❦ Gravitation & Cosmology home problems 3.1 ❦

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Home Problem 11

Three-dimensional anti-de Sitter space (AdS_3) is given by the metric

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - \mathrm{d}v^2 + \mathrm{d}x^2 + \mathrm{d}y^2, \tag{1}$$

where the coordinates are constrained by

$$-u^2 - v^2 + x^2 + y^2 = -L^2. \tag{2}$$

This constraint can be solved by

$$u = \sqrt{L^2 + r^2} \cos \frac{t}{L}, \tag{3}$$

$$v = \sqrt{L^2 + r^2} \sin \frac{t}{L}, \tag{4}$$

$$x = r \cos \phi, \tag{5}$$

$$y = r \sin \phi. \tag{6}$$

a)

Find the metric in the coordinates (t, r, ϕ)

Solution

We plug (3) to (6) into (1) and obtain

$$\mathrm{d}s^2 = -\frac{(L^2 + r^2)\mathrm{d}t^2}{L^2} + \frac{L^2 \mathrm{d}r^2}{L^2 + r^2} + r^2 \mathrm{d}\phi^2 = \begin{bmatrix} \mathrm{d}t & \mathrm{d}r & \mathrm{d}\phi \end{bmatrix} \begin{bmatrix} -\frac{(L^2+r^2)}{L^2} & 0 & 0 \\ 0 & \frac{L^2}{L^2+r^2} & 0 \\ 0 & 0 & r^2 \end{bmatrix} \begin{bmatrix} \mathrm{d}t \\ \mathrm{d}r \\ \mathrm{d}\phi \end{bmatrix}. \tag{7}$$

See the Mathematica script below for this calculation.

In[1]:=
```
ClearAll["Global`*"]
$Assumptions = (L ∈ Reals && L > 0);
```

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};(*\ ds^2 = -du^2 - dv^2 + dx^2 + dy^2\ *)$$

$$u = \sqrt{L^2 + r[\bullet]^2}\ \mathrm{Cos}\left[\frac{t[\bullet]}{L}\right];$$

$$v = \sqrt{L^2 + r[\bullet]^2}\ \mathrm{Sin}\left[\frac{t[\bullet]}{L}\right];$$

$$x = r[\bullet]\ \mathrm{Cos}[\phi[\bullet]];$$

$$y = r[\bullet]\ \mathrm{Sin}[\phi[\bullet]];$$

$$\mathrm{Print}\left["ds^2 = ", \mathrm{FullSimplify}\left[(\partial_\bullet u\ \ \partial_\bullet v\ \ \partial_\bullet x\ \ \partial_\bullet y).g.\begin{pmatrix} \partial_\bullet u \\ \partial_\bullet v \\ \partial_\bullet x \\ \partial_\bullet y \end{pmatrix}\right][[1, 1]]\right]$$

$$ds^2 = \frac{L^2\, r'[\bullet]^2}{L^2 + r[\bullet]^2} - \frac{\left(L^2 + r[\bullet]^2\right) t'[\bullet]^2}{L^2} + r[\bullet]^2\, \phi'[\bullet]^2$$

b)

Compute the proper distance from the origin $r = 0$ to spacial infinity $r \to \infty$.

Solution

From Weinberg, the proper distance is

$$\begin{aligned} d &= \int_0^\infty \sqrt{g_{rr}} \mathrm{d}r \\ &= \int_0^\infty \sqrt{\frac{L^2}{L^2 + r^2}} \mathrm{d}r \\ &= \infty \end{aligned} \tag{14.2.21}$$

c)

Find the time it takes for a photon to travel the distance in **b)**.

