MATHEMATICS University of Gothenburg
Examination in commutative algebra MMA 330, 2020-10-26
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1a) Determine the nilpotents in $\mathbf{Z}_{2020}=\mathbf{Z}/(2020)$. 2p

b) Determine the idempotents in $\mathbf{Z}_{2020}$. 3p
(Hint: One gets less computations in Z_{2020} by using that $(1-e)^2=1-e$ if $e^2=e$.)

2) Let K be a field and A be a subring such that $x\in A$ or $x^{-1}\in A$ for each $x\in K\setminus\{0\}$. Show that A is a local ring. 4p
(Hint: Use a criterion for local rings and that a/b or b/a is in A for $a,b\in A\setminus\{0\}$.)

3) Ler A be a normal integral domain and S be a multiplicative subset of $A\setminus\{0\}$. Prove that $S^{-1}A$ is a normal integral domain. 4p
(Hint: Clear denominators in a clever way for integral relations over $S^{-1}A$.)

4) Let $\tilde{\mathbf{Z}}$ be the integral closure of $\mathbf{Z}$ in $\mathbf{C}$.
a) Prove that there are no irreducible elements in $\tilde{\mathbf{Z}}$.
b) Prove that $\tilde{\mathbf{Z}}$ is not a Noetherian ring.

5) Let N_1 and N_2 be submodules of an A-module M. Show that if M/N_1 and M/N_2 are Noetherian A-modules, then so is $M/(N_1\cap N_2)$. 4p

6) Let S be a multiplicative set in a commutative ring A. Let M be an A-module, N be an $S^{-1}A$-module and f be an A-linear map from M to N viewed as an A-module. Show that exists a unique $S^{-1}A$–linear map $g: S^{-1}M\to N$ with $g(\frac{m}{1})=f(m)$ for all $m\in M$. 4p

(Do not forget to prove that the map g you introduce is well defined!)

You may use all the theorems in Reid's book and the Chinese remainder theorem on $\mathbf{Z}_n$ as formulated in Wikipedia. But all claims must be motivated.