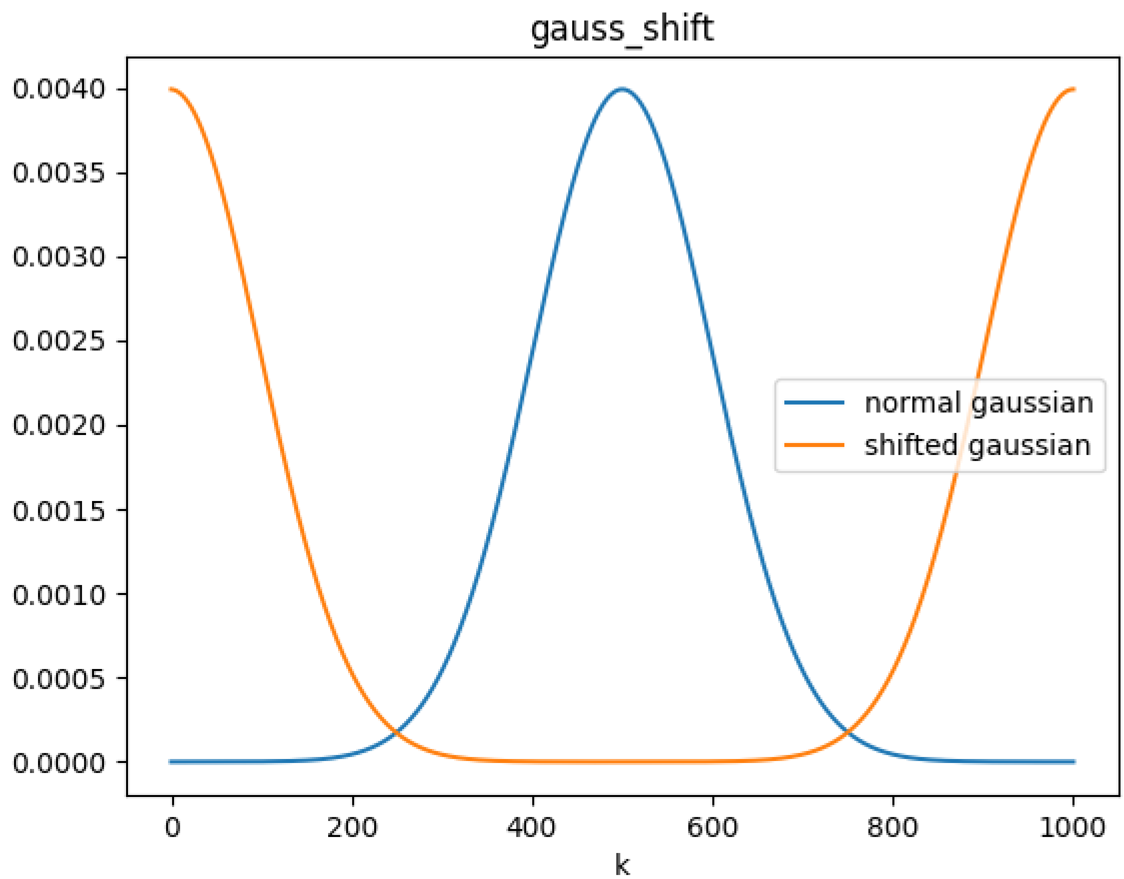
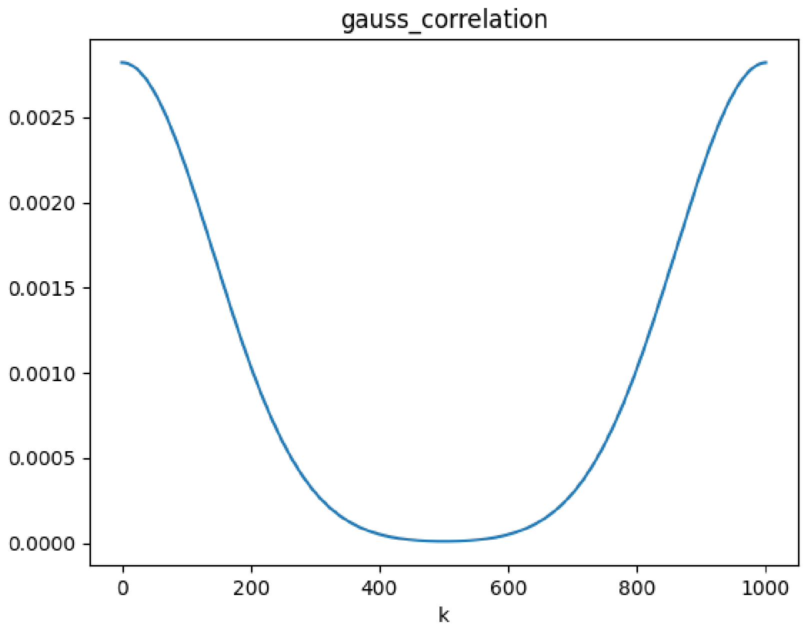
PHYS512 Assignment 5
Simon Harms
260841508

