❦ Home assignment 2 — Symmetry ❦

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1

A non-linear realisation. Consider the quotient space $SL(\mathbb{R}^2)/U(1)$, defined as equivalence classes of elements in $SL(2,\mathbb{R})$ modulo the right action of a $U(1)$. Two elements g and g' in $SL(2,\mathbb{R})$ (2×2 real matrices with unit determinant) are considered equivalent if they are related by a $U(1)$ transformation as $g' = gh$, where

$$h = \mathrm{e}^{\theta j}, \qquad j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Show that almost all elements in $SL(2,\mathbb{R})$ are in the same equivalence class as an element of the form

$$g = \frac{1}{\sqrt{y}} \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}. \tag{1}$$

Use this parametrisation to derive the transformation of the complex number $z = x + iy$ for such a representative of the equivalence class under the left action $g \mapsto Mg$ with

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R}), \tag{2}$$

$$ad - bc = 1. \tag{3}$$

Show that the metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(\mathrm{Im}\, z)^2} \tag{4}$$

is invariant under $SL(2,\mathbb{R})$. (This is the so called Poincaré upper half plane, describing a 2-dimensional hyperbolic space with constant curvature.) Discuss what the $SL(2,\mathbb{R})$ isometry means. Is this a maximally symmetric space? Also, examine what the the word ”almost” above means.

Solution

Calculations with Mathematica are presented in Appendix A.

Set

$$
\begin{aligned}
\theta &= \arctan\left[\frac{d}{\sqrt{c^2+d^2}}, \frac{c}{\sqrt{c^2+d^2}}\right] \\
x &= \frac{ac+bd}{c^2+d^2} && (5) \\
y &= \frac{1}{c^2+d^2}. && (6)
\end{aligned}
$$

This is allowed since either c or d must be nonzero. Then

$$
M\mathrm{e}^{\theta j} = \begin{bmatrix} \frac{1}{\sqrt{c^2+d^2}} & \frac{ac+bd}{\sqrt{c^2+d^2}} \\ 0 & \sqrt{c^2+d^2} \end{bmatrix} = \frac{1}{\sqrt{y}} \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}.
$$

Thus we have shown that almost all elements in $SL(2,\mathbb{R})$ are in the same equivalence class as (1).

Define

$$
g' = \frac{1}{\sqrt{y'}} \begin{bmatrix} y' & x' \\ 0 & 1 \end{bmatrix}
$$

by $g' \sim_{U(1)} Mg$ with M as in (2). Since

$$
Mg = \frac{1}{\sqrt{y}} \begin{bmatrix} ay & b+ax \\ cy & d+cx \end{bmatrix}
$$

we have to use (5) and (6) to transform Mg to the form (1), from which we obtain

$$
\begin{aligned}
x' &= \frac{(Mg)_{11}(Mg)_{21} + (Mg)_{12}(Mg)_{22}}{(Mg)_{21}^2 + (Mg)_{22}^2} = \frac{(ax+b)(cx+d) + acy^2}{c^2y^2 + (cx+d)^2} \\
y' &= \frac{1}{(Mg)_{21}^2 + (Mg)_{22}^2} = \frac{y}{c^2y^2 + (cx+d)^2}.
\end{aligned}
$$

In order to evaluate $\mathrm{d}s'^2$, we can use that

$$
\begin{aligned}
\mathrm{d}x' &= \frac{\partial x'}{\partial x}\mathrm{d}x + \frac{\partial x'}{\partial y}\mathrm{d}y \\
\mathrm{d}y' &= \frac{\partial y'}{\partial x}\mathrm{d}x + \frac{\partial y'}{\partial y}\mathrm{d}y
\end{aligned}
$$

by the chain rule. Hence

$$
\mathrm{d}s'^2 = \frac{\mathrm{d}x'^2 + \mathrm{d}y'^2}{y'^2} =
$$

$$
\begin{aligned}
&\Big[a^2d^2\left(-c^2\mathrm{d}x\,y^2+\mathrm{d}x\,(cx+d)^2+2c\mathrm{d}y\,y(cx+d)\right)^2-\\
&2abcd\left(-c^2\mathrm{d}x\,y^2+\mathrm{d}x\,(cx+d)^2+2c\mathrm{d}y\,y(cx+d)\right)^2+\\
&b^2c^2\left(-c^2\mathrm{d}x\,y^2+\mathrm{d}x\,(cx+d)^2+2c\mathrm{d}y\,y(cx+d)\right)^2+\\
&\left(c^2\mathrm{d}y\,y^2+2c\mathrm{d}x\,y(cx+d)-\mathrm{d}y\,(cx+d)^2\right)^2\Big]\Big/\left(c^2y^3+y(cx+d)^2\right)^2
\end{aligned}
$$

which luckily simplifies to

$$\frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}=\mathrm{d}s^2$$

by (3). The metric (4) is thus invariant under $SL(2,\mathbb{R})$.

We have now shown that there is an isometry between $\mathbb{C}$ and the Poincaré upper half plane.

2

Consider the rank 2 simple Lie algebra $C_2 \sim \mathfrak{sp}(4)$. This algebra has a 4-dimensional and a 5-dimensional representation. Describe these in a tensor language. Also, construct them as highest weight representations by acting with lowering operators on some highest weight states.

Solution

Any weight of the module V_Λ is of the form $\lambda = \Lambda - \beta$ where β is a non-negative integral combination of simple roots.

