Tutorial 1 Problems

for discussion on Week 46

Sets, countability, events and σ-fields

1. Describe the sample space Ω for the following experiments:

   (a) Toss of 3 indistinguishable coins;

   (b) 3 tosses of one coin;

   (c) Drawing 2 balls from an urn containing 2 black and 2 white balls;

   (d) A distance from a landed dart to the 'bullseye' (the centre of the dartboard);

   (e) Picking a random ark from a circle (give at least two ways of defining Ω!)

2. Prove the second de Morgan's law: $(A \cap B)^{\mathbf{c}} = A^{\mathbf{c}} \cup B^{\mathbf{c}}$.

3. Prove that a Cartesian product of two countable sets is countable.

4. Prove that a Cartesian product of countably many finite sets with at least two elements in each is uncountable.

5. Prove *inclusion–exclusion formula* from Problem 1.3.4 from GS [1] (i.e. Problem 4 for Section 1.3 – see page 8), for $n = 3$ first. Try for general n by induction.

6. Describe the σ-field of subsets of $\Omega = [0, 1]$ generated by the system of singletons, i.e. the sets $\{x\}$, $x \in [0, 1]$.

7. Prove that the intersection of two (actually, of any system of) σ-fields of the same sample space Ω is a σ-field. This enables one to define the σ-field $\sigma(\mathcal{A})$ generated by a system $\mathcal{A}$ of subsets as the intersection of all σ-fields containing $\mathcal{A}$.

[1] In what follows, GS stands for the course book: Geoffrey Grimmett and David Stirzaker, *Probability and Random Processes*, Oxford University Press, 3rd edition, 2001. Problem x.y.z means Problem z for Section x.y in this book

8. Prove that the Borel σ-field of subsets of $\mathbb{R}$ is countably generated, i.e. there is a countable system of subsets of $\mathbb{R}$ which also generates it.
   NB. Prove that any open interval can be represented as a countable union of intervals with rational endpoints and use Question 3.

9. Prove Problem 1.2.4 from GS.
   NB. Such $\mathcal{G}$ is called the σ-field *induced* on B.

Probability and its main properties

10. Make sure you can easily do Classwork>Study 1.1: Elementary probability on the VLE `vle.math.chalmers.se`

11. *Continuity of a measure.* Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Using the countable additivity property of a measure, prove that

    (a) for any growing system of sets $G_i \in \mathcal{F}$, i.e. $G_i \subseteq G_{i+1}$ for all i,
    $$\mathbf{P}(\cup_i G_i) = \lim_{i\to\infty} \mathbf{P}(G_i).$$

    (b) for any decaying system of sets $F_i \in \mathcal{F}$, i.e. $F_{i+1} \subseteq F_i$ for all i,
    $$\mathbf{P}(\cap_i F_i) = \lim_{i\to\infty} \mathbf{P}(F_i);$$

    (c) Show that the continuity property above implies the contable additivity, so they are actually equivalent.

12. Given a probabilioty measure $\mathbf{P}$ on the Borel sets of $\mathbb{R}$, using its continuity, show that the cumulative distribution function (c.d.f.) $F(x) = \mathbf{P}((-\infty, x])$ is continuous from the right.

13. It was shown at the lecture that a right continuous non-decaying function $F(x)$, such that $F(-\infty) = 0$ and $F(+\infty) = 1$, defines quantities $\mathbf{P}((a, b]) = F(b) - F(a)$ for all real $a < b$. Complete the definition of $\mathbf{P}$ for the ranges of types $(a, b], [a, b]$ and (a, b) and show that such defined $\mathbf{P}$ is continuous (hence countably additive by Prob. 11c) on the semi-ring of such intervals so that it can be extended to a measure of all Borel sets. Such F is then called the c.d.f. of $\mathbf{P}$.

14. Show that if $\mathbf{P}(B) > 0$, then the conditional probability $\mathbf{P}(A \mid B)$ as a function of $A \in \mathcal{F}$ is a probability measure. Using the setting Question 9, show that $\mathbf{P}(C \mid B)$ as a function of $C \in \mathcal{G}$ is a probability measure on $\mathcal{G}$.

15. Our experiment is to choose a random ark on the unit circle in the plane centred at the origin which *starts* at $(1, 0)$ and goes along the upper unit semi-circle in the contra-clockwise direction. One way to describe such an ark is by its length $l \in [0, \pi] = \Omega_1$. The other one is by identifying the end point by its first coordinate: x (so that the end point has coordinates $(x, \sqrt{1 - x^2})$). Then $x \in [-1, 1] = \Omega_2$. A *randomly selected ark* corresponds in both models to the uniform measure on the sample space, i.e. the Lebesgue measure scaled to have total mass of Ω to be 1. Let B be the event that the chosen ark has length at most $\pi/3$. Indicate graphically event B as a subset of Ω_1 and of Ω_2 and compute the probability of B in both models. Are these the same? Can you explain this?

16. Problem 1.3.1 from GS.

17. Show that if two events A and B cannot happen at the same time and their probabilities are non-zero, then they are dependent (i.e. not independent).

18. Problem 1.8.7 from GS.

19. Problem 1.7.4 from GS.
    NB. Notice that often a correct mathematical formulation of the problem is half its solution! Imagine an experiment when someone asks you every morning if you prefer a cup of coffee (x) to tea (y), and presumably your answer may depend on whether you slept well tonight (C). Denote by X an event that at a randomly selected day you prefer coffee to tea. Is $\mathbf{P}(X) = 1$ under the given conditions?

Symmetric experiment and combinatorics

20. A cube with all its faces coloured is cut into 1000 equal size small cubes. Find the probability that a small ball chosen at random has exactly 2 coloured faces.

21. Problem 1.8.1 from GS