MMA320 Introduction to Algebraic Geometry

Exercises for Chapter 2

2.1. What points in $\mathbb{P}^2$ do not belong to two of the three sets $\mathbb{A}^2_0$, $\mathbb{A}^2_1$, $\mathbb{A}^2_2$?

2.2. a) Describe the curve $C_1\colon 2X+Y^2=1$ in the other two standard coordinate charts on $\mathbb{P}^2(\mathbb{C})$. Hint: first homogenise the equation with the coordinate Z.
b) Let C_2 be defined by the equation $Y=X^3$ in the affine chart $Z=1$. What does C_2 look like at infinity? Give its equation and draw its real part.
c) Find all the points of $\mathbb{P}^2$ which lie on both curves C_1 and C_2.

2.3. a) Let $C_1\colon y=x^2+1$ and $C_2\colon y=0$. What is $C_1\cap C_2$ in $\mathbb{A}^2(\mathbb{R})$ respectively $\mathbb{A}^2(\mathbb{C})$? Does anything change if we make the equations homogeneous and think of the curves as lying in $\mathbb{P}^2$. Explain this in terms of 'asymptotic directions'.
b) Let C_k be the circle $x^2+y^2=k^2$ in $\mathbb{A}^2(\mathbb{R})$. Show that $C_1\cap C_2=\emptyset$. What happens if we replace $\mathbb{R}$ with $\mathbb{C}$? What about $\mathbb{P}^2(\mathbb{C})$?

2.4. Two conics in $\mathbb{P}^2(\mathbb{C})$ have four intersection points (counted with multiplicity). Give an example of two conics in $\mathbb{P}^2(\mathbb{R})$ which only intersect in 2 points.
Find the extra intersection points when you use the same equations to define conics in $\mathbb{P}^2_{\mathbb{C}}$. [This may or may not be difficult depending on your choice of equation.] Why can you deduce that these intersection points are distinct (even without calculating them)?

2.5. Let $k=\mathbb{Z}/(2)$ be the field with two elements. How many points has $\mathbb{P}^2(k)$? How many lines pass through $P=(1:0:0)$? How many points lie on each of these lines? Draw all points and lines. Hint: choose $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$ as vertices in a triangle and $(1:1:1)$ as interior point.

2.6. **Duality**. Let $\{e_0,e_1,e_2\}$ be a basis of $V\cong\mathbb{R}^3$ and let $(X_0:X_1:X_2)$ be corresponding homogeneous coordinates on $\mathbb{P}(V)\cong\mathbb{P}^2(\mathbb{R})$. Let $\{e_0^*,e_1^*,e_2^*\}$ be the dual basis in V^*, and $(U_0:U_1:U_2)$ corresponding homogeneous coordinates on $\mathbb{P}(V^*)\cong\mathbb{P}^2(\mathbb{R})$. Let $P=(a_0:a_1:a_2)$ be a point in $\mathbb{P}(V)$. Describe the pencil of all lines in $\mathbb{P}(V)$ through P in the coordinates $(U_0:U_1:U_2)$.

2.7. Let l_1 and l_2 be two disjoint lines in $\mathbb{P}^3$, and let $P\in\mathbb{P}^3\setminus(l_1\cup l_2)$ be a point. Show that there is a unique line $l\subset\mathbb{P}^3$ through P, intersecting $l_1$and l_2 and P.

2.8. Let P_0, P_1, P_2 (resp. Q_0, Q_1, Q_2) be three points in $\mathbb{P}^2$ not lying on a line. Show that there is a projective change of coordinates $T\colon\mathbb{P}^2\to\mathbb{P}^2$ such that $T(P_i)=Q_i$, $i=0,1,2$. Extend this to n points in $\mathbb{P}^n$, not lying on a hyperplane.

2.9. Let l_0, l_1, l_2 (resp. m_0, m_1, m_2) be lines in $\mathbb{P}^2$ that do not all pass through one and the same point. Show that there is a projective change of coordinates $T\colon\mathbb{P}^2\to\mathbb{P}^2$ such that $T(l_i)=m_i$, $i=0,1,2$. (Hint: Let $P_i=L_j\cap L_k$, $Q_i=m_j\cap m_k$.) .

2.10. Let $f\in k[X_0,\dots,X_n]$. Define the (formal) derivative $\frac{\partial f}{\partial X_i}\in k[X_0,\dots,X_n]$, for any field k. (Hint: product rule).

2.11. Let $f \in k[X_0, \dots, X_n]$ be a homogeneous polynomial of degree m. Show Euler's formula

$$\sum_{i=0}^{n} X_i \frac{\partial f}{\partial X_i} = mf \; . \tag{*}$$

When does the converse hold: for which fields does $(*)$ imply that $f \neq 0$ is homogeneous of degree m?

2.12. Let $I \subset k[X_0, \dots, X_n]$ be a homogeneous ideal. Show that I is prime if and only if for every two homogeneous elements $f, g \in I$ we have that $fg \in I$ implies $f \in I$ or $g \in I$.

2.13. The sum, product, intersection and radical of homogeneous ideals are also homogeneous ideals.

2.14. Show that an ideal $I \subset k[X_1, \dots, X_n]$ is prime if and only if its homogenisation $\bar{I} \subset k[X_0, \dots, X_n]$ is prime.

2.15. Find I^{sat}, where $I = (X^2, XY) \subset k[X, Y]$.

2.16. Let $k = \mathbb{Z}/(2)$ be the field with two elements. Determine for $V(x^2 + yz) \subset \mathbb{A}^3(\mathbb{Z}/(2))$ the ideal $I(V(x^2 + yz)) \subset \mathbb{Z}/(2)[x, y, z]$. Now consider the same equation in the projective plane: $V(X^2 + YZ) \subset \mathbb{P}^2$ and find the homogeneous ideal $J(V(X^2 + YZ)) \subset \mathbb{Z}/(2)[X, Y, Z]$.

