MANGALORE INSTITUTE OF TECHNOLOGY & ENGINEERING
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Affiliated to V.T.U., Belagavi, Approved by AICTE, New Delhi.

DEPARTMENT OF MATHEMATICS

MODULE -1: CALCULUS

CONTENTS:

Introduction to polar coordinates and curvature relating to Compute Science & Engineering

- Polar coordinates, Polar curves,
- angle between the radius vector and the tangent, angle between two curves.
- Pedal equations.
- Curvature and Radius of curvature - Cartesian, Parametric, Polar and Pedal forms.

Self-study: Center and circle of curvature, Evolutes and Involutes.

Applications: Communication signals, Manufacturing of microphones, and Image processing.

(RBT Levels: L1, L2 and L3)

LEARNING OBJECTIVES:

After Completion of this module, student will be able to:

- Understand the fundamentals of the differential calculus of functions of one variable.
- Transform the coordinates from Cartesian to Polar and vice versa.
- Apply concepts of calculus to find angle between polar curves
- Apply concepts of calculus to find Curvature and Radius of Curvature for curves defined in different forms

RECAPITULATION: CALCULUS

Calculus is a branch mathematics concerned with the calculation of instantaneous rates of change (Differential calculus) and the summation of infinitely many small factors to determine some whole (Integral Calculus).

Differential calculus is concerned with the study of rate of change of a continuous function. It is the study of the rate of change of dependent variable with respect to independent variable. The fundamental tool of differential calculus is derivative. Graphically, derivative represents the slope of the tangent at a point on the curve

RECAPITULATION: FUNCTIONS OF SINGLE VARIABLE:

The concept of functions is important in calculus because they play a key role in describing the real world problems in mathematical terms. The distance travelled by an object from an initial location along a straight line path depends on its speed. The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. In each case, the value of one variable quantity, which might be called as *y*, depends on the value of another variable quantity, which might be called *x*. Since the value of *y* is completely determined by the value of *x*, it's said that *y* is a function of *x*. Often the value of *y* is given by a *rule* or formula that says how to calculate it from the variable *x*. For instance, $A = \pi r^2$, the equation is a rule that calculates the area *A* of a circle from its radius *r*.

A symbolic way to say '*y* is a function of *x*' is by writing $y = f(x)$. In this notation, the symbol f represents the function. The variable *x* is independent variable, represents the input value of f, and *y*, the dependent variable, represents the corresponding output value of f at *x*.

REQUIREMENT OF NEW COORDINATE SYSTEMS:

Cartesian coordinate system for specifies a point in the XY – plane in two-dimensional geometry and XYZ- space in three- dimensional geometry. The requirement to define any new coordinate system is two-fold.

- One is based on geometry of the problem of practical situation wherein a more suitable coordinate system has to be chosen. For ex., the study of dispersion of a medicine injected in blood flow requires cylindrical coordinate system as the veins are cylindrical in nature. Use of Cartesian system may not be very suitable as it represents a rectangular channel and the corner effects have to be taken care.
- The second requirement is more of theoretical in nature. A mathematical expression which cannot be simplified in one coordinate system may be solved in simple way by transforming to other coordinate systems. For ex., $log(x+y)$ cannot be further simplified in Cartesian system whereas it's easier to solve in Polar coordinates.

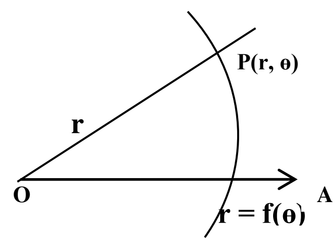
INTRODUCTION TO POLAR COORDINATES AND POLAR CURVES:

A new coordinate system is introduced to understand the concept of polar curves and their properties.

Any point P can be located on a plane with co-ordinates (r, θ) called **polar coordinates** of P where r = **radius vector** OP, (with pole/origin 'O'), θ = angle made by OP with the initial line OA.

The equation $r = f(\theta)$ or $\theta = f(r)$

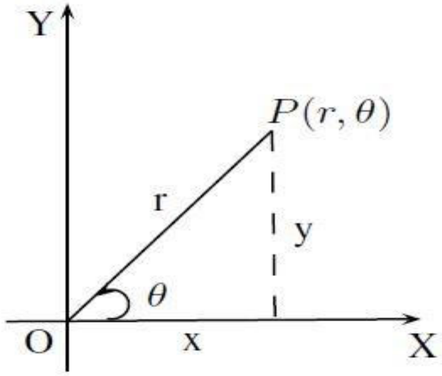
or $f(r, \theta) = c$ are known as a **polar curve**.



Polar curve

RELATION BETWEEN CARTESIAN AND POLAR COORDINATES:

Consider a point P in the xy-plane. Join the points O (origin) and P. Let r be the length of OP and θ be the angle which OP makes with the (positive) x-axis. The (r, θ) are called the polar coordinates of the point P, and we write $P = (r, \theta)$, or $P(r, \theta)$. In particular, r is called the radial distance and θ is called the polar angle. Also, O is called the pole, the x-axis is called the inital line and OP is called the radius vector.



Let (x, y) be the Cartesian coordinates of the point P. Then we find that

$$x = r\cos\theta\,, \qquad y = r\sin\theta \qquad --(1)$$

$$=> r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad --(2)$$

Relations (1) enables us to find the Cartesian coordinates when the polar coordinates are known. Relations (2) enables us to find the polar coordinates (r, θ) when the Cartesian coordinates (x, y) are known. Thus, relations (1) define the transformation from the polar coordinates to Cartesian coordinates and relations (2) defines the inverse transformation.

APPLICATIONS OF POLAR CURVES

- Polar coordinates are used often in navigation as the destination or direction of travel can be given as an angle and distance from the object being considered.
- Apart from the mechanical systems, polar coordinates are extended to three dimensions which helps in doing calculations on fields for example electric fields and magnetic fields and the temperature fields.
- The GPS information contained within a gpx file specify the locations of points along a route in polar coordinates, whose axes are latitude, longitude, and elevation.

Many more ….

