Fast Multidimensional Signal Processing with Shearlab.jl

Héctor Andrade Loarca

TU Berlin, BMS

2nd of July, 2017

What is a signal?

Our definition

Function (or something that can be represented as) that contains information about the behavior or attributes of some phenomenon. It can be digital (discrete) or analog (continuous).

What is a signal?

Our definition

Function (or something that can be represented as) that contains information about the behavior or attributes of some phenomenon. It can be digital (discrete) or analog (continuous).

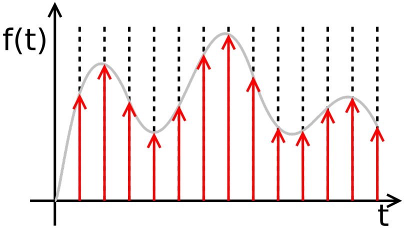


Figure: Digital and continuous one-dimensional signals

What is a signal?

Our definition

Function (or something that can be represented as) that contains information about the behavior or attributes of some phenomenon. It can be digital (discrete) or analog (continuous).

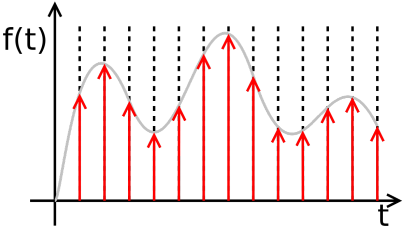


Figure: Digital and continuous one-dimensional signals

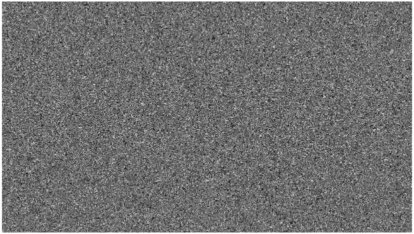


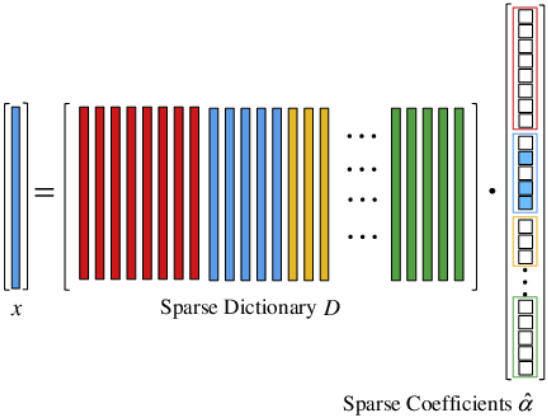
Figure: White noise, not a signal

Sparse representations of signals

- Relevant information in structured data is sparse, due the high correlation of its elements.

Sparse representations of signals

- Relevant information in structured data is sparse, due the high correlation of its elements.
- Goal: Find the right dictionary to represent optimally our data.

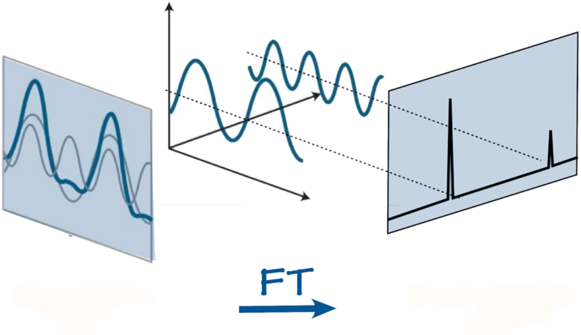


Fourier Transform (Fourier, 1822)

$$\hat{f}(\omega) := \int_{\mathbb{R}^n} f(x) e^{-i\langle x,\omega\rangle} dx$$

Fourier Transform (Fourier, 1822)

$$\hat{f}(\omega) := \int_{\mathbb{R}^n} f(x) e^{-i\langle x, \omega \rangle} dx$$

