

Solving inverse problems in imaging with Shearlab.jl

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Notebook and Beamer:

<https://github.com/arsenal997/Shearlab.jl/presentations/SIAM-IS>

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Inverse problems in Imaging

Goal

Recover parameters characterizing a system under investigation from measurements (recover image from data).

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Recover $f_{\text{true}} \in X$ from data

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where $y \in Y, \mathcal{T} : X \longrightarrow Y$ and $\delta g \in Y$.

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- **Classical solution:** Minimization of the miss-fit against data:

$$\min_{f \in X} \mathcal{L}(\mathcal{T}(f), g)$$

$\mathcal{L} : Y \times Y \longrightarrow \mathbb{R}$ is a transformation of the negative data log-likelihood, e.g. $L(f) = \|\mathcal{T}(f) - g\|_2^2$.



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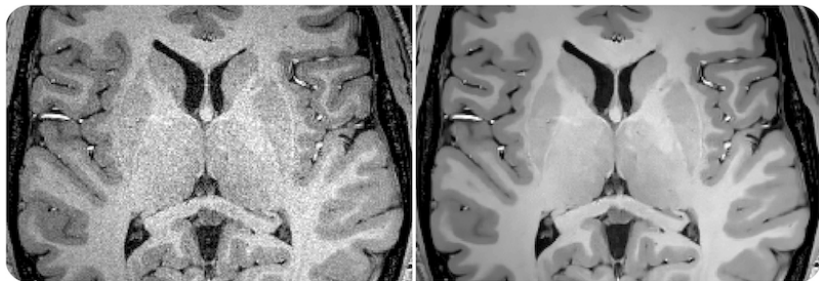
- ▶ Ill-posed problems produce overfitting when minimizing the data miss-fit, but they are the most common in applications (CT, EEG, MRI,...).
- ▶ **Regularization:** Set of methods to avoid overfitting by slightly modify the original problem to increase its regularity.
- ▶ **Variational regularization:** Introduces a "regularization functional" $\mathcal{S} : X \longrightarrow \mathbb{R}$ to encode a priori information about f_{true} , obtaining a new objective functional to minimize:

$$\min_{f \in X} [\mathcal{L}(\mathcal{T}(f), g) + \lambda \mathcal{S}(f)] \quad \text{for a fixed } \lambda \geq 0$$



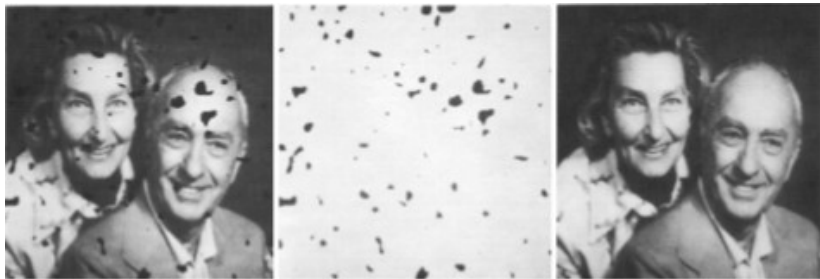
Some examples

$$\mathcal{T}(f)(x) = f(x) + \delta g(x)$$



Inpainting

$$\mathcal{T}(f) = P_K(f)$$

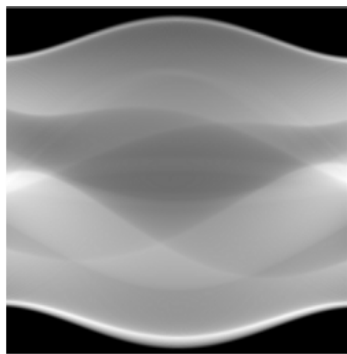


$$\mathcal{T}(f)(x) = (h \circledast f)(x)$$



Computarized Tomography (CT)

$$\mathcal{T}(f)(\theta, s) = \int_{-\infty}^{\infty} f(x_1, x_2) \delta(x_1 \cos(\theta) + x_2 \sin(\theta) - s) dx_1 dx_2$$



Magnetic Resonance Imaging (MRI)

$$\mathcal{T}(f) = (\text{k-space sampling})f$$

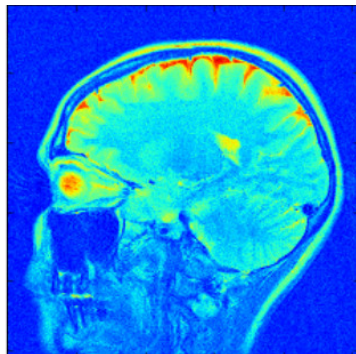
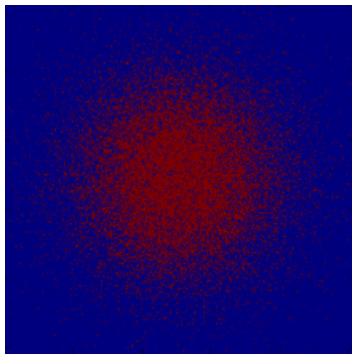


Image denoising

Goal

Recover an image $f \in X = \mathcal{E}^2(\mathbb{R}^2)$ from noisy data:

$$g = f + \delta g$$

where δg is Gaussian white noise, with sd. σ .