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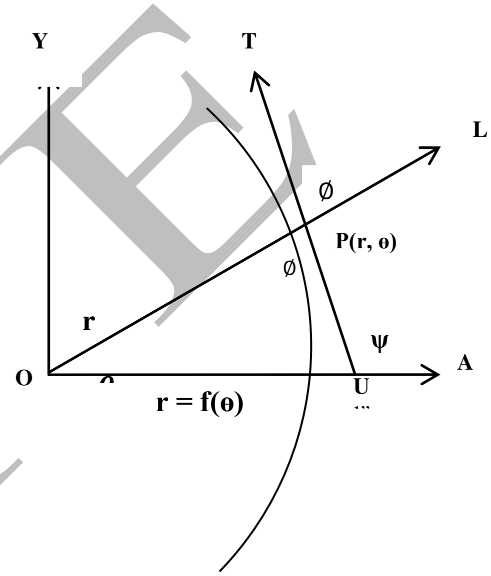
ANGLE BETWEEN THE RADIUS VECTOR AND THE TANGENT:

WITH USUAL NOTATION PROVE THAT $\boxed{\tan\phi = r\dfrac{d\theta}{dr}}$

Let "ϕ" be the angle between the radius vector OPL and the tangent TPU at the point 'P' on the polar curve $r = f(\theta)$.



From Fig.

$\psi = \theta + \phi$ (Exterior angle is equal to interior opposite angles)

$$\tan\psi = \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

i.e. $\dfrac{dy}{dx} = \dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$ ……………(1)

On the other hand, we have $x = r\cos\theta$ and $y = r\sin\theta$ differentiating these, w.r.t θ,

$$\frac{dx}{d\theta} = r(-\sin\theta) + \cos\theta\left(\frac{dr}{d\theta}\right) \quad \& \quad \frac{dy}{d\theta} = r(\cos\theta) + \sin\theta\left(\frac{dr}{d\theta}\right)$$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{r(\cos\theta)+\sin\theta\left(\frac{dr}{d\theta}\right)}{r(-\sin\theta)+\cos\theta\left(\frac{dr}{d\theta}\right)}$ dividing the numerator and denominator by $\dfrac{dr}{d\theta}\cos\theta$

$$\frac{dy}{dx} = \frac{r\frac{d\theta}{dr}+\tan\theta}{-r\frac{d\theta}{dr}\tan\theta+1} = \frac{\tan\theta + r\frac{d\theta}{dr}}{1-\tan\theta\left(r\frac{d\theta}{dr}\right)} \text{ ……………. (2)}$$

Comparing equations (1) and (2)

we get $\tan\phi = r\dfrac{d\theta}{dr}$

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NOTE: (i) $\cot\phi = \left(\frac{1}{r}\frac{dr}{d\theta}\right)$

(ii) If ϕ_1 and ϕ_2 are the angles between the radius vector and the tangents at the point of intersection of two curves $r = f_1(\theta)$ and $r = f_2(\theta)$ then the angle of intersection of the curves is given by

$\alpha = |\phi_1 - \phi_2|$

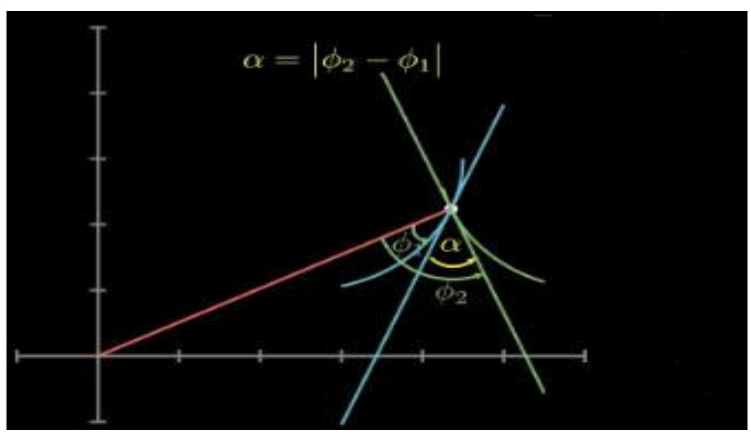
(iii) Suppose we are not able to obtain ϕ_1 and ϕ_2 explicitly then

$$\tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}$$

(iv) If $\tan\phi_1 . \tan\phi_2 = -1 \Rightarrow \phi_1 - \phi_2 = \frac{\pi}{2}$ (condition for the orthogonality of two polar curves)

FIND THE ANGLE BETWEEN TWO CURVES FOR THE FOLLOWING:

(1) $\boldsymbol{r = 2(1+cos\theta)}$ **and** $\boldsymbol{r^2 = 4cos2\theta}$



Solution: Consider $r^2 = 4cos2\theta$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-sin\theta}{1+cos\theta}$$

$$cot\phi_1 = \frac{-2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2cos^2\left(\frac{\theta}{2}\right)}$$

$$cot\phi_1 = -\tan\left(\frac{\theta}{2}\right) = \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\phi_1 = \frac{\pi}{2}+\frac{\theta}{2}$$

Consider $r^2 = 4cos2\theta$

$$\frac{dr}{d\theta} = -\frac{4sin2\theta}{r}$$

$$tan\phi_2 = -\tan\left(\frac{\pi}{2} + 2\theta\right) \quad => \phi_2 = \frac{\pi}{2} + 2\theta\,.$$

We have to eliminate θ between the given curves,

$r = 2(1 + cos\theta)$d $r^2 = 4cos2\theta$

Solving for θ, we get

$$\theta = 1 + \sqrt{3} => \phi_1 - \phi_2 = \frac{3(1+\sqrt{3})}{2}$$

(2) $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

Solution: Consider $r = \dfrac{a\theta}{1+\theta}$

$$\frac{1}{r} = \frac{1+\theta}{a\theta} = \frac{1}{a}\left(\frac{1}{\theta} + 1\right)$$

$$2\theta = -\frac{a}{r^2}\frac{dr}{d\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{r}{a\theta^2}$$

$$r\frac{d\theta}{dr} = \frac{a\theta^2}{r}$$

$$tan\phi = \frac{a\theta^2}{\frac{a\theta}{(1+\theta)}} \quad \tan\phi = \theta(1+\theta)$$

Consider $r = \frac{a}{1+\theta^2}$

$$logr = loga - \log(1+\theta^2)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\theta}{1+\theta^2}$$

$$cot\phi = \frac{-2\theta}{1+\theta^2}$$

$tan\phi = \frac{1+\theta^2}{-2\theta}$.

