❦ Quantum Bield Theory Problem 2 ❦

simjac

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Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int \mathrm{d}^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right).$$

It is easiest to analyze this theory by considering $\phi(x)$ and $\phi^*(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

a)

(i)

Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations.

Solution

We have

$$\begin{aligned} \mathcal{L} &= \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \\ &= \partial_0 \phi^* \partial_0 \phi - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi \end{aligned} \tag{1}$$

$$\Longrightarrow$$

$$\begin{aligned} \pi(x) &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^* \\ \pi^*(x) &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} = \partial_0 \phi. \end{aligned} \tag{2}$$

The commutation relations that Peskin & Schroeder imposes on ϕ and π are

$$[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\delta^3(\mathbf{x} - \mathbf{x}'), \tag{2.20a}$$

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = 0. \tag{2.20b}$$

We will further impose that

$$[\phi(\mathbf{x}), \phi^*(\mathbf{x}')] = 0. \tag{3}$$

From these relations it follows that

$$[\phi^*(\mathbf{x}, t), \pi^*(\mathbf{x}', t)] = [\pi(\mathbf{x}', t), \phi(\mathbf{x}, t)]^* \overset{(2.20a)}{=} i\delta^3(\mathbf{x} - \mathbf{x}')$$

and

$$\begin{aligned} [\pi(\mathbf{x}), \pi^*(\mathbf{x}')] &\overset{(2)}{=} [\partial_0 \phi^*(\mathbf{x}), \partial_0' \phi(\mathbf{x}')] \\ &= \partial_0 \phi^*(\mathbf{x}) \partial_0' \phi(\mathbf{x}') - \partial_0' \phi(\mathbf{x}') \partial_0 \phi^*(\mathbf{x}) \\ &= \partial_0 \partial_0' (\phi^*(\mathbf{x}) \phi(\mathbf{x}') - \phi(\mathbf{x}') \phi^*(\mathbf{x})) \\ &= \partial_0 \partial_0' [\phi(\mathbf{x}), \phi^*(\mathbf{x}')] \\ &\overset{(3)}{=} 0 \end{aligned} \tag{4}$$

and

$$\begin{aligned}
[\phi(\mathbf{x}), \pi^*(\mathbf{x}')] &\overset{(2)}{=} [\phi(\mathbf{x}), \partial_0'\phi(\mathbf{x}')] \\
&= \phi(\mathbf{x})\partial_0'\phi(\mathbf{x}') - \partial_0'\phi(\mathbf{x}')\phi(\mathbf{x}) \\
&= \partial_0'(\phi(\mathbf{x})\phi(\mathbf{x}') - \phi(\mathbf{x}')\phi(\mathbf{x})) \\
&= \partial_0'[\phi(\mathbf{x}), \phi(\mathbf{x}')] \\
&\overset{(2.20b)}{=} 0
\end{aligned} \tag{5}$$

and

$$[\phi^*(\mathbf{x}), \pi(\mathbf{x}')] = [\pi^*(\mathbf{x}'), \phi(\mathbf{x})]^* \overset{(5)}{=} 0.$$

(ii)

Show that the Hamiltonian is

$$H = \int \mathrm{d}^3\mathbf{x} \left(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi\right).$$

Solution

The Hamiltonian is given by

$$H = \int \mathrm{d}^3\mathbf{x} \left(\sum_i \pi_i \dot{\phi}_i - \mathcal{L}\right). \tag{2.5}$$

Substituting $\pi_1 = \pi$, $\pi_2 = \pi^*$, $\phi_1 = \phi$, $\phi_2 = \phi^*$, and $\mathcal{L}$ from (1) yields

$$\begin{aligned}
H &= \int \mathrm{d}^3\mathbf{x} \left(\pi\partial_0\phi + \pi^*\partial_0\phi^* - \partial_\mu\phi^*\partial^\mu\phi + m^2\phi^*\phi\right) \\
&= \int \mathrm{d}^3\mathbf{x} \left(\pi\partial_0\phi + \pi^*\partial_0\phi^* - \partial_0\phi^*\partial_0\phi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi\right) \\
&\overset{(2)}{=} \int \mathrm{d}^3\mathbf{x} \left(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi\right)
\end{aligned}$$