Solution

For a photon, $\mathrm{d}s = 0$, so (7) gives (assume $\mathrm{d}\varphi = 0$ as well)

$$\begin{aligned} 0 &= -\frac{L^2 + r^2}{L^2}\mathrm{d}t^2 + \frac{L^2}{L^2 + r^2}\mathrm{d}r^2 \\ &\implies \\ \frac{\mathrm{d}t^2}{\mathrm{d}r^2} &= \left(\frac{L^2}{L^2 + r^2}\right)^2. \end{aligned}$$

The time it takes for the photon is

$$\begin{aligned} T &= \int \mathrm{d}t \\ &= \int_0^\infty \frac{\mathrm{d}t}{\mathrm{d}r}\mathrm{d}r \\ &= \int_0^\infty \frac{L^2}{L^2 + r^2}\mathrm{d}r \\ &= \frac{\pi}{2}L \end{aligned}$$

d)

Repeat the calculations in **b)** and **c)** for the Schwarzschild metric but now between $r = \infty$ and the event horizon.

Solution

The Schwarzschild metric is

$$\mathrm{d}\tau^2 = \left[1 - \frac{r_\mathrm{s}}{r}\right]\mathrm{d}t^2 - \left[1 - \frac{r_\mathrm{s}}{r}\right]^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\varphi^2 \tag{8.2.12}$$

where $r_\mathrm{s} = 2MG$ is the *Schwarzschild radius.*

The proper distance is

$$d = \int_{r_s}^{\infty} \sqrt{g_{rr}} \mathrm{d}r$$
$$= \int_{r_s}^{\infty} \left[1 - \frac{r_s}{r}\right]^{-\frac{1}{2}} \mathrm{d}r$$
$$= \infty.$$

A similar argument as before ($\mathrm{d}\tau = \mathrm{d}\theta = \mathrm{d}\varphi = 0$) shows that

$$\frac{\mathrm{d}t^2}{\mathrm{d}r^2} = \left[1 - \frac{r_s}{r}\right]^{-2}.$$

Thus the time it takes for the photon is

$$T = \int \mathrm{d}t$$
$$= \int_{r_s}^{\infty} \left[1 - \frac{r_s}{r}\right]^{-1} \mathrm{d}r$$
$$= \infty$$

e)

What is the (maximal) symmetry of AdS_3?

Solution

(2) is the definition of the *special indefinite orthogonal group* $SO(2,2)$.

Home Problem 12

Consider a massive particle moving in the geometry of the Schwarzschild metric.

a)

Derive an equation for the radial coordinate $r(\tau)$ that resembles Newton's second law $m\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = F(r)$ where $F(r) = -\frac{\mathrm{d}V(r)}{\mathrm{d}r}$ is the force due to the potential $V(r)$. Determine the so obtained effective potential $V_{\mathrm{eff}}(r)$.

Solution

Consider again the Schwarzschild metric (8.2.12). Since the problem is spherically symmetric, every position, velocity, and force vector will lie in the same plane. We may choose our coordinates so that $\theta = \frac{\pi}{2}$ and $\mathrm{d}\theta = 0$. Then (8.2.12) simplifies to

$$\mathrm{d}\tau^2 = \left[1 - \frac{r_s}{r}\right] \mathrm{d}t^2 - \left[1 - \frac{r_s}{r}\right]^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\varphi^2, \qquad (8.2.12')$$

We now follow Home Problem 7 d), where we created a Lagrangian $L(x^i, \dot{x}^i)$ by replacing $\mathrm{d}x^i$ by $\dot{x}^i$ in $\mathrm{d}\tau^2(x^i, \mathrm{d}x^i)$. We obtain the Lagrangian

$$\mathcal{L} = \left[1 - \frac{r_\mathrm{s}}{r}\right] \dot{t}^2 - \left[1 - \frac{r_\mathrm{s}}{r}\right]^{-1} \dot{r}^2 - r^2\dot{\varphi}^2.$$

