❦ Gravitation & Cosmology home problems 2 ❦

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Home Problem 6

Derive the metric on S^2 (the two-sphere) from its definition $x^2 + y^2 + z^2 = a^2$:

a)

in (standard) polar coordinates (θ, ϕ)

Solution

WLOG, set $a = 1$. With

$$\begin{aligned} x &= \sin\theta\cos\phi \\ y &= \sin\theta\sin\phi \\ z &= \cos\theta, \end{aligned}$$

the metric induced on S^2 by the Euclidian metric is

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 = \begin{bmatrix}\mathrm{d}\theta & \mathrm{d}\phi\end{bmatrix} \begin{bmatrix}1 & 0 \\ 0 & \sin^2\theta\end{bmatrix} \begin{bmatrix}\mathrm{d}\theta \\ \mathrm{d}\phi\end{bmatrix} \tag{1}$$

b)

and by eliminating z (set $x^2 + y^2 = r^2$).

Solution

With

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= \sqrt{1 - r^2}, \end{aligned}$$

the metric takes the form

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = r^2\mathrm{d}\theta^2 + \frac{1}{1 - r^2}\mathrm{d}r^2 = \begin{bmatrix}\mathrm{d}r & \mathrm{d}\theta\end{bmatrix} \begin{bmatrix}\frac{1}{1-r^2} & 0 \\ 0 & r^2\end{bmatrix} \begin{bmatrix}\mathrm{d}r \\ \mathrm{d}\theta\end{bmatrix}$$

The following Mathematica script was used when solving a) and b).

In[1]:=
```
ClearAll["Global`*"]

x = Sin[θ[t]] Cos[ϕ[t]];
y = Sin[θ[t]] Sin[ϕ[t]];
z = Cos[θ[t]];
FullSimplify[D[x, t]^2 + D[y, t]^2 + D[z, t]^2]
```

Out[5]= $\theta'[t]^2 + \mathrm{Sin}[\theta[t]]^2\,\phi'[t]^2$

In[6]:=
```
ClearAll["Global`*"]

x = r[t] Cos[θ[t]];
y = r[t] Sin[θ[t]];
z = Sqrt[1 - r[t]^2];
FullSimplify[D[x, t]^2 + D[y, t]^2 + D[z, t]^2]
```

Out[10]= $-\frac{r'[t]^2}{-1 + r[t]^2} + r[t]^2\,\theta'[t]^2$

c)

Compute the affine connection in both of these sets of coordinates from its definition.

Solution

We allow ourselves to use

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right) \tag{3.3.7}$$

from Weinberg. The results are

$$\begin{aligned} \Gamma^{\theta}_{\phi\phi} &= -\frac{1}{2}\sin 2\theta \\ \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} &= \cot\theta \\ \text{the rest} &= 0 \end{aligned}$$

for the first set of coordinates, and

$$\begin{aligned} \Gamma^{r}_{rr} &= \frac{r}{1-r^2} \\ \Gamma^{r}_{\theta\theta} &= -r(1-r^2) \\ \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} &= \frac{1}{r} \\ \text{the rest} &= 0 \end{aligned}$$

for the second set of coordinates. The following Mathematica script was used for calculations on the first set and it's completely trivial to see how it was used on the second set by changing the metric.

```
In[1]:= ClearAll["Global`*"]
coordinateList = {θ, ϕ}; (*(θ,ϕ) in terms of (θ,ϕ)*)

(* Define g_μν *)
g = (1   0
     0   Sin[θ]^2);

(* Initialize Γ^μ_νρ as rank 3 tensor *)
tmp[a_, b_, c_] := 0;
Γ = Array[tmp, {2, 2, 2}];

(* Loop over indices in Γ^μ_νρ *)
Do[
 Do[
  Do[
   Do[
    xλ = coordinateList[[λ]];
    xμ = coordinateList[[μ]];
    xν = coordinateList[[ν]];

    Γ[[σ, λ, μ]] += 1/2 (Inverse[g])[[ν, σ]] (∂_xλ g[[μ, ν]] + ∂_xμ g[[λ, ν]] - ∂_xν g[[μ, λ]]);
    (* Γ^μ_νρ = 1/2 g_νσ (∂g_μν/∂x^λ + ∂g_λν/∂x^μ - ∂g_μλ/∂x^ν) *),
    {ν, {1, 2}}],
   {μ, {1, 2}}],
  {λ, {1, 2}}],
 {σ, {1, 2}}]

Print["Γ^θ_νρ = ", MatrixForm[FullSimplify[Γ[[1]]]]]
Print["Γ^ϕ_νρ = ", MatrixForm[FullSimplify[Γ[[2]]]]]
```

