Stellar Clocks – Introducing Maelstrom

Daniel Foreman-Mackey,[1★] Simon J. Murphy,[2†] David W. Hogg,[3‡]
[1] Flatiron Institute, NYC, USA.
[2] Sydney Institute for Astronomy (SIfA), School of Physics, University of Sydney, NSW 2006, Australia
[1] NYU, NYC, USA.

Accepted XXX. Received YYY; in original form ZZZ

ABSTRACT
Just some notes.

1 INTRODUCTION

Pulsating stars with coherent modes can be used as clocks. In a binary system, the pulsating star's orbit around the barycentre leads to a change in path length for starlight travelling to Earth. The time-dependent luminosity variations due to pulsation therefore include a time-delay term, τ:

$$y(t) = \sum_{j=1}^{J} \left[A_j \cos(\omega_j [t-\tau]) + B_j \sin(\omega_j [t-\tau])\right] + \epsilon \tag{1}$$

where A_j is the amplitude and $\omega_j = 2\pi\nu_j$ is the angular frequency of mode j, and ϵ describes variance unaccounted for by our pulsation model of J modes. We assume that ϵ is Gaussian distributed with variance σ^2 (encapsulating both measurement uncertainty and model mis-specification). Note that τ is defined to be positive when the star is on the far side of the barycentre. Following Murphy et al. (2016), the time delay can be expressed as a function of the true anomaly, f,

$$\tau = -\frac{a_1 \sin i}{\mathrm{c}} \frac{1-e^2}{1+e\cos f} \sin(f+\varpi) \tag{2}$$

in units of seconds, corresponding to a change in path length measured in light-seconds. We follow the previous convention that $a_1 \sin i$ denotes the projected semimajor axis, e is the eccentricity, ϖ is the angle between the node and the periastron, and c is the speed of light.

In previous papers in this series (Murphy et al. 2014; Murphy & Shibahashi 2015; Murphy et al. 2016), time-delay curves were extracted for pulsating binaries (PBs) by segmenting the *Kepler* light curve. The measured time delays were the average over the corresponding segments, which for highly eccentric binaries led to undersampling near periastron. Although a correction for undersampling was formulated (Murphy et al. 2016), it leads to decreased discriminatory power between orbits at high eccentricity (Murphy et al. 2018). Here, we describe an approach that forward-models the time delays directly on the light curve, mitigating the sampling problem. It has the added benefit that computationally expensive series expansions in Bessel functions are no longer required.

2 NEW APPROACH

A major strength of the existing method is the independent calculation of $\tau_{i,j}$ for each j-th oscillation mode in each i-th segment. This allows the response of each pulsation mode to the orbit to be visualised and checked for mutual agreement (e.g. figures 1–5 of Murphy et al. 2014). A weighted-average time delay is calculated across all j modes in each segment i, with the relative mode amplitudes A_j and B_j acting as weights (stronger modes are weighted more heavily). In our formulation here, the mode amplitudes will also act as weights since the contribution of each j-th mode to the luminosity variations is scaled by the amplitude A_j of that mode (Eq. 1), except that τ_j is now calculated at each individual observation t.

We begin by separating the time-delay equation (Eq. 2) into a shape and an amplitude component

$$\tau_{t,j} = \mathcal{A}_j \psi_t, \tag{3}$$

where

$$\psi_t = -\frac{1-e^2}{1+e\cos f} \sin(f+\varpi) \tag{4}$$

and $\mathcal{A}_j$ is evaluated for each j-th mode. Modes can be grouped together by their $\mathcal{A}_j$, which is particularly useful when there are two pulsators in the same binary system (e.g. figure 6 of Murphy et al. 2014).

We construct the design matrix, D_j, for each j-th mode and each observation time t_n, and we combine these into the master design matrix D, which has N rows and $2J$ columns:

$$D_j = \begin{pmatrix} \cos(\omega_j[t_1 - \tau_{1,j}]) & \sin(\omega_j[t_1 - \tau_{1,j}]) \\ \cos(\omega_j[t_2 - \tau_{2,j}]) & \sin(\omega_j[t_2 - \tau_{2,j}]) \\ \vdots & \vdots \\ \cos(\omega_j[t_n - \tau_{n,j}]) & \sin(\omega_j[t_n - \tau_{n,j}]) \\ \vdots & \vdots \\ \cos(\omega_j[t_N - \tau_{N,j}]) & \sin(\omega_j[t_N - \tau_{N,j}]) \end{pmatrix} \tag{5}$$

and

$$D = (D_1 \; D_2 \; \ldots \; D_j \; \ldots \; D_J). \tag{6}$$

The amplitude coefficients A_j and B_j are collected in the column matrices w_j

$$w_j = \begin{pmatrix} A_j \\ B_j \end{pmatrix}, \tag{7}$$

which are combined into the matrix of weights, w

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_j \\ \vdots \\ w_J \end{pmatrix}. \tag{8}$$