We also note that if we do the same in the LHS of (8.2.12'), we obtain

$$\mathcal{L} = 1 \tag{8}$$

The Euler-Lagrange equations + (8) now read

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left[1 - \frac{r_\mathrm{s}}{r}\right]\dot{t}\right) = 0 \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\dot{\varphi}\right) = 0 \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{2\dot{r}}{1 - \frac{r_\mathrm{s}}{r}}\right) = -\frac{r_\mathrm{s}\dot{r}^2}{r^2\left(1 - \frac{r_\mathrm{s}}{r}\right)^2} - \frac{r_\mathrm{s}\dot{t}^2}{r^2} + 2r\dot{\varphi}^2 \tag{11}$$

$$\left[1 - \frac{r_\mathrm{s}}{r}\right] \dot{t}^2 - \left[1 - \frac{r_\mathrm{s}}{r}\right]^{-1} \dot{r}^2 - r^2\dot{\varphi}^2 = 1. \tag{12}$$

In (9), we see that $\left[1 - \frac{r_\mathrm{s}}{r}\right]\dot{t}$ is constant, we'll call it $\frac{E}{m}$,

$$\left[1 - \frac{r_\mathrm{s}}{r}\right]\dot{t} = \frac{E}{m}. \tag{13}$$

Similarly, from (10), we see that

$$\frac{J}{m} = r^2\dot{\varphi} \tag{14}$$

is a constant.

We may now solve (11) for $\ddot{r} = \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2}$ in terms of the constants and r using (12) to (14):

$$m\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \frac{2J^2 r - r_\mathrm{s}\left(3J^2 + m^2r^2\right)}{2mr^4}, \tag{15}$$

which becomes

$$V(r) = \frac{J^2 r - \left(J^2 + m^2 r^2\right) r_\mathrm{s}}{2mr^3}$$

when integrated with respect to r.

It is nice that in (15), $\ddot{r}$ does not depend on E, since we have interpreted it to be the energy of the particle.

The Mathematica script below was used to aid these calculations.

In[9]:=
```
ClearAll["Global`*"]
$Assumptions = (r_s ∈ Reals && r_s > 0);

(* Print Euler-Lagrange equations *)
```

$$L = -\left(1 - \frac{r_s}{r[\tau]}\right) t'[\tau]^2 + \left(1 - \frac{r_s}{r[\tau]}\right)^{-1} r'[\tau]^2 + r[\tau]^2\,\phi'[\tau]^2;$$

$$\text{Print}\left[\text{"}\frac{d}{d\tau}\{\text{"}, \partial_{t'[\tau]} L, \text{"}\} = \text{"}, \partial_{t[\tau]} L\right]$$

$$\text{Print}\left[\text{"}\frac{d}{d\tau}\{\text{"}, \partial_{\theta'[\tau]} L, \text{"}\} = \text{"}, \partial_{\theta[\tau]} L\right]$$

$$\text{Print}\left[\text{"}\frac{d}{d\tau}\{\text{"}, \partial_{\phi'[\tau]} L, \text{"}\} = \text{"}, \partial_{\phi[\tau]} L\right]$$

$$\text{Print}\left[\text{"}\frac{d}{d\tau}\{\text{"}, \partial_{r'[\tau]} L, \text{"}\} = \text{"}, \partial_{r[\tau]} L\right]$$

$$\frac{d}{d\tau}\left\{2\left(-1 + \frac{r_s}{r[\tau]}\right) t'[\tau]\right\} = 0$$

$$\frac{d}{d\tau}\{0\} = 0$$

$$\frac{d}{d\tau}\left\{2\, r[\tau]^2\, \phi'[\tau]\right\} = 0$$

$$\frac{d}{d\tau}\left\{\frac{2\, r'[\tau]}{1 - \frac{r_s}{r[\tau]}}\right\} = -\frac{r_s\, r'[\tau]^2}{r[\tau]^2 \left(1 - \frac{r_s}{r[\tau]}\right)^2} - \frac{r_s\, t'[\tau]^2}{r[\tau]^2} + 2\, r[\tau]\, \phi'[\tau]^2$$

