Tomographic Reconstruction and Wavefront Set

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Applied Functional Analysis Group
Technische Universität Berlin

CT Reconstruction

Forward model

Given by the X-ray transform $\mathcal{R} : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{S}^1 \times \mathbb{R})$:

$$g = \mathcal{R}f(\theta, s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - s)dxdy$$

where $f \in \mathcal{D}'(\mathbb{R}^2)$.

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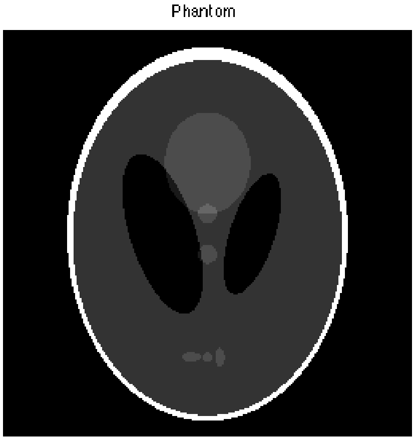
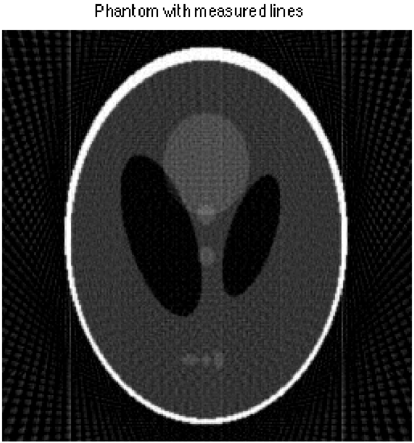
Ill-posedness:

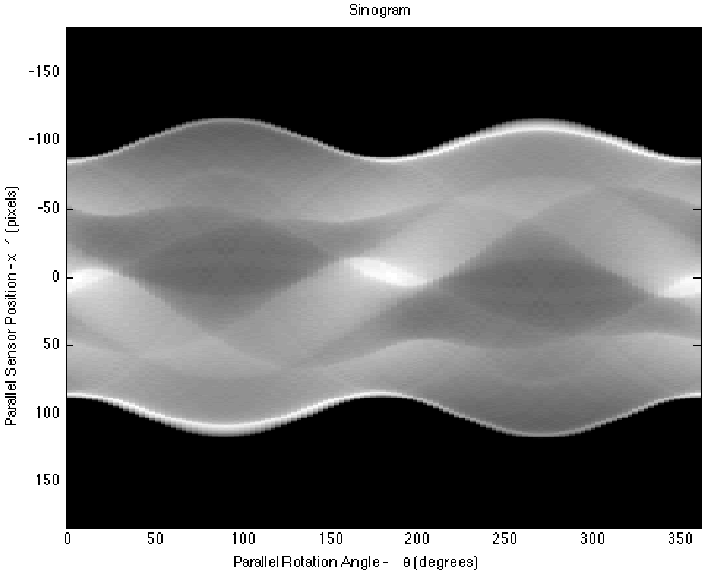
- Filtered back projection (R^{-1}) involves differentiation $\longrightarrow$ increases singularities and noise.
- R^{-1} is unbounded $\longrightarrow$ two far apart images can have very close X-ray transform.

Shepp-Logan phantom





Sinogram


Sinogram
-150
-100
-50
0
50
100
150
Parallel Sensor Position - x ′ (pixels)
0
50
100
150
200
250
300
350
Parallel Rotation Angle - θ (degrees)

Solving inverse problems

First approach

Minimizing the miss-fit against data:

$$\min_f \mathcal{L}(\mathcal{R}(f), g)$$

e.g. $\mathcal{L}(\mathcal{R}(f), g) = ||\mathcal{R}(f) - g||_2^2$. **Downside:** Ill-posedness results on overfitting.

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Approaches to address overfitting?:

- **Knowledge-driven regularization:** Model prescribed beforehand using first principles, data used to calibrate the model.
- **Data-driven regularization:** Model learned from data, without any prior first principles.
- **Hybrid:** Best of both worlds.

Knowledge-driven regularization

Pros ☺

- Guided by first principles (laws encoded by equations), tested and validated independently.
- Not much data required.
- Simple concepts, aiding the understanding.

