❦ Komplexanalys i flera variabler ❦
Assignment 3

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Prove that subharmonicity is a local property. That is, given an open set $U \subset \mathbb{C}$, a function $f\colon U \to \mathbb{R} \cup \{ -\infty \}$ is subharmonic if and only if for every $p \in U$ there exists a neighbourhood W of p, $W \subset U$, such that $f|_W$ is subharmonic. Hint: Perhaps try to use the maximum principle and Exercise 2.4.10.

Solution

Definition 1. A function $f\colon U \to \mathbb{R} \cup \{ -\infty \}$ is *subharmonic* if it is upper-semicontinuous and for every ball $B_r(a) \subset U$, and every function g continuous on $\overline{B}_r(a)$ and harmonic on $B_r(a)$, such that $f(x) \leq g(x)$ for $x \in \partial B_r(a)$, we have $f(x) \leq g(x)$, for all $x \in B_r(a)$.

Lemma 1 (Exercise 2.4.10 in Lebl)**.** *Suppose $U \subset \mathbb{C}$ is open and $g\colon U \to \mathbb{R}$ is harmonic. Then $f\colon U \to \mathbb{R} \cup \{ -\infty \}$ is subharmonic if and only if $f - g$ is subharmonic.*

Proof. Let h be continuous on some ball $\overline{B}_r(a)$ and harmonic on $B_r(a) \subset U$. If $f - g$ is subharmonic, then $f - g \leq h - g$ on $\partial B_r(a)$ implies $f - g \leq h - g$ on $B_r(a)$ (since $h - g$ is harmonic), and so $f \leq h$ on $\partial B_r(a)$ implies $f \leq h$ on $B_r(a)$. f is hence subharmonic.

If $f - g$ is not subharmonic, there exists some h, continuous on $\overline{B}_r(a)$ and harmonic on $B_r(a)$, such that $f - g \leq h$ on $\partial B_r(a)$, but $(f - g)(z) > h(z)$ form some $z \in B_r(a)$. But then we have found a function, $h + g$, continuous on $\overline{B}_r(a)$ and harmonic on $B_r(a)$, such that $f \leq h + g$ on $\partial B_r(a)$ but $f(z) > (h + g)(z)$ for some $z \in B_r(a)$. So f is not subharmonic. □

The if part Assume that, for every $p \in U$, there exists a neighbourhood W of p such that $f|_W$ is subharmonic. Then, for every $p \in U$ and such neighbourhood W of p, and for every ball $B_{r_p}(a_p) \subset W$ and g continuous on $\overline{B}_{r_p}(a_p)$ and harmonic on $B_{r_p}(a_p)$, $f(z) \leq g(z)$ on $\partial B_{r_p}(a_p)$ implies that $f(z) \leq g(z)$ on $B_{r_p}(a_p)$.

We want to show that, for $B_r(a) \subset U$ and g continuous on $\overline{B}_r(a)$ and harmonic on $B_r(a)$, $f(z) \leq g(z)$ on $\partial B_r(a)$ implies that $f(z) \leq g(z)$ on $B_r(a)$. So, for a contradiction, assume f is not subharmonic on U. Then there exists g continuous on $\overline{B}_r(a)$ and harmonic on $B_r(a)$ such that $f(z) \leq g(z)$ on $\partial B_r(a)$ and $f(p_0) > g(p_0)$ for some p_0 in $B_r(a)$. By lemma 1, f is subharmonic iff $f - g$ is subharmonic, so we have that $(f - g)(z) \leq 0$ on $\partial B_r(a)$, but $(f - g)(p_0) > 0$.

Firstly, we note that since $\overline{B}_r(a)$ is compact, the open cover defined as all of the neighbourhoods W of all of the points p in $\overline{B}_r(a)$ has a finite subcover. Call that subcover $\{ W_n \}$. We may also, since if a function h is subharmonic on A it is subharmonic on $B \subset A$, define $\{ V_n \}$ as $V_n = W_n \cap B_r(a)$. This we do in order to avoid trouble with g as it is not defined outside of $\overline{B}_r(a)$.

