Bhabha scattering

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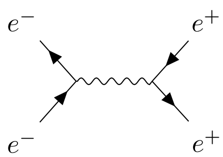
We have the interaction term

$$\mathcal{H}_{int} = -\mathcal{L}_{int} = e\bar{\Psi}\not{A}\Psi \tag{1}$$

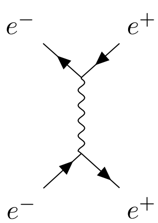
for QED. Letting the electrons have momentum p and p' while the positrons have momentum k and k'. The non-zero contributions are then the contractions

$$\langle p'k'|\bar{\Psi}\not{A}\Psi\bar{\Psi}\not{A}\Psi|pk\rangle, \qquad \langle p'k'|\bar{\Psi}\not{A}\Psi\bar{\Psi}\not{A}\Psi|pk\rangle. \tag{2}$$

Most importantly when imposing normal ordering we get a sign difference in the terms. We represent the processes as Feynman diagrams and apply the Feynman rules for QED to get



$$= \bar{u}^{s'}(p')(-ie\gamma^\mu)u^s(p)\frac{-ig_{\mu\nu}}{(p'-p)^2}\bar{v}^r(k)(-ie\gamma^\nu)v^{r'}(k') = ie^2\bar{u}^{s'}(p')\gamma^\mu u^s(p)\frac{1}{t}\bar{v}^r(k)\gamma_\mu v^{r'}(k') \tag{3}$$



$$= \bar{u}^{s'}(p')(-ie\gamma^\mu)v^{r'}(k')\frac{-ig_{\mu\nu}}{(p+k)^2}\bar{v}^r(k)(-ie\gamma^\nu)u^s(p) = ie^2\bar{u}^{s'}(p')\gamma^\mu v^{r'}(k')\frac{1}{s}\bar{v}^r(k)\gamma_\mu u^s(p) \tag{4}$$

where we introduced the Mandelstam variables

$$\begin{cases} t = (p'-p)^2 = (k'-k)^2 = -2p\cdot p' = -2k\cdot k' \\ s = (p+k)^2 = (p'+k')^2 = 2p\cdot k = 2p'\cdot k' \\ u = (k'-p)^2 = (k-p')^2 = -2k'\cdot p = -2k\cdot p', \end{cases} \tag{5}$$

where the last equalities are true neglecting the electron mass.

Thus

$$M = M_t - M_s = ie^2\left[\bar{u}_{p'}\gamma^\mu u_p\frac{1}{t}\bar{v}_k\gamma_\mu v_{k'} - \bar{u}_{p'}\gamma^\mu v_{k'}\frac{1}{s}\bar{v}_k\gamma_\mu u_p\right]. \tag{6}$$

Noting that $|M_s+M_t|^2 = |M_s|^2+|M_t|^2-2\mathrm{Re}(M_sM_t^*)$ we calculate each of these terms in turn. Starting with t-channel we have

$$|M_{\mathrm{s}}|^2 = \frac{e^4}{t^2}(\bar{u}_{p'}\gamma^\mu u_p\bar{v}_k\gamma_\mu v_{k'})^*(\bar{u}_{p'}\gamma^\nu u_p\bar{v}_k\gamma_\nu v_{k'}) = \frac{e^4}{t^2}(\bar{u}_p\gamma^\mu u_{p'}\bar{v}_{k'}\gamma_\mu v_k)(\bar{u}_{p'}\gamma^\nu u_p\bar{v}_k\gamma_\nu v_{k'}). \tag{7}$$