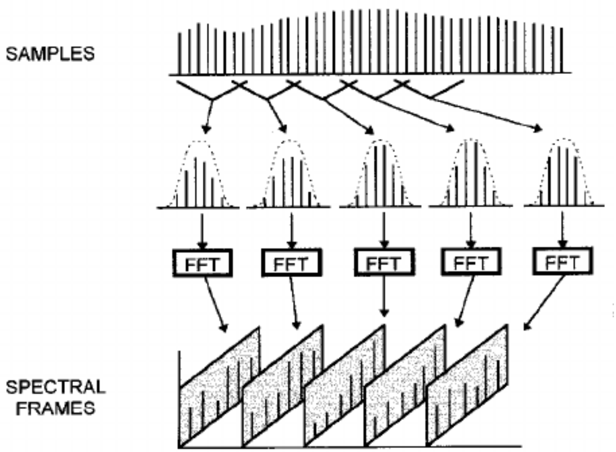


Short Time Fourier Transform (Gabor,1946)

$$S_g f(t,\omega) = \int_{\mathbb{R}} f(x)\overline{g(x-t)}e^{-ix\omega}dx$$

Short Time Fourier Transform (Gabor,1946)

$$S_g f(t,\omega) = \int_{\mathbb{R}} f(x)\overline{g(x-t)}e^{-ix\omega}dx$$



Wavelet Transform (Morlet and Grossman, 1984)

$$\begin{aligned}\mathcal{W}_\psi f(a,b) &= \int_{\mathbb{R}} f(t) a^{-\frac{1}{2}} \overline{\psi\left(\frac{t-b}{a}\right)} dt \\ &= (f * D_a \overline{\psi}^*)(b), \quad (a,b) \in \mathbb{R}^+ \times \mathbb{R}\end{aligned}$$

where

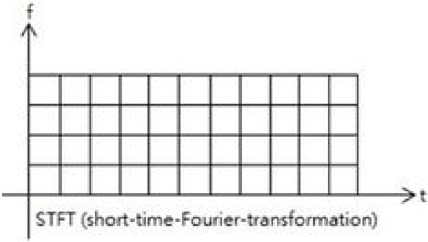
$$\int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty$$

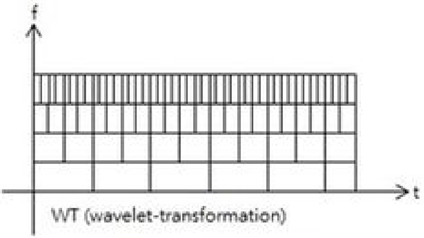
Wavelet Transform (Morlet and Grossman, 1984)

$$\begin{aligned} \mathcal{W}_\psi f(a,b) &= \int_{\mathbb{R}} f(t) a^{-\frac{1}{2}} \overline{\psi\left(\frac{t-b}{a}\right)} dt \\ &= (f * D_a \overline{\psi}^*)(b), \quad (a,b) \in \mathbb{R}^+ \times \mathbb{R} \end{aligned}$$

where

$$\int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty$$





Cartoon-like functions

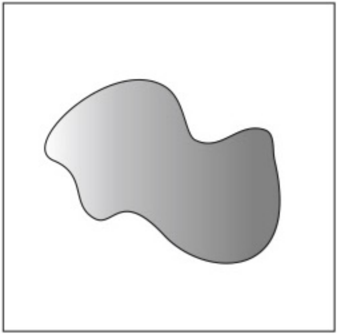
Definition

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$, $f \in \mathcal{E}^2(\mathbb{R}^2)$ if $f = f_0 + \chi_B f_1$, with $B \subset [0,1]^2$, $\partial B \in C^2$ and with bounded curvature. Moreover, $f_i \in C^2(\mathbb{R}^2)$ with $||f_i||_{C^2} \leq 1$ and supp$f_i \subset [0,1]^2$ for $i = 0, 1$.

Cartoon-like functions

Definition

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$, $f \in \mathcal{E}^2(\mathbb{R}^2)$ if $f = f_0 + \chi_B f_1$, with $B \subset [0,1]^2$, $\partial B \in C^2$ and with bounded curvature. Moreover, $f_i \in C^2(\mathbb{R}^2)$ with $||f_i||_{C^2} \leq 1$ and $\text{supp} f_i \subset [0,1]^2$ for $i = 0, 1$.



