❦ Home assignment 1 — Symmetry ❦

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1

Show that the real Lie algebra $\mathfrak{so}(4)$ of rotations in 4 euclidean dimensions is $\mathfrak{so}(4) \sim \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

Solution

Consider the basis

$$\begin{aligned} X_1 &= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, & X_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, & X_3 &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ Y_1 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, & Y_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, & Y_3 &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \end{aligned} \tag{1}$$

for $\mathfrak{so}(4)$. This basis has the structure constants,

$$[X_i, X_j] = 2\epsilon_{ijk} X^k, \quad [Y_i, Y_j] = 2\epsilon_{ijk} Y^k, \quad [X_i, Y_j] = 0, \tag{2}$$

which is the same as for the basis

$$\begin{bmatrix} -i\sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -i\sigma_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -i\sigma_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -i\sigma_1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & -i\sigma_2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & -i\sigma_3 \end{bmatrix}$$

for $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. See section 6 for this calculation. Thus there is a map $\phi : \{X_1, X_2, X_3, Y_1, Y_2, Y_3\} \to \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ such that

$$\phi([A, B]) = [\phi(A), \phi(B)], \quad A, B \in \{X_1, X_2, X_3, Y_1, Y_2, Y_3\}.$$

We can linearly extend this map uniquely so that

$$
\begin{aligned}
\phi(A+B) &= \phi(A)+\phi(B), && A,B\in\mathfrak{so}(4)\\
\phi(cA) &= c\phi(A), && A\in\mathfrak{so}(4),\ c\in\mathbb{R}.
\end{aligned}
$$

Since ϕ is bijective (maps the 6 basis elements of $\mathfrak{so}(4)$ to the 6 basis elements of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$), it is an isomorphism between $\mathfrak{so}(4)$ and $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$.

Remarks

$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ is now viewed as an algebra over $\mathbb{R}$.

Note also that the basis (1) for $\mathfrak{so}(4)$ is not taken from thin air as it can be written as

$$
\begin{aligned}
X_1 &= \begin{bmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{bmatrix}, \quad X_2 &&= \begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{bmatrix},\\
Y_1 &= \begin{bmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix}, \quad Y_2 &&= \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{bmatrix}.
\end{aligned}
$$

From this one could, if one was inclined to, proceed to derive (2) by

$$
\begin{aligned}
[X_1, X_2] &= \begin{bmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{bmatrix}\\
&= \begin{bmatrix} \sigma_3\sigma_1-\sigma_1\sigma_3 & 0 \\ 0 & \sigma_3\sigma_1-\sigma_1\sigma_3 \end{bmatrix} = \begin{bmatrix} -2i\sigma_2 & 0 \\ 0 & -2i\sigma_2 \end{bmatrix}\\
&= 2X_3
\end{aligned}
$$

and so on for all of the possible combinations of basis elements.

2

In $2n$ dimensions with signature (n,n), consider the matrix $\gamma = \frac{1}{(2n)!}\varepsilon^{M_1\dots M_{2n}}\gamma_{M_1\dots M_{2n}} = \gamma_1\dots\gamma_{2n}$. What is γ^2? How does it, γ, anti-commute with the γ matrices? Use γ to form projection operators on the two chiralities. If ψ is a spinor of definite chirality, what is the chirality of $v^M\gamma_M\psi$? How are the properties of γ affected if the signature is changed? (Hint: one may think of multiplying some of the γ-matrices by i.)

Solution

From the lecture, we know that

$$
\{\gamma_M, \gamma_N\} = 2\eta_{MN}I. \tag{3}
$$

Thus $\gamma_M\gamma_N = 2\eta_{MN} - \gamma_M\gamma_N$, and

$$
\begin{aligned}
\{\gamma_K, \gamma\} &= \gamma_K\gamma_1\cdots\gamma_{2n} + \gamma_1\cdots\gamma_{2n}\gamma_K \\
&= \gamma_K\gamma_1\cdots\gamma_{2n} + \gamma_1\cdots\gamma_{2n-1}2\eta_{2n,K} - \gamma_1\cdots\gamma_{2n-1}\gamma_K\gamma_{2n} \\
&\vdots \\
&\overset{\star}{=} \gamma_K\gamma_1\cdots\gamma_{2n} \\
&+ \gamma_1\cdots\gamma_{2n-1}2\eta_{2n,K} \\
&+ \gamma_1\cdots\gamma_K 2\eta_{K+1,K}\gamma_{K+2}\cdots\gamma_{2n} \\
&+ \gamma_1\cdots\gamma_{K-2}2\eta_{K-1,K}\gamma_K\cdots\gamma_{2n} + \ldots \\
&+ 2\eta_{1K}\gamma_2\cdots\gamma_{2n} \\
&- \gamma_K\gamma_1\cdots\gamma_{2n} \\
&\overset{\star\star}{=} 0
\end{aligned}
\tag{4}
$$

