❦ Quantum Field Theory Problem 4 ❦

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The interactions of pions at low energy can be described by a phenomenological model called the *linear sigma model.* Essentially, this model consists of N real scalar fields coupled by a ϕ^4 interaction that is symmetric under rotations of the N fields. More specifically, let $\Phi^i(x)$, $i = 1, \ldots, N$ be a set of N fields governed by the Hamiltonian

$$H = \int \mathrm{d}^3\mathbf{x} \left(\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\Phi)^2 + V(\Phi^2) \right), \tag{1}$$

where $\Phi^2 = \Phi^i\Phi^i$, and

$$V(\Phi^2) = \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4}(\Phi^2)^2 \tag{2}$$

is a symmetric function under rotations of Φ^i. For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to H; hence, V is the field potential energy.

(What does this Hamiltonian have to do with the strong interactions? There are two types of light quarks, u and d. These quarks have identical strong interactions, but different masses. If these quarks are massless, the Hamiltonian of the strong interactions is invariant to unitary transformations of the 2-component object (u, d):

$$\begin{pmatrix} u \\ d \end{pmatrix} \mapsto \exp\{i\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}/2\} \begin{pmatrix} u \\ d \end{pmatrix}.$$

This transformation is called an *isospin* rotation. If, in addition, the strong interactions are described by a vector "gluon" field (as is true in QCD), the strong interaction Hamiltonian is invariant to the isospin rotations done separately on the left-handed and right-handed components of the quark fields. Thus, the complete symmetry of QCD with two massless quarks is $SU(2) \times SU(2)$. It happens that $SO(4)$, the group of rotations in 4 dimensions, is isomorphic to $SU(2) \times SU(2)$, so for $N = 4$, the linear sigma model has the same symmetry group as the strong interactions.)

a)

Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly N copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter λ.

a.i)

Show that the propagator is

$$\overbracket{\Phi^i(x)\Phi^j}(y) = \delta^{ij} D_{\mathrm{F}}(x - y),$$

where D_{F} is the standard Klein-Gordon propagator for mass m,

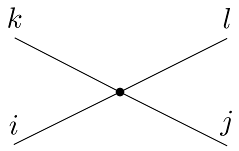
Solution

If $\lambda = 0$, (1) is just the Klein-Gordon Hamiltonian. Using the Peskin and Schroeder equation numbering, we have that

$$
\begin{aligned}
\contraction{}{\Phi}{^i(x)}{\Phi}\Phi^i(x)\Phi^j(y) &:= \begin{cases} [\Phi^{i+}(x), \Phi^{j-}(y)], & x^0 > y^0 \\ [\Phi^{i+}(y), \Phi^{j-}(x)], & x^0 < y^0 \end{cases} \\
&\overset{(4.32)}{=} \int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^6} \frac{1}{\sqrt{2 \cdot 2E_\mathbf{p}E_\mathbf{q}}} \left[a^i_\mathbf{p}, a^{j\dagger}_\mathbf{q}\right] \begin{cases} \exp\{i(-p^\alpha x_\alpha + q^\alpha y_\alpha)\}, & x^0 > y^0 \\ \exp\{i(p^\alpha x_\alpha - q^\alpha y_\alpha)\}, & x^0 < y^0 \end{cases} \\
&\overset{(2.29)}{=} \int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^6} \frac{1}{\sqrt{2 \cdot 2E_\mathbf{p}E_\mathbf{q}}} \delta^{ij}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) \begin{cases} \exp\{i(-p^\alpha x_\alpha + q^\alpha y_\alpha)\}, & x^0 > y^0 \\ \exp\{i(p^\alpha x_\alpha - q^\alpha y_\alpha)\}, & x^0 < y^0 \end{cases} \\
&= \delta^{ij} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \begin{cases} \exp\{ip^\alpha(-x+y)_\alpha\}, & x^0 > y^0 \\ \exp\{ip^\alpha(x-y)_\alpha\}, & x^0 < y^0 \end{cases} \\
&\overset{(2.50)}{=} \begin{cases} D(x-y), & x^0 > y^0 \\ D(x-y), & x^0 < y^0 \end{cases} \\
&\overset{(2.60)}{=} D_\mathrm{F}(x-y).
\end{aligned}
$$

a.ii)

and that there is one type of vertex given by



$$= -2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right).$$

(That is, a vertex between two Φ^1:s and two Φ^2:s has the value $(-2i\lambda)$; that between four Φ^1:s has the value $(-6i\lambda)$.)

