Proof of error bound on manifold given error bound on the tangent space

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Theorem 1. *Let M be a Riemannian manifold with sectional curvature bounded from below by H and let $f\colon R \to M$ where R is a subset of $\mathbb{R}^n$ such that its image fits in a single normal coordinate chart S of M. Let $p \in M$ and define $g = \log_p \circ f\colon R \to S$. Let $\widehat{g}\colon R \to S$ be an approximation of g such that*

$$|g(x) - \widehat{g}(x)| \leq \epsilon \tag{1}$$

and define $\widehat{f} = \exp_p \circ \widehat{g}$. If $H > 0$ and $|g|$, $|\widehat{g}| < \frac{\pi}{\sqrt{H}}$, then

$$\left|f(x) - \widehat{f}(x)\right| \leq \epsilon. \tag{2}$$

If $H < 0$, then

$$\left|f(x) - \widehat{f}(x)\right| \leq \epsilon \sqrt{\frac{\sinh^2\left(\sqrt{|H|}\epsilon/2\right)}{(\sqrt{|H|}\epsilon/2)^2} + 2\frac{\sinh\left(\sqrt{|H|}|g|\right)\sinh\left(\sqrt{|H|}|\widehat{g}|\right)}{|H||g||\widehat{g}|}}. \tag{3}$$

Furthermore, if the norms of vectors in S are bounded by σ, then

$$\left|f(x) - \widehat{f}(x)\right| \leq \epsilon \sqrt{\frac{\sinh^2\left(\sqrt{|H|}\epsilon/2\right)}{(\sqrt{|H|}\epsilon/2)^2} + 2\frac{\sinh^2\left(\sqrt{|H|}\sigma\right)}{|H|\sigma^2}} \tag{4}$$

Remark $\frac{\sinh x}{x}$ is a locally bounded increasing (on $\mathbb{R}^+$) function that approaches 1 as $x \to 0$.

Let's first state some things that we need in the proof of theorem 1.

Theorem 2 (Toponogov)**.** *Let M be a complete Riemannian manifold with sectional curvature $K \geq H$. Let γ_1 and γ_2 be geodesics on M such that $\gamma_1(0) = \gamma_2(0)$. If $H > 0$, assume $|\gamma_1|$, $|\gamma_2| \leq \frac{\pi}{\sqrt{H}}$. Let λ_1 and λ_2 be geodesics on a model manifold N of constant curvature H such that*

$$\lambda_1(0) = \lambda_2(0), \tag{5}$$
$$|\lambda_1| = |\gamma_1|, \tag{6}$$
$$|\lambda_2| = |\gamma_2|, \tag{7}$$
$$\angle(\lambda_1, \lambda_2) = \angle(\gamma_1, \gamma_2). \tag{8}$$

Then

$$d_M(\gamma_1(x), \gamma_2(y)) \leq d_N(\lambda_1(x), \lambda_2(y)) \tag{9}$$

for all x, y.

Lemma 1 (Spherical and hyperbolic law of cosines). *Let N be a manifold of constant curvature H. Let T be a geodesic triangle on N with side lengths A, B, C and angles a, b, c. Then*

$$\cos C\sqrt{H} = \cos A\sqrt{H}\cos B\sqrt{H} + \sin A\sqrt{H}\sin B\sqrt{H}\cos c \tag{10}$$

if $H > 0$, and

$$\cosh C\sqrt{|H|} = \cosh A\sqrt{|H|}\cosh B\sqrt{|H|} - \sinh A\sqrt{|H|}\sinh B\sqrt{|H|}\cos c \tag{11}$$

if $H < 0$.

proof of theorem 1. Let N be the manifold of constant curvature H and let $q \in N$. Define $h = \exp_{N,q} \circ g$ and $\widehat{h} = \exp_{N,q} \circ \widehat{g}$. This is well-defined by requiring that $|g|$, $|\widehat{g}| < \frac{\pi}{\sqrt{H}}$ when $H > 0$. By theorem 2, we have that

$$d_M(f(x), \widehat{f}(y)) \leq d_N(h(x), \widehat{h}(y)). \tag{12}$$

Now our task is to bound the right-hand side of this inequality when $x = y$.