where in $\star$ we have used that we do not need to commute γ_K past γ_K, so we only have to do $2n-1$ commutations in total, hence the minus sign in front of $\gamma_K\gamma_1\cdots\gamma_{2n}$. In $\star\star$, we have also assumed that the metric can be diagonalized, so that $\eta_{NM} = 0$ whenever $N \neq M$.

From (3), it follows that $(\gamma_K)^2 = \eta_{KK}I$, where η_{KK} is either 1 or -1 depending on if $K \leq n$ or $K > n$. We can do similarly as before and commute the γ-matrices across γ:

$$
\begin{aligned}
\gamma^2 &= \gamma_1\cdots\gamma_{2n}\gamma_1\cdots\gamma_{2n} \\
&= -\gamma_1\gamma_1\cdots\gamma_{2n}\gamma_2\cdots\gamma_{2n} \\
&= -\eta_{11}\gamma_2\cdots\gamma_{2n}\gamma_2\cdots\gamma_{2n} \\
&= -\eta_{11}\gamma_2\gamma_2\cdots\gamma_{2n}\gamma_3\cdots\gamma_{2n} \\
&= -\eta_{11}\eta_{22}\cdots\gamma_{2n}\gamma_3\cdots\gamma_{2n} \\
&\vdots \\
&= (-1)^n\eta_{11}\cdots\eta_{2n,2n}I \\
&= (-1)^n(-1)^nI \\
&= I
\end{aligned}
\tag{5}
$$

Since γ squares to I, and its eigenvectors $\{\Omega^{(p)} \text{ s.th } p = 1, \ldots, n\}$ span the whole 2^n-dimensional space, γ has 2^n eigenvalues in $\{1, -1\}$. If we can show that γ is traceless, it follows that it has n eigenvalues 1, and n eigenvalues -1. γ is traceless since

$$\begin{aligned}
\operatorname{tr}\,\gamma &= \operatorname{tr}\,\gamma I \\
&\overset{(3)}{=} \operatorname{tr}\,\gamma \frac{\gamma^L \gamma^L}{\eta^{LL}} \\
&= \frac{1}{\eta^{LL}} \begin{cases} \operatorname{tr}\,\gamma^L \gamma \gamma^L & \text{by cyclic permutation} \\ -\operatorname{tr}\,\gamma^L \gamma \gamma^L & \text{by (4)} \end{cases},
\end{aligned}$$

so $\operatorname{tr}\,\gamma$ must equal 0. We can now construct the projection operators

$$P_\pm = \frac{\gamma \pm 1}{\sqrt{2}}.$$

To show that these are projection operators, we want to show that $P_\pm$ is an idempotent, but $P_\pm$ has eignenvalues 0 and 1 due to γ having eigenvalues ± 1, so this is clear.

We change the metric from η with signature (n, n) to η' with signature (a, b) such that $a + b = 2n$ (again, assuming a diagonalizable invertible metric). (3) now reads

$$\{\gamma_M, \gamma_N\} = 2\eta'_{MN} I. \tag{3'}$$

Since we still assume the metric to be diagonalizable, the derivation of (4) still holds. In (5), in the last term, we'd obtain $(-1)^a(-1)^b = (-1)^{2n} = 1$. Thus (5) still holds as well.

3

Consider the Maxwell field strength 2-form

$$F = \frac{1}{4\pi r^3} \left(x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y \right),$$

which is well defined outside the origin. What is the corresponding B-field? Show that F satisfies Maxwell's equations for $r > 0$. Calculate the surface integral $\int_S F = \int_S \vec{B} \cdot\ \mathrm{d}\vec{S}$, where S is a surface enclosing $r = 0$, and conclude that there is a magnetic monopole at $r = 0$. Find a 1-form A such that $\mathrm{d}A = F$. Is it well defined everywhere outside the origin?

