Fast Multidimensional Signal Processing with Shearlab.jl

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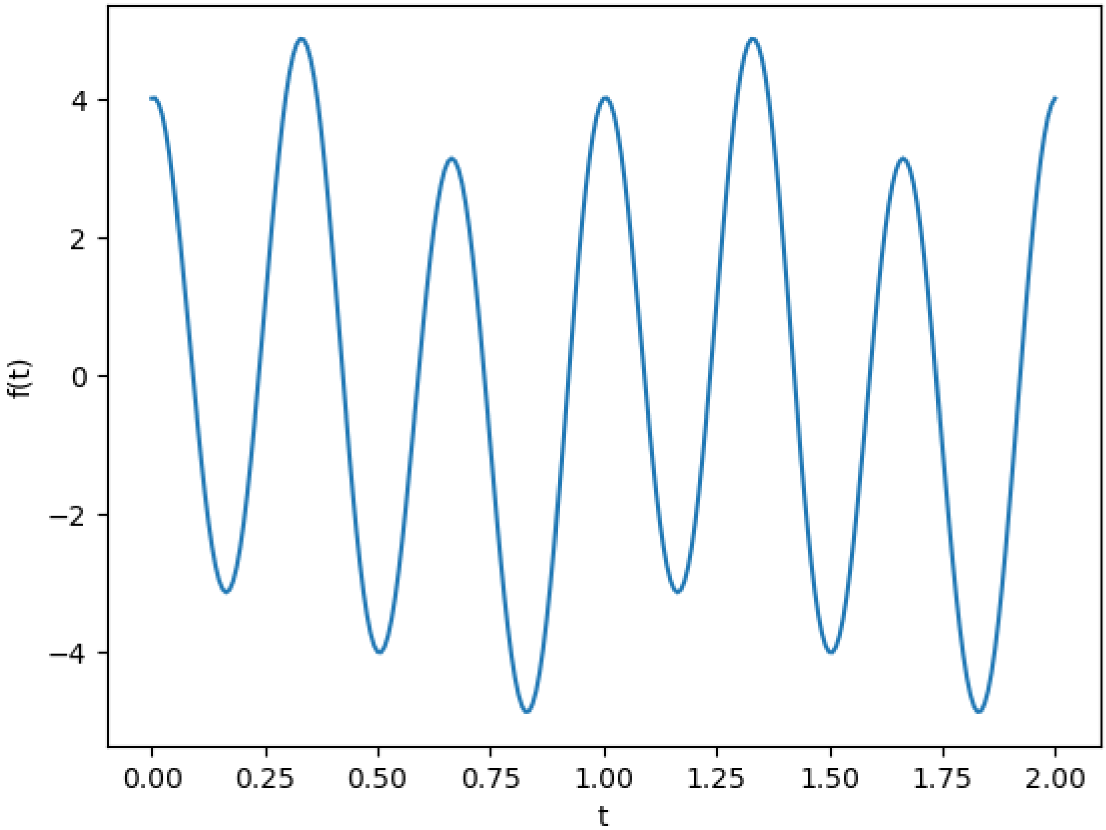
Fourier Transform.

In [1]:
```
using PyPlot
```

In [2]:
```
# Create and plot signal
t=0:2/256:2;
f=sin(2*pi*t)+2*(cos(6*pi*t))*2;
plot(t,f)
xlabel("t")
ylabel("f(t)")
```

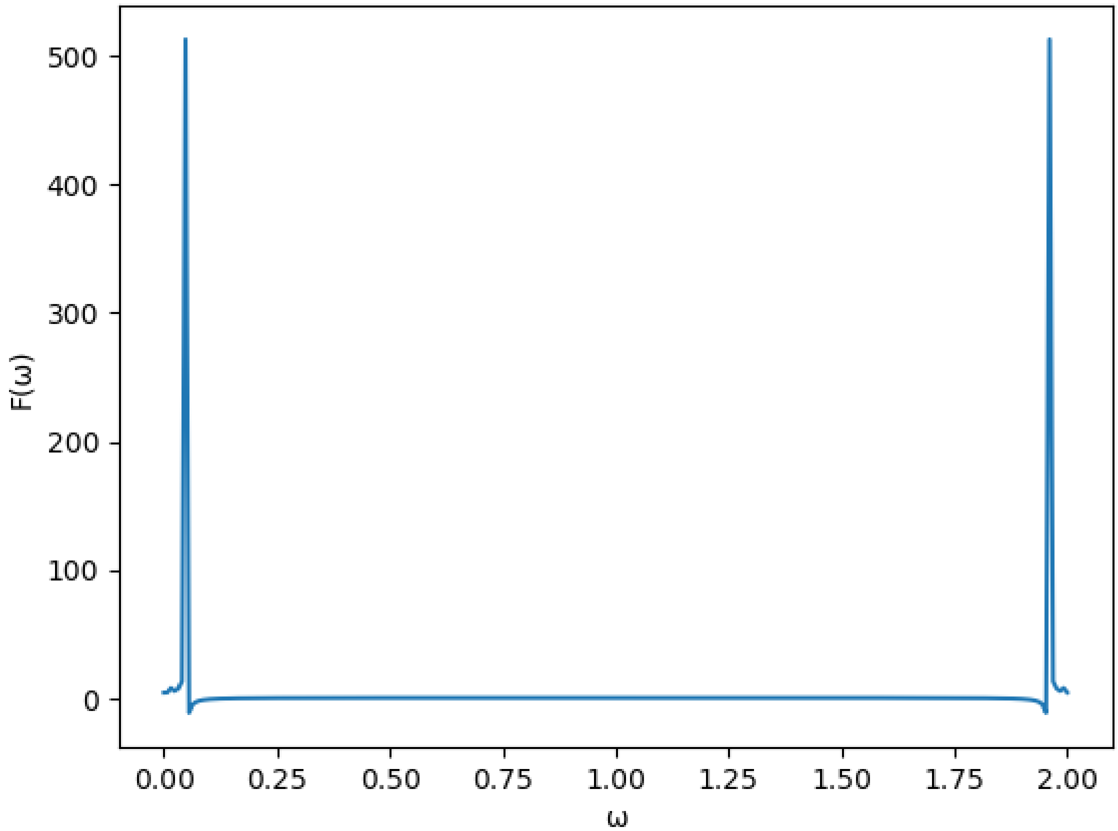



```
Out[2]: PyObject <matplotlib.text.Text object at 0x31ff4c550>
```

In [3]:
```
# Compute fast fourier transform
@time F = fft(f);
```

```
0.511862 seconds (414.51 k allocations: 17.621 MB, 1.52% gc time)
```