We have to eliminate θ between the given curves,

$r = \dfrac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

$\theta^3 = 1, \quad \theta = 1$

$\Longrightarrow \tan(\phi_1 - \phi_2) = \frac{2-(-1)}{1+(-2)} => \phi_1 - \phi_2 = \dfrac{3(1+\sqrt{3})}{2}$

FIND THE ANGLE OF INTERSECTION FOR THE FOLLOWING PAIRS OF CURVES:

i. $r = a\cos\theta$ and $2r = a$ **Ans:** $\frac{\pi}{3}$

ii. $r = a\,(1-\cos\theta)$ and $r = 2a\cos\theta$ **Ans:** $\frac{\pi}{2} + \frac{\cos^{-1}\left(\frac{1}{3}\right)}{2}$

iii. $r = 2(1+\cos\theta)$ and $r = 6\cos\theta$ **Ans:** $\frac{\pi}{2}$

PROVE THAT THE FOLLOWING POLAR CURVES INTERSECT ORTHOGONALLY

i. $r = a\sec^2\left(\frac{\theta}{2}\right)$ and $r = b\,cosec^2\left(\frac{\theta}{2}\right)$

ii. $r^n = a^n\cos n\theta$ and $r^n = b^n\sin n\theta$

iii. $re^{\theta} = a$ and $r = be^{\theta}$

LENGTH OF THE PERPENDICULAR FROM POLE ON THE TANGENT:

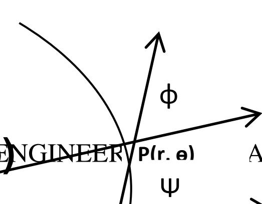
WITH USUAL NOTATION PROVE THAT

(i) $p = r\sin\phi$ **or** **(ii)** $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

From the Fig., let ON = p, the length of the perpendicular from the pole to the tangent at p on $r = f(\theta)$.

From the right angled triangle OPN,

$\sin\phi = \dfrac{ON}{OP} \quad \Rightarrow ON = OP\sin\phi$



i.e. $p = r\sin\phi$..............(i)

$$\frac{1}{p} = \frac{1}{r\sin\phi} = \frac{1}{r}\operatorname{cosec}\phi$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}\left(1+\cot^2\phi\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right]$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \text{..........}(ii)$$

NOTE: If $u = \frac{1}{r}$, we get $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$.

WORKING RULES TO FIND PEDAL EQUATIONS:

(i) Eliminate r and ϕ between the Equations (i) $r = f(\theta)$ & $p = r\sin\phi$

(ii) Eliminate only θ between the Equations (i) $r = f(\theta)$ & $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

Examples:

1. Determine the pedal equation for the Parabola $\frac{2a}{r} = 1 - \cos\theta$

Solution: Consider $\frac{2a}{r} = 1 - \cos\theta$ ....(i)

Differentiating with respect to θ

$$2a\left(-\frac{1}{r^2}\right)\frac{dr}{d\theta} = \sin\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-r\sin\theta}{2a}$$

iv.

$$r\frac{d\theta}{dr} = -\frac{2a}{r}\frac{1}{\sin\theta}$$

$$\tan\phi = -\frac{(1-\cos\theta)}{\sin\theta} = -\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} = -\tan\left(\frac{\theta}{2}\right)$$

$$\tan\phi = \tan\left(\pi - \frac{\theta}{2}\right) \Rightarrow \phi = \pi - \frac{\theta}{2}$$

Using the value of ϕ is $p = r\sin\phi$, we get

$$p = r\sin\left(-\frac{\theta}{2}\right) = -r\sin\left(\frac{\theta}{2}\right)\ldots\ldots\ldots\ldots(ii)$$

Eliminating 'θ' between (i) and (ii)

$$p^2 = r^2\sin^2\left(\frac{\theta}{2}\right) = r^2\left(\frac{1-\cos\theta}{2}\right) = \frac{r^2}{2}\left(\frac{2a}{r}\right)$$

$p^2 = ar$ is the required pedal equation

2. Determine the pedal equation for the equiangular spiral $r = e^{\theta cot\alpha}$

Solution: Consider $= e^{\theta cot\alpha}$, Differentiating with respect to θ

$$\frac{dr}{d\theta} = e^{\theta\cot\alpha}(\cot\alpha) = r\cot\alpha \quad \left(\because r = e^{\theta\cot\alpha}\right)$$

We know that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$$= \frac{1}{r^2} + \frac{1}{r^4}(r\cot\alpha)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4}\left(\cot^2\alpha\right) = \frac{1}{r^2}\left(1+\cot^2\alpha\right) = \frac{1}{r^2}\mathrm{cosec}^2\alpha$$

$$\frac{1}{p^2} = \frac{1}{r^2}\mathrm{cosec}^2\alpha$$

$$p^2 = \frac{r^2}{\mathrm{cosec}^2\alpha}$$

$r^2 = p^2\,\mathrm{cosec}^2\alpha$ is the required pedal equation.

