❦ Quantum Field Theory Problem 6 ❦

simjac

May 2020

Consider the pseudoscalar Yukawa Lagrangian,

$$\mathcal{L}_{\text{Yukawa}} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_\phi^2\phi^2 - \overline{\psi}(i\partial\!\!\!/ - m_\psi)\psi - ig\overline{\psi}\gamma^5\psi\phi, \tag{1}$$

where ϕ is a real scalar field and ψ is a Dirac fermion. Notice that this Lagrangian is invariant under the parity transform

$$\psi(t, \mathbf{x}) \mapsto \gamma^0\psi(t, -\mathbf{x}) \tag{2}$$
$$\phi(t, \mathbf{x}) \mapsto -\phi(t, -\mathbf{x}) \tag{3}$$

in which the field ϕ carries odd parity.

a)

a.i)

Determine the superficially divergent amplitude.

Solution

We have the interaction Hamiltonian

$$\mathcal{H}_{\text{I}} = ig\overline{\psi}\gamma^5\psi\phi \tag{4}$$

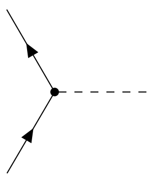
We want to calculate the two-point correlation function

$$\langle\Omega| T\phi(x)\phi(y) |\Omega\rangle = \langle 0| T\left\{\phi_{\text{I}}(x)\phi_{\text{I}}(y)\exp\left\{-i\int_{-T}^{T} \mathrm{d}t\, H_{\text{I}}(t)\right\}\right\} |0\rangle \tag{5}$$

The non-zero Feynman rules for the pseudoscalar Yukawa theory are

$$\phi \text{ - - - - - - } \phi \;=\; \frac{i}{p^2 - m_\phi^2} \tag{6}$$

$$\psi \longrightarrow \overline{\psi} \;=\; \frac{i(p\!\!\!/ + m_\psi)}{p^2 - m_\psi^2} \tag{7}$$



$$= g\gamma^5. \tag{8}$$

Consider, similarly to what Peskin does on page 316, a general Feynman diagram in this theory. Define

$$\begin{aligned} N_\phi &= \# \text{ of external } \phi \text{ lines} \\ N_\psi &= \# \text{ of external } \psi \text{ lines} \\ P_\phi &= \# \text{ of } \phi \text{ propagators} \\ P_\psi &= \# \text{ of } \psi \text{ propagators} \\ V &= \# \text{ of vertices} \\ L &= \# \text{ of loops.} \end{aligned}$$

The expression corresponding to a general diagram would then be proportianal to

$$g^V \int \frac{\mathrm{d}^4 p_1 \cdots \mathrm{d}^4 p_{P_\phi}}{(p_1^2 - m_\phi^2) \cdots (p_{P_\phi}^2 - m_\phi^2)} \frac{(\not{p}_1 + m_\psi)\,\mathrm{d}^4 q_1 \cdots (\not{p}_{P_\phi} + m_\psi)\,\mathrm{d}^4 q_{P_\psi}}{(q_1^2 - m_\psi^2) \cdots (p_{P_\psi}^2 - m_\psi^2)}. \tag{9}$$

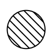
Counting powers of momenta in this integral gives us the superficial degree of divergence

$$D = 4L - P_\psi - 2P_\phi. \tag{10}$$

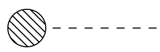
Using Euler's formula for planar graphs ($L = P_\phi + P_\psi - V + 1$) and $V = 2P_\psi + N_\psi = P_\phi + \frac{1}{2}N_\phi$ (since (8) is the only nonzero vertex, we are guaranteed that there are two ψ and one ϕ propagator per vertex), we rewrite this as

$$D = 4 - N_\phi - \frac{3}{2}N_\psi. \tag{11}$$

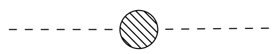
There are seven configurations satisfying $D \geq 0$ and hence there are seven superficially divergent Feynman diagrams. These diagrams are shown in figure 1.



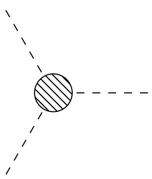
(a) $D = 4$



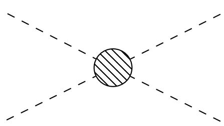
(b) $D = 3$



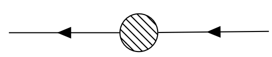
(c) $D = 2$



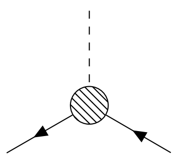
(d) $D = 1$



(e) $D = 0$



(f) $D = 1$



(g) $D = 0$

Figure 1

The first diagram, figure 1a, will add an infinite contribution to the vacuum energy. This will not be measurable. By (3), any diagram with an odd number of external ϕ-legs will be zero for symmetry reasons. Thus figure 1b and figure 1d are zero.

