**MMA320 Introduction to Algebraic Geometry**
Exercises for Chapter 3

**3.1**. Let $f(X)$ and $G(X)$ be polynomials in $A[X]$, $A$ an UFD. Let $f(X) = a_0(X - \alpha_1)\dots(X-\alpha_m)$ and $g(X) = b_0(X-\beta_1)\dots(X-\beta_n)$ with $b_0 \neq 0$ be the factorisation of the polynomials into linear factors (in an extension of the field of fractions $Q(A)$). Show that $R(f,g) = a_0^n b_0^m \prod_{i=1}^m \prod_{j=1}^n (\alpha_i - \beta_j) = a_0^n \prod_{i=1}^m g(\alpha_i) = (-1)^{mn} b_0^m \prod_{j=1}^n f(\beta_j)$.

**3.2**. Show that $R(f,gh) = R(f,g)R(f,h)$. Use this to show that $I_P(C,E) = I_P(C,D) + I_P(C,E)$.

**3.3**. Show that $R(f, g+af) = R(f,g)$ (suppose $\deg f \leq \deg g$). Let $C=(f)$, $D=(g)$ and $E=(g+af)$ be plane curves. Show that $I_P(C,E) = I_P(C,D)$ for all $P \in C \cap D$.

To compute intersection multiplicities, one can use the results of the previous two exercises together with proposition 3.15.

**3.4**. Let $C_1$, $C_2 \subset \mathbb{P}^2(K)$, $K$ algebraically closed of characteristic zero, be cubic curves with homogeneous equations

$$\begin{aligned} C_1\colon &\qquad (X+Z)^3 + 3Y^3 - Z^3 = 0, \\ C_2\colon &\qquad X^3 + X^2(Y+Z) + XZ^2 + Y^3 = 0. \end{aligned}$$

Compute the intersection points of $C_1$ and $C_2$ with their multiplicities.

**3.5**. Find all points of intersection of the following pairs of curves, and the intersection numbers at these points:
(a) $Y^2Z - X(X-2Z)(X+Z)$ and $Y^2 + X^2 - 2XZ$.
(b) $(X^2+Y^2)Z + X^3 + Y^3$ and $X^3 + Y^3 - 2XYZ$.
(c) $Y^5 - X(Y^2 - XZ)^2$ and $Y^4 + Y^3Z - X^2Z^2$.
(d) $(X^2+Y^2)^2 + 3X^2YZ - Y^3Z$ and $(X^2+Y^2)^3 - 4X^2Y^2Z^2$.

**3.6**. Suppose the intersections of the opposite sides of a hexagon lie on a straight line. Show that the vertices lie on a conic.

**3.7**. Let $C = (f)$ be a projective plane curve. Show that a point $P$ is a multiple point of $C$ if and only if $\frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial Z} = 0$.

**3.8**. Let P be a simple point on $C = (f)$. Show that the tangent line to $C$ at $P$ has the equation $\frac{\partial f}{\partial X}(P)X + \frac{\partial f}{\partial Y}(P)Y + \frac{\partial f}{\partial Z}(P)Z = 0$.

**3.9**. Show that $m_P(\frac{\partial f}{\partial X}) \geq m_P(f) - 1$ for all $f$ and $P$ with $f(P) = 0$.

**3.10**. Show that the following curves are irreducible; find their multiple points, and the multiplicities and tangents at the multiple points.
(a) $XY^4 + YZ^4 + XZ^4$.
(b) $X^2Y^3 + X^2Z^3 + Y^2Z^3$.
(c) $Y^2Z - X(X-Z)(X-\lambda Z)$, $\lambda \in k$.
(d) $X^n + Y^n + Z^n$, $n > 0$.

**3.11**. Let $C = (f)$ be an irreducible curve.
(a) Show that one of $\frac{\partial f}{\partial X}$, $\frac{\partial f}{\partial Y}$ and $\frac{\partial f}{\partial Z}$ is not the zero polynomial.
(b) Show that $C$ has only finitely many multiple points.

**3.12**. Prove that an irreducible cubic is either nonsingular or has at most one double point.

**3.13**. Show that every nonsingular curve in a projective plane over an algebraically closed field is irreducible. Is this true for affine curves?

**3.14**. A line $L$ is a component of the tangent cone of a curve $C$ at a point $P$ if and only if $I_P(C, L) > m_P(C)$.

**3.15**. (char$(k) = 0$) Let $C$ be an irreducible curve of degree n in $\mathbb{P}^2$. Suppose $P \in \mathbb{P}^2$, with $m_P(C) = r \geq 0$. Then for all but a finite number of lines $L$ through $P$, $L$ intersects $C$ in $n - r$ distinct points other than $P$.