3. Determine the pedal equation for the polar curve $r^m = a^m \sin m\theta + b^m \cos m\theta$.

Solution: Consider $r^m = a^m \sin m\theta + b^m \cos m\theta$

Differentiating with respect to θ

$$mr^{m-1}\frac{dr}{d\theta} = a^m(m\cos m\theta) + b^m(-m\sin m\theta)$$

$$\frac{r^m}{r}\frac{dr}{d\theta} = a^m \cos m\theta - b^m \sin m\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a^m \cos m\theta - b^m \sin m\theta}{a^m \sin m\theta + b^m \cos m\theta}$$

$$\cot\phi = \frac{a^m \cos m\theta - b^m \sin m\theta}{a^m \sin m\theta + b^m \cos m\theta}$$

Consider $p = r\sin\phi$, $\frac{1}{p} = \frac{1}{r}\operatorname{cosec}\phi$

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}\left(1+\cot^2\phi\right) = \frac{1}{r^2}\left[1+\left(\frac{a^m \cos m\theta - b^m \sin m\theta}{a^m \sin m\theta + b^m \cos m\theta}\right)^2\right]$$

$$= \frac{1}{r^2}\left[\frac{\left(a^m \sin m\theta + b^m \cos m\theta\right)^2 + \left(a^m \cos m\theta - b^m \sin m\theta\right)^2}{\left(a^m \sin m\theta + b^m \cos m\theta\right)^2}\right]$$

On simplification, we get

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{a^{2m}+b^{2m}}{r^{2m}}\right] \Rightarrow p^2 = \frac{r^{2(m+1)}}{a^{2m}+b^{2m}}$$ is the required pedal equation.

4. Find the pedal equation for the polar curve $\frac{l}{r} = 1 + e\cos\theta$.

Solution: Consider $\frac{l}{r} = 1 + e\cos\theta$

Differentiating with respect to θ

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$$l\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)=-e\sin\theta \Rightarrow \frac{l}{r}\left(\frac{1}{r}\frac{dr}{d\theta}\right)=e\sin\theta$$

$$\frac{l}{r}(\cot\phi)=e\sin\theta$$

$$\therefore \cot\phi=\left(\frac{r}{l}\right)e\sin\theta$$

We know that $\frac{1}{p^2}=\frac{1}{r^2}\left(1+\cot^2\phi\right)$

$$\frac{1}{p^2}=\frac{1}{r^2}\left[\frac{l^2+e^2r^2\sin^2\theta}{l^2}\right]=\frac{1}{r^2}\left(1+\frac{e^2r^2}{l^2}\sin^2\theta\right) \qquad \left[1+e\cos\theta=\frac{l}{r}\Rightarrow e\cos\theta=\frac{l-r}{r}\right]$$

$$\frac{1}{p^2}=\frac{1}{r^2}\left[\frac{l^2+e^2r^2\left\{1-\left(\frac{l-r}{re}\right)^2\right\}}{l^2}\right] \qquad \left[\cos\theta=\left(\frac{l-r}{re}\right)\right]$$

on simplification

$$\frac{1}{p^2}=\left(\frac{e^2-1}{e^2}\right)+\frac{2}{lr}=1-\left(\frac{l-r}{re}\right)^2 \qquad \left[\sin^2\theta=1-\cos^2\theta\right]$$

EXERCISE:

Find the pedal equations of the following polar curves

1. $r^n\cos n\theta=a^n$

2. $r^n\sin n\theta=b^n$

3. $r=a\theta$

4. $r=\frac{a}{\theta}$

5. $r=\cos\theta$

6.$r^m = a^m \cos m\theta$

7.$r^m = a^m \sin m\theta$

Answers:

1. $pr^{n-1} = a^n$ 2. $pr^{n-1} = -b^n$ 3. $\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4}$

4. $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{a^2}$ 5. $ap = r^2$ 6. $pa^m = r^{m+1}$

7. $pb^m = r^{m+1}$

CURVATURE AND RADIUS OF CURVATURE:

Curvature is the amount by which a geometric object deviates from being *flat*, or *straight* in the case of a line, but this is defined in different ways depending on the context. In geometry, the **radius of curvature**, *R*, of a curve at a point is a measure of the radius of the circular arc which best approximates the curve at that point. It is the reciprocal of the curvature. The distance from the center of a circle or sphere to its surface is its radius. For other curved lines or surfaces, the **radius of curvature** at a given point is the radius of a circle that mathematically best fits the curve at that point. In the case of a surface, the radius of curvature is the radius of a circle that best fits a *normal section*.

Imagine driving a car on a curvy road on a completely flat plain (so that the geographic plain is a geometric plane). At any one point along the way, lock the steering wheel in its position, so that the car thereafter follows a perfect circle. The car will, of course, deviate from the road, unless the road is also a perfect circle. The circle that the car makes is the circle of curvature, radius and the center of the circle are radius of curvature and center of curvature of the curvy road at the point at which the steering wheel was locked. The more sharply curved the road is at the point you locked the steering wheel, the smaller the radius of curvature.

Let P be a point on the curve $y = f(x)$ at the length's' from a fixed point A on it. Let the tangent at 'P' makes are angle ψ with positive direction of x – axis. As the point 'P' moves along curve, both s and ψ vary.

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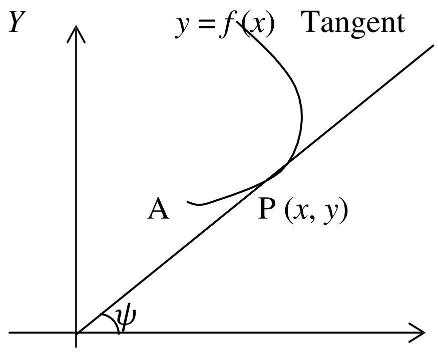

The rate of change ψ w.r.t s, $\frac{d\psi}{ds}$ is called the Curvature of the curve at 'P'.

