❦ Komplexanalys i flera variabler ❦
Assignment 1

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1a

Suppose Δ is a polydisc, Γ its distinguished boundary, and $f : \overline{\Delta} \to \mathbb{C}$ is continuous on $\overline{\Delta}$ and holomorphic on Δ. Prove $|f(z)|$ achieves its maximum on Γ.

Solution

$$\Delta := \Delta_1 \times \cdots \times \Delta_n,$$
$$\Gamma := \partial\Delta_1 \times \cdots \times \partial\Delta_n.$$

Since $\overline{\Delta}$ is compact and $|f|$ continuous, $|f|$ achieves its maximum on $\overline{\Delta}$. Let $p = (p_1, \dots, p_n)$ denote such a point in $\overline{\Delta}$ where $|f|$ achieves its maximum.

Assume now that $p \notin \Gamma$, i.e. $p_i \notin \partial\Delta_i$ for some i. Define $g : \overline{\Delta}_i \to \mathbb{C}$ by $g(w) = f(p_1, \dots p_{i-1}, w, p_{i+1}, \dots, p_n)$. g is then continuous on $\overline{\Delta}_i$ and holomorphic on Δ_i and achieves its max at $p_i \in \Delta_i$. By the one-variable maximum modulus principle (Lebl thm 0.1.2), g is constant. Thus, for any $p'_i \in \partial\Delta_i$, $g(p_i) = g(p'_i)$ and thus $f(p_1, \dots p_{i-1}, p_i, p_{i+1}, \dots, p_n) = f(p_1, \dots p_{i-1}, p'_i, p_{i+1}, \dots, p_n)$. Define p' as the vector where we have performed this procedure for all i s.th $p_i \notin \partial\Delta_i$. Then $f(p) = f(p')$ and $p' \in \Gamma$, so $|f|$ achieves its maximum on Γ.

1b

A ball is different from a polydisc. Prove that for every $p \in \partial\mathbb{B}_n$ there exists a continuous $f : \overline{\mathbb{B}_n} \to \mathbb{C}$, holomorphic on $\mathbb{B}_n$ such that $|f(z)|$ achieves a strict maximum at p.

Solution

$$\mathbb{B}_n := \{z \in \mathbb{C}^n \text{ s.th } |z| < 1\},$$
$$\overline{\mathbb{B}_n} := \{z \in \mathbb{C}^n \text{ s.th } |z| \le 1\},$$
$$\partial\mathbb{B}_n := \{z \in \mathbb{C}^n \text{ s.th } |z| = 1\}.$$

Consider the function $f : \overline{\mathbb{B}_n} \to \mathbb{C}$ defined by $z \mapsto \mathrm{e}^{z_1}$. f is clearly continuous on $\overline{\mathbb{B}_n}$ and holo on $\mathbb{B}_n$. Since $|f(z)| = \mathrm{e}^{\operatorname{Re} z_1}$ and since $\exp : \mathbb{R} \to \mathbb{R}$ is strictly increasing, we have that

$$\max_{z \in \overline{\mathbb{B}_n}} |f(z)| = \max_{z \in \overline{\mathbb{B}_n}} \mathrm{e}^{\operatorname{Re} z_1} = \mathrm{e}^1.$$

This means that $|f(z)|$ achieves its (strict) max at $z = (1 + 0i, 0, \dots, 0) \in \partial\mathbb{B}_n$.

For any $p \in \partial\mathbb{B}_n$ that is not equal to $(1+0i, 0, \dots, 0)$, we may define f by composing the exponential of z_1 with a unitary transform U which maps $p \mapsto (1+0i, 0, \dots, 0)$. The property of U being unitary guarantees that $U(\overline{\mathbb{B}_n}) = \overline{\mathbb{B}_n}$, so the composition does not change the domain of definition, as well as guaranteeing U is one-to-one, so the unique element $z \in \partial\mathbb{B}_n$ maximizing $|f|$ is p. Thus we have shown the statement of the exercise.

2

Show that $\mathcal{O}(U)$ is an integral domain (has no zero divisors) if and only if U is connected. That is, show that U being connected is equivalent to showing that if $h(z) = f(z)g(z)$ is identically zero for $f, g \in \mathcal{O}(U)$, then either f or g is identically zero.

Solution

Let $U \in \mathbb{C}^n$ be an open set.
For the "only if" direction, assume that U is connected. Then, if $h := fg$ is identically 0, for each $z \in U$, either $f(z) = 0$ or $g(z) = 0$. Since f and g are holo, they are continuous, and thus $f^{-1}(\{z \text{ s.th } |z| > 0\})$ and $g^{-1}(\{z \text{ s.th } |z| > 0\})$ are open since the set $\{z \text{ s.th } |z| > 0\}$ is open. Consider $f^{-1}(\{z \text{ s.th } |z| > 0\})$, either this set is the empty set, and $g \equiv 0$ by $h \equiv 0$, or this set is a nonempty open set and $f \equiv 0$ by the identity theorem (Lebl thm 1.2.6). Thus, if U is connected, either $f \equiv 0$ or $g \equiv 0$.