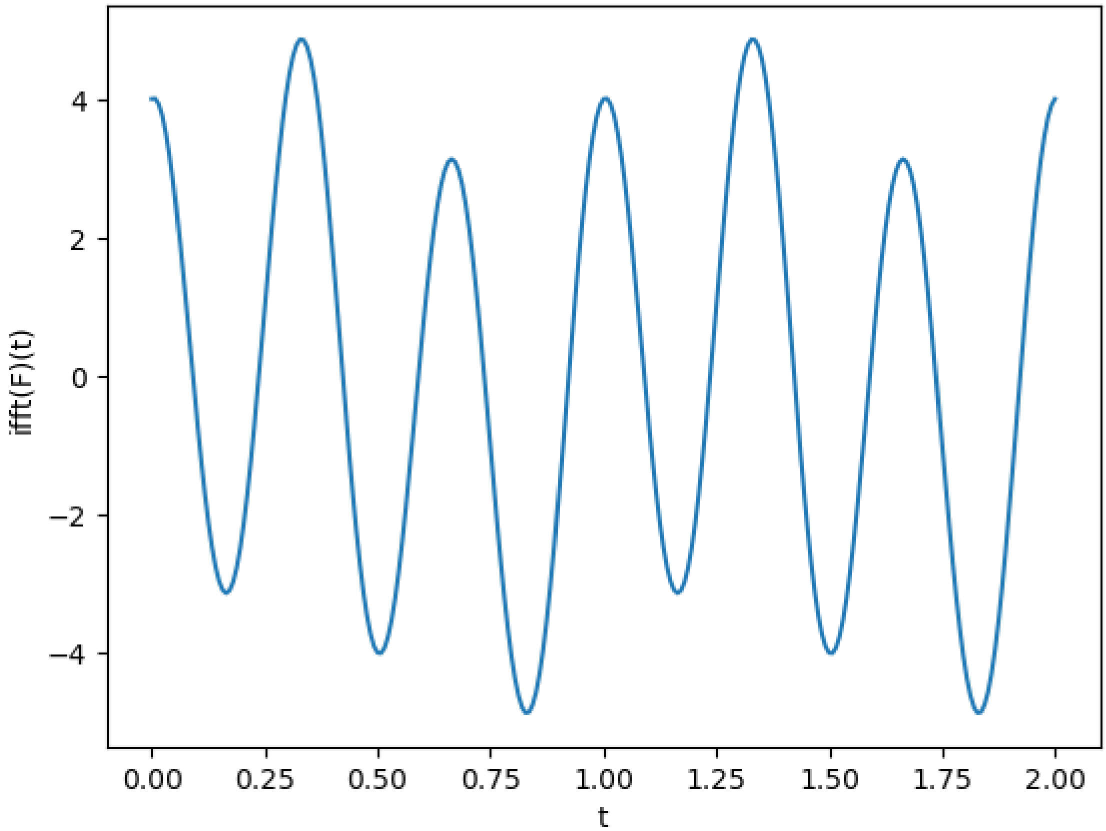
In [4]:
```
plot(t,F)
xlabel("ω")
ylabel("F(ω)")
```




```
/Users/hector/.julia/v0.5/Conda/deps/usr/lib/python2.7/site-packages/nump
y/core/numeric.py:531: ComplexWarning: Casting complex values to real dis
cards the imaginary part
  return array(a, dtype, copy=False, order=order)
```

Out[4]:
```
PyObject <matplotlib.text.Text object at 0x320064d90>
```

In [5]:
```
plot(t,ifft(F))
xlabel("t")
ylabel("ifft(F)(t)")
```



Out[5]: PyObject <matplotlib.text.Text object at 0x31feaa990>

Wavelet Transform

In [6]:
```
# using Shearlab
# Loard the Pkg
push!(LOAD_PATH,pwd()*"/../../../Shearlab.jl/src")
import Shearlab
reload("Shearlab")
```

```
WARNING: replacing module Shearlab
```

The 2D wavelet transform

$$d_j^k[n] = \langle f, \psi_{j,n}^k \rangle$$

for scales $j \in \mathbb{Z}$, position $n \in \mathbb{Z}^2$ and orientation $k \in \{H, V, D\}$.

And wavelet atoms defined by scaling and translating three mother atoms $\{\psi^H, \psi^V, \psi^D\}$:

$$\psi_{j,n}^k(x) = \frac{1}{2^j}\psi^k\left(\frac{x-2^j n}{2^j}\right)$$

Defined by tensor product of a 1-D wavelet function $\psi(t)$ and a 1-D scaling function $\phi(t)$

$$\psi^H(x) = \phi(x_1)\psi(x_2),\ \psi^V(x) = \psi(x_1)\phi(x_2) \text{ and } \psi^D(x) = \psi(x_1)\psi(x_2).$$

Translated into high-pass and low-pass filters

$$g[n] = \frac{1}{\sqrt{2}}\langle\psi(t/2), \phi(t-n)\rangle \text{ and } h[n] = \frac{1}{\sqrt{2}}\langle\phi(t/2), \phi(t-n)\rangle.$$

1D

Compute the low pass signal $a \in \mathbb{R}^{N/2}$ and the high pass signal $d \in \mathbb{R}^{N/2}$ as

$$a = (f * h) \downarrow 2 \text{ and } d = (f * g) \downarrow 2$$

where the sub-sampling is defined as

$$(u \downarrow 2)[k] = u[2k].$$

where $g[n] = (-1)^{1-n} h[1-n]$.

When the filters are orthogonal transform the inverse will be

$$(a \uparrow 2) * \tilde{h} + (d \uparrow 2) * \tilde{g} = f$$

where $\tilde{h}[n] = h[-n]$ (computed modulo N) and $(u \uparrow 2)[2n] = u[n]$ and $(u \uparrow 2)[2n+1] = 0$.

2D

One needs to perform filtering/downsampling in the different directions

$$\tilde{a}_{j-1} = (a_j *^H h) \downarrow^{2,H} \text{ and } \tilde{d}_{j-1} = (a_j *^H g) \downarrow^{2,H}.$$

Here, the operator $*^H$ and $\downarrow^{2,H}$ are defined by applying $*$ and $\downarrow^2$ to each column of the matrix.

$$a_{j-1} = (\tilde{a}_j *^V h) \downarrow^{2,V} \text{ and } d^V_{j-1} = (\tilde{a}_j *^V g) \downarrow^{2,V},$$

$$d^H_{j-1} = (\tilde{d}_j *^V h) \downarrow^{2,V} \text{ and } d^D_{j-1} = (\tilde{d}_j *^V g) \downarrow^{2,V}.$$

In [7]:
```
using Wavelets
```

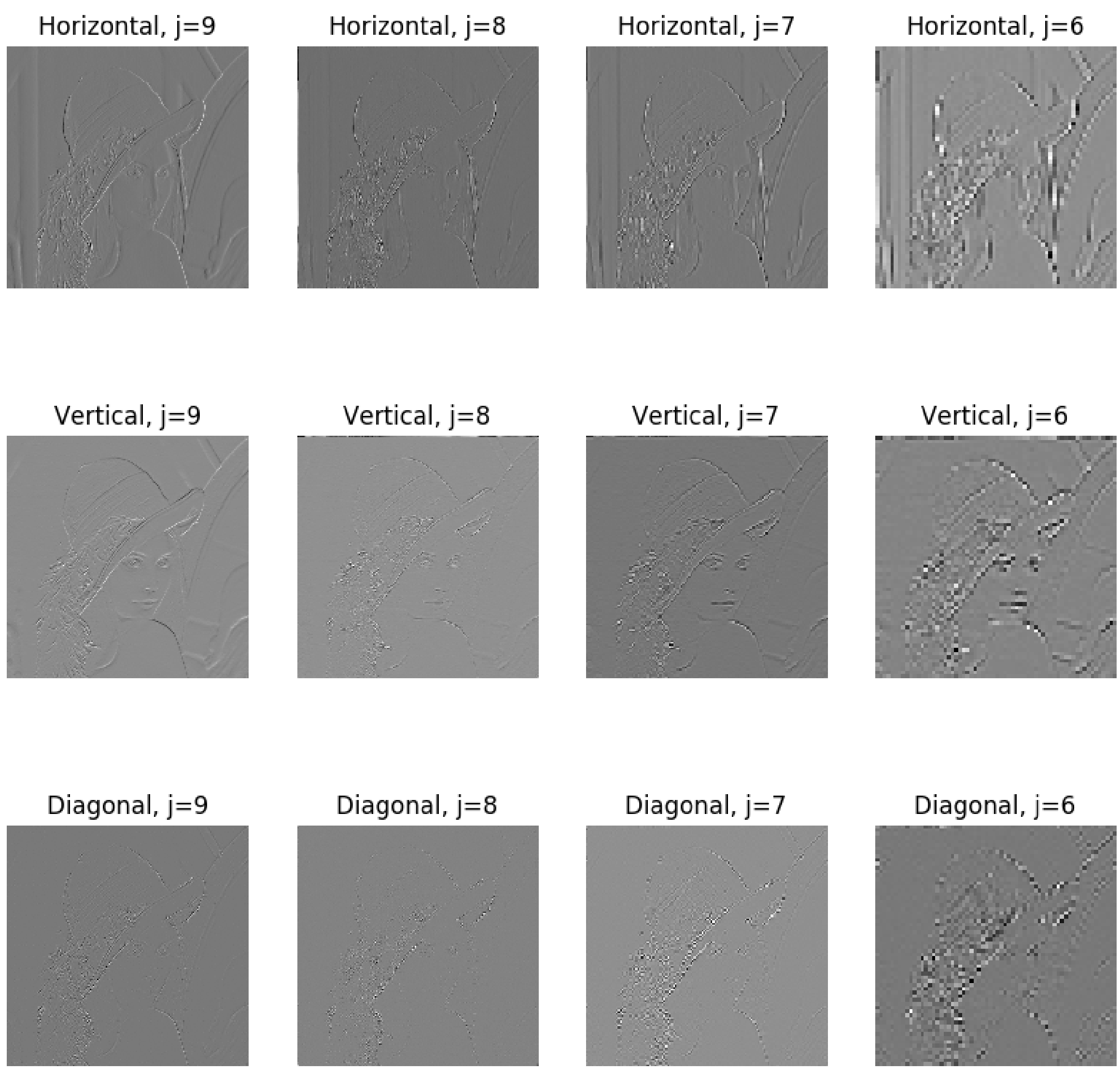
In [8]:
```
# Pick Daubechie filter and its mirror
h = Shearlab.filt_gen(WT.db2)
g = Shearlab.mirror(h)
```