The reciprocal of the Curvature at P is called the radius of curvature at P and is denoted by ρ.

$$\rho = \frac{ds}{d\psi}$$

RADIUS OF CURVATURE FOR CARTESIAN CURVE $y = f(x)$:

We know that

$tan\psi = \frac{dy}{dx}$ or $\psi = \tan^{-1}(y_1)$, Differentiating both sides w.r.t. s,

$$\frac{d(tan\,\psi)}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right)$$

i.e., $sec^2\psi\frac{d\psi}{ds} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds}$

But $\frac{dx}{ds} = cos\psi$ and by the definition $\frac{d\psi}{ds} = \frac{1}{\rho}$

Therefore $sec^2\psi\frac{1}{\rho} = \frac{d^2y}{dx^2}cos\psi$ or $sec^3\psi = \rho\frac{d^2y}{dx^2}$

Hence $\rho = \frac{sec^3\psi}{\frac{d^2y}{dx^2}}$

Ie., $\rho = \frac{(sec^2\psi)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+tan^2\psi)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left\{1+\left(\left(\frac{dy}{dx}\right)^2\right)\right\}^{3/2}}{\frac{d^2y}{dx^2}}$

Denoting $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$, we have

$$\rho = \frac{(1+(y')^2)^{3/2}}{y''}$$

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1. Find the radius of curvature of the curve $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

Solution: Given, $x^3 + y^3 = 3axy$

Differentiating w.r.t. x we have

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y\right)$$

i.e., $3(y^2 - ax)\frac{dy}{dx} = 3(ay - x^2)$ therefore $\frac{dy}{dx} = y' = \frac{ay-x^2}{y^2-ax}$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right), y' = \frac{3a^2/_2 - 9a^2/_4}{9a^2/_4 - 3a^2/_2} = -1$

$$\frac{d^2y}{dx^2} = y'' = \frac{(y^2-ax)(ay'-2x)-(ay-x^2)(2yy'-a)}{(y^2-ax)^2}$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $y'' = \frac{\left(3a^2/_2\right)(-a-3a)-\left(-3a^2/_4\right)(-3a-a)}{\left(3a^2/_4\right)^2}$

$$y'' = \frac{-3a^3 - 3a^3}{9a^4/_{16}} = \frac{16(-6a^3)}{9a^4} = \frac{-32}{3a}$$

We have $\rho = \frac{\left(1+(y')^2\right)^{3/2}}{y''}$

Hence $\rho = \frac{(1+1)^{3/2}}{-32/_{3a}} = \frac{2\sqrt{2}\cdot 3a}{-32} = \frac{-\sqrt{2}\cdot 3a}{16} = \frac{-3a}{8\sqrt{2}}$

Thus $|\rho| = \frac{3a}{8\sqrt{2}}$

2. Find the radius of curvature of $y = \frac{ax}{a+x}$ and show that $\left(\frac{2\rho}{a}\right)^{\frac{2}{3}} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$.

$$\rho = \frac{[1 + (y')^2]^{\frac{3}{2}}}{y''}$$

$y = \frac{ax}{a+x}$ ---------(1)

Differentiate with respect to x

$$y' = \frac{(a+x)(a) - ax(0+1)}{(a+x)^2}$$

$$y' = \frac{a^2 + ax - ax}{(a+x)^2}$$

$y' = \frac{a^2}{(a+x)^2}$ -------------(2)

Again, Differentiate with respect to x

$$y'' = \frac{(a+x)^2(0) - a^2[2(a+x)(1)]}{(a+x)^4}$$

$$y'' = \frac{-2a^2(a+x)}{(a+x)^4}$$

$$y'' = \frac{-2a^2}{(a+x)^3} \text{ -------(3)}$$

Hence, $\rho = \frac{\left[1+(y')^2\right]^{\frac{3}{2}}}{y''}$

Substitute y' and y'' values in the above equation, then

$$\rho = \frac{\left[1+\left(\frac{a^2}{(a+x)^2}\right)^2\right]^{\frac{3}{2}}}{\frac{-2a^2}{(a+x)^3}}$$

$$\rho = \frac{[(a+x)^4 + a^4]^{\frac{3}{2}}}{(a+x)^3}\left(\frac{(a+x)^3}{-2a^2}\right)$$

$$-2\rho = \frac{[(a+x)^4 + a^4]^{\frac{3}{2}}}{(a+x)^3}\left(\frac{1}{a^2}\right)$$

$$(-2\rho)^{\frac{2}{3}} = \frac{(a+x)^4 + a^4}{(a+x)^2}\left(\frac{1}{a^{\frac{4}{3}}}\right)$$

$$(2\rho)^{\frac{2}{3}} = \frac{1}{a^{\frac{4}{3}}}\left[\frac{(a+x)^4 + a^4}{(a+x)^2}\right]$$

We have,

$$y = \frac{ax}{a+x}$$

$$a + x = \frac{ax}{y}$$

$$(2\rho)^{\frac{2}{3}} = \frac{1}{a^{\frac{4}{3}}}\left[\frac{\left(\frac{ax}{y}\right)^4}{\left(\frac{ax}{y}\right)^2} + \frac{a^4}{\left(\frac{ax}{y}\right)^2}\right]$$

$$(2\rho)^{\frac{2}{3}} = \frac{1}{a^{\frac{4}{3}}}\left[a^2\left(\frac{x}{y}\right)^2 + a^2\left(\frac{y}{x}\right)^2\right]$$

$$(2\rho)^{\frac{2}{3}} = \frac{a^2}{a^{\frac{4}{3}}}\left[\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2\right]$$

$$(2\rho)^{\frac{2}{3}} = \frac{1}{a^{\frac{-2}{3}}}\left[\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2\right]$$

$$\left(\frac{2\rho}{a}\right)^{\frac{2}{3}} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

3. Find radius of curvature of the curve $y^2 = \frac{4a^2(2a-x)}{x}$ at (2a, 0)

$xy^2 = 8a^3 - 4a^2x$

Differentiate w.r.t. x

$2xyy' + y^2 = 0 - 4a^2$

$2xyy' = -4a^2 - y^2$

$y' = \frac{-4a^2-y^2}{2xy}$ -----------(1)

At (2a, 0)

$$y' = \frac{-4a^2 - 0}{2x(0)}$$

$y' = \infty$

$\Rightarrow x' = 0$

$$(1) \Rightarrow x' = -\frac{2xy}{4a^2 - y^2}$$

Differentiate w.r.t. y

$$x'' = -\left[\frac{4a^2 - y^2(2x + 2yx') - (2xy)(0 - 2y)}{(4a^2 - y^2)^2}\right]$$

At (2a, 0)

$$x'' = -\left[\frac{4a^2 - 0 - 0}{(4a^2 - 0^2)^2}\right]$$

$$x'' = -\left[\frac{4a^2}{(4a^2)^2}\right]$$

$$x'' = -\frac{1}{4a^2}$$

We have,

$$\rho = \frac{[1 + (x')^2]^{\frac{3}{2}}}{x''}$$

$$\rho = \frac{[1 + (0)^2]^{\frac{3}{2}}}{-\frac{1}{4a^2}}$$

$$\rho = -4a^2$$

$$|\rho| = 4a^2$$

EXERCISE

FIND THE RADIUS OF CURVATURE OF THE FOLLOWING CURVE

1. $x^4 + y^4 = 2$ at $(1, 1)$.