Solution

We make the ansatz

$$A = (0, 0, \frac{1 - \cos\theta}{4\pi r \sin\theta})$$

in spherical coordinates r, θ, ϕ. We can verify this ansatz using Mathematica, see section 7. The corresponding B-field is the (component wise) same as F since we have no time dependence or E-field. The surface integral of B over the unit sphere thus becomes

$$\int_S \vec{B} \cdot \mathrm{d}\vec{S} = \int_S \mathrm{d}\Omega \frac{1}{4\pi} \hat{r} \cdot \hat{r} = 1.$$

This implies that the magnetic field has a source. Since A is singular at $r = 0$, we cannot integrate over the whole volume and say something like "we have a net megnetic charge".

4

Construct the two 3-dimensional $\mathfrak{sl}(3)$-modules by starting from the highest weights with Dynkin labels (10) and (01) and acting with lowering operators. If we denote a representation by the Dynkin labels of its highest weight, we can write $\mathbf{3} = (10)$, $\bar{\mathbf{3}} = (01)$. Determine, by some method, the tensor products $(10) \otimes (10)$ and $(10) \otimes (01)$ as direct sums of irreducible representations. Illustrate with sums of weights in a picture.

Solution

$\mathfrak{sl}(3)$ is a semisimple Lie algebra. The Killing metric is thus invertible and the root space $\mathfrak{h}^*$ is isomorphic to the weight space $\mathfrak{h}$. This is the motivation for drawing roots and weights in the same picture.

By $[h_i, e_i] = \alpha_i e_i$, if a state v has h_i-eigenvalue λ_i, then $e_i v$ has h_i-eigenvalue $\lambda_i + \alpha_i$ since

$$h_i e_i v = (\alpha_i e_i + e_i h_i) v = (\alpha_i + \lambda_i) e_i v.$$

This is why adding roots in the root lattice corresponds to acting with a raising operator. The Cartan matrix tells us the amount of times we can act with each operator before getting the null state through

$$\left(\mathrm{ad}(e_i)\right)^{1-A_{ij}} e_j = 0$$
$$\left(\mathrm{ad}(f_i)\right)^{1-A_{ij}} f_j = 0.$$

Consider the $\mathfrak{sl}(3)$ roots shown in Figure 1a. From these roots, and the condition $\Lambda_j(\alpha_i) = \delta_{ij}$, the weights must be placed as in Figure 1b in the root lattice.