I decided to write up q5 and q6 on paper because writing lots of equations digitally is annoying

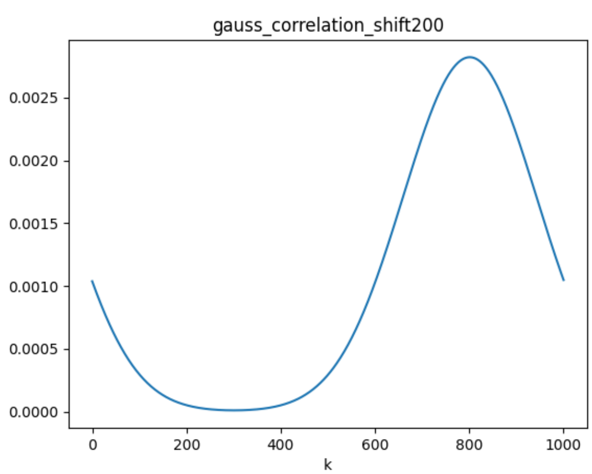
1)

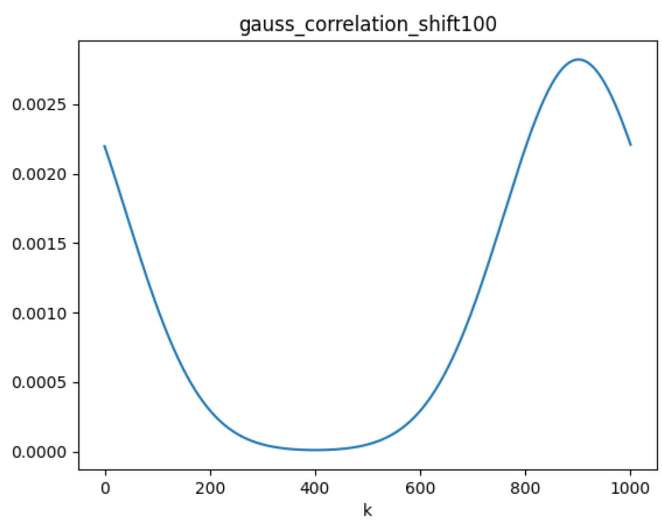


2)



3) Looking at the graph of the 2nd gaussian shifted by 100 and 200 the correlation function is shifted by the same amount as the second gaussian, this isn't surprising because we effectively have ift(fft(f) + conj(fft(f) + fft(delta(n)))) = ift(fft(f) + conj(fft(f)) + fft(delt(n))) = ift(fft(f ★ f) + fft(delta(n))) which is just the correlation function shifted by n.





4) Wraparound only becomes an issue when f and g are different lengths because then their fourier transforms will have different periods so adding zeros to whichever one is smaller to make them the same length should avoid this problem. The output array is then the length of whichever input array is longer

5. a) $\sum_{x=0}^{N-1} (\exp(-2\pi i k/N))^x$

partial sum of a geometric series:

$$\sum_{x=0}^{n} \alpha^x = \frac{1-\alpha^{n+1}}{1-\alpha}$$

$$\Rightarrow \sum_{x=0}^{N-1} (\exp(-2\pi i k/N))^x = \frac{1-\exp(-2\pi i k/N)^{N-1+1}}{1-\exp(-2\pi i k/N)}$$

$$= \frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)}$$

b) $\lim_{k\to 0} \frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)} = \frac{0}{0}$ so we can we l'hopital's rule

$$= \lim_{k\to 0} \frac{2\pi i k \exp(-2\pi i k)}{2\pi i k/N \exp(-2\pi i k/N)}$$

$$= N \lim_{k\to 0} \frac{\exp(-2\pi i k)}{\exp(-2\pi i k/N)} = N \frac{1}{1} = N$$

for $k \in \mathbb{Z}$:

$$\frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)} = \frac{1-1}{1-\exp(-2\pi i k/N)} = \frac{0}{1-\exp(-2\pi i k/N)}$$

