MANGALORE INSTITUTE OF TECHNOLOGY & ENGINEERING
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Affiliated to V.T.U., Belagavi, Approved by AICTE, New Delhi.

DEPARTMENT OF MATHEMATICS

MODULE -3

ORDINARY DIFFERENTIAL EQUATIONS (ODES) OF FIRST ORDER

CONTENTS:

Introduction to first-order ordinary differential equations pertaining to the applications for EC & EE engineering.

- Linear and Bernoulli's differential equations
- Exact and reducible to exact differential equations – Integrating factors on $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ and $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$.
- Orthogonal trajectories, L-R and C-R circuits.
- Non-linear differential equations:
  - Introduction to general and singular solutions
  - Solvable for p only
  - Clairaut's equations, reducible to Clairaut's equations.

Self-study: Applications of ODEs, Solvable for x and y

Applications of ordinary differential equations: Rate of Growth or Decay, Conduction of heat.

(RBT Levels: L1, L2 and L3)

LEARNING OBJECTIVES:

After Completion of this module, student will be able to:

- Solve Linear, Bernoulli's, Exact and reducible to exact differential equations
- Find orthogonal trajectories of differential equations
- Solve problems that arise in L-R and C-R circuits
- Understand general and singular solutions and solve Non-Linear differential equations – Solvable for P, Clairaut's equation, & reducible to Clairaut's equation

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INTRODUCTION TO DIFFERENTIAL EQUATION

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms, the relations are equations and the rates are derivatives. Equations containing derivatives are differential equations. Differential equations arise in an attempt to describe physical phenomena in mathematical terms. A single differential equation can serve as a mathematical model for many different processes. Therefore, to understand and to investigate problems involving the motion of fluids, the flow of current in electric circuits, the dissipation of heat in solid objects, the propagation and detection of seismic waves, or the increase or decrease of populations, mixing problems, draining tank/Torricelli's Law problems, projectile motion, Newton's Law of Cooling, orthogonal trajectories, melting snowball type problems, certain basic circuits. among many others, it is necessary to know something about differential equations. A differential equation that describes some physical process is often called a Mathematical model of the process.

DIFFERENTIAL EQUATION:

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be differential equation.

The ‘order’ of a differential equation is the order of the highest derivative in the equation and the power of highest order derivative after clearing fractional powers from derivatives in the equation gives the ‘degree’ of differential equation.

CLASSIFICATION OF DIFFERENTIAL EQUATION:

Differential equations are classified as:

i) Ordinary differential equation (ODE)

ii) Partial differential equations (PDE)

If the differential equation contains ordinary derivatives of one or more dependent variables with respect to single independent variable, equation is known as ODE.

Examples:

i) $\ddot{x} = -n^2 x$

ii) $\frac{[1+(y')^2]^{3/2}}{y''} = a$

If the differential equation contains partial derivatives of two or more independent variables, equation is known as PDE.

Examples:

i) $u_{xx} + u_{yy} = 0$

ii) $c^2 u_{xx} = u_t$

SOLUTION FOR 1^{st} ORDER DIFFERENTIAL EQUATIONS

The General form of the 1^{st} order and 1^{st} degree of differential equation is $\frac{dy}{dx} = b(x,y)$ and which can be solved by various method.

(1) LINEAR AND BERNOULLI'S DIFFERENTIAL EQUATIONS

A first-order ODE is said to be linear if it can be brought to the form

$$\frac{dy}{dx} + P(x)y = Q(x)\ (Linear\ in\ y)$$

The function $Q(x)$ on the right may be a force, and the solution $y(x)$ a displacement in a motion or an electrical current or some other physical quantity. In engineering, $Q(x)$ is frequently called the input, and $y(x)$ is called the output or the response to the input. The general solution is given by

$$y \times IF = \int Q(x) \times IF dx + C \text{ where } IF = e^{\int P(x)dx}$$

In a similar manner Linear equation in x can be solved.