Optimal error for 2D signals

Best N-term approx. error (Donoho, 2001)

Let $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset L^2(\mathbb{R}^2)$ a frame. The optimal best N-Term approximation error for any $f \in \mathcal{R}^2(\mathbb{R}^2)$ is

$$\sigma_N(f, \{\psi_\lambda\}_{\lambda\in\Lambda}) = O(N^{-1})$$

Optimal error for 2D signals

Best N-term approx. error (Donoho, 2001)

Let $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset L^2(\mathbb{R}^2)$ a frame. The optimal best N-Term approximation error for any $f \in \mathcal{R}^2(\mathbb{R}^2)$ is

$$\sigma_N(f, \{\psi_\lambda\}_{\lambda\in\Lambda}) = O(N^{-1})$$

Error of 2D-wavelets

$$\sigma_N(f, \{\psi_{j,m}\}_{j,m}) \sim N^{-1/2}$$

Optimal error for 2D signals

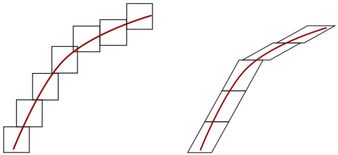
Best N-term approx. error (Donoho, 2001)

Let $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset L^2(\mathbb{R}^2)$ a frame. The optimal best N-Term approximation error for any $f \in \mathcal{R}^2(\mathbb{R}^2)$ is

$$\sigma_N(f, \{\psi_\lambda\}_{\lambda\in\Lambda}) = O(N^{-1})$$

Error of 2D-wavelets

$$\sigma_N(f, \{\psi_{j,m}\}_{j,m}) \sim N^{-1/2}$$



How to solve this? Scaling and Shearing

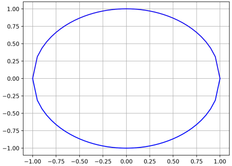
- *Scaling*

$$A_j := \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}$$

How to solve this? Scaling and Shearing

- *Scaling*

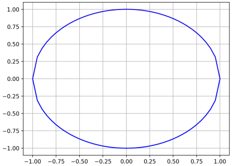
$$A_j := \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}$$

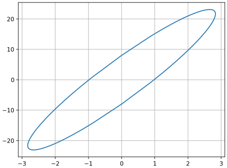


How to solve this? Scaling and Shearing

- *Scaling*

  $$A_j := \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}$$

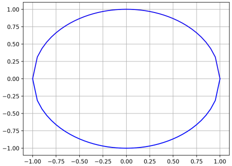


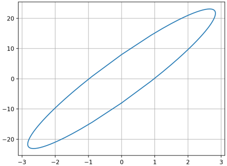


How to solve this? Scaling and Shearing

- *Scaling*

$$A_j := \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}$$





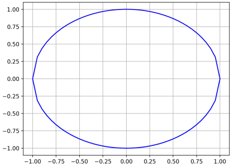
- *Shearing*

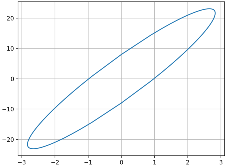
$$S_k := \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

How to solve this? Scaling and Shearing

- *Scaling*

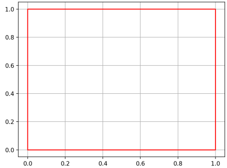
$$A_j := \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}$$





- *Shearing*

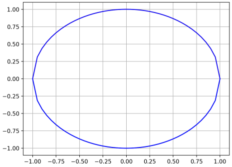
$$S_k := \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

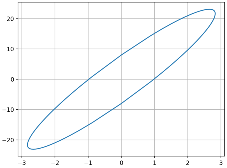


How to solve this? Scaling and Shearing

- *Scaling*

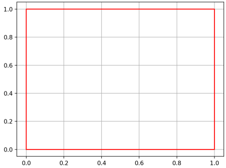
  $$A_j := \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}$$

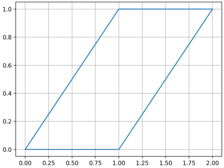




- *Shearing*

  $$S_k := \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$





Shearlet Transform (Kutyniok, Guo, Labate, 2005)

Classical Shearlet Transform

$$\langle f, \psi_{j,k,m} \rangle = \int_{\mathbb{R}^2} f(x) \overline{\psi_{j,k,m}(x)} dx$$

Shearlet Transform (Kutyniok, Guo, Labate, 2005)