$= 0$ unless $1-\exp(-2\pi i k/N)$ which is only true if k is a multiple of N

5. c) $F[\sin(\tfrac{2\pi k'}{N}x)](k) = \sum_x^{N-1} \sin(\tfrac{2\pi k'}{N}x)\exp(-2\pi i k x/N)$

$$= \sum_x^{N-1} \frac{e^{i\frac{2\pi k'}{N}x} - e^{-i\frac{2\pi k'}{N}x}}{2i}\exp(-2\pi i k x/N)$$

$$= \frac{1}{2i}\left(\sum_x^{N-1}\exp(-2\pi i(k-k')x/N) - \sum_x^{N-1}\exp(-2\pi i(k+k')x/N)\right)$$

$$= \frac{1}{2i}\left(\frac{1-\exp(2\pi i(k'-k))}{1-\exp(2\pi i\frac{(k'-k)}{N})} - \frac{1-\exp(2\pi i(-k'-k))}{1-\exp(2\pi i\frac{(k'-k)}{N})}\right)$$

This approximation is accurate looking at the residuals. We're pretty close to a delta function.

d) Windowed the dft is closer to a delta function

e) $F[0.5 - 0.5\cos(2\pi x/N)]$

$$= 0.5\,\delta(k) - 0.5\,(0.5\,(\delta(k-1) + \delta(k+1)))$$

$$= 0.5\,\delta(k) - 0.25\,\delta(k-1) - 0.25\,\delta(k-N)$$

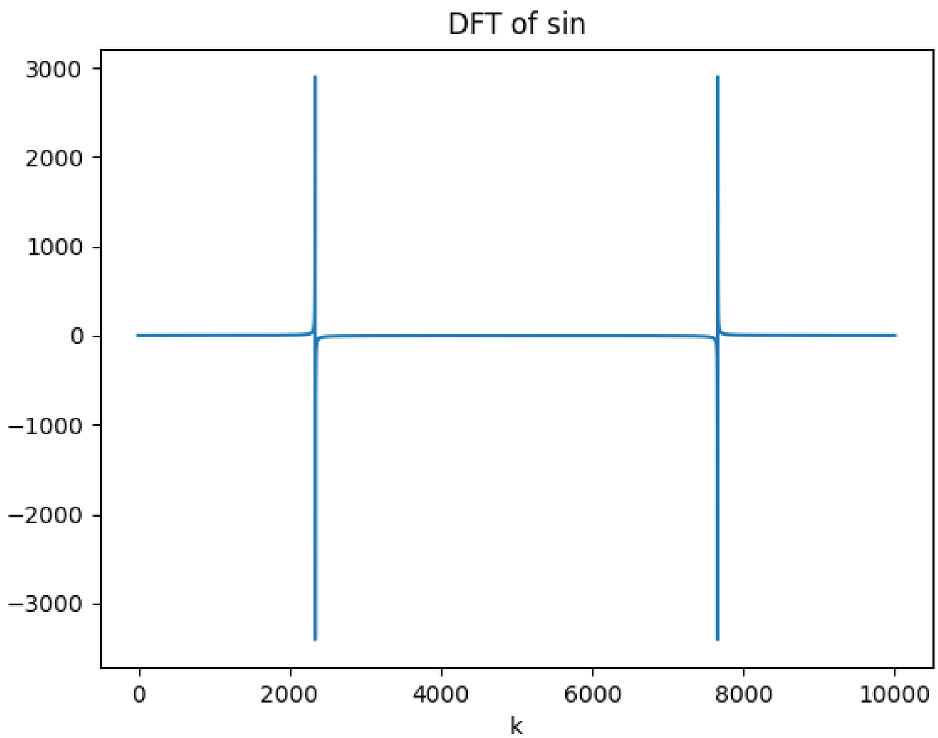
dft is cyclical

$$F[f(x)\,\text{window}(x)](k) = \sum_x F[f](y-x)\,F[w](x)$$

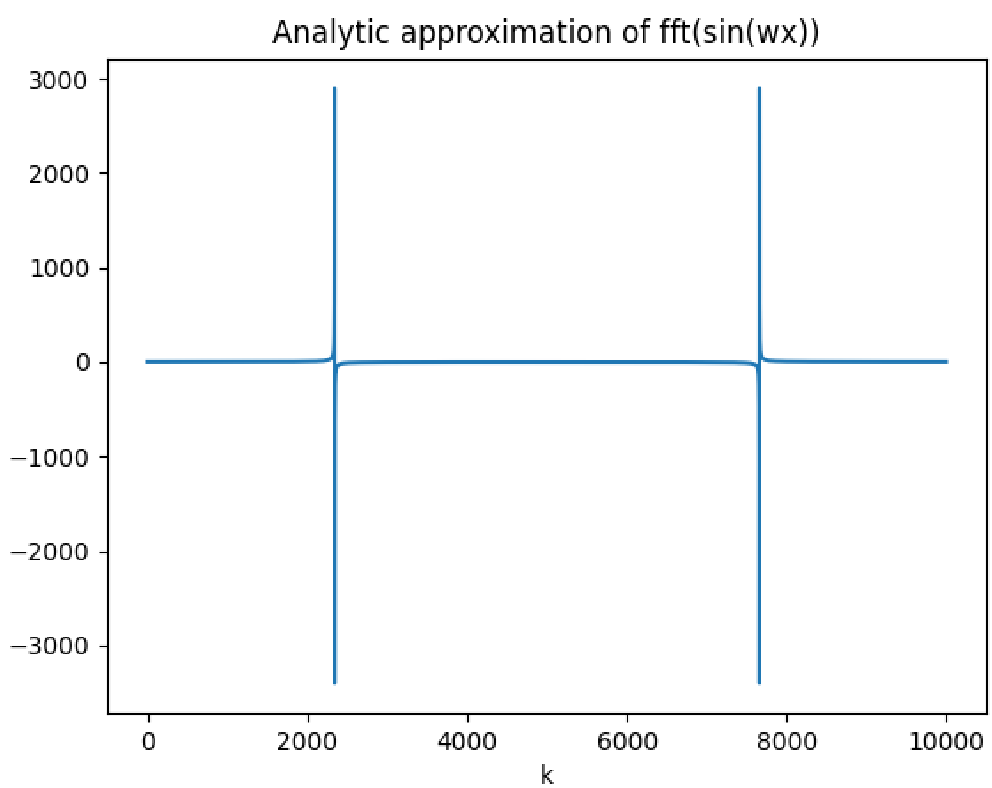
$$= F[f](y)\,\frac{N}{2} - F[f](y-1)\,\frac{N}{4} - F[f](y-(N-1))\,\frac{N}{4}$$

$$= \frac{N}{2}\left(F[f](y) - \frac{1}{2}(F[f](y+1) + F[f](y-1))\right)$$

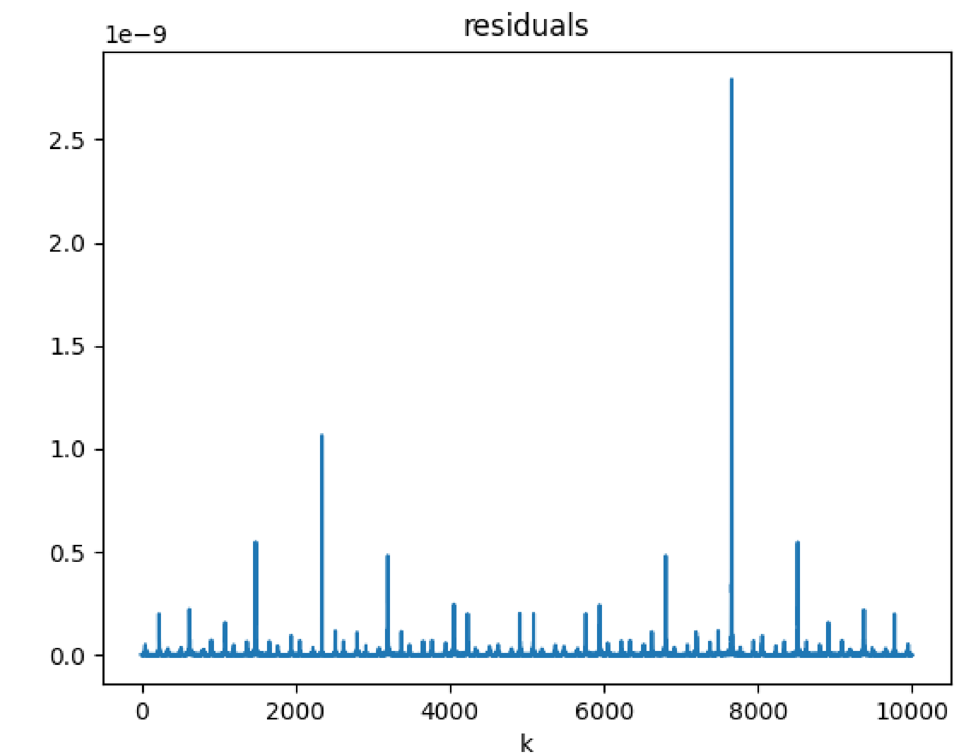
Graphs for 5)