Secondly, since $f - g$ is upper-semicontinuous on the compact set $\overline{B}_r(a)$, it attains its maximum there. Since this maximum evidently is not on the boundary, it is at some inner point q_0. We may just as well let $q_0 \in V_0$. But since $f - g$ is subharmonic on V_0, and since $f - g$ attains its max at an inner point of V_0, $f-g$ is by the Maximum Principle constant on V_0. And thus also constant on $\overline{V_0}$ by upper-semicontinuity and $(f-g)(q_0)$ being a global maximum on $\overline{B}_r(a)$. We may now go on and chose a point q_1 on ∂V_0 and there will exist some other neighbourhood V_1 from our open cover containing q_1. Likewise, since $f-g$ will attain its max at the inner point q_1 of V_1, it is constant on $\overline{V_1}$. The constantness of $f-g$ will thus spread like a disease through $B_r(a) = \bigcup\{V_n\}$ (since there are only finitely many V_n, this logic holds). But then $f = g + \text{constant}$ is harmonic, contradicting our assumption about f not being subharmonic.

The only if part If f is harmonic on U, then, since U is a neighbourhood of each point $p \in U$, we have that

> for every $p \in U$ there exists a neighbourhood W of p, $W \subset U$ such that $f|_W$ is subharmonic

is true if we just chose $W = U$ for each p.

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a)

Assume u and v are subharmonic on $U \subseteq \mathbb{C}$. Prove that $\log\left(\mathrm{e}^u + \mathrm{e}^v\right)$ is subharmonic.

Solution

Restating definition 1, we want to show that, for each $B_r(a) \subset U$,

> if $\log\left(\mathrm{e}^{u(z)} + \mathrm{e}^{v(z)}\right) \le g(z)$ on $\partial B_r(a)$ for any g continuous on $\overline{B_r(a)}$ and harmonic on $B_r(a)$, then $\log\left(\mathrm{e}^{u(z)} + \mathrm{e}^{v(z)}\right) \le g(z)$ on $B_r(a)$. (1)

We begin by stating

$$\begin{aligned} &\log\left(\mathrm{e}^{u(z)} + \mathrm{e}^{v(z)}\right) \le g(z) && (2)\\ \iff\ &\mathrm{e}^{u(z)} + \mathrm{e}^{v(z)} \le \mathrm{e}^{g(z)} \\ \iff\ &\begin{cases} u(z) \le g(z) \\ v(z) - \log\left(1 - \mathrm{e}^{u(z)-g(z)}\right) \le g(z) \end{cases} . && (3)\end{aligned}$$

$-\log\left(1 - \mathrm{e}^{u(z)-g(z)}\right)$ is subharmonic since it is a strictly increasing convex function of a subharmonic function:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(-\log(1-\mathrm{e}^x)\right) &= \frac{\mathrm{e}^x}{1-\mathrm{e}^x} > 0 \quad \text{on } [-\infty, 0) \quad \text{(where } u-g \text{ lives)}, \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(-\log(1-\mathrm{e}^x)\right) &= \frac{\mathrm{e}^x}{(1-\mathrm{e}^x)^2} > 0. \end{aligned}$$

So since both $u(z)$ and $v(z) - \log\left(1 - \mathrm{e}^{u(z)-g(z)}\right)$ are subharmonic, we have that "(3) true on $\partial B_r(a)$" implies "(3) true on $B_r(a)$", and hence also "(2) true on $\partial B_r(a)$" implies "(2) true on $B_r(a)$", which is what (1) says.

b)

Show that if $F = (F_1, \dots, F_m)$ is a tuple of holomorphic functions on $U \subseteq \mathbb{C}^n$, then $\log|F|^2$ is plurisubharmonic.

Solution

By proposition 2.4.8 in Lebl, we want to show that the Hessian matrix

$$\frac{\partial^2 \log |F|^2}{\partial \overline{z}_j \partial z_k}$$

is positive semidefinite at all points $p \in U$. Firstly, we make the observation that $|F|^2 = |TF|^2$ for any unitary matrix T. We may thus assume that F has only nonzero values in the F_1-direction at p:

$$F(p) = (F_1(p), 0, \ldots, 0) \tag{4}$$
$$|F(p)|^2 = |F_1(p)|^2. \tag{5}$$

Then, by using the holomorphicity of F (and suppressing the p dependence),

$$\begin{aligned}
\frac{\partial^2 \log |F|^2}{\partial \overline{z}_j \partial z_k} &= \frac{\partial}{\partial \overline{z}_j} \frac{1}{|F|^2} \frac{\partial |F|^2}{\partial z_k} \\
&= -\frac{1}{(|F|^2)^2} \frac{\partial |F|^2}{\partial \overline{z}_j} \frac{\partial |F|^2}{\partial z_k} + \frac{1}{|F|^2} \frac{\partial^2 |F|^2}{\partial \overline{z}_j \partial z_k} \\
&= -\frac{1}{(|F|^2)^2} \left(\frac{\partial \overline{F}_1}{\partial \overline{z}_j} F_1 + \cdots + \frac{\partial \overline{F}_m}{\partial \overline{z}_j} F_m \right) \left(\overline{F}_1 \frac{\partial F_1}{\partial z_k} + \cdots + \overline{F}_m \frac{\partial F_m}{\partial z_k} \right) \\
&+ \left(\frac{\partial \overline{F}_1}{\partial \overline{z}_j} \frac{\partial F_1}{\partial z_k} + \cdots + \frac{\partial \overline{F}_m}{\partial \overline{z}_j} \frac{\partial F_m}{\partial z_k} \right) \\
&\overset{(4)}{=} -\frac{1}{(|F|^2)^2} \overline{F}_1 F_1 \frac{\partial \overline{F}_1}{\partial \overline{z}_j} \frac{\partial F_1}{\partial z_k} + \frac{1}{|F|^2} \left(\frac{\partial \overline{F}_1}{\partial \overline{z}_j} \frac{\partial F_1}{\partial z_k} + \cdots + \frac{\partial \overline{F}_m}{\partial \overline{z}_j} \frac{\partial F_m}{\partial z_k} \right) \\
&\overset{(5)}{=} \frac{1}{|F|^2} \left(\frac{\partial \overline{F}_2}{\partial \overline{z}_j} \frac{\partial F_2}{\partial z_k} + \cdots + \frac{\partial \overline{F}_m}{\partial \overline{z}_j} \frac{\partial F_m}{\partial z_k} \right).
\end{aligned} \tag{6}$$

To show that (6) is positive semidefinite, we note that we may disregard the $\frac{1}{|F|^2}$-factor, and that if we can show that each term

$$\frac{\partial \overline{F}_\alpha}{\partial \overline{z}_j} \frac{\partial F_\alpha}{\partial z_k} \tag{7}$$

is positive semidefinite, then the sum is positive semidefinite and we are done. But (7) is positive semidefinite since if $w \in \mathbb{C}^n \setminus \{ 0 \}$,

$$\sum_{j,k} \overline{w}_j \frac{\partial \overline{F}_\alpha}{\partial \overline{z}_j} \frac{\partial F_\alpha}{\partial z_k} w_k = \overline{\heartsuit} \heartsuit = |\heartsuit|^2 \geq 0$$

with

$$\heartsuit = \sum_k \frac{\partial F_\alpha}{\partial z_k} w_k.$$

3

Give an example of a harmonic function on $\mathbb{C}^2$, which is not the real part of a holomorphic function. Make sure to provide an explanation of why the function has this property.

Also give an example of a harmonic function u on a domain $V \subseteq \mathbb{C}^2$ and a holomorphic change of coordinates, i.e., a biholomorphism $F \colon U \to V$, $U \subseteq \mathbb{C}^2$ a domain, such that u is harmonic but $u \circ F$ is not.

Solution

We have that f is *harmonic* if

$$\frac{\partial^2 f}{\partial \overline{z}_j \partial z_j} = 0$$

for $j = 1, 2$. Consider $f : z \mapsto (z_1 + \overline{z}_1)(z_2 + \overline{z}_2)$. It is evidently harmonic since $\frac{\partial f}{\partial z_1}$ is constant in $\overline{z}_1$ and $\frac{\partial f}{\partial z_2}$ is constant in $\overline{z}_2$. But it is not the real part of some holomorphic function since if

$$f(z) = g(z) + \overline{g(z)}$$

for some holomorphic g, expanding f in power series of z_1 and z_2 yields

$$f(z) = g(0) + \overline{g(0)} + g'(0)z_1 + \overline{g'(0)}\overline{z}_2 + \frac{1}{2}g''(0)z_1^2 + \frac{1}{2}\overline{g''(0)}\overline{z}_2^2 + \mathcal{O}(|z|^3). \tag{8}$$