Out[8]:
```
5-element Array{Float64,1}:
 -0.0
 -0.12941
 -0.224144
  0.836516
 -0.482963
```

In [9]:
```
# Read Data
n = 1024;
# The path of the image
name = "../../../ShearLab.jl/data_samples/lena.jpg";
f = Shearlab.load_image(name, n,n,1);
```

In [10]:
```
#Rescale image in [0,1] summing the 3 arrays in the RGB format
f = Shearlab.rescale(sum(f,3));
# Reduce one dimension
f = f[:,:,1];
Shearlab.imageplot(f);
```

In [11]:
```
# Different scales
Jmax = round(Int64,log2(n))-1;
Jmin = 1;
fW = copy(f);
clf;
figure(figsize=(10,10));
for j=Jmax:-1:Jmin
    A = fW[1:2^(j+1),1:2^(j+1)];
    for d=1:2
        Coarse = Shearlab.subsampling(Shearlab.cconvol(A,h,d),d);
        Detail = Shearlab.subsampling(Shearlab.cconvol(A,g,d),d);
        A = cat(d, Coarse, Detail );
    end
    fW[1:2^(j+1),1:2^(j+1)] = A;
    j1 = Jmax-j;
    if j1<4
        Shearlab.imageplot(A[1:2^j,2^j+1:2^(j+1)], "Horizontal, j=$j", 3,4,
        Shearlab.imageplot(A[2^j+1:2^(j+1),1:2^j], "Vertical, j=$j", 3,4, j1
        Shearlab.imageplot(A[2^j+1:2^(j+1),2^j+1:2^(j+1)], "Diagonal, j=$j",
    end
end
```



In [12]:

```
f1 = copy(fW);
clf;
figure(figsize=(10,10));
for j=Jmin:Jmax
    A = f1[1:2^(j+1),1:2^(j+1)];
    for d=1:2
        if d==1
            Coarse = A[1:2^j,:];
            Detail = A[2^j+1:2^(j+1),:];
        else
            Coarse = A[:,1:2^j];
            Detail = A[:,2^j+1:2^(j+1)];
        end
        Coarse = Shearlab.cconvol(Shearlab.upsampling(Coarse,d),Shearlab.rev
        Detail = Shearlab.cconvol(Shearlab.upsampling(Detail,d),Shearlab.rev
        A = Coarse + Detail;
        j1 = Jmax-j;
        if j1>0 && j1<5
            Shearlab.imageplot(A, "Partial reconstruction, j=$j", 2,2,j1);
        end
    end
    f1[1:2^(j+1),1:2^(j+1)] = A;
end
```

Partial reconstruction, j=8

Partial reconstruction, j=7

Partial reconstruction, j=6

Partial reconstruction, j=5

In [13]:
```
# number of kept coefficients
m = round(n^2/16);
# compute the threshold T
Jmin = 1;
fW = Shearlab.perform_wavortho_transf(f,Jmin,+1, h);
# select threshold
v = sort(abs(fW[:]));
if v[1]<v[n^2]
    v = Shearlab.reverse(v);
end
# inverse transform
T = v[Int(m)];
fWT = fW .* (abs(fW).>T);2
# inverse
fnlin = Shearlab.perform_wavortho_transf(fWT,Jmin,-1, h);
# display
clf;
figure(figsize=(10,10));
u1 = @sprintf("Original");
u2 = @sprintf("Thresholding, SNR=%.3fdB", Shearlab.snr(f,fnlin));
Shearlab.imageplot(Shearlab.clamp(f),u1, 1,2,1 );
Shearlab.imageplot(Shearlab.clamp(fnlin),u2, 1,2,2 );
```

Non-separable Shearlet Transform

$$\mathrm{DST}^{2D}_{j,k,m}(f) = \left(\overline{\psi^d_{j,k}} * f_J\right)\left(2^J A^{-1}_{2^j} M_{c_j} m\right) \text{ for } j = 0, 1, \ldots, J-1$$

with generating function

$$\hat{\psi}(\xi) = P(\xi_1/2, \xi_2/2)\widehat{\psi_1 \otimes \phi_1}(\xi)$$

where

$$\psi^d_{j,k} = S^d_{k/2^{j/2}}(p_j * W_j)$$

A_{2^j} is the parabolic scaling matrix given by

$$A_{2^j} = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}$$

and the Shearing transform is given by

$$S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

The scaling and wavelet filters now in each scale are:

$$\hat{h}_j(\xi_1) = \prod_{k=0}^{j-1} \hat{h}(2^k \xi_1)\ ,\ \hat{g}_j(\xi_1) = \hat{g}\left(\frac{2^j \xi_1}{2}\right) \hat{h}_{j-1}(\xi_1)$$

and the directional filter transform

$$p_j(\xi_1, \xi_2) = \widehat{P(2^{J-j-1}\xi_1, 2^{J-j/2}\xi_2)} \Longrightarrow p_j * W_j = p_j * g_{J-j} \otimes h_{J-j}$$

The digital shearing to mantain the domain grid will be

$$S_{k/2^{j/2}}(x) = \left(\left(x_{\uparrow 2^{j/2}} *_1 h_{j/2} \right) (S_k \cdot) *_1 \overline{h_{j/2}} \right)_{\downarrow 2^{j/2}}$$

for $j \in \{0, J-1\}$ and $|k| \leq \lceil 2^{j/2} \rceil$.

