Choosing boundry condition for the 0'th order spherical bessel function of the first kind

S.H. Albrechtsen

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All formulas used in this text can be found on wikipedia, see [1] and [2].

The differential equation for the spherical bessel function can be written as:

$$y'' = -\frac{1}{x^2}(2xy' + (x^2 - n(n+1))) \tag{1}$$

And the naive boundry condition one could use to integrate this numerically would be:

$$\text{For } x = 0, \quad y = 1, \quad y' = 0 \tag{2}$$

However, it is not possible to numerically evaluate the differential equation at $x = 0$. A different starting position is needed, and thereby different boundry condition values must be evaluated.

Let's choose the starting point a, which is a small number. By choosing a small shift from zero, using a series expansion of the starting conditions becomes a good approximation. However, the solution of the differential equation can obviously not be used to calculate these values.

Luckily, the differential equation itself gives a series expansion of the j_0 function, which is good for small displacements from 0. This is actually based on the series expansion of the generalized bessel function $J_{\frac{1}{2}}(x)$.[1] Here the two first terms of the expansion is included:

$$j_0(x) = \sqrt{\frac{\pi}{2x}} J_{1/2}(x) \tag{3}$$

$$\approx \sqrt{\frac{\pi}{2x}} \left(\frac{1}{\Gamma(3/2)}(x/2)^{1/2} - \frac{1}{\Gamma(5/2)}(x/2)^{5/2} \right) \tag{4}$$

[1]See the wikipedia page for bessel functions to find more series expansions and do this for other spherical bessen functions than j_0

Using $\Gamma(3/2) = \sqrt(\pi)/2$ and $\Gamma(5/2) = 3\sqrt{\pi}/4$:

$$j_0(x) \approx 1 - \sqrt{\frac{\pi}{2x}} \frac{1}{3\sqrt{\pi}/4} \frac{x^{5/2}}{2^{3/2}} \tag{5}$$

$$= 1 - \frac{1}{3}x^2 \tag{6}$$

Thereby the first derivative of j_0 is:

$$j_0'(x) \approx -\frac{2}{3}x \tag{7}$$

And thereby, for a given small displacement from 0 called a, better boundry condition can be given:

$$\text{For } x = a, \quad y = 1 - \frac{a^2}{2}, \quad y' = -\frac{2}{3}a \tag{8}$$

The accuracy of the boundry condition needs to be at least as good as the accuracy of the differential equation solver. It can be seen that the correction to y is in second order in a. Thereby, to have an absolute accuracy acc, a needs to satisfy the relation:

$$a \geq \sqrt{acc} \tag{9}$$

Of course, choosing a too small a might very well introduce numerical errors, like it would in the differential equation given in the start of this text. If this is the case, the series expansion of j_0 must be taken to higher terms, so that the lower limit of a can be a cubic- or higher-order-root of acc. Thereby a can be set higher without loss of precision.

References

[1] Wikipedia on bessel functions. https://en.wikipedia.org/wiki/Bessel_function.

[2] Wikipedia on the gamma function. https://en.wikipedia.org/wiki/Gamma_function.