(iii)

Compute the Hesienberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

Solution

We begin by calculating some commutators:

$$\begin{aligned}
[\phi(\mathbf{x}), \pi(\mathbf{x}')\pi^*(\mathbf{x}')] &= [\phi(\mathbf{x}), \pi(\mathbf{x}')]\pi^*(\mathbf{x}') + \pi(\mathbf{x}')[\phi(\mathbf{x}), \pi^*(\mathbf{x}')] \\
&\overset{(5)\text{ and }(2.20a)}{=} i\delta^3(\mathbf{x} - \mathbf{x}')\pi^*(\mathbf{x}')
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
[\phi(\mathbf{x}), \phi(\mathbf{x}')\phi^*(\mathbf{x}')] &= [\phi(\mathbf{x}), \phi(\mathbf{x}')]\phi^*(\mathbf{x}') + \phi(\mathbf{x}')[\phi(\mathbf{x}), \phi^*(\mathbf{x}')] \\
&\overset{(3)\text{ and }(2.20b)}{=} 0
\end{aligned} \tag{7}$$

and

$$
\begin{aligned}
[\phi(\mathbf{x}), \nabla\phi(\mathbf{x}')\cdot\nabla\phi^*(\mathbf{x}')] &= [\phi(\mathbf{x}), \nabla\phi(\mathbf{x}')]\cdot\nabla\phi^*(\mathbf{x}') + \nabla\phi(\mathbf{x}')\cdot[\phi(\mathbf{x}), \nabla\phi^*(\mathbf{x}')] \\
&= \nabla'\left([\phi(\mathbf{x}), \phi(\mathbf{x}')]\right)\cdot\nabla\phi^*(\mathbf{x}') + \nabla\phi(\mathbf{x}')\cdot\nabla'\left([\phi(\mathbf{x}), \phi^*(\mathbf{x}')]\right) \\
&\overset{(3)\text{ and }(2.20\text{b})}{=} 0
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
[\pi(\mathbf{x}), \pi(\mathbf{x}')\pi^*(\mathbf{x}')] &= [\pi(\mathbf{x}), \pi(\mathbf{x}')]\pi^*(\mathbf{x}') + \pi(\mathbf{x}')[\pi(\mathbf{x}), \pi^*(\mathbf{x}')] \\
&\overset{(4)\text{ and }(2.20\text{b})}{=} 0
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
[\pi(\mathbf{x}), \phi(\mathbf{x}')\phi^*(\mathbf{x}')] &= [\pi(\mathbf{x}), \phi(\mathbf{x}')]\phi^*(\mathbf{x}') + \phi(\mathbf{x}')[\pi(\mathbf{x}), \phi^*(\mathbf{x}')] \\
&\overset{(3)\text{ and }(2.20\text{a})}{=} -i\delta^3(\mathbf{x}-\mathbf{x}')\phi^*(\mathbf{x}')
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
[\pi(\mathbf{x}), \nabla\phi(\mathbf{x}')\cdot\nabla\phi^*(\mathbf{x}')] &= \nabla'\left([\pi(\mathbf{x}), \phi(\mathbf{x}')]\right)\cdot\nabla\phi^*(\mathbf{x}') + \nabla\phi(\mathbf{x}')\cdot\nabla'\left([\pi(\mathbf{x}), \phi^*(\mathbf{x}')]\right) \\
&\overset{(3)\text{ and }(2.20\text{a})}{=} -i\nabla'\delta^3(\mathbf{x}-\mathbf{x}')\cdot\nabla\phi^*(\mathbf{x}').
\end{aligned} \tag{11}
$$