In Figure 1a, the roots of $\mathfrak{sp}(4)$ are shown. From here, the Cartan matrix can be read off by identifying $(,)$ with the geometrical scalar product:

$$A=\begin{bmatrix}2\frac{(\alpha_1,\alpha_1)}{(\alpha_1,\alpha_1)} & 2\frac{(\alpha_1,\alpha_2)}{(\alpha_1,\alpha_1)}\\ 2\frac{(\alpha_2,\alpha_1)}{(\alpha_2,\alpha_2)} & 2\frac{(\alpha_2,\alpha_2)}{(\alpha_2,\alpha_2)}\end{bmatrix}=\begin{bmatrix}2 & -2\\ -1 & 2\end{bmatrix}.$$

By the Chevalley-Serre Relations,

$$[h_i,f_j]=-A_{ij}f_j \tag{7}$$
$$[e_i,f_j]=\delta_{ij}h_j \tag{8}$$
$$\mathrm{ad}(f_i)^{1-A_{ij}}(f_j)=0. \tag{9}$$

Consider Figure 1b, in which we start in the highest weight state Λ_1. From this state, we can act by either f_1 or f_2. Since

$$e_1\left|\lambda_1\right\rangle = e_2\left|\lambda_1\right\rangle = 0$$

by Λ_1 being a highest weight state,

$$0 = f_j e_i \left|\Lambda_1\right\rangle \overset{(8)}{=} (e_i f_j - h_i \delta_{ij}) \left|\Lambda_1\right\rangle = (e_i f_j - \delta_{ij}\Lambda_1(\alpha_i^\vee)) \left|\Lambda_1\right\rangle = (e_i f_j - \delta_{ij}\delta_{1i}) \left|\Lambda_1\right\rangle .$$

Thus

$$\begin{aligned}
&e_i f_1 \left|\Lambda_1\right\rangle = \delta_{i1}\delta_{1i} \left|\Lambda_1\right\rangle \text{ for } i = 1, 2 \\
&\quad\Longrightarrow \\
&e_1 f_1 \left|\Lambda_1\right\rangle = \left|\Lambda_1\right\rangle \\
&\quad\Longrightarrow \\
&f_1 \left|\Lambda_1\right\rangle \neq 0.
\end{aligned}$$

Also

$$\begin{aligned}
&e_i f_2 \left|\Lambda_1\right\rangle = \delta_{i2}\delta_{1i} \left|\Lambda_1\right\rangle \text{ for } i = 1, 2 \\
&\quad\Longrightarrow \\
&e_i f_2 \left|\Lambda_1\right\rangle = 0 \\
&\quad\Longrightarrow \\
&\text{either } f_2 \left|\Lambda_1\right\rangle \text{ is a highest weight state, or } f_2 \left|\Lambda_1\right\rangle = 0.
\end{aligned}$$

But $f_2 \left|\Lambda_1\right\rangle$ cannot be a highest weight state since it's below $\left|\Lambda_1\right\rangle$, so $f_2 \left|\Lambda_1\right\rangle = 0$. Hence we can only act with f_1 from the Λ_1-state.

Now consider

$$\begin{aligned}
f_i f_1 e_j \left|\Lambda_1\right\rangle &\overset{(8)}{=} f_i(e_j f_1 - h_1 \delta_{j1}) \left|\Lambda_1\right\rangle \\
&= f_i(e_j f_1 - \delta_{j1}) \left|\Lambda_1\right\rangle \\
&\overset{\substack{(8)\text{ and} \\ f_i\left|\Lambda_1\right\rangle = \delta_{i1} f_1 \left|\Lambda_1\right\rangle}}{=} (e_j f_i - \delta_{ij} h_j f_1 - \delta_{j1}\delta_{i1} f_1) \left|\Lambda_1\right\rangle \\
&\quad \left\{ h_j f_1 \left|\Lambda_1\right\rangle \overset{(7)}{=} (f_1 h_j - A_{j1} f_1) \left|\Lambda_1\right\rangle = (\delta_{j1} - A_{j1}) f_1 \left|\Lambda_1\right\rangle \right\} \\
&= \Big(e_j f_i - \delta_{ij}(\delta_{j_1} - a_{j1}) - \delta_{j1}\delta_{i1}\Big) f_1 \left|\Lambda_1\right\rangle \\
&\Longrightarrow \\
e_j f_i f_1 \left|\Lambda_1\right\rangle &= \Big(\delta_{ij}(\delta_{j1} - A_{j1}) + \delta_{j1}\delta_{i1}\Big) f_1 \left|\Lambda_1\right\rangle .
\end{aligned}$$

Then, similarly as before,

$$\begin{aligned}
e_2 f_2 f_1 \left|\Lambda_1\right\rangle = -A_{21} f_1 \left|\Lambda_1\right\rangle \neq 0 \\
\Longrightarrow \\
f_2 f_1 \left|\Lambda_1\right\rangle \neq 0.
\end{aligned}$$

Also, as before,

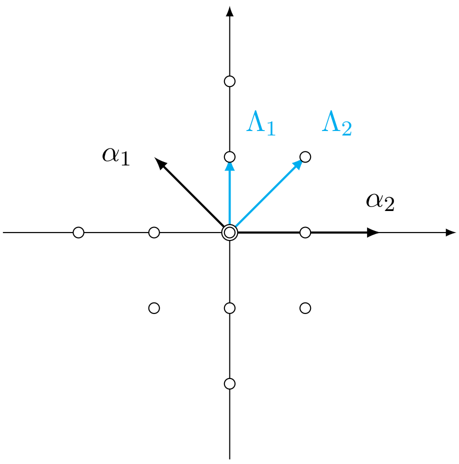
$$
\begin{aligned}
e_j f_1 f_1 \left|\Lambda_1\right\rangle &= \Big(\delta_{1j}(\delta_{j1} - A_{j1}) + \delta_{j1}\delta_{11}\Big) f_1 \left|\Lambda_1\right\rangle \\
&= \begin{cases} (1 - A_{11}) + 1 = (1-2) + 1 = 0 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases} \\
&\quad \{\text{same argument as before about } f_1 f_1 \left|\Lambda_1\right\rangle \text{ not being highest weight}\} \implies \\
f_1 f_1 \left|\Lambda_1\right\rangle &= 0.
\end{aligned}
$$

We have can hence only act with f_2 from the state $f_1 \left|\Lambda_1\right\rangle$. From here, we can use (9) to say that $[f_1, [f_1, f_2]] \neq 0$ (since $\mathrm{ad}(f_i)^{1-A_{ij}} f_j = \mathrm{ad}(f_i)^3 f_j = 0$) and the Jacobi identity to say that $f_1 f_2 f_1 \left|\Lambda_1\right\rangle \neq 0$ (it is the only surviving term in $[f_1, [f_1, f_2]]$):

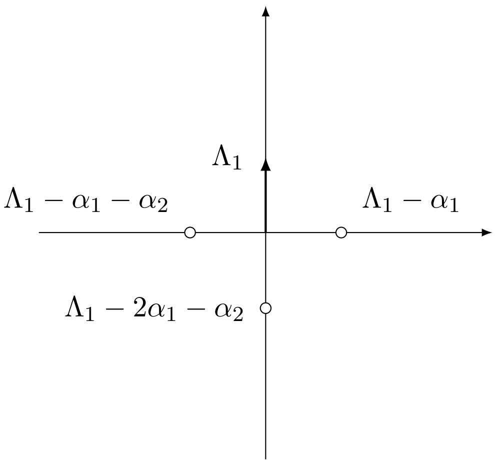
$$
[f_1, [f_2, f_1]] + \underbrace{[f_1, [f_1, f_2]]}_{\neq 0} + [f_2, \underbrace{[f_1, f_1]}_{=0}] = 0.
$$

From here there are no other options though.

