❦ Komplexanalys i flera variabler ❦
Assignment 4

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Determine whether $U = \mathbb{C}^2 \setminus \{ 0 \}$ is holomorphically convex (by using the definition of holomorphic convexity, not that this is equivalent to other properties).

Solution

Lemma 1 (Exercise 1.2.19 in Lebl)**.** *Zeros (and so poles) are never isolated in $\mathbb{C}^n$ for $n \geq 2$.*

Proof. (for $n = 2$, which is what we will this it for) We want to use Hurwitz Theorem (Theorem B.21 in Lebl) which states that if a sequence $f_n : V \to \mathbb{C}$ of holomorphic functions on a domain $V \subset \mathbb{C}$ converges uniformly to $f : V \to \mathbb{C}$, and if f is not identically zero and p is a zero of f, then there exists a disc $\Delta_r(p)$ and an N, such that for all $n \geq N$, f_n has the same number of zeros in $\Delta_r(p)$ as f.

So let $f : U \to \mathbb{C}$ function holo on $\mathbb{C}^2$ and let $p \in U$ be a zero of f. Either f is constantly zero and p is not isolated, or f is not identically zero and we have to argue using Hurwitz theorem. Since U is open, there exists some $\overline{\Delta}_R(p) = \overline{B}_R(p_1) \times \overline{B}_R(p_2) \subset U$. Take the sequence $f_n(z) = f(p_1 + \frac{1}{n}, z)$, $n > \frac{1}{R}$. Then we have, as in Hurwitz theorem, a sequence $f_n : B_R(p_2) \to \mathbb{C}$ which converges uniformly (f holo implies its derivatives are holo, and since $\overline{\Delta}_R(p)$ is compact f:s derivatives will be bounded on $\overline{\Delta}_R(p)$) to $z \mapsto f(p_1, z)$. Hence there exists some $\Delta_r(p)$ and N such that for all $n \geq N$, $f_n(z) = f(p_1 + \frac{1}{n}, z)$ has the same number of zeros in $\Delta_r(p)$ as $z \mapsto f(p_1, z)$. Thus the zero p is not isolated (since we may choose R arbitrarily small).

If f is meromorphic, then $f = \frac{g}{h}$ with g and h holo in some neighbourhood U' of p. By the definition of *pole* in Lebl p. 23, f has a pole at p if $h(p) = 0$. But we just proved that $h : U' \to \mathbb{C}$ cannot have an isolated zero. Hence poles are also not isolated. $\square$

To show that U is holomorphically convex, we'd need to show that if $K \subset\subset U$, then $\widehat{K}_U \subset\subset U$, where

$$\widehat{K} \coloneqq \left\{ z \in U \quad \text{s. th.} \quad |f(z)| \leq \sup_{w \in K} |f(w)| \text{ for all } f \in O(U) \right\}.$$

So take for example the unit sphere $K = \partial\mathbb{B}_2$. K is then relatively compact in U. Consider then $\widehat{K}$. I want to argue that $\widehat{K} = \overline{\mathbb{B}}_n \setminus \{ 0 \}$. For that, I'd need to argue firstly that if $z \in \overline{\mathbb{B}}_2 \{ 0 \}$, then

$$|f(z)| \leq \sup_{w \in K} |f(w)| \quad \text{for all} \quad f, \tag{1}$$

and secondly that if $z \notin \overline{\mathbb{B}}_2 \setminus \{ 0 \}$, then

$$|f(z)| > \sup_{w \in K} |f(w)| \quad \text{for some} \quad f. \tag{2}$$

For the first part, take some $f \in O(U)$. Then f may be extended to be meromorphic on U. Then, by lemma 1, f can be extended to to be holo on $\mathbb{C}^2$ and especially on $\overline{\mathbb{B}}_n$. Hence (1) follows by the maximum principle.

For the second part, if $z \notin \overline{\mathbb{B}}_2 \setminus \{ 0 \}$, then $|z| > 1$ and hence choosing $f(z) = z$ will satisfy (2).

Now that we have argued that $\widehat{K} = \overline{\mathbb{B}}_n \setminus \{ 0 \}$, it is straightforward to see that $\widehat{K} \not\subset\subset U$ by considering the sequence $n \mapsto \frac{1}{n}$. Hence U is not holomorphically convex.

