❦ Quantum Field Theory Problem 1 ❦

simjac

March 2020

Consider scalar QED, i.e., the theory consisting of a Maxwell field coupled to a complex scalar field $\Phi(x)$ with charge e, mass m and quartic self-interactions. The theory is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\Phi D^\mu\Phi^* - m^2\Phi\Phi^* - \frac{\lambda}{2}(\Phi\Phi^*)^2. \tag{1}$$

a)

Determine the exact form of the covariant derivative D_μ so that the Lagrangian is invariant under the gauge transformation $\Phi \mapsto \Phi' = \mathrm{e}^{ie\alpha(x)}\Phi$. Note the charge in the exponent!

Solution

A_μ transforms like $A_\mu \mapsto A'_\mu = A_\mu - \partial_\mu\alpha$. Thus $D_\mu = \partial_\mu + ieA_\mu$ solves $\mathcal{L}' = \mathcal{L}$ since

(i)

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\ &= \partial_\mu A_\nu + \partial_\mu\partial_\nu\alpha - \partial_\nu A_\mu - \partial_\nu\partial_\mu\alpha \\ &= F_{\mu\nu}. \end{aligned}$$

(ii)

$$\begin{aligned} [D'_\mu\Phi'][(D'^\mu\Phi')^*] &= \big[(\partial_\mu + ieA'_\mu)\mathrm{e}^{ie\alpha}\Phi\big]\big[(\partial^\mu - ieA'^\mu)\mathrm{e}^{-ie\alpha}\Phi^*\big] \\ &= \big[\big(\partial_\mu + ie(A_\mu - \partial_\mu\alpha)\big)\mathrm{e}^{ie\alpha}\Phi\big]\big[\big(\partial^\mu - ie(A^\mu - \partial^\mu\alpha)\big)\mathrm{e}^{-ie\alpha}\Phi^*\big] \\ &= \mathrm{e}^{ie\alpha}\big[\big(ie\partial_\mu\alpha + \partial_\mu + ieA_\mu - ie\partial_\mu\alpha\big)\Phi\big]\mathrm{e}^{-ie\alpha}\big[\big(-ie\partial_\mu\alpha + \partial_\mu - ieA_\mu + ie\partial_\mu\alpha\big)\Phi^*\big] \\ &= \big[\big(\partial_\mu + ieA_\mu\big)\Phi\big]\big[\big(\partial_\mu - ieA_\mu\big)\Phi^*\big] \\ &= [D_\mu\Phi][(D^\mu\Phi)^*]. \end{aligned}$$

(iii)

$$\begin{aligned} \Phi'\Phi'^* &= \mathrm{e}^{ie\alpha}\Phi\mathrm{e}^{-ie\alpha}\Phi^* \\ &= \Phi\Phi^*. \end{aligned}$$

This accounts for all the terms in (1).

b)

Express $\Phi(x)$ as a real scalar $R(x)$ times a phase factor $\mathrm{e}^{i\phi(x)}$ and show that if $\Phi(x)$ gets a VEV $\langle\Phi(x)\rangle = v \in \mathbb{R}$, the theory will contain one massive scalar boson (Higgs) and one massless one (Goldstone).

Solution

Let $\Phi(x) = R(x)\mathrm{e}^{i\phi(x)} = (v + r(x))\mathrm{e}^{i\phi(x)}$. Then

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu + ieA_\mu)\Phi(\partial^\mu - ieA^\mu)\Phi^* - m^2\Phi\Phi^* - \frac{\lambda}{2}(\Phi\Phi^*)^2 \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + 2e^2A_\mu A^\mu vr + e^2A_\mu A^\mu r^2 + e^2A_\mu A^\mu v^2 + 4eA^\mu vr\partial_\mu\phi \\ &+ 2eA^\mu r^2\partial_\mu\phi + 2eA_\mu v^2\partial_\mu\phi - 2m^2vr - m^2r^2 - m^2v^2 + \partial_\mu r\partial^\mu r - 2\lambda v^3 r \\ &- 3\lambda v^2r^2 - 2\lambda vr^3 + 2vr\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\lambda r^4 + r^2\partial_\mu\phi\partial^\mu\phi - \frac{\lambda v^4}{2} + v^2\partial_\mu\phi\partial^\mu\phi. \end{aligned} \tag{2}$$

We identify the mass squared of the r-field with the coefficient, $3\lambda v^2 + m^2$, of the r^2-term. Similarly, since there is no ϕ^2-term, the ϕ-field is massless.

c)

Assume that the scalar potential $V(\Phi,\bar{\Phi})$ is the *Mexican hat* type which requires m^2 to be negative (set $m^2 = -\mu^2$) Show that there is a stable vacuum away from $\Phi = 0$ and determine v in terms of the parameters of the Lagrangian.