2. $x^2y = a(x^2 + y^2)$ at $(-2a, +2a)$.

3. $xy^2 = a^4 - a^3x$ at $(a, 0)$.

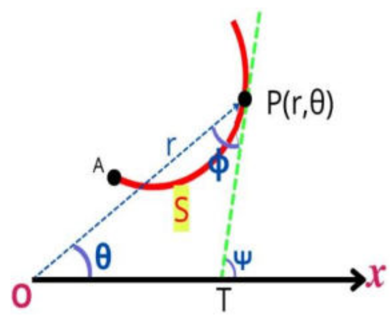
4. $y = 4\sin x \sin 2x$ at $x = \frac{\pi}{2}$.

5. $a^2y = x^3 - a^3$ at the point where the curve meets the x-axis.

6. Prove that ρ at any point (x, y) on the curve $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ is given by $\rho = \frac{2(ax+by)^{3/2}}{ab}$

DEPARTMENT OF MATHEMATICS

RADIUS OF CURVATURE IN POLAR FORM: $\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$.



Let r be the radius vector and φ be the angle made by the radius vector with the tangent at. Let ψ be the angle made by the tangent at P with the initial line. Let A be a fixed point on the curve and let $AP = s$. We have $\psi = \theta + \phi$

Let $OP = r$ be the radius vector and φ be the angle made by the radius vector with the tangent at $P(r, \theta)$. Let ψ be the angle made by the tangent at P with the initial line. Let A be a fixed point on the curve and let $AP = s$. We have $\psi = \theta + \phi$

Therefor $\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$

i.e., $\frac{1}{\rho} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right)$

Or $\rho = \frac{\left(\frac{ds}{d\theta}\right)}{1+\frac{d\phi}{d\theta}}$ ……………(1)

We know that $\tan\varphi = r\frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$

Ie., $\tan\varphi = \frac{r}{r_1}$ $r_1 = \left(\frac{dr}{d\theta}\right)$

Differentiating w. r. t. θ we get

$sec^2\varphi\frac{d\phi}{d\theta} = \frac{r_1 \cdot r_1 - r \cdot r_2}{r_1^2}$, where $r_2 = \frac{d^2r}{d\theta^2}$

Or $\frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2 sec^2\varphi} = \frac{r_1^2 - rr_2}{r_1^2(1+tan^2\varphi)}$

i.e., $\frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2\left(1+\left(\frac{r^2}{r_1^2}\right)\right)} = \frac{r_1^2 - rr_2}{r_1^2 + r^2}$

Hence $1 + \frac{d\phi}{d\theta} = 1 + \frac{r_1^2 - rr_2}{r_1^2 + r^2} = \frac{r_1^2 + r^2 + r_1^2 - rr_2}{r_1^2 + r^2}$

Ie., $1 + \frac{d\phi}{d\theta} = \frac{r^2 + 2r_1^2 - rr_2}{r_1^2 + r^2}$ ………………..(2)

Also we know that $\frac{ds}{d\theta} = \sqrt{r_1^2 + r^2}$ ……………(3)

Using (2) and (3) in (1) we get

$$\rho = \sqrt{r_1^2 + r^2} \cdot \frac{r_1^2 + r^2}{r^2 + 2r_1^2 - rr_2}$$

Thus $\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$ in the polar form.

EXERCISE

1. Find the radius of curvature of the curve $r = a\sin n\theta$ at the pole.

Solution: Given $r = a\sin n\theta$

Differentiate with respect to θ

$r_1 = \frac{dr}{d\theta} = an\cos n\theta,\ r_2 = \frac{d^2r}{d\theta^2} = -an^2\sin n\theta$

At the pole we have $\theta = 0$. When $\theta = 0$: $r = 0, r_1 = an, r_2 = 0$.

We have $\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$

Therefor $\rho = \frac{(a^2n^2)^{3/2}}{2a^2n^2} = \frac{a^3n^3}{2a^2n^2} = \frac{an}{2}$.

Thus $\rho = \frac{an}{2}$ at the pole.

DEPARTMENT OF MATHEMATICS

2. Find the radius of curvature of curve $r = a(1-\cos\theta)$

Apply log on both side
$\log r = \log a + \log(1-\cos\theta)$
Differentiate w.r.t. θ

$$\frac{r_1}{r} = 0 + \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{r_1}{r} = 0 + \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$\frac{r_1}{r} = \cot\frac{\theta}{2}$$

$$r_1 = r\cot\frac{\theta}{2}$$

Differentiate w.r.t. θ

$$r_2 = \frac{-r}{2}cosec^2\frac{\theta}{2} + \cot\frac{\theta}{2}r_1$$

$$r_2 = \frac{-r}{2}cosec^2\frac{\theta}{2} + r\,cot^2\frac{\theta}{2}$$

$$r_2 = r\left[cot^2\frac{\theta}{2} - \frac{1}{2}\left(1 + cot^2\frac{\theta}{2}\right)\right]$$

$$r_2 = r\left[cot^2\frac{\theta}{2} - \frac{1}{2} - \frac{1}{2}cot^2\frac{\theta}{2}\right]$$

$$r_2 = r\left[\frac{2cot^2\frac{\theta}{2} - 1 - cot^2\frac{\theta}{2}}{2}\right]$$

$$r_2 = r\left[\frac{cot^2\frac{\theta}{2} - 1}{2}\right]^2$$

$$\therefore \quad \rho = \frac{[r^2 + r_1^2]^{\frac{3}{2}}}{r^2 + 2r_1 - rr_2}$$