In[16]:=
```
tEqn = {1/2 ∂_{t'[τ]} L == -e/m};(* Since d/dτ{(-1+r_s/r[τ])t'[τ]}=0, E/m=(-1+r_s/r[τ])t'[τ] is a constant *)
ϕEqn = {1/2 ∂_{ϕ'[τ]} L == J/m};(* Since d/dτ{r[τ]^2 ϕ'[τ]}=0, J/m=r[τ]^2 ϕ'[τ] is a constant *)
rEqn = {D[∂_{r'[τ]} L, τ] == ∂_{r[τ]} L};

(* Solve Euler-Lagrange equations for t'[τ], ϕ'[τ], r'[τ], & r''[τ] *)
tpSolution = Solve[tEqn, t'[τ]][[1, 1]]
ϕpSolution = Solve[ϕEqn, ϕ'[τ]][[1, 1]]
rpSolution = Solve[L == 1, r'[τ]][[1, 1]] /. {ϕpSolution, tpSolution}
rppSolution = Solve[rEqn, r''[τ]][[1, 1]]
```

Out[19]= $t'[\tau] \to \dfrac{e\, r[\tau]}{m\,(r[\tau] - r_s)}$

Out[20]= $\phi'[\tau] \to \dfrac{J}{m\, r[\tau]^2}$

Out[21]= $r'[\tau] \to -\dfrac{\sqrt{1 - \frac{r_s}{r[\tau]}}\ \sqrt{-\frac{J^2}{m^2\, r[\tau]} + r[\tau] + \frac{e^2\, r[\tau]^3}{m^2\,(r[\tau]-r_s)^2} - \frac{e^2\, r[\tau]^2\, r_s}{m^2\,(r[\tau]-r_s)^2}}}{\sqrt{r[\tau]}}$

Out[22]= $r''[\tau] \to \dfrac{1}{2}\left(1 - \dfrac{r_s}{r[\tau]}\right)\left(\dfrac{r_s\, r'[\tau]^2}{r[\tau]^2\left(1 - \frac{r_s}{r[\tau]}\right)^2} - \dfrac{r_s\, t'[\tau]^2}{r[\tau]^2} + 2\, r[\tau]\, \phi'[\tau]^2\right)$

In[23]:=
```
(* Substitute t'[τ], ϕ'[τ], & r'[τ] into expression for r''[τ] *)
rppExpression = ReplaceAll[rppSolution][r''[τ]];
rppExpression = ReplaceAll[ϕpSolution][rppExpression];
rppExpression = ReplaceAll[tpSolution][rppExpression];
rppExpression = ReplaceAll[rpSolution][rppExpression];

Print["m d^2 r/dτ^2 = ", FullSimplify[m rppExpression]]
```

$m\dfrac{d^2\, r}{d\tau^2} \;=\; \dfrac{2\, J^2\, r[\tau] - 3\, J^2\, r_s + m^2\, r[\tau]^2\, r_s}{2\, m\, r[\tau]^4}$

In[28]:=
```
(* Integrate F(r) with respect to r,
which we now have to call ρ as to not confuse our symbolic integration routine *)
F[ρ_] := m rppExpression /. {r[τ] → ρ};
Print["V(ρ) = ", FullSimplify[-Integrate[F[ρ], ρ]]]
```

$$V(\rho) = \frac{J^2\,\rho + (-J^2 + m^2\,\rho^2)\,r_s}{2\,m\,\rho^3}$$

b)

Are there any values of $\frac{J}{m} = r^2\dot{\phi}$ for which it is possible for the particle to be in a stable circular orbit? If so, what is the radius of this orbit?

Solution

In a circular orbit, $\dot{r} = \ddot{r} = 0$. Solving $\dot{r} = 0$ yields

$$\left(\frac{J}{m}\right)^2 = \frac{\left(\frac{E}{m}\right)^2 r^3 + m^2 r^3 - m^2 r^2 r_\mathrm{s}}{r - r_\mathrm{s}}$$

which has a solution when $r > r_\mathrm{s}$. Solving $\ddot{r} = 0$ yields

$$\left(\frac{J}{m}\right)^2 = \frac{r^2 r_\mathrm{s}}{2r - 3r_\mathrm{s}}$$

which has a solution when $r > \frac{3}{2}r_\mathrm{s}$. These are all of the constraints, and so all circular orbits with $r > \frac{3}{2}r_\mathrm{s}$ are allowed by some specific J.

