Class Lectures (for Chapter 7)

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Deeper aspects of measure theory.

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We will need a lot of preliminary work, including the so-called Hahn and Jordan Decomposition theorems.

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a. A measure is a signed measure.
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c. Condition (ii) is there to avoid having $\infty - \infty$.

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This is the picture we want in general.

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Then a Hahn decomposition is given by $([0, \frac{3}{4}], (\frac{3}{4}, 1])$.

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Read lecture notes.

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If we can show that P^c is a negative set, we would be done.

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Then $P \cup F$ would be a positive set with ν-measure larger than m. Contradiction. QED

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Let ν be Lebesgue measure restricted to $[0, 1/2]$ plus a unit point mass at $3/4$.

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Example: The Cantor measure and Lebesgue measure. $E = C$ and $F = C^c$.

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No. μ_1 and μ_2 are not mutually singular.

Instead one should take ν^+ to be Lebesgue measure restricted to $[0, 1/4]$ and ν^- to be Lebesgue measure restricted to $[3/4, 1]$.

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Also

$$(\nu^+ - \nu^-)(A) = \nu^+(A) - \nu^-(A) = \nu(A \cap P) + \nu(A \cap N) = \nu(A).$$

QED

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Let ν and μ be two measures on $(X, \mathcal{M})$ with $\nu \ll \mu$ and with ν and μ being σ-finite.

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This is false if one does not assume σ-finiteness.

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3. (Kolmogorov) The Radon-Nikodym Theorem is crucial in advanced probability when one deals with the subtle concept of conditioning.

Proof of The Radon-Nikodym Theorem for finite measure spaces

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Note that $m \leq \nu(X) (< \infty)$.

claim: There exists $f_0 \in \mathcal{F}$ for which $\int f_0(x) d\mu(x) = m$; i.e. the supremum above is achieved.

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This f_0 will turn out to be our Radon Nikodym derivative.

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Then $\mu|_{\mathcal{A}}$ is atomic, $\mu|_{\mathcal{A}^c}$ is continuous and these measures are mutually singular.

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Proposition: Let μ and ν be measures with ν finite. Then $\nu \ll \mu$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ so that

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If $\mu(A) = 0$, then $\mu(A) < \delta$ for every $\delta > 0$ and hence $\nu(A) < \epsilon$ for every $\epsilon > 0$. So $\nu(A) = 0$.

The Lebesgue Decomposition Theorem

If the statement on the RHS fails, then there would exist an $\epsilon_0 > 0$ and sets (A_n) with $\mu(A_n) \leq 1/2^n$ and $\nu(A_n) \geq \epsilon_0$.

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Now $n \to \infty$ using continuity from above for ν (ν is a finite measure) gives $\nu(A) \geq \epsilon_0$.
QED