2.17. Let $C \subset \mathbb{P}^3$ be the rational normal curve of degree 3, given by the parametrization

$$\mathbb{P}^1 \to \mathbb{P}^3, \qquad (S : T) \mapsto (X : Y : Z : W) = (S^3 : S^2T : ST^2 : T^3) \; .$$

Let $P = (0 : 0 : 1 : 0) \in \mathbb{P}^3$, and let H be the hyperplane defined by $Z = 0$. Let π be the projection from P to H, i.e. the map associating to a point Q of C the intersection point of H with the unique line through P and Q.
a) Show that π is a morphism.
b) Determine the equation of the curve $\pi(C)$ in $H \cong \mathbb{P}^2$.
c) Is $\pi \colon C \to \pi(C)$ an isomorphism onto its image?

2.18. Show that the curve $C = V(X^3 - ZY^2) \subset \mathbb{P}^2$, defined over an algebraically closed field k, is birational to P^1. Consider the affine chart $U_0 = \{Z \neq 0\}$. Are the coordinate rings of the affine curve $C \cap U_0$ and $\mathbb{A}^1$ isomorphic?
Does there exist an affine chart U_1 such that $C \cap U_1$ has coordinate ring, isomorphic to $k[t]$?

2.19. Show that there is only one conic passing through the five points $(0 : 0 : 1)$, $(0 : 1 : 0)$, $(1 : 0 : 0)$, $(1 : 1 : 1)$, and $(1 : 2 : 3)$; show that it is nonsingular, if $\operatorname{char} k \neq 2, 3$.

2.20. Consider the nine points $(0 : 0 : 1)$, $(0 : 1 : 1)$, $(1 : 0 : 1)$, $(1 : 1 : 1)$, $(0 : 2 : 1)$, $(2 : 0 : 1)$, $(1 : 2 : 1)$, $(2 : 1 : 1)$, and $(2 : 2 : 1)$ in $\mathbb{P}^2$; it might help to make a picture. Show that there are infinitely many cubics passing through these points (if the field k is infinite).

2.21. Let L be the vector space of homogeneous polynomials of degree 2 in $k[X:Y:Z]$ and let $\mathbb{P}(L)$ be the linear system of conics.
a) Let P_1, P_2, P_3 and P_4 be four points in $\mathbb{P}^2$ and let $\mathbb{P}(L(P_1,P_2,P_3,P_4))$ be the linear system of conics passing through these points.
Show that $\dim \mathbb{P}(L(P_1,P_2,P_3,P_4)) = 2$ if the four points lie on a line, and that $\dim \mathbb{P}(L(P_1,P_2,P_3,P_4)) = 1$ otherwise.
b) Show that the space of irreducible conics in $\mathbb{P}^2$ is an open subset $U \subset \mathbb{P}(L)$. What geometric objects can be associated to the points in $\mathbb{P}(L) \setminus U$?

2.22. Show that the plane curves $C_1 = V(ZY^2 - X^3 + XZ^2)$ and $C_2 = V(X^3Z - Y^2Z^2 + X^2Y^2)$ are birational (hint: standard Cremona transformation). Describe occurring singularities.

2.23. Let $H_d \subset \mathbb{P}^n$ be a hypersurface of degree d. Show that the complement $\mathbb{P}^n \setminus H_d$ is an affine variety.

2.24. Let $f(X_0,\dots,X_n) = X_0 g(X_1,\dots,X_n) + h(X_1,\dots,X_n)$, where g is a homogeneous polynomial of degree $d-1$ and h is a homogeneous polynomial of degree d. Assuming that f is irreducible, prove that the variety $V(f)$ is rational.

2.25. Show that every isomorphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a projective transformation.

2.26. Is the union (resp. the intersection) of quasi-projective algebraic sets a quasiprojective algebraic set?

2.27. Let V be a projective variety over an algebraically closed field k. Show that $\mathcal{O}(V) = k$, that is, every rational function, regular on the whole of V, is constant. Hints: show that on each standard open affine set V_i an $r \in \mathcal{O}(V)$ has the form $r = f_i/X_i^{N_i}$ with $f_i \in S_{N_i}(V)$ homogeneous of degree N_i. Show that for N sufficiently large multiplication with r is an endomorphism of the vector space $S_N(V)$. Use the Theorem of Cayley–Hamilton to conclude that r is a root of a polynomial with coefficients in k; alternatively, consider an eigen vector and show that r equals the eigen value.

2.28. Show that every regular map from a projective variety to an affine variety maps to a point.

2.29. Let $C = V(f) \subset \mathbb{A}^2(\mathbb{C})$ and consider the blow up $\sigma\colon \mathrm{Bl}_0\mathbb{A}^2 \to \mathbb{A}^2$. Put $\tilde{f} = f \circ \sigma$. If $0 \in V(f)$, then the exceptional curve E is an irreducible component of $V(\tilde{f})$, and the closure of $V(\tilde{f}) \setminus E$ is the strict transform $\bar{C}$ of C. Compute an equation for $\bar{C}$. If $\bar{C}$ is singular, blow-up again until the strict transform of C is smooth. Resolve in this way the following curves:
A_n) $x^2 - y^{n+1}$, $n \geq 1$
D_n) $yx^2 - y^{n-1}$, $n \geq 4$,
E_6) $x^3 - y^4$,
E_7) $x^3 - xy^3$,
E_8) $x^3 - y^5$