In[1]:=
```
ClearAll["Global`*"]
$Assumptions = (rs ∈ Reals && rs > 0 &&
    r ∈ Reals && r > 0 &&
    m ∈ Reals && m > 0 &&
    J ∈ Reals && J > 0 &&
    e ∈ Reals && e > 0);
```

$$\text{rpp} = \frac{2 J^2 r - (3 J^2 + m^2 r^2)\,\text{rs}}{2 m r^4};$$ `(* r''[τ] from task 13a *)`

$$\text{rp} = -\frac{\sqrt{1-\frac{\text{rs}}{r}}\sqrt{-\frac{J^2}{m^2 r} + r + \frac{e^2 r^3}{m^2 (r-\text{rs})^2} - \frac{e^2 r^2 \text{rs}}{m^2 (r-\text{rs})^2}}}{\sqrt{r}};$$

```
(* Solve for J *)
Solve[rp == 0, J, Reals]
Solve[rpp == 0, J, Reals]
```

Out[5]=
$$\left\{\left\{J \to \text{ConditionalExpression}\left[-\sqrt{\frac{e^2 r^3 + m^2 r^3 - m^2 r^2 \text{rs}}{r-\text{rs}}}, (r > \text{rs} \,\&\&\, \text{rs} > 0) \,||\, (\text{rs} < 0 \,\&\&\, r > 0)\right]\right\},\right.$$
$$\left.\left\{J \to \text{ConditionalExpression}\left[\sqrt{\frac{e^2 r^3 + m^2 r^3 - m^2 r^2 \text{rs}}{r-\text{rs}}}, (r > \text{rs} \,\&\&\, \text{rs} > 0) \,||\, (\text{rs} < 0 \,\&\&\, r > 0)\right]\right\}\right\}$$

Out[6]=
$$\left\{\left\{J \to \text{ConditionalExpression}\left[-\sqrt{\frac{m^2 r^2 \text{rs}}{2 r - 3 \text{rs}}}, \left(r > \frac{3 \text{rs}}{2} \,\&\&\, \text{rs} > 0\right) \,||\, \left(r < \frac{3 \text{rs}}{2} \,\&\&\, \text{rs} < 0\right)\right]\right\},\right.$$
$$\left.\left\{J \to \text{ConditionalExpression}\left[\sqrt{\frac{m^2 r^2 \text{rs}}{2 r - 3 \text{rs}}}, \left(r > \frac{3 \text{rs}}{2} \,\&\&\, \text{rs} > 0\right) \,||\, \left(r < \frac{3 \text{rs}}{2} \,\&\&\, \text{rs} < 0\right)\right]\right\}\right\}$$

c)

Specialise to the case of a photon in a circular orbit. Are there any stable or unstable such orbits? If so, what is the radius?

Solution

For a photon in circular orbit, $\mathrm{d}\tau = 0$ and $\mathrm{d}r = 0$, so (8.2.12') reads

$$0 = \left[1 - \frac{r_\mathrm{s}}{r}\right] \mathrm{d}t^2 - r^2 \mathrm{d}\varphi^2 \tag{8.2.12''}$$
$$\frac{\mathrm{d}\varphi^2}{\mathrm{d}t^2} = \frac{1}{r^2}\left[1 - \frac{r_\mathrm{s}}{r}\right]$$

The geodesic equation of r with t as parameter is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} + \Gamma^r_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\frac{\mathrm{d}x^\rho}{\mathrm{d}t} = 0.$$

With the affine connection (17) which we calculated in Home Problem 13 and $\ddot{r} = 0$, we get

$$\frac{r_\mathrm{s}(r - r_\mathrm{s})}{2r^3} + (r_\mathrm{s} - r)\frac{\mathrm{d}\varphi^2}{\mathrm{d}t^2} = 0$$
$$\frac{r_\mathrm{s}}{2r^3} = \frac{\mathrm{d}\varphi^2}{\mathrm{d}t^2}.$$