In [14]:
```
# Read Data
n = 512;
# The path of the image
name = "../../../ShearLab.jl/data_samples/lena.jpg";
data_nopar = Shearlab.load_image(name, n);
data_par = Shearlab.load_image(name, n,n,1);
```

In [15]:
```
# Reduce one dimension
data_nopar = data_nopar[:,:,1];
data_par = data_par[:,:,1]
Shearlab.imageplot(data_nopar);
```

In [16]:
```
Shearlab.imageplot(data_par);
```

In [17]:
```
# Size of the images
sizeX_nopar = size(data_nopar,1);
sizeY_nopar = size(data_nopar,2);
sizeX_par = size(data_par,1);
sizeY_par = size(data_par,2);
```

In [18]:
```
# Set the variables for the Shearlet trasform
rows_nopar = sizeX_nopar;
cols_nopar = sizeY_nopar;
rows_par = sizeX_par;
cols_par = sizeY_par;
X_nopar = data_nopar;
X_par = data_par;
```

In [19]:
```
# No. of scales
nScales = 4;
shearLevels = ceil((1:nScales)/2)
scalingFilter = Shearlab.filt_gen("scaling_shearlet");
directionalFilter = Shearlab.filt_gen("directional_shearlet");
waveletFilter = Shearlab.mirror(scalingFilter);
scalingFilter2 = scalingFilter;
full = 0;
```

Generation of Shearlet System

In [24]:
```
# Compute the corresponding shearlet system without gpu
@time shearletSystem_nopar= Shearlab.getshearletsystem2D(rows_nopar,cols_nop
                                directionalFilter,
                                scalingFilter,0);
```

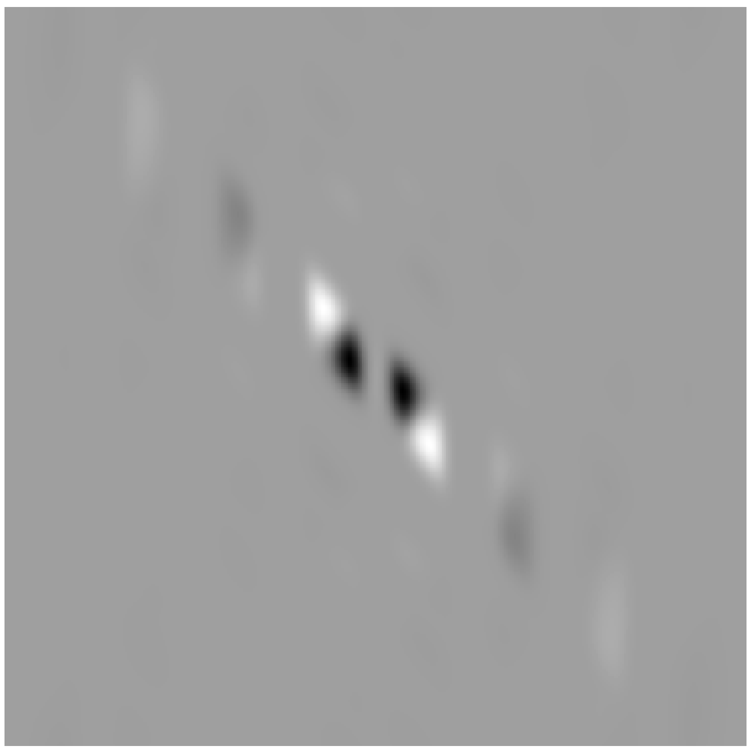
```
  8.314009 seconds (154.89 k allocations: 5.024 GB, 45.95% gc time)
```

In [21]:
```
using ArrayFire
```

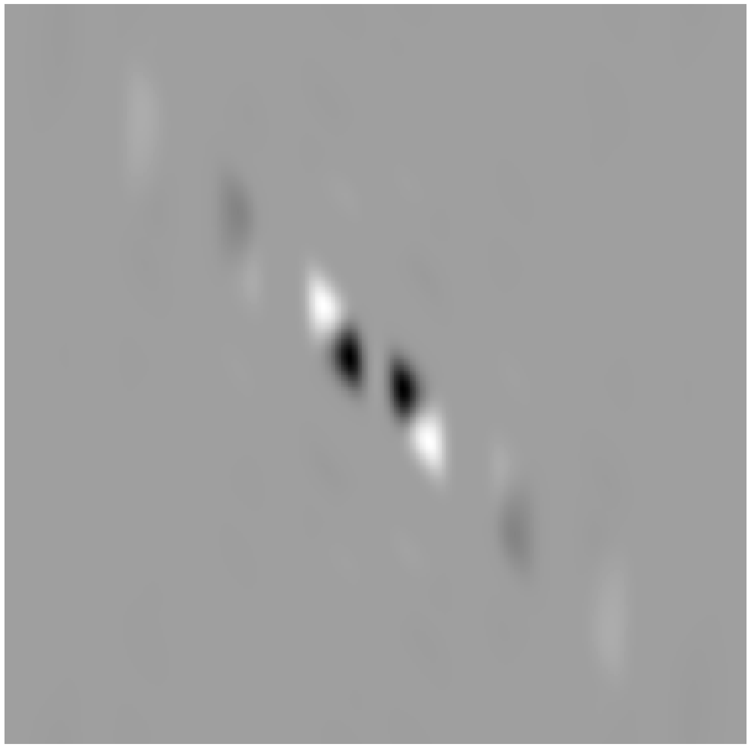
In [25]:
```
# Compute the corresponding shearlet system without gpu
@time shearletSystem_par = Shearlab.getshearletsystem2D(rows_par,cols_par,nS
                                directionalFilter,
                                scalingFilter,1);
```

```
  2.425434 seconds (173.99 k allocations: 506.944 MB, 15.23% gc time)
```

In [26]:
```
shearlet1 = Array(shearletSystem_nopar.shearlets[:,:,1]);
Shearlab.imageplot(real(shearlet1));
```

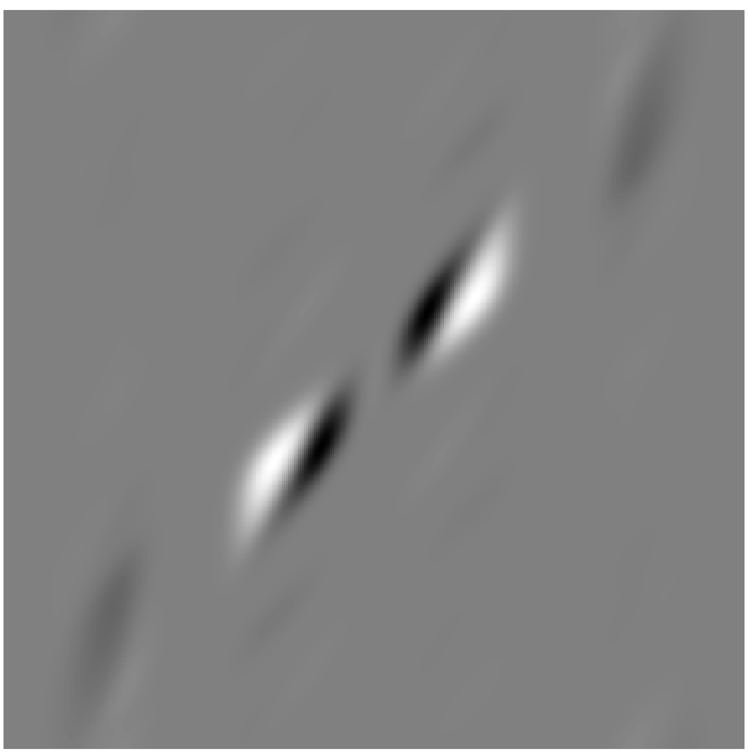


In [27]:
```
# Comparison with the generated by arrayfire
shearlet1 = Array(shearletSystem_par.shearlets[:,:,1]);
Shearlab.imageplot(real(shearlet1));
```



In [28]:
```
shearlet3 = Array(shearletSystem_nopar.shearlets[:,:,3]);
Shearlab.imageplot(real(shearlet3))
```

In [29]:
```
shearlet10 = Array(shearletSystem_nopar.shearlets[:,:,10]);
Shearlab.imageplot(real(shearlet10))
```



In [30]:
```
shearlet17 = Array(shearletSystem_nopar.shearlets[:,:,49]);
Shearlab.imageplot(real(shearlet17))
```

