6.3 Ideals in A and $S^{-1}A$ for a mutiplicative set S in A

Notation If I is an ideal in A and S a multiplicative set in A, then $S^{-1}I \subseteq S^{-1}A$ is the subset of all fractions of the form i/s where $i \in I$ and $s \in S$.

It follows from the subgroup criterion that $S^{-1}I$ is an additive subgroup of $S^{-1}A$. It is even an ideal in $S^{-1}A$ as $(a/s)(i/t)=ai/st \in S^{-1}I$ for all $a/s \in S^{-1}A$ and $i/t \in S^{-1}I$.

We obtain conversely an ideal I of A from an ideal J of $S^{-1}A$ by letting I be the preimage $\phi^{-1}(J)$ under the ring homomorphism $\phi\colon A \to S^{-1}A$ where $\phi(a)=a/1$

Definition If I is an ideal in A and S a multiplicative set in A, then the *saturation* I^ of I with respect to S is the set of all $a \in A$ such that $as \in I$ for some $s \in S$.

Note that $I \subseteq I$^. It is easy to see that I^ is an ideal and this will also follow from proposition 6.3(b) below. Reid considers also the set I^ in his proposition 6.3(b), but does not refer to I^ as the saturation of I.

Proposition 6.3. *Let A be a ring, S be a multiplicative set in A and $\phi\colon A \to S^{-1}A$ be the evident ring homomorphism which sends a to $a/1$. Then*

(a) *for any ideal J of $S^{-1}A$, then $S^{-1}I = J$ for $I=\phi^{-1}(J)$.*

(b) *for any ideal I of A, then $\phi^{-1}(S^{-1}I)= I$^.*

Proof (a) $J \subseteq S^{-1}I$. This is clear as $b/s \in J \Rightarrow b/1=(s/1)(b/s) \in J \Rightarrow b \in I$.

$S^{-1}I \subseteq J$. $S^{-1}I$ consists of fractions i/s with $s \in S$ and $\phi(i)=i/1 \in J$. Hence as $i/s=(1/s)(i/1)$ with $1/s \in S^{-1}A$ we get that $i/s \in J$ as J is an ideal in $S^{-1}A$

(b) $\phi^{-1}(S^{-1}I) \subseteq I$^. If $a \in \phi^{-1}(S^{-1}I)$, then $\phi(a)=a/1 \in S^{-1}I$ such that $a/1=b/t$ in $S^{-1}A$ for some $b \in I$ and $t \in S$. But then $\exists u \in S$ with $uta=ub \in I$ such that $a \in I$^ as $ut \in S$.

I^$\subseteq \phi^{-1}(S^{-1}I)$. Suppose $a \in I$^. Then we may find $s \in S$ with $sa \in I$. Therefore. $\phi(a) = a/1= sa/s \in S^{-1}I$, which means that $a \in \phi^{-1}(S^{-1}I)$.

Definition An ideal I of A is said to be saturated if $I = I$^. (Reid says here instead that I satisfies (*).)

Example : If I is the inverse image $\phi^{-1}(J)$ of an ideal $J \subseteq S^{-1}A$ under ϕ, then I is saturated by proposition 6.3 as $J = S^{-1}\phi^{-1}(J)) \Rightarrow I = \phi^{-1}(J) = \phi^{-1}(S^{-1}\phi^{-1}(J)) = \phi^{-1}(S^{-1}I) = I$^

Corollary 6.3. (i) *For an ideal I of A, I is saturated if and only if* $I=\phi^{-1}(S^{-1}I)$
(ii) *There is a bijection between the set of saturated ideals I of A and the set of all ideals J of* $S^{-1}A$ *given by* $J=S^{-1}I$ *and* $I=\phi^{-1}(J)$.
(iii) *If I an ideal, then I^=A* $\Leftrightarrow S^{-1}I = S^{-1}A \Leftrightarrow I \cap S \neq \emptyset$.