$$\Gamma^{\theta}{}_{\nu\rho} = \begin{pmatrix} 0 & 0 \\ 0 & -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta] \end{pmatrix}$$

$$\Gamma^{\phi}{}_{\nu\rho} = \begin{pmatrix} 0 & \mathrm{Cot}[\theta] \\ \mathrm{Cot}[\theta] & 0 \end{pmatrix}$$

Home Problem 7

Consider the variation of the path length between A and B, i.e., $S[x] = \int_A^B \mathrm{d}\tau$.

a)

Show that the terms with a derivative on the metric in $\delta S[x] = 0$ gives $\Gamma^\mu_{\nu\rho}$.

Solution of a)

We first state

$$\frac{\partial g_{\sigma\lambda}}{\partial x^\mu} + \frac{\partial g_{\mu\lambda}}{\partial x^\sigma} - \frac{\partial g_{\sigma\mu}}{\partial x^\lambda} = 2g_{\lambda\nu}\Gamma^\nu_{\mu\sigma} \tag{3.3.5}$$

from Weinberg. Then, following Weinberg,

$$\begin{aligned}
\delta S[x] &= \delta \int_A^B \mathrm{d}\tau \\
&= \delta \int_A^B \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} \\
&\left\{\delta(\mathrm{d}x^\mu \mathrm{d}x^\nu) = \mathrm{d}\delta x^\mu \mathrm{d}x^\nu + \mathrm{d}x^\mu \mathrm{d}\delta x^\nu = 2\mathrm{d}\delta x^\mu \mathrm{d}x^\nu\right\} \\
&= \int_A^B \frac{1}{2\sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}} \left[-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda \mathrm{d}x^\mu \mathrm{d}x^\nu - 2g_{\mu\nu}\mathrm{d}\delta x^\mu \mathrm{d}x^\nu\right] \\
&\overset{\mathrm{d}\tau=\sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}}{=} \int_A^B \frac{1}{2\mathrm{d}\tau} \left[-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda \mathrm{d}x^\mu \mathrm{d}x^\nu - 2g_{\mu\nu}\mathrm{d}\delta x^\mu \mathrm{d}x^\nu\right] \\
&= \int_A^B \frac{1}{2} \left[-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} - 2g_{\mu\nu}\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right]\mathrm{d}\tau \qquad (2) \\
&\overset{\text{PI on 2nd term}}{=} \int_A^B \frac{1}{2} \left[-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + 2\frac{\mathrm{d}g_{\lambda\nu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + 2g_{\lambda\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}\right]\delta x^\lambda \mathrm{d}\tau \\
&= \int_A^B \frac{1}{2} \left[-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + 2\frac{\partial g_{\lambda\nu}}{\partial x^\sigma}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + 2g_{\lambda\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}\right]\delta x^\lambda \mathrm{d}\tau
\end{aligned}$$

$$\left\{\begin{array}{c}\text{by renaming some indices being summed over and using that } g \text{ is symmetric, we see that} \\ -\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + 2\frac{\partial g_{\lambda\nu}}{\partial x^\sigma}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \left(-\frac{\partial g_{\sigma\mu}}{\partial x^\lambda} + \frac{\partial g_{\sigma\lambda}}{\partial x^\mu} + \frac{\partial g_{\mu\lambda}}{\partial x^\sigma}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \overset{(3.3.5)}{=} 2g_{\lambda\nu}\Gamma^\nu_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\end{array}\right\}$$

$$= \int_A^B \left[\Gamma^\nu_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} + \frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}\right] g_{\lambda\nu}\delta x^\lambda \mathrm{d}\tau.$$