The variance in the light curve, y (Eq. 1), is then expressed as

$$y = D \cdot w \tag{9}$$

and the maximum likelihood values for the weights can be found by linear regression

$$\hat{w} = (D^T \cdot D)^{-1}(D^T \cdot y_n) \tag{10}$$

where D^T is the transpose of D and y_n is the luminosity variation at time t_n. The value of the likelihood at this maximum is

$$\hat{L} = \sum_{n=1}^{N} \frac{(y_n - (D \cdot \hat{w})_n)^2}{\sigma^2} \tag{11}$$

where σ is the standard deviation of each measurement, y_n. SJM: this can't be the maximum likelihood, since it gets larger when one strays from $\hat{w}$. This is χ^2, which is related to $\hat{L}$ in some other way, such as $\hat{L} = \exp[-\chi^2/2]$.

2.1 Calculating the uncertainties

For a given parameter θ in the model, whose optimum value is $\hat{\theta}$, the log-likelihood is expressed as

$$\log L = -\frac{1}{2}\frac{(\theta - \hat{\theta})^2}{\sigma_\theta^2}, \tag{12}$$

so the uncertainty σ_θ^2 can be found by taking the second derivative of the log-likelihood

$$\begin{aligned} \frac{d \log L}{d\theta} &= -\frac{(\theta - \hat{\theta})}{\sigma_\theta^2} && (13) \\ \frac{d^2 \log L}{d\theta^2} &= -\frac{1}{\sigma_\theta^2} && (14) \\ \sigma_\theta^2 &= -\frac{1}{\frac{d^2 \log L}{d\theta^2}}. && (15) \end{aligned}$$

The arbitrary number of model parameters M are collected together in the Hessian, H,

$$H = \begin{pmatrix} \sigma_{\theta_1}\sigma_{\theta_1} & \sigma_{\theta_1}\sigma_{\theta_2} & \cdots & \sigma_{\theta_1}\sigma_{\theta_M} \\ \sigma_{\theta_2}\sigma_{\theta_1} & \sigma_{\theta_2}\sigma_{\theta_2} & \cdots & \sigma_{\theta_2}\sigma_{\theta_M} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{\theta_M}\sigma_{\theta_1} & \sigma_{\theta_M}\sigma_{\theta_2} & \cdots & \sigma_{\theta_M}\sigma_{\theta_M} \end{pmatrix} \tag{16}$$

which is a square matrix with M rows and columns. In addition to this, we define this beast called Σ_θ, which is the negative inverse of the Hessian,

$$\Sigma_\theta = -(H)^{-1}. \tag{17}$$

3 TO DO:

We only produce keplerian orbits.
Could have spline fitted to individual $\tau(t_n)$ measurements.
Note integration time is assumed to be zero.
Adding in RV data (be careful – use same parametrization).

3.1 Examples

Short period binaries:

(i) KIC 10080843 – 15-d PB2.
(ii) KIC 6780873 – 9.15-d PB1.
(iii) Simulated TESS subdwarfs, possibly super-Nyquist of 2-min cadence.

Other examples we might consider:

(i) *Kepler* δ Sct using the Nyquist aliases.

ACKNOWLEDGEMENTS

This research was supported by the Simons Foundation and the Australian Research Council.

REFERENCES

Murphy S. J., Shibahashi H., 2015, MNRAS, 450, 4475
Murphy S. J., Bedding T. R., Shibahashi H., Kurtz D. W., Kjeldsen H., 2014, MNRAS, 441, 2515
Murphy S. J., Shibahashi H., Bedding T. R., 2016, MNRAS, 461, 4215
Murphy S. J., Moe M., Kurtz D. W., Bedding T. R., Shibahashi H., Boffin H. M. J., 2018, MNRAS, 474, 4322

This paper has been typeset from a TEX/LATEX file prepared by the author.