Proof of error bound on manifold given error bound on the tangent space

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Theorem 1. *Let M be a Riemannian manifold with sectional curvature bounded from below by H and let $f\colon R \to M$ where R is a subset of $\mathbb{R}^n$ such that its image fits in a single normal coordinate chart S around $p \in M$, which fits inside some geodesic ball of radius σ. Define $g = \log_p \circ f\colon R \to S$. Let $\widehat{g}\colon R \to S$ be an approximation of g such that*

$$|g(x) - \widehat{g}(x)| \leq \epsilon \tag{1}$$

and define $\widehat{f} = \exp_p \circ \widehat{g}$.

If $H > 0$ and $\sigma < \frac{\pi}{\sqrt{H}}$, then

$$d_M(f, \widehat{f}) \leq \epsilon. \tag{2}$$

If $H < 0$, then

$$d_M(f, \widehat{f}) \leq \epsilon + \frac{2}{\sqrt{|H|}} \operatorname{arcsinh}\left(\frac{\epsilon}{2} \cdot \frac{\sinh \sqrt{|H|}\sigma}{\sigma}\right). \tag{3}$$

In terms of elementary functions, (3) can in turn be upper bounded by

$$d_M(f, \widehat{f}) \leq \epsilon + \frac{2}{\sqrt{|H|}} \log\left(\frac{\epsilon}{2} \cdot \frac{\mathrm{e}^{\sqrt{|H|}\sigma} - 1}{\sigma} + 1\right). \tag{4}$$

Many manifolds have known sectional curvature. Some examples of manifolds with sectional curvature bounded from below are

- Compact Lie groups (and their products with $\mathbb{R}^n$) like $\mathrm{O}(n)$ and G_2, where

$$K(P, Q) = \frac{\|[P, Q]\|_F^2}{4} \geq 0, \tag{5}$$

  (Cheeger 2008 corollary 3.19 + proposition 3.34).

- TODO.

Let's state some things that we need in the proof of theorem 1. How to deal with completeness?...

Theorem 2 (Toponogov). *Let M be a complete Riemannian manifold with sectional curvature $K \geq H$. Let γ_1 and γ_2 be geodesics on M such that $\gamma_1(0) = \gamma_2(0)$. If $H > 0$, assume $|\gamma_1|$, $|\gamma_2| \leq \frac{\pi}{\sqrt{H}}$. Let λ_1 and λ_2 be geodesics on a model manifold N of constant curvature H such that*

$$\lambda_1(0) = \lambda_2(0), \tag{6}$$
$$|\lambda_1| = |\gamma_1|, \tag{7}$$
$$|\lambda_2| = |\gamma_2|, \tag{8}$$
$$\angle(\lambda_1, \lambda_2) = \angle(\gamma_1, \gamma_2). \tag{9}$$

Then

$$d_M(\gamma_1(x), \gamma_2(y)) \leq d_N(\lambda_1(x), \lambda_2(y)) \tag{10}$$

for all x, y.

Lemma 1 (Spherical and hyperbolic law of cosines). *Let N be a manifold of constant curvature H. Let T be a geodesic triangle on N with side lengths A, B, C and angles a, b, c. Then*

$$\cos C\sqrt{H} = \cos A\sqrt{H}\cos B\sqrt{H} + \sin A\sqrt{H}\sin B\sqrt{H}\cos c \tag{11}$$

if $H > 0$, and

$$\cosh C\sqrt{|H|} = \cosh A\sqrt{|H|}\cosh B\sqrt{|H|} - \sinh A\sqrt{|H|}\sinh B\sqrt{|H|}\cos c \tag{12}$$

if $H < 0$.

proof of theorem 1. Let N be the manifold of constant curvature H and let $q \in N$. Define $h = \exp_{N,q} \circ g$ and $\widehat{h} = \exp_{N,q} \circ \widehat{g}$. This is well-defined by requiring that $|g|$, $|\widehat{g}| < \frac{\pi}{\sqrt{H}}$ when $H > 0$. By theorem 2, we have that

$$d_M(f(x), \widehat{f}(y)) \leq d_N(h(x), \widehat{h}(y)). \tag{13}$$

Now our task is to bound the right-hand side of this inequality when $x = y$.