Now, putting these two expressions for $\frac{\mathrm{d}\varphi^2}{\mathrm{d}t^2}$ equal, we have

$$\frac{1}{r^2}\left[1 - \frac{r_\mathrm{s}}{r}\right] = \frac{r_\mathrm{s}}{2r^3}$$
$$r = \frac{3}{2}r.$$

Home Problem 13

The death-defying spaceman Spiff lands on the hypothetical neutron star Buster. His spacecraft measures $H = 20\,\mathrm{m}$ in height. Buster's mass is $M = 2 \times 10^{30}\,\mathrm{kg}$ and its radius is $R = 5\,\mathrm{km}$. How far can Spiff walk away from the ship and still see it? According to Earthly standards, Spiff is a short man, being only $h = 1\,\mathrm{m}$ tall. Compare with the classical result (i.e., without gravity). Any approximations must be motivated.

Solution

We assume the simplified Schwarzschild metric (8.2.12') from before.

Consider a photon traveling from Buster to Spiff. At the very limit of how far Spiff can walk and still see Buster, just barely miss the ground at some point. At this point the photon will be traveling parallel to the ground. We will assume the star to be completely spherical, so that this means that $\dot{r} = 0$ there. This assumption is motivated by neutron stars being very large and dense objects, so they are not very oblate or prolate. They also have massive gravitational forces at their surface, resulting in mountains not being higher than millimeters.

We may obtain a Lagrangian parametrized by φ by dividing (8.2.12') by $\mathrm{d}\varphi^2$ to obtain a Lagrangian

$$\mathcal{L} = \left[1 - \frac{r_\mathrm{s}}{r}\right]\left(\frac{\mathrm{d}t}{\mathrm{d}\varphi}\right)^2 - \left[1 - \frac{r_\mathrm{s}}{r}\right]^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 - r^2.$$

Since $\mathrm{d}\tau = 0$ for a photon, we have that

$$\mathcal{L} = 0,$$

from which we may derive an expression

$$\frac{\mathrm{d}t}{\mathrm{d}\varphi} = \sqrt{\left[1 - \frac{r_\mathrm{s}}{r}\right]^{-2} \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \left[1 - \frac{r_\mathrm{s}}{r}\right]^{-1} r^2}$$

for $\frac{\mathrm{d}t}{\mathrm{d}\varphi}$. From here, we may derive a geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\varphi^2} + \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\varphi} \frac{\mathrm{d}x^\rho}{\mathrm{d}\varphi} = 0. \tag{16}$$

For our Schwarzschild metric, the affine connection is

$$\Gamma^r_{\nu\rho} = \begin{bmatrix} \frac{r_\mathrm{s}(r-r_\mathrm{s})}{2r^3} & 0 & 0 \\ 0 & -\frac{r_\mathrm{s}}{2r^2 - 2rr_\mathrm{s}} & 0 \\ 0 & 0 & r_\mathrm{s} - r \end{bmatrix}. \tag{17}$$

See Figure 1 the calculation of this affine connection.

Plugging this (16) with $\mu = r$, we get a differential equation

$$\begin{aligned} \frac{\mathrm{d}^2 r}{\mathrm{d}\varphi^2} + \frac{r_\mathrm{s}(r - r_\mathrm{s})}{2r^3} \left(\frac{\mathrm{d}t}{\mathrm{d}\varphi}\right)^2 - \frac{r_\mathrm{s}}{2r^2 - 2rr_\mathrm{s}} \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + (r_\mathrm{s} - r)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\varphi}\right)^2 &= 0 \\ \frac{\mathrm{d}^2 r}{\mathrm{d}\varphi^2} + \frac{r_\mathrm{s}(r - r_\mathrm{s})}{2r^3} \left(\left[1 - \frac{r_\mathrm{s}}{r}\right]^{-2} \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \left[1 - \frac{r_\mathrm{s}}{r}\right]^{-1} r^2\right) - \frac{r_\mathrm{s}}{2r^2 - 2rr_\mathrm{s}} \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + (r_\mathrm{s} - r) &= 0 \\ \frac{\mathrm{d}^2 r}{\mathrm{d}\varphi^2} - r + \frac{3}{2} r_\mathrm{s} &= 0. \end{aligned}$$

