Module-01

Quantum Mechanics:

de Broglie Hypothesis and Matter Waves, de Broglie wavelength and derivation of expression by analogy, Phase Velocity and Group Velocity, Heisenberg's Uncertainty Principle and its application (Nonexistence of electron inside the nucleus-Non Relativistic), Principle of Complementarity, Wave Function, Time independent Schrodinger wave equation, Physical Significance of a wave function and Born Interpretation, Expectation value, Eigen functions and Eigen Values, Motion of a particle in a one dimensional potential well of infinite depth , Waveforms and Probabilities. Numericals.

08 Hours

Quantum mechanics or Quantum physics is the science of very small. It explains the behavior of matter and its interactions with energy on the scale of atoms and subatomic particles.

Wave-particle duality and de-Broglie hypothesis

The phenomenon of interference, diffraction and polarization requires the presence of two or more waves at the same time and at the same place. It is very clear that two or more particles cannot simultaneously occupy the same position. So, one has to conclude that radiation behaves like a wave.

Black body radiation spectrum, photoelectric effect, Compton effect could not be explained on wave nature. These phenomena established that radiant energy has the particle nature. Thus, radiation sometimes behaves as a wave and at some other times as a particle. This is wave–particle duality of radiation.

In the year 1924 Luis de-Broglie extended the idea of dual nature of radiation to the matter and proposed that material particles possess wave characteristics. According to de-Broglie in the universe, whole of energy is in the form of radiation and mass. Since nature loves symmetry if radiation which normally behaves as wave can behave as a particle. Then one can even expect that, entities like electrons, protons etc, which ordinarily behave as particle exhibits properties of waves under appropriate circumstances and be termed as matter waves.

Matter waves:

The wave associated with the moving particle is called matter wave or de-Broglie wave (or pilot wave). The wavelength associated with particles with mass m and moving with certain velocity v and momentum p is given by,

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

Consider a photon with frequency ν and wavelength λ. Its energy according to Plank's theory is $E = h\nu$

$$E = h\nu = \frac{hc}{\lambda}$$

where,   h - Plank's constant

If a photon is considered as a particle of mass 'm' moving with a velocity 'c', then Energy of the photon

$$E = mc^2$$

$$hv = mc^2$$

$$\frac{hc}{\lambda} = mc^2$$

$$momentum = mc = p = \frac{h}{\lambda}$$

$$\therefore \quad \lambda = \frac{h}{p} = \frac{h}{mv}$$

This relation is called as de – Broglie wave equation.

de – Broglie wavelength of an electron

Consider an electron of mass 'm' which is at rest is subjected to a potential difference of V. the electrical work done (e×V) will appear as kinetic energy of the electron.

$$i.e. \quad E = eV$$

$$and \quad E = \frac{1}{2}mv^2$$

$$\therefore \quad m^2v^2 = 2mE$$

$$\Rightarrow \quad mv = p = \sqrt{2mE}$$

wavelength of electron wave $\lambda = \dfrac{h}{p} = \dfrac{h}{mv}$

$$\therefore \lambda = \frac{h}{\sqrt{2mE}}$$

$$\text{But} \quad E = eV$$

$$\therefore \lambda = \frac{h}{\sqrt{2meV}}$$

Instead of an electron, if a particle of charge 'q' is accelerated through a potential difference V, then

$$\lambda = \frac{h}{\sqrt{2mqV}}$$

Characteristic properties of matter waves

- Matter waves are the waves that are associated with moving particles.
- Matter waves are neither electromagnetic nor acoustic waves in nature.
- They can travel through vacuum and do not require any material medium for their propagation.

- The wavelength associated with matter waves decreases with increase in the mass of the particles.

- The velocity of matter waves depends on the velocity of the material particle and is not constant quantity.

- The phase velocity (wave velocity) of matter wave is inversely proportional to its wavelength.

Phase velocity and Group velocity

Phase velocity: The velocity with which a definite phase of each individual wave travels is called a phase velocity or wave velocity.

(a) Individual waves   (b) Resultant wave after superposition
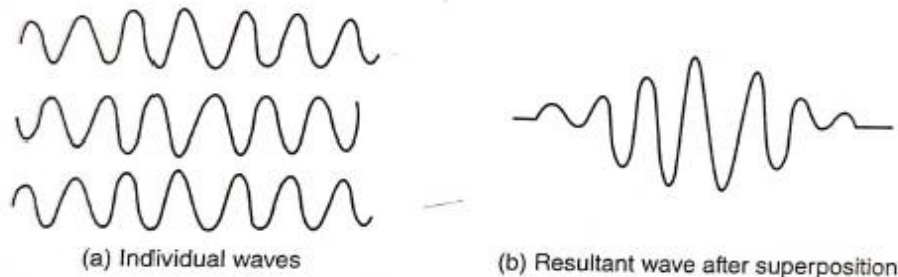
Consider a non-relativistic free particle of mass *m* having phase velocity v_p. The phase velocity is given by

$$v_p = \frac{\omega}{k}$$

Group velocity: The velocity with which the resultant envelop of the group of waves travels is called group velocity, denoted by v_g and is equal to the particle velocity, v. The velocity of the individual waves forming the wave packet is the phase velocity v_p.

$$v_g = \frac{d\omega}{dk}$$

Heisenberg's Uncertainty principle

Statement: In any simultaneous determination of the position and momentum of a particle, the product of the corresponding uncertainties inherently present in the measurement is equal to $\left(\frac{h}{4\pi}\right)$.

OR

The product of uncertainties in the simultaneous determination of the position and momentum of a particle is equal to or greater than $\left(\dfrac{h}{4\pi} \right)$

If Δx and Δp are the uncertainties inherently present in the measurement of position and momentum, then

$$\Delta x . \Delta p \geq \frac{h}{4\pi}$$

Other forms of Uncertainties are,

$$\Delta E . \Delta t \geq \frac{h}{4\pi}$$

$$\Delta L . \Delta \theta \geq \frac{h}{4\pi}$$

Where, ΔE, Δt are the uncertainties in the measurement of energy and time, and ΔL, $\Delta \theta$ are the uncertainties in the measurement of angular momentum and angular position.

Physical significance of Heisenberg's Uncertainty Principle

The physical significance of this principle is, one should not think of the exact position, or an accurate value for momentum of a particle. Instead, think of the probability of finding the particle at a certain position, or of the probable value for the momentum of the particle. The estimation of these probabilities is expressed in quantum mechanics by means of mathematical functions called probability density function.

Application of Uncertainty Principle

Non-existence of electron inside the nucleus (Non-Relativistic case)

The K.E. of electron is $\qquad E_K = \dfrac{p^2}{2m}$ $\qquad\qquad\qquad$ (1)

where   m is mass of electron $= 9.11 \times 10^{-31}$ kg and p is the momentum of electron.

We have from Heisenberg's uncertainty principle

$$\Delta x . \Delta p \geq \frac{h}{4\pi}$$

$$\Delta p \geq \frac{h}{4\pi . \Delta x} \qquad\qquad\qquad (2)$$

We know that the typical value of diameter of the nucleus is of the order of 10^{-14} m, thus if an electron is to exist inside the nucleus, then the uncertainty in its position Δx must not be greater than this value.

$$\Delta x \leq 10^{-14} m$$

Substitute this in equation (2)

$$\Delta p \geq \frac{6.634 \times 10^{-34}}{4 \times 3.142 \times 10^{-14}}$$

$$\Delta p \geq 5.27 \times 10^{-21} Ns$$

Which is the uncertainty in the momentum of the electron. Since the momentum of the electron must at least be equal to the uncertainty in the momentum. i.e.

$$\Delta p \geq 5.27 \times 10^{-21} Ns$$

From equation (1)

$$E_K = \frac{p^2}{2m}$$

$$E \geq \frac{\left(5.27 \times 10^{-21}\right)^2}{2 \times 9.11 \times 10^{-31}}$$

$$E \geq 1.524 \times 10^{-11} J$$

expressing it in eV

$$E \geq 95.13 MeV$$

This indicates that suppose if an electron is to be exist inside the nucleus, its energy must be greater than or equal to 95.13 MeV. However, the experimental results on β-decay emission reveal that the KE of the electrons is of the order of 3 to 4 MeV. This value being very low compared to that of calculated value of 95.13 MeV, confirms that the electrons do not exist inside the nucleus.

Complementarity Principle

In 1927, Niels Bohr realized that our models, or pictures, of matter and light are based upon their behaviour in various experiments in our laboratories. In some experiments, such as the photoelectric effect or the Compton effect, light behaves as if it consists of particles; in other experiments, such as the double-slit experiment, light behaves as if it consists of waves. Similarly, in experiments such as J.J. Thomson's cathode-ray studies, electrons behave as if

they are particles; in other experiments, such as George Paget Thomson diffraction studies, electrons behave as if they are waves. But light and electrons never behave simultaneously as if they consist of both particles and waves. In each specific experiment they behave either as particles or as waves, but never as both.

This suggested to Bohr that the particle and wave descriptions of light and of matter are both necessary even though they are logically incompatible with each other. They must be regarded as being "complementary" to each other—that is, like two different sides of the same coin. This led Bohr to formulate what is called the Principle of Complementarity:

"The wave and particle models are both required for a complete description of matter and of electromagnetic radiation. Since these two models are mutually exclusive, they cannot be used simultaneously".

Bohr showed that this principle is a fundamental consequence of quantum mechanics. He handled the wave–particle duality, not by resolving it in favour of either waves or particles, but by absorbing it into the foundations of quantum physics.

Wave function

- The variable quantity that characterizes the de–Broglie wave or matter wave is called the wave function.
- Mathematically it describes the motion of a particle and is usually denoted as $\Psi = (r, t)$ or $\Psi = (x, y, z, t)$.
- It gives complete information about the state of a physical system at a particular time. It is also called the state function and represents the probability amplitude. If Ψ is large the probability of finding the particle is also large and if Ψ is small, then the probability of finding the particle is small.
- The wave function gives the likelihood of finding the particle at a given instant and at a given position.

Max Born interpretation (Probability density)

- A Probability interpretation of the wave function was given by Max Born in 1926.
- He suggested that "the square of the magnitude of wave function i.e., $|\Psi|^2$ evaluated in a particular region represents the probability of finding the particle in that region.

- If P be the probability of finding the particle in an infinitesimal small volume dv = (dx, dy, dz) then

$$P = |\Psi|^2 \, dv \quad \text{or} \quad P = |\Psi(x, y, z)|^2 \, dx.dy.dz$$

Where $|\Psi|^2$ is probability density and Ψ is the probability amplitude.

- Probability of occurrence of an event is real and positive quantity but wave functions are complex. So, in order to get a positive and real value while evaluating $|\Psi|^2$ the wave function Ψ is multiplied by its complex conjugate Ψ^*.

- The product of $\Psi\Psi^*$ is always a positive real quantity and corresponds meaningfully to the definition of probability.

    Thus, probability density is $|\Psi|^2 = \Psi\Psi^*$

- If the particle exists, it must be present somewhere in the space. Then as per statistical rule the sum of the probabilities over all values of x, y, z must be unity.

    Hence Ψ must satisfies the relation,

$$\int_{-\infty}^{+\infty} |\Psi|^2 \, dv = 1 \quad \text{or} \quad \int_{-\infty}^{\infty} |\Psi(x, y, z)|^2 \, dx.dy.dz = 1$$

$$\int_{-\infty}^{\infty} \Psi\Psi^* \, dx.dy.dz = 1$$

    This is known as **Normalization** of wave function.

Properties of wave function

1) *The wave function Ψ is single valued everywhere.*

- From the figure it can be seen that a function f(x) is not single valued over certain interval.

- Here f(x) has 3 values f_1, f_2 and f_3 for the same value of P at x = P. Since $f_1 \neq f_2 \neq f_3$.

- The probability of finding the particle has 3 different values at the same location, which is impossible, hence it is not considered as wave function.
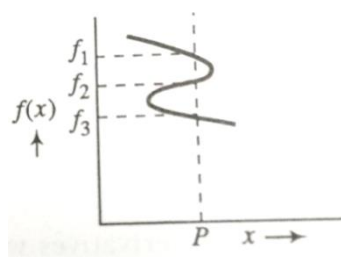
2) *The wave function Ψ must be finite everywhere.*

- A function f(x) is not finite at x = R as shown in figure. At x = R f(x) = ∞.
- It indicates a large probability of finding the particle at a single location, violating the uncertainty principle Thus f(x) is not acceptable as wave function.
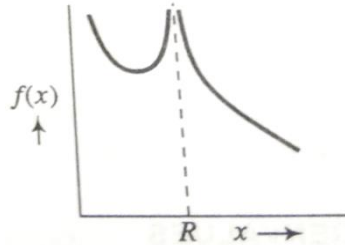
3) *The wave function Ψ and its first order derivatives are continuous everywhere.*

- A function f(x) is discontinuous at Q as shown in figure.
- At x = Q, f(x) is truncated at A and restarts at B, between A & B it is not defined. Thus, the state of the system at x = Q cannot be ascertained. Hence f(x) is not considered as wave function.
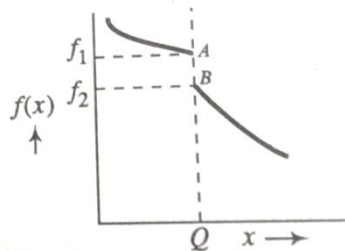
4) *For bound states, wave function Ψ must vanish at infinity. If Ψ is a complex function, then ΨΨ* must vanish at infinity.*

Physical significance of wave function:

- The wave function Ψ has no direct physical significance as it is not an observable quantity. Its physical significance could be realized through its probabilistic nature.
- It gives a statistical relationship between the particle and wave nature.
- It is a complex quantity and hence it may be positive or negative.
- It is a function of wave and time coordinate.
- It describes the behaviour of a single particle or photon and wave nature and not for number of particles.
- It is of probabilistic nature and can be expressed in terms of probability density.

Expectation value

The expectation value in quantum mechanics is the probabilistic expected value of the result or measurement of an experiment. It can be thought of as an average of all the possible outcomes of a measurement as weighted by their likelihood, and as such it is not the most probable value of a measurement; indeed, the expectation value may have zero probability of occurring. It is a fundamental concept in all areas of quantum physics.

Let us consider a particle moving along the x-axis. The result of the measurement of the position x is a continuous random variable. Consider a wave function Ψ (x, t) the $|\Psi(x,t)|^2$ value is a probability density for the position observable and $|\Psi(x,t)|^2 dx$ is the probability of finding the particle between x and $x+dx$ at a time t. Thus, if a measurement of position is repeated many times in an identical way on an identical particle in identical circumstances, many possible outcomes are possible and the expectation value of these outcomes is as per the following equation,

$$\langle x \rangle = \int_{-\square}^{+\square} x \, |\Psi(x,t)|^2 dx$$

Schrodinger's wave equation

- The wave function in quantum mechanics account for the wave like properties of a particle and it is obtained by solving a fundamental equation called Schrodinger equation.

- Erwin Schrodinger in 1926 developed a wave equation that describes the quantum mechanical behaviour of matter waves under different physical situations.

Time independent Schrodinger wave equation

The wave function describing the de-Broglie wave can be written in complex notation as,

$$\psi = Ae^{i(kx-\omega t)} \tag{1}$$

where,

     $\psi \longrightarrow$ Total wave function (function of 'x' and 't')

     $\omega \longrightarrow$ The angular frequency of the wave

     A$\longrightarrow$ The constant representing the wave amplitude

The time independent in equation (1) can be represented by another wave function,

$$\Psi = Ae^{ikx} \tag{2}$$

$\therefore$ equation (1) can be written as $\qquad \psi = \Psi e^{-i\omega t}$ $\qquad\qquad$ (3)

Differentiate equation (3) with respect to 'x' twice we get

$$\frac{d^2\psi}{dx^2} = e^{-i\omega t}\frac{d^2\Psi}{dx^2} \qquad\qquad (4)$$

Differentiate equation (3) with respect to 't' twice we get

$$\frac{d^2\psi}{dt^2} = -\omega^2 e^{-i\omega t}\Psi \qquad\qquad (5)$$

we know that the equation for a travelling wave is given by

$$\frac{d^2 y}{dx^2} = \frac{1}{v^2}\frac{d^2 y}{dt^2}$$

By analogy we can write the wave equation for de-Broglie wave for the motion of a free particle as,

$$\frac{d^2\psi}{dx^2} = \frac{1}{v^2}\frac{d^2\psi}{dt^2} \qquad\qquad (6)$$

This represents the de-Broglie wave propagating along x-direction with a velocity 'v' and Ψ is the displacement.

Substituting equations (4) and (5) in (6),

$$e^{-i\omega t}\frac{d^2\Psi}{dx^2} = \frac{1}{v^2}\left(-\omega^2 e^{-i\omega t}\Psi\right)$$

$$\frac{d^2\Psi}{dx^2} = \frac{-\omega^2}{v^2}\Psi$$

If λ and γ are the wavelength and frequency of the wave, then $\omega = 2\pi\gamma$ and $v = \gamma\lambda$. Substitute this in above equation

$$\frac{d^2\Psi}{dx^2} = -\frac{(2\pi\gamma)^2}{(\gamma\lambda)^2}\Psi$$

$$\frac{d^2\Psi}{dx^2} = -\frac{4\pi^2}{\lambda^2}\Psi$$

$$\frac{1}{\lambda^2} = -\frac{1}{4\pi^2\Psi}\frac{d^2\Psi}{dx^2} \qquad\qquad (7)$$

The kinetic energy of a particle of mass 'm' moving with velocity 'v' is given by

$$E_K = \frac{p^2}{2m} \qquad\qquad (8)$$

The de-Broglie wavelength is given by $\lambda = \dfrac{h}{p}$ or $p = \dfrac{h}{\lambda}$ substitute in equation (8)

$$E_K = \frac{1}{2m}\frac{h^2}{\lambda^2}$$

$$E_K = \frac{h^2}{2m}\frac{1}{\lambda^2}$$

Substitute for $\dfrac{1}{\lambda^2}$ from equation (7)

$$E_K = \frac{h^2}{2m}\left(-\frac{1}{4\pi^2\Psi}\frac{d^2\Psi}{dx^2}\right)$$

$$E_K = -\frac{h^2}{8\pi^2 m}\frac{1}{\Psi}\frac{d^2\Psi}{dx^2} \qquad\qquad (9)$$

Let there be field where the particle is present. Depending on its position in the field, the particle will possess a certain potential energy (V), then

Total Energy (E) = Kinetic Energy (E_K) + Potential Energy (V)

$\therefore$
$$E = -\frac{h^2}{8\pi^2 m}\frac{1}{\Psi}\frac{d^2\Psi}{dx^2} + V$$

$$E - V = -\frac{h^2}{8\pi^2 m}\frac{1}{\Psi}\frac{d^2\Psi}{dx^2}$$

$$-\frac{8\pi^2 m}{h^2}(E-V)\Psi = \frac{d^2\Psi}{dx^2}$$

$$\frac{d^2\Psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E-V)\Psi = 0$$

The above equation is the time independent Schrodinger's wave equation

<u>Note:</u>

Time independent Schrodinger's wave equation can also be written as

$$\frac{d^2\Psi}{dx^2} + \frac{2m}{\hbar^2}(E-V)\Psi = 0$$

In three dimensions it is given by

$$\frac{d^2\Psi}{dx^2} + \frac{d^2\Psi}{dy^2} + \frac{d^2\Psi}{dz^2} + \frac{8\pi^2 m}{h^2}(E-V)\Psi = 0$$

$$\nabla^2\Psi + \frac{8\pi^2 m}{h^2}(E-V)\Psi = 0$$

Eigen functions and Eigen values

    For a given system when we solve the Schrodinger's equation substituting the proper value of potential energy, we get the solutions of the differential equation which are called **Eigen functions** and the corresponding energy values are called **Eigen energy values**.

Application of Schrödinger's wave equation

Particle in 1-dimensional potential well of infinite height. (Particle in a box)
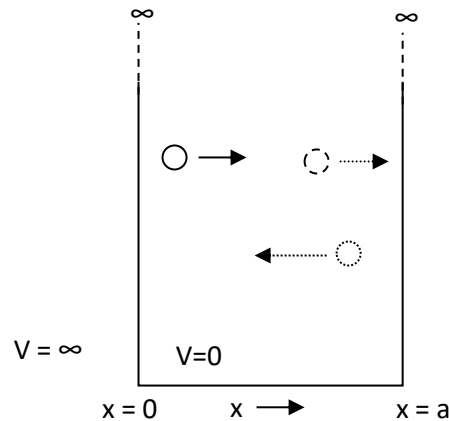
Consider a particle of mass 'm' is freely moving in x-direction in the region from x=0 to x=a (As shown in figure). Outside this region potential energy 'V' is infinity and within this region V = 0.

The time independent Schrodinger's wave equation is given by

$$\frac{d^2\Psi}{dx^2} + \frac{8\pi^2 m}{h^2}\left(E - V\right)\Psi = 0 \tag{1}$$

Outside the well i.e., V=∞ the Schrodinger's equation is

$$\frac{d^2\Psi}{dx^2} + \frac{8\pi^2 m}{h^2}\left(E - \infty\right)\Psi = 0 \tag{2}$$

This equation holds good only if ψ = 0 for all points outside the box i.e., $\left|\Psi\right|^2 = 0$, which means that the particle cannot be found at all outside the box.

Inside the box, i.e., V=0 the Schrödinger's equation is given by,

$$\frac{d^2\Psi}{dx^2} + \frac{8\pi^2 m}{h^2} E\Psi = 0 \tag{3}$$

$$\frac{d^2\Psi}{dx^2} + k^2\Psi = 0 \tag{4}$$

where,
$$k^2 = \frac{8m\pi^2 E}{h^2}$$
(5)

Discussion of solution

The general solution equation (4) is given by

$$\Psi = A\cos kx + B\sin kx$$
(6)

Where A & B are constants depending on the boundary conditions. Let us apply boundary conditions,

<u>Condition I:</u>

At x = 0 then $\Psi = 0$

Equation (6) $\Rightarrow$ $0 = A\cos 0 + B\sin 0$

i.e. $A = 0$
(7)

<u>Condition II:</u>

At x = a then $\Psi = 0$

Equation (6) $\Rightarrow$ $0 = A\cos ka + B\sin ka$

$$0 = B\sin ka$$

Here B $\neq$ 0

$\therefore$ $\sin ka = 0$

or $ka = n\pi$ $(\because \sin n\pi = 0)$

$$k = \frac{n\pi}{a}$$
(8)

$$k^2 = \frac{n^2\pi^2}{a^2}$$

Where, n = 1,2, 3.........

Substitute the values of A and k in equation (6) we get,

$$\Psi = B\sin\left(\frac{n\pi}{a}\right)x$$
(9)

Substituting the value of k in equation (5)

$$\frac{8m\pi^2 E}{h^2} = \frac{n^2\pi^2}{a^2}$$

$$E = \frac{n^2 h^2}{8ma^2}$$
(10)

This is the energy Eigen equation for the particle in 1-D potential well. Here n can take only integer values, the corresponding values that E takes are the "**energy Eigen values**".

When n = 0, $\Psi = 0$ which means to say that the electron is not present inside the box, which is not true. Hence the lowest value of 'n' is 1.

∴ The lowest allowed energy corresponds to n =1 is called the zero-point energy or ground state energy and all the states n >1 is called excited states.

$$E_{zero-point} = \frac{h^2}{8ma^2}$$

To evaluate B in equation (9) we have to perform normalization of wave function.

Consider, $\Psi = B \sin\left(\frac{n\pi}{a}\right) x$

The integral of the wave function over the entire space in the well must be equal to unity because there is only one particle within the well, the probability of finding the particle is 1.

$$\int_0^a |\Psi|^2 \, dx = 1$$

$$\int_0^a B^2 \sin^2\left(\frac{n\pi}{a}\right) x \, dx = 1$$

We know that $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$\frac{B^2}{2} \int_0^a \left(1 - \cos\frac{2n\pi x}{a}\right) dx = 1$$

$$\frac{B^2}{2}\left[\int_0^a dx - \int_0^a \cos\frac{2n\pi x}{a} dx\right] = 1$$

$$\frac{B^2}{2}\left[x - \frac{a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right]_0^a = 1$$

$$\frac{B^2}{2}\left[a - \frac{a}{2n\pi}\sin 2n\pi - 0\right] = 1$$

$$\therefore \frac{B^2 a}{2} = 1$$

$$\Rightarrow B = \sqrt{\frac{2}{a}}$$

Thus, the normalized wave function of a particle in a one-dimensional box is given by,

$$\Psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}\right) x$$

Where n=1,2,3……………

Since the particle in a box is a quantum mechanical problem, we need to evaluate the most probable location of the particle in a box and its energies at different permitted state.

Let us consider first three cases

Case 1: n =1

This is the ground state, and the particle is normally found in this state. For n =1, the Eigen

function is
$$\Psi_1 = B \sin\left(\frac{\pi}{a}\right) x$$

In the above equation $\Psi = 0$ for both x = 0 & x = a. but Ψ_1 has maximum value for x = a/2. and $|\Psi_1|^2 = 0$ at x = 0 and x = a, and $|\Psi_1|^2$ is maximum at x = (a/2).

A plot of Ψ_1 and $|\Psi_1|^2$, the probability density versus 'x' is as shown in the figure.

From the figure, it is clear that at ground state the probability of finding the particle is max at the centre and the particle cannot be found at the walls of the potential well.
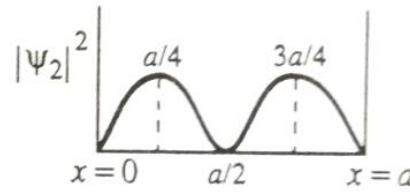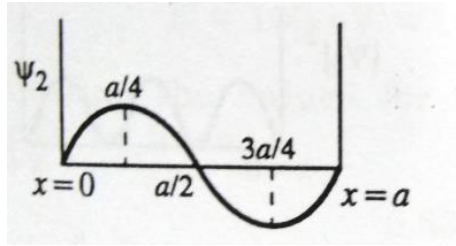
Energy of the particle in the ground state is given by, $E_0 = \dfrac{h^2}{8ma^2}$

Case 2: n = 2

This is the first excited state. The Eigen function for this state is given by

$$\Psi_2 = B \sin\left(\frac{2\pi}{a}\right) x$$

Now, $\Psi_2 = 0$ for the values $x = 0, \dfrac{a}{2}, a$ and Ψ_2 reaches maximum at $x = \dfrac{a}{4}, \dfrac{3a}{4}$

From the figure it can be seen that $|\Psi_2|^2 = 0$ at x = 0, $a/2, a$. It means that in the first excited state the particle cannot be observed either at the walls or at the centre.

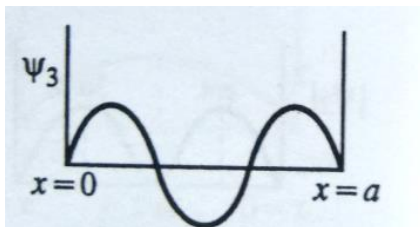The energy obtained for n = 2, $E_2 = 4\left(\dfrac{h^2}{8ma^2}\right)$

$$E_2 = 4E_0$$

Thus, the energy in the first excited state is 4 times the zero point energy.

Case 3: n = 3

This is the second excited state and the Eigen function for this state is given by

$$\Psi_3 = B\sin\left(\frac{3\pi}{a}\right)x$$

Now, $\Psi_3 = 0$ for the values $x = 0, \dfrac{a}{3}, \dfrac{2a}{3}, a$ and Ψ_3 reaches maximum at $x = \dfrac{a}{6}, \dfrac{a}{2}, \dfrac{5a}{6}$

The plot of $|\Psi_3|^2$ versus 'x' has maxima at $x = \dfrac{a}{6}, \dfrac{a}{2}, \dfrac{5a}{6}$ at which the particle is most likely to be found.

The energy obtained for n = 3, $E_3 = 9\left(\dfrac{h^2}{8ma^2}\right)$

$$E_3 = 9E_0$$