```
ClearAll["Global`*"]
$Assumptions = (θ ∈ Reals);
```

```
(*Find θ, x, and y s.th Me^(θj) = 1/√y (y x; 0 1)*)
```

$$(*\text{Find } \theta,\ x,\ \text{and } y \text{ s.th } Me^{\theta j} = \frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} *)$$

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

```
Print["e^(θj) = ", MatrixForm [MatrixExp[θj]]]
```

$$e^{\theta j} = \begin{pmatrix} \mathrm{Cos}[\theta] & \mathrm{Sin}[\theta] \\ -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] \end{pmatrix}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

```
Solve[(M.MatrixExp[θj])[[2, 1]] == 0, θ] (*We want Me^(θj) to be of the form (* *; 0 *)*)
```

$$\mathrm{Solve}\left[(M.\mathrm{MatrixExp}[\theta j])[[2,1]] == 0, \theta\right] (*\text{We want } Me^{\theta j} \text{ to be of the form } \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}*)$$

$$\left\{\left\{\theta \to \mathrm{ConditionalExpression}\left[\mathrm{ArcTan}\left[-\frac{d}{\sqrt{c^2+d^2}}, -\frac{c}{\sqrt{c^2+d^2}}\right] + 2\pi C[1],\ C[1] \in \mathrm{Integers}\right]\right\},\right.$$
$$\left.\left\{\theta \to \mathrm{ConditionalExpression}\left[\mathrm{ArcTan}\left[\frac{d}{\sqrt{c^2+d^2}}, \frac{c}{\sqrt{c^2+d^2}}\right] + 2\pi C[1],\ C[1] \in \mathrm{Integers}\right]\right\}\right\}$$

$$\theta = \mathrm{ArcTan}\left[\frac{d}{\sqrt{c^2+d^2}}, \frac{c}{\sqrt{c^2+d^2}}\right];$$
$$x = \frac{a\,c + b\,d}{c^2+d^2};$$
$$y = \frac{1}{c^2+d^2};$$

```
Print["Me^(θj) = ", MatrixForm [FullSimplify [M.MatrixExp[θj], a d - b c == 1]]]
Print["1/√y (y x; 0 1) = ", MatrixForm [FullSimplify [1/√y (y x; 0 1), a d - b c == 1]]]
```

$$Me^{\theta j} = \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & \frac{a\,c+b\,d}{\sqrt{c^2+d^2}} \\ 0 & \sqrt{c^2+d^2} \end{pmatrix}$$

$$\frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{c^2+d^2}} & (a\,c+b\,d)\sqrt{\frac{1}{c^2+d^2}} \\ 0 & \frac{1}{\sqrt{\frac{1}{c^2+d^2}}} \end{pmatrix}$$

```
(*Find the action of M on z=x+iy*)
Clear[x, y]
```

$$g = \frac{1}{\sqrt{y}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix};$$

$$\text{Print}\left[\text{"}\frac{1}{\sqrt{y}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mapsto \text{ ", MatrixForm [M.g]}\right]$$

$$\frac{1}{\sqrt{y}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a\sqrt{y} & \frac{b}{\sqrt{y}} + \frac{a x}{\sqrt{y}} \\ c\sqrt{y} & \frac{d}{\sqrt{y}} + \frac{c x}{\sqrt{y}} \end{pmatrix}$$

```
xp = ((M.g)[[1, 1]] (M.g)[[2, 1]] + (M.g)[[1, 2]] (M.g)[[2, 2]]) /
    ((M.g)[[2, 1]]^2 + (M.g)[[2, 2]]^2); (*x↦xp*)
yp = 1/((M.g)[[2, 1]]^2 + (M.g)[[2, 2]]^2); (*y↦yp*)
Print["x' = ", FullSimplify[xp]]
Print["y' = ", FullSimplify[yp]]
```

$$x' = \frac{(b + a x)\,(d + c x) + a c y^2}{(d + c x)^2 + c^2 y^2}$$

$$y' = \frac{y}{(d + c x)^2 + c^2 y^2}$$

```
dxp = D[xp, x] dx + D[xp, y] dy; (*Chain rule*)
dyp = D[yp, x] dx + D[yp, y] dy;
```

$$ds2 = \frac{dx^2 + dy^2}{y^2};$$

$$dsp2 = \frac{dxp^2 + dyp^2}{yp^2};$$

$$\text{Print}\left[\text{"}ds^2 = \text{", ds2}\right]$$

$$\text{Print}\left[\text{"}ds'^2 = \text{", FullSimplify}\left[dsp2,\, a d - b c == 1\right]\right]$$

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

$$ds'^2 = \frac{dx^2 + dy^2}{y^2}$$