The Heisenberg equation of motion says

$$
\begin{aligned}
i\partial_0\phi(\mathbf{x}) &= [\phi(\mathbf{x}), H] \\
&= [\phi(\mathbf{x}), \int \mathrm{d}^3\mathbf{x}'\left(\pi(\mathbf{x}')\pi^*(\mathbf{x}') + \nabla\phi(\mathbf{x}')\cdot\nabla\phi^*(\mathbf{x}') + m^2\phi(\mathbf{x}')\phi^*(\mathbf{x}')\right)] \\
&\overset{(6),\,(7)\text{ and }(8)}{=} \int \mathrm{d}^3\mathbf{x}' i\delta^3(\mathbf{x}-\mathbf{x}')\pi^*(\mathbf{x}') \\
&= i\pi^*(\mathbf{x}).
\end{aligned}
$$

Iterating the Heisenberg equation of motion once more yields

$$
\begin{aligned}
-\partial_0^2\phi(\mathbf{x}) &= i\partial_0(i\partial_0\phi(\mathbf{x})) \\
&= i\partial_0(i\pi^*(\mathbf{x})) \\
&= [i\pi^*(\mathbf{x}), H] \\
&= [i\pi^*(\mathbf{x}), \int \mathrm{d}^3\mathbf{x}'\left(\pi(\mathbf{x}')\pi^*(\mathbf{x}') + \nabla\phi(\mathbf{x}')\cdot\nabla\phi^*(\mathbf{x}') + m^2\phi(\mathbf{x}')\phi^*(\mathbf{x}')\right)] \\
&\overset{(9),\,(10)\text{ and }(11)}{=} i\int \mathrm{d}^3\mathbf{x}'\left(i\nabla'\delta^3(\mathbf{x}-\mathbf{x}')\nabla\phi(\mathbf{x}') + m^2 i\delta^3(\mathbf{x}-\mathbf{x}')\phi(\mathbf{x}')\right) \\
&= -\nabla^2\phi(\mathbf{x}) - m^2\phi(\mathbf{x})
\end{aligned}
$$

which is the Klein-Gordon equation. Note: I know the sign is wrong but I fail to see where it went wrong.

b)

(i)

Diagonalize H by introducing creation and annihilation operators.

Solution

We introduce the operators such that

$$\phi(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{E_\mathbf{p}}\left(a_\mathbf{p}\mathrm{e}^{-ip^\alpha x_\alpha} + b^\dagger_\mathbf{p}\mathrm{e}^{ip^\alpha x_\alpha}\right),$$
$$\phi^*(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{E_\mathbf{p}}\left(b_\mathbf{p}\mathrm{e}^{-ip^\alpha x_\alpha} + a^\dagger_\mathbf{p}\mathrm{e}^{ip^\alpha x_\alpha}\right)$$
$$\pi(\mathbf{x}) = \partial_0\phi^*(\mathbf{x}) = i\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left(-b_\mathbf{p}\mathrm{e}^{-ip^\alpha x_\alpha} + a^\dagger_\mathbf{p}\mathrm{e}^{ip^\alpha x_\alpha}\right),$$
$$\pi^*(\mathbf{x}) = \partial_0\phi(\mathbf{x}) = i\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left(-a_\mathbf{p}\mathrm{e}^{-ip^\alpha x_\alpha} + b^\dagger_\mathbf{p}\mathrm{e}^{ip^\alpha x_\alpha}\right)$$

Where

$$[a_\mathbf{p}, a^\dagger_{\mathbf{p}'}] = [b_\mathbf{p}, b^\dagger_{\mathbf{p}'}] = E_\mathbf{p}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}'), \tag{12}$$
$$[a_\mathbf{p}, a^\dagger_{\mathbf{p}'}] = [b_\mathbf{p}, b_{\mathbf{p}'}] = 0. \tag{13}$$
$$[a_\mathbf{p}, b^\dagger_{\mathbf{p}'}] = [a_\mathbf{p}, b_{\mathbf{p}'}] = 0. \tag{14}$$

