The exam questions will be based on the following sections in Miles Reid's book "Undergraduate Commutative Algebra".

Chapter 1, Chapter 2, Chapter 3
Chapter 4 (except sections 4.7, 4.8 and the second half of section 4.6)
Chapter 6 (except sections 6.7 and 6.8)
Chapter 8 (only sections 8.1-8.4)

The proofs in 1.5, 1.8, 2.7, 2.8, 3.2, 3.6, 4.2, 6.3 and 8.3 are especially important. But ideas from the other proofs like those in 4.9 or 8.4 may be needed to solve the problems. For the most difficult proofs e.g. 1.5 or 8.4, I will not ask you to reproduce the *full* proofs. Instead, I will then see if you can handle some special case or if you can reproduce some step of the proof if you get a hint.

The problems will be modeled on the following exercises in Reid's book, which in most cases will be solved at the problem sessions:

0.3, 0.7, 0.8, 0.9
2.1, 2.2, 2.6-2.10, 2.14
4.1-4.6
8.1, 8.2, 8.4
1.1-1.17
3.1-3.6, 3.8
6.1-6.5, 6.11-6.13

You should also know the following result used in step 2 of the proof of theorem 8.4

Theorem. Let $A$ be a commutative Noetherian ring and $M \neq \{0\}$ be an $A$-module. Then there exists an element $z \in M$ such that the annihilator $\mathrm{Ann}(z) = \{a \in A : az = 0\}$ of $z$ is a prime ideal.

Proof. $\mathrm{Ann}(x)$ is an additive subgroup for any $x \in M$. This follows from the subgroup criterion since $0 \in \mathrm{Ann}(x)$ and

$$f, g \in \mathrm{Ann}(x) \Rightarrow (f+g)x = fx + gx = 0 \Rightarrow f+g \in \mathrm{Ann}(x)$$
$$f, g \in \mathrm{Ann}(x) \Rightarrow (f-g)x = fx - gx = 0 \Rightarrow f-g \in \mathrm{Ann}(x)$$

$\mathrm{Ann}(x)$ is even an ideal in $A$ since

$$a \in A,\ f \in \mathrm{Ann}(x) \Rightarrow (af)x = a(fx) = a0 = 0 \Rightarrow af \in \mathrm{Ann}(x).$$

This ideal is proper since $1 \notin \mathrm{Ann}(x)$ for $x \neq 0$. We also note that $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(bx)$ for any $b \in A$ as

$$a \in \mathrm{Ann}(x) \Rightarrow ax = 0 \Rightarrow a(bx) = (ab)x = (ba)x = b(ax) = b0 = 0 \Rightarrow a \in \mathrm{Ann}(bx).$$

Now let $S$ be the partially ordered set of all ideals of the form $\mathrm{Ann}(x)$, $x \neq 0$. Then, as $A$ is Noetherian, there exists a maximal element $I = \mathrm{Ann}(z)$, $z \neq 0$ in $S$ (see Proposition 3.2 in Reid). In particular, we have then by the maximality of $I$ that $\mathrm{Ann}(bz) = \mathrm{Ann}(z)$ for any $b \in A$ outside $\mathrm{Ann}(z)$.

To show that $I$ is a prime ideal, let $a \in A$, $b \in A$ be elements with $ab \in \mathrm{Ann}(z)$. Then, for $b \notin \mathrm{Ann}(z)$ we have that $a \in \mathrm{Ann}(bz) = \mathrm{Ann}(z)$ as $a(bz) = (ab)z = 0$. So $a \in I$ or $b \in I$, as desired.

Remark. This result is applied in the proof of 8.4 when $A$ is a DVR with maximal ideal $m$ and $M = m/(x)$ for some $x \in m \setminus m^2$. If now $M \neq \{0\}$, then $\mathrm{Ann}(z)$ is a prime ideal $P$ for some $z \in M \setminus \{0\}$. Also, , $P \neq \{0\}$ as $x \in \mathrm{Ann}(z)$. Hence $P = m$ as $\{0\}$ and $m$ are the only prime ideals in $A$. If $y \in m \setminus (x)$ represents $z \in m/(x)$, we have thus that $my \subseteq (x)$ as asserted in step 2 of 8.4.