❦ Quantum Field Theory Problem 4 ❦

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The interactions of pions at low energy can be described by a phenomenological model called the *linear sigma model*. Essentially, this model consists of N real scalar fields coupled by a ϕ^4 interaction that is symmetric under rotations of the N fields. More specifically, let $\Phi^i(x)$, $i = 1, \ldots, N$ be a set of N fields governed by the Hamiltonian

$$H = \int \mathrm{d}^3\mathbf{x} \left(\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\Phi)^2 + V(\Phi^2) \right), \tag{1}$$

where $\Phi^2 = \Phi^i\Phi^i$, and

$$V(\Phi^2) = \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4}(\Phi^2)^2 \tag{2}$$

is a symmetric function under rotations of Φ^i. For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to H; hence, V is the field potential energy.

(What does this Hamiltonian have to do with the strong interactions? There are two types of light quarks, u and d. These quarks have identical strong interactions, but different masses. If these quarks are massless, the Hamiltonian of the strong interactions is invariant to unitary transformations of the 2-component object (u, d):

$$\begin{pmatrix} u \\ d \end{pmatrix} \mapsto \exp\{i\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}/2\} \begin{pmatrix} u \\ d \end{pmatrix}.$$

This transformation is called an *isospin* rotation. If, in addition, the strong interactions are described by a vector "gluon" field (as is true in QCD), the strong interaction Hamiltonian is invariant to the isospin rotations done separately on the left-handed and right-handed components of the quark fields. Thus, the complete symmetry of QCD with two massless quarks is $SU(2) \times SU(2)$. It happens that $SO(4)$, the group of rotations in 4 dimensions, is isomorphic to $SU(2) \times SU(2)$, so for $N = 4$, the linear sigma model has the same symmetry group as the strong interactions.)

a)

Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly N copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter λ.

a.i)

Show that the propagator is

$$\Phi^i(x)\Phi^j(y) = \delta^{ij} D_{\mathrm{F}}(x - y),$$

where D_{F} is the standard Klein-Gordon propagator for mass m,

Solution

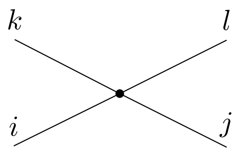
If $\lambda = 0$, (1) is just the Klein-Gordon Hamiltonian. Using the Peskin and Schroeder equation numbering, we have that

$$
\begin{aligned}
\overline{\Phi^i(x)\Phi^j}(y) &:= \begin{cases} [\Phi^{i+}(x), \Phi^{j-}(y)], & x^0 > y^0 \\ [\Phi^{i+}(y), \Phi^{j-}(x)], & x^0 < y^0 \end{cases} \\
&\stackrel{(4.32)}{=} \int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^6} \frac{1}{\sqrt{2 \cdot 2E_{\mathbf{p}}E_{\mathbf{q}}}} \left[a^i_{\mathbf{p}}, a^{j\dagger}_{\mathbf{q}}\right] \begin{cases} \exp\{i(-p^\alpha x_\alpha + q^\alpha y_\alpha)\}, & x^0 > y^0 \\ \exp\{i(p^\alpha x_\alpha - q^\alpha y_\alpha)\}, & x^0 < y^0 \end{cases} \\
&\stackrel{(2.29)}{=} \int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^6} \frac{1}{\sqrt{2 \cdot 2E_{\mathbf{p}}E_{\mathbf{q}}}} \delta^{ij}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) \begin{cases} \exp\{i(-p^\alpha x_\alpha + q^\alpha y_\alpha)\}, & x^0 > y^0 \\ \exp\{i(p^\alpha x_\alpha - q^\alpha y_\alpha)\}, & x^0 < y^0 \end{cases} \\
&\stackrel{*}{=} \delta^{ij} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \begin{cases} \exp\{ip^\alpha(-x+y)_\alpha\}, & x^0 > y^0 \\ \exp\{ip^\alpha(x-y)_\alpha\}, & x^0 < y^0 \end{cases} \\
&\stackrel{(2.50)}{=} \begin{cases} D(x-y), & x^0 > y^0 \\ D(x-y), & x^0 < y^0 \end{cases} \\
&\stackrel{(2.60)}{=} D_{\mathrm{F}}(x-y).
\end{aligned}
$$

In *, we have used that if $\mathbf{p} = \mathbf{q}$, then $p^\alpha = q^\alpha$.

a.ii)

and that there is one type of vertex given by



$$= -2i\lambda \left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right). \tag{3}$$

(That is, a vertex between two Φ^1:s and two Φ^2:s has the value $(-2i\lambda)$; that between four Φ^1:s has the value $(-6i\lambda)$.)