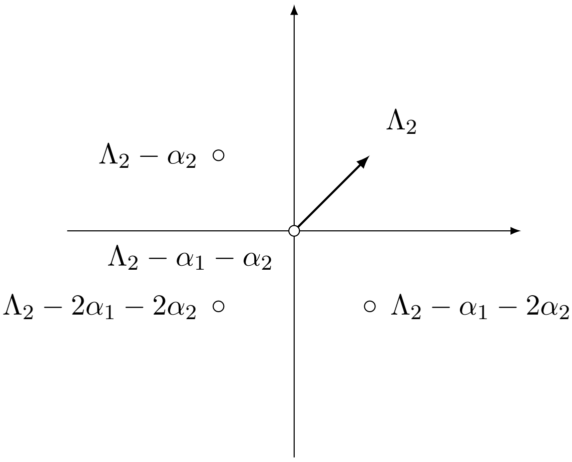
Continuing in this manner yields Figures 1b and 1c which are the 4- and 5-dimensional representations we are after.



(a) $\mathfrak{sp}(4)$ *fundamental roots and weights.*



(b) *Representation generated by* Λ_1.



(c) *Representation generated by* Λ_2.

Figure 1

3

Consider an "inversion" of space (or space-time), defined by

$$x^m \mapsto x'^m = \frac{x^m}{x^2}.$$

Show that a special conformal transformation, on the infinitesimal form given in the lecture notes, is obtained by first performing an inversion, then a translation, and finally an inversion.

Solution

By (3.6) in the lecture notes,

$$\delta_v x^m = (x \cdot v) x^m - \frac{1}{2} x^2 v^m. \tag{3.6}$$

is a special conformal transformation.

Consider such a mapping described: an inversion followed by a translation followed by an inversion,

$$x^m \mapsto \frac{\frac{x^m}{x^2}+a^m}{\left(\frac{x^m}{x^2}+a^m\right)^2} = \frac{x^m+x^2a^m}{\frac{1}{x^2}\left(x^s+x^2a^s\right)\left(x_s+x^2a_s\right)} = \frac{x^m+x^2a^m}{\frac{1}{x^2}\left(x^sx_s+x^2a^sx_s+x^sx^2a_s+x^4a^2\right)}$$
$$= \frac{x^m+x^2a^m}{\frac{1}{x^2}\left(x^2+2x^2(a\cdot x)+x^4a^2\right)} = \frac{x^m+x^2a^m}{(1+2(a\cdot x)+x^2a^2)}$$

Then Taylor expanding $\frac{1}{1-x} = 1+x+\mathcal{O}(x^2)$, we get

$$\begin{aligned}\frac{x^m+x^2a^m}{(1+2(a\cdot x)+x^2a^2)} &= \left(x^m+x^2a^m\right)\left(1-2(a\cdot x)+\mathcal{O}(a^2)\right)\\ &= x^m-2(a\cdot x)x^m+x^2a^m.\end{aligned}$$

Thus

$$\delta_a x^m = -2(a\cdot x)x^m + x^2a^m$$

which is on the same form as (3.6).

4

The Lie algebra F_4. There is a simple 52-dimensional Lie algebra F_4, with Dynkin diagram in the picture in the lecture notes. It has a subalgebra $\mathfrak{so}(9)$. The adjoint representation of F_4 becomes, seen as an $\mathfrak{so}(9)$ representation, the direct sum of the (36-dimensional) adjoint representation and a (16-dimensional) spinor representation. Construct the F_4 Lie bracket in an $\mathfrak{so}(9)$-covariant way, and check the Jacobi identities. (Information which can possibly be of help, and which may be used: The completely antisymmetric tensor product of 3 $\mathfrak{so}(9)$ spinors does not contain a spinor as one of its irreducible parts.)

Solution

$\mathfrak{so}(n)$ has a basis $\{J_{ij} = E_{[ij]}\}$ (where E_{ij} is a matrix with a 1 on row i column j and zeros everywhere else) with commutation relations

$$[J_{ij}, J_{kl}] = -4\delta_{[k[i}J_{j]l]}$$

(the -4 comes from the convention that symmetrizing has a factor $\frac{1}{2}$ built in). We can make some qualified guesses for the rest of the commutation relations based on the fact that we only may use invariant tensors and J_{ij} is antisymmetric is i and j.

$$\begin{aligned}[][J_{ij}, S_k] &= \alpha\delta_{k[i}S_{j]}\\ [S_i, S_j] &= \beta J_{ij}\end{aligned}$$

where α and β are scalar coefficients.

The Jacobi identities we need to check are

$$[J_{ij},[J_{kl},J_{mn}]]+[J_{kl},[J_{mn},J_{ij}]]+[J_{kl},[J_{mn},J_{ij}]]=0 \tag{10}$$
$$[J_{ij},[J_{kl},S_m]]+[J_{kl},[S_m,J_{ij}]]+[S_m,[J_{ij},J_{kl}]]=0 \tag{11}$$
$$[J_{ij},[S_k,S_l]]+[S_l,[S_k,J_{ij}]]+[S_k,[J_{ij},S_l]]=0 \tag{12}$$
$$[S_i,[S_j,S_k]]+[S_j,[S_k,S_i]]+[S_k,[S_i,S_j]]=0. \tag{13}$$