Since we are only interested in one-loop diagrams, and there is only one type of vertex, there is only one type of diagram that we are interested in for each Feynman diagram in figure 1. These are shown in figure 2.

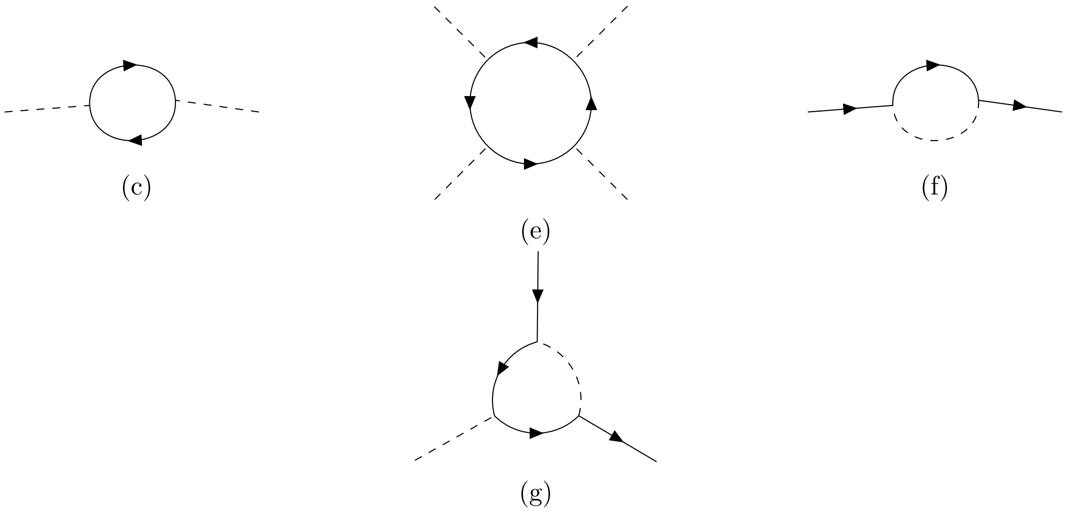


Figure 2

These have the following divergences (compare with Peskin section 10.1):

$$\text{figure 2c} \propto a_0\Lambda^2 + a_1 p^2 \log\Lambda \tag{12}$$
$$\text{figure 2e} \propto \log\Lambda \tag{13}$$
$$\text{figure 2f} \propto a_0\Lambda + \not{p}\log\Lambda \tag{14}$$
$$\text{figure 2g} \propto \log\Lambda. \tag{15}$$

a.ii)

Work out the Feynman rules for renormalized perturbation theory for this Lagrangian. Include all necessary counterterm vertices.

Solution

We follow Peskin section 10.2. Since figure 2e is divergent, but (1) contains no ϕ^4-term, we see that we will have to add such a counterterm

$$\frac{\lambda}{4!}\phi^4. \tag{16}$$

Define the renormalized fields ϕ_r and ψ_r by

$$\phi = Z_\phi^{\frac{1}{2}}\phi_r \tag{17}$$
$$\psi = Z_\psi^{\frac{1}{2}}\psi_r. \tag{18}$$

Putting (16), (17), and (18) into (1), we get

$$\mathcal{L} = \frac{1}{2}Z_\phi(\partial_\mu\phi_r)^2 - \frac{1}{2}Z_\phi m_\phi^2\phi_r^2 + Z_\psi\overline{\psi}_r(i\not{\partial} - m_\psi)\psi_r - iZ_\psi Z_\phi^{\frac{1}{2}} g\overline{\psi}_r\gamma^5\psi_r\phi_r - \frac{\lambda}{4!}Z_\phi^2\phi_r^4. \tag{19}$$