Solution

We have that

$$H_{\mathrm{I}} = \frac{\lambda}{4}(\Phi^2)^2.$$

Sandwiching this between two states $\langle p_1^k p_2^l|$ and $|p_3^i p_4^j\rangle$ and integrating, we get (3).

a.iii)

Compute, to leading order in λ, the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$, in the center-of-mass frame, for the scattering processes

$$\Phi^1\Phi^1 \to \Phi^1\Phi^2, \quad \Phi^1\Phi^1 \to \Phi^2\Phi^2, \quad \text{and} \quad \Phi^1\Phi^1 \to \Phi^1\Phi^1$$

as a function of their center-of-mass energy.

Solution

From (3), it follows that

$$
\begin{aligned}
i\mathcal{M}(\Phi^1\Phi^2 \to \Phi^1\Phi^2) &= -2i\lambda \\
i\mathcal{M}(\Phi^1\Phi^1 \to \Phi^2\Phi^2) &= -2i\lambda \\
i\mathcal{M}(\Phi^1\Phi^1 \to \Phi^1\Phi^1) &= -6i\lambda
\end{aligned}
$$

From Peskin & Schroeder's we have

$$
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\mathrm{cm}}^2}, \tag{4.85}
$$

which gives us

$$
\begin{aligned}
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} (\Phi^1\Phi^2 \to \Phi^1\Phi^2) &= \frac{\lambda^2}{16\pi^2 E_{\mathrm{cm}}^2} \\
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} (\Phi^1\Phi^1 \to \Phi^2\Phi^2) &= \frac{\lambda^2}{16\pi^2 E_{\mathrm{cm}}^2} \\
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} (\Phi^1\Phi^1 \to \Phi^1\Phi^1) &= \frac{9\lambda^2}{16\pi^2 E_{\mathrm{cm}}^2}.
\end{aligned}
$$

b)

Now consider the case $m^2 < 0$: $m^2 = -\mu^2$. In this case, V has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since V is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting Φ^i toward the minimum of V. By rotational invariance, we can consider this shift to be in the Nth direction. Write, then,

$$
\begin{aligned}
\Phi^i(x) &= \pi^i(x), \quad i = 1, \dots, N-1 \\
\Phi^N(x) &= v + \sigma(x),
\end{aligned}
$$

where v is a constant chosen to minimize V. (The notation π^i suggests a pion field and should not be confused with a canonical momentum.)

b.i)

Show that, in these new coordinates (and substituting for v its expression in terms of λ and μ), we have a theory of a massive σ field and $N-1$ *massless* pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \to 0$.

Solution

As in Home Problem 1 c), we have that

$$
v = \frac{\mu}{\sqrt{\lambda}}
$$

minimizes the potential. Substituting this and $\Phi^i(x) = (\pi^j(x), v + \sigma(x)), j = 1, \dots, N-1$, into (2), we get (see Figure 1 for this calculation)

$$
V(\Phi^2) = -\frac{\mu^4}{4\lambda} + \sqrt{\lambda}\mu\sigma^3 + \pi^2\sqrt{\lambda}\mu\sigma + \frac{\lambda(\sigma^2)^2}{4} + \frac{1}{2}\pi^2\lambda\sigma^2 + \frac{\pi^4\lambda}{4} + \mu^2\sigma^2, \tag{4}
$$

where $\pi^2 = \pi^j\pi^j$. We see that the σ field has a mass term, $\mu^2\sigma^2$, while the π fields have no mass terms. The fields are also evidently "interacting through cubic and quartic potential energy terms which all become small as $\lambda \to 0$" since all those terms have a factor of λ or $\sqrt{\lambda}$ in them.