Computation of shearlet coefficients

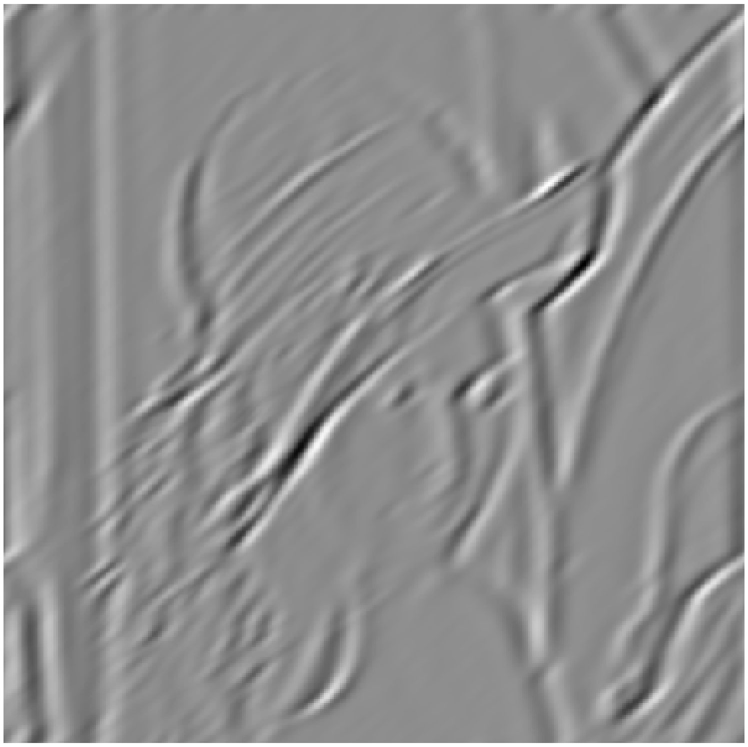
In [33]:
```
# Compute the coefficients
@time coeffs_nopar = Shearlab.SLsheardec2D(X_nopar,shearletSystem_nopar);
```

```
  1.205391 seconds (7.12 k allocations: 1.354 GB, 26.23% gc time)
```

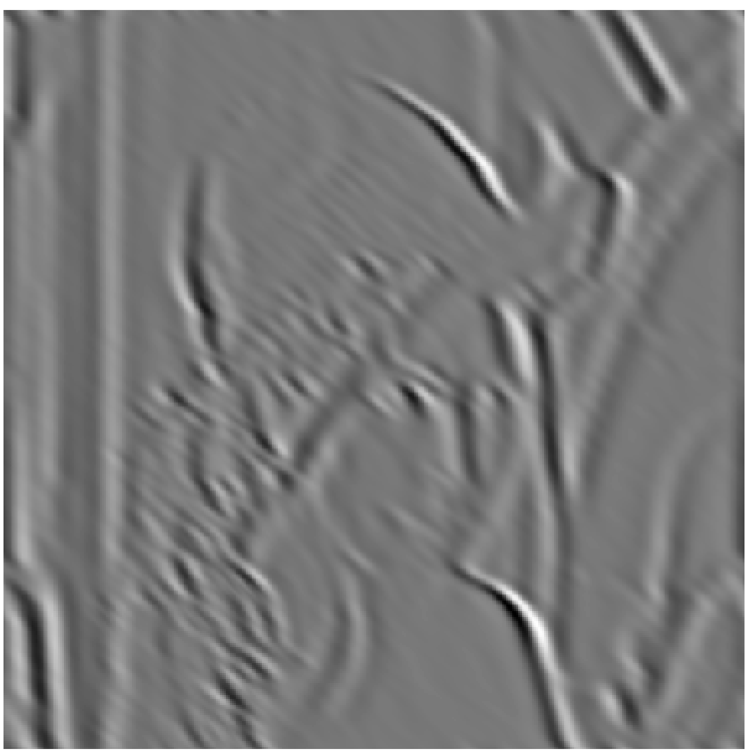
In [34]:
```
# Compute the coefficients in parallel
X_par = AFArray(convert(Array{Float32},X_par));
@time coeffs_par = Shearlab.SLsheardec2D(X_par,shearletSystem_par);
```

```
  0.273929 seconds (5.06 k allocations: 98.115 MB, 3.37% gc time)
```

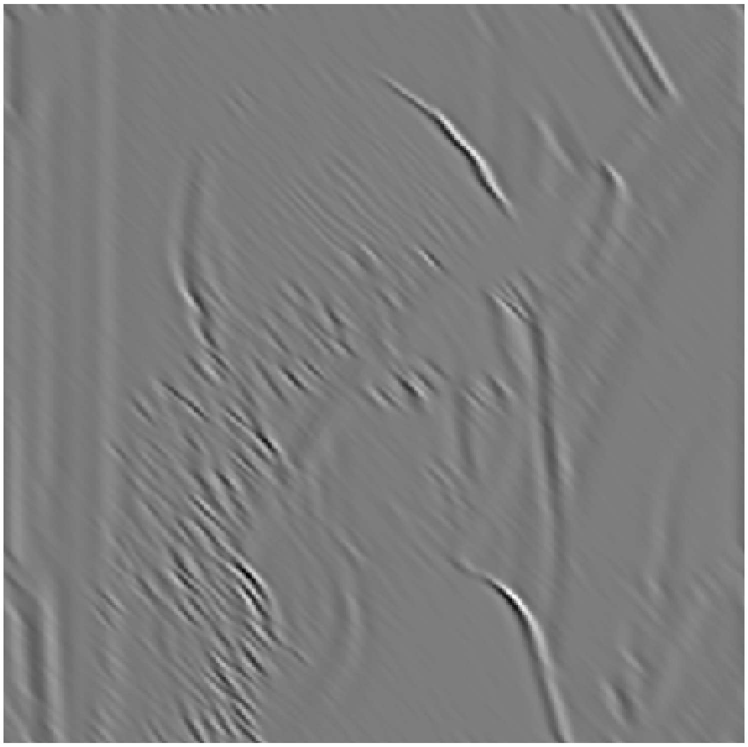
In [35]:
```
Shearlab.imageplot(real(Array(coeffs_nopar[:,:,1])))
```



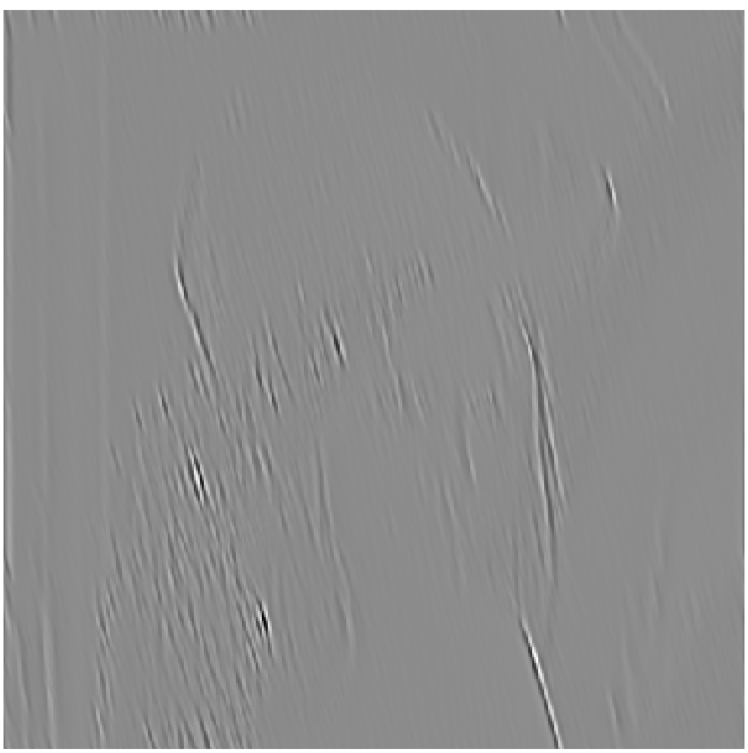
In [36]:
```
Shearlab.imageplot(real(Array(coeffs_nopar[:,:,5])))
```



