

# The Fresnel integrals and the Euler curve

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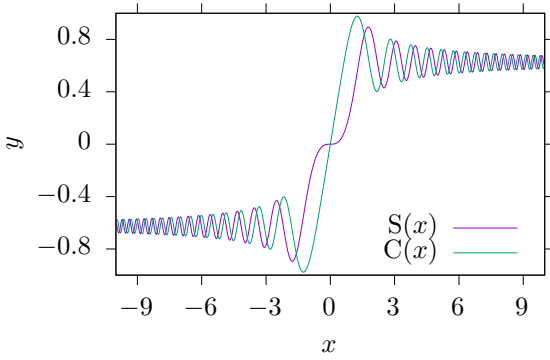


Figure 1: The  $S(x)$  and  $C(x)$  functions.

All information in this text is taken from Wikipedia[1].

## 1 The Fresnel integrals

The Fresnel Integrals are two transcendental functions used in optics. They are defined by the following integrals:

$$S(x) = \int_0^x \sin(t^2) dt \quad (1)$$

$$C(x) = \int_0^x \cos(t^2) dt \quad (2)$$

They appear in the description of near-field Fresnel diffraction. A plot of the integrals can be seen in figure 1.

As an alternative to equation 1 and 2, the functions can be computed by the power series expansion, which converges for all  $x$ :

$$S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)!(4n+3)}. \quad (3)$$

$$C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!(4n+1)}. \quad (4)$$

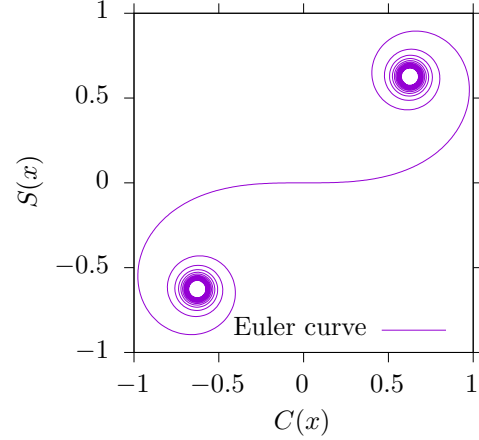


Figure 2: The Euler curve for a parameter  $t$  in the interval  $[-10, 10]$ .

## 2 Mathematical properties

- Both of the Fresnel integrals are odd functions of their parameter  $x$ .
- Using the power series, see equation 3 and ??, the Fresnel integrals can be extended to the complex plane.
- In the limit of  $x$  going to infinity:  $C(\infty) = S(\infty) = \sqrt{\frac{\pi}{8}} \approx 0.6267$

## 3 The Euler Spiral

The following parametric function is called the Euler spiral, and is plotted in figure 2

$$(x, y) = (C(t), S(t)). \quad (5)$$

The Euler spiral has the mathematical property that the curvature at any point is proportional to the length of the spiral, measured from the origin. Thereby, a vehicle following the spiral at a constant speed will experience a constant rate of angular acceleration.

## References

- [1] Wikipedia: Fresnel integral.  
[https://en.wikipedia.org/wiki/Fresnel\\_integral](https://en.wikipedia.org/wiki/Fresnel_integral),  
Retrieved 06/03/2020.