Notes: Canonical Transform

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Abstract

In this notes we present the canonical shearlet transform associated to the space of the two dimensional tomography. The main reason of this construction is the characterization of the Wavefront set of the sinogram of a tomographic image by studying the decay rate of its shearlet coefficients, which at the same time characterizes the Wavefront set of the original image.

1 Two dimensional electron tomography

Tomography is a process of imaging by sections or sectioning of a speciment of interest, through the use of any kind of penetrating wave. This method is used in radiology, archaeology, biology, geophysics, materials science, and other areas of science. In many cases the production of the images based on tomography arise after solving an inverse problem, known as tomographic reconstruction, generally ill-posed.

There exist different methods of tomography acquisition, in this work we will study the measurment setting associated with the so called X-ray transform (or parallel beam transform), that measures the attenuation coefficient of an specimen $f \in L^2(\mathbb{R}^n)$ by the values of its integral along different straight lines. In two dimensions the X-ray transform coincides with the Radon transform.

Definition 1 (Radon Transform)**.** *Let $f \in L^2(\mathbb{R}^2)$, $\theta \in [0, 2\pi)$ and $s \in \mathbb{R}$ then the Radon transform of f along the line $L(\theta, s) = \{(x, y) \in \mathbb{R}^2 : xcos\theta + y\sin\theta = s\}$ is given by the integral*

$$\mathcal{R}f(L) = \mathcal{R}f(\theta, s) = \iint_{(x,y)\in L} f(x, y)dS = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\delta(x\cos\theta + y\sin\theta - s)dxdy \qquad (1)$$

to this definition of Radon transform we refer as the sinogram and is the one we are gonna focus on. Although one could also use the next definition

$$\mathcal{R}f(u, s) := \int_{x_2\in\mathbb{R}} f(u - sx_2, x_2)dx_2 \qquad (2)$$

In Figure 1 one can see the Sheep-Logan phantom, which is a standard test image, that serves as the model of a human head in the developing and testing of image reconstruction algorithms. Figure 2 show the measured lines of the phantom used for the Radon transform, and finally in Figure 3 one can finally see the sinogram o Radon transform of the phatom.

As one can see the sinogram and the original picture are form of singularities of different types. This problem is ill-posed since its associated inverse (Filtered Back Projection) is unbounded, so one would need to perform a regularization method and extra information of f can play and important role when trying to regularize [9]. We would like to use information about the regularity of f as extra information in the regularization of the problem. The concepts of singularity and regularity can be generalized to all distributions, the theory behind this is known as microlocal regularity theory. In this work we will explain a method to extract regularity information of a function using shearlet frames, when one knows just its Radon transform.

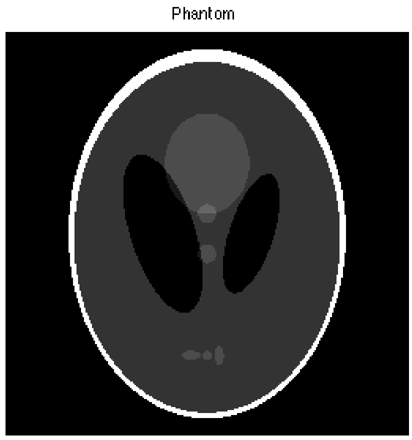

Figure 1: Shepp-Logan phantom

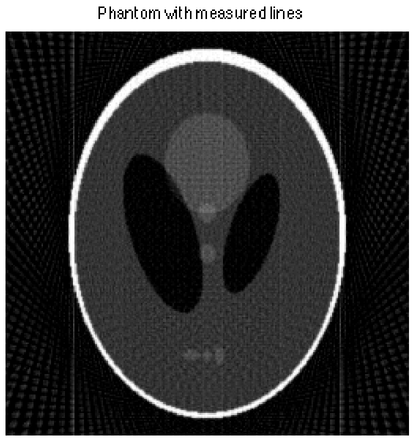

Figure 2: Shepp-Logan phantom with measured lines

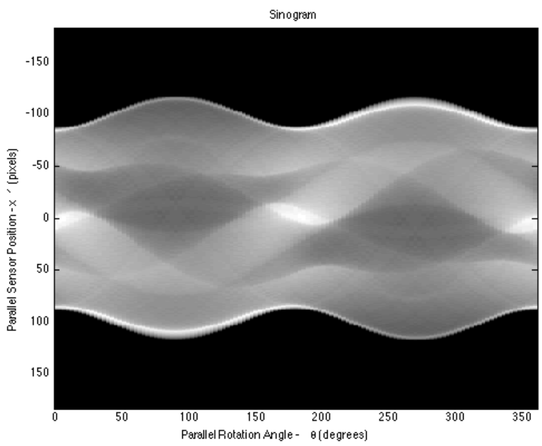

Figure 3: Sinogram

2 Microlocal regularity

Definition 2 (N-regular point and N-regular directed point). *Let $N \in \mathbb{R}$ and f a distribution on $\mathbb{R}^2$. We say that $x \in \mathbb{R}^2$ is a $N-$regular point if there exists a neighbourhood U_x of x such that $\Phi\psi \in C^N$, where Φ is a smooth cutoff function with $\Phi \equiv 1$ on U_x.*

Furthermore we call (x, λ) and N-regular directed point if there exists a neighbourhood U_x of x, a smooth cutoff function Φ with $\Phi \equiv 1$ on U_x and a neighbourhood V_λ of λ such that

$$(\Phi f)^\wedge(\eta) = O((1 - |\eta|)^{-N}) \quad \textit{for all} \quad \eta = (\eta_1, \eta_2) \quad \textit{such that} \quad \frac{\eta_2}{\eta_1} \in V_\lambda \tag{3}$$

The $N-$Wavefront Set $WF^N(f)$ is defined as

$$WF^N(f) := \{(x, \lambda) \in \mathbb{R}^2 \times (\mathbb{R}^2 \backslash \{0\}) : (x, \lambda) \quad \text{is not a } N\text{−regular directed point of f}\} \tag{4}$$

The Wavefront Set $WF(f)$ is defined as

$$WF(f) = \bigcup_{N>0} WF^N(f) \tag{5}$$

Notice that if $(x, \lambda) \in WF(f)$ then x is a singular point of f, then the Wavefront Set ha the information of the singular domain of f and the direction to where the singularities move. The directional information that one can extract using the Radon transform, can be used to characterize the Wavefront Set of a function, this using the definition at equation 2.