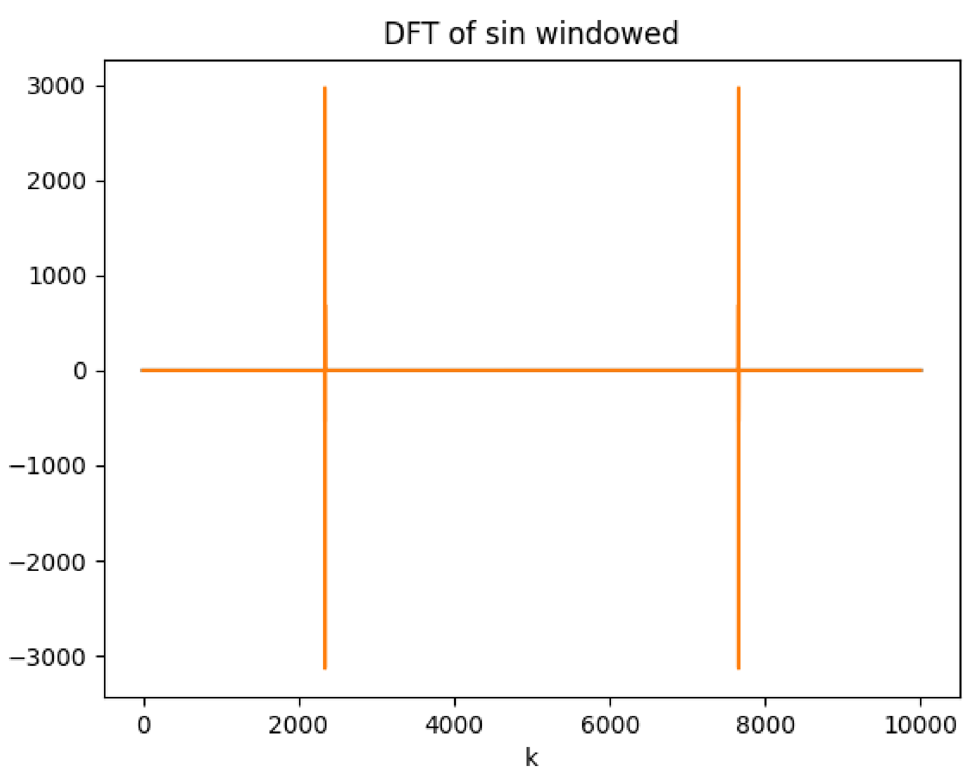


DFT of sin
3000
2000
1000
0
−1000
−2000
−3000
0
2000
4000
6000
8000
10000
k



Analytic approximation of fft(sin(wx))
3000
2000
1000
0
−1000
−2000
−3000
0
2000
4000
6000
8000
10000
k


residuals
1e−9
2.5
2.0
1.5
1.0
0.5
0.0
0
2000
4000
6000
8000
10000
k


DFT of sin windowed
3000
2000
1000
0
−1000
−2000
−3000
0
2000
4000
6000
8000
10000
k