$\delta S = 0$ gives us the geodesic equation

$$\Gamma^\nu_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} + \frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} = 0. \tag{3}$$

The way Weinberg does this is that he introduces an integration variable σ which is independent of the path, so he doesn't need to consider the variation of the differential in the integral. The way I have done it is maybe a bit less mathematically justified since things like $\delta\mathrm{d}x = \mathrm{d}\delta x$ are used without motivation.

b)

Show that the equations obtained from the variation of S' where (here $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$)

$$S'[x] = \int \mathrm{d}\tau L' = \int \mathrm{d}\tau(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu), \tag{4.8}$$

(note that there is *no* square root) are the same as those coming from $S[x] = \int_A^B \mathrm{d}\tau$.

Solution

Let the path of x be parametrized by a variable σ, then

$$\begin{aligned}\delta S'[x] &= \delta\int_A^B (-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma})\mathrm{d}\sigma \\ &= \int_A^B \left[-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} - 2g_{\mu\nu}\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\right]\mathrm{d}\sigma \\ &= \int_A^B \left[-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} - 2g_{\mu\nu}\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right]\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}\mathrm{d}\tau\end{aligned}$$

but this integrand is just $2\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}$ times the one in (2). It will thus also give (3).

c)

What is the basic property that is possessed by S but not by S'?

Solution

The action S is independent of our parametrization of the path, while S' isn't.

d)

The geodesic equations obtained in a) and b) arise as the so called Euler-Lagrange equations (EL eqs). The EL eqs are usually expressed in terms of a Lagrangian L as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0. \tag{EL}$$

Construct a Lagrangian (like the one in b) above) by turning the metric $\mathrm{d}s^2(x^i, \mathrm{d}x^i)$ into a Langrangian $L(x^i, \dot{x}^i)$ by replacing $\mathrm{d}x^i$ by $\dot{x}^i$ and derive the affine connection for the metric on the 2-sphere in both coordinate systems obtained in home problem 6.

Solution

Let $a = 1$.

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \mapsto \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2 = L.$$

The Euler Lagrange equation reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{x}^\nu}\right) - \frac{\partial L}{\partial x^\nu} = 0. \tag{4}$$

For $\nu = \theta$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left(2\dot{\theta}\right) - 2\sin\theta\cos\theta\dot{\phi}^2 = 0 \\ \implies &\ddot{\theta} = \frac{1}{2}\sin 2\theta\dot{\phi}^2 \end{aligned}\tag{5}$$

and for $\nu = \phi$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left(2\sin^2\theta\dot{\phi}\right) = 0 \\ \implies &\ddot{\phi} = -2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi}. \end{aligned}\tag{6}$$

Plugging this into the Geodesic equation

$$\Gamma^{\nu}_{\mu\sigma}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\sigma}}{\mathrm{d}\tau}+\frac{\mathrm{d}^2x^{\nu}}{\mathrm{d}\tau^2}=0, \tag{7}$$

we get (using that $\Gamma^{\nu}_{\mu\sigma}$ is symmetric in μ and σ)

$$\Gamma^{\theta}_{\theta\theta}\dot{\theta}^2+2\Gamma^{\theta}_{\theta\phi}\dot{\theta}\dot{\phi}+\Gamma^{\theta}_{\phi\phi}\dot{\phi}^2+\ddot{\theta}=0$$

and

$$\Gamma^{\phi}_{\theta\theta}\dot{\theta}^2+2\Gamma^{\phi}_{\theta\phi}\dot{\theta}\dot{\phi}+\Gamma^{\phi}_{\phi\phi}\dot{\phi}^2+\ddot{\phi}=0.$$