(10) is trivially satisfied since we have a matrix basis for $\mathfrak{so}(9)$. (13) is also fairly easy to show by expressing the LHS as $[S_{[i},[S_j,S_{k]}]]$,

$$[S_{[i},[S_j,S_{k]}]]=[S_{[i},\beta J_{jk]}]=-\alpha\beta\delta_{[i[j}S_{k]]}$$

which $=0$ since δ is symmetric, so antisymmetrizing it yields 0. (11) can be shown by

$$\begin{aligned}
&[J_{ij},[J_{kl},S_m]]+[J_{kl},[S_m,J_{ij}]]+[S_m,[J_{ij},J_{kl}]]\\
&=[J_{ij},\alpha\delta_{m[k}S_{l]}]+[J_{kl},-\alpha\delta_{m[i}S_{j]}]+[S_m,-4\delta_{[k[i}J_{j]l]}]\\
&=\alpha\delta_{m[k}[J_{ij},S_{l]}]-\alpha\delta_{m[i}[J_{kl},S_{j]}]-4\delta_{[k[i}[S_m,J_{j]l]}]\\
&=\frac{\alpha}{2}\Big(\delta_{mk}[J_{ij},S_l]-\delta_{ml}[J_{ij},S_k]\Big)-\frac{\alpha}{2}\Big(\delta_{mi}[J_{kl},S_j]-\delta_{mj}[J_{kl},S_i]\Big)\\
&\quad-\delta_{ki}[S_m,J_{jl}]+\delta_{li}[S_m,J_{jk}]+\delta_{kj}[S_m,J_{il}]-\delta_{lj}[S_m,J_{ik}]\\
&=\frac{\alpha^2}{2}\Big(\delta_{mk}\delta_{l[i}S_{j]}-\delta_{ml}\delta_{k[i}S_{j]}\Big)-\frac{\alpha^2}{2}\Big(\delta_{mi}\delta_{j[k}S_{l]}-\delta_{mj}\delta_{i[k}S_{l]}\Big)\\
&\quad+\alpha\delta_{ki}\delta_{m[j}S_{l]}-\alpha\delta_{li}\delta_{m[j}S_{k]}-\alpha\delta_{kj}\delta_{m[i}S_{l]}+\alpha\delta_{lj}\delta_{m[i}S_{k]}\\
&=\frac{\alpha^2}{4}\Big(\delta_{mk}(\delta_{li}S_j-\delta_{lj}S_i)-\delta_{ml}(\delta_{ki}S_j-\delta_{kj}S_i)\Big)-\frac{\alpha^2}{4}\Big(\delta_{mi}(\delta_{jk}S_l-\delta_{jl}S_k)-\delta_{mj}(\delta_{ik}S_l-\delta_{il}S_k)\Big)\\
&\quad+\frac{\alpha}{2}\delta_{ki}(\delta_{mj}S_l-\delta_{ml}S_j)-\frac{\alpha}{2}\delta_{li}(\delta_{mj}S_k-\delta_{mk}S_j)-\frac{\alpha}{2}\delta_{kj}(\delta_{mi}S_l-\delta_{ml}S_i)+\frac{\alpha}{2}\delta_{lj}(\delta_{mi}S_k-\delta_{mk}S_i)\\
&=\frac{\alpha}{2}\left(-\frac{\alpha}{2}\delta_{mk}\delta_{lj}+\frac{\alpha}{2}\delta_{ml}\delta_{kj}+\delta_{kj}\delta_{ml}-\delta_{lj}\delta_{mk}\right)S_i\\
&\quad+\frac{\alpha}{2}\left(\frac{\alpha}{2}\delta_{mk}\delta_{li}-\frac{\alpha}{2}\delta_{ml}\delta_{ki}+\delta_{ki}\delta_{ml}+\delta_{li}\delta_{mk}\right)S_j\\
&\quad+\frac{\alpha}{2}\left(\frac{\alpha}{2}\delta_{mi}\delta_{jl}-\frac{\alpha}{2}\delta_{mj}\delta_{il}-\delta_{li}\delta_{mj}+\delta_{lj}\delta_{mi}\right)S_k.
\end{aligned}$$

This is indeed 0 if we chose $\alpha=-2$. Finally, (12) can be shown by

$$\begin{aligned}
&[J_{ij},[S_l,S_k]]+[S_l,[S_k,J_{ij}]]+[S_k,[J_{ij},S_l]]\\
&=\beta[J_{ij},J_{lk}]-\alpha[S_l,\delta_{k[i}S_{j]}]+\alpha[S_k,\delta_{l[i}S_{j]}]\\
&=\beta[J_{ij},J_{lk}]-\frac{\alpha}{2}\Big(\delta_{ki}[S_l,S_j]-\delta_{kj}[S_l,S_i]\Big)+\frac{\alpha}{2}\Big(\delta_{li}[S_k,S_j]-\delta_{lj}[S_k,S_i]\Big)\\
&=-\beta4\delta_{[l[i}J_{j]k]}-\frac{\alpha}{2}\Big(\delta_{ki}\beta J_{lj}-\delta_{kj}\beta J_{li}\Big)+\frac{\alpha}{2}\Big(\delta_{li}\beta J_{kj}-\delta_{lj}\beta J_{ki}\Big)\\
&\overset{\alpha=-2}{=}\beta\left[-\delta_{li}J_{jk}+\delta_{ki}J_{jl}+\delta_{lj}J_{ik}-\delta_{kj}J_{il}+\Big(\delta_{ki}J_{lj}-\delta_{kj}J_{li}\Big)-\Big(\delta_{li}J_{kj}-\delta_{lj}J_{ki}\Big)\right]\\
&\overset{J_{ij}=-J_{ji}}{=}0.
\end{aligned}$$

This places no condition on β, but it is nice to see that it all cancels ut anyways.

To summarize, the Lie bracket we choose is

$$
\begin{aligned}
[J_{ij}, J_{kl}] &= -4\delta_{[k[i}J_{j]l]} \\
[J_{ij}, S_k] &= -2\delta_{k[i}S_{j]} \\
[S_i, S_j] &= \beta J_{ij}.
\end{aligned}
$$

5

The energy-momentum (or stress-energy) tensor can be derived as the variation of an action with respect to the metric as

$$T^{mn} = 2\frac{\partial \mathscr{L}}{\partial g_{mn}},$$

where $\mathscr{L}$ is the Lagrangian density. This applies also for a theory defined in flat space, but then the metric has to be reinstated so that coordinate invariance of the action is manifest. Use this definition to derive the energy-momentum tensor for Maxwell theory, and identify the usual forms of the energy density as T^{00} and the Poynting vector as T^{0i}. Show that the Maxwell energy-momentum tensor is traceless precisely in $d = 4$, and relate this property to the invariance of the action under a rescaling of the metric. This is a sign of conformal invariance.