Theorem 1 (Projection Slice Theorem, [7]). *Let f be a tempered distribution, then*

$$(\mathcal{R}f(u, s))^\wedge(\omega) = \hat{f}(w(1, s))$$

Therefore, (x, λ) is an $N-$regular directed point is that

$$(\mathcal{R}\Phi f(u, s))^\wedge(\omega) = O(|\omega|^{-N}) \quad \textit{and} \quad s \in V_\lambda$$

In [1] Ozan Ötkem, et al. presented a relation between the $N-$Wavefront Set of a function and the $(N + 1/2)-$Wavefront Set of its Radon transform. Also, in [7] Philipp Grohs presented a method of Wavefront Set Resolution using Continuous Shearlet Frames. The next is our goal:

We want to transfer the information of the Wavefront set of a function using Shearlet Frames, when we just know its Radon transform.

3 Shearlets in Sinogram Space

As one can refer to Equation 1, the Radon transform of a function $\mathcal{R}f(x_1, x_2)$, where $(x_1, x_2) \in [0, 2\pi) \times \mathbb{R}$ and it is 2π-periodic in the x_1-direction, so in order to be able to use Shearlet Frames in the Radon Transform of a function, we would need first to construct a Shearlet Transform on $L^2_{2\pi-x_1}([0, 2\pi) \times \mathbb{R})$ of square integrable functions on $[0, 2\pi) \times \mathbb{R}$ that are also 2π-periodic in x_1-direction.

We recall that given $(a, s, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^2$, the classical shearlet transform of a function $f \in L^2(\mathbb{R}^2)$ associated to the generating function $\psi \in L^2(\mathbb{R}^2)$ is given by

$$\langle f, \psi_{a,s,t} \rangle = \int_{\mathbb{R}^2} f(x)\overline{\psi_{a,s,t}(x)}dx \tag{6}$$

where the shearlet system is given by

$$\mathcal{SH}(\psi) = \{\psi_{a,s,t}(x) := a^{-3/4}\psi(S_s A_a x - t) : (a, s, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^2\}$$

and the scaling matrix A_a and shearing matrix S_s are given by

$$A_a := \begin{pmatrix} a & 0 \\ 0 & a^{1/2} \end{pmatrix}, \quad S_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

Under some conditions on the generating function ψ, like the case of ***classical shearlets*** [6, p. 19], the shearlet system will form a frame , in some cases (band-limited shearlets) one has even tight frames.

Our approach will be based on the so called ***cone-adapted shearlet system*** generated by three functions $\phi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$, parameters $(a, s, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^2$, the corresponding scaling and shearing matrices A_a and S_s and a sampling matrix $M_c = \text{diag}(c_1, c_2)$, where $c_1, c_2 \in \mathbb{R}$.

In this setting the frequency domain is divided in specific areas distributed in two high-frequency cones and one low-frequency squared section; the horizontal cone is covered by elements $\psi_{a,s,t}(x) = a^{-3/4}\psi(S_s A_a x - M_c t)$, similarly the vertical cone is covered by elements of the form $\tilde{\psi}_{a,s,t}$, and finally the low-frequency section covered by a low pass filter elements of the form $\varphi_t = \varphi(x - c_1 t)$. As well as the classical shearlets, the cone-adapted shearlets form a frame of $L^2(\mathbb{R}^2)$ under some conditions on the generating functions; the big difference between this two systems is that the later have a more optimal tiling of the frequency domain. The cone-adapted shearlet system generated by the functions $\varphi, \psi, \tilde{\psi}$ and sampling vector $c = (c_1, c_2)$ will be denote as $\mathcal{SH}(\varphi, \psi, \tilde{\psi}; c)$, for a deeper analysis of this construction we refer to [5].

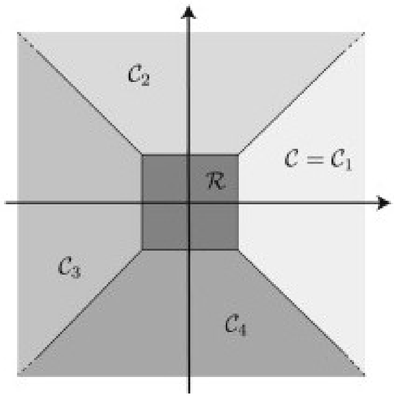

Figure 4: Frequency domain cones

As we already mentioned at the beginning of the section, the space associated to the sinogram of a function can be represented by the space

$$L^2_{2\pi - x_1}([0, 2\pi) \times \mathbb{R}) := \{f : [0, 2\pi) \times \mathbb{R} \longrightarrow \mathbb{R} \Big| \int_{[0,2\pi)\times\mathbb{R}} |f(x_1, x_2)|^2 dx_1 dx_2 < \infty, \\ f(x_1, x_2) = f(x_1 + 2\pi, x_2) \forall (x_1, x_2) \in [0, 2\pi) \times \mathbb{R}\}$$

We would like to construct a shearlet system that forms a frame for this space, the domain of the functions in this space is bounded (constraint in the infinte band $[0, 2\pi] \times \mathbb{R}$), moreover one needs to have periodic conditions in the boundaries $x_1 = 0$ and $x_1 = 2\pi$. Our construction will be based mainly on three concepts:

- Compactly supported shearlet frames.
- Periodic summation.
- Shearlets in bounded domains.

Broadly speaking we need to construct a shearlet system of periodic functions on a bounded domain, and for that one need first to construct compactly supported elements which form a frame. In the following subsections we are gonna explain this concepts to finally construct the desired system.