Substituting $\ddot{\theta}$ and $\ddot{\phi}$ using (5) and (6), we get

$$\begin{aligned}\Gamma^{\theta}_{\theta\theta}\dot{\theta}^2+2\Gamma^{\theta}_{\theta\phi}\dot{\theta}\dot{\phi}+\Gamma^{\theta}_{\phi\phi}\dot{\phi}^2+\frac{1}{2}\sin 2\theta\dot{\phi}^2=0\\ \Gamma^{\phi}_{\theta\theta}\dot{\theta}^2+2\Gamma^{\phi}_{\theta\phi}\dot{\theta}\dot{\phi}+\Gamma^{\phi}_{\phi\phi}\dot{\phi}^2-2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi}=0.\end{aligned}$$

Here, $\dot{\theta}$ and $\dot{\phi}$ are independet variables so the uniqueness of polynomial coefficients implies that

$$\begin{aligned}\Gamma^{\theta}_{\phi\phi}&=-\frac{1}{2}\sin 2\theta\\ \Gamma^{\phi}_{\theta\phi}&=\frac{\cos\theta}{\sin\theta}\end{aligned}$$

while the rest of the coefficients are 0. This agrees with the previous result.

Now,

$$\mathrm{d}s^2=r^2\mathrm{d}\theta^2+\frac{1}{1-r^2}\mathrm{d}r^2\mapsto r^2\dot{\theta}^2+\frac{1}{1-r^2}\dot{r}^2=L.$$

Then, by (4), for $\nu=\theta$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left(2r^2\dot{\theta}\right)=0\\ &\implies \ddot{\theta}=-\frac{2\dot{\theta}\dot{r}}{r}\end{aligned} \tag{8}$$

and for $\nu=r$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{1-r^2}2\dot{r}^2\right)-2r\dot{\theta}^2+\frac{2r}{(1-r^2)^2}\dot{r}^2=0\\ &\implies \ddot{r}=r(1-r^2)\dot{\theta}^2-\frac{r}{1-r^2}\dot{r}^2.\end{aligned} \tag{9}$$

Plugging this into (7), we get

$$\Gamma^{\theta}_{\theta\theta}\dot{\theta}^2+2\Gamma^{\theta}_{\theta r}\dot{\theta}\dot{r}+\Gamma^{\theta}_{rr}\dot{r}^2+\ddot{\theta}=0$$

and

$$\Gamma^{r}_{\theta\theta}\dot{\theta}^2+2\Gamma^{r}_{\theta r}\dot{\theta}\dot{r}+\Gamma^{r}_{rr}\dot{r}^2+\ddot{r}=0.$$

Substituting $\ddot{\theta}$ and $\ddot{\phi}$ using (8) and (9), we get

$$\begin{aligned}\Gamma^{\theta}_{\theta\theta}\dot{\theta}^2+2\Gamma^{\theta}_{\theta r}\dot{\theta}\dot{r}+\Gamma^{\theta}_{rr}\dot{r}^2-\frac{2\dot{\theta}\dot{r}}{r}=0\\ \Gamma^{r}_{\theta\theta}\dot{\theta}^2+2\Gamma^{r}_{\theta r}\dot{\theta}\dot{r}+\Gamma^{r}_{rr}\dot{r}^2+r(1-r^2)\dot{\theta}^2-\frac{r}{1-r^2}\dot{r}^2=0\end{aligned}$$

with the solution

$$\begin{aligned}
\Gamma^{\theta}_{\theta r} &= \frac{1}{r} \\
\Gamma^{r}_{\theta\theta} &= -r(1-r^2) \\
\Gamma^{r}_{rr} &= \frac{r}{1-r^2}
\end{aligned}$$

with the rest of the coefficients $= 0$.

It is nice to see that all of this agrees with home problem 6.

Home Problem 8

Write out explicitly the Laplacian acting on a scalar field, i.e.,

$$\Box\phi = \nabla_\mu \nabla^\mu \phi, \tag{4.11}$$

on a flat two-dimensional space in polar coordinates. This operator can also be written $\nabla^\mu \nabla_\mu \phi$ where you should note the change in the position of the upper and lower indices. Why are these two expressions for the $\Box$-operator equivalent?