Average over all spins and applying the completeness relations (PS 5.3)

$$\sum_s u_p^s\bar{u}_p^s = \not{p}+m = \not{p}, \qquad \sum_s v_p^s\bar{v}_p^s = \not{p}-m = \not{p} \tag{8}$$

neglecting the electron mass we get

$$\overline{|M_{\rm s}|}^2 = \frac{e^4}{4t^2}\sum_{s,s',r,r'}(\bar{u}_p\gamma^\mu u_{p'}\bar{v}_{k'}\gamma_\mu v_k)(\bar{u}_{p'}\gamma^\nu u_p\bar{v}_k\gamma_\nu v_{k'}) = \frac{e^4}{4t^2}\sum_{s,s',r,r'}(\bar{u}_p\gamma^\mu u_{p'})(\bar{u}_{p'}\gamma^\nu u_p)(\bar{v}_k\gamma_\nu v_{k'})(\bar{v}_{k'}\gamma_\mu v_k) =$$
$$\frac{e^4}{4t^2}\sum_{s,r}(\bar{u}_p\gamma^\mu \not{p}'\gamma^\nu u_p)(\bar{v}_k\gamma_\nu \not{k}'\gamma_\mu v_k) = \frac{e^4}{4t^2}\sum_{s,r}[(\bar{u}_p)_a(\gamma^\mu\not{p}'\gamma^\nu)_{ab}(u_p)_b][(\bar{v}_k)_c(\gamma_\nu\not{k}'\gamma_\mu)_{cd}(v_k)_d] =$$
$$[(\gamma^\mu\not{p}'\gamma^\nu)_{ab}\not{p}_{ba}][(\gamma_\nu\not{k}'\gamma_\mu)_{cd}\not{k}_{dc}] = \mathrm{tr}[\gamma^\mu\not{p}'\gamma^\nu\not{p}]\mathrm{tr}[\gamma_\nu\not{k}'\gamma_\mu\not{k}]. \quad (9)$$

From the section Trace technology in PS we know that

$$\mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma) = 4(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}). \quad (10)$$

Applying this we have

$$\overline{|M_{\rm s}|}^2 = \frac{4e^4}{t^2}[g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}](p')_\rho p_\sigma[g_{\nu\xi}g_{\mu\eta} - g_{\nu\mu}g_{\xi\eta} + g_{\nu\eta}g_{\mu\xi}](k')^\xi k^\eta =$$
$$\frac{4e^2}{t^2}[(p')^\mu p^\nu - g^{\mu\nu}p'\cdot p + (p')^\nu p^\mu][(k')_\nu k_\mu - g_{\mu\nu}k'\cdot k + (k')_\mu k_\nu] =$$
$$\frac{4e^2}{t^2}\left[2(p'\cdot k)(p\cdot k') + 2(p'\cdot k')(p\cdot k) + (g^{\mu\nu}g_{\mu\nu} - 4)(p'\cdot p)(k'\cdot k)\right] = \frac{8e^2}{t^2}\left[(p'\cdot k)(p\cdot k') + (p'\cdot k')(p\cdot k)\right] = \frac{2e^2(u^2+s^2)}{t^2}. \quad (11)$$

By crossing symmetry the s-channel is the same letting $p\to p, k'\to k'$, $k\to -p', p'\to -k$ implying $s\leftrightarrow t$ giving

$$\overline{|M_{\rm t}|}^2 = 2e^4\frac{u^2+t^2}{s^2}. \quad (12)$$

We are now left with computing the interference term. We get that

$$\sum_{s,s',r,r'}(\bar{u}_{p'}\gamma^\mu u_p\bar{v}_k\gamma_\mu v_{k'})(\bar{u}_{p'}\gamma^\mu v_{k'}\frac{1}{s}\bar{v}_k\gamma_\mu u_p)^* = \sum_{s,s',r,r'}(\bar{u}_{p'}\gamma^\mu u_p\bar{v}_k\gamma_\mu v_{k'})(\bar{v}_{k'}\gamma^\nu u_{p'}\bar{u}_p\gamma_\nu v_k) =$$
$$\sum_{s,s',r,r'}(\bar{u}_{p'}\gamma^\mu u_p)(\bar{u}_p\gamma_\nu v_k)(\bar{v}_k\gamma_\mu v_{k'})(\bar{v}_{k'}\gamma^\nu u_{p'}) = \sum_{s'}\bar{u}_{p'}\gamma^\mu\not{p}\gamma_\nu\not{k}\gamma_\mu\not{k}'\gamma^\nu u_{p'} = \sum_{s'}(\bar{u}_{p'})_a(\gamma^\mu\not{p}\gamma_\nu\not{k}\gamma_\mu\not{k}'\gamma^\nu)_{ab}(u_{p'})_b$$
$$= (\gamma^\mu\not{p}\gamma_\nu\not{k}\gamma_\mu\not{k}'\gamma^\nu)_{ab}\not{p}'_{ba} = \mathrm{tr}(\not{p}'\gamma^\mu\not{p}\gamma_\nu\not{k}\gamma_\mu\not{k}'\gamma^\nu) = -2\mathrm{tr}(\not{p}'(\not{k}\gamma_\nu\not{p})\not{k}'\gamma^\nu) = -8\mathrm{tr}(\not{p}'\not{k})(p\cdot k') = -32(p'\cdot k)(p\cdot k') \quad (13)$$

