MANGALORE INSTITUTE OF TECHNOLOGY & ENGINEERING
Accredited by NAAC with A+ Grade, An ISO 9001: 2015 Certified Institution
(A Unit of Rajalaxmi Education Trust®, Mangalore - 575001)
Affiliated to V.T.U., Belagavi, Approved by AICTE, New Delhi.

DEPARTMENT OF MATHEMATICS

MODULE -2:

SERIES EXPANSION AND MULTIVARIABE CALCULUS

CONTENTS:

Introduction to series expansion and partial differentiation in Computer Science & Engineering applications.

- Taylor's and Maclaurin's series expansion for one variable (Statement only)- Problems
- Indeterminate forms – L'Hospital's rule - Problem
- Partial differentiation
- Total derivative – Differentiation of Composite functions, Jacobian and problems. Maxima and minima for a function of two variables. Problems.

Self-study: Euler's Theorem and Problems. Method of Lagrange's undetermined multipliers with single constraint.

Applications: Series expansion in computer programming, Computing errors and approximations.

(RBT Levels: L1, L2 and L3)

LEARNING OBJECTIVES:

After Completion of this module, student will be able to:

- Expand the function in power series using Taylor's and Maclaurin's series.
- Evaluate Indeterminate forms by L'Hospital's Rule
- Understand the fundamentals of the differential calculus of functions of multiple variable.
- Find Maxima and Minima for a function of two variables

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TAYLOR'S SERIES EXPANSION FOR FUNCTION OF ONE VARIABLE:
(ENGLISH MATHEMATICIAN BROOK TAYLOR 1685-1731)

Taylor's series expansion for the given function f(x) in powers of $(x-a)$ or about the point 'a' is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \ldots + \frac{(x-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \ldots. \quad \text{--------}(1)$$

$$= f(a) + \sum_{n=1}^{\infty} \frac{(x-a)^n}{\lfloor n} f^{(n)}(a) \to (4)$$

MACLAURIN'S SERIES EXPANSION FOR FUNCTION OF ONE VARIABLE:
(SCOTTISH MATHEMATICIAN COLIN MACLAURIN 1698-1746)

When a = 0, expression (1) reduces to a Maclaurin's expansion given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots + \frac{x^{n-1}}{\lfloor n-1} f^{(n-1)}(0) + \ldots.$$

$$= f(0) + \sum_{n=1}^{\infty} \frac{x^n}{\lfloor n} f^{(n)}(0) \to (5)$$

ADVANTAGES OF TAYLOR'S SERIES AND MACLAURIN'S SERIES

- Taylor series are studied because polynomial functions are easy and if one could find a way to represent complicated functions as series (infinite polynomials) then one can easily study the properties of difficult functions.

- Evaluating definite Integrals: Some functions have no antiderivative which can be expressed in terms of familiar functions. This makes evaluating definite integrals for some functions difficult because the Fundamental Theorem of Calculus cannot be used. If we have a polynomial representation of a function, we can oftentimes use that to evaluate a definite integral.

- Understanding asymptotic behavior: Sometimes, a Taylor series can tell us useful information about how a function behaves in an important part of its domain.

- Understanding the growth of functions
- Solving differential equations

EXAMPLES

1. Obtain a Maclaurin's series for $f(x) = \sin x$ up to the term containing x^5

Solution: The Maclaurin's series for $f(x)$ is

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{(4)}(0) + \frac{x^5}{\lfloor 5} f^{(5)}(0) \ldots \rightarrow (1)$$

Here $f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0 \;\; f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0 \;\; f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$

$f^{(iv)}(x) = \sin x \Rightarrow f^{(iv)}(0) = \sin 0 = 0 \;\; f^{(v)}(x) = \cos x \Rightarrow f^{(v)}(0) = \cos 0 = 1$

Substituting these values in (1), we get the Maclaurin's series for $f(x) = Sin\ x$ as

$$f(x) = \sin x = 0 + x\,(1) + \frac{x^2}{\lfloor 2}(0) + \frac{x^3}{\lfloor 3}(-1) + \frac{x^4}{\lfloor 4}(0) + \frac{x^5}{\lfloor 5}(1) \ldots \Rightarrow \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \ldots$$

2. Using Maclaurin's series expand the function $\sqrt{1 + sin2x}$ up to term x^3

Sol: Let y(x) = $\sqrt{1 + sin2x}$

Consider the Maclaurin's series expansion,

$$y(x) = y(0) + x.y^{I}(0) + \frac{x^2}{2!}y^{II}(0) + \frac{x^3}{3!}y^{III}(0)$$

y(x)= $\sqrt{1 + sin2x} \Rightarrow y(0) = \sqrt{1 + 0} = 1$

y (x)=$sinx + cosx$

Differentiate above equation w.r.to x we get

$y^{I}(x) = cosx - sinx \Rightarrow y^{I}(0) = 1$

$y^{II}(x) = -sinx - cosx \Rightarrow y^{II}(0) = -1$

$y^{III}(x) = -cosx + sinx \Rightarrow y^{III}(0) = -1$

Therefore,

$$y(x) = y(0) + x.y^I(0) + \frac{x^2}{2!}y^{II}(0) + \frac{x^3}{3!}y^{III}(0) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!}$$

3. Obtain Maclaurin's series of the function $e^{\sin x} upto\ x^4$.

$By\ Maclaurin's\ series\ we\ have$

$$y(x) = y(0) + x.y^I(0) + \frac{x^2}{2!}y^{II}(0) + \frac{x^3}{3!}y^{III}(0) + \frac{x^4}{4!}y^{IV}(0)$$

$let y(x) = e^{\sin x} \Rightarrow y(0)e^{\sin 0} = e^0 = 1$

Differentiate above equation w.r.to x we get

$y^I(x) = e^{\sin x}.\cos x \Rightarrow y^I(x) = y.\cos x\ \ (\because y(x) = e^{\sin x})$
$y^I(0) = y(0).\cos(0) = 1 \times 1 = 1$
$y^{II}(x) = y(-\sin x) + \cos x\,(y^I) = -y\sin x + y^I\cos x$
$y^{II}(0) = -y(0)\sin(0) + y^I(0)\cos(0) = 1 = 0 + 1 \times 1 = 1$
$y^{III}(x) = -y\cos x - \sin x\,(y^I) - y^I\sin x + y^{II}\cos x$
$y^{III}(0) = -y(0)\cos(0) - 2\sin(0)y^I(0) + \cos(0)y^{II}(0)$
$y^{III}(0) = -1 \times 1 + 1 \times 1 = -1 + 1$
$y^{III}(0) = 0$

$y^{IV}(x) = -y(-\sin x) - \cos x\,(y^I) - 2y^I\cos x - 2\sin x\,(y^{II}) + y^{II}(-\sin x) + \cos x\,(y^{III})$
$y^{IV}(x) = y\sin x - 3y^I\cos x - 3y^{II}\sin x + y^{III}\cos x$
$y^{IV}(0) = y(0)\sin(0) - 3y^I(0)\cos(0) - 3y^{III}(0)\sin(0) + y^{III}(0)\cos(0)$
$y^{IV}(0) = -3 \times 1 \times 1 = -3$