We expand each term of the Hamiltonian:

$$\pi^*\pi = i^2\int\frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'}{(2\pi)^6}\left(a_\mathbf{p}b_{\mathbf{p}'}\mathrm{e}^{i(-p-p')^\alpha x_\alpha} - a_\mathbf{p}a^\dagger_{\mathbf{p}'}\mathrm{e}^{i(-p+p')^\alpha x_\alpha} - b^\dagger_\mathbf{p}b_{\mathbf{p}'}\mathrm{e}^{i(p-p')^\alpha x_\alpha} + b^\dagger_\mathbf{p}a^\dagger_{\mathbf{p}'}\mathrm{e}^{i(p+p')^\alpha x_\alpha}\right)$$
$$\nabla\phi^*\cdot\nabla\phi = \int\frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'}{(2\pi)^6}\frac{\mathbf{p}\cdot\mathbf{p}'}{E_\mathbf{p}E_{\mathbf{p}'}}\left(b_\mathbf{p}a_{\mathbf{p}'}\mathrm{e}^{i(-p-p')^\alpha x_\alpha} - b_\mathbf{p}b^\dagger_{\mathbf{p}'}\mathrm{e}^{i(-p+p')^\alpha x_\alpha} - a^\dagger_\mathbf{p}a_{\mathbf{p}'}\mathrm{e}^{i(p-p')^\alpha x_\alpha} + a^\dagger_\mathbf{p}b^\dagger_{\mathbf{p}'}\mathrm{e}^{i(p+p')^\alpha x_\alpha}\right)$$
$$\phi^*\phi = \int\frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'}{(2\pi)^6}\frac{1}{E_\mathbf{p}E_{\mathbf{p}'}}\left(b_\mathbf{p}a_{\mathbf{p}'}\mathrm{e}^{i(-p-p')^\alpha x_\alpha} + b_\mathbf{p}b^\dagger_{\mathbf{p}'}\mathrm{e}^{i(-p+p')^\alpha x_\alpha} + a^\dagger_\mathbf{p}a_{\mathbf{p}'}\mathrm{e}^{i(p-p')^\alpha x_\alpha} + a^\dagger_\mathbf{p}b^\dagger_{\mathbf{p}'}\mathrm{e}^{i(p+p')^\alpha x_\alpha}\right).$$

Since $p^\alpha = (E_\mathbf{p}, \mathbf{p})$, and since $\int\frac{\mathrm{d}^3\mathbf{x}}{(2\pi)^3}\mathrm{e}^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} = \delta^3(\mathbf{p}-\mathbf{p}')$, we have that

$$\begin{aligned}\int\pi^*\pi\,\mathrm{d}^3\mathbf{x} &= i^2\int\frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'}{(2\pi)^3}\Bigg(\left[a_\mathbf{p}b_{\mathbf{p}'}\mathrm{e}^{i(-E_\mathbf{p}-E_{\mathbf{p}'})t} + b^\dagger_\mathbf{p}a^\dagger_{\mathbf{p}'}\mathrm{e}^{i(E_\mathbf{p}+E_{\mathbf{p}'})t}\right]\delta^3(\mathbf{p}+\mathbf{p}')\\ &\quad + \left[-a_\mathbf{p}a^\dagger_{\mathbf{p}'}\mathrm{e}^{i(-E_\mathbf{p}+E_{\mathbf{p}'})t} - b^\dagger_\mathbf{p}b_{\mathbf{p}'}\mathrm{e}^{i(E_\mathbf{p}-E_{\mathbf{p}'})t}\right]\delta^3(\mathbf{p}-\mathbf{p}')\Bigg)\\ &= \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\Bigg(-a_\mathbf{p}b_{-\mathbf{p}}\mathrm{e}^{i(-E_\mathbf{p}-E_{-\mathbf{p}})t} - b^\dagger_\mathbf{p}a^\dagger_{-\mathbf{p}}\mathrm{e}^{i(E_\mathbf{p}+E_{-\mathbf{p}})t} + a_\mathbf{p}a^\dagger_\mathbf{p} + b^\dagger_\mathbf{p}b_\mathbf{p}\Bigg)\end{aligned}$$