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- **Examples:** Analytic pseudoinverse (e.g. FBP), Iterative methods with early stopping (e.g. ART), Variational methods (e.g. TV, ℓ^1).

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- Computationally exhaustive.

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- **Basic idea:** Parametrized $\mathcal{R}_\theta^\dagger : L^2(\mathbb{S}^1 \times \mathbb{R}) \longrightarrow L^2(\mathbb{R}^2)$ s.t. $\mathcal{R}_\theta^\dagger(g) = f_{\text{true}}\ \forall \theta \in \Theta$ whenever $\mathcal{R}(f_{\text{true}}) = g$. It estimates θ minimizing a loss $L : \Theta \longrightarrow \mathbb{R}_0^+$.

Hybrid methods

Motivation

- If $\mathcal{R}$ is **local** (e.g. deblurring, denoising) $\longrightarrow$ CNN and sufficient known pairs (g, f_{true}) are enough to reconstruct (Jin et al., 2016).
- If $\mathcal{R}$ is **global** (e.g. CT, MRI) $\longrightarrow$ CNN does not work, it becomes unfeasible to work with fully connected layers.

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Possible solutions

- **Learned post-processing:** First an initial (not learned) reconstruction (e.g. FBP), and denoise with CNN (Chen et al., 2017).
- **Learned regularizer:** Learn a regularization functional (e.g. dictionary learning) and perform variational regularization (Xu et al. 2012).
- **Learned iterative schemes:** Using as model a classical optimization iterative method and learn the best update in each iteration using a-priori information (Öktem et al. 2017).

Primal-dual algorithm

Minimization problem:

$$\min_{f \in L^2(\mathbb{R}^2)} \mathcal{L}(\mathcal{R}(f), g) + \mathcal{S}(f)$$

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Algorithm 2 Non-linear primal-dual algorithm

Given $\sigma, \tau > 0$ s.t. $\sigma\tau||\mathcal{R}|| < 1$, $\gamma \in [0,1]$ and $f_0 \in L^2(\mathbb{R}^2)$, $h_0 \in L^2(\mathbb{S}^1 \times \mathbb{R})$:
for $i = 1, \dots, I$ **do**
  $h_{i+1} \longleftarrow \mathrm{prox}_{\sigma\mathcal{L}}(h_i + \sigma\mathbb{R}(\overline{f}_i))$
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Proximal operator:

$$\mathrm{prox}_{\tau\mathcal{S}}(f) = \underset{f' \in L^2(\mathbb{R}^2}{\mathrm{argmin}} \left[\mathcal{S}(f') + \frac{1}{2\tau}||f' - f||_2^2 \right]$$

Learned primal-dual algorithm (Öktem et al. 2017)

Algorithm 4 Learned primal-dual algorithm

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Primal and dual operators: Γ_{θ^d} and Λ_{θ^p} are learned CNN-ResNets of the form:

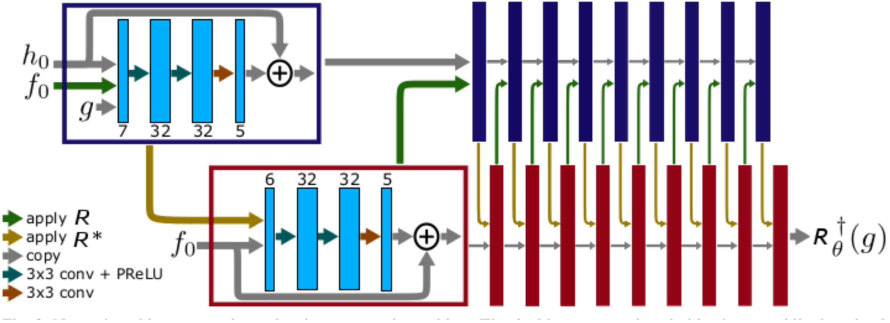
$$Id + \mathcal{W}_{w_d,b_d} \circ \mathcal{A}_{c_d} \circ \ldots \circ \mathcal{W}_{w_2,b_2} \circ \mathcal{A}_{c_2} \circ \mathcal{W}_{w_1,b_1} \circ \mathcal{A}_{c_1}$$

where,

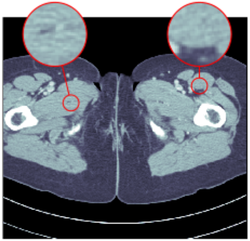
$$\mathcal{W}_{w_i,b_i}(f') = b_i + w_i * f'$$

$$\mathcal{A}_{c_i}(x) = \mathrm{PReLU}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -c_i & \text{else} \end{cases}$$