$\therefore By\ Maclaurin's\ series\ we\ have$

$$y(x) = y(0) + x.y^I(0) + \frac{x^2}{2!}y^{II}(0) + \frac{x^3}{3!}y^{III}(0) + \frac{x^4}{4!}y^{IV}(0)$$

$$e^{\sin x} = 1 + x(1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(0) + \frac{x^4}{24}(-3)$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

4. **Obtain Maclaurin's series expansion of log(1+e^x) upto 3 non-vanishing terms**

Solution:-

$\therefore By\ Maclaurin's\ series\ we\ have$

$$y(x) = y(0) + x.y^{I}(0) + \frac{x^2}{2!}y^{II}(0) + \frac{x^3}{3!}y^{III}(0) + \frac{x^4}{4!}y^{IV}(0)$$

Let $y(x) = log\,(1 + e^x)$

$y\,(0) = log\,(1 + e^0) = log\,2$

Differentiate above equation w.r.to x we get

$y'(\mathrm{x}) = \frac{1}{1+e^x}(e^x)$, at $x = 0, y'(0) = 1/(1+1) = 1/2$

$(1 + e^x)\,y'(x) = e^x$,

$(1+e^x)y''(x)+y'(x)e^x = e^x$ at $x = 0, (1+e^0)y''(0) + y'(0)e^0 = e^0$

$2\times y''(0) + \frac{1}{2}(1) = 1 \quad => y''(0) = \frac{1-\frac{1}{2}}{2} = \frac{1}{4}$

$(1+e^x)y'''(x)+y''(x)e^x+y'(x)e^x + e^x y''(x) = e^x$

$(1+e^x)y'''(x)+2y''(x)e^x+e^x y'(x)=e^x$

$(1+1)y'''(0)+2y''(0)\times 1+1\times y'(0)=1 \quad => y'''(0) = 0$

$(1+e^x)y^{\prime v}(x)+y'''(x)e^x+2e^x y'''(x)+2y''(x)e^x+e^x y''(x)+y'(x)e^x = e^x$

$(1+e^x)y^{\prime v}(x)+3e^x y'''(x)+3e^x y''(x)+e^x y'(x) = e^x$

$(1+e^x)y^{\prime v}(0)+3e^x y'''(0) + 3\times(1)\times y''(0)+1\times y'(0) = 1$

$(1+1)y^{\prime v}(0)+3\times 1\times 0+3\times 1\times \frac{1}{4}+1\times 1/2=1$

$2y^{\prime v}(0)+\frac{5}{4} = 1$

$2y^{\prime v}(0) = \frac{-1}{4}$

$y^{\prime v}(0)= \frac{-1}{8}$

By Maclaurin's series expression:

$y(x)=y(0)+xy'(0)+\frac{x^2}{2!}y''(0)+\frac{x^3}{3!}y'''(0)+\frac{x^4}{4!}y^{\prime v}(0)$

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EXERXISE

1. Expand using Maclaurins's series, the function $\log(1+e^x)$ upto the term containing x^4.
2. Expand using Maclaurins's series, the function $\log(secx)$ upto the term containing x^6
3. Expand using Maclaurins's series, the function $\sin(e^x-1)$ upto the term containing x^4.
4. Expand using Maclaurins's series, the function $\log(1+sinx)$ upto x^5
5. Expand using Maclaurins's series, the function e^{xcosx} upto the term containing x^4.
6. Expand using Maclaurins's series, the function $e^{asin^{-1}x}$ upto the term containing x^4.

INDETERMINATE FORMS: L'HOSPITAL'S RULE

WHAT ARE INDETERMINATE FORMS?

When evaluating limits, we come across situations where the basic rules for evaluating limits might fail. For example, we can apply the quotient rule in case of rational functions:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{(x\to a)}f(x)}{\lim_{(x\to a)}g(x)}\quad , \quad if\ lim(x\to a)g(x)\neq 0$$

The above rule can only be applied if the expression in the denominator does not approach zero as x approaches a.

A more complicated situation arises if both the numerator and denominator both approach zero as x approaches a. This is called an indeterminate form of type $0/0$.

Similarly, there are indeterminate forms of the type ∞/∞, given by:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{(x\to a)}f(x)}{\lim_{(x\to a)}g(x)}\ when\ \lim_{(x\to a)}f(x)=\infty\ and\ \lim_{(x\to a)}g(x)=\infty$$

WHAT IS L'HOSPITAL'S RULE?

The L'Hospital rule states the following:

If we have an indeterminate form of the type $\frac{0}{0}$ $or\ \infty/\infty$

i.e., $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{(x\to a)} f(x)}{\lim\limits_{(x\to a)} g(x)}\ is\ \frac{0}{0}\ \mathbf{OR}\ \frac{\infty}{\infty}$

Then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f^1(x)}{g^1(x)}$

WHEN TO APPLY L'HOSPITAL'S RULE

An important point to note is that L'Hospital's rule is only applicable when the conditions for f(x) and g(x) are met. i.e., $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{(x\to a)} f(x)}{\lim\limits_{(x\to a)} g(x)}\ is\ \frac{0}{0}\ \mathbf{OR}\ \frac{\infty}{\infty}$

For example:

$\lim\limits_{x\to 0}\dfrac{sin(x)}{1/x^2}$ Cannot apply L'Hospital's rule as it's not $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form

$\lim\limits_{x\to 0}\dfrac{sin(x)}{x}$ Can apply the rule as it's 0/0 form

$\lim\limits_{x\to \infty}\dfrac{e^x}{\frac{1}{x}+1}$ Cannot apply L'Hospital's rule as it's not $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form

$\lim\limits_{x\to \infty}\dfrac{e^x}{x}$ Can apply L'Hospital's rule as it is ∞/∞ form

MORE INDETERMINATE FORMS

The L'Hospital's rule only tells us how to deal with $0/0$ or ∞/∞ forms. However, there are more indeterminate forms such as $0^0, 0^\infty, \infty^0 \, and\ 1^\infty$. So how do we deal with the rest? We can use some methods in mathematics to convert the above indeterminate forms to $0/0$ or ∞/∞. This will enable us to easily apply L'Hospital's rule to almost all indeterminate forms.