This is a second order linear differential equation whose solutions are

$$A\mathrm{e}^{\varphi} + B\mathrm{e}^{-\varphi} + \frac{3}{2} r_\mathrm{s}.$$

Our boundary conditions are

$$\begin{aligned} r(\varphi = 0) &= R, \\ \frac{\mathrm{d}r}{\mathrm{d}\varphi}(\varphi = 0) &= 0, \end{aligned}$$

when defining $\varphi = 0$ as the point where the photon just barely misses the ground. These boundary conditions give us

$$r(\varphi) = \left(\frac{1}{2}R - \frac{3}{4}r_\mathrm{s}\right) \cosh\varphi + \frac{3}{2} r_\mathrm{s}.$$

Solving $r(\varphi_1) = R + h$ and $r(\varphi_2) = R + H$ we get

$$\begin{aligned} \varphi_1 &= \cosh^{-1} \frac{R + h - \frac{3}{2}r_\mathrm{s}}{\frac{1}{2}R - \frac{3}{4}r_\mathrm{s}} = 0.270\,264\,\mathrm{rad} \\ \varphi_2 &= \cosh^{-1} \frac{R + H - \frac{3}{2}r_\mathrm{s}}{\frac{1}{2}R - \frac{3}{4}r_\mathrm{s}} = -0.060\,607\,7\,\mathrm{rad}. \end{aligned}$$

Notice how we know which inverse of cosh to take from Figure 2, where we have defined φ_1 to be positive and φ_2 to be negative. The maximum distance Spiff can walk while still seeing the top of his ship is thus (with more significant digits that appropriate)

$$R\Delta\phi = 1.654\,36\,\mathrm{km}.$$

Without gravity, the distance would have been a simple trigonometry problem where $\varphi_1 = \cos^{-1}\frac{R}{R+H} = 0.089\,294\,\mathrm{rad}$ and $\varphi_2 = \cos^{-1}\frac{R}{R+h} = -0.019\,998\,3\,\mathrm{rad}$ would imply

$$R\Delta\phi = 546.5\,\mathrm{m},$$

which is alot shorter.