Case 1: $H > 0$. Let $\cos' t = \cos t\sqrt{H}$ and $\sin' t = \sin t\sqrt{H}$ be reparametrized trigonometric functions and consider the geodesic triangle $(h(x), \widehat{h}(x), q)$ on N. By lemma 1,

$$\cos' d(h, \widehat{h}) = \cos' d(h, q)\cos' d(\widehat{h}, q) + \sin' d(h, q)\sin' d(\widehat{h}, q)\cos\left\{\angle(h, \widehat{h})\right\} \tag{14}$$
$$= \cos'|g|\cos'|\widehat{g}| + \sin'|g|\sin'|\widehat{g}|\cos\{\angle(g, \widehat{g})\}. \tag{15}$$

Since $|g|$, $|\widehat{g}| < \frac{\pi}{\sqrt{H}}$,

$$\sin'|g|\sin'|\widehat{g}| \geq 0. \tag{16}$$

Hence using

$$\cos\{\angle(g, \widehat{g})\} \leq 1 \tag{17}$$

in (15) gives us

$$\cos' d(h, \widehat{h}) \geq \cos'|g|\cos'|\widehat{g}| + \sin'|g|\sin'|\widehat{g}| \tag{18}$$
$$= \cos(|g| - |\widehat{g}|). \tag{19}$$

And so

$$d(h, \widehat{h}) \leq ||g| - |\widehat{g}|| \tag{20}$$
$$\leq |g - \widehat{g}| \tag{21}$$

Case 2: $H < 0$. Let $\cosh' t = \cosh t\sqrt{|H|}$ and $\sinh' t = \sinh t\sqrt{|H|}$ and consider again the geodesic triangle $(h(x), \widehat{h}(x), q)$. Again, by lemma 1,

$$\cosh' d(h, \widehat{h}) = \cosh' |g| \cosh' |\widehat{g}| - \sinh' |g| \sinh' |\widehat{g}| \cos\{\angle(g, \widehat{g})\}. \tag{22}$$

Since $\sinh'$ is positive for positive arguments,

$$\sinh' |g| \sinh' |\widehat{g}| \geq 0. \tag{23}$$

Hence using

$$\begin{aligned} \cos\{\angle(g, \widehat{g})\} &= \frac{\langle g, \widehat{g}\rangle}{|g||\widehat{g}|} && (24) \\ &= \frac{|g|^2 + |\widehat{g}|^2}{2|g||\widehat{g}|} - \frac{|g - \widehat{g}|^2}{2|g||\widehat{g}|} && (25) \\ &\geq 1 - \frac{|g - \widehat{g}|^2}{2|g||\widehat{g}|} && (26) \end{aligned}$$

in (22) gives us

$$\begin{aligned} \cosh' d(h, \widehat{h}) &\leq \cosh' |g| \cosh' |\widehat{g}| - \sinh' |g| \sinh' |\widehat{g}| \left(1 - \frac{|g - \widehat{g}|^2}{2|g||\widehat{g}|}\right) && (27) \\ &= \cosh' (|g| - |\widehat{g}|) + \sinh' |g| \sinh' |\widehat{g}| \frac{|g - \widehat{g}|^2}{2|g||\widehat{g}|} && (28) \\ &\quad \{\text{inverse triangle inequality}\} \\ &\leq \cosh' (|g - \widehat{g}|) + \sinh' |g| \sinh' |\widehat{g}| \frac{|g - \widehat{g}|^2}{2|g||\widehat{g}|}, && (29) \end{aligned}$$

and so

$$\begin{aligned} 1 + 2\sinh'^2 \frac{d(h, \widehat{h})}{2} &\leq 1 + 2\sinh'^2 \frac{|g - \widehat{g}|}{2} + \sinh' |g| \sinh' |\widehat{g}| \frac{|g - \widehat{g}|^2}{2|g||\widehat{g}|} && (30) \\ d(h, \widehat{h}) &\leq \frac{2}{\sqrt{|H|}} \operatorname{arcsinh} \sqrt{\sinh'^2 \frac{|g - \widehat{g}|}{2} + \sinh' |g| \sinh' |\widehat{g}| \frac{|g - \widehat{g}|^2}{4|g||\widehat{g}|}} && (31) \end{aligned}$$