3.1 Compactly supported shearlets

We will to follow the construction of shearlet frames on bounded domains proposed by Kutyniok et al. [3] and for that we first need to construct a compactly supported shearlet frame. As we already mentioned before there are known cases of shearlet systems on $L^2(\mathbb{R}^2)$ that also form a frame for this space. For instance, the shearlet systems generated by classical shearlets and form a Parseval frame (therefore tight), but their elements are band-limited, therefore they cannot be compactly supported in the spatial domain. One would like then to have a general result of cone-adapted shearlet systems which are compactly supported and form a frame for $L^2(\mathbb{R}^2)$.

Following the construction proposed by Kutyniok et al. [5] for $\varphi, \psi, \tilde{\psi} \in L^2(\mathbb{R})^2$, lets define $\Theta : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$\Theta(\xi, \omega) := |\hat{\varphi}(\xi)||\hat{\varphi}(\xi + \omega)| + \Omega_1(\xi, \omega) + \Theta_2(\xi, \omega)$$

where

$$\Theta_1(\xi, \omega) = \sum_{j \geqslant 0} \sum_{|k| \leqslant \lceil 2^{j/2} \rceil} |\hat{\psi}(S_k^{\intercal} A_{2^{-j}} \xi)||\hat{\psi}(S_k^{\intercal} A_2^{-j} \xi + \omega)|$$

and

$$\Theta_2(\xi, \omega) = \sum_{j \geqslant 0} \sum_{|k| \leqslant \lceil 2^{j/2} \rceil} |\hat{\tilde{\psi}}(S_k \tilde{A}_{2^{-j}} \xi)||\hat{\tilde{\psi}}(S_k \tilde{A}_{2^{-j}} \xi + \omega)|$$

Also, for $c = (c_1, c_2) \in (\mathbb{R}_+)^2$, let

$$R(c) = \sum_{m \in \mathbb{Z}^2 \setminus \{0\}} (\Gamma_0(c_1^{-1} m)\Gamma_0(-c_1^{-1} m))^{1/2} + (\Gamma_1(M_c^{-1} m)\Gamma_1(-M_c^{-1} m))^{1/2}$$
$$+ (\Gamma_2(\tilde{M}_c^{-1} m)\Gamma_2(-\tilde{M}_c^{-1} m))^{1/2},$$

where

$$\Gamma_0(\omega) = \underset{\xi \in \mathbb{R}^2}{\text{esssup}} |\hat{\varphi}(\xi)||\hat{\varphi}(\xi + \omega)| \quad \text{and} \quad \Gamma_i(\omega) = \underset{\xi \in \mathbb{R}^{\not\in}}{\text{esssup}} \Theta_i(\xi, \omega) \quad \text{for i=1,2}$$

Where esssup is the essential supremum, which for a function $f \in L^2(\mathbb{R}^2)$ is defined as

$$\underset{\xi \in \mathbb{R}^2}{\text{ess sup}} f(x) = \inf\{a \in \mathbb{R} : \mu(\{\xi \in \mathbb{R}^2 : f(\xi) > a\}) = 0\}$$

where μ is the standard measure of $L^2(\mathbb{R}^2)$. Intuitively the essential supremum is the supremum of a function when we ignore sets of measure zero where the function might be unbounded. Similarly one can define the essential infimum as

$$\underset{\xi \in \mathbb{R}^2}{\text{ess inf}} f(x) = \sup\{b \in \mathbb{R} : \mu(\{\xi \in \mathbb{R}^2 : f(\xi) < b\}) = 0\}$$

With this concepts already defined we are ready to state general sufficient conditions for the construction of shearlet frames.

Theorem 2 ([5]). *Let $\varphi, \psi \in L^2(\mathbb{R}^2)$ be such that*

$$|\hat{\varphi}(\xi_1, \xi_2)| \leqslant C_1 \cdot \min\{1, |\xi_1|^{-\gamma}\} \cdot \min\{1, |\xi_2|^{-\gamma}\}$$

and

$$|\hat{\psi}(\xi_1, \xi_2)| \leqslant C_2 \cdot \min\{1, |\xi_1|^{\alpha}\} \cdot \min\{1, |\xi_2|^{-\gamma}\} \cdot \min\{1, |\xi_2|^{-\gamma}\}$$

for some positive constants $C_1, C_2 < \infty$ and $\alpha > \gamma > 3$. Define $\tilde{\psi}(x_1, x_2) = \psi(x_2, x_1)$, and let $L_{\inf}, L_{\sup}$ be defined by

$$L_{\inf}(\omega) = \underset{\xi \in \mathbb{R}^2}{\mathit{ess}\inf} \Theta(\xi, 0) \quad \mathit{and} \quad L_{\sup} = \underset{\xi \in \mathbb{R}^2}{\mathit{ess}\sup} \Theta(\xi, 0)$$

Then there exists a sampling parameter $c = (c_1, c_2) \in (\mathbb{R}^+)^2$ *with* $c_1 = c_2$ *such that* $\mathcal{SH}(\varphi, \psi, \tilde{\psi}; c)$ *forms a frame for* $L^2(\mathbb{R}^2)$ *with frame bounds* A *and* B *satisfying*

$$0 < \frac{1}{|detM_c|}[L_{\inf} - R(c)] \leqslant A \leqslant B \leqslant \frac{1}{|detM_c|}[L_{\sup} + R(c)] \leqslant \infty$$

For the proof of this theorem we refer to [5]. It is easy to show that band-limited shearlets obey this theorem, although it is harder in general to construct compactly supported shearlets that satisfy Theorem 2 and that form a frame for $L^2(\mathbb{R}^2)$. One can find various examples of compactly supported shearlets in [5], for its characteristics and simplicity we are gonna use the construction presented in the following theorem.