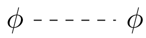
Defining

$$\begin{aligned}
\delta_{m_\phi} &= Z_\phi m_\phi^2 - m_{\phi_r}^2 \\
\delta_{m_\psi} &= Z_\psi m_\psi - m_{\psi_r} \\
\delta_{Z_\phi} &= Z_\phi - 1 \\
\delta_\lambda &= \lambda Z_\phi^2 - \lambda_r \\
\delta_g &= \frac{g}{g_r} Z_\psi Z_\phi^{\frac{1}{2}} - 1 \\
\delta_{Z_\psi} &= Z_\psi - 1,
\end{aligned}$$

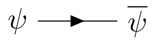
where the variables with an r-index will correspond to observables. This gives us the renormalized Lagrangian on the form

$$\begin{aligned}
\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{1}{2} m_{\phi_r}^2 \phi_r^2 + \overline{\psi}_r (i\partial\!\!\!/ - m_{\psi_r})\psi_r - i g_r \overline{\psi}_r \gamma^5 \psi_r \phi_r - \frac{\lambda}{4!}\phi_r^4 \\
+ \frac{1}{2}\delta_{Z_\phi}(\partial_\mu \phi_r)^2 - \frac{1}{2}\delta_{m_\phi} m_{\phi_r}^2 \phi_r^2 + \overline{\psi}_r (i\delta_{Z_\psi}\partial\!\!\!/ - \delta_{m_\psi})\psi_r - i g_r \delta_g \overline{\psi}_r \gamma^5 \psi_r \phi_r - \frac{\delta_\lambda}{4!}\phi_r^4.
\end{aligned}$$

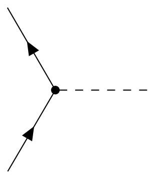
The second line here are all the counterterms. Reading off their Feynman rules from the Lagrangian (the symmetry factors from Peskin page 93 determine the overall constant), we have



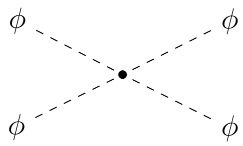
$$= \frac{i}{p^2 - m_{\phi_r}^2} \tag{20}$$



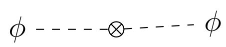
$$= \frac{i(p\!\!\!/ + m_{\psi_r})}{p^2 - m_{\psi_r}^2} \tag{21}$$



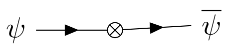
$$= g_r \gamma^5 \tag{22}$$



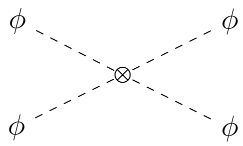
$$= -i\lambda_r \tag{23}$$



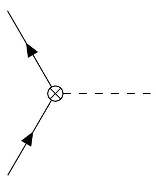
$$= i(p^2 \delta_{Z_\phi} - \delta_{m_\phi}) \tag{24}$$



$$= i(p\!\!\!/ \delta_{Z_\psi} - \delta_{m_\psi}) \tag{25}$$



$$= -i\delta_\lambda \tag{26}$$



$$= g_r \delta_g \gamma^5 \tag{27}$$

The diagrams with an $\otimes$ are the counterterms. We see that (20), (21), and (22) are basically the same as (6), (7), and (8) but with the renormalized fields instead.

a.iii)

Show that the theory contains a superficially divergent 4ϕ amplitude.

Solution

This is what (13) states.

a.iv)

This means that the theory cannot be renormalized unless one includes a scalar self-interaction

$$\delta\mathcal{L} = \frac{\lambda}{4!}\phi^4, \tag{28}$$

and a counterterm of the same form. It is of course possible to set the renormalized value of this coupling to zero, but that is not a natural choice, since the counterterm will still be nonzero. Are any further interactions required.

Solution

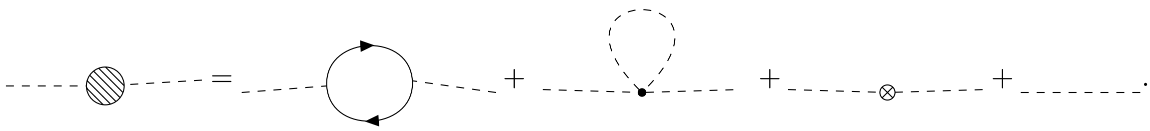
No, I don't think so.

b)

Compute the divergent part (the pole as $d \to 4$) of each counterterm, to the one-loop order of perturbation theory, implementing a sufficient set of renormalization conditions. You need not worry about finite parts of the counterterms. Since the divergent parts must have a fixed dependence on the external momenta, you can simplify this calculation by choosing the momenta in the simplest possible way.