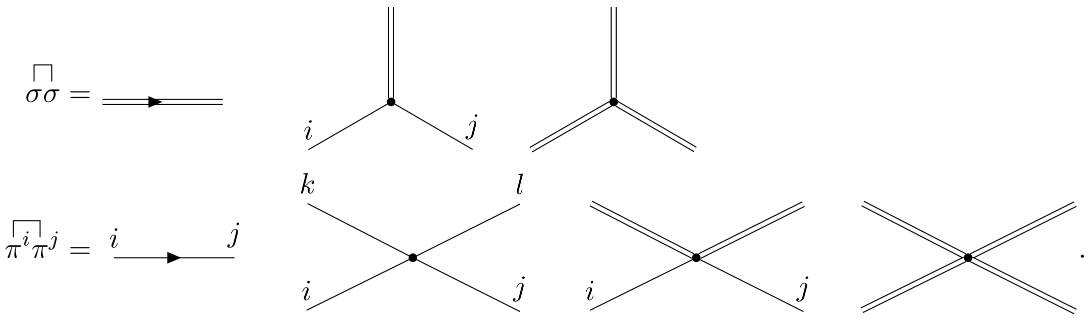
```
In[1]:= ClearAll["Global`*"]

        V = -1/2 μ^2 (π^2 + (v + σ)^2) + λ/4 (π^2 + (v + σ)^2)^2;

        V /. {v → μ/Sqrt[λ]} // FullSimplify // Expand
```

$$\text{Out[3]=}\quad \frac{\pi^4 \lambda}{4} - \frac{\mu^4}{4\lambda} + \pi^2 \sqrt{\lambda}\, \mu \sigma + \frac{1}{2} \pi^2 \lambda \sigma^2 + \mu^2 \sigma^2 + \sqrt{\lambda}\, \mu \sigma^3 + \frac{\lambda \sigma^4}{4}$$

Figure 1: Mathematica code for (4). Since everything here commutes, we might as well pretend that everything is c-numbers.

b.ii)

Construct the Feynman rules by assigning values to the propagators and vertices:



Solution

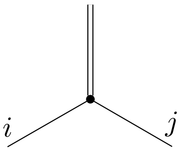
$\overline{\sigma\sigma}$ is just the propagator for a free σ particle is,

$$\overline{\sigma\sigma} = \frac{i}{p^2 - 2\mu^2}.$$

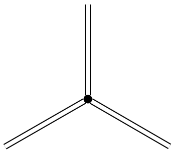
Likewise, $\overline{\pi^i\pi^j}$ is, since the pion is massless,

$$\overline{\pi^i\pi^j} = \frac{i\delta^{ij}}{p^2}.$$

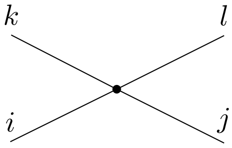
From these and (3), we get



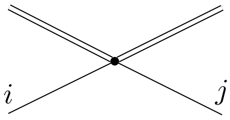
$$= -2i\lambda v\delta^{ij}$$



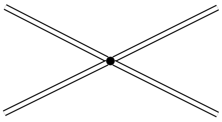
$$= -6i\lambda v$$



$$= -2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right)$$



$$= -2i\lambda\delta^{ij}$$



$$= -6i\lambda.$$

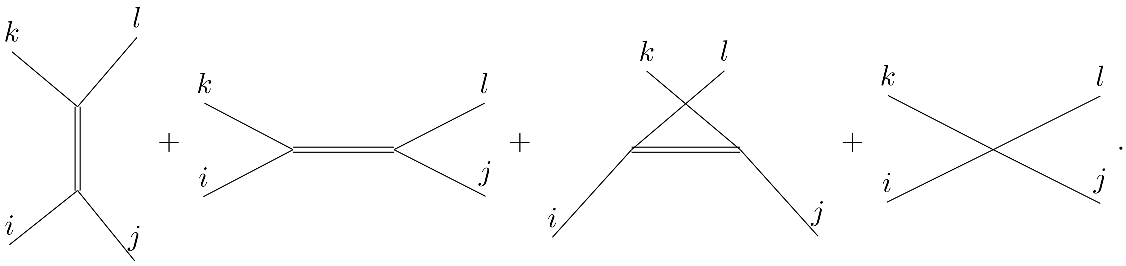
c)

c.i)

Compute the scattering amplitude for the process

$$\pi^i(p_1)\pi^j(p_2) \to \pi^k(p_3)\pi^l(p_4)$$

to leading order in λ. There are now four Feynman diagrams that contribute:



Solution

Multiplying together the values we derived in section b.ii) for each vertex, we get

$$\begin{aligned}
\mathcal{M}_{ij\to kl} &= -4i\lambda^2 v^2 \delta^{ij}\delta^{kl}\frac{1}{p^2 - 2\mu^2} - 2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right)\\
\mathcal{M}_{ik\to jl} &= -4i\lambda^2 v^2 \delta^{jl}\delta^{ik}\frac{1}{p^2 - 2\mu^2}\\
\mathcal{M}_{ij\to lk} &= -4i\lambda^2 v^2 \delta^{ij}\delta^{lk}\frac{1}{p^2 - 2\mu^2}
\end{aligned}$$

whose sum is

$$\mathcal{M} = -4i\lambda^2 v^2\left[\frac{\delta^{ij}\delta^{kl}}{p^2 - 2\mu^2} + \frac{\delta^{jl}\delta^{ik}}{q^2 - 2\mu^2} + \frac{\delta^{il}\delta^{jk}}{k^2 - 2\mu^2}\right] - 2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{lj}\delta^{ki}\right). \qquad (5)$$

c.ii)

Show that, at treshold ($\mathbf{p} = 0$), these diagrams sum to *zero*. (Hint: It may be easiest to first consider the specific process $\pi^1\pi^1 \to \pi^2\pi^2$, for which only the first and fourth diagrams are nonzero, before tackling the general case.)

Solution

Since the pions are massles, $p^\alpha = (\mathbf{p}, 0)$ and $\mathbf{p} = 0 \implies p^2 = 0$. Plugging $p^2 = q^2 = k^2 = 0$ into (5), we get (also using $v = \frac{\mu}{\sqrt{\lambda}}$)

$$\mathcal{M} = 2i\lambda\left[\delta^{ij}\delta^{kl} + \delta^{jl}\delta^{ik} + \delta^{il}\delta^{jk}\right] - 2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{lj}\delta^{ki}\right) = 0. \tag{6}$$

c.iii)

Show that, in the special case $N = 2$ (1 species of pion), the term of $\mathcal{O}(p^2)$ also cancels.

Solution

If $N = 2$, we have one pion and one σ. Since the indices $i = j = k = l$, (5) reduces to

$$\begin{aligned}
\mathcal{M} &= -4i\lambda^2 v\left[\frac{1}{p^2 - 2\mu^2} + \frac{1}{q^2 - 2\mu^2} + \frac{1}{k^2 - 2\mu^2}\right] - 6i\lambda \\
&= -2i\lambda\left[\frac{2\mu^2 + p^2 - 2\mu^2}{p^2 - 2\mu^2} + \frac{2\mu^2 + q^2 - 2\mu^2}{q^2 - 2\mu^2} + \frac{2\mu^2 + k^2 - 2\mu^2}{k^2 - 2\mu^2}\right] \\
&= -2i\lambda\left[\frac{p^2}{p^2 - 2\mu^2} + \frac{q^2}{q^2 - 2\mu^2} + \frac{k^2}{k^2 - 2\mu^2}\right] \\
&= \text{“}\mathcal{O}(p^2) + \mathcal{O}(q^2) + \mathcal{O}(k^2)\text{”}.
\end{aligned}$$

I don't really know why these would cancel, exept for maybe $p^2 = 0$ on-shell since the pion is massless.

d)

Add to V a symmetry-breaking term,

$$\Delta V = -a\Phi^N,$$

where a is a (small) constant. (In QCD, a term of this form is produced if the u and d quarks have the same nonvanishing mass.)

d.i)

Find the new value of v that minimizes V,

Solution

Minimizing

$$V(\Phi^2) = \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4}(\Phi^2)^2 - a\Phi^N$$

with respect to Φ^i yields $\Phi^j = 0$, $j = 1, \ldots N-1$, and $\Phi^N = v$ with

$$v = \frac{a}{2\mu^2} + \frac{\mu}{\sqrt{\lambda}}.$$

This is true since $\Phi^j \neq 0$ can only increase V and v is the solution to

$$\frac{\partial}{\partial \Phi^N}\left(\frac{1}{2}m^2(\Phi^N)^2 + \frac{\lambda}{4}\left((\Phi^N)^2\right)^2 - a\Phi^N\right) \;=\; 0.$$

d.ii)

and work out the content of the theory about that point.