$$\rho = \frac{\left[r^2 + r^2 \cot^2\frac{\theta}{2}\right]^{\frac{3}{2}}}{r^2 + 2r^2\cot^2\frac{\theta}{2} - r^2\left[\frac{\cot^2\frac{\theta}{2} - 1}{2}\right]}$$

$$\rho = \frac{r^3 cosec^3\frac{\theta}{2}}{\frac{2r^2 + 4r^2\cot^2\frac{\theta}{2} - r^2\cot^2\frac{\theta}{2} + r^2}{2}}$$

$$\rho = \frac{2r^3 cosec^3\frac{\theta}{2}}{r^2\left(2 + 4\cot^2\frac{\theta}{2} - \cot^2\frac{\theta}{2} + 1\right)}$$

$$\rho = \frac{2r^3 cosec^3\frac{\theta}{2}}{r^2\left(2 + 4\cot^2\frac{\theta}{2} - \cot^2\frac{\theta}{2} + 1\right)}$$

$$\rho = \frac{2r cosec^3\frac{\theta}{2}}{3 + 3\cot^2\frac{\theta}{2}}$$

$$\rho = \frac{2r cosec^3\frac{\theta}{2}}{3\left(1 + \cot^2\frac{\theta}{2}\right)}$$

$$\rho = \frac{2r cosec^3\frac{\theta}{2}}{3\left(cosec^2\frac{\theta}{2}\right)}$$

$$\rho = \frac{2r}{3}\frac{1}{\sin\frac{\theta}{2}}$$

$$\rho = \frac{2\big(a(1-\cos\theta)\big)}{3\sin\frac{\theta}{2}}$$

$$\rho = \frac{2a\left(2\sin^2\frac{\theta}{2}\right)}{3\sin\frac{\theta}{2}}$$

$$\rho = \frac{4a}{3}\sin\frac{\theta}{2}$$

DEPARTMENT OF MATHEMATICS

3. Find the radius of curvature of curve $r = a(1 + cos\theta)$

Solution : $\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$

Given : $r = a(1 + \cos\theta)$

Apply log on both sides

$$\log r = \log a + \log(1 + \cos\theta)$$

$$\frac{1}{r}r_1 = 0 + \frac{1}{1+\cos\theta}(-\sin\theta)$$

$$r_1 = -r\frac{\sin\theta}{1+\cos\theta}$$

$$r = -r\tan\left(\frac{\theta}{2}\right)$$

Differentiate with respect to θ

$$r_2 = \left[-r\frac{\sec^2\frac{\theta}{2}}{2} + tan\frac{\theta}{2}r_1\right]$$

We have $\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$

$$(r^2 + r_1^2)^{\frac{3}{2}} = \left[r^2 + r^2\tan^2\frac{\theta}{2}\right]$$

$$= r^3\left[1 + \tan^2\frac{\theta}{2}\right]$$

$$= r^3\sec^3\frac{\theta}{2}$$

$$r^2 + 2r_1^2 - rr_2 = r^2 + 2r^2\tan^2\frac{\theta}{2} - r\left(-\frac{r}{2}\sec^2\frac{\theta}{2} + r\tan^2\frac{\theta}{2}\right)$$

$$= r^2\left[1 + \frac{\sec^2\frac{\theta}{2}}{2} + \tan^2\frac{\theta}{2}\right]$$

$$\rho = \frac{r^3\sec^3\frac{\theta}{2}}{r^2\left[1 + \frac{\sec^2\frac{\theta}{2}}{2} + \tan^2\frac{\theta}{2}\right]}$$

$$= \frac{r \sec^3 \frac{\theta}{2}}{\left[\frac{\sec^2 \frac{\theta}{2}}{2} + \sec^2 \frac{\theta}{2}\right]}$$

$$= \frac{2r}{3} sec \frac{\theta}{2}$$

We have $r = a(1 + \cos\theta)$

$$\frac{r}{a} = 2\cos^2\frac{\theta}{2}$$

$$\sqrt{\frac{r}{2a}} = \cos\frac{\theta}{2}$$

$$\sec\frac{\theta}{2} = \sqrt{\frac{2a}{r}}$$

$$\therefore \quad \rho = \frac{2r}{3} \times \sqrt{\frac{2a}{r}}$$

or

$$\rho^2 = \frac{8ar}{9}$$

EXERCISE

Find the radius of curvature of the curve $r^n = a^n \sin n\theta$.

Find the radius of curvature of the curve $r^n = a^n \cos n\theta$.

RADIUS OF CURVATURE IN PARAMETRIC FORM x=x(t), y=y(t).

$$\rho = \frac{((x')^2 + (y')^2)^{3/2}}{x'y'' - y'x''}$$

EXAMPLES

1. Find ρ at the point t of the curve $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$.

Solution: We know that

$$\rho = \frac{((x')^2 + (y')^2)^{3/2}}{x'y'' - y'x''}$$

DEPARTMENT OF MATHEMATICS

Considering $x = a(\cos t + t\sin t)$

Differentiating w. r. t. t

$$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t)$$

$$x' = at\cos t$$

Differentiate x'' w. r. t. t

$$x'' = a(t\,(-\sin t) + \cos t)$$

$$x'' = a(\cos t - t\sin t)$$

Consider $y = a(\sin t - t\cos t)$

Differentiate y' w. r. t. t

$$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t)$$

$$y' = at\sin t$$

Differentiate y'' w. r. t. t

$$y'' = a(t\cos t + \sin t)$$

$$\rho = \frac{((at\cos t)^2 + (at\sin t)^2)^{3/2}}{at\cos t\, a(t\cos t + \sin t) - at\sin t a(\cos t - t\sin t)}$$

$$\rho = \frac{\left(a^2t^2(sin^2t + cos^2t)\right)^{3/2}}{at\cos t\, a(t\cos t + \sin t) - at\sin t a(\cos t - t\sin t)}$$

$$\rho = \frac{(a^2t^2)^{3/2}}{a^2t^2cos^2t + a^2t\ cost\ sint + a^2t^2sin^2t - a^2t\ cost\ sint}$$

$$\rho = \frac{(a^2t^2)^{3/2}}{a^2t^2cos^2t + a^2t\ cost\ sint + a^2t^2sin^2t - a^2t\ cost\ sint}$$

$$\rho = \frac{((at)^2)^{3/2}}{a^2t^2cos^2t + a^2t^2sin^2t}$$

$$\rho = \frac{(at)^3}{a^2t^2(cos^2t + sin^2t)}$$

$$\rho = \frac{(at)^3}{(at)^2}$$

$$\rho = at.$$

2. Find the radius of curvature of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Solution: The parametric equation of astroid is given by