Theorem 3 ([5])**.** *Let* $K, L > 0$ *be such that* $L \geqslant 10$ *and* $\frac{3L}{2} \leqslant K \leqslant 3L - 2$*, and define a shearlet* $\psi \in L^2(\mathbb{R}^2)$ *by*

$$\hat{\psi}(\xi) = m_1(4\xi_1)\hat{\varphi}(\xi_1)\hat{\varphi}(2\xi_2), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

where m_0 *is the low-pass filter satisfying*

$$|m_0(\xi_1)|^2 = (cos(\pi\xi_1))^{2K} \sum_{n=0}^{L_1} \binom{K-1+n}{n} (sin(\pi\xi))^{2n}, \quad \xi_1 \in \mathbb{R}$$

m_1 *is the associated bandpass filter defined by*

$$|m_1(\xi_1)|^2 = |m_0(\xi_1 + 1/2)|^2, \quad \xi_1 \in \mathbb{R}$$

and φ *is the scaling function given by*

$$\hat{\varphi}(\xi_1) = \prod_{j=0}^{\infty} m_0(2^{-j}\xi_1), \quad \xi \in \mathbb{R}$$

Then there exists a sampling constant $\hat{c}_1 > 0$ *such that the shearlet system* $\Psi(\psi; c)$ *forms a frame for* $\check{L}^2(\mathcal{C}_1 \cup \mathcal{C}_3) := \{f \in L^2(\mathbb{R}^2) : supp\hat{f} \subset \mathcal{C}_1 \cup \mathcal{C}_3\}$ *(where* $\mathcal{C}_1$ *and* $\mathcal{C}_3$ *are the horizontal cones in Figure 4) for any sampling matrix* M_c *with* $c = (c_1, c_2) \in (\mathbb{R}_+)^2$ *and* $c_2 \leqslant c_1 \leqslant \hat{c}_1$*.*

Since the proof of this theorem is not the main interest of this work we refer to [5] for the proof. It is worth to notice that Theorem 3 but Theorem 2 assumes that one has uniform sampling, i.e. the sampling constants c_1 and c_2 coincides, but this results is easy to generalize to non-uniform sampling ($c_1 \neq c_2$), we refer again to [5] for the specific bounds estimates.

By construction the shearlet generators proposed in Theorem 3 are compactly supported (and therefore all the elements of the generated system), but they are frame just for $\check{L}^2(\mathcal{C}_1 \cup \mathcal{C}_3)$; even though one can easily construct shearlet frames for the whole space $L^2(\mathbb{R}^2)$ as follows.

Theorem 4 ([5])**.** *Let* $\psi \in L^2(\mathbb{R}^2)$ *be the shearlet with associated scaling function* $\varphi_1 \in L^2(\mathbb{R})$ *both introduced in Theorem 2, and set* $\varphi(x_1, x_2) = \varphi(x_1)\varphi(x_2)$ *and* $\tilde{\psi}(x_1, x_2) = \psi(x_2, x_1)$*. Then the corresponding shearlet system* $\mathcal{SH}(\varphi, \psi, \tilde{\psi}; c)$ *forms a frame for* $L^2(\mathbb{R}^2)$ *for any sampling matrices* M_c *and* $\tilde{M}_c$ *with* $c = (c_1, c_2) \in (\mathbb{R}_+)^2$ *and* $c_2 \leqslant c_1 \leqslant \hat{c}_1$*.*

Finally, we would like to mention that there is a trade-off between compact support of the shearlet generators. The known constructions of tight frames do not use separable generators, and these constructions can be shown to not be applicable to compactly supported generators. Tightness is difficult to obtain while allowing for compactly supported generators, but we can gain separability and hence fast algorithmic realizations. On the other hand, when allowing non-compactly supported generators, tightness is possible, but separability seems to be out of reach, which makes fast algorithmic realizations very difficult. In any case, we have already knowledge of fast implementation of non-separable shearlets and the theory of Wavefront set resolution using shearlet frames proposed by Grohs in [7] does not require tight frames. In the next subsection we will show how to construct compactly supported 2π periodic shearlet frames.

3.2 Periodic shearlets

For some applications one would like to define shearlets on bounded domain with periodic boundary conditions, in our case we have periodic conditions in the vertical boundaries given by $x_1 = 0$ and $x_1 = 2\pi$, so we should construct a cone-adapted shearlet system that forms a frame for $L^2_{2\pi - x_1}([0, 2\pi) \times \mathbb{R})$; in particular, its elements need to be 2π-periodic in the x_1 direction.

Lets start with the compactly supported system $\mathcal{SH}(\phi, \psi, \tilde{\psi})$ defined at 3 and 4, notice that without loss of generality one can assume that the support of φ, ψ and $\tilde{\psi}$ is contained on the band $[0, 2\pi) \times \mathbb{R}$.

In order to impose 2π-periodicity of our target system, following Daubechies approach in [2], we will first perform 2π-periodic summation of φ, ψ and $\tilde{\psi}$, in this case we will just use three elements in the sum to maintain the compact support of the functions, lets define then for all $(a, s, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^2$ the functions $\varphi^{2\pi - x_1}_{a,s,t}$, $\psi^{2\pi - x_1}_{a,s,t}$ and $\tilde{\psi}^{2\pi - x_1}_{a,s,t}$ given by

$$\begin{aligned}
\varphi^{2\pi - x_1}_{a,s,t}(x_1, x_2) &:= \sum_{\ell \in \{-1,0,1\}} \varphi_{a,s,t}(x_1 + 2\pi\ell, x_2) \\
\psi^{2\pi - x_1}_{a,s,t}(x_1, x_2) &:= \sum_{\ell \in \{-1,0,1\}} \psi_{a,s,t}(x_1 + 2\pi\ell, x_2) \\
\tilde{\psi}^{2\pi - x_2}_{a,s,t}(x_1, x_2) &:= \sum_{\ell \in \{-1,0,1\}} \tilde{\psi}_{a,s,t}(x_1 + 2\pi, x_2)
\end{aligned}$$