Noting that $\frac{\sinh x}{x}$ is increasing on $\mathbb{R}^+$, we have that

$$d(h, \widehat{h}) \leq \frac{2}{\sqrt{|H|}} \operatorname{arcsinh} \sqrt{\sinh'^2 \frac{\epsilon}{2} + \frac{\epsilon^2}{4} \cdot \frac{\sinh'^2 \sigma}{\sigma^2}}. \tag{32}$$

Concave functions are subadditive: for example $\sqrt{x^2 + y^2} \leq x + y$ for positive x and y, so

$$\begin{aligned} d(h, \widehat{h}) &\leq \frac{2}{\sqrt{|H|}} \operatorname{arcsinh} \left(\sinh' \frac{\epsilon}{2} + \frac{\epsilon}{2} \cdot \frac{\sinh' \sigma}{\sigma}\right) && (33) \\ &\quad \{\text{arcsinh is also concave}\} && (34) \\ &\leq \epsilon + \frac{2}{\sqrt{|H|}} \operatorname{arcsinh} \left(\frac{\epsilon}{2} \cdot \frac{\sinh' \sigma}{\sigma}\right). && (35) \end{aligned}$$

□

When $H \to 0^-$, (3) tends to

$$d(h, \widehat{h}) \leq 2\epsilon, \tag{36}$$

which shows that it is an imperfect bound. To explain why, first note that (26) is an equality when $|g| = |\widehat{g}|$, but we also used the inverse triangle inequality in (29) which is an equality when g and $\widehat{g}$ are colinear. Furthermore, in (33) we used subadditivity for the square root, which is only an equality when one of the terms is 0. But for small curvatures and small ϵ, the two terms in (33) are approximately equal. One could say that we lost a factor $\sqrt{2}$ in each of these steps.

In some regions, (3) gives a terrible bound for the error on the manifold. For example, such innocent looking input as $H = -10$, $\sigma = 10$, $\epsilon = 1 \times 10^{-12}$ give an error bound of ≈ 0.7 for $\widehat{f}$. In general, if we want (3) to give a small error, we should have $\sigma < \frac{1}{\sqrt{|H|}}$.

Even though the bound (3) is imperfect, there is, in the following sense, no bound with better properties.

Proposition 1. *Let M be a manifold of constant negative curvature H and let $\widehat{g}$ be an approximation to g such that the maximum error ϵ is attained at a point $x \in S$ such that $|\widehat{g}(x)| = |g(x)| = \sigma$, then*

$$d_M(f(x), \widehat{f}(x)) = \frac{2}{\sqrt{|H|}} \operatorname{arcsinh}\left(\frac{\epsilon}{2} \cdot \frac{\sinh \sqrt{|H|}\sigma}{\sigma}\right). \tag{37}$$

Proof. If $|g| = |\widehat{g}|$, then (26) is an equality. We then do not need to use the inverse triangle inequality in (29), and so (30) reduces to

$$1 + 2\sinh'^2 \frac{d(f, \widehat{f})}{2} = 1 + 2\sinh'^2 \sigma^2 \cdot \frac{\epsilon^2}{2\sigma^2}, \tag{38}$$

$$d(f, \widehat{f}) = \frac{2}{\sqrt{|H|}} \operatorname{arcsinh}\left(\sinh'^2 \sigma \cdot \frac{\epsilon}{2\sigma}\right). \tag{39}$$

□