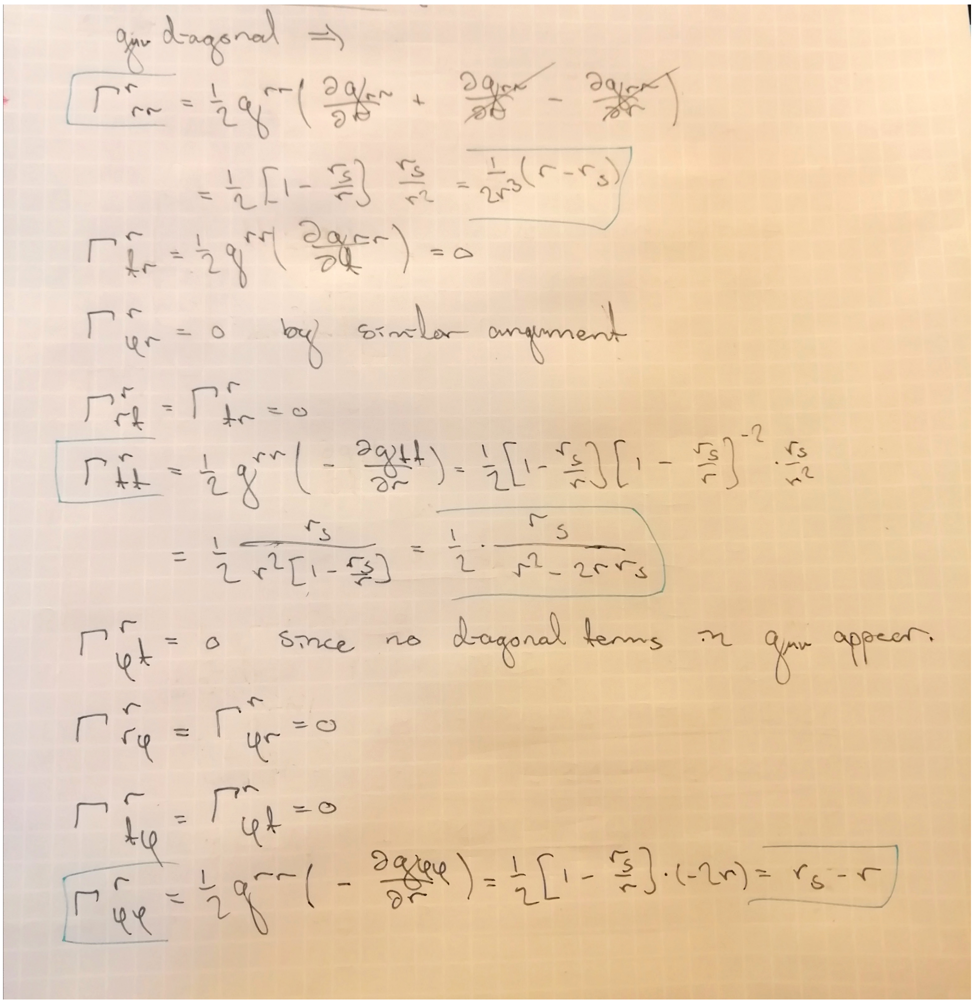

Figure 1: Calculating the affine connection for the Schwarzschild metric.

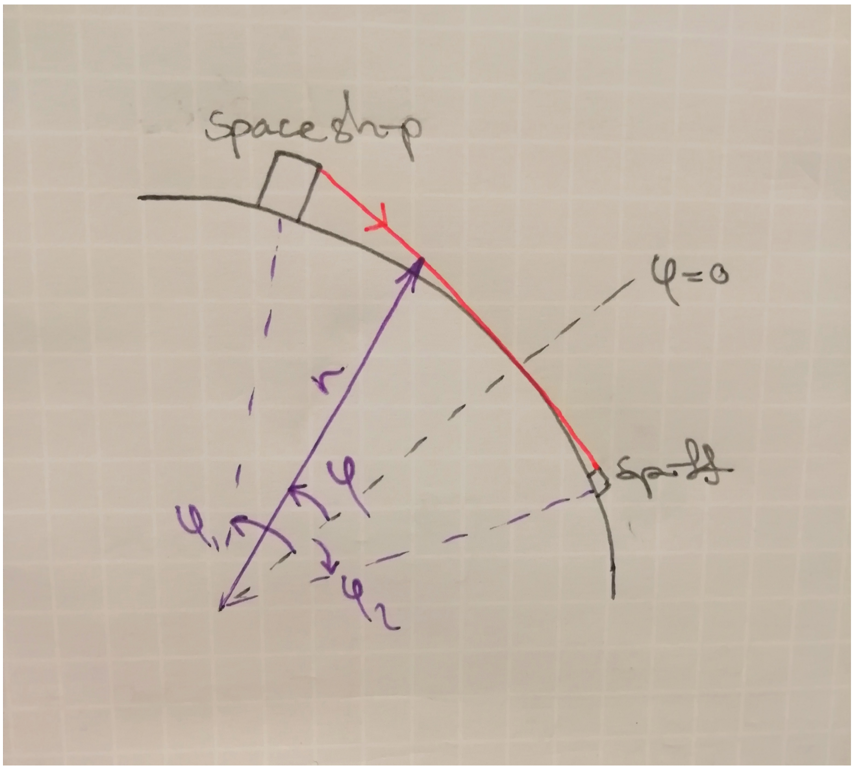

Figure 2: Illustration of the surface of Buster