Classical Shearlet Transform

$$\langle f, \psi_{j,k,m} \rangle = \int_{\mathbb{R}^2} f(x)\overline{\psi_{j,k,m}(x)}dx$$

where

$$\mathcal{SH}(\psi) = \{\psi_{j,k,m}(x) = 2^{3j/4}\psi(S_k A_j x - m) : (j,k) \in \mathbb{Z}^2, m \in \mathbb{Z}^2\}$$

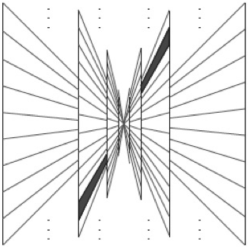
Shearlet Transform (Kutyniok, Guo, Labate, 2005)

Classical Shearlet Transform

$$\langle f, \psi_{j,k,m} \rangle = \int_{\mathbb{R}^2} f(x)\overline{\psi_{j,k,m}(x)}dx$$

where

$$\mathcal{SH}(\psi) = \{\psi_{j,k,m}(x) = 2^{3j/4}\psi(S_k A_j x - m) : (j,k) \in \mathbb{Z}^2, m \in \mathbb{Z}^2\}$$

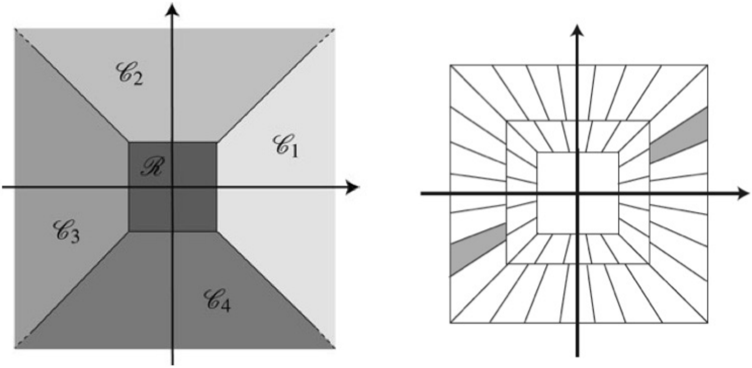


Cone-based shearlet transform

$$\mathcal{SH}(\phi, \psi, \tilde{\psi}, c) := \mathcal{P}_{\mathcal{R}}\Phi(\phi, c1) \cup \mathcal{P}_{\mathcal{C}_1}\Psi(\psi, c) \cup \mathcal{P}_{\mathcal{C}_2}\tilde{\Psi}(\tilde{\psi}, c)$$

Cone-based shearlet transform

$$\mathcal{SH}(\phi, \psi, \tilde{\psi}, c) := \mathcal{P}_{\mathcal{R}}\Phi(\phi, c1) \cup \mathcal{P}_{\mathcal{C}_1}\Psi(\psi, c) \cup \mathcal{P}_{\mathcal{C}_2}\tilde{\Psi}(\tilde{\psi}, c)$$



(a) (b)

Separable and non-separable generator

- *Separable*

  $\psi^{\mathsf{sep}}(x_1, x_2) = \psi_1(x_1)\phi_1(x_2)$

Separable and non-separable generator

- *Separable*

$$\psi^{\mathsf{sep}}(x_1, x_2) = \psi_1(x_1)\phi_1(x_2)$$

- *Non-separable*

$$\hat{\psi}^{\mathsf{non}}(\xi) = P\left(\frac{\xi_1}{2}, \xi_2\right)\hat{\psi}^{\mathsf{sep}}(\xi)$$

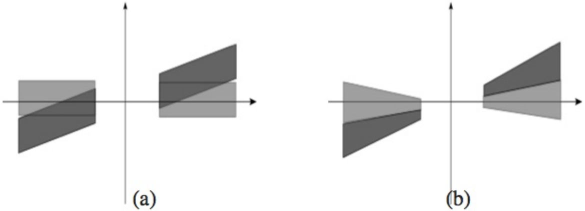
Separable and non-separable generator

- *Separable*

$$\psi^{\mathrm{sep}}(x_1, x_2) = \psi_1(x_1)\phi_1(x_2)$$

- *Non-separable*

$$\hat{\psi}^{\mathrm{non}}(\xi) = P\left(\frac{\xi_1}{2}, \xi_2\right)\hat{\psi}^{\mathrm{sep}}(\xi)$$



