Tutorial 2 Problems

for discussion on Week 47

Conditional, Full probability and Bayes formula

1. What is the probability that the sum of numbers shown on two symmetric dice is 8 if we know that the sum is even? The same question, given the sum is odd.

2. There are N lottery tickets lying on the table, among them $n < N$ are winning. The first person comes and picks one ticket at random, followed by the second and the third. Who has the largest chance of getting a winning ticket: the first, second or third person? Answering this question may help you to decide when it is better to come to an exam where you supposed to take a variant at random and you know answer to only 'lucky' few. Waiting could mean that your lucky tickets been already taken by other students who came earlier...

3. Work out the VLE studies 1.2-1.4.

Independence

4. Problem 1.5.9 from GS[*]

5. Problem 1.5.3 from GS.

Random variables and their distributions

6. The density of a probability measure $\mathbf{P}$ on the line is a constant c on the interval $[a, b]$ and 0 anywhere else. Find this constant and evaluate the measure $\mathbf{P}$ of $[a, (a + b)/2]$, $(a, (a + b)/2)$ and $[a + (b - a)/4, b - (b - a)/4]$.

7. The density of a probability measure $\mathbf{P}$ on the line has the form ax for $x \in [0, 5]$ and 0 anywhere else. Find the constant a and find the *median* of $\mathbf{P}$ which is such a point m that $\mathbf{P}(-\infty, m] = 1/2$[†] and evaluate the measure $\mathbf{P}$ of $[0, 2.5]$, $(2.5, 5)$.

8. Show that any mapping $\Omega \mapsto \mathbb{R}$ is always measurable with respect to 2^Ω – the σ-field of its all subsets. This would imply that on discrete spaces when we usually consider 2^Ω as a standard σ-field, any mapping to $\mathbb{R}$ is a random variable.

[*] In what follows, GS stands for the course book: Geoffrey Grimmett and David Stirzaker, *Probability and Random Processes*, Oxford University Press, 3rd edition, 2001. Problem x.y.z means Problem z for Section x.y in this book

[†] More exactly, the median is $\sup\{t : \mathbf{P}(-\infty, t] \leq 1/2\}$ which is the same as given above for continuous distributions.

9. Let $[X, \mathcal{X}]$ be a measurable space and let ξ be a mapping from some set Ω into X. Shown that the system of inverse images $\sigma(\xi) = \{\xi^{-1}(B) : \ B \in \mathcal{X}\}$ is a σ-field. It is called the σ-field *generated by* ξ. Show that ξ is $[\sigma(\xi), \mathcal{X}]$-measurable. In particular, when $X = \mathbb{R}$, ξ it is a random variable on $[\Omega, \sigma(\xi)]$. Describe the σ-field generated by ξ taking only two values: 0 and 1.

10. The following example shows that random variables defined on different sample spaces may, however, have the same distribution. Consider rolling a symmetric die. An evident chice is $\Omega = \{1, 2, 3, 4, 5, 6\}$ and ξ_0, the number shown on the face of the die, is the identity mapping into $\mathbb{R}$: $\xi_0(k) = k$ for $k \in \Omega$. Alternatively, take $\Omega = [0, 1]$ with the Borel σ-field of its subsets and the probability measure $\mathbf{P}$ which is the restriction of the Lebesgue measure onto $[0, 1]$ (so just the ordinary length). Define $\xi_1(\omega) = [6\omega] + 1, \ \omega \in [0, 1]$, where $[x]$ is the integer part of x. Check that the distributions $\mathbf{P}_{\xi_0}$ and $\mathbf{P}_{\xi_1}$ are the same. What is this measure?

11. A random variable ξ has equal chances to take the following 3 values: $-1, 0$ and 1. Find its Probability Mass Function (p.m.f.), draw its Cumulative Probability Function (c.d.f.) and find p.m.f.'s for the following random variables:

    (a) $|\xi|$;
    (b) $\xi^2 + 1$;
    (c) 2^ξ;
    (d) $\xi_+ = \max\{\xi, 0\}$.

12. Problem 2.1.2 from GS

13. Problem 2.3.2 from GS

14. Problem 2.3.3 from GS. This property is the basis of computer simulations of random variables known as Monte-Carlo methods.

Lebesgue integral

15. Given two measures μ_1, μ_2 on a measurable space $[X, \mathcal{X}]$, define the measure $\mu_1 + \mu_2$ by means of $(\mu_1 + \mu_2)(B) = \mu_1(B) + \mu_2(B)$ for any $B \in \mathcal{X}$. By employing the Monotone Class Argument (MCA), show that

$$\int f \, d(\mu_1 + \mu_2) = \int f \, d\mu_1 + \int f \, d\mu_2.$$

16. Show that if $f \geq 0$ a.e. (a.e. stands for *almost everywhere*, meaning that $\mu\{x : \ f(x) \not\geq 0\} = 0$) is such that $\mu\{x : \ f(x) > 0\} > 0$, then $\int f \, d\mu > 0$.
    Hint. The MCA does not help here since the strict inequality for members of a sequence implies only a *non-strict* inequality for the limit! Instead, represent $\{x : \ f(x) > 0\} = \cup_n A_n$, where $A_n = \{x : \ f(x) \geq 1/n\}$ is a growing sequence of sets and estimate from below the integral over A_n.