For the "if" direction, it will suffice to find one example of a disconnected U where $h \equiv 0$ but neither $f \equiv 0$ nor $g \equiv 0$. It is true that any U which is union of two open disjoint sets A and B with

$$f(z) = \begin{cases} 1, & z \in A \\ 0, & z \in B \end{cases},$$

$$g(z) = \begin{cases} 0, & z \in A \\ 1, & z \in B \end{cases}$$

is such an example.

3

Suppose $U \subset \mathbb{C}^n$ is a domain and $f \in \mathcal{O}(U)$. Show that the complement of the zero set, $U \setminus f^{-1}(0)$, is connected.

Bonus: you might also try to show that $U \setminus f^{-1}(0)$ is not simply connected.

Solution

Assume the opposite, that $U \setminus f^{-1}(0)$ is disconnected, i.e. it is the union of two nonempty open disjoint sets A and B. By the Riemann extension theorem (Lebl thm 1.6.1), any locally bounded holo function g defined on $U \setminus f^{-1}(0)$ can be extended uniquely to a holo function G on U. Since A and B are disconnected,

$$g(z) = \begin{cases} 0, & z \in A \\ 1, & z \in B \end{cases}$$

is locally bounded and holo on $U \setminus f^{-1}(0)$. Thus there exists some holo function G on U which which agrees with g on $U \setminus f^{-1}(0)$. But since U is a domain and A is an open subset on which $G \equiv g \equiv 0$, the identity theorem (Lebl thm 1.2.6) implies that $G \equiv 0$ on the whole of U. This is a contradiction since $g \equiv 1$ on B and thus $U \setminus f^{-1}(0)$ must be connected.

4

Find the domain of convergence of $\sum_{j,k} c_{j,k} z_1^j z_2^k$ and draw the corresponding picture if $c_{k,k} = 2^k$, $c_{0,k} = c_{j,0} = 1$, and $c_{j,k} = 0$ otherwise.

Solution

Since power series converge absolutely,

$$\sum_{j,k} c_{j,k} z_1^j z_2^k = \sum_j z_1^j + \sum_j z_2^j + \sum_j 2^j (z_1 z_2)^j. \tag{1}$$

The first term converges when $|z_1| < 1$, the second term converges when $|z_2| < 1$, and the third term converges when $|z_1 z_2| < \frac{1}{2}$. The LHS converges when all three terms converge. This intersection is illustrated in Figure 1.

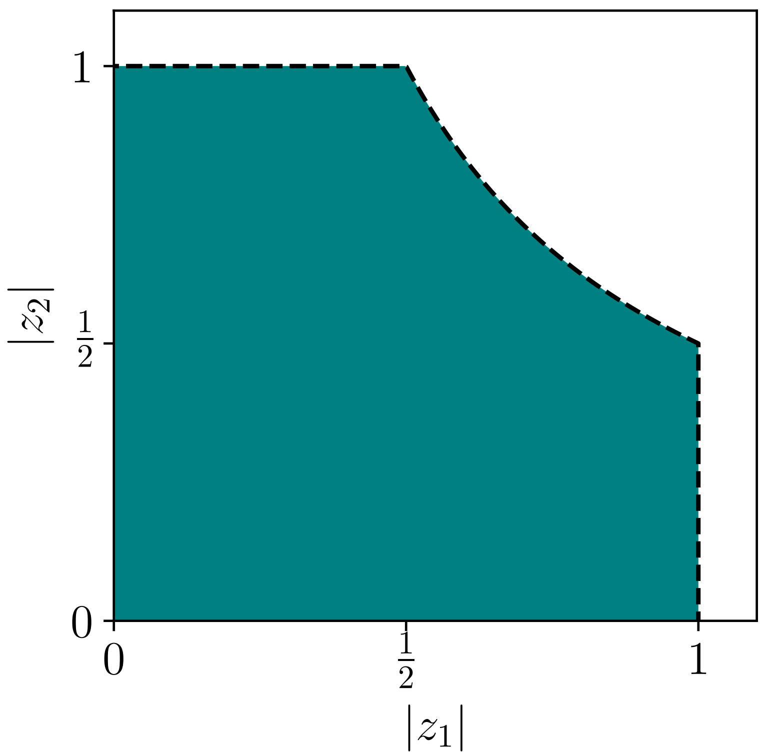

Figure 1: The domain of convergence of (1).

5

Assume that a complete Reinhardt domain $D \subseteq \mathbb{C}^2$ is given by

$$D = \{(w_1, w_2) \in \mathbb{C}^2 \text{ s.th } |w_1| \le r, |w_2| \le h(|w_1|)\}$$

for some $h : [0, r] \to \mathbb{R}_+$. Characterize the logarithmic convexity of D in terms of some property of h. Use this characterization to show that a ball in $\mathbb{C}^2$ with center 0 is logarithmically convex.