NOTE: By applying log we can convert $0^0, 0^\infty, \infty^0 \, and\ 1^\infty$ to $0/0$ or ∞/∞

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Evaluate the following Indeterminate forms

1. $\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$

$$let\ \ k=\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}=\left(\frac{3}{3}\right)^{\frac{1}{0}}\quad [1^\infty]$$

taking log on bothsides

$$log\,k=\lim_{x\to 0}\frac{1}{x}log\left(\frac{a^x+b^x+c^x}{3}\right)$$

$$=\lim_{x\to 0}log\frac{\left(\frac{a^x+b^x+c^x}{3}\right)}{x}=\frac{0}{0}$$

$apply\ L-Hospital\ rule$:

$$log\,k=\lim_{x\to 0}\frac{\frac{3}{a^x+b^x+c^x}\times\frac{1}{3}\{a^x\log a+b^x\log b+c^x\log c\}}{1}$$

$$log\,k=\frac{3}{3}\times\frac{1}{3}\{log\,a+log\,b+log\,c\}$$

$$=\frac{1}{3}log(abc)\Rightarrow log\,k=log(\sqrt[3]{abc})$$

$$k=(abc)^{\frac{1}{3}}$$

2) $\lim\limits_{x\to\frac{\pi}{2}}(sin\,x)^{tan\,x}$

$$let\quad k=\lim_{x\to\frac{\pi}{2}}(sin\,x)^{tan\,x}=1^\infty$$

taking log on bothsides

$$log\,k=\lim_{x\to\frac{\pi}{2}}(tan\,x)\,log\,x\,(sin\,x)=\infty\times 0$$

$$\lim_{x\to\frac{\pi}{2}}\frac{log(\,sin\,x)}{cot\,x}=\frac{0}{0}$$

$applyL-Hrule$:

$$log\,k=\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{sin\,x}\times cos\,x}{-cos\,e\,c^2x}=\lim_{x\to\frac{\pi}{2}}\frac{cot\,x}{-cos\,e\,c^2x}=\frac{0}{1}$$

$$log\,k=0\Rightarrow k=e^0$$

$$k=1$$

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EXERCISE

1. Evaluate $\lim\limits_{x\to 0}\left(\frac{sinx}{x}\right)^{\frac{1}{x^2}}$
2. Evaluate $\lim\limits_{x\to 0}\left(\frac{tax}{x}\right)^{\frac{1}{x^2}}$
3. Evaluate $\lim\limits_{x\to 0}(secx)^{cotx}$
4. Evaluate $\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{\frac{1}{x}}$
5. Evaluate $\lim\limits_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$
6. Evaluate $\lim\limits_{x\to 0} cosx^{cot^2x}$
7. Evaluate $\lim\limits_{x\to 0}(a^x+x)^{\frac{1}{x}}$
8. Evaluate $\lim\limits_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}}$
9. Evaluate $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{2sinx}$
10. Evaluate $\lim\limits_{x\to 0}(cotx)^{tanx}$
11. Evaluate $\lim\limits_{x\to \infty}\left(\frac{\pi}{2}-tan^{-1}x\right)^{\frac{1}{x}}$

PARTIAL DIFFERENTIATION

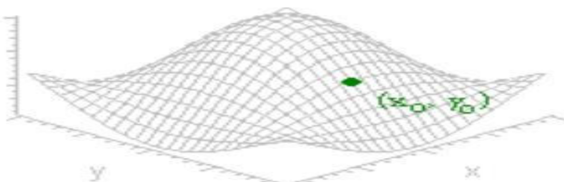
INTRODUCTION

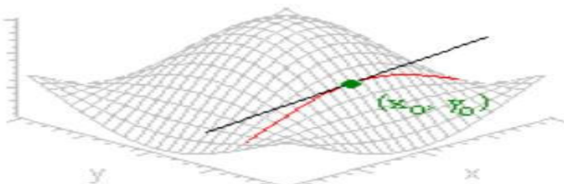
Partial differential equations abound in all branches of science and engineering and many areas of business. The number of applications is endless. Partial derivatives have many important uses in math and science.

So far, we had been dealing with functions of a single independent variable. We will now consider functions which depend on more than one independent variable. Such functions are called functions of several variables.

GEOMETRICAL MEANING :

Suppose the graph of $z = f(x, y)$ is the surface shown. Consider the partial Derivative of f with respect to x at a point (x_0, y_0). Holding y constant and varying x, we trace out a curve that is the intersection of the surface with the vertical plane $y = y_0$. The partial derivative $f_x(x_0, y_0)$ measures the change in z per unit increase in x along this curve. That is, $f_x(x_0, y_0)$ is just the slope of the curve at (x_0, y_0). The geometrical interpretation of $f_y(x_0, y_0)$ is analogous.





NEED FOR PARTIAL DIFFERENTIAION

- In studying a real-world phenomenon, a quantity being investigated usually depends on two or more independent variables.
- So we need to extend the basic ideas of the calculus of functions of a single variable to functions of several variables.
- Although the calculus rules remain essentially the same, the calculus is even richer. The derivatives of functions of several variables are more varied and more interesting because of the different ways in which the variables can interact. Their integrals lead to a greater variety of applications.
- The studies of probability, statistics, fluid dynamics, and electricity, to mention only a few, all lead in natural ways to functions of more than one variable.

In engineering, it sometimes happens that the variation of one quantity depends on changes taking place in two, or more, other quantities. For example, the volume V of a cylinder is given by $V = \pi r^2 h$. The volume will change if either radius r or height h is changed. The formula for volume may be stated mathematically as $V = f\,(r,\,h)$ which means V is some function of r and h. Some other practical examples include:

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- **Time of oscillation,** $t = 2\pi\sqrt{\frac{l}{g}}$ **i.e.** $t = f(l, g)$.
- **Torque** $T = I\alpha$, **i.e.** $T = f(I, \alpha)$.
- **Pressure of an ideal gas** $p = \frac{mRT}{V}$ **i.e.** $p = f(T, V)$.

When differentiating a function having two variables, one variable is kept constant and the differential coefficient of the other variable is found with respect to that variable. The differential coefficient obtained is called a partial derivative of the function

REAL-WORLD APPLICATIONS:

- **SHAPE PROCESSING USING PDE:**

Shape processing refers to operations such as denoising, fairing, feature extraction, segmentation, simplification, classification, and editing. Such operations are the basic building blocks of many applications in computer graphics, animation, computer vision, and shape retrieval. Many shape processing operations can be achieved by means of partial differential equationsor PDEs. The desired operation is described as a (set of) PDE(s) that act on surface information, such as area, normals, curvature, and similar quantities.

- **PDES ARE A VERY ATTRACTIVE INSTRUMENT:** They allow complex manipulations to be described precisely, compactly, and measurably, and come with efficient and effective numerical methods for solving them.
- **PARTIAL DERIVATIVE IN ECONOMICS:**

In economics the demand of quantity and quantity supplied are affected by several factors such as selling price, consumer buying power and taxation which means there are multi variable factors that affect the demand and supply. In economics marginal analysis is used to find out or evaluate the change in value of a function resulting from 1-unit increase in one of its variables. For example Partial derivative is used in marginal Demand to obtain condition for determining whether two goods are substitute or complementary. Two goods are said to be substitute goods if an increase in the demand for either result in a decrease for the other. While two goods are said to be complementary goods if a decrease of either result in a decrease in the demand.

For example complementary goods are mobile phones and phone lines. If there is more demand for mobile phone, it will lead to more demand for phone line too.