Solution

The Lagrangian density is

$$\mathscr{L} = -\frac{1}{4}\sqrt{||g||}g^{mp}g^{nq}F_{mn}F_{pq}, \tag{14}$$

where $|g|$ denotes the determinant of the metric $g : V \to V^* \sim V$, and $||g||$ the absolute value of the determinant. When differentiating (14) we will need to know how to differentiate each of the factors.

Firstly, Jacobi's formula says that

$$\frac{\partial |g|}{\partial g_{mn}} = |g| \operatorname{Tr}\left\{(g^{-1})^{pq}\frac{\partial g_{qr}}{\partial g_{mn}}\right\} = |g|g^{pq}\frac{\partial g_{qp}}{\partial g_{mn}} = |g|g^{pq}\delta_{q\,p}^{\,m\,n} = |g|g^{mn}. \tag{15}$$

Secondly, $\frac{\partial g^{pq}}{\partial g_{mn}}$ can be evaluated by

$$\begin{aligned}
\frac{\partial g^{pq}}{\partial g_{mn}} &= \frac{\partial}{\partial g_{mn}} \left(g^{pr}\delta_r^{\,q}\right) \\
&= \frac{\partial}{\partial g_{mn}} \left(g^{pr} g_{rs} g^{sq}\right) \\
&= \frac{\partial g^{pr}}{\partial g_{mn}} \underbrace{g_{rs} g^{sq}}_{=\delta_r^q} + g^{pr} \underbrace{\frac{\partial g_{rs}}{\partial g_{mn}}}_{=\delta_{r\,s}^{\,m\,n}} g^{sq} + \underbrace{g^{pr} g_{rs}}_{=\delta_s^p} \frac{\partial g^{sq}}{\partial g_{mn}} \\
&= \frac{\partial g^{pq}}{\partial g_{mn}} + g^{pm} g^{nq} + \frac{\partial g^{pq}}{\partial g_{mn}} \\
\Longrightarrow & \\
\frac{\partial g^{pq}}{\partial g_{mn}} &= -g^{pm} g^{nq}. \qquad (16)
\end{aligned}$$

Thirdly, for convenience, we state

$$\frac{\partial\sqrt{||g||}}{\partial g_{mn}} = \frac{1}{2\sqrt{||g||}} \frac{\partial\,\mathrm{sgn}(|g|)|g|}{\partial g_{mn}} \overset{(15)}{=} \frac{1}{2\sqrt{||g||}}\,\mathrm{sgn}(|g|)|g| g^{mn} = \frac{\sqrt{||g||}}{2} g^{mn}. \qquad (17)$$

By (15), (16) and (17),

$$\begin{aligned}
T^{\mu\nu} &= -\frac{1}{2}\frac{\partial}{\partial g_{\mu\nu}} \left(\sqrt{||g||} g^{mp} g^{nq} F_{mn} F_{pq}\right) \\
&= -\frac{1}{2}\left(\frac{\partial\sqrt{||g||}}{\partial g_{\mu\nu}} g^{mp} g^{nq} F_{mn} F_{pq} + \sqrt{||g||}\frac{\partial g^{mp}}{\partial g_{\mu\nu}} g^{nq} F_{mn} F_{pq} + \sqrt{||g||} g^{mp} \frac{\partial g^{nq}}{\partial g_{\mu\nu}} F_{mn} F_{pq}\right) \\
&= -\frac{1}{2}\left(\frac{\sqrt{||g||}}{2} g^{\mu\nu} g^{mp} g^{nq} F_{mn} F_{pq} - \sqrt{||g||} g^{m\mu} g^{\nu q} g^{nq} F_{mn} F_{pq} - \sqrt{||g||} g^{mp} g^{n\mu} g^{\nu q} F_{mn} F_{pq}\right) \\
&= -\frac{1}{2}\left(\frac{\sqrt{||g||}}{2} g^{\mu\nu} g^{mp} g^{nq} F_{mn} F_{pq} - \sqrt{||g||} g^{m\mu} g^{\nu q} g^{nq} F_{mn} F_{pq} - \sqrt{||g||} g^{mp} g^{n\mu} g^{\nu q} F_{mn} F_{pq}\right) \\
&= -\frac{\sqrt{||g||}}{2}\left(\frac{1}{2} g^{\mu\nu} F^{pq} F_{pq} - 2F^{p\mu} F_p^{\ \nu}\right) \\
&= \sqrt{||g||}\left(F^{p\mu} F_p^{\ \nu} - \frac{1}{4} g^{\mu\nu} F^{pq} F_{pq}\right).
\end{aligned}$$

6

Symmetries of the Kepler problem. Consider the motion of a Newtonian particle with mass m in the central potential $V(\vec{r}) = -\frac{k}{r}$. Show that the components of the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ fulfil $\{L_i, H\} = 0$, and are conserved charges. Which is the Lie algebra generated by these charges? Consider the Runge-Lenz vector

$$\vec{A} = \vec{p} \times \vec{L} - km\hat{r}.$$

The dimensionless vector $\frac{\vec{A}}{km}$ is the so called eccentricity vector. Show that $\vec{A}$ is conserved. It is convenient to rescale the Runge-Lenz vector to

$$\vec{B} = \frac{\vec{A}}{\sqrt{2m|E|}},$$

where E is the energy, for $E \neq 0$. Investigate the algebra of conserved charges under the Poisson bracket. It may be different in the cases $E < 0$, $E = 0$, and $E > 0$. Such "hidden symmetries" may be used to relate solutions to the equations of motion with the same energy to each other.

Solution

I was unsure of the aptness of upper and lower indices in this problem, so the rule is just that indices appearing twice in the same term are summed over.