In [37]:
```
Shearlab.imageplot(real(Array(coeffs_nopar[:,:,10])))
```



In [38]:
```
Shearlab.imageplot(real(Array(coeffs_nopar[:,:,16])))
```



Reconstruction of the image with the shearlet system and coefficients

In [41]:
```
# Make the recovery
@time Xrec_nopar=Shearlab.SLshearrec2D(coeffs_nopar,shearletSystem_nopar);
```

```
  0.969499 seconds (6.17 k allocations: 1.364 GB, 21.72% gc time)
```

In [42]:
```
# Make the recovery in parallel
@time Xrec_par=Shearlab.SLshearrec2D(coeffs_par,shearletSystem_par);
```

```
  0.010846 seconds (5.55 k allocations: 2.126 MB)
```

In [43]:
```
# The recovery is very good
Shearlab.imageplot(Array(Xrec_nopar));
```

In [44]:
```
# The recovery is very good
Shearlab.imageplot(Array(Xrec_par));
```

Benchmarks with matlab version

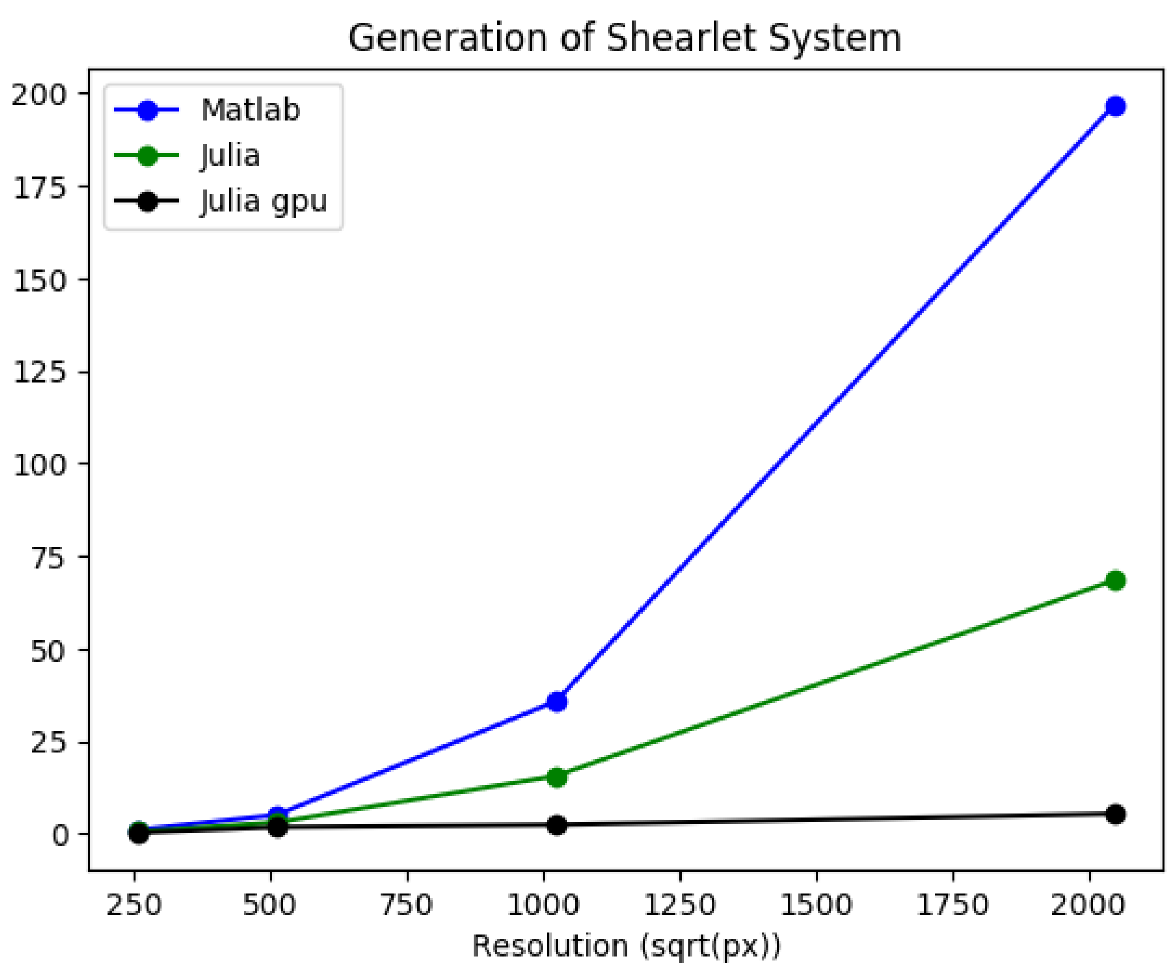
- **2D version.**

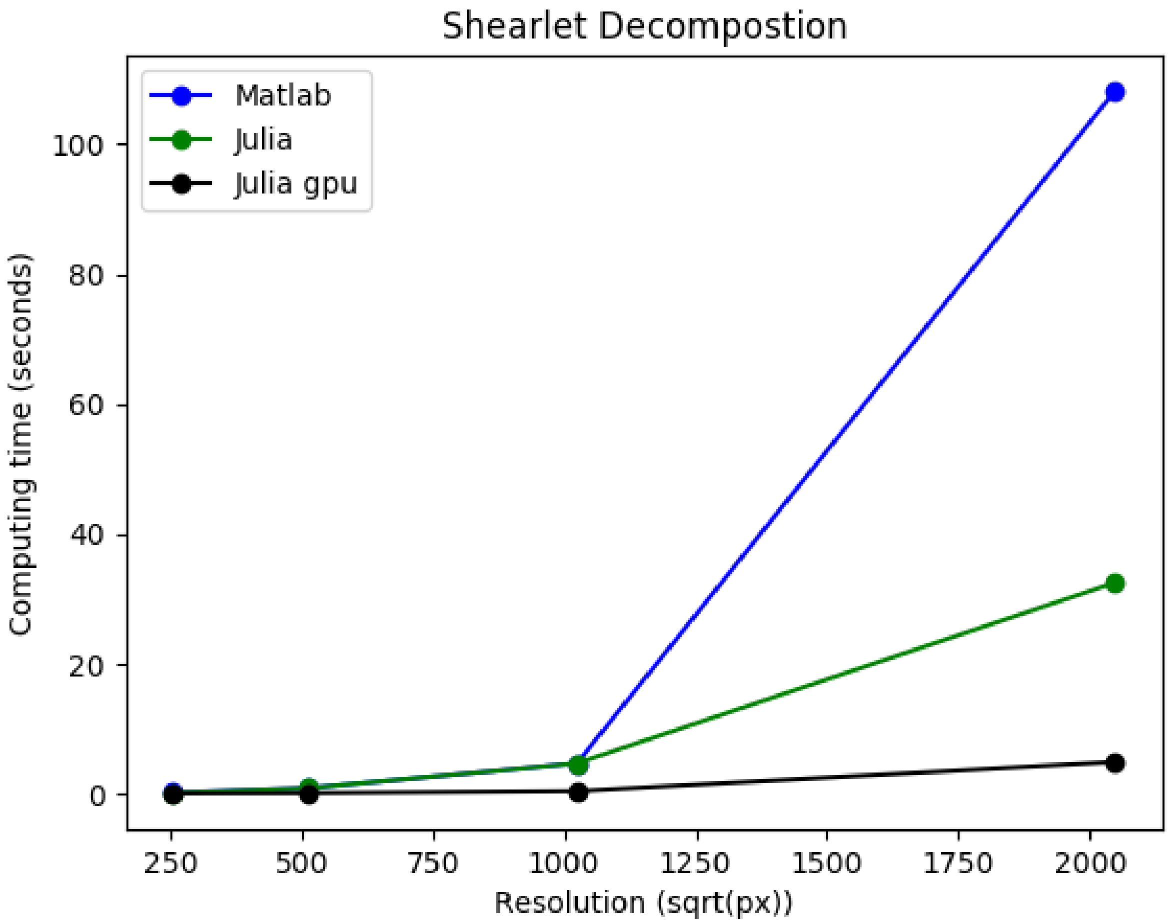
Benchmark	Matlab(seconds)	Julia no gpu(seconds)	Julia gpu (seconds)	Improvement rate no gpu	Improvement rate gpu
Shearlet System 256x256	1.06	0.61	0.31	1.73	3.42
Decoding 256x256	0.18	0.15	0.043	1.2	4.19
Reconstruction 256x256	0.18	0.12	0.018	1.5	8.57
Shearlet System 512x512	5.15	3.07	1.8	1.22	2.08
Decoding 512x512	0.96	0.87	0.09	1.10	10.66
Reconstruction 512x512	0.84	0.52	0.021	1.62	14.00