- **PARTIAL DERIVATIVE IN ENGINEERING:**

In image processing edge detection algorithm is used which uses partial derivatives to improve edge detection. Grayscale digital images can be considered as $2D$ sampled points of a graph of a function $u(x, y)$ where the domain of the function is the area of the image.

Partial Derivatives are used in basic laws of Physics for example Newton's Law of Linear Motion, Maxwell's equations of Electromagnetism and Einstein's equation in General Relativity.

- **In Chemistry**. One use of derivatives in chemistry is when you want to find the concentration of an element in a product.
- PDEs are used to model many systems in many different fields of science and engineering. For Example:
- **Laplace Equation:** It is used to describe the steady state distribution of heat in a body. Also used to describe the steady state distribution of electrical charge in a body.

$$\frac{\partial^2 u(x,\ y,\ z)}{\partial x^2}+\frac{\partial^2 u(x,\ y,\ z)}{\partial y^2}+\frac{\partial^2 u(x,\ y,\ z)}{\partial z^2}=0$$

- **Heat Equation :** The function $u(x, y, z, t)$ is used to represent the temperature at time t in a physical body at a point with coordinates (x, y, z). If $\boldsymbol{\alpha}$ is the thermal diffusivity.

$$\frac{\partial u(x,y,z,t)}{\partial t}=\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

It is sufficient to consider the case $\boldsymbol{\alpha} = 1$.

- **Wave Equation :** The function $u(x, y, z, t)$ is used to represent the displacement at time 't' of a particle whose position at rest is (x, y, z).

$$\frac{\partial^2 u(x,y,z,t)}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

The constant c represents the propagation speed of the wave.

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WHAT IS PARTIAL DIFFERENTIATION?

A partial differential equation is an equation involving a function of two or more variables and some of its partial derivatives. Therefore, a partial differential equation contains one dependent variable and more than one independent variable.

i.e. Let $Z = f(x, y)$ be a function of two independent variable x and y

And z will be taken as the dependent variable. Then we will use the following standard notations to denote the partial derivatives.

$$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q,\ \frac{\partial^2 z}{\partial x^2} = r,\ \frac{\partial^2 z}{\partial x \partial y} = s,\ \frac{\partial^2 z}{\partial y^2} = t$$

PARTIAL DERIVATIVES OF FIRST ORDER

Let $z = f\,(x, y)$ is a function of two variables. The first order partial derivative of z or f with respect to x at a point (x, y) is

$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x,\ y) - f(x,\ y)}{\delta x}$ *provided the limit exists.*

$\Rightarrow \frac{\partial z}{\partial x}$ is the first order partial derivative of z with respect to x, treating y as constant.

It is denoted by $\frac{\partial z}{\partial x},\ z_x,\ \frac{\partial f}{\partial x},\ f_x,\ D_x f$ **or** p.

Similarly Let $z = f\,(x, y)$ is a function of two variables. The first order partial derivative of z or f with respect to y at a point (x, y) is

$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x,\ y+\delta y) - f(x,\ y)}{\delta y}$ *provided the limit exists.*

$\Rightarrow \frac{\partial z}{\partial y}$ is the first order partial derivative of z with respect to y, treating x as constant.

It is denoted by $\frac{\partial z}{\partial y},\ z_y,\ \frac{\partial f}{\partial y},\ f_y,\ D_y f$ **or** q.

PARTIAL DERIVATIVES OF HIGHER ORDER

Each of the first order partial derivatives being functions of x and y they can be further differentiated partially with respect to both x and y resulting in second order partial derivatives.

i.e. Let $z = f(x, y)$ is a function of two variables. Then $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are also functions of two variables and their partials can be taken. Hence we can differentiate with respect to x and y again to find second order and higher order partial derivatives. They are

- $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x^2}$ or z_{xx} or f_{xx} or $r \Rightarrow 2nd$ derivative of z w.r.t. x
- $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$ or $\frac{\partial^2 f}{\partial x \partial y}$ or z_{xy} or f_{xy} or $s \Rightarrow 2nd$ derivative of z w.r.t. y and x
- $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$ or $\frac{\partial^2 f}{\partial y \partial x}$ or z_{yx} or f_{yx} or $t \Rightarrow 2nd$ derivative of z w.r.t. x and y
- $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$ or $\frac{\partial^2 f}{\partial y^2}$ or z_{yy} or f_{yy} or $t \Rightarrow 2nd$ derivative of z w.r.t. y

The third and higher order partial derivatives of $f(x, y)$ are defined in an analogous

it can be verified that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

All the rules of differentiation applicable to functions of a single independent variable are applicable in partial differentiation also, the only difference is while differentiating partially with respect to one independent variable all other independent variables are treated as constants.

TOTAL DERIVATIVES

Let $z = f(x, y)$ be a differentiable function of two variables x and y, then **Total differential (or Exact differential)** $\boldsymbol{dz}$ is defined by

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \qquad (1)$$

Further, if $z = f(x, y)$ where $x = x(t)$, $y = y(t)$ i.e. x and y are themselves functions of an independent variable 't', then total **derivative of** z is given by

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad (2)$$

Similarly, the total differential of a function $u = f(x, y, z)$ is defined by

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{3}$$

Further, if $u = f(x, y, z)$ and if $x = x(t)$, $y = y(t)$, $z = z(t)$, then the total derivative of u is given by $\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$. (4)

DIFFERENTIATION OF COMPOSITE FUNCTIONS:

Let $z = f(x, y)$ and $x = \phi(u, v)$ and $y = \varphi(u, v)$ are functions of u and v then,

i.e, $z = f[x(u, v), y(u, v)]$ then

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

Similarly, if $z = f(u, v)$ are functions of u and v and if $u = \phi(x, y)$ and $y = \varphi(x, y)$ are functions of x and y then,

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \quad \& \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}.$$

NOTE:

1) The above formulae can be extended to functions of three or more variables and expressions are called Chain rule for partial differentiation.
2) The second and higher order partial derivatives of $z = f(x, y)$ can be obtained by repeated applications of the above expressions.

