_2 \leq b_3$

6. for which right-hand sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

null space of a matrix

7. construct a matrix whose null space is all combinations of the two vectors

$$\begin{bmatrix}2\\2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\1\\0\\1\end{bmatrix}$$

8. reduce the matrices A, B, and C to their row-reduced echelon forms R. what is the null space of each matrix? we are looking for all solutions to e.g. $A\mathbf{x} = \mathbf{0}$.

$$A = \begin{bmatrix}1 & 2 & 2 & 4 & 6\\1 & 2 & 3 & 6 & 9\\0 & 0 & 1 & 2 & 3\end{bmatrix}, \quad B = \begin{bmatrix}2 & 4 & 2\\0 & 4 & 4\\0 & 8 & 8\end{bmatrix}, \quad C = \begin{bmatrix}-1 & 3 & 5\\-2 & 6 & 10\end{bmatrix}$$

9. fill out the matrix so it has rank 1.

$$B = \begin{bmatrix} & 9 & \\ 1 & & \\ 2 & 6 & -3\end{bmatrix}$$

10. choose vectors $\mathbf{u}$ and $\mathbf{v}$ such that rank one matrix $A = \mathbf{u}\mathbf{v}^\intercal$. this is the natural form for every matrix that is rank one.

$$A = \begin{bmatrix}2 & 2 & 6 & 4\\-1 & -1 & -3 & -2\end{bmatrix}$$

11. set up a system of equations and find the null space to solve for the proper coefficients in the chemical equation:

$C_2H_5OH + O_2 \longrightarrow CO_2 + H_2O$

this is a nice example of where a null space comes into play and why there are infinite solutions (some of which are physically plausible). see A. Downey's textbook "Think linear algebra for the answer, here `https://allendowney.github.io/ThinkLinearAlgebra/nullspace.html#stoichiometry`.

the complete/general solution to $A\mathbf{x} = \mathbf{b}$

12. find the complete/general solution (i.e., all solutions) to the system:

$$\begin{bmatrix}1 & 3 & 1 & 2\\2 & 6 & 4 & 8\\0 & 0 & 2 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}1\\3\\1\end{bmatrix}$$

13. under what condition on b_1, b_2, b_3 is this system solvable? tag on $\mathbf{b}$ as the fourth column to the matrix and do elimination to find out.

$$\begin{aligned} x + 2y - 2z &= b_1\\ 2x + 5y - 4z &= b_2\\ 4x + 9y - 8z &= b_3 \end{aligned}$$

14. construct a 2 by 3 system of equations with a particular solution $[2, 4, 0]$ and a null space consisting of multiples of $[1, 1, 1]$.

15. reduce to an upper-triangular form $U\mathbf{x} = \mathbf{c}$ (Gaussian elimination) then further to row reduced echelon form $R\mathbf{x} = \mathbf{d}$ (Gauss-Jordan). write all solutions to the system using $R\mathbf{x} = \mathbf{d}$.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$$

independence, basis, dimension

16. decide the dependence or independence of the following vectors, by looking for ways to combine them to arrive at the zero vector.
    (a) $[1, 3, 2]$, $[2, 1, 3]$, $[3, 2, 1]$
    (b) $[1, -3, 2]$, $[2, 1, -3]$, $[-3, 2, 1]$
    (c) $[1, 0, 0]$, $[1, 1, 0]$, $[1, 1, 1]$
    (d) $[2, 0, 0, 4]$, $[3, 6, 0, 6]$, $[4, 7, 0, 8]$, $[1, 0, 9, 2]$

17. precisely describe the row space and column space of U.

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

18. find basis vectors for the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$. then find a basis for the intersection of that plane with the xy-plane. find a basis for all vectors perpendicular to the plane, too.

19. find a basis for all 2 by 3 matrices whose columns add to zero. what is the dimension of this abstract space?

20. find a basis for all 2 by 3 matrices whose null space contains $[2, 1, 1]$. what is the dimension of this abstract space?

the four fundamental subspaces of a matrix

21. find a basis for each of the four fundamental subspaces of the following matrices. note, the CR factorization of D is provided for you (no need to multiply the matrices...). what are the dimensions of the subspaces? (check they satisfy the fundamental theorem of linear algebra.)

$$A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

22. without multiplying matrices, find bases for the row and column space of A. is the square matrix A invertible?

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

(*source* for linear algebra problems: "Introduction to Linear Algebra" by Strang, 5th Ed., 2016)