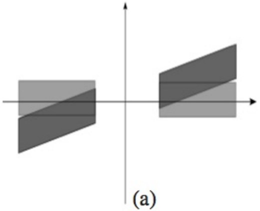
Separable and non-separable generator

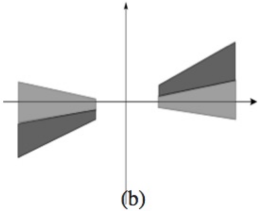
- *Separable*

$$\psi^{\text{sep}}(x_1, x_2) = \psi_1(x_1)\phi_1(x_2)$$

- *Non-separable*

$$\hat{\psi}^{\text{non}}(\xi) = P\left(\frac{\xi_1}{2}, \xi_2\right)\hat{\psi}^{\text{sep}}(\xi)$$





- Best N-term approximation error

$$\sigma_N(f, \{\psi_{j,k,m}\}_{j,k,m}) \sim N^{-1}(\log(N))^{3/2}$$

Current software

- Matlab
  - FFST- Fast Finite Shearlet Transform (Häuser, Steidl,TU Keiserlautern)
    http://www.mathematik.uni-kl.de/imagepro/software/ffst/
  - 2D/3D Shearlet Toolbox (D. Labate, University of Houston)
    https://www.math.uh.edu/~dlabate/software.html
  - **Shearlab3D** (G. Kutyniok, W.-Q.Lim, R. Reisenhoffer, TU Berlin)
    http://www.shearlab.org/

Current software

- Matlab
  - FFST- Fast Finite Shearlet Transform (Häuser, Steidl,TU Keiserlautern)
    http://www.mathematik.uni-kl.de/imagepro/software/ffst/
  - 2D/3D Shearlet Toolbox (D. Labate, University of Houston)
    https://www.math.uh.edu/~dlabate/software.html
  - **Shearlab3D** (G. Kutyniok, W.-Q.Lim, R. Reisenhoffer, TU Berlin)
    http://www.shearlab.org/
- Python
  - pyShearLab (Stefan Loock, U Götingen)
    http://na.math.uni-goettingen.de/pyshearlab/

Current software

- Matlab
  - FFST- Fast Finite Shearlet Transform (Häuser, Steidl,TU Keiserlautern)
    http://www.mathematik.uni-kl.de/imagepro/software/ffst/
  - 2D/3D Shearlet Toolbox (D. Labate, University of Houston)
    https://www.math.uh.edu/~dlabate/software.html
  - **Shearlab3D** (G. Kutyniok, W.-Q.Lim, R. Reisenhoffer, TU Berlin)
    http://www.shearlab.org/
- Python
  - pyShearLab (Stefan Loock, U Götingen)
    http://na.math.uni-goettingen.de/pyshearlab/
- **Julia**
  - **Shearlab.jl** (H. Andrade, TU Berlin)
    https://github.com/arsenal9971/Shearlab.jl

Why Julia?

- Extensive use of `fft`, well implemented in Julia.

Why Julia?

- Extensive use of fft , well implemented in Julia.
- Fast vectorization and loops as well as JIT-compilation.

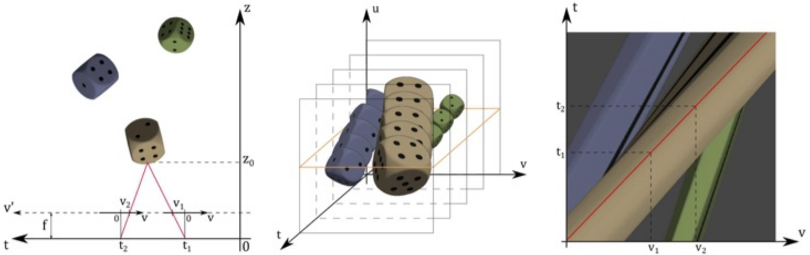
Why Julia?

- Extensive use of fft , well implemented in Julia.
- Fast vectorization and loops as well as JIT-compilation.
- Plenty of image filtering, import and rescaling functions with Images. jl , Wavelets. jl .