From the Lagrangian

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{r}_i\dot{r}_i - V(\vec{r})$$

we obtain the conjugate momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{r}_i} = m\dot{r}_i.$$

Then the Hamiltonian

$$\mathcal{H} = \frac{p_i p_i}{2m} + V(\vec{r}).$$

We also have that

$$L_i = \epsilon_{ijk} r_j p_k.$$

Evaluating the Poisson bracket, we get

$$\{L_i, \mathcal{H}\} = \frac{\partial L_i}{\partial r_l}\frac{\partial \mathcal{H}}{\partial p_l} - \frac{\partial \mathcal{H}}{\partial r_l}\frac{\partial L_i}{\partial p_l} = \epsilon_{ijk}\delta_{lj}p_k\frac{p_l}{m} + \frac{r_l k}{r^3}\epsilon_{ijk}r_j\delta_{lk} = \epsilon_{ilk}p_k\frac{p_l}{m} + \frac{r_l k}{r^3}\epsilon_{ijl}r_j, \tag{18}$$

but the first term is both symmetric and antisymmetric in k and l, while the second term is symmetric and antisymmetric in l and j, they are thus both 0. Hence $\dot{L}_i = \{L_i, \mathcal{H}\} = 0$.

We now want to calculate the structure constants for the Lie algebra spanned by $\{L_i\}$. We do this by identifying the Lie bracket with the Poisson bracket:

$$\begin{aligned}\{L_i, L_m\} &= \frac{\partial L_i}{\partial r_l}\frac{\partial L_m}{\partial p_l} - \frac{\partial L_m}{\partial r_l}\frac{\partial L_i}{\partial p_l} = \epsilon_{ijk}\delta_{lj}p_k\epsilon_{mj'k'}r_{j'}\delta_{lk'} - \epsilon_{mjk}\delta_{lj}p_k\epsilon_{ij'k'}r_{j'}\delta_{lk'} \\ &= \epsilon_{ilk}p_k\epsilon_{mj'l}r_{j'} - \epsilon_{mlk}p_k\epsilon_{ij'l}r_{j'} = -r_{j'}p_k(\delta_{im}\delta_{kj'} - \delta_{ij'}\delta_{km}) + r_{j'}p_k(\delta_{mi}\delta_{kj'} - \delta_{ki}\delta_{mj'}) \\ &= r_i p_m - r_m p_i = \epsilon_{qim}L_q. \end{aligned} \tag{19}$$

These are the same constants as $\mathfrak{so}(3)$ (with basis $\{\tau_i\}$). It is very nice to see that the Lie algebra associated with angular momentum is the algebra to the group of rotations of ordinary 3D space.

A similar calculation as in (18) yields

$$\dot{A} = \{A, H\} = 0.$$

A similar calculation as in (19) yields

$$\{B_i, B_m\} = -\epsilon_{qim} L_q \operatorname{sgn} E \tag{20}$$
$$\{B_i, L_m\} = \epsilon_{qim} B_q. \tag{21}$$

See Appendix B for these calculations.

Using

$$\sqrt{\operatorname{sgn} E} = \begin{cases} i & \text{if } E < 0 \\ 1 & \text{if } E > 0 \end{cases},$$

we can perform a basis change

$$\begin{aligned}
e_1 &= L_1 + \sqrt{\operatorname{sgn} E} i B_1 \\
e_2 &= L_2 + \sqrt{\operatorname{sgn} E} i B_2 \\
e_3 &= L_3 + \sqrt{\operatorname{sgn} E} i B_3 \\
f_1 &= L_1 - \sqrt{\operatorname{sgn} E} i B_1 \\
f_2 &= L_2 - \sqrt{\operatorname{sgn} E} i B_2 \\
f_3 &= L_3 - \sqrt{\operatorname{sgn} E} i B_3.
\end{aligned}$$

Using (19), (20) and (21) we get the commutation relations

$$\begin{aligned}
\{e_i, e_m\} &= \{L_i + \sqrt{\operatorname{sgn} E} i B_i, L_i + \sqrt{\operatorname{sgn} E} i B_i\} \\
&= \{L_i, L_m\} + \sqrt{\operatorname{sgn} E} i \{L_i, B_m\} + \sqrt{\operatorname{sgn} E} i \{B_i, L_m\} - \operatorname{sgn} E \{B_i, B_m\} \\
&= \epsilon_{qim} \left(L_q + \sqrt{\operatorname{sgn} E} i B_q + \sqrt{\operatorname{sgn} E} i B_q + (\operatorname{sgn} E)^2 L_q\right) \\
&= 2\epsilon_{qim} e_i \\
\{f_i, f_m\} &= \{L_i - \sqrt{\operatorname{sgn} E} i B_i, L_i - \sqrt{\operatorname{sgn} E} i B_i\} \\
&= \{L_i, L_m\} - \sqrt{\operatorname{sgn} E} i \{L_i, B_m\} - \sqrt{\operatorname{sgn} E} i \{B_i, L_m\} - \operatorname{sgn} E \{B_i, B_m\} \\
&= \epsilon_{qim} \left(L_q - \sqrt{\operatorname{sgn} E} i B_q - \sqrt{\operatorname{sgn} E} i B_q + (\operatorname{sgn} E)^2 L_q\right) \\
&= 2\epsilon_{qim} f_i \\
\{e_i, f_m\} &= \{L_i + \sqrt{\operatorname{sgn} E} i B_i, L_i - \sqrt{\operatorname{sgn} E} i B_i\} \\
&= \epsilon_{qim} \left(L_q + \sqrt{\operatorname{sgn} E} i B_q - \sqrt{\operatorname{sgn} E} i B_q - (\operatorname{sgn} E)^2 L_q\right) \\
&= 0.
\end{aligned}$$

for our new basis. This is clearly $\mathfrak{so}(3) \times \mathfrak{so}(3) \sim \mathfrak{so}(4)$.