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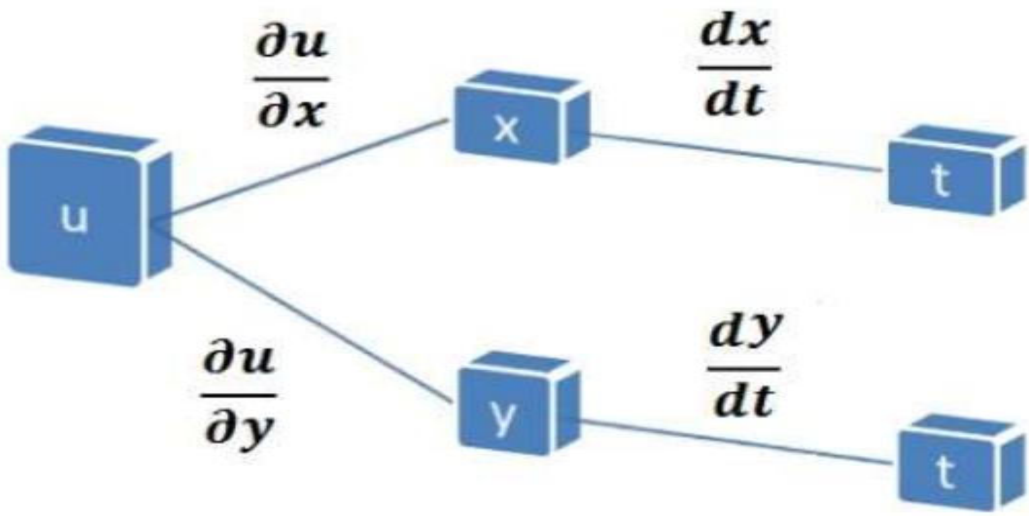
DEPARTMENT OF MATHEMATICS

Total Derivative and Chain rule:



1. Find $\frac{du}{dt}$ if $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}cos3t$, $z = e^{2t}sin3t$.

Solution: $u \longrightarrow (x, y, z) \longrightarrow t$

$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$

Given $u = x^2 + y^2 + z^2$

$\frac{\partial u}{\partial x} = 2x \Longrightarrow \frac{\partial u}{\partial x} = 2e^{2t}$, $\quad \frac{\partial u}{\partial y} = 2y \Longrightarrow \frac{\partial u}{\partial y} = 2e^{2t}cos3t$, $\quad \frac{\partial u}{\partial z} = 2z \Longrightarrow \frac{\partial u}{\partial z} = 2e^{2t}sin3t$

$\frac{dx}{dt} = 2e^{2t}$, $\quad \frac{dy}{dt} = 2e^{2t}cos3t - 3e^{2t}sin3t$, $\quad \frac{dz}{dt} = 2e^{2t}sin3t + 3e^{2t}cos3t$

$\therefore \frac{du}{dt} = 2e^{2t}(2e^{2t}) + 2e^{2t}cos3t(2e^{2t}cos3t - 3e^{2t}sin3t) + 2e^{2t}sin3t((2e^{2t}sin3t + 3e^{2t}cos3t)$

$\frac{du}{dt} = 4e^{4t} + 2e^{4t}(2\cos^2 3t - 3cos3t \sin3t) + 2e^{4t}(2\sin^2 3t + 3cos3t \sin3t)$

$\frac{du}{dt} = 4e^{4t} + 2e^{4t}(2\cos^2 3t - 3cos3t \sin3t + 2\sin^2 3t + 3cos3t \sin3t)$

$\frac{du}{dt} = 4e^{4t} + 4e^{4t}$

$\frac{du}{dt} = 8e^{4t}$

2. If $z = f(x, y)$, $x = u - v$ and $y = uv$, Prove the following

(a) $(u+v)\frac{\partial z}{\partial x} = u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v}$ (b) $(u+v)\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$

$z \longrightarrow (x, y) \longrightarrow (u, v)$

Given $x = u - v \Longrightarrow \frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = -1$

$y = uv \Longrightarrow \frac{\partial y}{\partial u} = v, \quad \frac{\partial y}{\partial v} = u$

Now $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \Longrightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \Longrightarrow \frac{\partial z}{\partial u} = -\frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y}$

(a) Consider $u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = u\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y} + v\frac{\partial z}{\partial x} - uv\frac{\partial z}{\partial y}$

$\Longrightarrow u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial x} \Longrightarrow \boldsymbol{u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = (u+v)\frac{\partial z}{\partial x}}$

(b) Now $\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} - \frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y}$

$\Longrightarrow \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = v\frac{\partial z}{\partial y} + u\frac{\partial z}{\partial y} \Longrightarrow (u+v)\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$

3. Find the total derivative of the function $z = xy^2 + x^2y$ where $x = at \; and \; y = 2at$

Solution:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$\frac{\partial z}{\partial x} = y^2 + 2xy \quad \frac{\partial z}{\partial y} = 2xy + x^2 \; and \quad \frac{dx}{dt} = a \quad \frac{dy}{dt} = 2a$

Therefore

$\frac{dz}{dt} = (y^2 + 2xy).a + \quad (2xy + x^2).2a$

$=((2at)^2 + 2(at)(2at)).a + (2(at)(2at) + (at)^2).2a$

$= 8\,a^3t^2 + 10\,a^3t^2$

$= 18a^3t^2$

4. If $\boldsymbol{z = f(x, y)}$ where $\boldsymbol{x = e^u + e^{-v}}$ & $\boldsymbol{y = e^{-u} - e^v}$ then $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Solution: $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u) + \frac{\partial z}{\partial y}(-e^{-u})$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{v}) + \frac{\partial z}{\partial y}(-e^{-v})$$

subtract the above equations we get,

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^{u} + e^{-v})\left(\frac{\partial z}{\partial x}\right) - (e^{-u} - e^{v})\left(\frac{\partial z}{\partial y}\right)$$

Therefore,

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x.\frac{\partial z}{\partial x} - y.\frac{\partial z}{\partial y}$$

5. If $\boldsymbol{u} = f(x-y, y-z, z-x)$ *show that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$=0

Solution: Here u is a composite function of x, y, z

Given, $r = x - y$, $s = y - z$, $t = z - x$

then $\boldsymbol{u} = f(x-y, y-z, z-x)$=f(r, s, t)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-1)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad \rightarrow 1$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \quad \rightarrow 2$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1)$$

$$\Rightarrow \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \quad \rightarrow 3$$

$(1) + (2) + (3)$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0$$

6. If $u = f(\frac{y-x}{xy}, \frac{z-x}{xz})$ then prove that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Solution:

Let $r = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$ $\qquad s = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left[\frac{(xy)(-1)-(y-x)(y)}{(xy)^2}\right] + \frac{\partial u}{\partial s}\left[\frac{(xz)(-1)-(z-x)z}{(xz)^2}\right]$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left[-\frac{1}{x^2}\right] + \frac{\partial u}{\partial s}\left[-\frac{1}{x^2}\right] \quad \rightarrow 1$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left[\frac{1}{y^2}\right] + \frac{\partial u}{\partial s}[0]$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left[\frac{1}{y^2}\right] \quad \rightarrow 2$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left[\frac{1}{z^2}\right]$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\left[\frac{1}{z^2}\right] \quad \rightarrow 3$$

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = x^2\left(\frac{\partial u}{\partial r}\left[-\frac{1}{x^2}\right] + \frac{\partial u}{\partial s}\left[-\frac{1}{x^2}\right]\right) + y^2\left(\frac{\partial u}{\partial r}\left[\frac{1}{y^2}\right]\right) + z^2\left(\frac{\partial u}{\partial s}\left[\frac{1}{z^2}\right]\right) = 0$$

EXERCISE:

1 If $u = x^3y^2 + x^2y^3$, $x = at^2$, $y = 2at$, then find $\frac{du}{dt}$.

2 If $u = tan^{-1}\left(\frac{x}{y}\right)$ where $x = e^t - e^{-t}$, $y = e^t + e^{-t}$ then find $\frac{du}{dt}$.

3 If $u = x^2 + y^2 + z^2$ where $x = e^{2t}$, $y = e^{2t}cos2t$, $z = e^{2t}sin2t$ find $\frac{du}{dt}$

4 If $u = e^x \sin(yz)$, where $x = t^2$, $y = t - 1$, $z = \frac{1}{t}$ find $\frac{du}{dt}$ at $t = 1$.

JACOBIANS

Changing variable is something we come across very often in Integration. There are many reasons for changing variables but the main reason for changing variables is to convert the integrand into something simpler and also to transform the region into another region which is easy to work with. When we convert into a new set of variables it is not always easy to find the limits. So, before we move into changing variables with multiple integrals we first need to see how the region may change with a change of variables. In order to change variables in an integration we will need the Jacobian of the transformation.