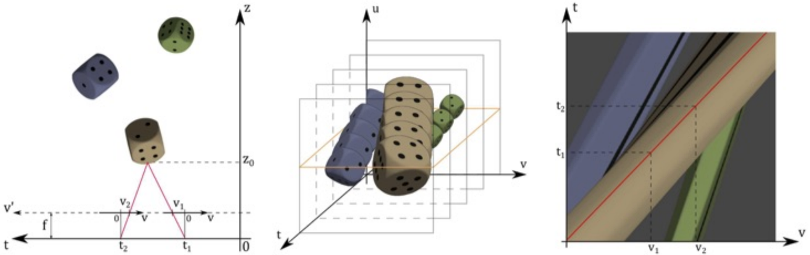
Why Julia?

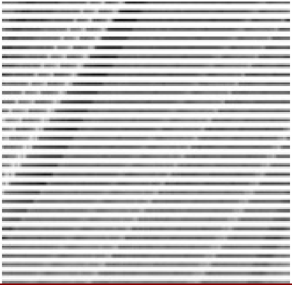
- Extensive use of fft , well implemented in Julia.
- Fast vectorization and loops as well as JIT-compilation.
- Plenty of image filtering, import and rescaling functions with Images. jl , Wavelets. jl .
- Support of multithreading and painless GPU processing with ArrayFire . jl .

Current application: Light Field Recovery

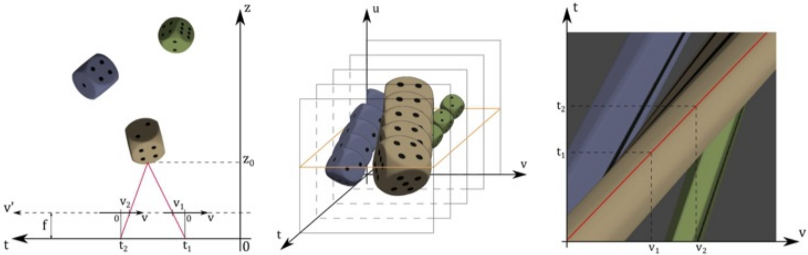


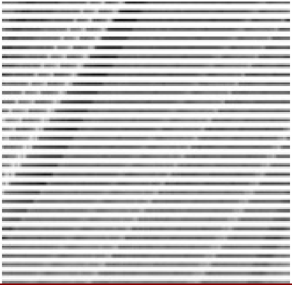
Current application: Light Field Recovery

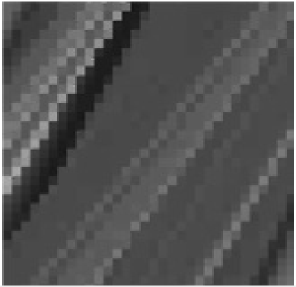




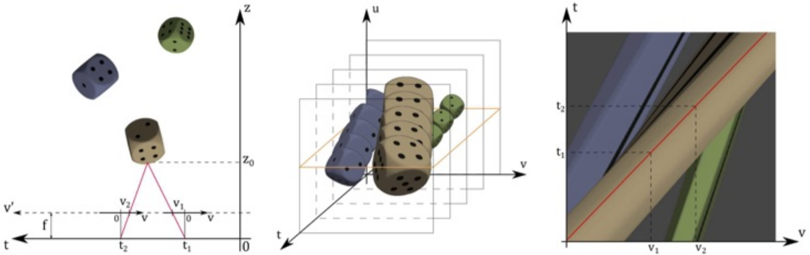
Current application: Light Field Recovery

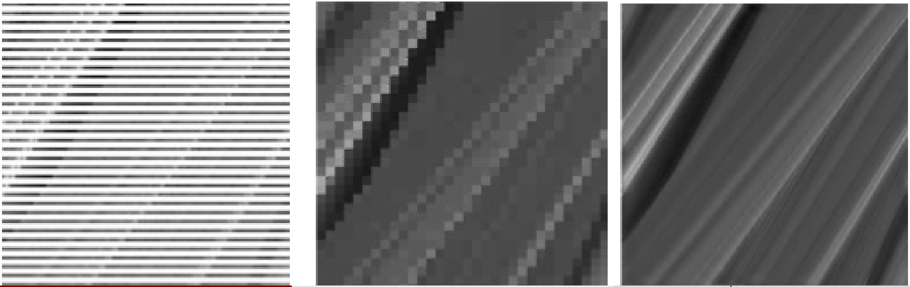






Current application: Light Field Recovery





Thanks!

Questions?