2.3 Spatial dynamics of a rigid body

2.3.1 Preliminaries: variation of rotational parameters vs. infinitesimal rotations

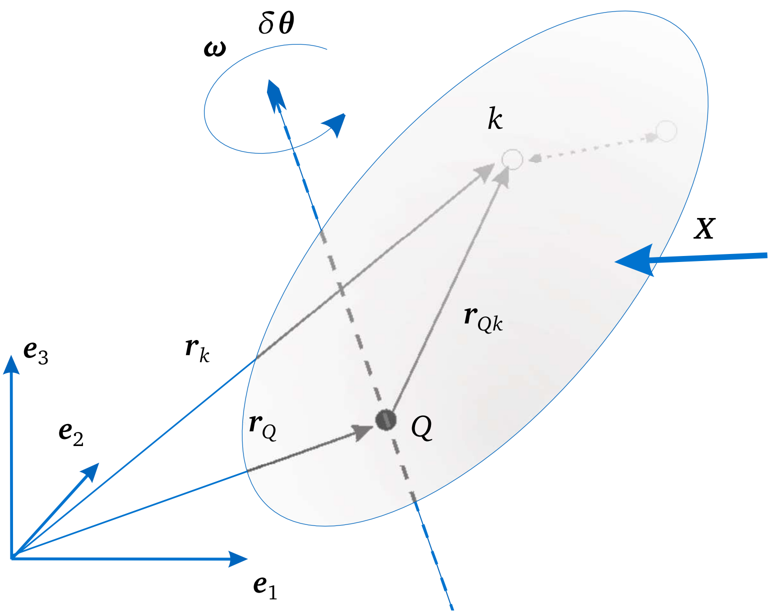

Figure 2.3: A rigid-body in space

When we derived the Newton-Euler equations for a two-dimensional rigid body from the principle of virtual work in section 2.1.1, we required the notion of "virtual rotation" (see $\delta\theta$ in equation (2.1.3)). In 2D, this virtual rotation can easily be obtained as the direct variation of a rotational degree of freedom (θ in that case).

In 3D however, we cannot use the concepts of "virtual rotation" and "variation of rotational degrees of freedom" interchangeably anymore. In other words, when writing "$\delta\boldsymbol{\theta}$" in 3D, we do not mean the variation of some "angle-vector" but rather a physical, infinitesimal rotation. In this section we wish to explain why this is the case and how "virtual rotations" are to be interpreted.

First, a note on rotational degrees of freedom, or *parametrizations* of rotations: The motion of any point P on a rigid-body can be written as ${}_I\boldsymbol{r}_P(t) = {}_I\boldsymbol{r}_Q(t) + {}_I\boldsymbol{A}_B(t)_K\boldsymbol{r}_{QP}$, where K is a body-fixed frame-of-reference, Q is a body-fixed reference point and ${}_I\boldsymbol{A}_B(t)$ is a rotation-matrix. We can then choose to *parametrize* the rotation ${}_I\boldsymbol{A}_B$ by decomposing it into a product of e.g. three elementary rotations, i.e. ${}_I\boldsymbol{A}_B(t) = \boldsymbol{A}_\gamma(t)\boldsymbol{A}_\beta(t)\boldsymbol{A}_\alpha(t)$, where $\boldsymbol{A}_\alpha$, etc. are simple rotations around coordinate axes by angles α, β and γ[6]. We define $\boldsymbol{\beta} := (\alpha\ \beta\ \gamma)^T$.

It can then be shown (see e.g. appendix A.4.2) that in general $\boldsymbol{\omega} \neq \dot{\boldsymbol{\beta}}$. Instead, $\boldsymbol{\omega} = \boldsymbol{J}(\boldsymbol{\beta})\dot{\boldsymbol{\beta}}$ (see e.g. (A.4.10) on page 183). **The angular velocity $\boldsymbol{\omega}$ is not equal to the time-derivative of the rotational degrees of freedom $\boldsymbol{\beta}$.** $\boldsymbol{J}(\boldsymbol{\beta})$ is a matrix that depends on the

[6] We are only using three angles here but note that the amount of rotational degrees of freedom is actually arbitrary and depends on the specific problem at hand.