**3.16**. (char$(k) = p > 0$) Let $C = V(X^{p+1} - Y^p Z)$, $P = (0 : 1 : 0)$. Find $L \cap C$ for all lines $L$ passing through $P$. Show that every line that is tangent to $C$ at a simple point passes through $P$.

**3.17**. If a curve $C$ of degree $n$ has a point $P$ of multiplicity $n$, show that $C$ consists of $n$ lines through $P$ (not necessarily distinct).

**3.18**. Let $f(x, y)$ be the affine equation of a real or complex plane curve, and $p = (a, b)$ a point on it; suppose that $\frac{\partial f}{\partial y}(p) \neq 0$, so by the implicit function theorem $y = \varphi(x)$ in a neighbourhood of $p$. Prove that $p$ is an inflection point (in the sense that $\varphi''(a) = 0$) if and only if:

$$\begin{vmatrix} f_{xx} & f_{xy} & f_x \\ f_{xy} & f_{yy} & f_y \\ f_x & f_y & 0 \end{vmatrix} = 0.$$

(Hint: differentiate $f(x, \varphi(x)) \equiv 0$ twice. Compute also the determinant.)
Use Euler's formula and $f(a, b) = 0$ to translate this condition into the condition on the vanishing of the Hessian of the associated homogeneous polynomial $F(X, Y, Z)$.

**3.19**. Let $C \subset \mathbb{P}^2(k)$, $\operatorname{char} k \neq 2$, be a plane cubic with inflection point $P$. Prove that a change of coordinates can be used to bring $C$ in the normal form $Y^2 Z = X^3 + aX^2 Z + bXZ^2 + cZ^3$.
Hint: take coordinates such that $P = (0 : 1 : 0)$, and its tangent is $Z = 0$; get rid of the linear term in $Y$ by completing the square.

**3.20**. A real plane cubic with one inflection point has two other inflection points.
Hint: use the result of the previous exercise in affine coordinates, express $y$ as function of $x$ and show that $y''(x)$ has to have a zero.

**3.21**. Find the inflection points of the singular cubics $ZY^2 = X^2(X+Z)$ and $Y^2 Z = X^3$, both in $\mathbb{P}^2(\mathbb{R})$ and in $\mathbb{P}^2(\mathbb{C})$.

**3.22**. Let $P$ and $Q$ be inflection points on a cubic curve $C$. Show that the third intersection point of the line $\overline{PQ}$ with $C$ is also an inflection point.
Hint: use coordinates in which $P = (0 : 1 : 0)$, $Q = (0 : 0 : 1)$ and the flexes are $Y = 0$ and $Z = 0$.

**3.23**. Consider the cubic curve $C$ in $\mathbb{P}^2(\mathbb{C})$ with equation:

$$X^3 + Y^3 + Z^3 - 3\lambda XYZ = 0,$$

where $\lambda^3 \neq 1$. Find its inflection points. Compute the inflectional tangents. What are all lines joining inflection points?

**3.24**. Give an example (in char $p$) of an irreducible curve in the projective plane with identically vanishing Hessian, but where not all points are inflection points.

**3.25**. For which points $P$ on a nonsingular cubic $C$ does there exist a nonsingular conic that intersects $C$ only at $P$?

**3.26**. Let $C$ be the cubic curve in $\mathbb{P}^2(\mathbb{Q})$ with affine equation:

$$Y^2 = X^3 - 2X^2 + 1 \,.$$

Take $(0:1:0)$, the inflection point at infinity, as neutral element of the group law. Compute all multiples $2p$, $3p$, $4p$, $5p$, $\dots$, of the point $p = (0,1)$.

**3.27**. The irreducible cubic curve $C = (Y^2Z - X^3)$ has a cusp in $Q = (0:0:1)$ and an inflection point in $(0:1:0)$. Show that the set $C^{\mathrm{reg}} = C \setminus Q$ has a group structure isomorphic to the additive group $K^+$ of the field $K$.
Hint: take the inflection point as neutral element and use a suitable parametrisation of $C^{\mathrm{reg}}$.

The last exercise can best be done by computer.

**3.28**. Find in two ways all points of intersection of the following pairs of curves, and the intersection numbers at these points:
(a) $(X+Z)^3 + 3Y^3 - Z^3$ and $X^3 + X^2(Y+Z) + XZ^2 + Y^3$.
(b) $Y^5 - X(Y^2 - XZ)^2$ and $Y^4 + Y^3Z - X^2Z^2$.
Determine first the resultant (for a suitable choice of distuinguished coordinate).
Hint: use the idea of remark 1.24 to form a 1-row matrix or ideal containing $m+n$ polynomials, determine the matrix of all coefficients using the command `coeffs` (Singular) or `coefficients` (Macaulay2) and compute the determinant.
A different approach is to compute the primary decomposition of the ideal generated by the two polynomials. Then compute the `degree` of the components found.