where the last line uses the contraction identities

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\nu = -2\gamma^\sigma\gamma^\rho\gamma^\nu, \quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}. \quad (14)$$

Thus we have the interference term

$$-2\mathrm{Re}(M_sM_t^*) = -2\frac{e^4}{4ts}(-32)(p'\cdot k)(p\cdot k') = \frac{4u^2e^4}{ts}. \quad (15)$$

Adding up the terms we then have

$$\overline{|M|}^2 = 2e^4\left(\frac{u^2+s^2}{t^2} + \frac{u^2+t^2}{s^2} + \frac{2u^2}{ts}\right) = 2e^4\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right]. \quad (16)$$

According to PS 4.85 the differential cross section is then (averaged over spin)

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CM} = \frac{\overline{|M|}^2}{64\pi^2E_{cm}^2} = \frac{e^4}{32\pi^2E_{cm}^2}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right] = \frac{\alpha^2}{2s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right] \quad (17)$$

introducing the fine structure constant $\alpha = \frac{e^2}{4\pi}$ and using $p = (E, E\hat{z}$, $k = (E, -E\hat{z})$ implying $E_{CM}^2 = 4E^2 = (p+k)^2 = 2p\cdot k = s$ in the CM frame.

Integrating over the solid angle $\mathrm{d}\Omega =$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)}\right)_{CM} = \frac{\pi\alpha^2}{s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right]. \tag{18}$$

We introduce $p' = (E,\mathbf{k})$, $k' = (E,-\mathbf{k})$ giving

$$\begin{cases} u = -2k\cdot p' = -2(E^2 + |k||k|\cos\theta) = -2E^2(1+\cos\theta) = -\frac{s}{2}(1+\cos\theta) \\ t = -2k\cdot k' = -2(E^2 - |k||k'|\cos\theta) = -2E^2(1-\cos\theta) = -\frac{s}{2}(1-\cos\theta). \end{cases} \tag{19}$$

Rewriting in terms of $\cos\theta$ we then have

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)}\right)_{CM} = \frac{\alpha^2}{8s}\left[(1+\cos\theta)^2\left(1-\frac{2}{1-\cos\theta}\right)^2 + (1-\cos\theta)^2 + \frac{16}{(1-\cos\theta)^2}\right]. \tag{20}$$

Plotting this as a function of $\cos\theta$ we see this diverges as $\cos\theta \to 1$ i.e. at $\theta = 0$. We see the divergence comes form

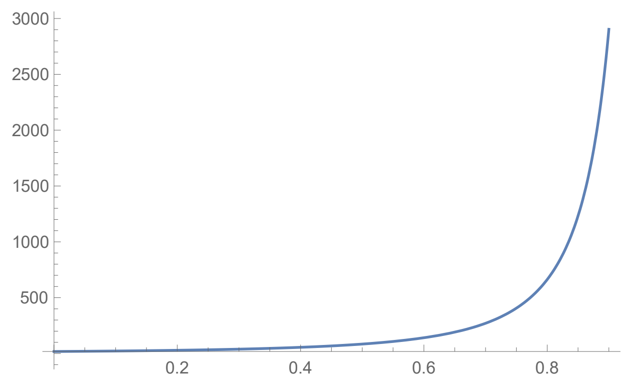


Figure 1: Differential cross section (without the prefactor) as a function of $\cos\theta$

the t-channel diagram. More specifically from the photon propagator in the t-channel, i.e. the divergence would still show if we didn't neglect the mass.