and similarly

$$\int\nabla\phi^*\cdot\nabla\phi\,\mathrm{d}^3\mathbf{x} = \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathbf{p}^2}{E^2_\mathbf{p}}\left(b_\mathbf{p}a_{-\mathbf{p}}\mathrm{e}^{i(-E_\mathbf{p}-E_{-\mathbf{p}})t} + a^\dagger_\mathbf{p}b^\dagger_{-\mathbf{p}}\mathrm{e}^{i(E_\mathbf{p}+E_{-\mathbf{p}})t} - b_\mathbf{p}b^\dagger_\mathbf{p} - a^\dagger_\mathbf{p}a_\mathbf{p}\right)$$
$$\int\phi^*\phi\,\mathrm{d}^3\mathbf{x} = \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{E^2_\mathbf{p}}\left(b_\mathbf{p}a_{-\mathbf{p}}\mathrm{e}^{i(-E_\mathbf{p}-E_{-\mathbf{p}})t} + a^\dagger_\mathbf{p}b^\dagger_{-\mathbf{p}}\mathrm{e}^{i(E_\mathbf{p}+E_{-\mathbf{p}})t} + b_\mathbf{p}b^\dagger_\mathbf{p} + a^\dagger_\mathbf{p}a_\mathbf{p}\right).$$

Using $\mathbf{p}^2 + m^2 = E^2_\mathbf{p}$, we get

$$\begin{aligned}H &= \int\pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi\,\mathrm{d}^3\mathbf{x}\\ &= \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\Bigg(\left(-a_\mathbf{p}b_{-\mathbf{p}} + b_\mathbf{p}a_{-\mathbf{p}}\right)\mathrm{e}^{i(-E_\mathbf{p}-E_{-\mathbf{p}})t} + \left(-b^\dagger_\mathbf{p}a^\dagger_{-\mathbf{p}} + a^\dagger_\mathbf{p}b^\dagger_{-\mathbf{p}}\right)\mathrm{e}^{i(E_\mathbf{p}+E_{-\mathbf{p}})t}\\ &\quad + a_\mathbf{p}a^\dagger_\mathbf{p} + b^\dagger_\mathbf{p}b_\mathbf{p} + \frac{-\mathbf{p}^2+m^2}{E^2_\mathbf{p}}\left(a^\dagger_\mathbf{p}a_\mathbf{p} + b_\mathbf{p}b^\dagger_\mathbf{p}\right)\Bigg).\end{aligned}$$

The first term in H is 0 since

$$\int \left(-a_{\mathbf{p}}b_{-\mathbf{p}} + b_{\mathbf{p}}a_{-\mathbf{p}}\right) \mathrm{e}^{i(-E_{\mathbf{p}}-E_{-\mathbf{p}})t}\mathrm{d}^3\mathbf{p} = \int \left(-a_{\mathbf{p}}b_{-\mathbf{p}} + b_{-\mathbf{p}}a_{\mathbf{p}}\right) \mathrm{e}^{i(-E_{\mathbf{p}}-E_{-\mathbf{p}})t}\mathrm{d}^3\mathbf{p} \overset{(14)}{=} 0 \qquad (15)$$

and since the second term is just the Hermitian conjugate of the first term, it too is 0. Thus the Hamiltonian is

$$H = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left(a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}} + \frac{-\mathbf{p}^2+m^2}{E^2_{\mathbf{p}}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b_{\mathbf{p}}b^\dagger_{\mathbf{p}}\right)\right).$$

Since the creation and annihilation operators now only appear in the pairs $a_{\mathbf{p}}a^\dagger_{\mathbf{p}}$, $b^\dagger_{\mathbf{p}}b_{\mathbf{p}}$, $a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$, and $b_{\mathbf{p}}b^\dagger_{\mathbf{p}}$, H is diagonal in the momentum basis.

(ii)

Show that the theory contains two sets of particles of mass m.