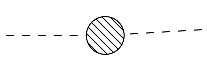
Solution

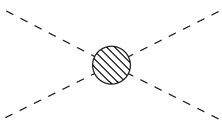
Now that we have one more vertex allowed, (23), there is no longer just one one-loop diagram corresponding to each diagram in figure 1. For example, we have that

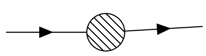


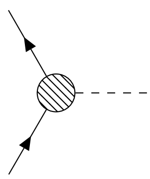
(29)

We may impose, as Peskin does in (10.19), the renormalization conditions


$$= \frac{i}{p^2 - m_{\phi_r}^2} \tag{30}$$


$$= -i\lambda_r \quad \text{at } s = 4m^2, t = u = 0 \tag{31}$$


$$= \frac{i}{\not{p} - m_{\psi_r}} \tag{32}$$


$$= g_r\gamma^5 \tag{33}$$

(34)

and, as Peskin does in (10.28) and (10.40), state that this is equivalent to

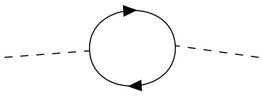
$$M^2(p^2)\Big|_{p^2=m_{\phi_r}^2} = 0 \tag{35}$$

$$\Sigma(\not{p})\Big|_{\not{p}=m_{\psi_r}} = 0 \tag{36}$$

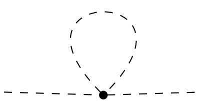
$$\frac{\mathrm{d}}{\mathrm{d}p^2}M^2(p^2)\Big|_{p^2=m_{\phi_r}^2} = 0 \tag{37}$$

$$\frac{\mathrm{d}}{\mathrm{d}p^2}\Sigma(\not{p})\Big|_{\not{p}=m_{\psi_r}} = 0. \tag{38}$$

Considering only the divergent terms of (29), we have (as on page 328–329 in Peskin)



$$\begin{aligned}
&\overset{(21)\text{ and }(22)}{=} (-ig_r)^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{tr}\left\{\frac{i}{\not{k}-m_{\psi_r}}\gamma^5 \frac{i}{(\not{k}-\not{p})-m_{\psi_r}}\gamma^5\right\} \\
&\overset{\text{figure 3}}{=} (-ig_r)^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{4\left(k\cdot(k+p)-m_{\psi_r}^2\right)}{(k^2-m_{\psi_r}^2)\left((k+p)^2-m_{\psi_r}^2\right)} \\
&\left\{\frac{1}{AB} = \int_0^1 \mathrm{d}x\, \frac{1}{(xA-(1-x)B)^2}\right\} \\
&= 4(-ig_r)^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k\cdot(k+p)-m_{\psi_r}^2}{x(k+p)^2 - xm_{\psi_r}^2 + k^2 - m_{\psi_r}^2 - xk^2 + xm_{\psi_r}^2} \\
&\left\{\text{define} \quad \begin{aligned} \ell &= k+xp \\ \Delta &= -x(1-x)p^2 + m_{\psi_r}^2 \end{aligned}\right\} \\
&= 4(-ig_r)^2 \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \int_0^1 \mathrm{d}x\, \frac{\ell^2 - x(1-x)p^2 - m_{\psi_r}^2}{(\ell^2-\Delta)^2} \\
&\overset{\text{Peskin (A.44) and (A.45)}}{=} 4(-ig_r)^2 \int_0^1 \mathrm{d}x\, \frac{1}{(4\pi)^{\frac{d}{2}}}\left[-\frac{d\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} - \frac{d\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}}\left(x(1-x)p^2 + m_{\psi_r}^2\right)\right] \\
&\overset{\text{Peskin (A.49) and (A-50)}}{=} \frac{-ig_r^2(m_{\psi_r}^2 - \frac{1}{2}p^2)}{2\pi^2(d-4)} + \mathcal{O}\left((d-4)^0\right).
\end{aligned} \tag{39}$$



$$\begin{aligned}
&\overset{(20)\text{ and }(23)}{=} -i\lambda_r \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2-m_{\phi_r}^2} \\
&\overset{(A.44)\text{ in Peskin}}{=} -i\lambda_r \frac{1}{(4\pi)^{\frac{d}{2}} m_{\phi_r}^{2-d}}\Gamma\left(1-\frac{d}{2}\right) \\
&\overset{(A.50)\text{ in Peskin}}{=} -i\lambda_r \frac{m_{\phi_r}^2}{16\pi^2(d-4)} + \mathcal{O}\left((d-4)^0\right)
\end{aligned} \tag{40}$$

