Proof of error bound on manifold given error bound on the tangent space

2022-11-23

Theorem 1 (Toponogov). *Let M be a complete Riemannian manifold with sectional curvature $K \geq H$. Let γ_1 and γ_2 be geodesics on M such that $\gamma_1(0) = \gamma_2(0)$. If $H > 0$, assume $|\gamma_1|$, $|\gamma_2| \leq \frac{\pi}{\sqrt{H}}$. Let λ_1 and λ_2 be geodesics on a model manifold N of constant curvature H such that*

$$\lambda_1(0) = \lambda_2(0), \tag{1}$$
$$|\lambda_1| = |\gamma_1|, \tag{2}$$
$$|\lambda_2| = |\gamma_2|, \tag{3}$$
$$\angle(\lambda_1, \lambda_2) = \angle(\gamma_1, \gamma_2). \tag{4}$$

Then

$$d_M(\gamma_1(x), \gamma_2(y)) \leq d_N(\lambda_1(x), \lambda_2(y)) \tag{5}$$

for all x, $y \in [0, 1]$.

Lemma 1 (Spherical and hyperbolic law of cosines). *Let N be a manifold of constant curvature H. Let T be a geodesic triangle on N with side lengths A, B, C and angles a, b, c. Then*

$$\cos C\sqrt{H} = \cos A\sqrt{H} \cos B\sqrt{H} + \sin A\sqrt{H} \sin B\sqrt{H} \cos c \tag{6}$$

if $H > 0$, and

$$\cosh C\sqrt{|H|} = \cosh A\sqrt{|H|} \cosh B\sqrt{|H|} - \sinh A\sqrt{|H|} \sinh B\sqrt{|H|} \cos c \tag{7}$$

if $H < 0$.

Lemma 2. *TODO: inequalities for cosine.*

Theorem 2. *Let M be a Riemannian manifold with sectional curvature bounded from below by H and let $f\colon R \to M$ where R is a subset of $\mathbb{R}^n$ such that its image fits in a single normal coordinate chart of M. Let $p \in M$ and define $g = \log_p \circ f\colon R \to \mathbb{R}^n$. Let $\widehat{g}$ be an approximation of g and define $\widehat{f} = \exp_p \circ \widehat{g}$. If $\widehat{g}$ satisfies $|g(x) - \widehat{g}(x)| \leq G$ for some sufficiently small G, then*

$$\left| f(x) - \widehat{f}(x) \right| \leq G \tag{8}$$

if $H < 0$, *and*

$$\left|f(x) - \widehat{f}(x)\right| \leq \frac{2G}{\sqrt{H}}\sqrt{b(G) + \frac{\sinh^2(\sqrt{H}|g|)}{2 \cdot |g|^2}} \tag{9}$$

if $H > 0$, *where* b *is a locally bounded function that approches* $\frac{1}{2}$ *as* $G \to 0$.

Proof. Let N be the manifold of constant curvature H and let $q \in N$. Define $h = \exp_{N,q} \circ g$ and $\widehat{h} = \exp_{N,q} \circ \widehat{g}$. By theorem 1, we have that

$$d_M(f(x), \widehat{f}(y)) \leq d_N(h(x), \widehat{h}(y)). \tag{10}$$

Case 1: $H > 0$**.** Let $\cos' t = \cos t\sqrt{H}$ and $\sin' t = \sin t\sqrt{H}$ be reparametrized trigonometric functions and consider the geodesic triangle $(h(x), \widehat{h}(x), q)$ on N. By lemma 1,

$$\begin{aligned}
\cos' d(h, \widehat{h}) &= \cos' d(h, q)\cos' d(\widehat{h}, q) + \sin' d(h, q)\sin' d(\widehat{h}, q)\cos\left\{\angle(h, \widehat{h})\right\} && (11)\\
&= \cos' |g| \cos' |\widehat{g}| + \sin' |g| \sin' |\widehat{g}| \cos\{\angle(g, \widehat{g})\} && (12)\\
&= \frac{1}{2}\big[\cos'(|g| - |\widehat{g}|) + \cos'(|g| + |\widehat{g}|)\big] && (13)\\
&\quad + \frac{1}{2}\big[\cos'(|g| - |\widehat{g}|) - \cos'(|g| + |\widehat{g}|)\big]\cos\{\angle(g, \widehat{g})\}. && (14)
\end{aligned}$$

