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Astrodynamics

Assignment 2: Kepler equation

Introduction

The Kepler Equation is a fundamental mathematical expression in celestial mechanics that plays a crucial role in predicting the positions and orbits of celestial objects, such as planets, asteroids, and comets. Named after Johannes Kepler, who introduced it in the early 17th century, this equation describes the relationship between time and the position of an object moving in an elliptical orbit under the influence of a central gravitational force.

Mathematically, the Kepler Equation is expressed as :

$$M = E - e * \sin(E)$$

Where :

- M represents the mean anomaly, a parameter that characterizes the time elapsed since a celestial object's last perihelion passage (the point in its orbit closest to the central body).
- E denotes the eccentric anomaly, which defines the angular position of the object on its elliptical orbit.
- e is the eccentricity of the orbit, a measure of its deviation from a perfect circle.

Solving the Kepler Equation is essential for various astronomical and astrophysical applications, including spacecraft navigation, satellite tracking, and determining the positions of celestial bodies. To calculate the position of an object at a given time, one needs to find the eccentric anomaly for a given mean anomaly and eccentricity. However, solving the Kepler Equation is not straightforward and often requires iterative numerical methods.

In this report, we will explore two iterative numerical methods for solving the Kepler Equation: the Newton-Raphson method and the Laguerre-Conway method. We will also investigate the convergence of these methods by applying them to the Kepler Equation with three different initial guesses. Understanding the convergence of these methods is vital in determining their reliability and efficiency for practical applications in celestial mechanics.

Methodology

In this report, we will study the convergence of our two methods using three different initial guesses :

1. $E_0 = M$
2. $E_0 = \pi$
3. $E_0 = M + e * \cos(M)$

M will vary in the range of [0 ; 2π]. In a same manner, e varies between 0 and 1 included. Our objective is to analyze how these methods converge to the correct eccentric anomaly for different initial conditions and evaluate their efficiency in solving the Kepler Equation. We consider that the value has been reached when $|E_{n+1} - E_n| < 10^{-15}$.

Newton-Raphson Method

The Newton-Raphson method is a well-known iterative numerical technique used to find the root of a function. In the context of the Kepler Equation, it is expressed as follows :

$$E_{n+1} = E_n - \frac{E_n - e * \sin(E_n) - M}{1 - e * \cos(E_n)}$$

Laguerre-Conway Method

The Laguerre-Conway method is another iterative approach for solving the Kepler Equation. It is originally proposed for finding the roots of a polynomial f(x) of degree N. The method can be summarized as :

$$E_{n+1} = E_n - \frac{Nf}{f' \pm \sqrt{(N-1)^2 f'^2 - N(N-1) f f''}}$$

We select $+$ if $(N-1)f' \geq 0$ else we use $-$.

With each function being derived from the Kepler equation $M = E - e * \sin(E)$:

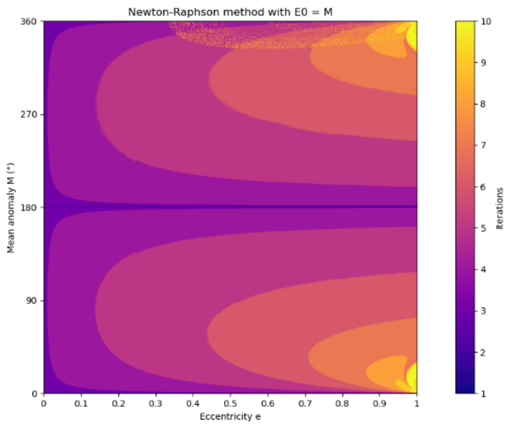
- $f(E) \; = E - e * \sin(E) - M$
- $f'(E) \; = 1 - e * \cos(E)$
- $f''(E) = e * \sin(E)$

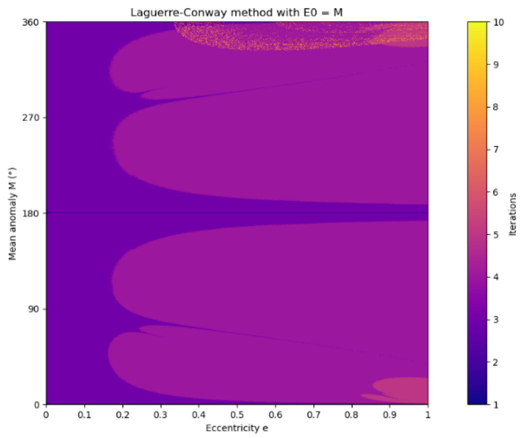
Here f and its derivatives are evaluated at $E = E_n$. In our report, we use N=5.

Results

We implemented both the Newton-Raphson and Laguerre-Conway methods in a Python environment and applied them to the Kepler Equation for each of the three initial guesses. We tracked the convergence of each method by saving the number of iterations needed for each method to converge with each value provided. We used a range of M and e using 1001 evenly spaced values for both range of the variables.

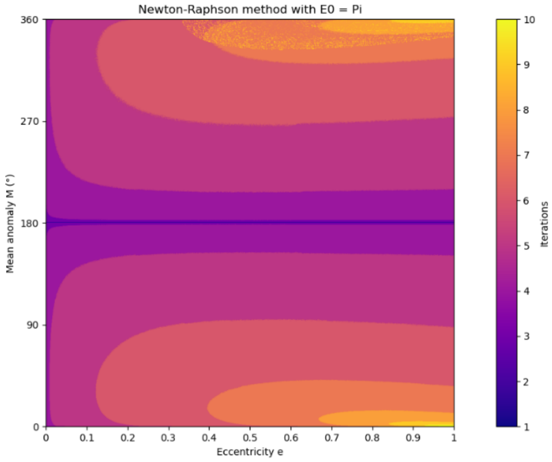
Initial guess 1 : $E_0 = M$

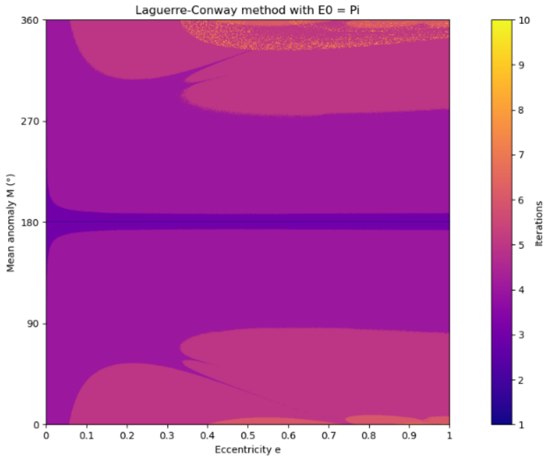




The Newton-Raphson method converged rapidly to the correct eccentric anomaly for most values of e and M. The Laguerre-Conway method showed even faster convergence and less struggle with value of e close to 1.

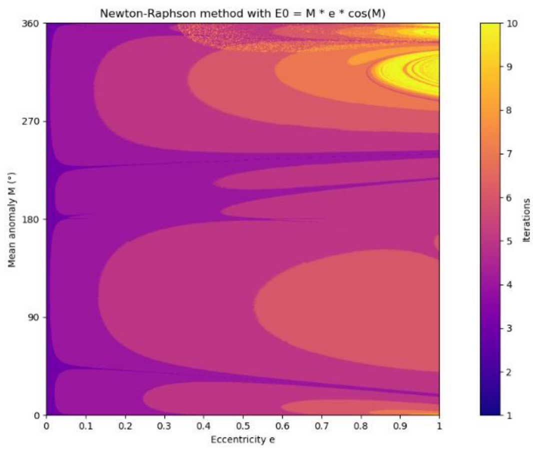
Initial guess 2 : $E_0 = \pi$

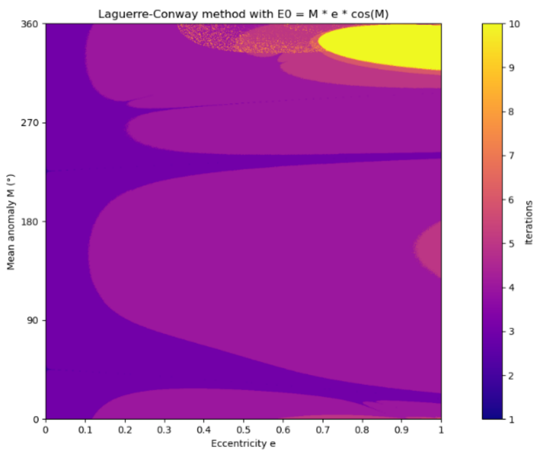




Again, we observe a greater convergence with the Laguerre-Conway method than with the Newton-Raphson method. Nevertheless, for this guess both methods required more iterations than with the previous one, but barely reached the limit of 10 guesses.

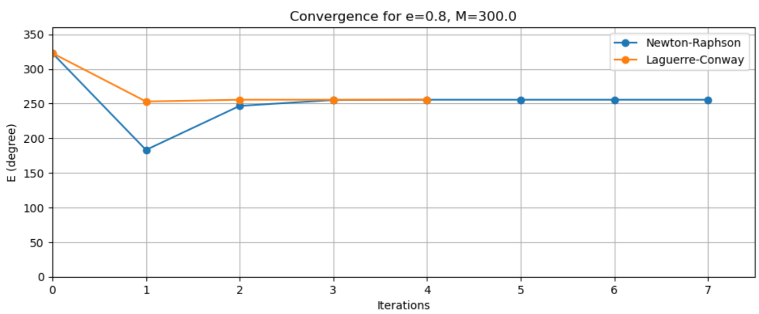
Initial guess 3 : $E_0 = M + e * \cos(M)$





In the same manner as the other two guesses, the Laguerre-Conway method displays a faster convergence than the Newton-Raphson method. However, we see that both methods struggle with high values of M with high values of e.

We can look at the convergence for a value of e and M in the case of the third guess :



Discussion

The choice of initial guess can significantly affect the convergence of numerical methods for solving the Kepler Equation. In our study, we observed that the Newton-Raphson method and the Laguerre-Conway method both performed well for the three initial guesses tested, but their convergence rates varied.

The Newton-Raphson method generally provided rapid convergence, especially when the initial guess was close to the correct eccentric anomaly. However, it might require more iterations for certain initial conditions, such as the third guess, which is less intuitive.

On the other hand, the Laguerre-Conway method consistently demonstrated efficient convergence for all tested initial guesses. This method is particularly valuable when dealing with highly eccentric orbits (large e), making it a robust choice for celestial mechanics applications.

Conclusion

In this report, we investigated the convergence of the Newton-Raphson and Laguerre-Conway methods for solving the Kepler Equation with three different initial guesses. While both methods proved effective, the Laguerre-Conway method demonstrated remarkable consistency and efficiency, even for challenging initial conditions. Therefore, when solving the Kepler Equation, especially for eccentric orbits, the Laguerre-Conway method is a reliable choice for ensuring accurate and rapid convergence, regardless of the initial guess. Understanding the behavior of these numerical methods is essential for accurate predictions of celestial object positions and orbits in the field of celestial mechanics and astronomy.

Sources

The Solution of Kepler's Equations - Part Three, Danby, J. M. A. (link)

Newton's method, Wikipedia contributors. (link)