Now, notice that if we define $\varphi^{\ell}_{a,s,t}(x_1, x_2) = \varphi_{a,s,t}(x_1 + 2\pi\ell, x_2)$ we have that $\widehat{\varphi^{\ell}_{a,s,t}}(\xi_1, \xi_2) = e^{i2\pi(2\pi\ell)\xi_1}\widehat{\varphi_{a,s,t}}(\xi_1, \xi_2)$, one has the same result for $\psi^{\ell}_{a,s,t}(x_1, x_2)$ and $\tilde{\psi}^{\ell}_{a,s,t}$. This leads to the next estimate

$$\begin{aligned}
|\widehat{\varphi^{2\pi - x_1}_{a,s,t}}(\xi_1, \xi_2)| = |\sum_{\ell \in \{-1,0,1\}} \widehat{\varphi^{\ell}_{a,s,t}}(\xi_1, \xi_2)| &\leqslant \sum_{\ell \in \{-1,0,1\}} |\widehat{\varphi^{\ell}_{a,s,t}}(\xi_1, \xi_2)| \\
&= \sum_{\ell \in \{-1,0,1\}} |e^{i4\pi^2\ell\xi_1}| \cdot |\widehat{\varphi_{a,s,t}}| = 3|\widehat{\varphi_{a,s,t}}(\xi_1, \xi_2)|
\end{aligned}$$

By definition of φ on Theorem 2, we have that there exists constants $0 < C_1 < \infty$ and $3 < \gamma < \infty$ such that

$$|\widehat{\psi^{2\pi - x_1}}| = |\widehat{\psi^{2\pi - x_1}_{1,0,(0,0)}}| \leqslant 3|\widehat{\varphi_{a,s,t}}(\xi_1, \xi_2)| \leqslant 3C_1 \cdot \min\{1, |\xi_1|^{-\gamma}\} \cdot \min\{1, |\xi_2|^{-\gamma}\}$$

Similarly one has that there exists constants $0 < C_2 < \infty$ and $\alpha < \infty$ with $\gamma < \alpha$, such that

$$|\widehat{\psi^{2\pi - x_1}}(\xi_1, \xi_2)| \leqslant 3C_2 \min\{1, |\xi_1|^{\alpha}\} \cdot \min\{1, |\xi_1|^{-\gamma}\} \cdot \min\{1, |\xi_2|^{-\gamma}\}$$

Finally, using this estimates and Theorem 2 there exists sampling parameters $c = (c_1, c_2) \in (\mathbb{R}_+)^2$ one has that the system $\mathcal{SH}(\varphi^{2\pi - x_1}, \psi^{2\pi - x_1}, \tilde{\psi}^{2\pi - x_1}; c)$ forms a frame of $L^2(\mathbb{R}^2)$.

It is clear that restricted to the space $\Omega = [0, 2\pi) \times \mathbb{R}$ the elements of the constructed system are 2π-periodic, we now just need the theory to define a system in the bounded domain Ω such that it forms a fram for $L^2(\Omega)$, this is explained in the next section.

3.3 Shearlets in bounded domains

Now we have all the tools to construct the system we are looking for. We will follow the construction of the system using the approach proposed by Kutyniok et al. in [3], we will focus in the next definition and proposition.

Definition 3 (Shearlets in bounded domains,[3]). *Let $\Omega \subset \mathbb{R}^2$, for some sampling constat $c \in (\mathbb{R}_+)^2$, the cone-adapted shearlet system $\mathcal{SH}_\Omega(\varphi, \psi, \tilde{\varphi}; c)$ for $L^2(\Omega)$ generated by a scaling function $\varphi \in L^2(\mathbb{R}^2)$ and shearlets $\psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$ is defined by*

$$\mathcal{SH}_\Omega(\varphi, \psi, \tilde{\psi}; c) = P_\Omega(\mathcal{SH}(\varphi, \psi, \tilde{\psi}; c)$$

where P_Ω is the orthogonal projection onto the subspace $L^2(\Omega)$.

Proposition 1 ([3]). *Let $c \in (\mathbb{R}_+)^2$ a sampling constant, let $\varphi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$, and let $\Omega \subset \mathbb{R}^2$ with positive measure. Then the following conditions are equivalent.*

(i) The shearlet system $\mathcal{SH}(\varphi, \psi, \tilde{\psi}; c)$ is a frame for $L^2(\mathbb{R}^2)$ with frame bounds A and B.

(ii) The shearlet system $\mathcal{SH}_\Omega(\varphi, \psi, \tilde{\psi}; c)$ is a frame for $L^2(\Omega)$ with frame bounds A and B.

The proof of this theorem can be found in [3].

We finally get the result we are looking for, a cone-adapted shearlet frame for the the sinogram space $L^2([0, 2\pi) \times \mathbb{R}$.

Theorem 5. *Let $\Omega = [0, 2\pi) \times \mathbb{R}$, $\varphi^{2\pi}, \psi^{2\pi - x_1}, \tilde{\psi}^{2\pi - x_1} \in L^2(\mathbb{R}^2)$ and the sampling constant $c \in (\mathbb{R}_+)^2$ defined as above; furthermore let $\mathcal{SH}(\varphi^{2\pi - x_1}, \psi^{2\pi - x_1}, \tilde{\psi}^{2\pi - x_1}; c)$ the cone-adapted shearlet frame of $L^2(\mathbb{R}^2)$ formed by the generators $\varphi^{2\pi}, \psi^{2\pi - x_1}, \tilde{\psi}^{2\pi - x_1} \in L^2(\mathbb{R}^2)$. Then using the notation of Definition 3, the system $\mathcal{SH}_\Omega(\varphi^{2\pi - x_1}, \psi^{2\pi - x_1}, \tilde{\psi}^{2\pi - x_1}; c)$ forms a frame of $L^2_{2\pi - x_1}(\Omega)$*

Proof. Clearly the proof is a direct consequence of the construction of the system $\mathcal{SH}(\varphi^{2\pi - x_1}, \psi^{2\pi - x_1}, \tilde{\psi}^{2\pi - x_1}; c)$ and Proposition 1. □

This ends with our construction, the next step is to use this system in order to characterize the Wavefront set of a function by knowing its Radon transform, this will be treated in the next section. Before we proceed to the next step, it is worthwhile to mention that even this approach of shearlets construction on bounded domains using orthogonal projections maintains the approximation properties for cartoon-like functions, it still have some drawbacks, in particular one loses smoothness properties and the number of vanishing moments of the elements intersecting the boundary; this will just affect in one direction since the domain Ω is just bounded in the x_1-direction, later on we will see how this wont affect the properties on Wavefront set resolution.

