Fast Multidimensional Signal Processing with Shearlab.jl

Héctor Andrade Loarca
(github: **arsenal9971**)
Notebook and Beamer:
https://github.com/arsenal997/Shearlab.jl/presentations/JuliaCon2017

TU Berlin, BMS

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What is a signal?

Our definition

Function (or something that can be represented as) that contains information about the behavior or attributes of some phenomenon. It can be digital (discrete) or analog (continuous).

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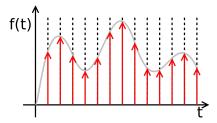
Figure: Digital and continuous one-dimensional signals

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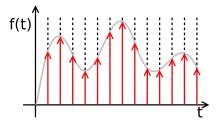

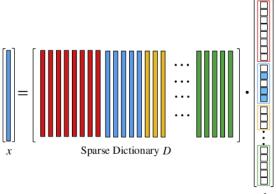Figure: Digital and continuous one-dimensional signals

Figure: White noise, not a signal

Sparse representations of signals

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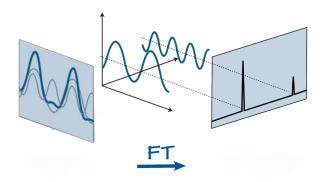- Goal: Find the right dictionary to represent optimally our data.

$$x = \text{Sparse Dictionary } D \cdot \text{Sparse Coefficients } \hat{\alpha}$$

Fourier Transform (Fourier, 1822)

$$\hat{f}(\omega) := \int_{\mathbb{R}^n} f(x)e^{-i\langle x,\omega \rangle}dx$$

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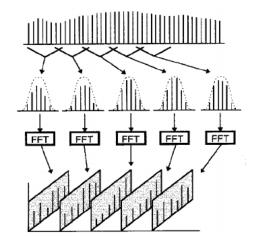

FT

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$$\mathcal{W}_\psi f(a,b) = \int_{\mathbb{R}} f(t) a^{-\frac{1}{2}} \overline{\psi\left(\frac{t-b}{a}\right)} dt$$

$$= (f * D_a \overline{\psi}^*)(b), \quad (a,b) \in \mathbb{R}^+ \times \mathbb{R}$$

where
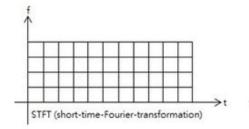
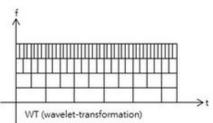$$\int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty$$

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STFT (short-time-Fourier-transformation)

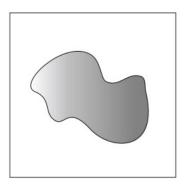WT (wavelet-transformation)

Cartoon-like functions

Definition

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$, $f \in \mathcal{E}^2(\mathbb{R}^2)$ if $f = f_0 + \chi_B f_1$, with $B \subset [0,1]^2$, $\partial B \in C^2$ and with bounded curvature. Moreover, $f_i \in C^2(\mathbb{R}^2)$ with $||f_i||_{C^2} \leq 1$ and $\mathrm{supp} f_i \subset [0,1]^2$ for $i = 0, 1$.

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Optimal error for 2D signals

Best N-term approx. error (Donoho, 2001)

Let $\{\psi_\lambda\}_{\lambda \in \Lambda} \subset L^2(\mathbb{R}^2)$ a frame. The optimal best N-Term approximation error for any $f \in \mathcal{R}^2(\mathbb{R}^2)$ is

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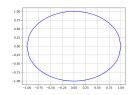
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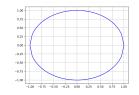 $$A_j := \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}$$

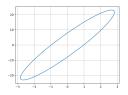# How to solve this? Scaling and Shearing

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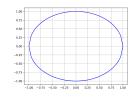
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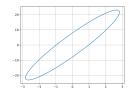- *Scaling*

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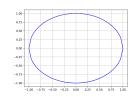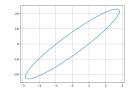
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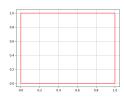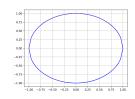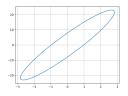$$S_k := \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$
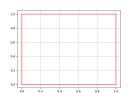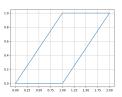
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$$\langle f, \psi_{j,k,m} \rangle = \int_{\mathbb{R}^2} f(x)\overline{\psi_{j,k,m}(x)}dx$$

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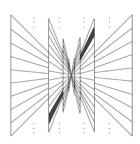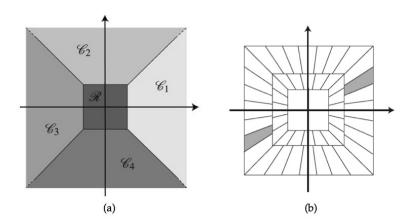
Cone-based shearlet transform

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Separable and non-separable generator

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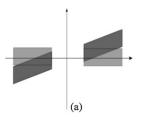
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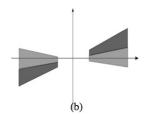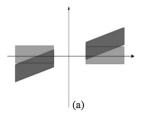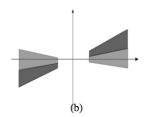

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Current software

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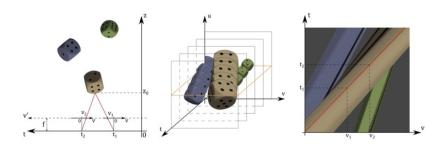   http://na.math.uni-goettingen.de/pyshearlab/

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- **Julia**
  - **Shearlab.jl** (H. Andrade, TU Berlin)
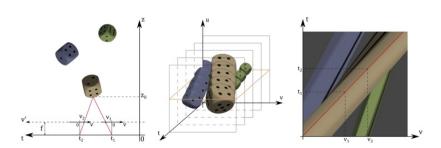   https://github.com/arsenal9971/Shearlab.jl

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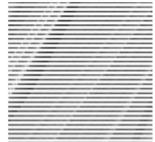
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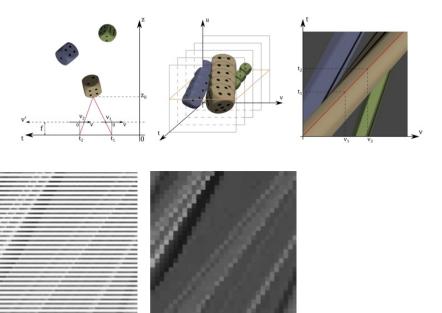
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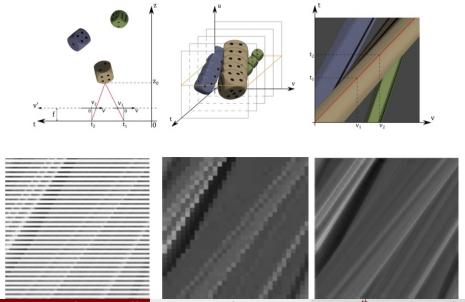- Plenty of image filtering, import and rescaling functions with Images. jl , Wavelets. jl .

- Support of multithreading and painless GPU processing with ArrayFire . jl .

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Thanks!

Questions?