Solution

I am unsure how to see this.

c)

(i)

Rewrite the conserved charge

$$Q = \int \mathrm{d}^3\mathbf{x}\frac{i}{2}\left(\phi^*\pi^* - \pi\phi\right)$$

in terms of creation and annihilation operators.

Solution

First, we calculate

$$\phi^*\pi^* = \int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'}{(2\pi)^6}\frac{i}{E_{\mathbf{p}}}\left(-b_{\mathbf{p}}a_{\mathbf{p}'}\mathrm{e}^{i(-p-p')^\alpha x_\alpha} + b_{\mathbf{p}}b^\dagger_{\mathbf{p}'}\mathrm{e}^{i(-p+p')^\alpha x_\alpha} - a^\dagger_{\mathbf{p}}a_{\mathbf{p}'}\mathrm{e}^{i(p-p')^\alpha x_\alpha} - a^\dagger_{\mathbf{p}}b^\dagger_{\mathbf{p}'}\mathrm{e}^{i(p+p)^\alpha x_\alpha}\right).$$

As before, when integrating over $\mathbf{x}$, we get a delta function in $\mathbf{p}-\mathbf{p}'$ and are able to simplify as

$$\int \phi^*\pi^*\,\mathrm{d}\mathbf{x} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{i}{E_{\mathbf{p}}}\left(-b_{\mathbf{p}}a_{-\mathbf{p}}\mathrm{e}^{i(-E_{\mathbf{p}}-E_{-\mathbf{p}})t} + b_{\mathbf{p}}b^\dagger_{\mathbf{p}} - a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}}\mathrm{e}^{i(E_{\mathbf{p}}+E_{-\mathbf{p}})t}\right). \qquad (16)$$

If we now add (16) to its negative complex conjugate, we get

$$\begin{aligned}\int \mathrm{d}^3\mathbf{x}\frac{i}{2}\left(\phi^*\pi^* - \pi\phi\right) = \frac{i^2}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{E_{\mathbf{p}}}\Bigg(&\left(-b_{\mathbf{p}}a_{-\mathbf{p}} + b_{-\mathbf{p}}a_{\mathbf{p}}\right)\mathrm{e}^{i(-E_{\mathbf{p}}-E_{-\mathbf{p}})t} + \left(-a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}} + a^\dagger_{-\mathbf{p}}b^\dagger_{\mathbf{p}}\right)\mathrm{e}^{i(-E_{\mathbf{p}}-E_{-\mathbf{p}})t}\\ &+ b_{\mathbf{p}}b^\dagger_{\mathbf{p}} - a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\Bigg)\end{aligned}$$

By the same argument as in (15) (except we won't have to appeal to (14)), the first two terms here are 0. Thus the charge is

$$Q = \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{E_{\mathbf{p}}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - b_{\mathbf{p}}b^\dagger_{\mathbf{p}}\right)$$

(ii)

Evaluate the charge of the particles of each type.

Solution

In analogy with Peskin & Schroeder's (2.32), we calculate

$$
\begin{aligned}
[Q, a^\dagger_{\mathbf{p}}] &= \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3}\frac{1}{E_{\mathbf{p}'}}\left([a^\dagger_{\mathbf{p}'}a_{\mathbf{p}'}, a^\dagger_{\mathbf{p}}] - [b_{\mathbf{p}'}b^\dagger_{\mathbf{p}'}, a^\dagger_{\mathbf{p}}]\right)\\
&\overset{(13)\text{ and }(14)}{=} \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3}\frac{1}{E_{\mathbf{p}'}} a^\dagger_{\mathbf{p}'}[a_{\mathbf{p}'}, a^\dagger_{\mathbf{p}}]\\
&\overset{(12)}{=} \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3}\frac{1}{E_{\mathbf{p}'}} a^\dagger_{\mathbf{p}'}E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\\
&= \frac{1}{2}a^\dagger_{\mathbf{p}}.
\end{aligned}
$$

Thus the charge of the particle created by $a^\dagger_{\mathbf{p}}$ is $+\frac{1}{2}$. By a similar computation, the charge of the particle created by $b^\dagger_{\mathbf{p}}$ is $-\frac{1}{2}$.

d)

(i)

Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, $a = 1, 2$. Show that there are now four conserved charges, one given by the generalization of **c)**, and the other three given by

$$
Q^i = \int \mathrm{d}^3\mathbf{x}\frac{i}{2}\left(\phi^*_a(\sigma^i)_{ab}\pi^*_b - \pi_a(\sigma^i)_{ab}\phi_b\right),
$$

where σ^i are the Pauli sigma matrices.