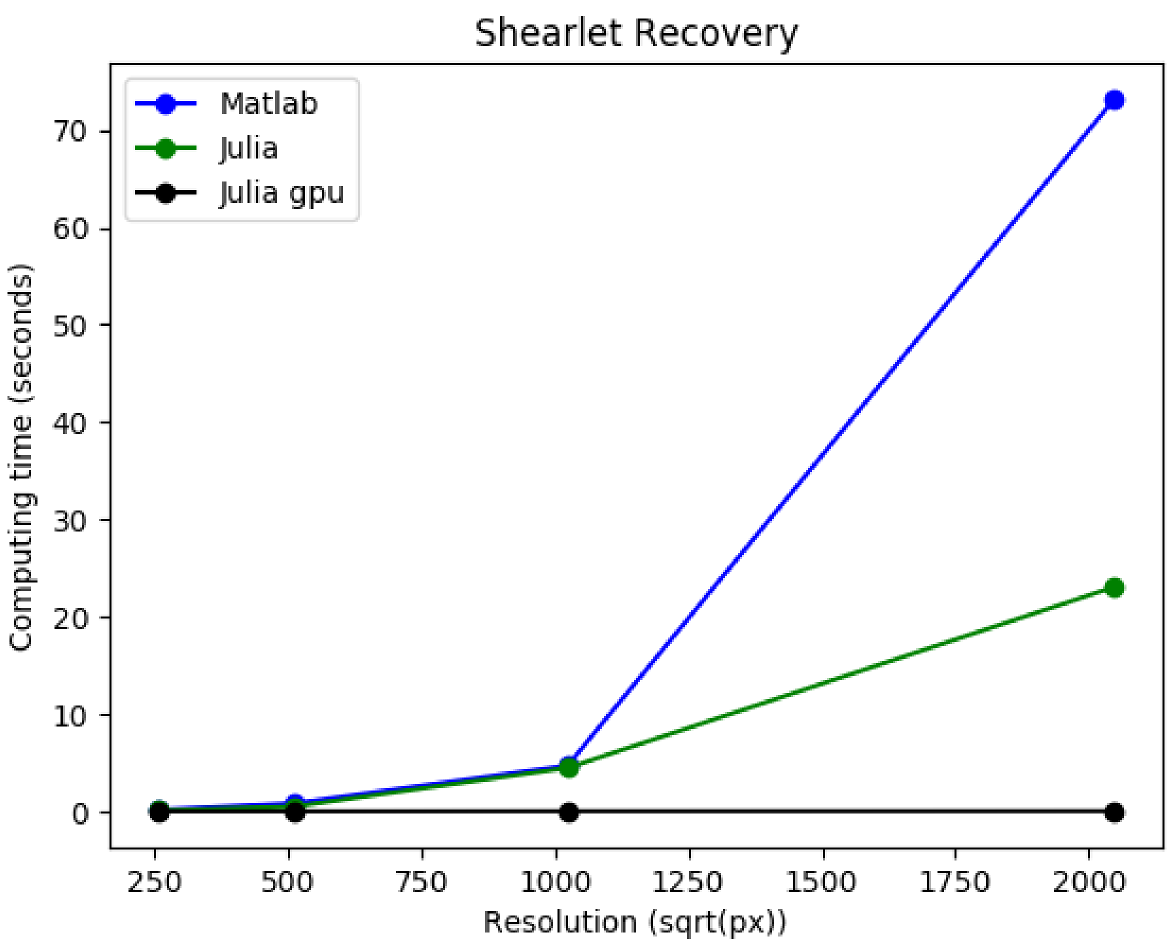
Benchmark	Matlab(seconds)	Julia no gpu(seconds)	Julia gpu (seconds)	Improvement rate no gpu	Improvement rate gpu
Shearlet System 1024x1024	35.84	15.65	2.44	2.29	14.68
Decoding 1024x1024	4.70	4.67	0.40	1.01	8.54
Reconstruction 1024x1024	4.72	4.48	0.037	1.05	127.56
Shearlet System 2048x2048	196.69	68.43	5.4	2.87	36.42
Decoding 2048x2048	108.19	32.50	5.88	3.33	18.39
Reconstruction 2048x2048	73.20	23.08	4.23	97.6	23.09

The benchmarks were made with 4 scales, in a Macbook pro with OSX 10.10.5, with 8GB memory, 2.7GHz Intel Core i5 processor and Graphic Card Intel Iris Graphics 6100 1536 MB.

- **Benchmarks plots 2D**.




Shearlet Decompostion
Matlab
Julia
Julia gpu
Computing time (seconds)
0
20
40
60
80
100
250
500
750
1000
1250
1500
1750
2000
Resolution (sqrt(px))


Shearlet Recovery
Matlab
Julia
Julia gpu
Computing time (seconds)
0
10
20
30
40
50
60
70
250
500
750
1000
1250
1500
1750
2000
Resolution (sqrt(px))

Applications

Image Denoising

Let $f \in \ell^2(\mathbb{Z}^2)$ and

$$f_{\text{noisy}}(i,j) = f(i,j) + e(i,j)$$

with noise $e(i,j) \sim \mathcal{N}(0,\sigma^2)$.
Then

$$f_{\text{denoised}} = S^* T_\delta S(f_{\text{noisy}})$$

where S is sparsifying transform (analysis operator of the shearlet system) and T_δ is the hard thersholding operator

$$(T_\delta x)(n) = \begin{cases} x(n) \text{ if } |x(n)| \geq \delta \\ 0 \text{ else.} \end{cases}$$

In [45]:
```
# settings
sigma = 30;
scales = 4;
thresholdingFactor = 3;
```

In [46]:
```
# Give noise to data
X_nopar = data_nopar;
Xnoisy_nopar = X_nopar + sigma*randn(size(X_nopar));
```

In [48]:
```
# Thresholding
@time coeffs_nopar = coeffs_nopar.*(abs(coeffs_nopar).> thresholdingFactor*r
        size(X_nopar,1),size(X_nopar,2),length(shearletSystem_nopar.RMS))*si
```

```
  0.534048 seconds (533.75 k allocations: 606.016 MB, 30.88% gc time)
```

In [49]:
```
# Reconstruction
Xrec_nopar = Shearlab.shearrec2D(coeffs_nopar, shearletSystem_nopar);
```

In [50]:
```
# display
clf;
figure(figsize=(10,10));
Shearlab.imageplot(Array(X_nopar), "Original", 1,2,1);
Shearlab.imageplot(Array(Xnoisy_nopar), "Noised", 1,2,2);
```