Solution

TODO: BOTH ERIC & ARVID SEEM TO THINK THIS IS TRIVIAL, BUT WHY THO? USE (A.2) FROM PESKIN & SCHROEDER?

a.iii)

Compute, to leading order in λ, the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$, in the center-of-mass frame, for the scattering processes

$$\Phi^1\Phi^1 \to \Phi^1\Phi^2, \quad \Phi^1\Phi^1 \to \Phi^2\Phi^2, \quad \text{and} \quad \Phi^1\Phi^1 \to \Phi^1\Phi^1$$

as a function of their center-of-mass energy.

Solution

$$\Phi^1\Phi^1 \to \Phi^1\Phi^2: \quad i\mathcal{M} = TODO \implies \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_\mathrm{CM} = TODO$$

b)

Now consider the case $m^2 < 0$: $m^2 = -\mu^2$. In this case, V has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since V is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting Φ^i toward the minimum of V. By rotational invariance, we can consider this shift to be in the Nth direction. Write, then,

$$\begin{aligned} \Phi^i(x) &= \pi^i(x), \quad i = 1, \ldots, N-1 \\ \Phi^N(x) &= v + \sigma(x), \end{aligned}$$

where v is a constant chosen to minimize V. (The notation π^i suggests a pion field and should not be confused with a canonical momentum.)

b.i)

Show that, in these new coordinates (and substituting for v its expression in terms of λ and μ), we have a theory of a massive σ field and $N-1$ *massless* pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \to 0$.

Solution

As in Home Problem 1 c), we have that

$$v = \frac{\mu}{\sqrt{\lambda}}$$

minimizes the potential. Substituting this and $\Phi^i(x) = (\pi^j(x), v + \sigma(x)), j = 1, \ldots, N-1$ into (2), we get (see Figure 1 for this calculation)

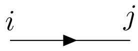
$$V(\Phi^2) = -\frac{\mu^4}{4\lambda} + \sqrt{\lambda}\mu\sigma^3 + \pi^2\sqrt{\lambda}\mu\sigma + \frac{\lambda\sigma^4}{4} + \frac{1}{2}\pi^2\lambda\sigma^2 + \frac{\pi^4\lambda}{4} + \mu^2\sigma^2, \tag{3}$$

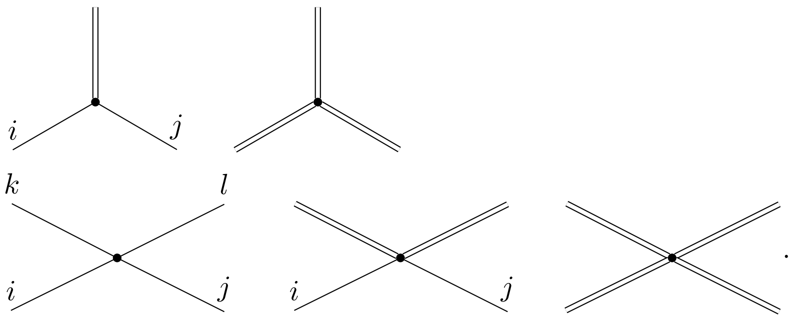
where $\pi^2 = \pi^j\pi^j$. We see that the σ field has a mass term, $\mu^2\sigma^2$, while the π fields have no mass terms. The fields are also evidently "interacting through cubic and quartic potential energy terms which all become small as $\lambda \to 0$" since all those terms have a factor of λ or $\sqrt{\lambda}$ in them.

b.ii)

Construct the Feynman rules by assigning values to the propagators and vertices:

$\overbracket{\sigma\sigma} =$

$\overbracket{\pi^i\pi^j} =$ 



Solution

$\overbracket{\sigma\sigma}$ is just the propagator for a free σ particle is,

$$\overbracket{\sigma\sigma} = \frac{i}{p^2 - 2\mu^2}.$$

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In[1]:= ClearAll["Global`*"]

        V = -1/2 μ^2 (π^2 + (v + σ)^2) + λ/4 (π^2 + (v + σ)^2)^2;
        V /. {v → μ/Sqrt[λ]} // FullSimplify // Expand

Out[3]= π^4 λ/4 - μ^4/(4 λ) + π^2 Sqrt[λ] μ σ + 1/2 π^2 λ σ^2 + μ^2 σ^2 + Sqrt[λ] μ σ^3 + λ σ^4/4
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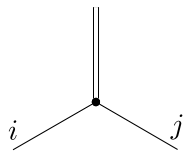
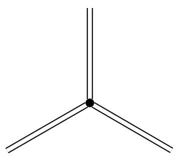
Figure 1: Mathematica code for (3). Since everything here commutes, we might as well pretend that everything is c-numbers.