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- ▶ The worst behaviour of the estimator is the supremum

$$\sup_{f \in X} \mathbb{E} \|f - \tilde{f}\|_2^2$$

the *Minimax* MSE will be

$$\inf_{\tilde{f}} \sup_{f \in X} \mathbb{E} \|f - \tilde{f}\|_2^2$$



Frame

A frame for $\mathcal{E}^2(\mathbb{R}^2)$ is a collection $\Psi = \{\psi_i\}_{i \in \mathcal{I}} \subset \mathcal{E}^2(\mathbb{R}^2)$ satisfying

$$A\|f\|_2^2 \leq \|\{\langle f, \psi_i \rangle\}_{i \in \mathcal{I}}\|_{\ell^2(\mathcal{I})}^2 \leq B\|f\|_2^2 \quad \forall f \in \mathcal{E}^2(\mathbb{R}^2)$$

for some $0 < A \leq B < \infty$.

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- ▶ It is proven (Labate et al., 2012): If an image is sparse within a frame $\{\psi_i\}_{i \in \mathcal{I}}$, one can obtain a *Minimax* MSE estimator by thresholding the coefficients in the expansion of the noisy data:

$$g = \sum_{i \in \mathcal{I}} \langle g, \psi_i \rangle \psi_i$$

Goal

Recover an image $f \in \mathcal{E}^2(\mathbb{R}^2)$ from known data:

$$g = P_K(f)$$

where P_K is an orthogonal projection onto the known subspace $(\mathcal{E}^2(\mathbb{R}^2))_K$.

Image inpainting

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- **Compressed sensing result:** If a signal (image) is sparse within a frame Ψ , it can be recovered from highly underdetermined, non-adaptive linear measurements by ℓ^1 -regularization (Davenport et al., 2012), i.e.

$$\min_{\tilde{f} \in \mathcal{E}^2(\mathbb{R}^2)} \|\{\langle \tilde{f}, \psi_i \rangle\}_{i \in \mathcal{I}}\|_{\ell^1(\mathcal{I})} \quad \text{s.t.} \quad P_K(\tilde{f}) = g = P_K(f)$$



Error estimate

Let $\delta > 0$ and $\Lambda \subset \mathcal{I}$ be a δ -**cluster** for f with respect to Ψ .

If $\mu_c(\Lambda, P_M \Psi) < 1/2$ and f^* is the minimizer of the problem, then

$$\|\{\langle f^* - f, \psi_i \rangle\}_{i \in \mathcal{I}}\|_{\ell^1(\mathcal{I})} \leq \frac{2\delta}{1 - \mu_c(\Lambda, P_M \Psi)}$$

where P_M is the projection onto the missing subspace and $\mu_c(\Lambda, P_M \Psi)$ the **cluster coherence**, defined by

$$\mu_c(\Lambda, P_M \Psi) := \max_{j \in \mathcal{I}} \sum_{i \in \Lambda} |\langle P_M \psi_i, P_M \psi_j \rangle|$$

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- **Conclusion:** A sparsifying frame for images allows you to perform image denoising and inpainting, the reconstruction quality depends on the sparsifying level. **Problem:** Pick a good frame.



Cartoon-like functions

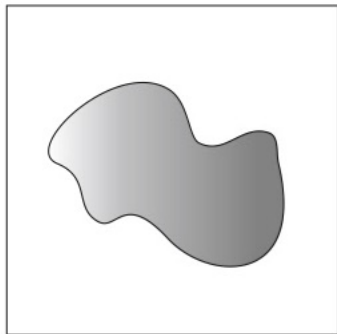
Definition

Let $f : \mathbb{R}^2 \rightarrow \mathbb{C}$, $f \in \mathcal{E}^2(\mathbb{R}^2)$ if $f = f_0 + \chi_B f_1$, with $B \subset [0, 1]^2$, $\partial B \in C^2$ and with bounded curvature. Moreover, $f_i \in C^2(\mathbb{R}^2)$ with $\|f_i\|_{C^2} \leq 1$ and $\text{supp} f_i \subset [0, 1]^2$ for $i = 0, 1$.

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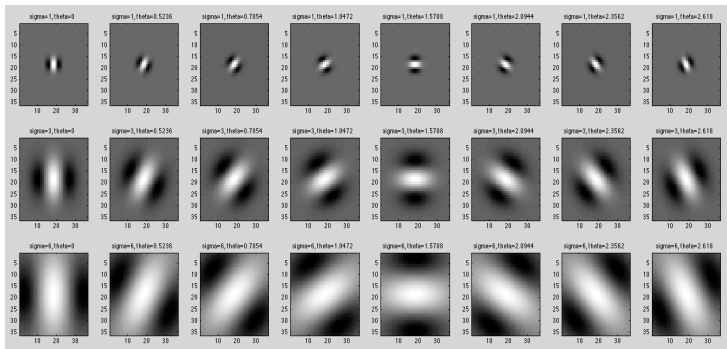
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Examples of frames for images

- ▶ Gabor frames (Gabor, 1946).
- ▶ Wavelet frames (Morlet et al., 1984).
- ▶ Curvelet frames (Candès et al., 1999).
- ▶ Shearlet frames (Kutyniok et al., 2005).



