❦ Gravitation & Cosmology home problems 3.2 ❦

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Home Problem 14

A plane-fronted gravitational wave in the positive z-direction may be described by the metric

$$\mathrm{d}\tau^2 = 2\mathrm{d}u\mathrm{d}v - a(u)^2\mathrm{d}x^2 - b(u)^2\mathrm{d}y^2, \quad \text{where } u = \frac{1}{\sqrt{2}}(t-z), v = \frac{1}{\sqrt{2}}(t+z). \tag{4.35}$$

a)

Compute the Ricci tensor using the coordinates (u, v, x, y). What is the condition on the functions $a(u)$ and $b(u)$ for the Ricci tensor to vanish?

Solution

$R_{uu} = -\frac{a''(u)}{a(u)} - \frac{b''(u)}{b(u)}$ from the Mathematica code below. It vanishes if $-\frac{a''(u)}{a(u)} = \frac{b''(u)}{b(u)}$.

```
In[1]:= ClearAll["Global`*"]

dim = 4;
coordinateList = {u, v, x, y};
g = (0  1    0       0
     1  0    0       0
     0  0  -a[u]^2   0
     0  0    0     -b[u]^2);
gInv = Inverse[g];

(* Initialize Γ^μ_νρ as rank 3 tensor *)
tmp[a_, b_, c_] := 0;
Γ = Array[tmp, {dim, dim, dim}];

(* Loop over indices in Γ^μ_νρ *)
Do[
 Do[
  Do[
   Do[
    (* Calculate Γ^μ_νρ *)
    xμ = coordinateList[[μ]];
    xν = coordinateList[[ν]];
    xρ = coordinateList[[ρ]];
    xσ = coordinateList[[σ]];

    Γ[[μ, ν, ρ]] += 1/2 gInv[[μ, σ]] (∂_xν g[[σ, ρ]] + ∂_xρ g[[ν, σ]] - ∂_xσ g[[ρ, ν]]),
    {σ, 1, dim}];

   (* Print nonzero values *)
   If[Γ[[μ, ν, ρ]] == 0, (* Do nothing *),
    Print["Γ"^xμ_(xν, xρ), " = ", Γ[[μ, ν, ρ]]], Print["Γ"^xμ_(xν, xρ), " = ", Γ[[μ, ν, ρ]]]],

   {ρ, 1, dim}],
  {ν, 1, dim}],
 {μ, 1, dim}]
```

$\Gamma^{v}{}_{x,x} = a[u]\, a'[u]$

$\Gamma^{v}{}_{y,y} = b[u]\, b'[u]$

$\Gamma^{x}{}_{u,x} = \dfrac{a'[u]}{a[u]}$

$\Gamma^{x}{}_{x,u} = \dfrac{a'[u]}{a[u]}$

$\Gamma^{y}{}_{u,y} = \dfrac{b'[u]}{b[u]}$

$\Gamma^{y}{}_{y,u} = \dfrac{b'[u]}{b[u]}$

In[9]:=
```
(* Initialize R^ρ_σμν as rank 4 tensor *)
tmp[a_, b_, c_, d_] := 0;
R = Array[tmp, {dim, dim, dim, dim}];

(* Loop over indices in R^ρ_σμν *)
Do[
 Do[
  Do[
   Do[
    (* Calculate R^ρ_σμν *)
    xμ = coordinateList[[μ]];
    xν = coordinateList[[ν]];
    xρ = coordinateList[[ρ]];
    xσ = coordinateList[[σ]];

    R[[ρ, σ, μ, ν]] = ∂_xμ Γ[[ρ, ν, σ]] - ∂_xν Γ[[ρ, σ, μ]] +
      Sum[Γ[[ρ, μ, τ]] × Γ[[τ, ν, σ]] - Γ[[ρ, ν, τ]] × Γ[[τ, μ, σ]], {τ, 1, dim}];

    (* Print nonzero values *)
    If[R[[ρ, σ, μ, ν]] == 0, (* Do nothing *), Print["R"^xρ_xσ,xμ,xν, " = ", R[[ρ, σ, μ, ν]]],
     Print["R"^xρ_xσ,xμ,xν, " = ", R[[ρ, σ, μ, ν]]]],

    {ν, 1, dim}],
   {μ, 1, dim}],
  {σ, 1, dim}],
 {ρ, 1, dim}]
```