2

For each $k \in \mathbb{N}_0$, let $\ell_m^k \in \mathbb{N}_0$ be the smallest non-negative integer such that $\ell_m^k \geq k\alpha_m$. Prove that the domain of convergence of the power series

$$\sum_{k=0}^{\infty} \mathrm{e}^{-k\beta} z_1^{\ell_1^k} \cdots z_n^{\ell_n^k} \tag{3}$$

is precisely the set

$$\{ z \in \mathbb{C}^n \quad \text{s.th} \quad |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} < \mathrm{e}^{\beta} \}. \tag{4}$$

Hint: That it diverges outside is easy, what is hard is that it converges inside. Perhaps useful is to notice $\frac{\ell_m^k}{k} - \alpha_m \leq \frac{1}{k}$, and furthermore notice that if z is in the set, there is some $\epsilon > 0$ such that $(1+\epsilon)|z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} = \mathrm{e}^{\beta}$.

Solution

$$k\alpha_m + 1 \geq \ell_m^k \geq k\alpha_m.$$

Fix z and let I be the set of indices i with $|z_i| \geq 1$ and J be the set of indices j with $|z_j| > 1$. Then $|z_i|^{\ell_m^k} \leq |z_i|^{(k+1)\alpha_m}$ and $|z_j|^{\ell_m^k} < |z_j|^{k\alpha_m}$, and so $|z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} = A < \mathrm{e}^{\beta}$ implies

$$\begin{aligned}
\left| \sum_{k=0}^{\infty} \mathrm{e}^{-k\beta} z_1^{\ell_1^k} \cdots z_n^{\ell_n^k} \right| &\leq \sum_{k=0}^{\infty} \mathrm{e}^{-k\beta} |z_1|^{\ell_1^k} \cdots |z_n|^{\ell_n^k} \\
&\leq \prod_i |z_i| \sum_{k=0}^{\infty} \mathrm{e}^{-k\beta} |z_1|^{k\alpha_1} \cdots |z_n|^{k\alpha_n} \\
&\leq \prod_i |z_i| \sum_{k=0}^{\infty} \left(\frac{A}{\mathrm{e}^{\beta}} \right)^k
\end{aligned}$$

which converges since $\frac{A}{\mathrm{e}^{\beta}} < 1$.

If $|z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} = \mathrm{e}^{\beta}$, then the norm of the terms in (3) will tend to 1, and so cannot possibly converge. The case $|z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} = \mathrm{e}^{\beta}$ will likewise diverge. We have thus shown that (4) is the domain of convergence.

3

Let $U \subseteq \mathbb{C}^n$ be a domain and assume $f_1, f_2 \in O(U)$ have no common zeros.

a)

Show that there are smooth functions g_1, g_2 on U such that $f_1 g_1 + f_2 g_2 = 1$.
Hint: There are smooth functions $\chi_i \colon U \to [0, 1]$ such that χ_i is identically 0 in a neighborhood of $\{ f_i = 0 \}$ for $i = 1, 2$, and such that $\chi_1 + \chi_2 = 1$, and you may take this for granted.

Solution

Let Ω_i be such neighbourhoods of $f_i^{-1}(\{0\})$. These neighbourhoods are necessarily disjoint since $\chi_1 + \chi_2 = 1$ implies χ_1 and χ_2 cannot be 0 at the same time. Then

$$g_1(z) = \begin{cases} \frac{\chi_1(z)}{f_1(z)}, & f_1(z) \neq 0 \\ 0, & f_1(z) = 0 \end{cases}$$

is smooth on U since χ_1 and $\frac{1}{f_1}$ are smooth on $U \setminus f_1^{-1}(\{0\})$ and χ_1 is 0 on a neighbourhood of $f_1^{-1}(\{0\})$. The same argument holds for

$$g_2(z) = \begin{cases} \frac{\chi_2(z)}{f_2(z)}, & f_2(z) \neq 0 \\ 0, & f_2(z) = 0 \end{cases}.$$

Noting that $f_i g_i = \chi_i$ holds on the whole of U, it is clear that

$$f_1 g_1 + f_2 g_2 = 1.$$

b)

Assume that for any smooth $(0,1)$-form α on U such that $\overline{\partial}\alpha = 0$, one can find a smooth function β on U such that $\overline{\partial}\beta = \alpha$. Show that one may choose g_1, g_2 in section a) to be holomorphic. Hint: Let $\widetilde{g}_1, \widetilde{g}_2$ be smooth solutions from section a). One can then take $g_1 = \widetilde{g}_1 - f_2\gamma, g_2 = \widetilde{g}_2 + f_1\gamma$ for an appropriate choice of smooth function γ.