Case 1: $H > 0$**.** Let $\cos' t = \cos t\sqrt{H}$ and $\sin' t = \sin t\sqrt{H}$ be reparametrized trigonometric functions and consider the geodesic triangle $(h(x), \widehat{h}(x), q)$ on N. By lemma 1,

$$\begin{aligned} \cos' d(h, \widehat{h}) &= \cos' d(h, q)\cos' d(\widehat{h}, q) + \sin' d(h, q)\sin' d(\widehat{h}, q)\cos\left\{\angle(h, \widehat{h})\right\} && (13) \\ &= \cos' |g|\cos' |\widehat{g}| + \sin' |g|\sin' |\widehat{g}|\cos\{\angle(g, \widehat{g})\}. && (14) \end{aligned}$$

Since $|g|$, $|\widehat{g}| < \frac{\pi}{\sqrt{H}}$,

$$\sin' |g|\sin' |\widehat{g}| \geq 0. \tag{15}$$

Hence using

$$\cos\{\angle(g, \widehat{g})\} \leq 1 \tag{16}$$

in (14) gives us

$$\begin{aligned} \cos' d(h, \widehat{h}) &\geq \cos' |g|\cos' |\widehat{g}| + \sin' |g|\sin' |\widehat{g}| && (17) \\ &= \cos(|g| - |\widehat{g}|). && (18) \end{aligned}$$

And so

$$\begin{aligned} d(h, \widehat{h}) &\leq ||g| - |\widehat{g}|| && (19) \\ &\leq |g - \widehat{g}| && (20) \end{aligned}$$

Case 2: $H < 0$. Let $\cosh' t = \cosh t\sqrt{|H|}$ and $\sinh' t = \sinh t\sqrt{|H|}$ and consider again the geodesic triangle $(h(x), \widehat{h}(x), q)$. Again, by lemma 1,

$$\cosh' d(h, \widehat{h}) = \cosh' |g| \cosh' |\widehat{g}| - \sinh' |g| \sinh' |\widehat{g}| \cos\{\angle(g, \widehat{g})\}. \tag{21}$$

Since $\sinh'$ is positive for positive arguments,

$$\sinh' |g| \sinh' |\widehat{g}| \geq 0. \tag{22}$$

Hence using

$$\begin{aligned} \cos\{\angle(g, \widehat{g})\} &= \frac{\langle g, \widehat{g}\rangle}{|g||\widehat{g}|} && (23)\\ &= \frac{|g|^2 + |\widehat{g}|^2}{2|g||\widehat{g}|} - \frac{|g - \widehat{g}|^2}{2|g||\widehat{g}|} && (24)\\ &\geq 1 - \frac{|g - \widehat{g}|^2}{2|g||\widehat{g}|} && (25) \end{aligned}$$

in (21) gives us

$$\cosh' d(h, \widehat{h}) \leq \cosh'(|g| - |\widehat{g}|) + \sinh' |g| \sinh' |\widehat{g}| \frac{|g - \widehat{g}|^2}{2|g||\widehat{g}|}. \tag{26}$$

The taylor expansion of $\cosh'$ contains only positive terms, so the LHS of (26) can be lower bounded by truncating its series. Since $\cosh'$ is increasing on $\mathbb{R}^+$ and symmetric, the RHS of (26) can be upper bounded using the inverse triangle inequality. We have

$$\begin{aligned} 1 + \frac{1}{2} d(h, \widehat{h})^2 |H| &\leq \cosh' |g - \widehat{g}| + \sinh' |g| \sinh' |\widehat{g}| \frac{|g - \widehat{g}|^2}{2|g||\widehat{g}|} && (27)\\ &= 1 + 2\sinh'^2 \frac{|g - \widehat{g}|}{2} + \frac{\sinh' |g| \sinh' |\widehat{g}|}{2|g||\widehat{g}|} |g - \widehat{g}|^2, && (28)\\ d(h, \widehat{h}) &\leq |g - \widehat{g}| \sqrt{4 \frac{\sinh'^2 \frac{|g - \widehat{g}|}{2}}{|H||g - \widehat{g}|^2} + 2 \frac{\sinh' |g| \sinh' |\widehat{g}|}{|H||g||\widehat{g}|}} && (29)\\ &\leq \epsilon \sqrt{4 \frac{\sinh'^2 \frac{\epsilon}{2}}{|H|\epsilon^2} + 2 \frac{\sinh' |g| \sinh' |\widehat{g}|}{|H||g||\widehat{g}|}}. && (30) \end{aligned}$$

□