Solution

Since $\nabla_\mu \nabla^\mu \phi$ is a scalar,

$$\nabla_\mu \nabla^\mu \phi \stackrel{\text{local inertial frame}}{=} \partial_\mu \partial^\mu \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}.$$

If ∇^μ was a tensor,

$$\nabla_\mu \nabla^\mu \phi = \nabla^\mu \nabla_\mu \phi$$

would be completely trivial since our representation module is isomorphic to its dual, but now the situation is a bit more complicated. I am not completely comfortable with what it means in general to contract indices between an operator and a tensor like this, so I will not try to argue about that, but my intuition tells me that $❦_\mu \clubsuit^\mu = ❦^\mu \clubsuit_\mu$ no matter what $❦$ and $\clubsuit$ are; operators, spinors, etc. For now, it suffices to note that both $\nabla_\mu \nabla^\mu \phi$ and $\nabla^\mu \nabla_\mu \phi$ are scalars:

$$\nabla_\mu \nabla^\mu \phi = \partial_\mu \partial^\mu \phi = \partial^\mu \partial_\mu \phi = \nabla^\mu \nabla_\mu \phi.$$

Home Problem 9.1

Consider the two-dimensional sphere with radius a. Compute the affine connection, Riemann tensor, Ricci tensor and curvature scalar for this two-sphere in polar coordinates (θ, ϕ).

Solution

There is no metric specified, so I will assume it's the same as in home problem 6. From home problem 6 c), we know that the nonzero components of the affine connection are

$$\begin{aligned}
\Gamma^{\theta}_{\phi\phi} &= -\frac{1}{2}\sin 2\theta \\
\Gamma^{\phi}_{\phi\theta} &= \Gamma^{\phi}_{\theta\phi} = \cot\theta.
\end{aligned}$$

We may calculate $R^\lambda{}_{\mu\nu\kappa}$, $R_{\mu\nu}$, and R from their definitions

$$R^\lambda{}_{\mu\nu\kappa} = \frac{\partial\Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} - \frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} + \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta},$$

(there was some slight confusion here since Weinberg seems to have the negative of this definition)

$$R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$$
$$R = R^\mu{}_\mu$$

by the Mathematica script below. We get

$$R^\theta{}_{\phi\theta\phi} = -R^\theta{}_{\phi\theta\phi} = \sin^2\theta$$
$$R^\phi{}_{\theta\theta\phi} = -R^\phi{}_{\theta\theta\phi} = 1.$$

It is easy to convince ourselves, using the metric (1) to lower the first index, that $R^\lambda{}_{\mu\nu\kappa}$ has the right symmetry properties.

Now, it is also straightforward to compute

$$R^{\mu\nu} = R^\theta_{\mu\theta\nu} + R^\phi_{\mu\phi\nu} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix},$$