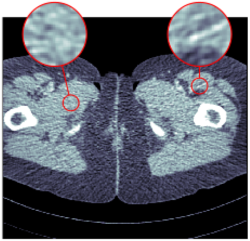
Architecture



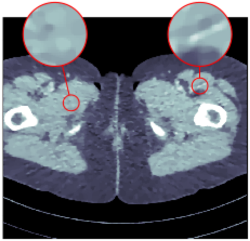
Benchmarks (Adler,Öktem)



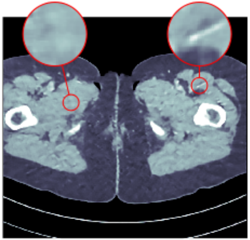
(a) 512 × 512 pixel human phantom



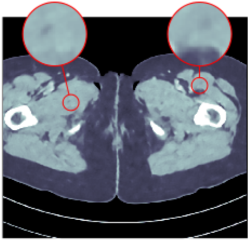
(b) Filtered back-projection (FBP)
PSNR 33.65 dB, SSIM 0.830, 423 ms



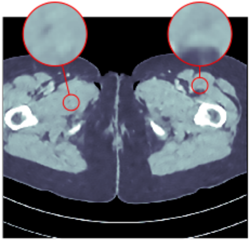
(c) Total variation (TV)
PSNR 37.48 dB, SSIM 0.946, 64 371 ms



(d) FBP + U-Net denoising
PSNR 41.92 dB, SSIM 0.941, 463 ms



(e) Primal-Dual, linear
PSNR 44.10 dB, SSIM 0.969, 620 ms



(f) Primal-Dual, non-linear
PSNR 43.91 dB, SSIM 0.969, 670 ms

Wavefront set as extra information

Definition (N-Wavefront set)

Let $N \in \mathbb{R}$ and f a distribution on $\mathbb{R}^2$. We say $(x, \lambda) \in \mathbb{R}^2 \times \mathbb{R}^2$ is a $N-$regular directed point if there exists a nbd. of U_x of x, a smooth cutoff function Φ with $\Phi \equiv 1$ on U_x and a nbd. V_λ of λ such that:

$$(\Phi f)^\wedge(\eta) = O((1 - |\eta|)^{-N}) \quad \text{for all} \quad \eta = (\eta_1, \eta_2) \quad \text{such that} \quad \frac{\eta_2}{\eta_1} \in V_\lambda$$

The $N-$Wavefront set $WF^N(f)$ is the complement of the $N-$regular directed point. The Wavefront Set $WF(f)$ is defined as

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- How can we compute the wavefront set?
- How can we incorporate it in the learned primal-dual reconstruction?

Answer: Continuous Shearlet frames and Canonical relation

Classical Shearlet Transform (Guo, Kutyniok, Labate; 2006)

$$\langle f, \psi_{a,s,t} \rangle = \int_{\mathbb{R}^2} f(x) \overline{\psi_{a,s,t}(x)} dx$$

where

$$\mathcal{SH}(\psi) = \{\psi_{a,s,t}(x) := a^{-3/4} \psi(S_s A_a x - t) : (a,s,t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^2\}$$

and

$$A_a := \begin{pmatrix} a^1 & 0 \\ 0 & a^{1/2} \end{pmatrix} \quad S_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

Resolution of the WF set with Shearlets

Theorem (Grohs, 2011)