Solution

Putting this new v into (2) as we did in section b.i), we get the mildly intimidating expression (see Figure 2 for this calculation)

$$\begin{aligned} V = {}& \frac{a^4\lambda}{64\mu^8} + \frac{a^3\lambda\sigma}{8\mu^6} + \frac{a^3\sqrt{\lambda}}{8\mu^5} + \frac{3a^2\lambda\sigma^2}{8\mu^4} + \frac{\pi^2a^2\lambda}{8\mu^4} + \frac{3a^2\sqrt{\lambda}\sigma}{4\mu^3} - \frac{a^2}{4\mu^2} + \frac{a\lambda\sigma^3}{2\mu^2} + \frac{\pi^2a\lambda\sigma}{2\mu^2} \\ & + \frac{3a\sqrt{\lambda}\sigma^2}{2\mu} + \frac{\pi^2a\sqrt{\lambda}}{2\mu} - \frac{a\mu}{\sqrt{\lambda}} - \frac{\mu^4}{4\lambda} + \sqrt{\lambda}\mu\sigma^3 + \pi^2\sqrt{\lambda}\mu\sigma + \frac{\lambda\sigma^4}{4} + \frac{1}{2}\pi^2\lambda\sigma^2 + \frac{\pi^4\lambda}{4} + \mu^2\sigma^2. \end{aligned} \tag{7}$$

In[1]:= `ClearAll["Global`*"]`

$$V = -\frac{1}{2}\mu^2\left(\pi^2 + (v+\sigma)^2\right) + \frac{\lambda}{4}\left(\pi^2 + (v+\sigma)^2\right)^2 - a\,(v+\sigma);$$

$$V \,/.\, \left\{v \to \frac{\mu}{\sqrt{\lambda}} + \frac{a}{2\,\mu^2}\right\} \;//\; \text{FullSimplify} \;//\; \text{Expand}$$

Out[3]= $$\frac{\pi^4\lambda}{4} + \frac{a^4\lambda}{64\mu^8} + \frac{a^3\sqrt{\lambda}}{8\mu^5} + \frac{a^2\pi^2\lambda}{8\mu^4} - \frac{a^2}{4\mu^2} + \frac{a\pi^2\sqrt{\lambda}}{2\mu} - \frac{a\mu}{\sqrt{\lambda}} - \frac{\mu^4}{4\lambda} + \frac{a^3\lambda\sigma}{8\mu^6} + \frac{3a^2\sqrt{\lambda}\sigma}{4\mu^3} + \frac{a\pi^2\lambda\sigma}{2\mu^2} + \pi^2\sqrt{\lambda}\mu\sigma + \frac{1}{2}\pi^2\lambda\sigma^2 + \frac{3a^2\lambda\sigma^2}{8\mu^4} + \frac{3a\sqrt{\lambda}\sigma^2}{2\mu} + \mu^2\sigma^2 + \frac{a\lambda\sigma^3}{2\mu^2} + \sqrt{\lambda}\mu\sigma^3 + \frac{\lambda\sigma^4}{4}$$

Figure 2

d.iii)

Show that the pion acquires a mass such that $m_\pi^2 \sim a$,

Solution

In (7), the pion clearly has a mass term. In fact, it has two, $\frac{\pi^2a^2\lambda}{8\mu^4}$ and $\frac{\pi^2a\sqrt{\lambda}}{2\mu}$. For small a, the second one of these will dominate and the pion mass will be proportinal to a.

d.iv)

and show that the pion scattering amplitude at treshold is now nonvanishing and also proportional to a.

Solution

In (6), we showed that the two terms in $\mathcal{M}$ canceled by substituting in v. Now v is different and so we will have $v_{\text{new}} = v_{\text{old}} + \frac{a}{2\mu^2}$. Thus, at treshold,

$$\begin{aligned}\mathcal{M} &= -4i\lambda^2 v_{\text{new}}^2 \left[\frac{\delta^{ij}\delta^{kl}}{-2\mu^2} + \frac{\delta^{jl}\delta^{ik}}{-2\mu^2} + \frac{\delta^{il}\delta^{jk}}{-2\mu^2}\right] - 2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{lj}\delta^{ki}\right) \\ &= 4i\lambda^2\left(2\frac{\mu}{\sqrt{\lambda}}\frac{a}{2\mu^2} + \frac{a^2}{4\mu^4}\right)\frac{1}{2\mu^2}\left[\delta^{ij}\delta^{kl} + \delta^{jl}\delta^{ik} + \delta^{il}\delta^{jk}\right].\end{aligned}$$

Again, for small a, this is proportianal to a.