Solution

Remark 1. *Since f_i are holo, $\overline{\partial} f_i \alpha = f_i \overline{\partial}\alpha$.*

Remark 2. $\overline{\partial} \circ \overline{\partial} = 0$.

We notice first that if $g_1 = \widetilde{g}_1 - f_2\gamma$ and $g_2 = \widetilde{g}_2 + f_1\gamma$, then

$$f_1 g_1 + f_2 g_2 = f_1\widetilde{g}_1 + f_2\widetilde{g}_2 + (-f_1 f_2 \gamma + f_2 f_1 \gamma) = 1.$$

Let

$$\alpha = \begin{cases} \dfrac{\overline{\partial}\widetilde{g}_1}{f_2}, & z \in \Omega_2^{\complement} \\ -\dfrac{\overline{\partial}\widetilde{g}_2}{f_1}, & z \in \Omega_1^{\complement} \end{cases}.$$

It is clear that α is well-defined on $\Omega_1^{\complement} \cap \Omega_2^{\complement}$ since if both f_1 and f_2 are non-zero,

$$\frac{\overline{\partial}\widetilde{g}_1}{f_2} = \overline{\partial}\frac{\chi_1}{f_1 f_2} = \overline{\partial}\frac{(1-\chi_2)}{f_1 f_2} = -\frac{\overline{\partial}\widetilde{g}_2}{f_1}.$$

α is also clearly smooth. By remark 2, it is also clear that $\overline{\partial}\alpha = 0$ on U. So, by assumption, there exists some smooth γ such that $\overline{\partial}\gamma = \alpha$. Putting this gamma into out definitions of g_1 and g_2, it is clear that

$$\overline{\partial} g_1 = \overline{\partial}(\widetilde{g}_1 - f_2\gamma) = \overline{\partial}\widetilde{g}_1 - f_2\alpha = 0 \quad \text{on } \Omega_2^{\complement} \tag{5}$$

$$\overline{\partial} g_2 = \overline{\partial}(\widetilde{g}_2 + f_1\gamma) = \overline{\partial}\widetilde{g}_2 + f_1\alpha = 0 \quad \text{on } \Omega_1^{\complement} \quad . \tag{6}$$

But $\overline{\partial}\widetilde{g}_1 = \frac{\overline{\partial}\chi_1}{f_1} = 0$ on Ω_2 since $\chi_1 = 1$ there. Since also $\widetilde{g}_2 = 0$ on Ω_2, (5) is true on the whole of U. The same argument goes for (6). g_1 and g_2 are hence holomorphic on U.

4

Assume that g is a smooth $(0,1)$-form and that ψ is a smooth solution to

$$\bar{\partial}\psi = g. \tag{7}$$

Explain why one cannot expect that the support of ψ is contained in the support of g, e.g., by finding an example where the support of ψ is strictly larger than the support of g.

Solution

Consider the $(0,0)$-form $\psi\colon (z,\bar{z}) \mapsto 1$, which is a solution to (7) with $g = 0$. Then the support of g is clearly not contained in the support of ψ.

Another solution

5

a)

Prove that if $n \geq 2$, no domain of the form $U = \mathbb{C}^n \setminus K$ for a compact K is biholomorphic to a bounded domain.

Solution

Assume, for contradiction, that $f\colon U \to V$, $V \subset \mathbb{C}^n$, is such a biholomorhism. Consider the first coordinate, $f_1\colon U \to \mathbb{C}$, of f. By theorem 4.3.1 in Lebl, since f_1 is holo on $\mathbb{C}^n \setminus K$ for a compact set K, f_1 can be extended analytically to $\mathbb{C}^n$. Then, since f_1 is continuous on $\mathbb{C}^n$ and K is compact, $\sup_{z\in K} f_1(z) = M < \infty$. But, by assumption, $\sup_{z\in K^\complement} f_1(z) = m < \infty$. f_1 is thus bounded on $\mathbb{C}^n$ by $\max(M, m)$ and is thus constant. This contradicts our assumption about f being bijective.

b)

Prove that every domain of the form $U = \mathbb{C} \setminus K$ for a compact K with nonempty interior is biholomorphic to a bounded domain.

Solution

Let $p \in \operatorname{int} K$, then the distance r from p to U is nonzero since $\operatorname{int} K$ is open. Then the map $f\colon z \mapsto \frac{1}{z-p}$ is biholo on some superset (for example $\mathbb{C} \setminus \mathbb{B}_{r/2}(p)$) of U. The set $f(U)$ is hence bounded by $\frac{1}{r}$.