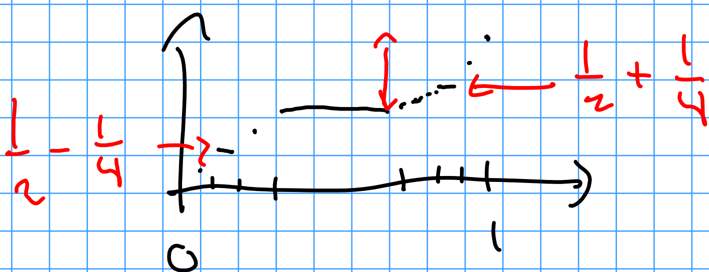
$x \in \limsup_i (A_i) = \bigcap_{k=1}^{\infty} \left(\bigcup_{i \geq k} A_i \right)$

& $x \in A_i$ for inf. A_i

$A_n = \cancel{[\ldots]}$

$[0, \frac{1}{n}]$



$x \sim y$ & $x - y \in \mathbb{Q}$

$[0, 1] \subseteq \bigcup_{q \in \mathbb{Q}} A_q \subseteq [-1, 1]$

$F(x) = \mu([0, x])$

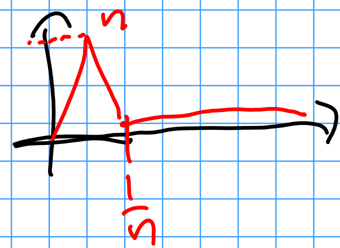
$\int f \, dm(x) = \int f^+ \, dm(x) - \int f^- \, dm(x)$

$\left| \int f \, dm(x) \right| < \infty$

$\int |f| \, dm(x)$

$\lim_{i \to \infty} \int f_i(x) \, dm(x) = \int \sup_i f(x) \, dm(x)$

$$\liminf_i \underbrace{\int f_i(x)\,dm(x)}_{=\frac{1}{2}} \geq \int \liminf_i f_i(x)\,dm(x)$$



$|f_n| < g \quad \forall n$

&

$f_n \to f$

then

$$\lim_{i\to\infty}\int f_i(x)\,dm(x) = \int f(x)\,dm(x)$$

(i) $f_n \to f$ a.e & $\{x \in X \text{ s.th } f_n(x) \not\to f(x)\}$ $\subseteq A$ w/ $m(A) = 0$

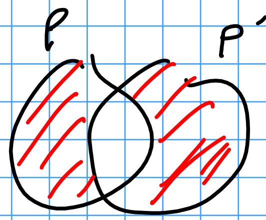
(ii) $f_n \to f$ in meas. & $m(\{x \in X \text{ s.th } |f_n(x) - f(x)| \geq \varepsilon\}) \xrightarrow{n\to\infty} 0 \quad \forall \varepsilon > 0$

$\Rightarrow$

(i) $\Leftarrow$ (ii)

$m(\{x \in X \text{ s.th } |f_n(x) - f(x)| \geq \frac{1}{2}\}) = 1$

$P \triangle P' = (P$



$$\nu = \nu^+ - \nu^-$$

$$\nu^+ \perp \nu^-$$

$\nu^+ \perp \nu^-$: if $\exists\, P \sqcup N = X$ &

$$\left| \begin{array}{l} \nu^+(N) = 0 \\ \nu^-(P) = 0 \end{array} \right.$$

$\underline{\mu \ll \nu}$ if $\nu(A) = 0 \Rightarrow \mu(A) = 0$

$\nu(A) = 0$ but $\mu(A) > 0$

$\exists f_0$ s.th $\underbrace{\mu(A)}_{>0} = \underbrace{\int_A f_0 \, d\nu(x)}_{\nu(A) \Rightarrow = 0}$

ν & μ meas.

$$\nu = \nu_{ac} + \nu_s$$

s.th $\underbrace{\nu_{ac}}_{=0} \ll \mu$ & $\nu_s \perp \mu$

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(x) \, dm(x)$$

$$Hf = \sup_{r>0} A_r |f|(x)$$

1.

2.