and

$$R = R^\mu{}_\mu = 2.$$

```
In[1]:= ClearAll["Global`*"]
coordinateList = {θ, ϕ};

(* Initialize and define Γ^μ_νρ as a rank 3 tensor *)
tmp[a_, b_, c_] := 0;
Γ = Array[tmp, {2, 2, 2}];
Γ[[1, 2, 2]] = -1/2 Sin[2 θ];
Γ[[2, 1, 2]] = Cot[θ];
Γ[[2, 2, 1]] = Cot[θ];

(* Initialize R^λ_μνκ as a rank 4 tensor *)
tmp[a_, b_, c_, d_] := 0;
R = Array[tmp, {2, 2, 2, 2}];

(* Loop over indices in R^ρ_μνκ *)
Do[
 Do[
  Do[
   Do[
    xκ = coordinateList[[κ]];
    xν = coordinateList[[ν]];

    (* R^λ_μνκ = ∂Γ^λ_μν/∂x^κ - ∂Γ^λ_μκ/∂x^ν + Γ^η_μνΓ^λ_κη - Γ^η_μκΓ^λ_νη *)
    R[[λ, μ, ν, κ]] += ∂_xν Γ[[λ, μ, κ]] - ∂_xκ Γ[[λ, μ, ν]];
    Do[
     R[[λ, μ, ν, κ]] += Γ[[η, μ, κ]] × Γ[[λ, ν, η]] - Γ[[η, μ, ν]] × Γ[[λ, κ, η]],
     {η, {1, 2}}],

    {κ, {1, 2}}],
   {ν, {1, 2}}],
  {μ, {1, 2}}],
 {λ, {1, 2}}]


Print["R^θ_θνκ = ", MatrixForm[FullSimplify[R[[1, 1]]]]]
Print["R^θ_ϕνκ = ", MatrixForm[FullSimplify[R[[1, 2]]]]]
```

```
Print["R^ϕ_θνκ = ", MatrixForm[FullSimplify[R[[2, 1]]]]]
Print["R^ϕ_ϕνκ = ", MatrixForm[FullSimplify[R[[2, 2]]]]]
```

$R^{\theta}{}_{\theta\nu\kappa} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$R^{\theta}{}_{\phi\nu\kappa} = \begin{pmatrix} 0 & \mathrm{Sin}[\theta]^2 \\ -\mathrm{Sin}[\theta]^2 & 0 \end{pmatrix}$

$R^{\phi}{}_{\theta\nu\kappa} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$R^{\phi}{}_{\phi\nu\kappa} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

In[15]:=
```
(* Initialize Ricci^μ_ν as rank 2 tensor *)
tmp[a_, b_] := 0;
Ricci = Array[tmp, {2, 2}];

(* Loop over indices in Ricci_μν *)
Do[
 Do[
  Do[
   Ricci[[μ, ν]] += R[[σ, μ, σ, ν]],
   {σ, 1, 2}],
  {ν, 1, 2}],
 {μ, 1, 2}]

Print["Ricci_μν = ", MatrixForm[FullSimplify[Ricci]]]
```

$\mathrm{Ricci}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{Sin}[\theta]^2 \end{pmatrix}$

In[19]:=
```
(* Calculate curvature scalar *)
gInv = {{1, 0}, {0, 1/Sin[θ]^2}};
Print["R = ", FullSimplify[Tr[gInv.Ricci]]]
```

R = 2

Home Problem 9.2

Consider the metrics

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - k\frac{r^2}{L^2}} + r^2\mathrm{d}\phi^2, \quad k = 1, 0, -1. \tag{4.15}$$

a)

Compute the Riemann tensor, the Ricci tensor and the curvature scalar.