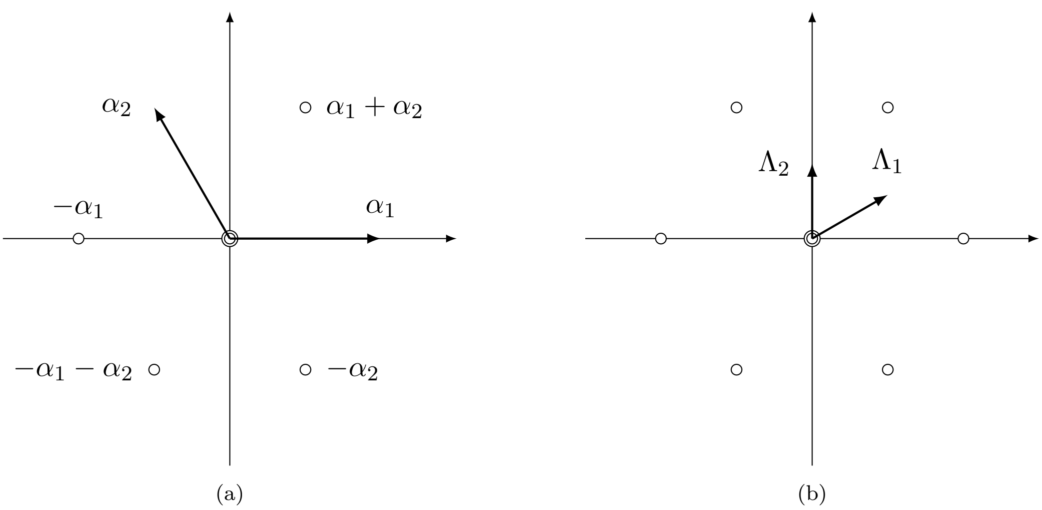
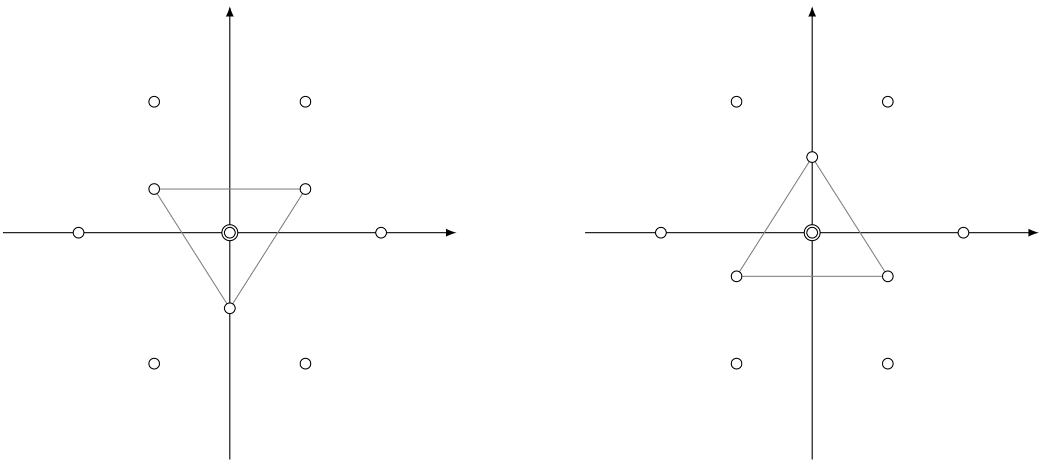


Figure 1: $\mathfrak{sl}(3)$ *roots and fundamental weights.*

Acting by the lowering operators on $(10) = \Lambda_1$ and $(01) = \Lambda_2$, we obtain Figure 2a and Figure 2b.



(a) *The (10) representaion of* $\mathfrak{sl}(3)$.

(b) *The (01) representaion of* $\mathfrak{sl}(3)$.

Figure 2

5

The Weyl group of a semi-simple Lie algebra is the discrete group generated by reflections in hyperplanes orthogonal to the simple roots. It is a symmetry of the root system. Reflection in the hyperplane orthogonal to α_i maps a vector β to $w_i(\beta) = \beta - \frac{2(\beta,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$. Describe the Weyl groups of $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ (number of elements, multiplication table).

Solution

$\mathfrak{sl}(2)$ has the Weyl group $\{e, w_1\}$ with trivial group operation. In $\mathfrak{sl}(3)$, we can reach not only reflections, but also rotations through composition of reflections in different axis. Denote by w_i the reflection in the line L_i in Figure 3. There are in total 3 different reflections and 3 rotations (e, r_1, and r_2) which are generated by $\{w_i\}$. This is the symmetry group of a equilateral triangle. $\mathfrak{sl}(3)$ thus has the Weyl group S_3, who's Cayley table is presented in Table 1.

Table 1

	w_1	w_2	w_3	r_1	r_2
w_1	e	r_1	r_2	w_2	w_3
w_2	r_2	e	r_1	w_3	w_3
w_3	r_1	r_2	e	w_1	w_2
r_1	w_3	w_1	w_2	r_2	e
r_2	w_2	w_3	w_1	e	r_1

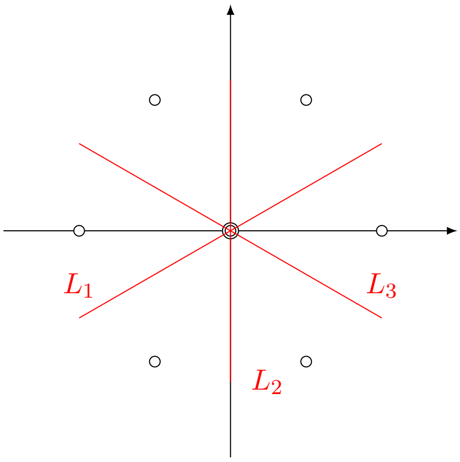


Figure 3: $\mathfrak{sl}(3)$ *roots.*

6 Calculating structure constants for section 1 with Mathematica

ClearAll["Global`*"]

lieBracket[A_, B_]:=$A.B - B.A$

$$\mathrm{A1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \mathrm{A2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \mathrm{A3} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\mathrm{B1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \mathrm{B2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \mathrm{B3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