$$\overset{(25)}{=} i(p^2\delta_{Z_\phi} - \delta_{m_\phi}). \tag{41}$$

Substituting in (39), (40), and (41) into (30) and matching the coefficients of the $\mathcal{O}\left((d-4)^{-1}\right)$-terms, we get

$$\delta_{Z_\phi} = \frac{g_r^2}{8\pi^2(d-4)}$$
$$\delta_{m_\phi} = \frac{1}{(d-4)}\left(\frac{\lambda_r m_{\phi_r}^2}{16\pi^2} - \frac{g_r^2 m_{\psi_r}^2}{2\pi^2}\right).$$

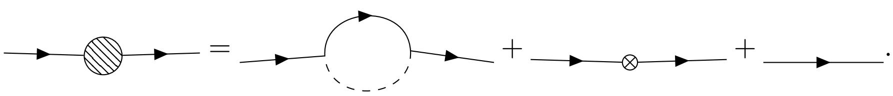
```
In[1]:= ClearAll["Global`*"]
        η = {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}};
        id = IdentityMatrix[4];
        γ0 = {{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}}; γ1 = {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, -1, 0, 0}, {-1, 0, 0, 0}}; γ2 = {{0, 0, 0, -I}, {0, 0, I, 0}, {0, I, 0, 0}, {-I, 0, 0, 0}}; γ3 = {{0, 0, 1, 0}, {0, 0, 0, -1}, {-1, 0, 0, 0}, {0, 1, 0, 0}};
        γ5 = I γ0.γ1.γ2.γ3;
        p = {{Ep}, {p1}, {p2}, {p3}}; k = {{Ek}, {k1}, {k2}, {k3}};
        γ = {γ0, γ1, γ2, γ3};
        pSlash = Sum[γ[[μ]] (η.p)[[μ, 1]], {μ, 1, 4}];
        kSlash = Sum[γ[[μ]] (η.k)[[μ, 1]], {μ, 1, 4}];

In[10]:= Tr[γ5.(kSlash + pSlash + mϕr id).γ5.(kSlash + mϕr id) / (((k + p)ᵀ.η.(k + p) - mϕr^2) (kᵀ.η.k - mϕr^2))[[1, 1]]] // FullSimplify

Out[10]= 4 (-Ek (Ek + Ep) + k1 (k1 + p1) + k2 (k2 + p2) + k3 (k3 + p3) + mϕr^2) / ((-Ek^2 + k1^2 + k2^2 + k3^2 + mϕr^2) (-(Ek + Ep)^2 + (k1 + p1)^2 + (k2 + p2)^2 + (k3 + p3)^2 + mϕr^2))
```

Figure 3

We may do likewise for the rest of (31), (32), and (33) to get the rest of the information about the renormalized theory. I do this without showing my calculations.



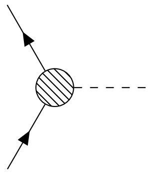
$$= \frac{ig_r^2}{16\pi^2(d-4)}\left(\frac{1}{2}\not{p} - m_{\psi_r}\right) + i(\not{p}\delta_{Z_\psi} - \delta_{Z_\psi}). \tag{42}$$

As before, matching powers and using our renormalization conditions, we get

$$\delta_{Z_\psi} = \frac{-g_r^2}{32\pi^2(d-4)} \tag{43}$$

$$\delta_{m_\psi} = \frac{-g_r^2 m_{\psi_r}}{16\pi^2(d-4)}. \tag{44}$$

Similarly, expanding (33), we get

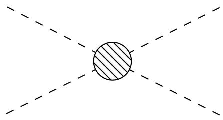


$$= \frac{-g_r^3}{4\pi^2(d-4)} + \delta_g \gamma^5 \tag{45}$$

$$\Longrightarrow \tag{46}$$

$$\delta_g = \frac{g^3}{4\pi^2(d-4)}. \tag{47}$$

And lastly,



$$= \frac{3ig_r^4}{4\pi^2(d-4)} + \frac{i\lambda_r^2}{16\pi^2} - \delta_\lambda \tag{48}$$

$$\Longrightarrow \tag{49}$$

$$\delta_\lambda = \frac{3ig_r^4}{4\pi^2(d-4)} + \frac{i\lambda_r^2}{16\pi^2}. \tag{50}$$