4 Wavefront set resolution using shearlet frames on the sinogram space

The first multiscale system used to analyse microlocal regularity of distributions was the wavelet system [2]; due its isotropic characteristic this system cannot be used to also capture directional information of a function, then the decaying rate of the wavelet transform of a function cannot be used to characterize its N-regular directed points and therefore the N-Wavefront set, it can just be used to characterize its N-regular points and its singular support.

There exist already previous work on atempting to characterize the wavefront set of a distribution using its shearlet transform, we would like to refer mainly to the work of Kutyniok and Labate [8] and the work of Grohs [7]. The first works presents a method to characterize the N-Wavefront set of a distribution by analysing the decay rate of its continuous shearlet transform with a system that forms a tight frame for $L^2(\mathbb{R}^2)$, in particular the elements of this system are band-limited. This characteristic limits a lot the possibility of shearlet system that one could use and in particular makes it hard to implement.

Our work will be based on the second approach proposed by Grohs. In this work he exposed a method of resolution of the Wavefront set of a distribution, using continuous shearlet frames with very general conditions; this allow us to use more general constructions of continuous shearlet frames, in particular, compactly supported shearlet frames in bounded domains as the system we presented in the last section.

The next definition and three results help us to characterize the N-Wavefront and the Wavefront set of a tempered distribution using shearlet frames.

Definition 4 (((n_1, n_2)-Sobolev space, [7]). *Let $(n_1, n_2) \in \mathbb{N}^2$, then the (n_1, n_2)-Sobolev space over $\mathbb{R}^2$, $H_{(n_1,n_2)}(\mathbb{R}^2)$ is defined by*

$$H_{(n_1,n_2)}(\mathbb{R}^2) := \{f \in L^2(\mathbb{R}^2) | \left(\frac{\partial}{\partial x_1}\right)^{n_1} \left(\frac{\partial}{\partial x_2}\right)^{n_2} f \in L^2(\mathbb{R}^2)\}$$

Lemma 1 ([7]). *Let $\psi \in L^2(\mathbb{R}^2)$ a shearlet function, $\Xi, \Gamma, u, v \in \mathbb{R}_+$ and $\xi \in \mathbb{R}^2$,then define the function $\Delta_{u,v}(\psi)(\xi)$ by*

$$\Delta_{u,v}(\psi)(\xi) := \chi_{C_{u,v}}(\xi) \int_{0<a<\Gamma, |s|<\Xi} |\hat{\psi}(a\xi_1, \sqrt{a}(\xi_2 - s\xi_1)|^2 a^{-3/2} dads,$$

where $C_{u,v} := \{\xi \in \mathbb{R}^2 | |\xi_1| \geqslant u, |\xi_2| \leqslant v|\xi_1|\}$.
Furthermore define W by

$$\Delta_{u,v}(\psi)(\xi) + |\hat{W}(\xi)|^2 = C_\psi \chi_{C_{u,v}}(\xi).$$

Assume that $\Xi > v$, $u \geqslant 0$ and that $\psi = \frac{\partial^M}{\partial x_1^M}\theta$ has M anisotropic moments, Fourier decay of order L_1 in the first variable and that θ has Fourier decay of order L_2 in the second variable such that

$$2M - 1/2 > L_2 > M > 1/2$$

Then

$$|\hat{W}(\xi)|^2 = O(|\xi|^{-2\min(L_1, L_2 - M)}).$$

In particular if ψ is sufficiently smooth and has sufficiently many vanishing moments then W is a smooth function (i.e. a useful window function).

Theorem 6 (Resolution of the N-Wavefront Set, [7]). *Let $f \in L^2(\mathbb{R}^2)$, $N \in \mathbb{R}$ and $\epsilon > 0$. Then there exist constants P, M, L, L_1, L_2 such that for all functions $\psi \in H_{(N,0)}(\mathbb{R}^2)$ with M vanishing moments in x_1-direction, decay of order P towards infinity, C^L in the second coordinate and L_1, L_2 as in Lemma 1 we have the following result: write $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, where*

$$\begin{aligned}\mathcal{D}_1 = \{(t_0, s_0) \in \mathbb{R}^2 \times [-1, 1] | &\text{for } (s, t) \text{ in a neighbourhood } U \text{ of } (s_0, t_0),\\ &|\mathcal{SH}_\psi f(a, s, t)| = O(a^N), \text{with the implied constant uniform over } U\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_2 = \{(t_0, s_0) \in \mathbb{R}^2 \times [1, \infty] | &\text{for } (1/s, t) \text{ in a neighbourhood } U \text{ of } (s_0, t_0),\\ &|\mathcal{SH}_{\tilde{\psi}} f(a, s, t)| = O(a^N), \text{with the implied constant uniform over } U\}\end{aligned}$$

where $\mathcal{SH}_\psi f(a, s, t)$ is the system generated just by ψ, and similarly $\mathcal{SH}_{\tilde{\psi}} f(a, s, t)$. Then

$$WF^{N+3/4+\epsilon}(f)^c \subseteq \mathcal{D} \subseteq WF^{N-11/4-\epsilon}(f)^c.$$

Theorem 7 (Resolution of Wavefront Set, [7]). *Let ψ be a Schwartz function with infinitely many vanishing moments in x_1-direction. Let f be a tempered distribution and $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, where*

$$\mathcal{D}_1 = \{(t_0, s_0) \in \mathbb{R}^2 \times [-1,1] | \textit{for } (s,t) \textit{ in a neighbourhood } U \textit{ of } (s_0,t_0),$$
$$|\mathcal{SH}_\psi f(a,s,t)| = O(a^k), \textit{for all } k \in \mathbb{N} \textit{ with the implied constant uniform over } U\}$$

and

$$\mathcal{D}_2 = \{(t_0, s_0) \in \mathbb{R}^2 \times [1,\infty] | \textit{for } (1/s,t) \textit{ in a neighbourhood } U \textit{ of } (s_0,t_0),$$
$$|\mathcal{SH}_{\tilde{\psi}} f(a,s,t)| = O(a^k), \textit{for all } k \in \mathbb{N} \textit{ with the implied constant uniform over } U\}$$

Then,

$$WF(f)^c = \mathcal{D}$$

The proof of this results is based on the extensive use of the characterization of N-regular directed points of tempered distributions by the Projection Slice Theorem (see Theorem 1) and comparing the decay rate of the Radon transform of the distribution with the decay rate of its continuous Shearlet transform. If the reader wants to go deeper on the details, we recommned to consult [7].