Let ψ be a Schwartz function with infinitely many vanishing moments in x_2-direction and $\{\psi_{a,s,t}\}$ a CSF. Let f be a tempered distribution and $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, where

$$\mathcal{D}_1 = \{(t_0, s_0) \in \mathbb{R}^2 \times [-1,1] | \textit{for } (s,t) \textit{ in a neighbourhood } U \textit{ of } (s_0,t_0), \\ |\mathcal{SH}_\psi f(a,s,t)| = O(a^k), \forall k \in \mathbb{N}\}$$

and

$$\mathcal{D}_2 = \{(t_0, s_0) \in \mathbb{R}^2 \times [1,\infty) | \textit{for } (1/s,t) \textit{ in a neighbourhood } U \textit{ of } (s_0,t_0), \\ |\mathcal{SH}_{\tilde{\psi}} f(a,s,t)| = O(a^k), \forall k \in \mathbb{N}\}$$

Then,

$$WF(f) = \mathcal{D}^c$$

Canonical relation (Öktem, Quinto; 2008)

Let $f \in D'(\mathbb{R}^2)$, and assume $\mathbb{R}(\theta, s)$ is given on an open set $U \subset [0, \pi) \times \mathbb{R}$. Let $(\theta_0, s) \in U$, let

$$\xi_0 = \xi_s e_s + \xi_\sigma \sigma(\theta_0) \perp (\cos(\theta_0), \sin(\theta_0))$$

Moreover, let $x_0 \in \ell(\theta_0, s)$. Then $(x_0, \xi_0 dx) \in WF(f)$ if and only if

$$((\theta_0, s), (\xi_\sigma x \cdot \omega(\theta_0))d\theta + \xi_s ds) \in WF(\mathcal{R}(f))$$

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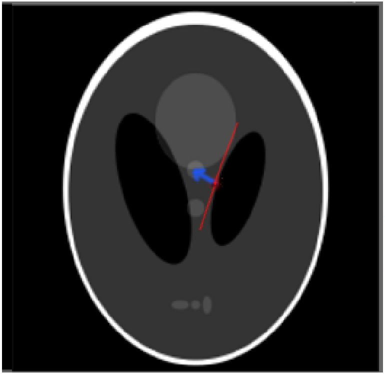

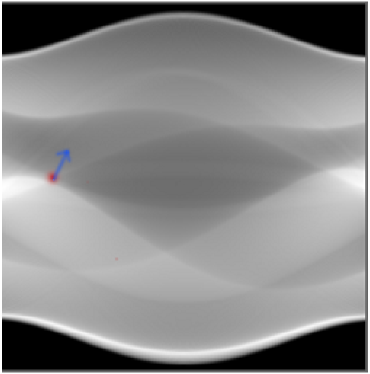

Shearlets on the sinogram

- Let $\{\psi_{a,s,t}\}_{(a,s,t)\in\mathbb{R}^+\mathbb{R}\times\mathbb{R}^2}$ continuous compactly compactly supported shearlet, with support on $[0,2\pi)\times\mathbb{R}$, that form a frame of $L^2([0,\pi),\mathbb{R})$.

  Then, the system $\{\tilde{\psi}_{a,s,t}\}_{(a,s,t)\in\mathbb{R}^+\mathbb{R}\times([0,\pi)\times\mathbb{R})}$ given by

$$\tilde{\psi}_{a,s,t}(\theta,s):=\sum_{n\in\mathbb{Z}}\psi_{a,s,t}(\theta+n\pi,s)$$

  is a Continuous Shearlet frame for $L^2(\mathbb{S}^1\times\mathbb{R})$.

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- We have all the tools!

Modified learned primal-dual

Modified problem:

$$\min_{f \in L^2(\mathbb{R}^2)} \mathcal{L}(\mathcal{R}(SH_\psi^{-1}\tilde{f}), g) \quad \text{s.t.} \ \mathbf{C}(WF(SH_\psi^{-1}\tilde{f}))) = WF(g)$$

Algorithm 6 Modified learned primal-dual algorithm

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5: $\mathbf{C}(WF(SH_{\psi}^{-1}\tilde{f}_{i+1})))=WF(h_{i+1})$
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Are we done? **Of course not!. Where is the code?!**

Implementation

- Learned primal dual on the Shearlet coefficients. ✓

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  - Learned Wavefront set extractor (in progress).
- **Lesson:** Don't assume the continuous theory will work in the computer ☹.

Thanks!

Questions?