X1 = (1/2)(A1 + B1); X2 = (1/2)(A2 + B2); X3 = (1/2)(A3 + B3);

Y1 = (1/2)(A1 − B1); Y2 = (1/2)(A2 − B2); Y3 = (1/2)(A3 − B3);

so4BasisList = {X1, X2, X3, Y1, Y2, Y3};

(*Write down su(2) $\oplus$ su(2) structure constants*)

su2StructureConstants[i_, j_, k_]:=LeviCivitaTensor[3][[i, j, k]]

(*Check if the same structure constants hold for so(4)*)

For[$i = 1, i \leq 3, i{+}{+}$,

For[$j = 1, j \leq 3, j$++,

bool = (lieBracket[so4BasisList[[i]], so4BasisList[[j]]] ==

Sum[su2StructureConstants[i, j, k]so4BasisList[[k]], {k, 1, 3}]);

Print[bool, " for ij=", i, j];

]

]

True for ij=11

True for ij=12

True for ij=13

True for ij=21

True for ij=22

True for ij=23

True for ij=31

True for ij=32

True for ij=33

For[$i = 1, i \leq 3, i$++,

For[$j = 1, j \leq 3, j$++,

bool = (lieBracket[so4BasisList[[$i + 3$]], so4BasisList[[$j + 3$]]] ==

Sum[su2StructureConstants[i, j, k]so4BasisList[[$k + 3$]], {k, 1, 3}]);

Print[bool, " for ij=", $i + 3, j + 3$];

]

]

True for ij=44

True for ij=45

True for ij=46

True for ij=54

True for ij=55

True for ij=56

True for ij=64

True for ij=65

True for ij=66

```
For[i = 1, i ≤ 3, i++,
For[j = 1, j ≤ 3, j++,
bool = (lieBracket[so4BasisList[[i]], so4BasisList[[j + 3]]] ==
ConstantArray[0, {4, 4}]);
Print[bool, " for ij=", i, j + 3];
]
]
```

True for ij=14

True for ij=15

True for ij=16

True for ij=24

True for ij=25

True for ij=26

True for ij=34

True for ij=35

True for ij=36

7 Verifying Maxwell field strength with Mathematica in section 3

```
ClearAll["Global`*"]
$Assumptions = (x ∈ Reals&&
      y ∈ Reals&&
      z ∈ Reals);
(*Define some shorthands*)
```

$r = \left(x^2 + y^2 + z^2\right)^{1/2};$

$\cos\theta = \frac{z}{r};$

$\sin\theta = \frac{\left(x^2+y^2\right)^{1/2}}{r};$

$\cos\phi = \frac{x}{\left(x^2+y^2\right)^{1/2}};$

$\sin\phi = \frac{y}{\left(x^2+y^2\right)^{1/2}};$

(*Make a qualified guess*)

$\mathrm{Ax} = -\frac{1-\cos\theta}{r\sin\theta}\sin\phi;$

$\mathrm{Ay} = \frac{1-\cos\theta}{r\sin\theta}\cos\phi;$

$\mathrm{Az} = 0;$

(*Check guess*)

$\mathrm{Fyz} = \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}};$

$\mathrm{Fzx} = \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}};$

$\mathrm{Fxy} = \frac{z}{\left(x^2+y^2+z^2\right)^{3/2}};$

Print["“”∂y(Az) - ∂z(Ay) == Fyz” = ”,

FullSimplify[D[Az, y] − D[Ay, z]] == Fyz];

Print["“”∂z(Ax) - ∂x(Az) == Fzx” = ”,

FullSimplify[D[Ax, z] − D[Az, x]] == Fzx];

Print["“”∂x(Az) - ∂y(Ax) == Fxy” = ”,

FullSimplify[D[Ay, x] − D[Ax, y]] == Fxy];

”∂y(Az) - ∂z(Ay) == Fyz” = True

”∂z(Ax) - ∂x(Az) == Fzx” = True

”∂x(Az) - ∂y(Ax) == Fxy” = True