One can notice that Theorems 6 and 7 use some extra assumptions that we haven't take into account on our construction of the system $\mathcal{SH}_\Omega(\varphi^{2\pi-x_1}, \psi^{2\pi-x_1}, \tilde{\psi}^{2\pi-x_1})$. In the case of Theorem 6 one needs that the shearlet function $\psi \in H_{(N,0)}(\mathbb{R}^2)$ with M vanishing moments in x_1-direction. In our case we need to work with the space $H^{2\pi-x_1}_{(N,0)}(\Omega)$ which is formed by functions in $H_{(N,0)}(\Omega)$ with periodic boundary conditions in the lines $x_1 = 0$ and $x_1 = 2\pi$, it is clear that by construction of $\psi^{2\pi-x_1}$, the generating shearlet may have lack of regularity or vanishing moments just in the x_1-direction.

The construction of the system and the proof of Theorem 6 are both symmetric with respect of the vertical and horizontal cones, so one can assume that $H^{2\pi-x_1}_{(0,N)}(\Omega)$ with M vanishing moments in x_2-direction and the result still holds; by definition $\psi^{2\pi-x_1}$ satisfy this two conditions.

In the case of Theorem 7 one needs ψ to be a Schwartz function with infinitely many vanishing moments in x_1-direction. The infinite vanishing moments condition can be fixed by changing the direction as the approach above. For the condition Schwartz function condition one can use the fact that a compactly supported smooth function is a Schwartz function; $\psi^{2\pi-x_1}$ is smooth in the interior of Ω and by construction using periodic summation the transtion in the periodic boundaries is also smooth, the reason of this is that the function ψ before projection into the subspace in Theorems 3 and 5 is smooth and compactly supported. This makes this result also hold for the system $\mathcal{SH}_\Omega(\varphi^{2\pi-x_1}, \psi^{2\pi-x_1}, \tilde{\psi}^{2\pi-x_1})$.

Another thing that my call our attention in Theorem 6 one does not have an equality result, Grohs explains in Remark 6.2 of [7] that this may be caused since the used notion of Wavefront set is not useful for any microlocal function space, this could be solved by generalizing the result to microlocal Sobolev spaces, but for our purposes this estimate is good enough.

This results let characterize the N-Wavefront set of a distribution on the Sinagram space, that is, given a function $f \in L^2(\mathbb{R}^2)$ we can characterize the N-Wavefront set of its Radon transform by analysing the decay of the shearlet transform of the sinogram using the constructed shearlet system in the sinogram space. For our application we actually need to have some information of the N-Wavefront set of the function itself, not of its Radon transform. In the next section we will present some results proposed by Öktem et al. in [1] that relate the Wavefront set of a function with the N-Wavefront set of its Radon transform.

5 N-Wavefront set resolution via microlocal analysis of Radon transform

Ozan et al. in [1] use a different notation and slightly different forward operator, for instance their results are based in the three-dimensional parallel beam transform, and we are working in two dimensions. We need then to translate their notation and dimensionality and finally change their results.

Definition 5 (Parallel beam transform, [1]). *Let $n \in \mathbb{N}$ and f an n-dimensional distribution, then its parallel beam transform $\mathcal{P}f$ (X-ray transform) is defined by:*

$$\mathcal{P}(y, \omega) := \int_{-\infty}^{\infty} f(y + t\omega)dt$$

where $\omega \in \mathbb{S}^{n-1}$ and $y \in \omega^{\perp}$.

In the three-dimensional case, one can parametrize the sampling curve S by a differentiable $\omega(\theta)$ where $\theta \in [0, 2\pi)$, and let

$$\sigma(\theta) := \frac{\omega'(\theta)}{||\omega(\theta)||}$$

be a unit tangent to the curve S at $\omega(\theta)$. In the case of single tilt axis sampling (one restrict the directions to a single tilt axis), using the x-axis as tilt axis. Then ω is given by

$$\omega(\theta) := (0, \cos(\theta), \sin(\theta)), \quad \theta \in (0, 2\pi]$$

Let $e_1 := (1, 0, 0)$ and $\sigma(\theta) = (0, -\sin(\theta), \cos(\theta))$ form an orthonormal basis of the plane $\omega(\theta)^{\perp}$, then

$$y = (y_1, y_\sigma) \mapsto y_1 e_1 + y_\sigma \sigma(\theta) \in \omega(\theta)^{\perp}$$

In these coordinates the set of lines is parameterized by

$$\begin{aligned} Y &:= \{(y, \theta) | y = (y_1, y_\sigma) \in \mathbb{R}^2, \theta \in (0, 2\pi)\} \\ (y, \theta) &\mapsto \ell(y, \theta) := \{y_1 e_1 + y_\sigma \sigma(\theta) | t \in \mathbb{R}\} \end{aligned}$$

Then the parallel beam transform in this parameterization will be given by

$$\mathcal{P}f(y, \theta) = \mathcal{P}f(y_1 e_1 + y_\sigma \sigma(\theta), \omega(\theta))$$

Under this notation the Wavefront set of a distribution is typically defiend as a subset in the cotangent bundle, the reason for this is to have a invariant definition under differmorphisms, in this case one can parameterize the cotangent bundle, in the case of $\mathbb{R}^3$, elements of the cotangent fiber at $x \in \mathbb{R}^3$ is given by elements of the form $\xi dx = \xi_1 dx_1 + \xi_2 dx_2 + \xi_3 dx_3$, where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and dx_i is the dual covector of $\frac{\partial}{\partial x_i}$, then the cotangent bundle will be given by

$$T^*(\mathbb{R}^3) := \{(x, \xi dx) | x \in \mathbb{R}^3, \xi \in \mathbb{R}^3\}$$

Once these concepts are defined, we can present the next results.