6. a) $P[f](k) = |F[f](k)|^2 = F[f](k)\overline{F[f](k)}$

$= F[f \star f](k) = F[E(\delta)](k)$ ← auto correlation

$P[f](k) \propto F[c - |\delta|](k)$

$= F[c - |\delta|^1](k)$ from wikipedia ⇓

$= c\delta(k) + \frac{2\sin(\frac{\pi 1}{2})\Gamma(1+1)}{|2\pi k|^{1+1}}$

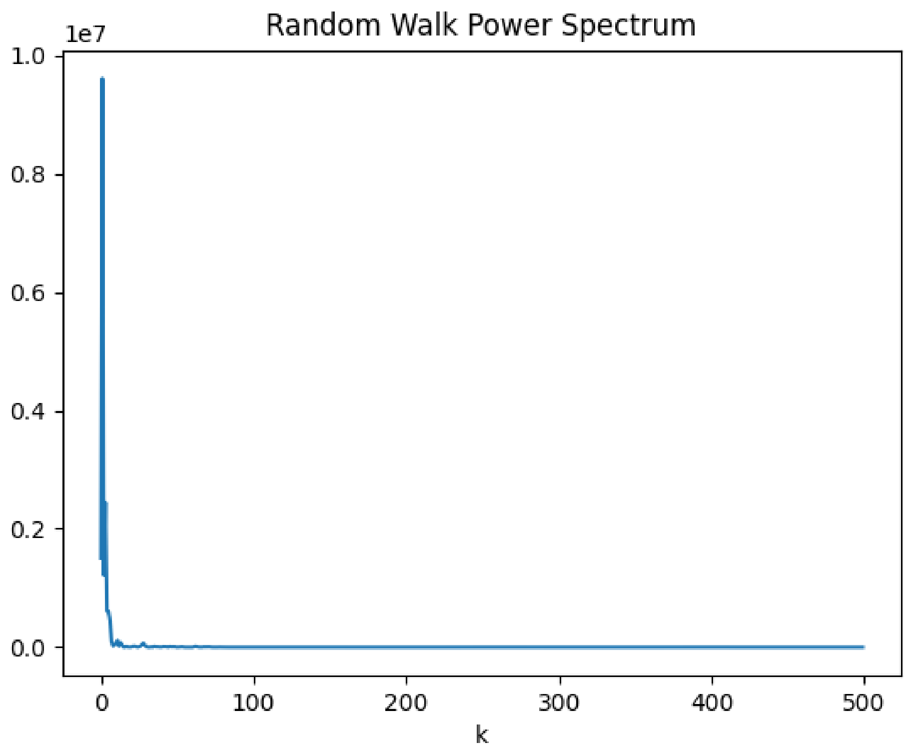
$= c\delta(k) + \frac{4}{4\pi^2 k^2}$

$= c\delta(k) + \pi^{-2} k^{-2}$

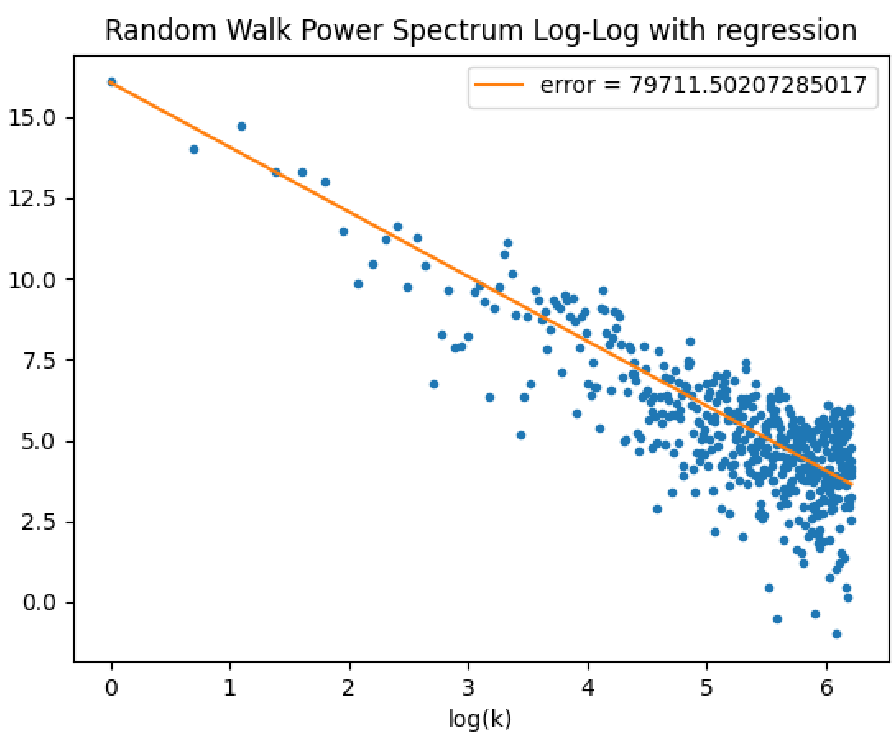
∴ the power spectrum goes as k^{-2}

b) The fit is pretty good judging by the log-log graph which had an error on the order 10^5 which is good given the scale and noise of our spectrum

Graphs for 6)



Random Walk Power Spectrum
1e7
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
k



Random Walk Power Spectrum Log-Log with regression
error = 79711.50207285017
15.0
12.5
10.0
7.5
5.0
2.5
0.0
0
1
2
3
4
5
6
log(k)