Solved exercises in Miles Reid's Undergraduate Commutative Algebra

Simon Stefanus Jacobsson

Friday 7th August, 2020

0.1

Let A be a ring and consider the polynomial ring $A[T]$. Prove that T is not a zero-divisor in $A[T]$. Generalize the argument to prove that a monic polynomial

$$f = T^n + a_{n-1}T^{n-1} + \ldots + a_0$$

is not a zero-divisor in $A[T]$.

Solution

Let $g = a_nT^n + \ldots + a_1T + a_0 \in A[T]$ and let $gT = 0$. Then

$$a_nT^{n+1} + \ldots + a_1T^2 + a_0T = 0$$

implies $a_n, \ldots, a_1, a_0 = 0$ by uniqueness of polynomial coefficients.

Similarly, if $g(T^n + b_{n-1}T^{n-1} + \ldots + b_1T + b_0)$, the T^{2n}-term gives that $a_n = 0$, the T^{2n-1}-term gives that $a_{n-1} = 0$, and so on. Hence $g = 0$ and any monic polynomial will not be a zero-divisor.

0.2

Let A be a ring, $a \in A$, and $f \in A[T]$. Prove that there exists an expression $f = (T - a)q + r$ with $q \in A[T]$ and $r \in A$. [Hint: subtract off a suitable multiple of $(T - a)$ to cancel the leading term, then use induction on $\deg f$.] By substituting $T = a$, show that $r = f(a)$. (this result is often called the *remainder theorem* in algebra textbooks.)

Solution

For induction, assume it holds for $\deg f = p$. Then, for $f = c_{p+1}T^{p+1} + \ldots + c_0$, we have $\deg\left(f - c_{p+1}(T - a)\right) = p$. Hence

$$\begin{aligned} f - c_{p+1}(T - a) &= (T - a)q' + r \quad \text{for some } q' \in A[T] \text{ and } r \in A \\ f &= (T - a)(q' + c_{p+1}) + r \end{aligned}$$

and hence it is true for $\deg f = p + 1$ (with $q = q' + c_{p+1}$).

For $\deg f = 0$, it is obviously true with $q = 0$ and $r = f$.

0.4

Let $A[T]$ be the polynomial ring over a ring A, and let B be a ring. Suppose that $\varphi\colon A \to B$ is a given ring homomorphism; show that ring homomorphisms $\psi\colon A[T] \to B$ extending φ are in one-to-one correspondence with elements in B.

Solution

This is clear since in

$$\begin{aligned}\psi\left(f(T)\right) &= \psi\left(c_n T^n + \ldots + c_1 T + c_0\right)\\ &= c_n \psi(T)^n + \ldots + c_1 \psi(T) + c_0,\end{aligned}$$

if we know $\psi(T)$, we know the whole expression, and $\psi(T)$ can be any element in B.

0.7

TODO

1

1.1

Give an example of a ring A and ideals I, J such that $I \cup J$ is not an ideal; in your example, what is the smallest ideal containing I and J?

Solution

Consider (6) and (10) in $\mathbb{Z}$. $10 \in (10)$ and $6 \in (6)$, but $10 + 6 = 16 \notin (10) \cup (6)$, so $(10) \cup (6)$ is not even a subring of $\mathbb{Z}$. The smallest ideal containing $(10) \cup (6)$ is $(\gcd(6, 10)) = (2)$.

1.2

The *product* of two ideals I and J is the set of all sums $\sum_i f_i g_i$ with $f_i \in I$ and $g_i \in J$. Give an example in which $IJ \neq I \cap I$.

Solution

$I = J = (2)$ gives $IJ = (4) \neq (2) = I \cap J$.

1.3

Let $A = k[X, Y]/(XY)$. Show that any element of A has a unique representation in the form

$$a + f(X)X + g(Y)Y \quad \text{with} \quad a \in k,\ f \in k[X], \text{ and } g \in k[Y].$$

How do you multiply two such elements?

Prove that A has exactly two minimal prime ideals. If possible, find ideals I, J, and K to contradict each of the following statements:

1. $IJ = I \cap J$
2. $(I + J)(I \cap J) = IJ$
3. $I \cap (J + K) = (I \cap J) + (I \cap K)$.

Solution

TODO

1.4

Two ideals I and J are *strongly coprime* if $I + J = A$. Check that this is the usual notion for coprime $A = \mathbb{Z}$ or $k[X]$. Prove that if I and J are strongly coprime, then

$$IJ = I \cap J \quad \text{and} \quad A/IJ \sim (A/I) \times (A/J).$$

Prove also that if I and J are strongly coprime then so are I^n and J^n for $n \geq 1$.

Solution

Since $\mathbb{Z}$ is a PID, let $I = (a)$ and $J = (b)$. Then

$$I + J = (a) + (b) = (a, b) = (\gcd(a, b)) = \begin{cases} (1) = A & \text{if } a \text{ and } b \text{ coprime in the usual sense} \\ \text{not } (1) & \text{if } a \text{ and } b \text{ not coprime in the usual sense} \end{cases}.$$

Similarly for $k[X]$.

In general, $IJ \subset I \cap J$ since $IJ \subset IA = I$ and $IJ \subset AJ = J$. TODO

1.5

Let $\varphi\colon A \to B$ be a ring homomorphism. Prove that φ^{-1} takes prime ideals of B to prime ideals of A. In particular, if $A \subset B$, and P is a prime ideal of B then $A \cap P$ is a prime ideal of A.

Solution

If P is a prime ideal in B, then $\varphi^{-1}(P)$ is an ideal since if $\varphi(a) \in P$, then $\varphi(ab) = \varphi(a)\varphi(b) \in P$ by P being an ideal. Furthermore, $\varphi^{-1}(P)$ is prime since if $\varphi(a)$, $\varphi(b) \in P^{\complement}$, then $\varphi(ab) = \varphi(a)\varphi(b) \in P^{\complement}$ by P being prime.

1.6

Prove or give a counterexample to

1. the intersection of two prime ideals is prime
2. the ideal $P_1 + P_2$ generated by two prime ideals P_1 and P_2 is prime
3. if $\varphi\colon A \to B$ is a ring homomorphism then φ^{-1} takes maximal ideals of B to maximal ideals of A
4. the map φ^{-1} of Proposition 1.2 (quotient homomorphism) takes maximal ideals of A/I to maximal ideals of A.

Solution

Item 1 is false since (2) and (3) in $\mathbb{Z}_6$ is a counterexample. $(2) \cap (3) = (0)$ is not prime in $\mathbb{Z}_6$ since $[2][3] = [0]$.

Item 2 is false since (2) and $(x^2 + 3)$ in $\mathbb{Z}[X]$ is a counterexample. Both 2 and $x^2 + 3$ are irreducible in $\mathbb{Z}[X]$, but $(2) + (x^2 + 3) = (2, x^2 + 3)$ is not prime since $x^2 - 1 \in (2, x^2 + 3)$ but $x^2 - 1 = (x + 1)(x - 1)$ while neither of those factors are in $(2, x^2 + 3)$.

Items 3 and 4 are also false by the same counterexample. Let $A = \mathbb{Z}[X, Y]$, let $I = (X - 2, Y - 3)$, and let φ be the quotient homomorphism. We can identify A/I with $\mathbb{Z}$ and say that $\varphi\colon X \mapsto 2$ and $Y \mapsto 3$. Then (2) is maximal in $\mathbb{Z}$ but $\varphi^{-1}((2)) = (2, X)$ is not maximal in $\mathbb{Z}[X, Y]$.