Solution

We generalize (2.20a) to (11) to for $a_{a\mathbf{p}}$, $a^\dagger_{a\mathbf{p}}$, $b_{a\mathbf{p}}$, and $b^\dagger_{a\mathbf{p}}$ by saying that only operators with the same index can fail to commute. The thing which we want to show is 0 is

$$
[Q^i, H] = \left[\int \mathrm{d}^3\mathbf{x}\frac{i}{2}\left(\phi^*_a(\sigma^i)_{ab}\pi^*_b - \pi_a(\sigma^i)_{ab}\phi_b\right), \int \mathrm{d}^3\mathbf{x}'\left(\pi^*_c\pi_c + \nabla\phi^*_c\cdot\nabla\phi_c + m^2\phi^*_c\phi_c\right)\right]
$$

The terms that will appear are:

$$
\begin{aligned}
[\phi^*_a, \pi^*_c\pi_c]\sigma^i_{ab}\pi^*_b &\overset{(6)}{=} i\delta^3(\mathbf{x}-\mathbf{x}')\pi_a(\mathbf{x})\sigma^i_{ab}\pi^*_b(\mathbf{x}')\\
\pi_a\sigma^i_{ab}[\phi_b, \pi^*_c\pi_c] &\overset{(6)}{=} i\delta^3(\mathbf{x}-\mathbf{x}')\pi_a(\mathbf{x})\sigma^i_{ab}\pi^*_b(\mathbf{x}')\\
\phi^*_a\sigma^i_{ab}[\pi^*_b, \nabla\phi^*_c\cdot\nabla\phi_c] &\overset{(11)}{=} -\phi^*_a(\mathbf{x})\sigma^i_{ab}i\nabla'\delta^3(\mathbf{x}-\mathbf{x}')\cdot\nabla\phi_b(\mathbf{x}')\\
[\pi_a, \nabla\phi^*_c\cdot\nabla\phi_c]\sigma^i_{ab}\phi_b &\overset{(11)}{=} -i\nabla'\delta^3(\mathbf{x}-\mathbf{x}')\cdot\nabla\phi^*_b(\mathbf{x}')\sigma^i_{ab}\phi_b(\mathbf{x})\\
\phi^*_a\sigma^i_{ab}[\pi^*_b, \phi^*_c\phi_c] &\overset{(10)}{=} -\phi^*_a(\mathbf{x})\sigma^i_{ab}i\delta^3(\mathbf{x}-\mathbf{x}')\phi^*_b(\mathbf{x}')\\
[\pi_a, \phi^*_c\phi_c]\sigma^i_{ab}\phi_b &\overset{(10)}{=} -\phi^*_a(\mathbf{x})\sigma^i_{ab}i\delta^3(\mathbf{x}-\mathbf{x}')\phi^*_b(\mathbf{x}').
\end{aligned}
$$

All of these terms will cancel when integrated. Thus $\partial_0 Q^i = [Q^i, H] = 0$.

(ii)

Show that these three charges have the commutation relations of angular momentum ($SU(2)$).

Solution

From the expression

$$[Q^i, Q^j]$$

and the fact that the Pauli sigma matrices satisfy the $\mathfrak{su}(2)$ algebra, since we generalized (2.20a) to (11) in such a way that fields with different indices will commute, these charges will have the $SU(2)$ commutation relations.

(iii)

Generalize these results to the case of n identical complex scalar fields.