4.19 f_n sequence of meas. funcs. $\in L^+$

then

$$\liminf_{n\to\infty} \int f_n(x)\,dm(x) \geq \int \liminf_{n\to\infty} f_n(x)\,dm(x)$$

Proof: fix integer k. Then $\forall j \geq k$

$$\inf_{n\geq k} f_n(x) \leq f_j(x)$$

$$\int \inf_{n\geq k} f_n(x)\,dm \leq \int f_j\,dm \quad \forall j \geq k$$

$$\lim \int \underbrace{\inf_{n\geq k} f_n(x)}_{\text{incr. in } k}\,dm \leq \inf_{j\geq k} \int f_j\,dm$$

$$\int \liminf_{k\to\infty} f_k(x)\,dm \leq \liminf_{k\to\infty} \int f_k\,dm$$

4.22. f_n sequence in $L^1(X, M, m)$ conv. to f

s.th $\exists g \in L^1$ s.th

$$|f_n| \leq g \quad \forall n.$$

Then $f \in L^1$ &

$$\lim_{n\to\infty} \int f_n\,dm = \int f\,dm.$$

Proof: $|f| \le g$ hence $f \in L^1$.

Consider

$$g + f_n \geq 0 \quad \& \quad g - f_n \geq 0.$$

$$\cancel{\int g\,dm} + \boxed{\int f\,dm} = \int g + f\,dm$$

$$= \int \liminf_{n\to\infty} (g + f_n)\,dm$$

$$\leq \liminf_{n\to\infty} \int g + f_n\,dm$$

$$= \cancel{\int g\,dm} + \boxed{\liminf_{n\to\infty} \int f_n\,dm}$$

$$\cancel{\int g\,dm} + \int f\,dm = \int g - f\,dm$$

$$= \int \liminf (g - f_n)\,dm$$

$$\leq \liminf \int g - f_n\,dm$$

$$\underset{\geq}{=} \cancel{\int g\,dm} + \limsup \int f_n\,dm$$

$$\int f\,dm \leq \liminf \int f_n\,dm \leq \limsup \int f_n\,dm \leq \int f\,dm$$

$$\lim \int f_n\,dm \dot{=} \int f\,dm$$

Thm 8.10 $f \in L^1_{loc}((\mathbb{R}^n, \mathcal{M}, m))$ then Lebesgue-a.e.

$$\lim_{r\to 0} A_r f(x) = f(x).$$

Proof: enough to consider f non-zero on $[-N, N]^n$

$$\mathbb{R}^n = \bigcup_{N \in \mathbb{N}} [-N, N]^n . \qquad f \in L^1 .$$

So for $\alpha > 0$, define

$$E_\alpha = \{ x \in \mathbb{R}^n \text{ s.th } \limsup_{r \to 0} |A_r f(x) - f(x)| > \alpha \}$$

$\to m(E_\alpha) = 0$

$$\{ x \text{ s.th Thm 8.10 fails} \} = \bigcup_{n \in \mathbb{N}} E_{\frac{1}{n}} .$$

Now, fix $\varepsilon > 0$. Then $\exists$ continuous $g \in L^1$ s.th

$$\int |f - g| \, dm < \varepsilon \longleftarrow$$

Then

$$\limsup_{r \to 0} |A_r f(x) - f(x)|$$

$$= \limsup |A_r f(x) - A_r g(x) + A_r g(x) - g(x) + g(x) - f(x)|$$

$$\leq \limsup \Big(\underbrace{|A_r (f(x) - g(x))|}_{\limsup \, \leq H(g-f)(x)} + \underbrace{|A_r g - g|}_{=0} + |g - f| \Big)$$

$$E_\alpha \subseteq \{ x \text{ s.th } H(g-f)(x) > \tfrac{\alpha}{2} \} \cup \{ x \text{ s.th } |g - f| > \tfrac{\alpha}{2} \}$$

$$m(E_\alpha) \leq \underbrace{m(\{ x \text{ s.th } H(g-f) > \tfrac{\alpha}{2} \})}_{< \frac{2(3^n)}{\alpha} \varepsilon} + \underbrace{m(\{ x \text{ s.th } |g - f| > \tfrac{\alpha}{2} \})}_{< \frac{2}{\alpha} \cdot \varepsilon}$$

$$\leq c \cdot \varepsilon \qquad \forall \varepsilon > 0 .$$