$R^{v}{}_{x,u,x} = a[u]\, a''[u]$

$R^{v}{}_{x,x,u} = -a[u]\, a''[u]$

$R^{v}{}_{y,u,y} = b[u]\, b''[u]$

$R^{v}{}_{y,y,u} = -b[u]\, b''[u]$

$$R^{x}{}_{u,u,x} = \frac{a''[u]}{a[u]}$$

$$R^{x}{}_{u,x,u} = -\frac{a''[u]}{a[u]}$$

$$R^{y}{}_{u,u,y} = \frac{b''[u]}{b[u]}$$

$$R^{y}{}_{u,y,u} = -\frac{b''[u]}{b[u]}$$

In[12]:=
```
(* Initialize Ricci tensor Ricci_μν as rank 2 tensor *)
tmp[a_, b_] := 0;
Ricci = Array[tmp, {dim, dim}];

(* Loop over indices in Ricci_μν *)
Do[
 Do[
  (* Calculate Ricci_μν *)
  Ricci[[μ, ν]] = Sum[R[[ρ, μ, ρ, ν]], {ρ, 1, dim}];

  (* Print nonzero values *)
  xμ = coordinateList[[μ]];
  xν = coordinateList[[ν]];
  If[Ricci[[μ, ν]] == 0, (* Do nothing *),
   Print["Ricci"_{xμ, xν}, " = ", Ricci[[μ, ν]]], Print["Ricci"_{xμ, xν}, " = ", Ricci[[μ, ν]]]],

  {ν, 1, dim}],
 {μ, 1, dim}]
```

$$\mathrm{Ricci}_{u,u} = -\frac{a''[u]}{a[u]} - \frac{b''[u]}{b[u]}$$

b)

Determine the Killing vectors and show that the infinitesimal transformations corresponding to them indeed leave the metric invariant. What is the physical interpretation of these Killing vectors?

Solution

In appendix.

Home Problem 15

Einstein originally introduced the cosmological constant into his equations so that they would allow for time-independent solutions corresponding to a static universe. Consider the Friedmann equations in the form

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p), \tag{4.40}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} + \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{4.41}$$

where $k = -1, 0, 1$ is the parameter describing the properties of the maximally symmetric universe. Assume that ρ and $p = \rho\omega$ describe ordinary "matter", i.e., $\rho > 0$ and $\omega \geq 0$.

a)

Give the condition for these equations to have static solutions. What does it imply for the geometry of the universe?

Solution

If $\dot{a} = \ddot{a} = 0$, then (4.40) implies Λ is positive, and (4.41) implies k is positive. This is the geometry of $S^3 \times \mathbb{R}$.

b)

For a dust-dominated universe with $p = 0$, in view of **a)**, what is the relation between a and ρ?

Solution

Subtracting (4.40) from (4.41) yields

$$0 = 4\pi G\rho - \frac{k}{a^2}$$

$$a^2 = \frac{k}{4\pi G\rho}$$

(a positive).

c)

Are the solutions in **a)** stable?

Solution

(4.40) is a linear ODE in a, define c by

$$\ddot{a} = \left[\frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p)\right] a = ca.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} a \\ \dot{a} \end{bmatrix} = \begin{bmatrix} \dot{a} \\ ca \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix}\begin{bmatrix} a \\ \dot{a} \end{bmatrix}.$$

So the solutions are stable if the matrix

$$\begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix} \tag{1}$$

only has eigenvalues with negative real parts. But (1) has eigenvalues $\pm\sqrt{c}$, at least one of which must have nonnegative real part. The solution is thus not stable.