In[9]:=
```
ClearAll["Global`*"]
$Assumptions = (r_s ∈ Reals && r_s > 0);

(* Print Euler-Lagrange equations *)
```

$$L = -\left(1 - \frac{r_s}{r[\tau]}\right) t'[\tau]^2 + \left(1 - \frac{r_s}{r[\tau]}\right)^{-1} r'[\tau]^2 + r[\tau]^2\, \phi'[\tau]^2;$$

$$\text{Print}\left["\frac{d}{d\tau}\{", \partial_{t'[\tau]} L, "\} = ", \partial_{t[\tau]} L\right]$$

$$\text{Print}\left["\frac{d}{d\tau}\{", \partial_{\theta'[\tau]} L, "\} = ", \partial_{\theta[\tau]} L\right]$$

$$\text{Print}\left["\frac{d}{d\tau}\{", \partial_{\phi'[\tau]} L, "\} = ", \partial_{\phi[\tau]} L\right]$$

$$\text{Print}\left["\frac{d}{d\tau}\{", \partial_{r'[\tau]} L, "\} = ", \partial_{r[\tau]} L\right]$$

$$\frac{d}{d\tau}\left\{2\left(-1 + \frac{r_s}{r[\tau]}\right) t'[\tau]\right\} = 0$$

$$\frac{d}{d\tau}\{0\} = 0$$

$$\frac{d}{d\tau}\{2\, r[\tau]^2\, \phi'[\tau]\} = 0$$

$$\frac{d}{d\tau}\left\{\frac{2\, r'[\tau]}{1 - \frac{r_s}{r[\tau]}}\right\} = -\frac{r_s\, r'[\tau]^2}{r[\tau]^2 \left(1 - \frac{r_s}{r[\tau]}\right)^2} - \frac{r_s\, t'[\tau]^2}{r[\tau]^2} + 2\, r[\tau]\, \phi'[\tau]^2$$

In[16]:=
```
tEqn = {1/2 ∂_{t'[τ]} L == -e/m};(* Since d/dτ{(-1+r_s/r[τ])t'[τ]}=0, E/m=(-1+r_s/r[τ])t'[τ] is a constant *)
ϕEqn = {1/2 ∂_{ϕ'[τ]} L == J/m};(* Since d/dτ{r[τ]^2 ϕ'[τ]}=0, J/m=r[τ]^2 ϕ'[τ] is a constant *)
rEqn = {D[∂_{r'[τ]} L, τ] == ∂_{r[τ]} L};

(* Solve Euler-Lagrange equations for t'[τ], ϕ'[τ], r'[τ], & r''[τ] *)
tpSolution = Solve[tEqn, t'[τ]][[1, 1]]
ϕpSolution = Solve[ϕEqn, ϕ'[τ]][[1, 1]]
rpSolution = Solve[L == 1, r'[τ]][[1, 1]] /. {ϕpSolution, tpSolution}
rppSolution = Solve[rEqn, r''[τ]][[1, 1]]
```

Out[19]= $$t'[\tau] \to \frac{e\, r[\tau]}{m\,(r[\tau] - r_s)}$$

Out[20]= $$\phi'[\tau] \to \frac{J}{m\, r[\tau]^2}$$

Out[21]= $$r'[\tau] \to -\frac{\sqrt{1 - \frac{r_s}{r[\tau]}}\ \sqrt{-\frac{J^2}{m^2\, r[\tau]} + r[\tau] + \frac{e^2\, r[\tau]^3}{m^2\,(r[\tau]-r_s)^2} - \frac{e^2\, r[\tau]^2\, r_s}{m^2\,(r[\tau]-r_s)^2}}}{\sqrt{r[\tau]}}$$

Out[22]= $$r''[\tau] \to \frac{1}{2}\left(1 - \frac{r_s}{r[\tau]}\right)\left(\frac{r_s\, r'[\tau]^2}{r[\tau]^2\left(1 - \frac{r_s}{r[\tau]}\right)^2} - \frac{r_s\, t'[\tau]^2}{r[\tau]^2} + 2\, r[\tau]\, \phi'[\tau]^2\right)$$

In[23]:=
```
(* Substitute t'[τ], ϕ'[τ], & r'[τ] into expression for r''[τ] *)
rppExpression = ReplaceAll[rppSolution][r''[τ]];
rppExpression = ReplaceAll[ϕpSolution][rppExpression];
rppExpression = ReplaceAll[tpSolution][rppExpression];
rppExpression = ReplaceAll[rpSolution][rppExpression];

Print["m d^2 r/dτ^2 = ", FullSimplify[m rppExpression]]
```

$$m\frac{d^2 r}{d\tau^2} = \frac{2 J^2\, r[\tau] - 3 J^2\, r_s + m^2\, r[\tau]^2\, r_s}{2\, m\, r[\tau]^4}$$

In[28]:=
```
(* Integrate F(r) with respect to r,
which we now have to call ρ as to not confuse our symbolic integration routine *)
F[ρ_] := m rppExpression /. {r[τ] → ρ};
Print["V(ρ) = ", FullSimplify[-Integrate[F[ρ], ρ]]]
```

$$V(\rho) = \frac{J^2 \rho + (-J^2 + m^2 \rho^2) r_s}{2 m \rho^3}$$