Solution

Let $a, b \in \operatorname{tr} D$, i.e.

$$(0,0) \le (a_1, a_2) \le (r, h(a_1)) \tag{2}$$
$$(0,0) \le (b_1, b_2) \le (r, h(b_1)). \tag{3}$$

Then, for D to be logarithmically convex, we must have

$$a_2^{1-\lambda} b_2^{\lambda} \le h(a_1^{1-\lambda} b_1^{\lambda}) \tag{4}$$

for all $\lambda \in [0,1]$ (the $|w_1|$-inequality is trivially satisfied). Since raising to a positive power is a strictly increasing function, (2) and (3) implies that

$$0 \le a_2^{1-\lambda} \le h(a_1^{1-\lambda})$$
$$0 \le b_2^{\lambda} \le h(b_1^{\lambda}).$$

Since everything here is positive, this implies that

$$a_2^{1-\lambda} b_2^{\lambda} \le h(a_1^{1-\lambda}) h(b_1^{\lambda}).$$

One way to clearly satisfy (4) is thus if

$$h(a_1^{1-\lambda}) h(b_1^{\lambda}) \le h(a_1^{1-\lambda} b_1^{\lambda}). \tag{5}$$

A unit ball in $\mathbb{C}^n$ with center 0 is a Reinhardt domain and has the trace

$$\operatorname{tr} \mathbb{B}_n = \{(r_1, r_2) \in \mathbb{R}^2_{\ge 0} \;\text{ s.th }\; r_1^2 + r_2^2 \le 1\}.$$

We may describe this domain as above by setting $h(r_1) = \sqrt{1 - r_1^2}$ and $r = 1$. I could not come up with any proof for this h satsfying (5) for all r_1, r_1', and λ, but I tested some cases on a $40 \times 40 \times 40$ grid and found that it seems reasonable. See Figure 2.

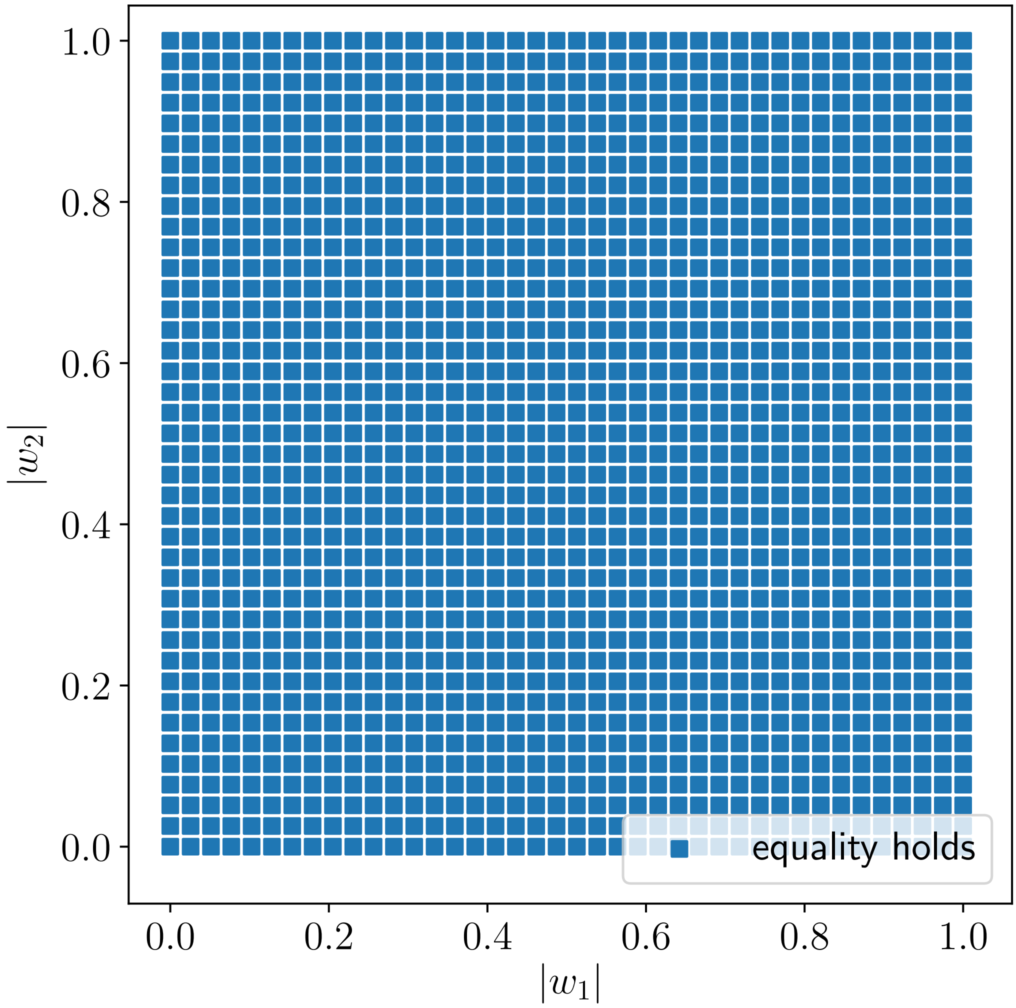

Figure 2