respectively, the virtual work principle (1.2.10) then writes $\delta \boldsymbol{u}_1 \cdot \boldsymbol{F}_1 + \delta \boldsymbol{u}_2 \cdot \boldsymbol{F}_2 = 0$. Hence,

$$
\begin{aligned}
&\begin{bmatrix} \delta\theta_1 & \delta\theta_2 \end{bmatrix} \begin{bmatrix} -\ell_1 \sin\theta_1 & \ell_1\cos\theta_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} m_1\ddot{u}_{11} - mg \\ m_1\ddot{u}_{21} \end{bmatrix} \\
&+ \begin{bmatrix} \delta\theta_1 & \delta\theta_2 \end{bmatrix} \begin{bmatrix} -\ell_1\sin\theta_1 - \ell_2\sin(\theta_1+\theta_2) & \ell_1\cos\theta_1 + \ell_2\cos(\theta_1+\theta_2) \\ -\ell_2\sin(\theta_1+\theta_2) & \ell_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} m_2\ddot{u}_{12} - mg \\ m_2\ddot{u}_{22} \end{bmatrix} = 0
\end{aligned}
$$

This relation being satisfied for any variation $\delta\theta_1$ and $\delta\theta_2$, we obtain the equations of motion

$$
\begin{aligned}
&\begin{bmatrix} -\ell_1 \sin\theta_1 & \ell_1\cos\theta_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} m_1\ddot{u}_{11} - mg \\ m_1\ddot{u}_{21} \end{bmatrix} \\
&\quad + \begin{bmatrix} -\ell_1\sin\theta_1 - \ell_2\sin(\theta_1+\theta_2) & \ell_1\cos\theta_1 + \ell_2\cos(\theta_1+\theta_2) \\ -\ell_2\sin(\theta_1+\theta_2) & \ell_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} m_2\ddot{u}_{12} - mg \\ m_2\ddot{u}_{22} \end{bmatrix} = 0
\end{aligned}
$$

Expressing then the accelerations in terms of the degrees of freedom θ_1 and θ_2 using the kinematic relations above, one obtains the two equations of motion to determine the generalized displacements.

1.3 Lagrange equations

The virtual work principle discussed in the previous chapter defines a procedure to obtain the equations of motion where all the unknown forces due to kinematical constraints vanish. The basic idea of the virtual work principle is to project the equations of motion of all individual particles onto a direction compatible with the constraints. However, as seen in the examples of the double pendulum, applying the virtual work principle still requires lengthy mathematical manipulations.

Therefore, in this section we introduce a different, though equivalent, way to derive the equation of motions in terms of the generalized coordinates: the Lagrange equations. These equations involve the energy of the system which often facilitates the set up of the equations of motion. However, in the energy description of the system and when applying the Lagrange equations, the physical interpretation in the sense of Newton's law is often lost.

Figure 1.8: Photo taken in Turino (courtesy of D. LoConte 2009): Joseph-Louis Lagrange, born as Giuseppe Lodovico Lagrangia, in Turino, Italy, in 1736. Mathematician and astronomer who worked in France and Prussia. He died in Paris on 1813.