Proof (i) This is an immediate consequence of part (b) of the proposition.
(ii) We have seen from the example that $I=\phi^{-1}(J)$ is saturated and from (a) that $S^{-1}I = J$ for $I=\phi^{-1}(J)$.
Conversely, if I is saturated, then $I=\phi^{-1}(S^{-1}I)$ by (i).
(iii) If I^$=A$, then $1 \in I$^ such that $s1 \in I$ for some $s \in S^{-}$. Hence $I \cap S \neq \emptyset$.
If $I \cap S \neq \emptyset$ then $i=s$ for some $i \in I$ and $s \in S$ and hence $S^{-1}I = S^{-1}A$ as $i/s = 1/1 \in S^{-1}I$.
Finally, if $S^{-1}I = S^{-1}A$, then I^$=A$^$=A$ by part (b) of proposition 6.3.

To compare prime ideals in A and $S^{-1}A$ we need the following lemma, where the first assertion is taken from Reid's corollary 6.3(iv) and the second assertion is his proposition 6.3(c).

Lemma *If P is a prime ideal with* $P \cap S = \emptyset$ *then P is saturated and* $S^{-1}P$ *a prime ideal in* $S^{-1}A$.

Proof To see that P is saturated suppose that $sa \in P$ for $s \in S$ and $a \in A$. Then $a \in P$ as $s \in P$ is false by the assumption. Hence $P=P$^.

To see that $S^{-1}P$ is a prime ideal, suppose $(a/s)(b/t) = p/v$ for $a,b \in A$, $p \in P$ and $s,t,v \in S$. Then $\exists u \in S$ with $uvab = ustp$ which means that $(uv)ab \in P$. Therefore, $ab \in P$^ as $uv \in S$ and $ab \in P$. But then $ab \in P$ as $P=P$^ an. $a \in P$ or $b \in P$ as P is a prime ideal. Hence $a/s \in S^{-1}P$ or $b/t \in S^{-1}P$, which proves that $S^{-1}P$ is a prime ideal in $S^{-1}A$.

Now recall that we have a natural map ϕ^*: Spec $B \to$ Spec A associated to any ring homomorphism $\phi : A \to B$ which sends a prime ideal Q in B to the prime ideal $P=\phi^{-1}(Q)$ in A. We will consider the case where $B = S^{-1}A$ and $\phi: A \to S^{-1}A$ is the map which sends a to $a/1$.

The following result is the last assertion in corollary 6.3(iv) in Reid´s book

<u>Theorem</u> ϕ^*: Spec $S^{-1}A \to$ Spec A *is injective and its image is the subset*

$\{P \in \text{Spec } A : P \cap S = \emptyset\}$ *of* Spec A.

<u>Proof</u> Let $Q \in \text{Spec } S^{-1}A$ and $P=\phi^{-1}(Q)$ its image in Spec A under ϕ^*. Then $Q = S^{-1}P$ by proposition 6.3(a) and $P \cap S = \emptyset$ by the second equivalence in corollary 6.3(iii). Hence ϕ^* is injective with image contained in $\{P \in \text{Spec } A : P \cap S = \emptyset\}$.

To show that ϕ^* maps Spec $S^{-1}A$ *onto* $\{P \in \text{Spec } A : P \cap S = \emptyset\}$, let P be a prime ideal in A with $P \cap S = \emptyset$. Then P is saturated and $Q = S^{-1}P$ a prime ideal in $S^{-1}A$ by the lemma. Therefore. $P=\phi^{-1}(Q)$ by proposition 6.3(b), which proves that any $P \in \text{Spec } A$ with $P \cap S = \emptyset$ is in the image of ϕ^*.

<u>Remark</u> We have actually proved that any $P \in \text{Spec } A$ with $P \cap S = \emptyset$ is equal to $\phi^{-1}(Q)$ for $Q = S^{-1}P \in \text{Spec } S^{-1}A$.