Jacobians were invented by German mathematician C.G. Jacob Jacobi (1804- 1851),who made significant contributions to mechanics, Partial differential equations and calculus of variations.

DEFINITION:

If u and v are functions of the two independent variables x and y, then the determinant

$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of $u,\ v$ with respect to x, y and is written as

$\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$ or $J(u,\ v)$

Similarly if $u,\ v$ and w be the functions of three independent variables x, y and z, then

$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$ is called the Jacobian of $u,\ v,\ w$ with respect to $x, y,\ z$ and is written as

$\frac{\partial(u,v,w)}{\partial(x,y,z)}$ or $J\left(\frac{u,v,\mathrm{w}}{x,y,z}\right)$ or $J(u,\ v,\ w)$

EXAMPLES

1. If $x = rsin\theta\cos\varphi,\ \ y = rsin\theta\sin\varphi,\ \ z = rcos\theta$. Find $J(x,\ y,\ z)$

Given $x = rsin\theta\cos\varphi \qquad \mathrm{y} = rsin\theta\sin\varphi \qquad z = rcos\theta$

$$J = J\left(\frac{x,y,z}{r,\theta,\varphi}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} \Rightarrow J = \begin{vmatrix} sin\theta\cos\varphi & rcos\theta\cos\varphi & -rsin\theta\sin\varphi \\ sin\theta\sin\varphi & rcos\theta\sin\varphi & rsin\theta\cos\varphi \\ cos\theta & -rsin\theta & 0 \end{vmatrix}$$

On Expanding

$$\begin{aligned} J &= sin\theta\cos\varphi\,(r^2\sin^2\theta\cos\varphi) \\ &\quad - rcos\theta\cos\varphi\,(-rsin\theta cos\theta\cos\varphi) \\ &\quad - rsin\theta\sin\varphi\,(-rsin^2\theta\sin\varphi - rcos^2\theta\sin\varphi) \\ &= r^2\sin^3\theta\cos^2\varphi + r^2\cos^2\theta\, sin\theta\cos^2\varphi \\ &\quad + r^2\sin^3\theta\sin^2\varphi + r^2\cos^2\theta\, sin\theta\cos^2\varphi \end{aligned}$$

$= r^2\sin^3\theta\,(\cos^2\varphi + \sin^2\varphi) + r^2\cos^2\theta\, sin\theta\,(\cos^2\varphi + sin^2\varphi)$

$= r^2\sin^3\theta + r^2\cos^2\theta\, sin\theta$

$= r^2 sin\theta(\sin^2\theta + \cos^2\theta)$

$= r^2 sin\theta$

2. If $u + v = e^x cosy, \quad u - v = e^x siny$, find $J\left(\frac{u,v}{x,y}\right)$.

Given $u + v = e^x cosy$,..........(1) $\quad u - v = e^x siny$........(2)

Adding (1) and (2), we get $u = \frac{e^x}{2}(cosy + siny)$

$(1) - (2)$ gives $v = \frac{e^x}{2}(cosy - siny)$

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}\Rightarrow J=\begin{vmatrix}\frac{e^x}{2}(cosy+siny) & \frac{e^x}{2}(-siny+cosy)\\ \frac{e^x}{2}(cosy-siny) & \frac{e^x}{2}(-siny-cosy)\end{vmatrix}$$

$$J=-\frac{e^{2x}}{4}(cosy+siny)^2-\frac{e^{2x}}{4}(cosy-siny)^2$$

$$J=-\frac{e^{2x}}{4}(\cos^2 y+\sin^2 y+2cosysiny+\cos^2 y+\sin^2 y-2cosysiny)$$

$$J=-\frac{e^{2x}}{4}(1+1)$$

$$J=-\frac{e^{2x}}{2}$$

3. If $u=\frac{yz}{x}$, $v=\frac{zx}{y}$, $w=\frac{xy}{z}$, prove that $\frac{\partial(u,v,w)}{\partial(x,y,z)}=4$.

Given $u=\frac{yz}{x}$, $v=\frac{zx}{y}$, $w=\frac{xy}{z}$

$$J=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}\Rightarrow J=\begin{vmatrix}-\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x}\\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y}\\ \frac{x}{z} & \frac{y}{z} & -\frac{xy}{z^2}\end{vmatrix}$$

Taking common factor $\frac{1}{x^2},\frac{1}{y^2},\frac{1}{z^2}$ from R_1, R_2 and R_3 respectively

$$J=\frac{1}{x^2}\cdot\frac{1}{y^2}\cdot\frac{1}{z^2}\begin{vmatrix}-yz & zx & xy\\ zy & -xz & xy\\ yz & xz & -xy\end{vmatrix}$$

Taking common factor $yz,\ zx,\ xy$ from C_1, C_2 and C_3 respectively

$$J=\frac{yz.zx.xy}{x^2y^2z^2}\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}\Rightarrow J=-1(1-1)-1(-1-1)+1(1+1)\Rightarrow \boldsymbol{J=4}$$

4. If $x=uv$, $y=\frac{u}{v}$, then find J

Given $x=uv$, ……. (1) $\quad y=\frac{u}{v}$…………(2)

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\Rightarrow J=\begin{vmatrix}v & u\\ \frac{1}{v} & -\frac{u}{v^2}\end{vmatrix}\Rightarrow J=v\left(-\frac{u}{v^2}\right)-\frac{u}{v}\Rightarrow J=-2\frac{u}{v}$$

5. If x=$\rho cos\emptyset$, $y=\rho sin\emptyset$ $and\ z=z\ find\ \frac{\partial(x,y,z)}{\partial(\rho,\emptyset,z)}$

Solution:

$$\frac{\partial(x,y,z)}{\partial(\rho,\emptyset,z)} = \begin{vmatrix} cos\emptyset & -\rho sin\emptyset & 0 \\ sin\emptyset & \rho cos\emptyset & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= Cos\emptyset(\rho cos\emptyset) + \rho sin\emptyset(sin\emptyset)$$