Original

Noised

In [51]:
```
elin = Shearlab.snr(Array(X_nopar),Array(Xrec_nopar));
# display
clf;
figure(figsize=(10,10));
Shearlab.imageplot(Array(X_nopar), "Original", 1,2,1);
u = @sprintf("Denoised, SNR=%.3f", elin);
Shearlab.imageplot(real(Array(Xrec_nopar)), u, 1,2,2);
```

Original

Denoised, SNR=27.405

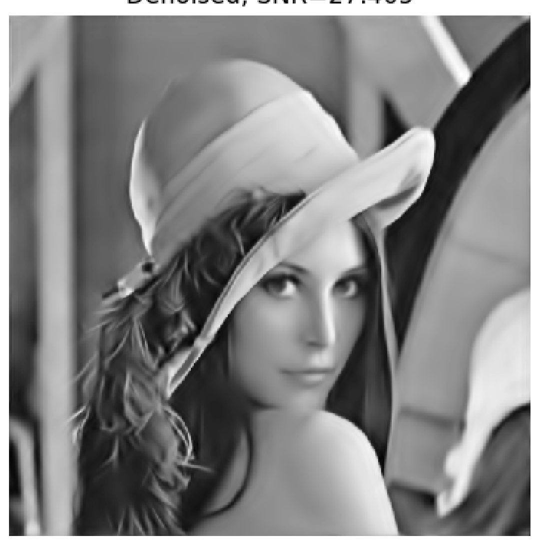


Image inpainting

- Inpainting = optimization problem.

Let $x \in \ell^2(\mathbb{Z}^2)$ be a grayscale image partially occluded by a binary mask $\mathbf{M} \in \{0,1\}^{\mathbb{Z}\times\mathbb{Z}}$, i.e.

$$y = \mathbf{M}x$$

One can recover x (inpaint) using an sparsifying transformation with the inverse problem

$$y^* = \min_{x\in\mathbb{R}^{N\times N}} \|S(x)\|_1 \text{ s.t. } y = \mathbf{M}x$$

By iterative thresholding algorithm

$$x_{n+1} = S^*(T_{\lambda_n}(S(x_n + \alpha_n(y - \mathbf{M}x_n))))$$

λ_n decreases with the iteration number lineraly in $[\lambda_{min}, \lambda_{max}]$.

In [67]:
```
function inpaint2D(imgMasked,mask,iterations,stopFactor,shearletsystem)
    coeffs = Shearlab.sheardec2D(imgMasked,shearletsystem);
    coeffsNormalized = zeros(size(coeffs))+im*zeros(size(coeffs));
    for i in 1:shearletsystem.nShearlets
        coeffsNormalized[:,:,i] = coeffs[:,:,i]./shearletsystem.RMS[i];
    end
    delta = maximum(abs(coeffsNormalized[:]));
    lambda=(stopFactor)^(1/(iterations-1));
    imgInpainted = zeros(size(imgMasked));
    #iterative thresholding
    for it = 1:iterations
        res = mask.*(imgMasked-imgInpainted);
        coeffs = Shearlab.sheardec2D(imgInpainted+res,shearletsystem);
        coeffsNormalized = zeros(size(coeffs))+im*zeros(size(coeffs));
        for i in 1:shearletsystem.nShearlets
            coeffsNormalized[:,:,i] = coeffs[:,:,i]./shearletsystem.RMS[i];
        end
        coeffs = coeffs.*(abs(coeffsNormalized).>delta);
        imgInpainted = Shearlab.shearrec2D(coeffs,shearletsystem);
        delta=delta*lambda;
    end
    imgInpainted
end
```

```
WARNING: Method definition inpaint2D(Any, Any, Any, Any, Any) in module M
ain at In[66]:2 overwritten at In[67]:2.
```

Out[67]:
```
inpaint2D (generic function with 1 method)
```

In [68]:
```
# Rename images
img_nopar = data_nopar
img_par = data_par;
```

In [69]:
```
# Import two masks
name = "../../../ShearLab.jl/data_samples/data_inpainting_2d/mask_rand.png";
mask_rand = Shearlab.load_image(name, n);
mask_rand = mask_rand[:,:,1];
name = "../../../ShearLab.jl/data_samples/data_inpainting_2d/mask_squares.pr
mask_squares = Shearlab.load_image(name, n);
mask_squares = mask_squares[:,:,1];
```

In [70]:
```
# Setting of data
imgMasked_rand_nopar = img_nopar.*mask_rand;
imgMasked_rand_par = AFArray(convert(Array{Float32},img_par.*mask_rand));
imgMasked_squares_nopar = img_nopar.*mask_squares;
imgMasked_squares_par = AFArray(convert(Array{Float32},img_par.*mask_squares
stopFactor = 0.005; # The highest coefficient times stopFactor
sizeX = size(imgMasked_rand_par,1);
sizeY = size(imgMasked_rand_par,2);
nScales = 4;
shearLevels = [1, 1, 2, 2];
```

In [71]:
```
tic()
imginpainted_rand50_nopar = inpaint2D(imgMasked_rand_nopar,mask_rand,50,stop
toc()
```

```
elapsed time: 158.482325353 seconds
```

Out[71]: 158.482325353

In [72]:
```
clf;
figure(figsize=(10,10));
Shearlab.imageplot(img_nopar, "Original", 1,2,1);
Shearlab.imageplot(imgMasked_rand_nopar, "Masked random", 1,2,2);
```

In [73]:
```
elin = Shearlab.snr(img_nopar,imginpainted_rand50_nopar);
# display
clf;
figure(figsize=(10,10));
Shearlab.imageplot(img_nopar, "Original", 1,2,1);
u = @sprintf("Inpainted random 50 iterations, SNR=%.3f", elin);
Shearlab.imageplot(real(imginpainted_rand50_nopar), u, 1,2,2);
```

In [74]:
```
tic()
imginpainted_squares50_nopar = inpaint2D(imgMasked_squares_nopar,mask_square
toc()
```

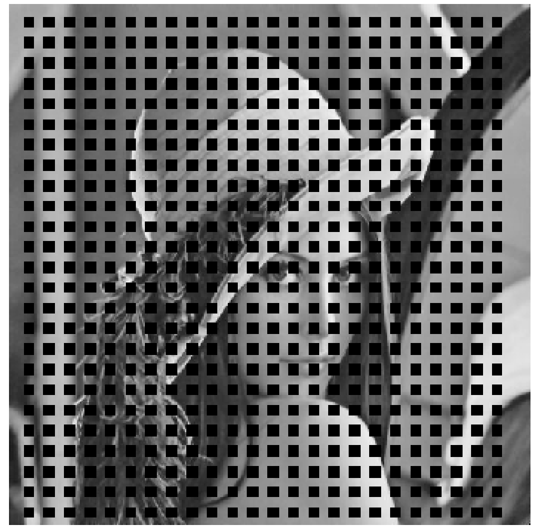
```
elapsed time: 153.952855039 seconds
```

Out[74]: 153.952855039

In [75]:
```
clf;
figure(figsize=(10,10));
Shearlab.imageplot(img_nopar, "Original", 1,2,1);
Shearlab.imageplot(imgMasked_squares_nopar, "Masked squares", 1,2,2);
```

Original

Masked squares



In [76]:
```
elin = Shearlab.snr(img_nopar,imginpainted_squares50_nopar);
# display
clf;
figure(figsize=(10,10));
Shearlab.imageplot(img_nopar, "Original", 1,2,1);
u = @sprintf("Inpainted squares 50 iterations, SNR=%.3f", elin);
Shearlab.imageplot(real(imginpainted_squares50_nopar), u, 1,2,2);
```

Original

Inpainted squares 50 iterations, SNR=28.182

In [ ]: