Synthesis Of Control Law For Non-Linear System (inverted Pendulum) Based On Metric Equivalence

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SYNTHESIS OF CONTROL LAW FOR NON-LINEAR SYSTEM (INVERTED PENDULUM) BASED ON METRIC EQUIVALENCE [1]

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Abstract: Stabilization (control system design) is one of fundamental areas investigated in the system theory. In this paper is presented method based on metric equivalence. The special attention is devoted of the application this technique for design a controller law on two examples.

Keywords: Stability, state equivalence, non-linear control, dissipation normal form

1. INTRODUCTION

The synthesis of the control design for linear systems fall into often solved problem, but for non-linear systems are shortly discussed. One of them consists in approximative linearizing of the non-linear system to be stabilized about an operating point, and then linear feedback control methods are used to design a controller. This approach is successful in case of a system trajectory is restricted to a small neighborhood about the chosen operating point. The other methods are based on a transformation of a non-linear system into a suitable form. This way is presented in for example (Zak and Maccarley, 1986), which method makes use of the transformation a "controllable like" canonical form.

The paper deals with the stabilization method which has a similar idea as the method mentioned in (Černý *et al.*, n.d.). This method is based on Dissipation normal form, where structure of a closed loop system is just chosen in this form. And because this form is non-linear we have more flexibility in the choice of the desired behaviour of a closed loop system than a linear one.

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2. PROBLEM FORMULATION

Consider the non-linear representation $\Re(S)$ of a system S in the form:

$$\Re(S): \dot{x} = f(x) + g(x)u \qquad y = h(x), \tag{1}$$

where $x \in R^n$ is a state vector, $u \in R^1$ is an input, y is an output, $f(x) \in C^\infty : R^n \to R^n$, $g(x) \in C^\infty : R^n \to R^n$ are vector fields and $h(x) \in C^\infty : R^n \to R^1$ is a scalar field.
Our aim is to design a control law

$$u = L(x), \tag{2}$$

where $L(x) : R^n \to R^1$, so that the representation $\Re(S_{cl})$ of a closed loop system S_{cl}:

$$\Re(S_{cl}): \dot{x} = f(x) + g(x)L(x) \qquad y = h(x) \tag{3}$$

is asymptotically stable.

3. DISSIPATION NORMAL FORM

Definition 1. (Dissipation normal form) Consider the representation $\Re(S)$ of a system S and suppose there exists a Lyapunov function $V(\cdot)$ defined on some domain Ω of the state space R^n. The representation $\Re(S)$ of a system S is named Dissipation normal form if the Lyapunov function $V(\cdot)$ fulfills the following conditions:

$$a)\ V(x) = \|x\|^2 \tag{4}$$
$$b)\ L_f(V(x)) = \beta(y) \leq 0. \tag{5}$$

Theorem 2. (Structure of the representation $\Re(S)$) If the representation $\Re(S)$ of a system S has the structure:

$$\dot{x} = \begin{bmatrix} \alpha_1(x_1) & \alpha_2 & 0 & . & 0 \\ -\alpha_2 & 0 & \alpha_3 & . & . \\ 0 & -\alpha_3 & . & . & 0 \\ . & . & . & . & \alpha_n \\ 0 & . & 0 & -\alpha_n & 0 \end{bmatrix} x$$
$$y = \mu(x_1),$$

where $\alpha_1(x_1)$, $\mu(x_1)$ are non-linear functions ($\alpha_1(x_1) < 0$ for $x_1 \neq 0$; let us suppose that the inverse μ^{-1} exists and $\mu(x_1) = 0 \Longleftrightarrow x_1 = 0$) and $\alpha_2, \ldots, \alpha_n \neq 0$ are real constants then the equilibrium state $x_e = 0$ is asymptotically stable and the corresponding Lyapunov function $V(\cdot)$ fulfills the conditions (4), (5).

It holds

$$\begin{aligned} L_f(V(x)) &= 2x_1^2\alpha_1(x_1) = \\ &= \beta(y) \leq 0. \end{aligned} \tag{6}$$

It results from (6) that

$$\alpha_1(x_1) < 0 \text{ for } x_1 \neq 0 \tag{7}$$

is the necessary and sufficient condition for the asymptotical stability. The conditions $\mu(x_1) = 0 \Longleftrightarrow x_1 = 0$ and existing μ^{-1} together with $\alpha_i \neq 0$ for $i = 2, \ldots, n$ are the necessary and sufficient conditions for some sort of an observability.

Remark 3. (metric equivalence vs. state equivalence) The structure of the representation $\Re(S)$ of a system S (6) is only one of possible structures which conform to the conditions (4), (5). We would obtain another one if we used an orthonormal transformation to apply it to the relation (6). We have chosen this one, but we could still choose another one.

4. PROBLEM SOLUTION

Let us transform the non-linear representation $\Re(S)$ of the system S (1) into the generalized observability normal form:

$$\begin{aligned} \bar{\Re}(S) : \dot{\bar{x}}_1 &= \bar{x}_2 \\ &\vdots \\ \dot{\bar{x}}_{n-1} &= \bar{x}_n \\ \dot{\bar{x}}_n &= \bar{f}_n(\bar{x}) + \bar{g}_n(\bar{x})u \\ y &= \bar{x}_1. \end{aligned} \tag{8}$$

The appropriate difeomorfism

$$T : R^n \to R^n,\ \bar{x} = T(x) \tag{9}$$

exists if the condition

$$rank\, \frac{\partial T(x)}{\partial x} = n \tag{10}$$

is fulfilled and then it is determined by the following relation:

$$\begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix} = \begin{bmatrix} L_f^0(h) \\ \vdots \\ L_f^{n-1}(h) \end{bmatrix}. \tag{11}$$

We put in a control law

$$u = \bar{L}(\bar{x}) \tag{12}$$

now, where $\bar{L} : R^n \to R^1$. Let us assume that the representation $\bar{\Re}(S_{cl})$ of a closed loop system S_{cl} is state equivalent with the representation $\Re^*(S_{cl})$ which has the dissipation normal form. The appropriate difeomorfism

$$T : R^n \to R^n,\ \bar{x} = T^*(x^*) \tag{13}$$

exists if the condition

$$rank\, \frac{\partial T^*(x^*)}{\partial x^*} = n \tag{14}$$

is fulfilled and then it is determined by the following relation:

$$\begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix} = \begin{bmatrix} L_{f^*}^0(h^*) \\ \vdots \\ L_{f^*}^{n-1}(h^*) \end{bmatrix}, \tag{15}$$

where

$$f^*(x^*) = \begin{bmatrix} f_1^*(x_1^*) & f_2^* & 0 & \ldots & 0 \\ -f_2^* & 0 & f_3^* & \ddots & \vdots \\ 0 & -f_3^* & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & f_n^* \\ 0 & \ldots & 0 & -f_n^* & 0 \end{bmatrix} x^*, \tag{16}$$

$$h^*(x^*) = x_1^*. \tag{17}$$

Then the control law $u = \bar{L}(\bar{x})$ is:

$$\bar{L}(\bar{x}) = \arg\{\bar{f}(\bar{x}) + \bar{g}(\bar{x})\bar{L}(\bar{x}) - \\ - L_{f^*}^n(h^*) = 0\}\Big|_{x^* = T^{*^{-1}}(\bar{x})} . \tag{18}$$

In the end we transform the designed control law to original co-ordinates:

$$L(x) = \bar{L}(\bar{x})\,\big|_{x = T^{-1}(\bar{x})} . \tag{19}$$

Remark 4. More information about this stabilization method of non-linear systems you can find in (Černý *et al.*, n.d.).

5. INVERTED PENDULUM WITH DC MOTOR

In this section we are going to use the stabilization method shortly mentioned above for inverted pendulum with DC motor control (Fig. 1). This example you can find for example in (Zak and Maccarley, 1986).

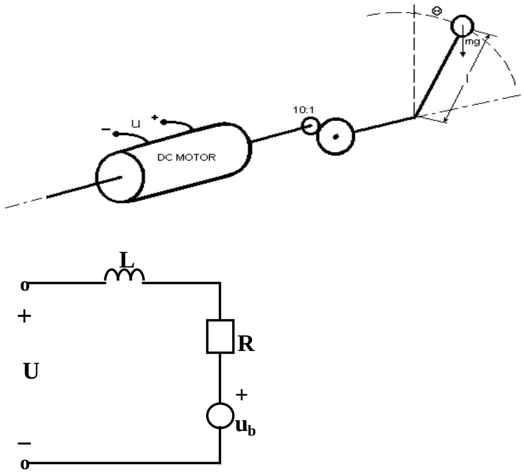

Fig. 1. Inverted pendulum with DC motor control

The representation $\Re(S)$ which describes this system S has the form:

$$\Re(S):\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}x_2\\ K_1\sin(x_1) + K_2x_3\\ K_3x_2 + K_4x_3\end{bmatrix} + \begin{bmatrix}0\\0\\K_5\end{bmatrix}u$$
$$y = x_1, \tag{20}$$

where the state variables are $x_1 = \varphi,\ x_2 = \dot{\varphi} = \omega,\ x_3 = I$ and the constants $K_1 = \frac{g}{l},\ K_2 = \frac{10K_m}{l^2m},\ K_3 = -\frac{10K_b}{L},\ K_4 = -\frac{R}{L},\ K_5 = \frac{1}{L}$.

At first we transform the representation $\Re(S)$ of the system S into the generalized observability normal form:

$$\bar{\Re}(S):\begin{bmatrix}\dot{\bar{x}}_1\\ \dot{\bar{x}}_2\\ \dot{\bar{x}}_3\end{bmatrix} = \begin{bmatrix}\bar{x}_2\\ \bar{x}_3\\ \bar{f}_3(\cdot)\end{bmatrix} + \begin{bmatrix}0\\0\\K_2K_5\end{bmatrix}u$$
$$y = \bar{x}_1, \tag{21}$$

where $\bar{f}_3(\cdot) = K_1\cos(\bar{x}_1)\bar{x}_2 + K_2K_3\bar{x}_2 + K_4\bar{x}_3 - K_1K_4\sin(\bar{x}_1)$.

The relevant difeomorfism $\bar{x} = T(x)$ is:

$$\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\\ K_1\sin(x_1) + K_2x_3\end{bmatrix}. \tag{22}$$

Now we choose design parameters in the structure of the representation $\Re^*(S_{cl})$ of the closed loop system S_{cl} which has the dissipation normal form:

$$\Re^*(S):\begin{bmatrix}\dot{x}_1^*\\ \dot{x}_2^*\\ \dot{x}_3^*\end{bmatrix} = \begin{bmatrix}f_1^*(x_1^*) & f_2^* & 0\\ -f_2^* & 0 & f_3^*\\ 0 & -f_3^* & 0\end{bmatrix}\begin{bmatrix}x_1^*\\ x_2^*\\ x_3^*\end{bmatrix}$$
$$y = x_1^*. \tag{23}$$

The design parameters are:

$$f_1^*(x_1^*) = -x_1^{*^2} - 10, \tag{24}$$
$$f_2^* = 1, \tag{25}$$
$$f_3^* = 1. \tag{26}$$

The relevant difeomorfism $\bar{x} = T(x^*)$ is:

$$\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix} = \begin{bmatrix}x_1^*\\ -x_1^{*^3} - 10x_1^* + x_2^*\\ 3x_1^{*^5} + 40x_1^{*^3} - 3x_1^{*^2}x_2^* + 99x_1^* - \\ -10x_2^* + x_3^*\end{bmatrix} \tag{27}$$

By using the relations (18), (19) we obtain the control law:

$$L(x) = \frac{1}{K_2K_5}\Big[- x_1^3 - 6x_1x_2^2 - 10x_1 - 2x_2 - \\ - (K_1\sin(x_1) + K_2x_3)(30x_1^2 + 1) - \\ - K_1\cos(x_1)x_2 - K_2K_3x_2 - \\ - K_2K_4x_3\Big] \tag{28}$$

The behaviour of the closed loop system is shown on Fig.2, where $K_1 = 9.8,\ K_2 = 1,\ K_3 = -10,\ K_4 = -10,\ K_5 = 10$.

6. INVERTED PENDULUM ON THE CART

This section refer how make a controller for the inverted pendulum with a cart. Employ a physical principle and write this equations:

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(s(t) + l\sin(\varphi(t))\right) = H(t) \tag{29}$$

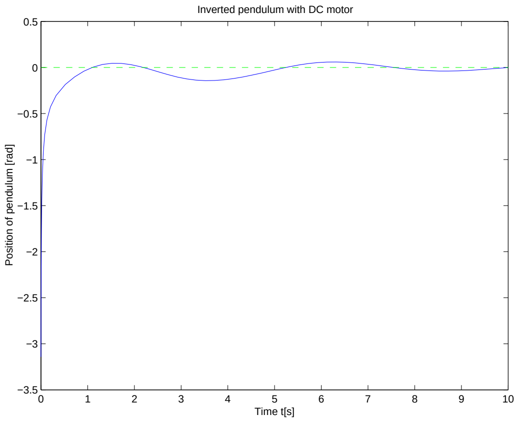

Fig. 2. Controlled inverted pendulum with DC motor

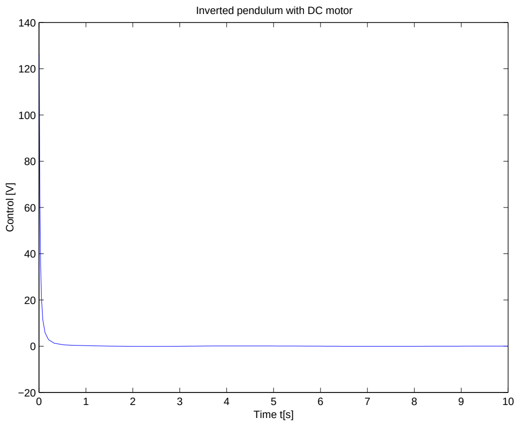

Fig. 3. Controlled inverted pendulum with DC motor – the control law

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(l\cos(\varphi(t))) = V(t) - mg \tag{30}$$

$$\begin{aligned} J\frac{\mathrm{d}^2\varphi(t)}{\mathrm{d}t^2} = \; & V(t)l\sin(\varphi(t)) - \\ & -H(t)l\cos(\varphi(t)) \end{aligned} \tag{31}$$

$$m_v\frac{\mathrm{d}^2 s(t)}{\mathrm{d}t^2} + B\frac{\mathrm{d}s(t)}{\mathrm{d}t} + H(t) = F(t) \tag{32}$$

where significance of the variables are visible from picture 4.

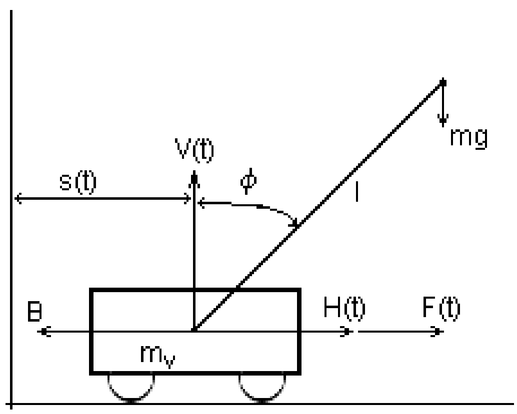

Fig. 4. Model inverted pendulum on the cart

From consequential modification we obtain the following equations:

$$\begin{aligned} \ddot{\varphi}(J + ml^2) \; & +D\dot{\varphi} - mgl\sin(\varphi) + \\ & +ml\cos(\varphi)\ddot{s} = 0 \end{aligned} \tag{33}$$

$$\begin{aligned} \ddot{s}(m_v + m) + B\dot{s} \; & +ml\ddot{\varphi}\cos(\varphi) - \\ & -ml\dot{\varphi}\sin(\varphi) = F(t) \end{aligned} \tag{34}$$

We can see that the change of location of the pendulum (33) influences of the move of the cart (34) and conversely, too. The first we investigate simplified case of this model, where we will be consider only equation (33). Alone input in this case is $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \ddot{s}$, another influence of the cart we doesn't reason.

6.1 *Simplified model*

If we mark state variables $\varphi = x_1$, $\dot{\varphi} = x_2$, constants $K_1 = \frac{gml}{J+ml^2}$, $K_2 = \frac{D}{J+ml^2}$, $K_3 = -\frac{ml}{J+ml^2}$ and $\ddot{s}$ as input u, then simplified model can be described by following representation in generalized observability normal form :

$$\begin{aligned} \Re(S) : \dot{x}_1 &= x_2 \\ \dot{x}_2 &= K_1\sin(x_1) - K_2 x_2 + K_3\sin(x_1)u \\ y &= x_1 \end{aligned} \tag{35}$$

Now we choose design parameters in the structure of the representation $\Re^*(S_{cl})$ of the closed loop system S_{cl} which has the dissipation normal form:

$$\begin{aligned} \Re^*(S) : \begin{bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \end{bmatrix} &= \begin{bmatrix} f_1^*(x_1^*) & f_2^* \\ -f_2^* & 0 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \\ y &= x_1^*. \end{aligned} \tag{36}$$

The design parameters are:

$$\begin{aligned} f_1^*(x_1^*) &= -x_1^{*^2} - 10, \\ f_2^* &= 1, \end{aligned} \tag{37}$$

The relevant difeomorfism $x = T(x^*)$ is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^* \\ -x_1^{*^3} - 10x_1^* + x_2^* \end{bmatrix} \tag{38}$$

Because the representation $\Re(S)$ is in generalized observability normal form, we use the relations (18) and than we obtain the control law:

$$\begin{aligned} L(x) = \frac{1}{K_3\cos(x_1)}\Big[& -3x_1^2 x_2 - 10x_2 - 4x_1 - \\ & -K_1\sin(x_1) + K_2 x_2\Big] \end{aligned} \tag{39}$$

The behaviour of the closed loop system is shown on Fig.5, where $K_1 = 8.9025$, $K_2 = 0.3895$, $K_3 = 0.9084$.

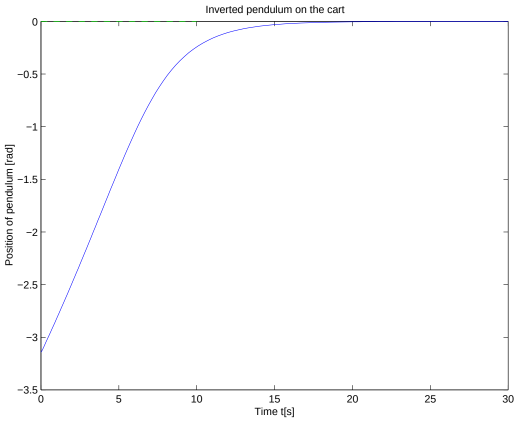

Fig. 5. Controlled inverted pendulum on the cart, simplified model

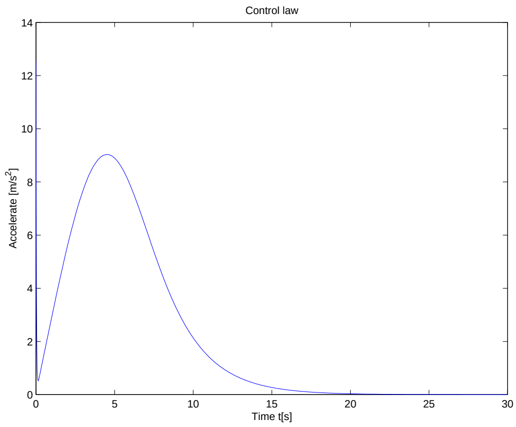

Fig. 6. Controlled inverted pendulum on the cart – control law

6.2 *More complicated model*

Now will be consider model inclusive of the influence of a cart, i.e. equations (33) and (34). As the first we must eliminated from equation (33) the variable $\ddot{s}$ and from equation (34) variable $\ddot{\varphi}$. Further marked state space variables as $\varphi = x_1$, $\dot{\varphi} = x_2$, $s = x_3$, $\dot{s} = v = x_4$, constants $K_1 = \frac{gml}{J+ml^2}$, $K_2 = \frac{D}{J+ml^2}$, $K_3 = -\frac{ml}{J+ml^2}$ and $K_4 = \frac{ml}{m_v+m}$, $K_5 = \frac{B}{m_v+m}$, $K_6 = \frac{1}{m_v+m}$. Finally we can write a new representation $\Re(S)$ of the system S as

$$\begin{aligned} \Re(S) : \dot{x}_1 &= x_2 \\ \dot{x}_2 &= F_1(x_1) - F_2(x_1)x_2 - \\ &\quad -F_3(x_1)x_4 + F_4(x_1)F(t) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -F_5(x_1) + F_6(x_1)x_2 - \\ &\quad -F_7(x_1)x_4 + F_8(x_1)F(t) \\ y &= x_1, \end{aligned} \tag{40}$$

where non-linear function $F_i(\cdot)$ are:

$$F_1(x_1) = \frac{K_1 \sin(x_1)}{1 + K_3K_4\cos^2(x_1)} \tag{41}$$

$$F_2(x_1) = \frac{K_2 - K_3K_4\cos(x_1)\sin(x_1)}{1 + K_3K_4\cos^2(x_1)} \tag{42}$$

$$F_3(x_1) = \frac{K_3K_5\cos(x_1)}{1 + K_3K_4\cos^2(x_1)} \tag{43}$$

$$F_4(x_1) = \frac{K_3K_6\cos(x_1)}{1 + K_3K_4\cos^2(x_1)} \tag{44}$$

$$F_5(x_1) = \frac{K_1K_4\cos(x_1)\sin(x_1)}{1 + K_3K_4\cos^2(x_1)} \tag{45}$$

$$F_6(x_1) = \frac{K_4(K_2\cos(x_1) + \sin(x_1))}{1 + K_3K_4\cos^2(x_1)} \tag{46}$$

$$F_7(x_1) = \frac{K_5}{1 + K_3K_4\cos^2(x_1)} \tag{47}$$

$$F_8(x_1) = \frac{K_6}{1 + K_3K_4\cos^2(x_1)} \tag{48}$$

See that the force $F(t)$ has a directly influence on the position of the pendulum. And because we want to find a control for position of the pendulum only, we restrict design a controller on reduced subsystem $\Re_{red}(S)$ of the representation $\Re(S)$. Then this reduced subsystem is:

$$\begin{aligned} \Re_{red}(S) : \dot{x}_1 &= x_2 \\ \dot{x}_2 &= F_1(x_1) - F_2(x_1)x_2 - \\ &\quad -F_3(x_1)\xi + F_4(x_1)F(t) \\ y &= x_1, \end{aligned} \tag{49}$$

where variable ξ is velocity of the cart (variable x_4) and now it is representing the second input into reduced representation $\Re_{red}(S)$.

We design a control law $L(\cdot)$ for closed loop representation $\Re_{red}(S_{cl})$ of a reduced system S_{cl}:

$$\begin{aligned} \Re_{red}(S_{cl}) : \dot{x}_1 &= x_2 \\ \dot{x}_2 &= F_1(x_1) - F_2(x_1)x_2 - \\ &\quad -F_3(x_1)\xi + F_4(x_1)L(x_1, x_2) \\ y &= x_1. \end{aligned} \tag{50}$$

The following computation is identical as in the previous section. The representation $\Re^*_{red}(S_{cl})$ is in the dissipation normal form (see 36) with the same parameters (37).

Then we obtain control law and can apply it on the representation $\Re(S)$ of the system S (40):

$$\begin{aligned} L(x) = \frac{1}{F_4(x_1)}\Big[&- 3x_1^2x_2 - 10x_1 - 4x_2 \\ &+ F_3(x_1)x_4 + F_2(x_1)x_2 - \\ &- F_1(x_1)\Big] \end{aligned} \tag{51}$$

7. CONCLUSIONS

The paper dealt with the stabilization method which has a similar idea as the method mentioned in (Černý *et al.*, n.d.). This method is based on Dissipation normal form, where structure of a closed loop system is just chosen in this form. And because this form is non-linear we have more flexibility in the choice of the desired behaviour of a closed loop system than a linear one. The representation in this form is given by two conditions as you can see in (Hrušák and Černý, 2000) or (Fialová, 2000). The other plus of this method is in it, that we know the Lyapunov function of the representation of a closed loop system and then deciding about asymptotical stability is easy for us.

8. REFERENCES

Fialová, A. (2000). Synthesis of nonlinear observer using by g.t.i.o.i.a algorithm. In: *CAO2000*. IFAC. pp. 111–114.

Hrušák, J. and V. Černý (2000). Generalization of nonlinear observer design via dissipation normal form. In: *Control applications of optimization*. St. Petersburg, Russia.

Černý, V., A. Fialová and J. Hrušák (n.d.). Stabilization of non-linear systems based on metric equivalence. accepted for publication on The 9th Mediterranean Conference on Control and Automation, Croatia, June 27-29, 2001.

Zak, S. H. and C. A. Maccarley (1986). State - feedback control of non-linear systems. *Int. J. Control* **43**(5), 1497–1514.