Solution

We combine some of the Mathematica scripts we've written thus far into one large Riemann tensor-, Ricci tensor-, and curvature scalar-calculator:

```
In[1]:= ClearAll["Global`*"]
coordinateList = {r, ϕ};

(* Loop over k *)
Do[
 Print["
------------ k = ", k, " ------------"];

 (* Define gμν *)
 g = ( 1/(1-k r^2/L^2)   0
       0                 r^2 );

 (* Initialize Γ^μ_νρ as rank 3 tensor *)
 tmp[a_, b_, c_] := 0;
 Γ = Array[tmp, {2, 2, 2}];

 (* Loop over indices in Γ^μ_νρ *)
 Do[
  Do[
   Do[
    Do[
     xμ = coordinateList[[μ]];
     xν = coordinateList[[ν]];
     xρ = coordinateList[[ρ]];

     Γ[[μ, ν, ρ]] += 1/2 (Inverse[g])[[σ, μ]] (∂_xν g[[ρ, σ]] + ∂_xρ g[[ν, σ]] - ∂_xσ g[[ν, ρ]]); ,
     {σ, {1, 2}}],
    {ρ, {1, 2}}],
   {ν, {1, 2}}],
  {μ, {1, 2}}];

 (* Initialize R^λ_μνκ as a rank 4 tensor *)
 tmp[a_, b_, c_, d_] := 0;
 R = Array[tmp, {2, 2, 2, 2}];
```

```
(* Loop over indices in R^ρ_μνκ *)
Do[
 Do[
  Do[
   Do[
    xκ = coordinateList[[κ]];
    xν = coordinateList[[ν]];

    (* R^λ_μνκ = ∂Γ^λ_μν/∂x^κ - ∂Γ^λ_μκ/∂x^ν + Γ^η_μνΓ^λ_κη - Γ^η_μκΓ^λ_νη *)
    R[[λ, μ, ν, κ]] += ∂_xν Γ[[λ, μ, κ]] - ∂_xκ Γ[[λ, μ, ν]];
    Do[
     R[[λ, μ, ν, κ]] += Γ[[η, μ, κ]] × Γ[[λ, ν, η]] - Γ[[η, μ, ν]] × Γ[[λ, κ, η]],
     {η, {1, 2}}],

    {κ, {1, 2}}],
   {ν, {1, 2}}],
  {μ, {1, 2}}],
 {λ, {1, 2}}];


Print["R^θ_θνκ = ", MatrixForm[FullSimplify[R[[1, 1]]]]];
Print["R^θ_ϕνκ = ", MatrixForm[FullSimplify[R[[1, 2]]]]];
Print["R^ϕ_θνκ = ", MatrixForm[FullSimplify[R[[2, 1]]]]];
Print["R^ϕ_ϕνκ = ", MatrixForm[FullSimplify[R[[2, 2]]]]];

(* Initialize Ricci^μ_ν as rank 2 tensor *)
tmp[a_, b_] := 0;
Ricci = Array[tmp, {2, 2}];

(* Loop over indices in Ricci_μν *)
Do[
 Do[
  Do[
   Ricci[[μ, ν]] += R[[σ, μ, σ, ν]],
   {σ, {1, 2}}],
  {ν, {1, 2}}],
 {μ, {1, 2}}];
```

```
Print["Ricci_μν = ", MatrixForm[FullSimplify[Ricci]]];

(* Calculate curvature scalar *)
Print["R = ", FullSimplify[Tr[Inverse[g].Ricci]]];,
{k, {-1, 0, 1}}]
```

------------ k = -1 ------------

$R^{\theta}{}_{\theta\nu\kappa} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$R^{\theta}{}_{\phi\nu\kappa} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$R^{\phi}{}_{\theta\nu\kappa} = \begin{pmatrix} 0 & \frac{2}{L^2+r^2} \\ -\frac{2}{L^2+r^2} & 0 \end{pmatrix}$

$R^{\phi}{}_{\phi\nu\kappa} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\mathrm{Ricci}_{\mu\nu} = \begin{pmatrix} -\frac{2}{L^2+r^2} & 0 \\ 0 & 0 \end{pmatrix}$

$R = -\frac{2}{L^2}$

------------ k = 0 ------------

$R^{\theta}{}_{\theta\nu\kappa} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$R^{\theta}{}_{\phi\nu\kappa} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$R^{\phi}{}_{\theta\nu\kappa} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$R^{\phi}{}_{\phi\nu\kappa} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\mathrm{Ricci}_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$R = 0$

------------ k = 1 ------------

$R^{\theta}{}_{\theta\nu\kappa} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$R^{\theta}{}_{\phi\nu\kappa} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$R^{\phi}{}_{\theta\nu\kappa} = \begin{pmatrix} 0 & -\frac{2}{L^2-r^2} \\ \frac{2}{L^2-r^2} & 0 \end{pmatrix}$

$$R^{\phi}{}_{\phi\nu\kappa} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{Ricci}_{\mu\nu} = \begin{pmatrix} \frac{2}{L^2-r^2} & 0 \\ 0 & 0 \end{pmatrix}$$

$$R = \frac{2}{L^2}$$

b)

Do the curvature scalars, R, come out as expected (their dependence on L and their sign)?

Solution

Yes, they come out as expected. Since these are the metrics of Home Problem 5 a) and c) and flat space.

c)

What is the geometry of the manifold in each case? Note that $r \leq L$ in the case $k = 1$. Why is this condition necessary?