But f cannot have the series expansion (8) since it doesn't contain any of the cross terms $z_1\overline{z}_2$ and $z_2\overline{z}_1$.

f is harmonic on $\mathbb{C}^2$, and $F : (z_1, z_2) \mapsto (z_1 + z_2, z_2)$ is a biholomorphism from $\mathbb{C}$ to $\mathbb{C}$, but

$$f \circ F(z) = (z_1 + z_2 + \overline{z}_1 + \overline{z}_2)(z_2 + \overline{z}_2)$$

is not harmonic since it has a term $2z_2\overline{z}_2$ which will not be killed by $\frac{\partial^2}{\partial z_2 \partial \overline{z}_2}$.

4

Show that every open set $U \subset \mathbb{R}^n$ is convex with respect to real polynomials.

Solution

U is convex with respect to real polynomials $\mathcal{P}(\mathbb{R}^n)$ on $\mathbb{R}^n$ if, for each $K \subset\subset U$, we have that $\widehat{K} \subset\subset U$, where

$$\widehat{K} := \left\{ x \in U \text{ s. th. } f(x) \leq \sup_{y \in K} f(y) \text{ for all } f \in \mathcal{P}(\mathbb{R}^n) \right\}.$$

So let $K \subset\subset U$. We want to show that $\widehat{K} \subset \overline{K}$ by showing that if x is in $\widehat{K} \setminus K$, then x will still be in $\overline{K}$. To that end, let $x \in \widehat{K} \setminus K$ and consider the real polynomial

$$p(y) = -\left[(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\right].$$

This polynomial has a global isolated maximum at $y = x$. So for this x to be in $\widehat{K}$, we must have that $p(x) = \sup_{y \in K} p(y)$, which can only happen if x is a limit point of K (by continuity and maximum being isolated).

Thus, since $\widehat{K} \subset \overline{K}$, we have that $K \subset\subset U$ implies $\widehat{K} \subset\subset U$.

5

a)

Let H be the hyperplane $H := \{ z_2 = 0 \} \subset \mathbb{C}^2_{(z1,z2)}$, show that $H^{\mathrm{C}} = \mathbb{C}^2 \setminus H$ is Hartogs pseudoconvex.

Solution

Let

$$f(z) = \max\big(-\log|z_2|, |z|^2\big).$$

This function is plurisubharmonic since

- $-\log|z_2| = -\operatorname{Re}\log z_2$ is the real part of a holomorphic function, so it is pluriharmonic and thus plurisubharmonic
- $|z|^2 = z_1\overline{z}_1 + z_2\overline{z}_2$ is plurisubharmonic since $\frac{\partial^2|z|^2}{\partial\overline{z}_j\partial z_k} = \delta_{jk}$ is positively definite
- the pointwise maximum of two plurisubharmonic functions is plurisubharmonic.

It is also continuous of H^{C}.

For a given $r \in \mathbb{R}$, we have that

$$\begin{aligned} A &= \left\{\, z \in H^{\mathrm{C}} \text{ s. th. } f(z) < r \,\right\} \\ &= \left\{\, z \in H^{\mathrm{C}} \text{ s. th. } -\log|z_2| < r \,\right\} \cap \left\{\, z \in \mathbb{C}^2 \setminus H \text{ s. th. } |z|^2 < r \,\right\} \\ &= \left\{\, z \in H^{\mathrm{C}} \text{ s. th. } |z_2| > \mathrm{e}^{-r} \,\right\} \cap \left\{\, z \in H^{\mathrm{C}} \text{ s. th. } |z|^2 < r \,\right\}. \end{aligned}$$

If $r < 0$, we can't take $\sqrt{r}$, and so $A = \emptyset$ (which is trivially compact). Otherwise, the set looks like the intersection shown in Figure 1. Since A does not intersect ∂H^{C}, its closure in H^{C} is the same as its closure in $\mathbb{C}^2$. Since A is bounded, it is relatively compact in $\mathbb{C}^2$ (Heine-Borel theorem).

b)

Let $B = \mathbb{R}^2 \subset \mathbb{C}^2$ be naturally embedded (that is, it is the set where z_1 and z_2 are real). Show that the set $\mathbb{C}^2 \setminus \mathbb{R}^2$ is not Hartogs pseudoconvex.

