Solutions to the exam in commutative algebra MMA300 2020-10-26

1a) $\mathbf{Z}_{2020}$ is naturally isomorphic to $\mathbf{Z}_4\times\mathbf{Z}_5\times\mathbf{Z}_{101}$ where $\mathbf{Z}_5$ and $\mathbf{Z}_{101}$ are integral domains. If (a, b, c) is a nilpotent in $\mathbf{Z}_4\times\mathbf{Z}_5\times\mathbf{Z}_{101}$ we will thus have that b=0, c=0 and that a is even. The corresponding elements n in $\mathbf{Z}_{2020}$ must therefore be divisible by 2×5×101=1010, which gives the nilpotents 1010 and 0 in $\mathbf{Z}_{2020}$

b) The only idempotents in an integral domain are 0 and 1 and this is also true for $\mathbf{Z}_4$. There are therefore eight idempotents (0,0,0) , (0,0,1), (0,1,0) , (0,1,1) , (1,0,0) , (1,0,1) , (1,1,0) and (1,1,1) in $\mathbf{Z}_4\times\mathbf{Z}_5\times\mathbf{Z}_{101}$, where (0,0,0) corresponds to 0 and (1,1,1) to 1 in $\mathbf{Z}_{2020}$. Further (0,0,1) corresponds to a multiple n of 20 with $n\equiv1$(mod 101), which gives that $n\equiv-100$(mod 2020) and n=1920 in $\mathbf{Z}_{2020}$. Similarily (0,1,0) corresponds to a multiple n of 404 with $n\equiv1$(mod 5), which gives that n=1616 in $\mathbf{Z}_{2020}$ while (0,0,1) corresponds to a multiple n of 505 with $n\equiv1$(mod 4) which gives that n=505 in $\mathbf{Z}_{2020}$. Finally,
(0,1,1)=(1,1,1) –(1,0,0) corresponds to 1–1920 =2021–1920=101 in $\mathbf{Z}_{2020}$. while (1,0,1)=(1,1,1) –(0,1,0) corresponds to 1– 1616 =2021–1616=405 in $\mathbf{Z}_{2020}$ and (1,1,0)=(1,1,1) –(0,0,1) corresponds to 1–505 = 2021–505 =1516 in $\mathbf{Z}_{2020}$. There are thus eight idempotents in $\mathbf{Z}_{2020}$ represented by 0,1,101,405, 505, 1516, 1616 and 1920.

2) Let a,b be non-units in $A\backslash\{0\}$. If $a/b\in A$, then $a\pm b=b(a/b\pm1)$ cannot be a unit as if $b(a/b\pm1)c=1$ for $c\in A$, we would have the inverse $(a/b\pm1)c\in A$ to b. Similarily, if $b/a\in A$, then $a\pm b= a(1\pm b/a)$ cannot be a unit in A. As $a/b\in A$ or $b/a\in A$, the non-units will thus form an additive subgroup I of A. Moreover, if $a\in A$ and $i\in I$, then $ai\in I$ as if $(ai)c=1$ for $c\in A$ we would have the inverse $ac\in A$ to i. By the criterion in section 1.13, A is thus a local ring with maximal ideal I.

3) Suppose that $x\in K$ is integral over $S^{-1}A$ with $x^n+c_{n-1}x^{n-1}+\ldots c_1x+c_0=0$ for $c_i= a_i/s_i$ in $S^{-1}A$. Let s be the product of all s_i. Then $a_{n-i}:=c_{n-i}s^i\in A$ for all $i\geq1$ and $(xs)^n+a_{n-1}(xs)^{n-1}+\ldots a_1(xs)+a_0=0$. Hence $xs\in K$ is integral over A and $xs\in A$ as A is normal. Therefore, $x= xs/s\in S^{-1}A$, as was to be proved.

4a) If the complex number $a\neq0$ is integral over $\mathbf{Z}$, then we may find a complex number $b\neq0$ with $b^2=a$, which is also integral over $\mathbf{Z}$. If now b is a unit in $\tilde{\mathbf{Z}}$, then $a=b^2$ is also a unit in $\tilde{\mathbf{Z}}$, while if b is a non-unit in $\tilde{\mathbf{Z}}$ $a=b^2$ will be reducible in $\tilde{\mathbf{Z}}$. Hence no complex number in $\tilde{\mathbf{Z}}$ can be irreducible in $\tilde{\mathbf{Z}}$.

b) $1/2\notin\tilde{\mathbf{Z}}$ as $\mathbf{Z}$ is integrally closed in $\mathbf{Q}$ by exercise 0.7. So 2 is not a unit in $\tilde{\mathbf{Z}}$. Let $b_0=2$ and $b_{k+1}=\sqrt{b_k}$ for $k\geq0$. Then $(b_k)\subseteq(b_{k+1})$ in $\tilde{\mathbf{Z}}$ but not $(b_k)=(b_{k+1})$ as otherwise $b_k/b_{k+1} = b_{k+1}$ would be a unit in $\tilde{\mathbf{Z}}$ just like all powers of b_{k+1} including 2. We have thus an infinite chain of different ideals

$$(b_0) \subset (b_1) \subset \dots \qquad\qquad \dots \subset (b_{k-1}) \subset (b_k) \subset (b_{k+1}) \subset ,,,$$

which proves that $\tilde{\mathbf{Z}}$ is not Noetherian.

5) Let $\varphi: M \to M/N_1 \oplus M/N_2$ be the A-linear map which sends $m \in M$ to $(m+N_1, m+N_2)$. As the kernel of φ is $N_1 \cap N_2$, we obtain then from prop 2.3(c) an A-linear isomorphism from $M/(N_1 \cap N_2)$ to im φ. Further, $M/N_1 \oplus M/N_2$ is Noetherian by corollary 3.5(i) just like its submodule im φ and $M/(N_1 \cap N_2)$ by the trivial part of prop 3.4.

6) If such an A-linear map $g: S^{-1}M \to N$ exists, then

$$g(\tfrac{m}{s}) = g(\tfrac{1}{s}\tfrac{m}{1}) = \tfrac{1}{s}g(\tfrac{m}{1}) = s^{-1}f(m)$$

To see that this gives a well defined map $g: S^{-1}M \to N$, suppose that $m/s = n/t$ in $S^{-1}M$. Then $utm = usn$ for some $u \in S$ such that

$$g(m/s) = g(utm/stu) = (stu)^{-1}f(utm) = (stu)^{-1}f(usn) = g(usn/stu) = g(n/t).$$

This map is $S^{-1}A$-linear as

$$g(\tfrac{m}{s} + \tfrac{n}{t}) = g(\tfrac{tm+sn}{st}) = \tfrac{f(tm+sn)}{st} = \tfrac{tf(m)+sf(n)}{st} = \tfrac{f(m)}{s} + \tfrac{f(n)}{t} = g(\tfrac{m}{s}) + g(\tfrac{n}{t})$$

$$\tfrac{a}{s}g(\tfrac{m}{t}) = \tfrac{a}{s}\tfrac{f(m)}{t} = \tfrac{af(m)}{st} = \tfrac{f(am)}{st} = g(\tfrac{am}{st}) = g(\tfrac{a}{s}\tfrac{m}{t}).$$