Solution

For $k = 1$, the geometry is that of a sphere with radius L. It has constant curvature $R = \frac{2}{L}$. For $k = -1$, the geometry is that of a pseudosphere, or hyperbolic space, with "radius" L. It has constant curvature $R = -\frac{2}{L}$. For $k = 0$, the geometry is that of flat space. It has constant curvature $R = 0$.

If $r = L$, the metric is infinite, this could be likened to being on the light cone. If $r > L$, the metric is negative. this could be likened to being timelike.

Home Problem 10

Consider a space-time whose Riemann tensor is

$$R_{\mu\nu\rho\sigma} = f(x)(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{4.16}$$

a)

Show that this tensor has the correct symmetry properties to be a Riemann tensor.

Solution

The correct symmetry properties are

A. $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$

B. $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$

C. $R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0$

We show A. by

$$R_{\mu\nu\rho\sigma} = f(x)(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \overset{g_{\mu\nu}=g_{\nu\mu}}{=} f(x)(g_{\rho\mu}g_{\sigma\nu} - g_{\nu\rho}g_{\mu\sigma}) = R_{\rho\sigma\mu\nu},$$

and B. by

$$R_{\mu\nu\rho\sigma} = f(x)(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) = f(x)(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) = -R_{\mu\nu\rho\sigma},$$

and C. by

$$\begin{aligned} R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} &= f(x)(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} + g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma} + g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu}) \\ &\overset{g_{\mu\nu}=g_{\nu\mu}}{=} 0. \end{aligned}$$

b)

Show that the function has to be constant in dimension $D \geq 3$.

Solution

Einstein's equation reads

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \quad R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}. \tag{10}$$

From (4.16),

$$\begin{aligned} R_{\mu\nu} &= f(x)(g^{\sigma}{}_{\sigma} g_{\mu\nu} - g^{\sigma}{}_{\nu} g_{\mu\sigma}) \\ &= f(x)(D-1) g_{\mu\nu} \end{aligned}$$

since $g_{\mu}{}^{\nu}$ is the identity. We also have that

$$R = R^{\mu}{}_{\mu} = f(x)(D^2 - D).$$

Putting this into (10) and taking the divergence, we get

$$\begin{aligned} 0 &= \nabla^{\mu} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - 8\pi G T_{\mu\nu} \right] \\ &\overset{T_{\mu\nu}\text{ conserved}}{=} (\nabla^{\mu} f)(x)(D-1) g_{\mu\nu} + f(x)(D-1)(\nabla^{\mu} g_{\mu\nu}) - \frac{1}{2}(\nabla^{\mu} g_{\mu\nu}) R - \frac{1}{2} g_{\mu\nu}(\nabla^{\mu} R) \\ &\overset{\substack{\nabla^{\mu} g_{\mu\nu} = \partial^{\mu}\eta_{\mu\nu} = 0 \\ \text{in local inertial frame}}}{=} \left[(D-1) - \frac{1}{2}(D^2 - D) \right] (\nabla^{\mu} f)(x) g_{\mu\nu}. \end{aligned}$$

This is 0 if either $(D-1) - \frac{1}{2}(D^2 - D) = 0$, which happens when $D = 1, 2$, or $(\nabla^{\mu} f)(x) =$, i.e. f is constant.

c)

Find the relation between the cosmological constant Λ and f by solving Einstein's equations in an empty spacetime.

Solution

Einstein's equation now reads

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{11}$$

In empty space, the $T_{\mu\nu} = 0$, so (11) gives

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R - \Lambda \eta_{\mu\nu} &= 0 \\ f(x)(D-1)\eta_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} f(x)(D^2 - D) - \Lambda \eta_{\mu\nu} &= 0 \\ \Lambda &= f(x) \left(D - 1 + \frac{1}{2}(D^2 - D) \right). \end{aligned}$$

We can also see that $\Lambda = 0$ when $D = 1$.