MMA320 Introduction to Algebraic Geometry

Exercises for Chapter 4

4.1. Every non empty Hausdorff space has Krull dimension 0.

4.2. An affine or projective algebraic set of dimension 0 consists of finitely many points.

4.3. A local ring of dimension 0 consists only of units and nilpotent elements.

4.4. Show that the ring $k[[X_1,\dots,X_n]]$ of *formal* power series over a field k is a local ring.

4.5. Let I be an ideal in a ring R. Show that $\dim I = \dim \sqrt{I}$.

4.6. Let $R = k[X,Y,Z]/(XZ,YZ)$. What is the dimension of R? Show that R contains maximal chains of prime ideals of different lengths. Let $P=(0,0,1)$. What is the dimension of the local ring $R_{\mathfrak{m}_P}$, where $\mathfrak{m}_P$ is the maximal ideal of P?

4.7. Let $R \subset S$ be a ring extension. Suppose that the set $S \setminus R$ is closed under multiplication. Show that R is integrally closed in S.

4.8. Let $R \subset S$ be an integral extension. If $x \in R$ is a unit in S, then x is a unit in R.

4.9. Find the normalisation $\widetilde{R}$ of the ring $R = k[X,Y,Z]/(X^2 - Y^2Z)$. The inclusion $R \subset \widetilde{R}$ induces a map from an affine variety to $V(X^2 - Y^2Z)$. Is it surjective? In the case $k = \mathbb{C}$, what can one say about the real points?

4.10. Find the normalisation $\widetilde{R}$ of the ring $R = k[X,Y]/(Y^2 - X^{2m+1})$.

4.11. Let $R \subset S$ be a ring extension and let T be the integral closure of R in S. Show that T is integrally closed in S.

4.12. Show that $k[Z] \subset k[X,Y,Z]/(X^2,Y^2,XYZ)$ is a Noether normalisation.

4.13. Let I be the ideal in $k[X_1,X_2,X_3,X_4]$ generated by the maximal minors of the matrix

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \end{pmatrix}.$$

Find a Noether normalisation $k[Y_1,Y_2,Y_3,Y_4] \subset k[X_1,X_2,X_3,X_4]$ such that $I \cap k[Y_1,Y_2,Y_3,Y_4] = (Y_1,\dots,Y_h)$ for suitable h.