A Mathematica code for problem 1

```
ClearAll["Global`*"]
$Assumptions = (θ ∈ Reals);
```

(*Find θ, x, and y s.th $Me^{\theta j} = \frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$*)

$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

Print["$e^{\theta j}$ = ", MatrixForm [MatrixExp[θj]]]

$e^{\theta j} = \begin{pmatrix} \text{Cos}[\theta] & \text{Sin}[\theta] \\ -\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}$

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$;

Solve[(M.MatrixExp[θj])[[2, 1]] == 0, θ] (*We want $Me^{\theta j}$ to be of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$*)

$\left\{\left\{\theta \to \text{ConditionalExpression}\left[\text{ArcTan}\left[-\frac{d}{\sqrt{c^2+d^2}}, -\frac{c}{\sqrt{c^2+d^2}}\right] + 2\pi C[1], C[1] \in \text{Integers}\right]\right\},\right.$
$\left.\left\{\theta \to \text{ConditionalExpression}\left[\text{ArcTan}\left[\frac{d}{\sqrt{c^2+d^2}}, \frac{c}{\sqrt{c^2+d^2}}\right] + 2\pi C[1], C[1] \in \text{Integers}\right]\right\}\right\}$

$\theta = \text{ArcTan}\left[\frac{d}{\sqrt{c^2+d^2}}, \frac{c}{\sqrt{c^2+d^2}}\right]$;

$x = \frac{a c + b d}{c^2 + d^2}$;

$y = \frac{1}{c^2 + d^2}$;

Print["$Me^{\theta j}$ = ", MatrixForm [FullSimplify [M.MatrixExp[θj], a d - b c == 1]]]

Print["$\frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ = ", MatrixForm [FullSimplify [$\frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$, a d - b c == 1]]]

$Me^{\theta j} = \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & \frac{a c + b d}{\sqrt{c^2+d^2}} \\ 0 & \sqrt{c^2+d^2} \end{pmatrix}$

$\frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{c^2+d^2}} & (a c + b d)\sqrt{\frac{1}{c^2+d^2}} \\ 0 & \frac{1}{\sqrt{\frac{1}{c^2+d^2}}} \end{pmatrix}$

```
(*Find the action of M on z=x+iy*)
Clear[x, y]
```

$$g = \frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix};$$

$$\text{Print}\left[\text{"}\frac{1}{\sqrt{y}}\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \mapsto \text{ ", MatrixForm [M.g]}\right]$$

$$\frac{1}{\sqrt{y}}\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \mapsto \begin{pmatrix} a\sqrt{y} & \frac{b}{\sqrt{y}} + \frac{a\,x}{\sqrt{y}} \\ c\sqrt{y} & \frac{d}{\sqrt{y}} + \frac{c\,x}{\sqrt{y}} \end{pmatrix}$$

```
xp = ((M.g) [[1, 1]] (M.g) [[2, 1]] + (M.g) [[1, 2]] (M.g) [[2, 2]]) /
      ((M.g) [[2, 1]]^2 + (M.g) [[2, 2]]^2); (*x↦xp*)
yp = 1 / ((M.g) [[2, 1]]^2 + (M.g) [[2, 2]]^2); (*y↦yp*)
Print["x' = ", FullSimplify [xp]]
Print["y' = ", FullSimplify [yp]]
```

$$x' = \frac{(b + a\,x)\,(d + c\,x) + a\,c\,y^2}{(d + c\,x)^2 + c^2\,y^2}$$

$$y' = \frac{y}{(d + c\,x)^2 + c^2\,y^2}$$

```
dxp = D[xp, x] dx + D[xp, y] dy; (*Chain rule*)
dyp = D[yp, x] dx + D[yp, y] dy;
```

$$ds2 = \frac{dx^2 + dy^2}{y^2};$$

$$dsp2 = \frac{dxp^2 + dyp^2}{yp^2};$$

```
Print["ds^2 = ", ds2]
Print["ds'^2 = ", FullSimplify [dsp2, a d - b c == 1]]
```

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

$$ds'^2 = \frac{dx^2 + dy^2}{y^2}$$

B Mathematica code for problem 6

In[1]:=
```
ClearAll["Global`*"]
$Assumptions = (M ∈ Reals && M > 0 && (*mass *)
                    e ∈ Reals && (*energy*)
                    x ∈ Reals && y ∈ Reals && z ∈ Reals &&
                    px ∈ Reals && py ∈ Reals && pz ∈ Reals);

(*Calculate {L,H}*)
rvec = {x, y, z};
pvec = {px, py, pz};
Lvec = Cross[rvec, pvec];
V[vec_] := -k/(Norm [vec]);
H = Dot[pvec, pvec]/(2 M) + V[rvec];
PoissonBraket[A_, B_] := D[A, x] D[B, px] - D[B, x] D[A, px] +
                    D[A, y] D[B, py] - D[B, y] D[A, py] +
                    D[A, z] D[B, pz] - D[B, z] D[A, pz];
```

In[9]:=
```
Print["{L,H} = ", FullSimplify [PoissonBraket[Lvec, H]]]
```

```
{L,H} = {0, 0, 0}
```

In[10]:=
```
(*Check {L_i,L_m }*)
ϵ = LeviCivitaTensor[3];
δ[i_, j_] := KroneckerDelta[i, j];
poissonBraketLiLm = IdentityMatrix[3]; (*Initiate*)
poissonBraketLiLm2 = IdentityMatrix[3]; (*Initiate*)

For[i = 1, i ≤ 3, i++,
  For[m = 1, m ≤ 3, m ++,
    poissonBraketLiLm [[i, m ]] = PoissonBraket[Lvec[[i]], Lvec[[m ]]];
    poissonBraketLiLm2 [[i, m ]] = Sum [ϵ[[q, i, m ]] Lvec[[q]], {q, 1, 3}];
    (*su(2) structure constants*)
  ]
]

Print["{L_i,L_m } = ", MatrixForm [poissonBraketLiLm ]]
Print["ϵ_qim  L_q = ", MatrixForm [poissonBraketLiLm2 ]]
```

{L_i,L_m } = $\begin{pmatrix} 0 & py\,x - px\,y & pz\,x - px\,z \\ -py\,x + px\,y & 0 & pz\,y - py\,z \\ -pz\,x + px\,z & -pz\,y + py\,z & 0 \end{pmatrix}$

ϵ_qim L_q = $\begin{pmatrix} 0 & py\,x - px\,y & pz\,x - px\,z \\ -py\,x + px\,y & 0 & pz\,y - py\,z \\ -pz\,x + px\,z & -pz\,y + py\,z & 0 \end{pmatrix}$

In[17]:=
```
(*Check {B_i,B_m }*)
Avec = Cross[pvec, Lvec] - k M rvec / Norm [rvec];
Bvec = Avec / Sqrt[2 M Abs[e]];
Print["{A,H} = ", FullSimplify [PoissonBraket[Avec, H]]]
```

```
{A,H} = {0, 0, 0}
```