Theorem 8 (Microlocal regularity theorem, [1]). *Let f be a distribution of compact support on $\mathbb{R}^3$, $(y_1, y_\sigma, \theta_0) = y \in Y$, and let $\xi \in \omega(\theta_0)^{\perp}$ be a nonzero vector where we write $\xi_0 = \xi_1 e_2 + \xi_\sigma(\theta_0)$. Finally, let $x_0 \in \ell(y_1, y_\sigma, \theta_0)$. If $\xi_\sigma \neq 0$, then there exists a corresponding covector in $T_y^*(Y)$ such that $(x_0, \xi_0 dx) \in WF(f)$ if and only if this covector is in $WF(\mathcal{P}f)$ (correspondence given in Theorem 9). If we also assume that $\mathcal{P}f$ is $\mathcal{C}^\infty$ near y, then $(x_0, \xi_0 dx) \notin WF(f)$*

Theorem 9 ([1]). *Let f be a distribution of compact support on $\mathbb{R}^3$, $\theta_{\max} \in [0, 2\pi)$, and assume $\mathcal{P}f(y,\theta)$ is given on an open set $U \subset Y$. Moreover, let $(y_1, y_\sigma, \theta_0) \in U$, let ξ_0 be a nonzero vector perpendicular to $\omega(\theta_0)$ written a $\xi_0 = \xi_1 e_1 + \xi_\sigma \sigma(\theta_0)$, and assume $\xi_\sigma \neq 0$ (i.e., ξ_0 is not parallel to e_1). Finally, let $x_0 \in \ell(y_1, y_\sigma, \theta_0)$. Then, $(x_0, \xi_0 dx) \in WF^\alpha(f)$ if and only if*

$$((y_1, y_\sigma, \theta_0), \xi_1 dy_1 + \xi_\sigma dy_\sigma + (\xi_\sigma x \cdot \omega(\theta_0))d\theta) \in WF^{\alpha+1/2}(\mathcal{P}f)$$

For the proof of this theorem we refer to [1]. In the next subsection we present finally the two-dimensional version of this results applied to the Radon transform.

5.1 Two-dimensional version applied to the Radon transform.

Let f a two dimensional distribution, then the parametrization of the curve mentioned above will be given by $\omega(\theta) = (\cos(\theta), \sin(\theta))$ with $\theta \in [0, 2\pi)$, and therefore $\sigma(\theta) = (-\sin(\theta), \cos(\theta))$ and let $y(\theta, s) = s\sigma(\theta)$. In these coordinates the set of lines where the Radon transform and the parallel beam transform and then the Radon transform will be given by

$$\begin{aligned} Y' &:= \{(\theta, s) | s \in \mathbb{R}, \theta \in [0, 2\pi)\} \\ (\theta, s) &\mapsto \ell'(\theta, s) := \{s\sigma\theta + t\omega(\theta) | t \in \mathbb{R}\} \end{aligned}$$

With this notation we can proof that in two dimensions the parallel beam transform and the Radon transform are the same,

$$\begin{aligned} \mathcal{P}(y(\theta, s), \omega(\theta)) &= \int_{-\infty}^{\infty} f(y(\theta) + t\omega(\theta))dt \\ &= \int_{-\infty}^{\infty} f(s\sigma(\theta) + t\omega(\theta))dt \\ &= \int_{-\infty}^{\infty} f(x_1, x_2)\delta(x_1\cos(\theta) + y_1\sin(\theta) - s)dx_1 dx_2 \\ &= \mathcal{R}f(\theta, s) \end{aligned}$$

We are now ready to rewrite the results presented in Theorems 8 and 9.

Theorem 10 (Alternative of Theorem 8). *Let f a distribution of compact support on $\mathbb{R}^2$, $(\theta, s) = y \in Y'$, and let $\xi_0 \in \omega(\theta_0)^\perp$ be a nonzero vector where we write $\xi_0 = \xi_\sigma \sigma(\theta_0)$. Finally, let $x_0 \in \ell'(\theta_0, s)$. If $\xi_\sigma \neq 0$, then there exists a corresponding covector in $T_y^*(Y')$ such that $(x_0, \xi_0 dx) \in WF(f)$ if and only if this covector is in $WF(\mathcal{R}f)$ (correspondence given in Theorem 11). If we also assume that $\mathcal{R}f$ is $\mathcal{C}^\infty$ near y, then $(x_0, \xi_0 dx) \notin WF(f)$.*

Theorem 11 (Alternative of Theorem 9). *Let f be a distribution of compact support on $\mathbb{R}^2$, $\theta_{\max} \in [0, 2\pi)$, and assume $\mathbb{R}f(\theta, s)$ is given in on an open set $U \subset Y'$. Moreover, let $(\theta_0, s) \in U$, let ξ_0 be a nonzero vector perpendicular to to $\omega(\theta_0)$, and assume $\xi_\sigma \neq 0$. Finally, let $x_0 \in \ell'(\theta_0, s)$. Then $(x_0, \xi_0 dx) \in WF^\alpha(f)$ if and only if*

$$((\theta_0, s), \xi_\sigma dy_\sigma + (\xi_\sigma x \cdot \omega(\theta_0))d\theta) \in WF^{\alpha+1/2}(\mathcal{R}f)$$

The proof of this result is the same as the proof of the original, just with the change of notation. This two results give us a way to compute the Wavefront set of a distribution by knowing the Wavefront set of its Radon which was our final goal.

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