Solution

By theorem 2.5.6 in Lebl, since B is a domain, B being Hartogs pseudoconvex is equivalent to $-\log\rho(z)$ being plurisubharmonic, where $\rho(z)$ is the distance from z to ∂B. In our case,

$$\rho(z) = \max\left(\operatorname{Im}|z_1|, \operatorname{Im}|z_2|\right).$$

Consider some z with $\operatorname{Im}|z_1| > \operatorname{Im}|z_2|$, then

$$\begin{aligned} \frac{\partial^2}{\partial\overline{z}_1\partial z_1}\left(-\log\rho(z)\right) &= -\frac{\partial^2}{\partial\overline{z}_1\partial z_1}\log\frac{|z_1 - \overline{z}_1|}{2} \\ &= -\frac{\partial}{\partial\overline{z}_1}\frac{1}{z_1 - \overline{z}_1} \\ &= -\frac{1}{(z_1 - \overline{z}_1)^2} \end{aligned}$$

while the rest of the entries in the complex Hessian are 0. Thus the complex Hessian is not positive semi-definite, implying $-\log\rho(z)$ is not plurisubharmonic, which in turn implies that B is not Hartogs pseudoconvex.

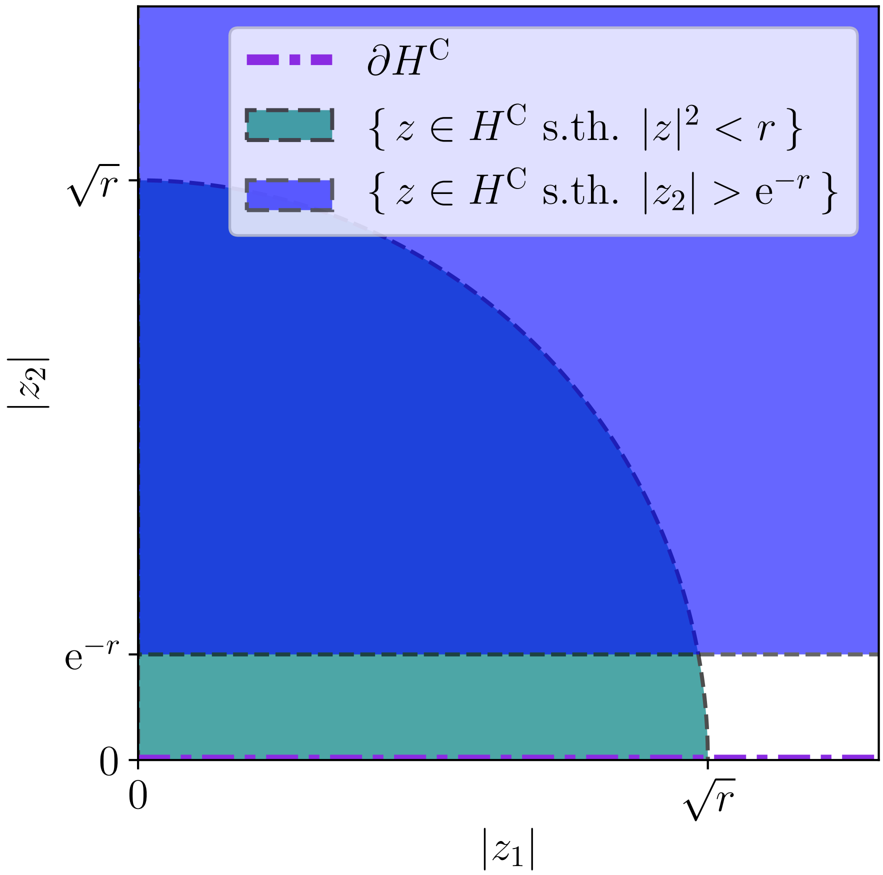
∂H^{C}
$\{ z \in H^{\mathrm{C}}$ s.th. $|z|^2 < r \}$
$\{ z \in H^{\mathrm{C}}$ s.th. $|z_2| > \mathrm{e}^{-r} \}$
$\sqrt{r}$
e^{-r}
0
0
$\sqrt{r}$
$|z_2|$
$|z_1|$

Figure 1