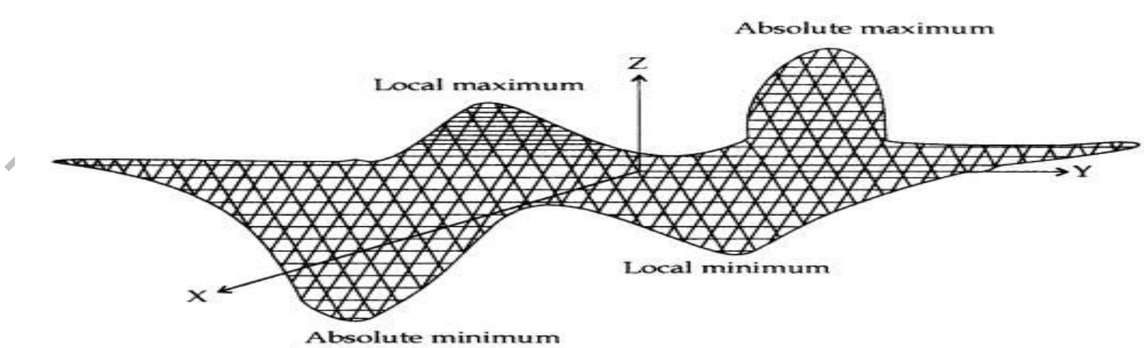
$$= \rho$$

EXERCISE

1. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = xy + yz + zx$ find $J\left(\frac{u,v,w}{x,y,z}\right)$.
2. If $x = e^v secu,\ \ y = e^v tanu$ find $\frac{\partial(x,y)}{\partial(u,v)}$.
3. If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$ prove that $J\left(\frac{u,v,w}{x,y,z}\right) = 20$ at $(1,-1,0)$.

MAXIMA AND MINIMA OF FUNCTION OF TWO VARIABLE

There are many practical situations in which it is necessary or useful to know the largest and smallest values of a function of two variables. For example, if we consider the plot of a function $f(x,\ y)$ of two variables to look like a mountain range, then the mountain tops, or the high points in their immediate vicinity, are called local maxima of $f(x,\ y)$ and the valley bottom, or the low points in their immediate vicinity, are called local minima of $f(x,\ y)$. The highest mountain and deepest valley in the entire mountain range are known as the absolute maxima and the absolute minimum respectively.



A function f of two variables has a local maximum at $(a,\ b)$ if $f(x,\ y)\ \leq f(a,\ b)$ when $(x,\ y)$ is in neighbourhood of (a, b). The number $f(a,\ b)$ is called local maximum value of f. If $f(x,\ y)\ \leq\ f(a,\ b)$ for all points $(x,\ y)$ in the domain of the function f, then the function has its absolute maximum at (a, b) and $f(a, b)$ is the absolute maximum value of f.

If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b) then $f(a, b)$ is the local minimum value of f. If $f(x, y) \geq f(a, b)$ for all points (x, y) in the domain of f then f has its absolute minimum value at (a, b).

A point (a, b) in the domain of a function $f(x, y)$ is called a critical point (or stationary point) of $f(x, y)$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or if one or both partial derivatives do not exist at (a, b).

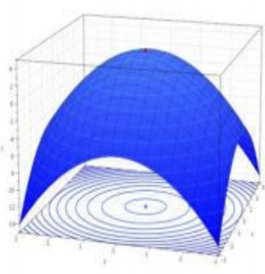
A point (a, b) where $f(x, y)$ has neither a maximum nor a minimum is called a saddle point to $f(x, y)$.

Let the sign of $f(a + h, b + k) - f(a, b)$ remain of the same for all values (positive or negative) of h, k. Then we have

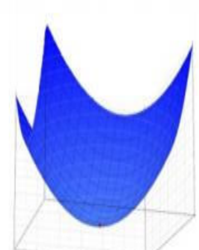
(i) For $f(a + h, b + k) - f(a, b) < 0$, $f(a, b)$ is maximum.
i.e. A function $f(x, y)$ is said to have a Maximum value at (a, b) if their exists a neighborhood point of (a, b) $(say\ (a + h, b + k))$ such that $f(a, b) > f(a + h, b + k)$.

(ii) For $f(a + h, b + k) - f(a, b) > 0$, $f(a, b)$ is minimum.
i.e. Minimum value at (a, b) if there exists a neighborhood point of (a, b) $(say\ (a + h, b + k))$ such that $f(a, b) < f(a + h, b + k)$.

A Maximum point on the graph is at the top (in red)



A Minimum point on the graph (in red) $f(x, y) = x^2 + y^2(1-x)^3$



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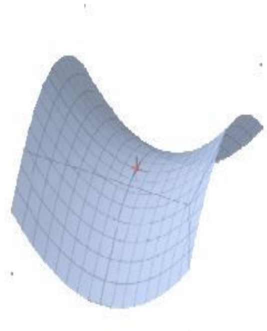
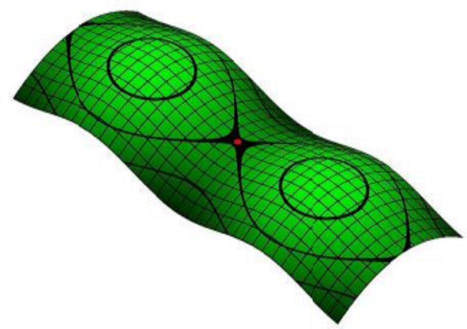
A saddle point on the graph of $z=x^2-y^2$ (in red)



Saddle point between two hills.



The problem of determining the maximum or minimum of a function is encountered in geometry, mechanics, physics, and other fields, and was one of the motivating factors in the development of the calculus in the seventeenth century.

A function of two variables can be written in the form $z = f(x, y)$. A critical point is a point (a, b) such that the two partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ are zero at the point (a, b). A relative maximum or a relative minimum occurs at a critical point.

A critical point is a maximum if the value of f at that point is greater than its value at all its sufficiently close neighboring points.

A critical point is a minimum if the value of f at that point is less than its value at all its sufficiently close neighboring points.

A critical point is a saddle point if the value of f at that point is greater than its value at some neighboring point and if the value of f at that point is less than its value at some other neighboring point. Saddle point is a point which is neither a maximum nor a minimum

MAIMA AND MINIMA FOR A FUNCTION OF TWO VARIABLES

Working procedure:

- If f is a function of two variables then find the respective derivatives $\boldsymbol{f_x}$ **and** $\boldsymbol{f_y}$
- Find the critical points of (a,b) i.e $\boldsymbol{f_x = f_y = 0}$
- Find the derivative of second order $\boldsymbol{A = f_{xx}}$, $\boldsymbol{B = f_{xy}}$, $\boldsymbol{C = f_{yy}}$
- If $\boldsymbol{AC - B^2 > O}$ & $\boldsymbol{A > 0}$ ***at*** $(\boldsymbol{a}, \boldsymbol{b})$ then $\boldsymbol{f}$ is minimum

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- If $AC - B^2 > O$ & $A < 0$ at (a, b) then f is maximum
- If $AC - B^2 < O$ then f is neither maximum nor minimum and is called as saddle point.