Optimal approximation error for images

Best N-term approx. error (Donoho, 2001)

Let $\{\psi_\lambda\}_{\lambda \in \Lambda} \subset L^2(\mathbb{R}^2)$ a frame. The optimal best N-Term approximation error for any $f \in \mathcal{E}^2(\mathbb{R}^2)$ is

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Error of 2D-wavelets

$$\sigma_N(f, \{\psi_{j,m}\}_{j,m}) \sim N^{-1/2}$$

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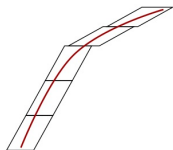
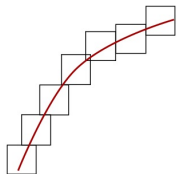
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$$A_j := \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}$$

$$S_k := \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

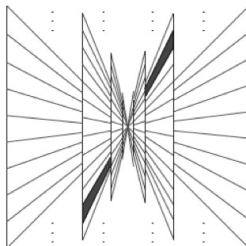
Shearlet Transform (Kutyniok, Guo, Labate, 2005)

Classical Shearlet Transform

$$\langle f, \psi_{j,k,m} \rangle = \int_{\mathbb{R}^2} f(x) \overline{\psi_{j,k,m}(x)} dx$$

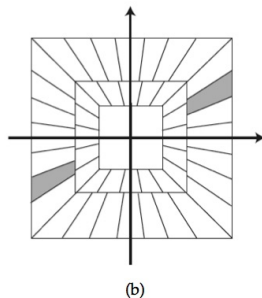
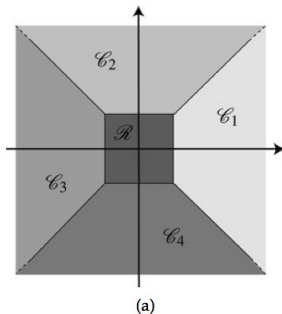
where

$$\mathcal{SH}(\psi) = \{\psi_{j,k,m}(x) = 2^{3j/4} \psi(S_k A_j x - m) : (j, k) \in \mathbb{Z}^2, m \in \mathbb{Z}^2\}$$



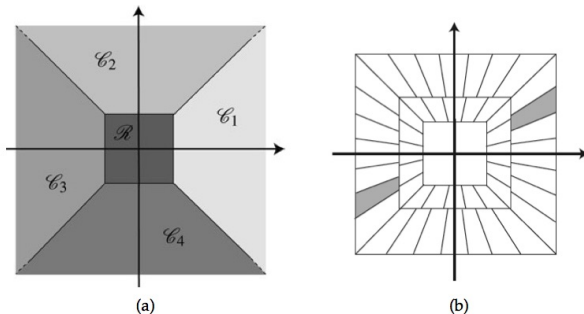
Cone-adapted shearlet transform

$$SH(\phi, \psi, \tilde{\psi}, c) := \mathcal{P}_{\mathcal{R}}\Phi(\phi, c1) \cup \mathcal{P}_{C_1}\Psi(\psi, c) \cup \mathcal{P}_{C_2}\tilde{\Psi}(\tilde{\psi}, c)$$



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- ▶ Best N -term approximation error

$$\sigma_N(f, \{\psi_{j,k,m}\}_{j,k,m}) \sim N^{-1}(\log(N))^{3/2}$$

▶ Matlab

- ▶ FFST- Fast Finite Shearlet Transform (Häuser, Steidl, TU Keiserlautern)
<http://www.mathematik.uni-kl.de/imagepro/software/ffst/>
- ▶ 2D/3D Shearlet Toolbox (D. Labate, University of Houston)
<https://www.math.uh.edu/~dlabate/software.html>
- ▶ **Shearlab3D** (G. Kutyniok, W.-Q.Lim, R. Reisenhoffer, TU Berlin)
<http://www.shearlab.org/>

▶ Python

- ▶ pyShearLab (Stefan Loock, U Göttingen)
<http://na.math.uni-goettingen.de/pyshearlab/>

▶ Julia

- ▶ **Shearlab.jl** (H. Andrade, TU Berlin)
<https://github.com/arsenal9971/Shearlab.jl>

Why Julia?

- ▶ Extensive use of `fft` , well implemented in Julia.
- ▶ Fast vectorization and loops as well as JIT-compilation.
- ▶ Plenty of image filtering, import and rescaling functions with `Images.jl` , `Wavelets.jl` .
- ▶ Support of multithreading and painless GPU processing with `ArrayFire.jl` .



Thanks!

Questions?