Solution

$$\begin{aligned} V(\Phi,\bar{\Phi}) &= -\mu^2\Phi^*\Phi + \frac{\lambda}{2}\left(\Phi^*\Phi\right)^2 \\ &= -\mu^2(v+r)^2 + \frac{\lambda}{2}(v+r)^4 \end{aligned}$$

This has a minimum at $v + r = \frac{\mu}{\sqrt{\lambda}}$.

d)

Explain now the masses of the Higgs and Goldstone scalar bosons found in **b)**.

Solution

Consider the Lagrangian (2), the Euler-Lagrange equation for r then implies

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu r))}\right) - \frac{\partial\mathcal{L}}{\partial r} = 0,$$

$$r\left(e^2A_\mu A^\mu + m^2 - \partial_\mu\phi\partial^\mu\phi\right) + A_\mu e^2A^\mu v + m^2v + \partial_\mu\partial^\mu r = v\partial_\mu\phi\partial^\mu\phi.$$

Solving this with the ansatz $\Phi = \mathrm{e}^{ip_\mu x^\mu}$ yields

$$p_\mu p^\mu = e^2A_\mu A^\mu + m^2 > 0$$

and thus this particle is massive.

The Euler-Lagrange equation for σ defined by $\partial\mu\sigma = r\partial_\mu\phi$ then implies

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\sigma))}\right) - \frac{\partial\mathcal{L}}{\partial\sigma} = 0,$$

$$(r+v)\left(2\partial_\mu r\partial^\mu\phi + (r+v)\partial_\mu\partial^\mu\phi\right) = 0.$$

This equation is reduces to $0 = 0$ with the ansatz $\Phi = \mathrm{e}^{ip_\mu x^\mu}$ which I don't know how to interpret.

e)

Show that the vector field can absorb ("eat") the Goldstone boson in which process the Goldstone boson disappears from the Lagrangian and the vector field becomes massive, i.e., it has now three degrees of freedom. Determine this mass and explain why its sign is physically correct.

Solution

Performing a gauge transform with $\alpha = \frac{1}{e}\phi$, (1) becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left[(\partial_\mu + ieA_\mu)(r+v)\right]\left[(\partial^\mu - ieA^\mu)(r+v)\right] - m^2(r+v)^2 - \frac{\lambda}{2}(r+v)^4.$$

As before, we identify the mass squared of the A_μ-field with the coefficient, e^2v^2, of the A^2-term. This field has three degrees of freedom, four (since 4-vector) minus one (from choosing gauge).

The following Mathematica script was used for some of the calculations

In[1]:=
```
ClearAll["Global`*"]
$Assumptions = (v ∈ Reals
    && r[x] ∈ Reals
    && ϕ[x] ∈ Reals);

Φ[x_] := (v + r[x]) Exp[I ϕ[x]];
L = -1/4 F[A] × F[A] + (∂x Φ[x] + I e A Φ[x]) (∂x Φ[x]* + I e A Φ[x]*) -
     m^2 Φ[x] × Φ[x]* - λ/2 (Φ[x] × Φ[x]*)^2 // ComplexExpand // FullSimplify;
Print["L = ", Expand[L]]

(* Euler-Lagrange equation of free r-field *)
ELeq = ∂x(∂r'[x] L) - ∂r[x] L == 0 /. {λ → 0} // FullSimplify
(* Substitute *)
rTmp[x_] := 1;
ϕTmp[x_] := p x;
ELeq /. {r → rTmp, ϕ → ϕTmp} // FullSimplify
```

$$\text{L} \;=\; -A^2 e^2 v^2 - m^2 v^2 - \frac{v^4 \lambda}{2} - \frac{F[A]^2}{4} - 2 A^2 e^2 v\, r[x] - 2 m^2 v\, r[x] - 2 v^3 \lambda\, r[x] - A^2 e^2 r[x]^2 - m^2 r[x]^2 - 3 v^2 \lambda\, r[x]^2 - 2 v \lambda\, r[x]^3 - \frac{1}{2} \lambda\, r[x]^4 + 2 i A e v\, r'[x] + 2 i A e\, r[x]\, r'[x] + r'[x]^2 + v^2 \phi'[x]^2 + 2 v\, r[x]\, \phi'[x]^2 + r[x]^2 \phi'[x]^2$$

Out[6]= $(v + r[x])\left(A^2 e^2 + m^2 - \phi'[x]^2\right) + r''[x] == 0$

Out[9]= $\left(A^2 e^2 + m^2 - p^2\right)(1 + v) == 0$

In[10]:=
```
(* Euler-Lagrange equation of ϕ-field *)
ELeq = ∂x(∂ϕ'[x] L) - ∂ϕ[x] L == 0 // FullSimplify

(* Substitute *)
rTmp[x_] := 1;
ϕTmp[x_] := p x;
ELeq /. {r → rTmp, ϕ → ϕTmp} // FullSimplify
```

Out[10]= $(v + r[x])\left(2\, r'[x]\, \phi'[x] + (v + r[x])\, \phi''[x]\right) == 0$

Out[13]= True