In[20]:=
```
poissonBraketBiBm = IdentityMatrix[3]; (*Initiate*)
poissonBraketBiBm2 = IdentityMatrix[3]; (*Initiate*)

For[i = 1, i ≤ 3, i++,
  For[m  = 1, m  ≤ 3, m  ++,
    poissonBraketBiBm [[i, m  ]] = PoissonBraket[Bvec[[i]], Bvec[[m ]]];
    poissonBraketBiBm2 [[i, m  ]] = Sum [ϵ[[q, i, m  ]] Lvec[[q]], {q, 1, 3}] Sign[e];
  ]
]

Print["{B_i,B_m } = ", MatrixForm [FullSimplify [poissonBraketBiBm e/H]]]
(*multiply  by 1=e/H to help FullSimplify *)
Print["ϵ_qim  L_q Sgn[e] = ", MatrixForm [FullSimplify [poissonBraketBiBm2 ]]]
```

$$\{B\_i,B\_m\} = \begin{pmatrix} 0 & (-py\,x+px\,y)\,\mathrm{Sign}[e] & (-pz\,x+px\,z)\,\mathrm{Sign}[e] \\ (py\,x-px\,y)\,\mathrm{Sign}[e] & 0 & (-pz\,y+py\,z)\,\mathrm{Sign}[e] \\ (pz\,x-px\,z)\,\mathrm{Sign}[e] & (pz\,y-py\,z)\,\mathrm{Sign}[e] & 0 \end{pmatrix}$$

$$\epsilon\_qim\ \ L\_q\ \mathrm{Sgn}[e] = \begin{pmatrix} 0 & (py\,x-px\,y)\,\mathrm{Sign}[e] & (pz\,x-px\,z)\,\mathrm{Sign}[e] \\ (-py\,x+px\,y)\,\mathrm{Sign}[e] & 0 & (pz\,y-py\,z)\,\mathrm{Sign}[e] \\ (-pz\,x+px\,z)\,\mathrm{Sign}[e] & (-pz\,y+py\,z)\,\mathrm{Sign}[e] & 0 \end{pmatrix}$$

In[25]:=
```
(*Check {B_i,L_m }*)
poissonBraketBiLm = IdentityMatrix[3]; (*Initiate*)
poissonBraketBiLm2 = IdentityMatrix[3]; (*Initiate*)

For[i = 1, i ≤ 3, i++,
  For[m  = 1, m  ≤ 3, m  ++,
    poissonBraketBiLm [[i, m  ]] = PoissonBraket[Bvec[[i]], Lvec[[m ]]];
    poissonBraketBiLm2 [[i, m  ]] = Sum [ϵ[[q, i, m  ]] Bvec[[q]], {q, 1, 3}];
  ]
]

Print["{B_i,L_m } = ", MatrixForm [FullSimplify [poissonBraketBiLm ]]]
Print["ϵ_qim  B_q = ", MatrixForm [FullSimplify [poissonBraketBiLm2 ]]]
```

$$\{\text{B\_i,L\_m }\} = \begin{pmatrix} 0 & \frac{-px\,pz\,x-py\,pz\,y+px^2\,z+py^2\,z-\frac{kMz}{\sqrt{x^2+y^2+z^2}}}{\sqrt{2}\,\sqrt{\frac{eM}{\text{Sign}[e]}}} & \frac{px\,py\,x-px^2\,y-pz^2\,y+py\,pz\,z+\frac{kMy}{\sqrt{x^2+y^2+z^2}}}{\sqrt{2}\,\sqrt{\frac{eM}{\text{Sign}[e]}}} \\ \frac{px\,pz\,x+py\,pz\,y-px^2\,z-py^2\,z+\frac{kMz}{\sqrt{x^2+y^2+z^2}}}{\sqrt{2}\,\sqrt{\frac{eM}{\text{Sign}[e]}}} & 0 & \frac{py^2\,x+pz^2\,x-px\,py\,y-px\,pz\,z-\frac{kMx}{\sqrt{x^2+y^2+z^2}}}{\sqrt{2}\,\sqrt{\frac{eM}{\text{Sign}[e]}}} \\ \frac{-px\,py\,x+px^2\,y+pz^2\,y-py\,pz\,z-\frac{kMy}{\sqrt{x^2+y^2+z^2}}}{\sqrt{2}\,\sqrt{\frac{eM}{\text{Sign}[e]}}} & \frac{-py^2\,x-pz^2\,x+px\,py\,y+px\,pz\,z+\frac{kMx}{\sqrt{x^2+y^2+z^2}}}{\sqrt{2}\,\sqrt{\frac{eM}{\text{Sign}[e]}}} & 0 \end{pmatrix}$$

$$\epsilon\text{\_qim B\_q} = \begin{pmatrix} 0 & \frac{-px\,pz\,x-py\,pz\,y+px^2\,z+py^2\,z-\frac{kMz}{\sqrt{x^2+y^2+z^2}}}{\sqrt{2}\,\sqrt{M}\,\sqrt{\text{Abs}[e]}} & \frac{px\,py\,x-px^2\,y-pz^2\,y+py\,pz\,z+\frac{kMy}{\sqrt{x^2+y^2+z^2}}}{\sqrt{2}\,\sqrt{M}\,\sqrt{\text{Abs}[e]}} \\ \frac{px\,pz\,x+py\,pz\,y-px^2\,z-py^2\,z+\frac{kMz}{\sqrt{x^2+y^2+z^2}}}{\sqrt{2}\,\sqrt{M}\,\sqrt{\text{Abs}[e]}} & 0 & \frac{py^2\,x+pz^2\,x-px\,py\,y-px\,pz\,z-\frac{kMx}{\sqrt{x^2+y^2+z^2}}}{\sqrt{2}\,\sqrt{M}\,\sqrt{\text{Abs}[e]}} \\ \frac{-px\,py\,x+px^2\,y+pz^2\,y-py\,pz\,z-\frac{kMy}{\sqrt{x^2+y^2+z^2}}}{\sqrt{2}\,\sqrt{M}\,\sqrt{\text{Abs}[e]}} & \frac{-py^2\,x-pz^2\,x+px\,py\,y+px\,pz\,z+\frac{kMx}{\sqrt{x^2+y^2+z^2}}}{\sqrt{2}\,\sqrt{M}\,\sqrt{\text{Abs}[e]}} & 0 \end{pmatrix}$$