A first-order ODE is said to be Bernoulli's differential equation if it can be brought to the form

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$$\frac{dy}{dx} + P(x)y = Q(x)y^n \rightarrow (1)$$

Equation (1) can be reduced to Linear by dividing equation (1) with y^n

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \longrightarrow (2)$$

Assume $y^{1-n} = u$ and differentiate w.r.t. x, we get

$$\frac{1}{y^n}\frac{dy}{dx} = -\frac{1}{1-n}\frac{du}{dx}$$

By the above Substitution equation (2) will reduce to linear DE of the form

$$\frac{du}{dx} + P'(x)u = Q'(x)$$

Similar procedure is followed to solve the equation $\frac{dx}{dy} + P(y)x = Q(y)x^n$

EXAMPLES

1. Solve $\frac{dy}{dx} = \frac{y}{x} + 2x^2$

Solution:

$$\frac{dy}{dx} + \left(\frac{-1}{x}\right)y = 2x^2 \rightarrow (1)$$

$$\therefore \frac{dy}{dx} + P(x)y = Q(x)$$

$$P = -\frac{1}{x}\ Q = 2x^2$$

$$\Rightarrow IF = e^{\int P(x)dx}$$

$$\Rightarrow IF = e^{-\int(1/x)dx}$$

$$= e^{-\log x}$$

$$= e^{\log_e\left(\frac{1}{x}\right)}$$

$$\Rightarrow IF = \frac{1}{x}$$

The solution is

$$y \times IF = \int Q(x) \times IF dx + c$$

$$\Rightarrow y\left(\frac{1}{x}\right) = \int 2x^2 \cdot \frac{1}{x} dx + c$$

$$\Rightarrow \frac{y}{x} = 2\int x dx + c$$

$$\Rightarrow \frac{y}{x} = \frac{2x^2}{2} + c$$

$$\frac{y}{x} = x^2 + c$$

2) Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x$

Solution:

$$\frac{dy}{dx} + \frac{y}{x} = y^2 x \rightarrow (1)$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\left(\frac{1}{y}\right) = x \rightarrow (2) \left(\text{take } \frac{1}{y} = u \rightarrow (3)\right.$$

diff u w.r.t x

$$\frac{-1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{du}{dx}$$

$$(2) \Rightarrow -\frac{du}{dx} + \frac{1}{x}u = x$$

$$\Rightarrow \frac{du}{dx} + \left(\frac{-1}{x}\right)u = -x \rightarrow (u)$$

$$\Rightarrow \frac{du}{dx} + P'(x)u = Q'(x)$$

$$P' = -\frac{1}{x}, Q' = -x$$

$$\Rightarrow IF = e^{\int -1/x dx} = e^{\log 1/x}$$

$$\Rightarrow IF = \frac{1}{x}$$

∴ The Solution is

$$u * IF = \int (Q'(x) \times IF)dx + c$$

$$\Rightarrow u\left(\frac{1}{x}\right) = -\int x \times \frac{1}{x} dx + c$$

$$\Rightarrow \frac{u}{x} = -x + c$$

$$\Rightarrow \frac{1}{xy} + x = c$$

EXERCISE

1. $\frac{dy}{dx} + y\tan x = y^3 \sec x$

2. $x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x\, x$

3. $\frac{dy}{dx} - y\tan x = \frac{\sin x \cos^2 x}{y^2}$

4. $xy(1 + xy^2)\frac{dy}{dx} = 1$

(2) EXACT AND REDUCIBLE TO EXACT DIFFERENTIAL EQUATION:

Step 1 : Write the given differential equation in the form of $M(x,y)dx + N(x,y)dy = 0$

Step 2 : Identify M and N and then find $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$

Step 3 : If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then it is called an exact differential equation and the solution is given by

$$\int M(x,y)dx + \int (\text{the term do not cantain } x \text{ in N})\, dy = c$$

If the given equation is not Exact, then it can be reduced to Exact by multiplying the given equation by an ***Integrating factor***.

Method of finding ***Integrating factor(IF)***

Step 1 : Find $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$

Step 2: If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is close to N, then $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$(function of x only),

and Integrating factor is given by $IF = e^{\int f(x)dx}$

Step 3: If $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}$ is close to M, then $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=g(y)$ (function of y only),

and Integrating factor is given by $IF=e^{-\int g(y)dy}$

EXAMPLES

(1) Solve $\frac{dy}{dx}+\frac{2x+3y-1}{3x+4y-2}=0$

Soln: Given

$$\frac{dy}{dx}+\frac{2x+3y-1}{3x+4y-2}=0$$

$$\Rightarrow (3x+4y-2)dy+(2x+3y-1)dx=0$$
$$\Rightarrow (2x+3y-1)dx+(3x+4y-2)dy=0$$
$$\Rightarrow M(x,y)dx+N(x,y)dy=0$$
$$\therefore M=2x+3y-1,\ N=3x+4y-2$$
$$\therefore \frac{\partial M}{\partial y}=3,\ \frac{\partial N}{\partial x}=3$$
$$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

∴ The given ODE is Exact Differential Equation

∴ The solution is

$$\int M(x,y)dx+\int(\text{term not containing } x \text{ in N})\, dy=c$$

$$\Rightarrow \int (2x+3y-1)dx+\int (4y-2)dy=c$$
$$\Rightarrow 2\int xdx+3y\int 1dx-\int 1dx+4\int ydy-2\int 1dy=0$$

$$\Rightarrow 2\frac{x^2}{2}+3yx-x+\frac{4y^2}{2}-2y=c$$
$$\Rightarrow x^2+3xy-x+2y^2-2y=c$$

2] Solve $(2x+y+1)dx+(x+2y+1)dy=0$

Solution: Given

$(2x + y + 1)dx + (x + 2y + 1)dy = 0 \rightarrow (1)$

$$M = (2x + y + 1) \quad N = (x + 2y + 1)$$
$$\frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = 1$$
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The given ODE is Exact Differential Equation

The Solution is

$$\int M(x,y)dx + \int (\text{term not containing } x \text{ in N})\, dy = c$$

$$\Rightarrow \int (2x + y + 1)dx + \int (2y + 1)dy = c$$
$$\Rightarrow 2\int xdx + y\int 1dx + 1\int dx + 2\int ydy + 1\int dy = c$$
$$\Rightarrow 2\frac{x^2}{2} + y(x) + x + \frac{2y^2}{2} + y = c$$
$$\Rightarrow x^2 + x + xy + y^2 + y = c$$

3] Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

Soln: : $(\sin x + x\cos y + x)dy + (y\cos x + \sin y + y)dx = 0$

$\Rightarrow M(x,y)dx + N(x,y)dy = 0$

$M = y\cos x + \sin y + y$

$N = \sin x + x\cos y + x$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1 \quad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The given $D.E$ is on EDE

The solution is

$$\int M(x,y)dx + \int (\text{term not cantaining } x \text{ in N})\, dy = c$$

$$\Rightarrow \int (y\cos x + \sin y + y)dx + \int 0dy = c$$

$$\Rightarrow y\int \cos x dx + \sin y \int 1dx + y\int 1dx = c$$

$$\Rightarrow y\sin x + x\sin y + xy = c$$

EXERCISE

1. $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

2. $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0$

3. $(x^2 + y^2 + x)dx + xydy = 0$

4. $y(2xy + 1)dx - xdy = 0$

5. $y(2xy - y + 1)dx + x(3x - 4y + 3)dy = 0$

6. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

ORTHOGONAL TRAJECTORIES

An important type of problem in physics or geometry is to find a family of curves that intersects a given family of curves at right angles. The new curves are called **Orthogonal trajectories** of the given curves (and conversely).

Examples are curves of equal temperature (isotherms) and curves of heat flow, curves of equal altitude (Contour lines) on a map and curves of steepest descent on that map, curves of equal potential (curves of equal voltage) and curves of electric force.

Here the angle of intersection between two curves is defined to be the angle between the tangents of the curves at the intersection point.

Consider a family of curves $f(x, y, c) = 0$ in the $xy - plane$, where C is the parameter

1. Eliminate the parameter and construct the differential equation $F\left(x, y, \frac{dy}{dx}\right) = 0$

2. To get orthogonal trajectory of the given curve replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the above equation

3. Solve the differential equation $G\left(x, y, -\frac{dx}{dy}\right) = 0$ to get the **Orthogonal trajectories**

POLAR FORM

To find the orthogonal trajectories of the curves $f(\mathrm{r}, \theta, c) = 0$

1. Form its differential equation in the form $f\left[r, \theta, \frac{dr}{d\theta}\right] = 0$, by eliminating c

2. Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in the above differential equation

3. We obtain differential equation in the form $f\left[r, \theta, -r^2\frac{d\theta}{dr}\right] = 0$

4. Solve the above differential equation to get the **Orthogonal trajectories.**

EXAMPLES

1) Find the orthogonal trajectory of the parabola $y^2 = 4ax$, where a is the parameter.

Soln: $y^2 = 4ax \to (1)$

diff (1) w.r.t ' x '

(1) $\Rightarrow 2y\frac{dy}{dx} = 4a \to (2)$

(1) $\Rightarrow y^2 = 2y\frac{dy}{dx} \cdot x$

$\Rightarrow y = 2x\frac{dy}{dx} \to (3)$

Let $\frac{dy}{dx} = -\frac{dx}{dy}$

(3) $\Rightarrow y = 2x\left(-\frac{dx}{dy}\right)$

$\Rightarrow ydy = -2xdx$

$\Rightarrow \int ydy = -2\int xdx$

$$\Rightarrow \frac{y^2}{2} = -2\frac{x^2}{2} + 15$$

$$x^2 + \frac{y^2}{2} = k$$

2) Find the orthogonal trajectory of $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ where λ if the parameter.

Solution: $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1 \to \infty$

$$\Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2+\lambda} = 0$$

$$\Rightarrow \frac{x}{a^2} + \frac{yy'}{b^4+\lambda} = 0$$

$$\frac{y^2}{b^2+\lambda} = 1 - \frac{x^2}{a^2}$$

$$\frac{y}{b^2+\lambda} = \frac{a^2-x^2}{a^2 y}$$

$$\frac{x}{a^2} = -\frac{(a^2-x^2)}{a^2 y}\frac{dy}{dx}$$

Put $\frac{dy}{dx} = -\frac{dx}{dy}$

$$x = \frac{-(a^2-x^2)}{y}\left(-\frac{dx}{dy}\right)$$

$$x = \frac{(a^2-x^2)}{y}\left(\frac{dx}{dy}\right)$$

$$y\,dy = \left(\frac{a^2-x^2}{x}\right)dx$$

$$\int y\,dy = \int \left(\frac{a^2}{x} - x\right)\,dx$$

$$\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + c_1$$

3. Find the orthogonal trajectory of the cardioid $r = a(1 - \cos\theta)$

Solution: $r = a(1 - \cos\theta)$

D. w.r.t. $'\,\theta\,'$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$a = \frac{r}{1 - \cos\theta}$$

$$\frac{dr}{d\theta} = \frac{r}{1 - \cos\theta}\sin\theta$$

$$Put\ \frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$$

$$-r^2\frac{d\theta}{dr} = \frac{r}{1 - \cos\theta}\sin\theta$$

$$-r\frac{d\theta}{dr} = \frac{\sin\theta}{1 - \cos\theta}$$

$$-\frac{(1 - \cos\theta)\ d\theta}{\sin\theta} = \frac{dr}{r}$$

$$\frac{dr}{r} + \frac{(1 - \cos\theta)\ d\theta}{\sin\theta} = 0$$

Integrate

$$\log r + 2\log\left(\sec\frac{\theta}{2}\right) = \log c_1$$

$$r\sec^2\left(\frac{\theta}{2}\right) = c_1$$

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EXERCISE

1. $y^2 = 4a(x+a)$

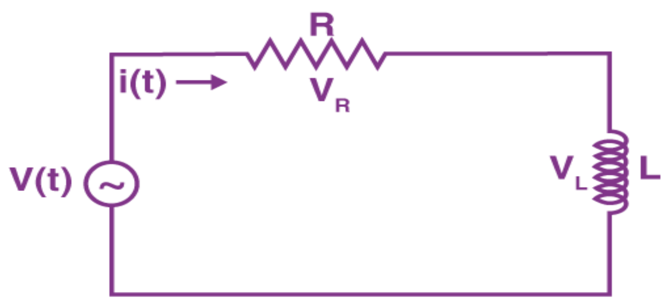
2. $r = a(1-\cos\theta)$

3. $r^n = a^n \cos n\theta$

4. $r^n \sin n\theta = a^n$

5. $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$

R-L SERIES CIRCUIT



Consider a circuit containing resistance R and inductance L in series with a voltage source (battery) V(t). Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's first law, we have sum of voltage drops across R and L is equal to V(t).

$$R\,i + L\frac{di}{dt} = E \quad OR$$

$$\frac{di}{dt} + \frac{R}{L}\,i = \frac{E}{L}\ , \quad \text{Linear D.E}$$

EXAMPLE

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1. Show that the differential for the current i in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force $E\ sin\ \omega t$ satisfies the equation $L\frac{di}{dt} + R\,i = E\ sin\ \omega t$.

Find the value of the current at any time t, if initially there is no current in the circuit.

Sol: By Kirchhoff's first law, we have sum of voltage drops across R and L is equal to $E \sin \omega t$

i.e., $R\,i + L\frac{di}{dt} = E\ sin\ \omega t$ ……. (1), which is the required D.E

(1) can be written as $\frac{di}{dt} + \frac{R}{L}\,i = \frac{E}{L}\ sin\ \omega t$

$I.F = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$

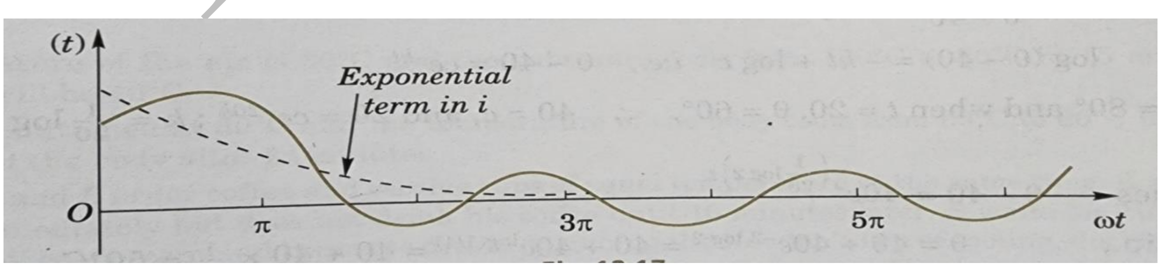
Solution: $i\ (I.F) = \int \frac{E}{L}\ sin\ \omega t\ (I.F)\ dt + C$

$i = \frac{E}{\sqrt{(R^2+\omega^2 L^2)}}\sin(\omega t - \phi) + c\,e^{-\frac{R}{L}t}$ where $tan\,\phi = L\omega/R$ ……….(2)

Initially when $t = 0, i = 0$. $\quad \therefore c = \frac{E \sin\phi}{\sqrt{(R^2+\omega^2 L^2)}}$

Thus (2) takes the form $i = \frac{E}{\sqrt{(R^2+\omega^2 L^2)}}\sin(\omega t - \phi) + \frac{E \sin\phi}{\sqrt{(R^2+\omega^2 L^2)}}\ e^{-\frac{R}{L}t}$

$i = \frac{E}{\sqrt{(R^2+\omega^2 L^2)}}\left[\sin(\omega t - \phi) + \sin\phi\, e^{-\frac{R}{L}t}\right]$ which gives the current i at any time t



Observation: As t increases indefinitely, the exponential term will approach zero. This implies that after sometime the current $i(t)$ will execute nearly harmonic oscillations only.

EXERCISE

1. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i build up at a rate given by $L\frac{di}{dt} + Ri = E$. Find i as a function of t. How long will it be, before the current has reached one-half its final value, if $E = 6\ volts,\ R = 100\ ohms$ $and\ L = 0.1\ henry$.

2. When a resistance $R\ ohms$ connected in series with an inductance $L\ henries$ with an emf of $E\ volts,$ the current $i\ amperes$ at time t is given by $L\frac{di}{dt} + Ri = E$. If $E = 100\sin t\ \ volts$ and $i = 0$ when $t = 0$, Find i as a function of t.

NON-LINEAR DIFFERENTIAL EQUATION

EQUATIONS SOLVABLE FOR P

A differential equation of the first order but of the nth degree is of the form

$$A_n p^n + A_1 p^{n-1} + A_2 p^{n-2} + A_3 p^{n-3} + \cdots .. + A_n = 0 \to (1)$$

Where $p = \frac{dy}{dz}$ the polynomial and $A_0, A_1, A_2, A_3, \ldots$. An are the functions of x and y

Splitting up the left hand side of (1) into n linear factors, we have

$$[p - f_1(x \cdot y)]\,[p - f_2(x,y)]\,[p - f_3(x,y)] \ldots [p - f_n(x,y)] = 0$$

Equating each of the factors to zero,

$$\Rightarrow p = f_1(x,y), \qquad p = f_2(x,y), \ldots .\ p = f_n(x,y)$$

Solving each of these equations of the first order and first degree, we get the solutions

$\Rightarrow F_1(x,y,c) = 0 \ldots\ldots F_2(x,y,c) = 0 \ldots F_n(x,y,c) = 0$

These n solutions constitute the general solution of (1) and is written as

$F_1(x,y,c), F_2(x,y,c) \ldots . F_n(x \cdot y, c) = 0$

EXAMPLES

1] Solve $\left(\frac{dy}{dx}\right)^2 - 7\left(\frac{dy}{dx}\right) + 12 = 0$

Soln:- $\left(\frac{dy}{dx}\right)^2 - 7\left(\frac{dy}{dx}\right) + 12 = 0 \rightarrow (1)$

Let, $\frac{dy}{dx} = P$

$$
\begin{aligned}
(1) \quad & \Rightarrow p^2 - 7p + 12 = 0 \\
\Rightarrow \quad & p^2 - 3p - 4p + 12 = 0 \\
& \Rightarrow p(p-3) - 4(p-3) = 0 \\
& \Rightarrow (p-3)(p-4) = 0 \\
& \Rightarrow p - 3 = 0, p - 4 = 0 \\
& \therefore p = 3 \text{ and } p = 4
\end{aligned}
$$

Case:-1) $P = 3$

case:-2] $P = 4$

$$
\begin{array}{ll}
\frac{dy}{dx} = 3 & \frac{dy}{dx} = 4 \\
\Rightarrow \int dy = \int 3 \cdot dx & \Rightarrow \int dy = \int 4dx \\
\Rightarrow \quad y = 3x + c_1 & \Rightarrow y = 4x + c_2 \\
\Rightarrow \quad y - 3x - c_1 = 0 & \Rightarrow y - 4x - c_2 = 0
\end{array}
$$

$\therefore$ The complete solution of equation (1) is :-

$(y - 3x - c_1)(y - 4x - c_2) = 0$

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2] Solve $y\left(\frac{dy}{dx}\right)^2+(x-y)\left(\frac{dy}{dx}\right)-x=0$

Given:- $y\left(\frac{dy}{dx}\right)^2+(x-y)\left(\frac{dy}{dx}\right)-x=0$…….(1)

Let $\frac{dy}{dx} \quad = p$

Equation (1)

$$\Rightarrow 4p^2+(x-y)p-x=0$$
$$\Rightarrow 4p^2+xp-yp-x=0$$
$$\Rightarrow p(4p+x)-1(yp+x)=0$$
$$\Rightarrow (p-1)(yp+x)=0$$
$$\Rightarrow p=1, p=-x/y$$

Case:1) $P=1$

$$\Rightarrow \frac{dy}{dx}=1$$

$$\Rightarrow \int dy=\int dx$$

$$\Rightarrow y=x+c_1$$

$$\Rightarrow \mathrm{y}-x-c_1=0$$

Case: 2) $P=-\frac{x}{y}$

$$\Rightarrow \frac{dy}{dx}=-\frac{x}{y}$$

$$\Rightarrow \int ydy=-\int x\cdot dx$$

$$\Rightarrow \frac{y^2}{2}=\frac{-x^2}{2}+c_2$$

$$\Rightarrow y^2=-x^2+2c_2$$

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$\Rightarrow y^2 + x^2 - 2C_2 = 0$

∴ The complete solution is for equation (1) is

$[y - x - c_1][y^2 + x^2 - 2c_2] = 0$

EXERCISE

1. $p^2 + 2pycotx - y^2 = 0$

2. $p(p+y)=x(x+y)$

3. $xyp^2 - (x^2 + y^2)p + xy = 0$

4. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

5. $p^2 - 2psinhx - 1 = 0$

6. $x\left(\frac{dy}{dx}\right)^2 - (2x + 3y)\left(\frac{dy}{dx}\right) + 6y = 0$

