MATHEMATICS University of Gothenburg
Examination in commutative algebra MMA 330, 2016-10-24
No books, written notes or any other aids are allowed.
Telephone : 031-772 35 80

1. Let A be an integral domain and p be an element which is prime in A. Let $S \subseteq A$ be a multiplicative subset such that p does not divide any element in S. Show that $p/1$ is prime in $S^{-1}A$. 4p

2. Let S be a multiplicative subset in a commutative ring A and M be a finitely generated A-module. Show that $S^{-1}M = 0$ if and only if some $s \in S$ annihilates M, that is if $sM = 0$. 4p

3. Let K be a finite field extension of $\mathbf{Q}$ and O_K be the integral closure of $\mathbf{Z}$ in K. 5p
a) Show that no integer $n \geq 2$ is invertible in O_K.
b) Let p be a prime. Show that there exists a prime ideal P in O_K with $P \cap \mathbf{Z} = (p)$.

4. Let I, J be ideals in a commutative ring A and P be a prime ideal in A. Suppose that $I \cap J \subseteq P$. Show that $I \subseteq P$ or $J \subseteq P$. 4p

5. Let t be a non-unit in a Noetherian integral domain A. Show that $\bigcap_{n=1}^{\infty} (t^n) = \{0\}$. 4p

6. Let A be a Noetherian local integral domain with principal maximal ideal and $K = \mathrm{Frac}(A)$ be its field of fractions. Show that there exists a discrete valuation $v: K^* \to \mathbf{Z}$ with $A = \{f \in K^* : v(f) \geq 0\} \cup \{0\}$. 4p

You may use any theorem in Reid's book to solve the first 4 exercises, but all claims should be motivated.

MATHEMATICS University of Gothenburg
Examination in commutative algebra MMA 330, 2017-01-03
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1 Prove or give a counterexample:
a) the intersection of two prime ideals is prime; 2p
b) the ideal $P + Q$ generated by 2 prime ideals P and Q is again prime. 2p

2. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of A-modules over a commutative ring A. Suppose thar L and N are finitely generated A-modules. Show that M is also finitely generated. 4p

3a). Show that the ring $\mathbf{Z}[\sqrt{-3}]$ is not integrally closed in its field of fractions. 3p
3b) Determine its integral closure. 2p

4a) Is there a discrete valuation v of $\mathbf{Q}(i)$ with $v(2)=1$? 2p
4b) Is there a discrete valuation v of $\mathbf{Q}(i)$ with $v(3)=1$? 3p

5. Prove that any ideal $I \neq A$ of a commutative ring is contained in a maximal ideal. 3p

6. Formulate and prove Hilbert's basis theorem. 4p

All claims that are made should be motivated.

MATHEMATICS University of Gothenburg
Examination in commutative algebra MMA 330, 2018-10-29
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1 Find a monic polynomial in $\mathbf{Z}[X]$ which vanishes at $\sqrt{2}+i$. 3p

2 Let A be a ring and S be the set of all $s\in A\setminus\{0\}$, which are not zero divisors.

a) Show that S is a multiplicative set. 2p

b) Show that $\frac{a}{s}=\frac{b}{t}$ in $S^{-1}A$ if and only if $at=bs$ in A. 1p

c) Show that every $\frac{a}{s}\neq\frac{0}{1}$ in $S^{-1}A$ is either a unit or a zero divisor. 2p

3. Let k be a field and $I\subset k[X_1,\ldots,X_n,\ldots]$ be the ideal generated by the monomials X_mX_n $m.n\in\mathbf{N}$. Let $A=k[X_1,\ldots,X_n,\ldots]/I$ and $x_n=X_n+I$.

a) Prove that $(x_1,\ldots,x_n,\ldots)$ is the only prime ideal in A. 3p

(Suggestion: Determine the intersection of all prime ideals.)

b) Determine if A is Artinian or not. 1p

c) Determine if A is Noetherian or not. 1p

4a) Let I be an ideal in a Noetherian ring A. Prove that there exists $x\in 1+I$ such that $xm=0$ for every $m\in\bigcap_{n=1}^{\infty}I^n$. (Hint: This is not hard if you just apply a suitable result in Reids's book.) 3p

4b) Show that $\bigcap_{n=1}^{\infty}I^n=\{0\}$ for any ideal $I\neq A$ in a local Noetherian ring A. 1p

5) Let I be a non-principal ideal of $\mathbf{Z}[X]$. Prove that $I\cap\mathbf{Z}\neq\{0\}$. 4p

6) Let A be a Noetherian integral domain and $t\in A$ a non-unit. Show that $\bigcap_{n=1}^{\infty}(t^n)=0$. 4p

All rings are commutative with a unit element 1. *You may use the results in Reid's book for the first* four *exercises. But all claims should be motivated.*

MATHEMATICS University of Gothenburg
Examination in commutative algebra MMA 330, 2019-01-07
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All rings are supposed to be commutative with 1.

1) Let I and J be two ideals in a ring.

a) Give an example of a ring and two ideals where $IJ \neq I \cap J$. 1p

b) Show that $IJ = I \cap J$ if I and J are coprime (i.e. if $I+J=A$). 3p

2) Let A be a ring and S be a multiplicative set in A. Let $\phi: A \to S^{-1}A$ 4p
be the map which sends $a \in A$ to $\frac{a}{1} \in S^{-1}A$. Show that ϕ is injective if
and only if S does not contain zero or a zero divisor in A.

3) Let A be a ring and M be a Noetherian A-module. Let $I=\mathrm{Ann}(M)$ 4p
be the ideal of A given by $I=\{a \in A: am=0, \forall m \in M\}$. Prove that A/I is
a Noetherian ring.

4) Let A be a ring with $\mathbf{C}[X] \subseteq A \subseteq \mathbf{C}(X)$ and $E \subseteq \mathbf{C}$ be the set of all 5p
$\alpha \in \mathbf{C}$ such that some $f \in A$ has a pole at α (i.e. $|f(X)| \to \infty$ when $X \to \alpha$).

a) Let $\alpha \in \mathbf{C}$. Show that $1/(X-\alpha) \in A$ if and only if $\alpha \in E$.

b) Show that A as ring is generated by $\mathbf{C}[X]$ and all $1/(X-\alpha)$ with $\alpha \in E$.

(Hint: The problem is similar to an exercise in Reid's book on subrings of $\mathbf{Q}$.)

5) Let $A \subset B$ be two rings and let $b \in B$. Show that b is integral over A 4p
if and only if $A[b]$ is finitely generated as A-module.

6) Formulate and prove Hilbert's basis theorem. 4p

All claims must be motivated.