EXAMPLES:

1. Find the extreme value of $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

Solution: $f_x = 3x^2 + 3y^2 - 6x$

$f_y = 6xy - 6y$

$fxx = 6x - 6 = A$

$6x - 6 = fyy = C$

$fxy = 6y = B$

Now we find the critical points, $f_x = 0$ *and* $f_y = 0$

$3x^2 + 3y^2 - 6x = 0$ and $6xy$ - $6y = 0$

Put $y=0$ in the above eqn,

$3x^2 - 6x = 0$

$3x(x-2)=0 \Rightarrow x = 0$ & $x - 2 = 0$

Therefore, x=0,2 and the points are (0,0) (2,0)

Again put $x=1$ we get, $3 + 3y^2 - 6 = 0 \Rightarrow y = \pm 1$

Therefore the points are (1,1) (1,-1)

The total critical points are, (0,0)(2,0)(1,1)(1,-1)

$Points$	$(0,0)$	$(2,0)$	$(1,1)$	$(1,-1)$
$fxx = 6x - 6$	$-6 < 0$	$6 > 0$	0	0
$fxy = 6y$	0	0	6	-6
$fyy = 6x - 6$	$-6 < 0$	$6 > 0$	0	0
$AC - B2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 > 0$
$Conclusion$	$Maxima$	$Minima$	$Saddle$	$saddle$

For maxima and minimum value,

$f(x, y) = x3 + 3xy2 - 3x2 - 3y2 + 4$

$f(0,0) = 4$ and $f(2,0) = 0$

2. Divide the number 24 into 3 parts such that product may be maximum.

Sol: Let (x, y, z) be the 3 parts of 24.

$x + y + z = 24$

Now, the product of three numbers is $f = xyz$

From the above equation, $f = xy(24 - x - y)$

$f = 24xy - x^2 y - xy^2$

$f_x = 24y - 2xy - y^2$ $\qquad$ $fy = 24x - x^2 - 2xy$

$A = f_{xx} = -2y$ & $C = f_{yy} = -2x$

$B = f_{xy} = 24 - 2x - 2y$

To find the critical points, $f_x = 0$ and $f_y = 0$

The points are (0,0) & (8,8)

Points	(0,0)	(8,8)
$A = -2y$	0	$-16 < 0$
$B = 24 - 2x - 2y$	$24 > 0$	$-8 < 0$
$C = -2x$	0	$-16 < 0$
$AC - B2$	$-576 < 0$	$192 > 0$
Conclusion	saddle	maxima

Therefore, f is maximum at (8,8)

i.e $x = y = z = 8$

3. Find extreme value of $x^3 + xy^2 + 21x - 12x^2 - 2y^2$.

$fx = 3x^2 + y^2 + 21 - 24x$

$fy = 2xy - 4y$

$A = fxx = 6x - 24$

$B = fxy = 2y$

$C = fyy = 2x - 4$

For extreme value $fx = 0$ and $fy = 0$.

$3x^2 + y^2 + 21 - 24x = 0$ ----- 2

$2xy - 4y = 0$ ----- 3

From 3,

$2xy - 4y = 0$

$xy - 2y = 0$

$Therefore\ y(x-2) = 0$

$Y = 0\ and\ x = 2$

$At\ x = 2, \qquad \Rightarrow\ 12 + y^2 + 21 - 48 = 0$

$y = 0, \qquad y^2 = 15 \Rightarrow y = \pm\sqrt{15}$

$3x^2 + 21 - 24x = 0$

$\Rightarrow x = 1,7$

Critical points are $(1,0)(7,0)(2,\sqrt{15})(2,-\sqrt{15})$

	$(1,0)$	$(7,0)$	$(2,\sqrt{15})$	$(2,-\sqrt{15})$
A	$-18 < 0$	$18 > 0$	$-12 < 0$	$-12 < 0$
B	0	0	$2\sqrt{15}$	$-2\sqrt{15}$
C	-2	10	0	0
AC-B^2	$36 < 0$	$180 > 0$	$-60 < 0$	$-60 < 0$
Conclusion	Max	Min	Saddle point	Saddle point

At $(1,0)$ the max value $f(1,0) = 10$

At $(7,0)$ the min value $f(7,0) = -9$

4. Find the extreme value for the function $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 - 4$

$fx = 6xy - 6y$

$f_{xx} = 6x - 6 = A$

$f_{yy} = 6x - 6 = C$

$f_{xy} = 6y = B$

Now we find the critical points, $f_x = 0$ and $f_y = 0$

$3x^2 + 3y^2 - 6x = 0$ and $6xy - 6y = 0$. Put $y = 0$ in the above equation,

$3x^2 - 6x = 0 \;=>\; 3x(x - 2) = 0$

$x = 0$ and $x - 2 = 0 \;=>\; x = 0,2$ and the points are $(0,0)(2,0)$

Again put $x = 1$ we get, $3 + 3y2 - 6 = 0$

$y = \pm 1$. Therefore the points are $(1,1)$ $(1,-1)$

The total critical points are $(0,0)(2,0)(1,1)(1,-1)$

$Points$	$(0,0)$	$(2,0)$	$(1,1)$	$(1,-1)$
$f_{xx} = 6x$	$-6 < 0$	$6 > 0$	0	0
$f_{xy} = 6y$	0	0	6	-6
$f_{yy} = 6x$	$-6 < 0$	$6 > 0$	0	0
$AC - B2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 > 0$
$Conclusion$	$Maxima$	$Minima$	$Saddle\ point$	$Saddle\ point$

For maxima and minima value: $f(x,y) = x3 + 3xy2 - 3x2 - 3y2 - 4$

$$f(0,0) = 4 \; and \; f(2,0) = 0$$

EXERCISE

MANGALORE INSTITUTE OF TECHNOLOGY & ENGINEERING
Accredited by NAAC with A+ Grade, An ISO 9001: 2015 Certified Institution
(A Unit of Rajalaxmi Education Trust®, Mangalore - 575001)
Affiliated to V.T.U., Belagavi, Approved by AICTE, New Delhi.

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1. Discuss the maxima and minima of $f(x,y) = x^3y^2(1-x-y)$.
2. Find the maximum and minimum value of $sinx\ siny\ \sin(x+y)$.
3. Examine the function $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ for extreme values.
4. Find the maximum and minimum value of $1 + \sin(x^2 + y^2)$. A rectangular box open at the top is to have volume of 32 cubic ft. find the dimensions of the box requiring least material of its construction.
5. Find the maximum and minimum distances of the point $(3,4,12)$ from the sphere $x^2 + y^2 + z^2 = 4$. The temperature T at any point (x,y,z) in space is $T = 400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.
6. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $ax + by + cz = p$
7. Find the maximum of $x^2y^2z^2$ subject to the condition $x^2 + y^2 + z^2 = a^2$.
8. Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.