Likewise, $\wick{\c{\pi}^i \c{\pi}^j}$ is, since the pion is massless,

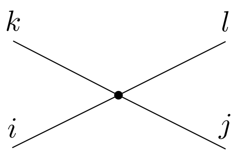
$$\wick{\c{\pi}^i \c{\pi}^j} = \frac{i\delta^{ij}}{p^2}.$$



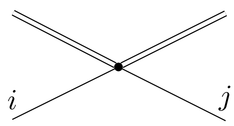
$$= -2i\lambda v \delta^{ij} \tag{4}$$



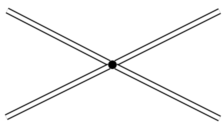
$$= -6i\lambda v \tag{5}$$



$$= -2i\lambda \left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right) \tag{6}$$



$$= -2i\lambda \delta^{ij} \tag{7}$$



$$= -6i\lambda \tag{8}$$

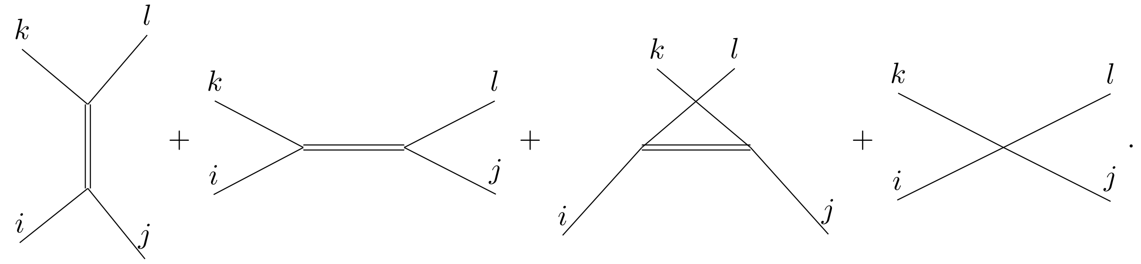
c)

c.i)

Compute the scattering amplitude for the process

$$\pi^i(p_1)\pi^j(p_2) \to \pi^k(p_3)\pi^l(p_4)$$

to leading order in λ. There are now four Feynman diagrams that contribute:



Solution

Multiplying the together values we derived in section b.ii) for each vertex, we get

$$\begin{aligned}
\mathcal{M}_{ij\to kl} &= -2i\lambda v\delta^{ij}\frac{i}{p^2-2\mu^2+i\epsilon}(-2i)\lambda v\delta^{kl} = -4i\lambda^2v^2\delta^{ij}\delta^{kl}\frac{1}{p^2-2\mu^2+i\epsilon}\\
\mathcal{M}_{ik\to jl} &= -2i\lambda v\delta^{jl}\frac{i}{p^2-2\mu^2+i\epsilon}(-2i)\lambda v\delta^{ik} = -4i\lambda^2v^2\delta^{jl}\delta^{ik}\frac{1}{p^2-2\mu^2+i\epsilon}\\
\mathcal{M}_{ij\to lk} &= -2i\lambda v\delta^{ij}\frac{i}{p^2-2\mu^2+i\epsilon}(-2i)\lambda v\delta^{lk} = -4i\lambda^2v^2\delta^{ij}\delta^{lk}\frac{1}{p^2-2\mu^2+i\epsilon}\\
\widetilde{\mathcal{M}}_{ij\to kl} &= -2i\lambda\left(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}\right).
\end{aligned}$$

c.ii)

Show that, at treshold ($\mathbf{p}=0$), these diagrams sum to *zero*. (Hint: It may be easiest to first consider the specific process $\pi^1\pi^1 \to \pi^2\pi^2$, for which only the first and fourth diagrams are nonzero, before tackling the general case.)

c.iii)

Show that, in the special case $N=2$ (1 species of pion), the term of $\mathcal{O}(p^2)$ also cancels.

Solution

d)

Add to V a symmetry-breaking term,

$$\Delta V = -a\Phi^N,$$

where a is a (small) constant. (In QCD, a term of this form is produced if the u and d quarks have the same nonvanishing mass.)

d.i)

Find the new value of v that minimizes V,

Solution

d.ii)

and work out the content of the theory about that point.

Solution

d.iii)

Show that the pion acquires a mass such that $m_\pi^2 \sim a$,

Solution

d.iv)

and show that the pion scattering amplitude at treshold is now nonvanishing and also proportional to a.