7. $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

CLAIRAUT'S EQUATION

An equation of the form $y = px + f(p)$ is known as Clairaut's equation

Differentiating the above equation w.r.t. 'x' we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

General solution of Clairaut's equation is obtained on replacing p by c, given by

$$y = cx + f(c) \text{.....(1)}$$

To obtain singular solution, we proceed as follows

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1. Find the general solution of given equation

2. Differentiate this w.r.t. ‘c’ giving $x + f(c) = 0$ ……(2)

3. Eliminate c from equation (1) and (2) which will give the singular solution

EXAMPLES

1. Find the General and singular solution of the Clairaut's equation. $y = px + \frac{a}{p}$

Soln:- Equation is in the form of $y = px + f(p)$

$\therefore$ The general solution.

$$(1) \Rightarrow y = cx + \frac{a}{c} \rightarrow (2)$$

Differentiating (2) $w \cdot r \cdot t$ to ' c '

$$(2) \qquad \begin{aligned} &\Rightarrow 0 = x - \frac{a}{c^2} \\ &\Rightarrow -x = -\frac{a}{c^2} \\ &\Rightarrow \frac{c^2}{a} = \frac{1}{x} \\ &\Rightarrow c^2 = \frac{a}{x} \\ &\Rightarrow c = \sqrt{\frac{a}{x}} \end{aligned}$$

$\therefore$ The sinqular solution is

$$\Rightarrow y = \sqrt{\frac{a}{x}} \cdot x + \frac{a\sqrt{x}}{\sqrt{a}}$$
$$\Rightarrow y = \sqrt{a} \cdot \sqrt{x} + \sqrt{a}\sqrt{x}$$
$$\Rightarrow y = 2\sqrt{ax}$$
$$y^2 = 4ax$$

2] Reduce the equation $(px-y)(py+x)=2p$ to Clairauts form taking the Substitution $X=x^2$ and $Y=y^2$ and hence find its solution.

Given: $(px-y)(py+x)=2p \rightarrow (1)$

and given X= $x^2 \rightarrow (2),\ Y=y^2 \rightarrow (3)$

Diff $(2),(3)$ wrt to x and y

$$\frac{dX}{dx}=2x \rightarrow (4),\quad \frac{dY}{dy}=2y \rightarrow (5)$$

w.k.t, $\frac{dy}{dx}=P$

$$\frac{(5)}{(4)} \quad \Rightarrow \frac{dY}{dX}=\frac{2ydy}{2xdx}$$

$$\Rightarrow \frac{dY}{dX}=\frac{y\cdot dy}{x\cdot dx}$$

$$\Rightarrow P=\frac{y}{x}.p$$

$$\Rightarrow p=\sqrt{\frac{X}{Y}}P$$

$$\text{Tn } eq(1), \left[\sqrt{\frac{X}{Y}}P\sqrt{X}-\sqrt{Y}\right]\left[\sqrt{\frac{X}{Y}}P\sqrt{Y}+\sqrt{X}\right]=2\sqrt{\frac{X}{Y}}P$$

$$\Rightarrow \left[\frac{PX-Y}{\sqrt{Y}}\right][P+1]\sqrt{X}=\frac{2\sqrt{X}}{\sqrt{Y}}P$$

$$\Rightarrow PX-Y=\frac{2P}{P+1}$$

$$\Rightarrow -Y=-PX+\frac{2P}{P+1}$$

$$\Rightarrow Y=PX+\left[\frac{-2P}{P+1}\right]$$

The reduced equation is

$$\Rightarrow Y = cX + \left[\frac{-2c}{c+1}\right]$$

$$\Rightarrow y^2 = cx^2 + \left[\frac{-2c}{c+1}\right]$$

EXERCISE

1. *p=log(px-y)*

2. $xp^2 + px - py + 1 - y = 0$

3. $xp^2 - py + kp + a = 0$

4. $xp^3 - yp^2 + 1 = 0$

5. $(px - y)(py + x) = a^2p \ taking \ x^2 = U, x^2 = V$

6. $e^{4x}(p-1) + e^{2y}p^2 = 0 \ taking \ e^{2x} = U, e^{2y} = V$