$x = a\ cos^3t\ and\ y = a\ sin^3t$

We know that

$$\rho = \frac{[(x')^2 + (y')^2]^{\frac{3}{2}}}{x'y'' - y'x''}$$

We have

$x = acos^3t$

Differentiate with respect to t

$x' = a \times 3 \times cos^2t \sin t$

Differentiate with respect to t

$$x'' = -3a[cos^2t \cos t + \sin t\,(2\cos t)(-\sin t)]$$
$$= -3a \times cos^3t + 6a \times sin^2t \times t \cos t$$

We have

$y = a\ sin^3t$

$y' = 3a \times sin^2t \times \cos t$

$$y'' = 3a[sin^2t(-\sin t) + \cos t\,(2\sin t)(\cos t)]$$
$$= -3a \times sin^3t + 6a \times cos^2t \times \sin t$$

$\therefore\ \rho$

$$= \frac{[(-3acos^2tsint)^2 + (3asin^2tcost)^2]^{\frac{3}{2}}}{(-3acos^2tsint)(-3asin^3t + 6acos^2tsint) - (3asin^2tcost)(+3acos^3t + 6asin^2tcost)}$$

$$\rho = \frac{[(-3acos^2tsint)^2 + (3asin^2tcost)^2]^{\frac{3}{2}}}{(-3acos^2tsint)(-3asin^3t + 6acos^2tsint) - (3asin^2tcost)(+3acos^3t + 6asin^2tcost)}$$

$$\rho = \frac{(9acos^4t\ sin^2t + 9asin^4tcos^2t)^{\frac{3}{2}}}{9a^2\ sin^4tcos^2t - 18a^2\ cos^4t\ sin^2t + 9a^2\ cos^4t\ sin^2t - 18a^2\ sin^4tcos^2t}$$

$$\rho = \frac{(9a^2 \cos^2 t \sin^2 t)^{\frac{3}{2}} (\cos^2 t + \sin^2 t)^{\frac{3}{2}}}{-9a^2 \sin^4 t\cos^2 t - 9a^2 \cos^4 t \sin^2 t}$$
$$\rho = \frac{27a^3 \cos^3 t \sin^3 t}{-9a^2 \sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)}$$
$$\rho = \frac{-3a \sin t \cos t}{1}$$
$$|\rho| = 3a \sin t \cos t$$

FIND THE RADIUS OF CURVATURE OF THE FOLLOWING PARAMETRIC CURVES

1. $x = a\log(\sec t + \tan t)$ and $y = a\ \sec t$ 2. $x = a(\cos t + \log(\tan {}^{t}/_{2})$ and $y = a\ \sin t$.

3. $x = a(t + \sin t)$ and $y = a\ (1 - \cos t)$. 4. $y^2 = 4ax$.

RADIUS OF CURVATURE IN THE CASE OF PEDAL FORM

$$\rho = r\frac{dr}{dp}$$

EXAMPLE

1. Find the radius of curvature of $r = a(1 - \cos\theta)$ and prove that $\rho \propto \sqrt{r}$.

Solution: Given $r = a(1 - \cos\theta)$

Applying log on both sides

$\log r = \log a + \log(1 - \cos\theta)$

Differentiate w. r. t. θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1 - \cos\theta}(0 - (-\sin\theta))$$

$$\cot\varphi = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin\theta^2}$$

$$\cot\varphi = \cot {}^{\theta}/_{2}$$

$$\varphi = {}^{\theta}/_{2}$$

DEPARTMENT OF MATHEMATICS

We have $p = r\sin\varphi$

$p = r\sin\frac{\theta}{2}$………………..(1)

By Given equation $r = a(1-\cos\theta)$

$$\frac{r}{a} = 2\sin\frac{\theta^2}{2}$$

$$\sin\frac{\theta^2}{2} = \frac{r}{2a}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{r}{2a}}$$

$$p = r\sqrt{\frac{r}{2a}}$$

$$p = \frac{r^{3/2}}{\sqrt{2a}}$$

Differentiate w.r.t. r

$$\frac{dp}{dr} = \frac{1}{\sqrt{2a}} \cdot \frac{3}{2} \cdot r^{1/2}$$

$$\frac{dp}{dr} = \frac{3}{2} \cdot \sqrt{\frac{r}{2a}}$$

$$\frac{dr}{dp} = \frac{2}{3} \cdot \sqrt{\frac{2a}{r}}$$

Therefore $\rho = r \cdot \frac{dr}{dp}$

$$\rho = r \cdot \frac{2}{3}\sqrt{\frac{2a}{r}}$$

$$\rho = \frac{2}{3}\sqrt{2ar}$$

$$\rho = \frac{2\sqrt{2a}}{3}\sqrt{r}$$

$\rho \propto \sqrt{r}$ and $\frac{2\sqrt{2a}}{3}$ is proportionality constant.

2. Using pedal formula of radius of curvature, prove that $\rho \propto \frac{r^3}{a^2}$ for $r^2 = a^2 \sec 2\theta$.

SOME OF THE APPLICATIONS:

- Radius of curvature is applied to measurements of the stress in the semiconductor structures.
- When engineers design trains track, they need to ensure the curvature of the track to be safe and provide a comfortable ride for the given speed of the trains.
- It is used in highway design of loops and somewhat sharp turns of the main road.
- The radius of curvature determines various factors:

  (a) The super elevation of the curve, i.e. the level by which the outer rail of the curve is elevated with reference to the inner rail.

  (b) Maximum permissible speed of train over the curve.

  (c) Transition length that is to be provided on either end of the curve.