Since $\cos'$ is decreasing on $[0, \frac{\pi}{\sqrt{H}}]$, and since $|g|, |\widehat{g}| < \frac{\pi}{2\sqrt{H}}$ (write down this condition in the thm) ,

$$\cos'(|g| - |\widehat{g}|) - \cos'(|g| + |\widehat{g}|) \geq 0, \tag{15}$$

so using $\cos\{\angle(g, \widehat{g})\} \leq 1$ in (14) gives us

$$\cos' d(h, \widehat{h}) \geq \cos(|g| - |\widehat{g}|). \tag{16}$$

And so

$$\begin{aligned}
d(h, \widehat{h}) &\leq ||g| - |\widehat{g}|| && (17)\\
&\leq |g - \widehat{g}| && (18)
\end{aligned}$$

Case 2: $H < 0$**.** Let $\cosh' t = \cosh t\sqrt{|H|}$ and $\sinh' t = \sinh t\sqrt{|H|}$ be reparametrized hyperbolic trigonometric functions and consider again the geodesic triangle $(h(x), \widehat{h}(x), q)$. Similarly to case 1,

$$\begin{aligned}
\cosh' d(h, \widehat{h}) &= \frac{1}{2}\big[\cosh'(|g| - |\widehat{g}|) + \cosh'(|g| + |\widehat{g}|)\big] && (19)\\
&\quad - \frac{1}{2}\big[-\cosh'(|g| - |\widehat{g}|) + \cosh'(|g| + |\widehat{g}|)\big]\cos\{\angle(g, \widehat{g})\}. && (20)
\end{aligned}$$

Since $\cosh'$ is increasing,

$$-\cosh'(|g| - |\widehat{g}|) + \cosh'(|g| + |\widehat{g}|) \geq 0, \tag{21}$$

so $\cos\{\angle(g,\widehat{g})\} \geq 1-\frac{1}{2}\angle(g,\widehat{g})^2 \geq 1-\frac{|g-\widehat{g}|^2}{|g|^2}$ (motivate these?) gives us

$$\cosh' d(h,\widehat{h}) \leq \cosh'(|g|-|\widehat{g}|) + \sinh'|g|\sinh'|\widehat{g}|\frac{|g-\widehat{g}|^2}{|g|^2}. \tag{22}$$

The taylor series of $\cosh'$ contains only positive terms, so the LHS can be lower bounded by truncating its series. Since $\cosh'$ is increasing, the RHS can be upper bounded by the inverse triangle inequality. We have

$$1+\frac{1}{2}d(h,\widehat{h})^2|H| \leq \cosh'|g-\widehat{g}| + \sinh'|g|\sinh'|\widehat{g}|\frac{|g-\widehat{g}|^2}{|g|^2} \tag{23}$$

$$= 1+2\sinh'^2\frac{|g-\widehat{g}|}{2} + \frac{\sinh'|g|\sinh'|\widehat{g}|}{|g|^2}|g-\widehat{g}|^2, \tag{24}$$

$$d(h,\widehat{h}) \leq \frac{2}{\sqrt{H}}\sqrt{\frac{\sinh'^2\frac{|g-\widehat{g}|}{2}}{|g-\widehat{g}|^2} + \frac{\sinh'|g|\sinh'|\widehat{g}|}{2|g|^2}}|g-\widehat{g}| \tag{25}$$

$$\leq \frac{2}{\sqrt{H}}\sqrt{\frac{\sinh'^2\frac{G}{2}}{G^2} + \frac{\sinh'|g|\sinh'|\widehat{g}|}{2|g|^2}}|g-\widehat{g}|. \tag{26}$$

Motivate why we can assume $|g|$ is the largest. □