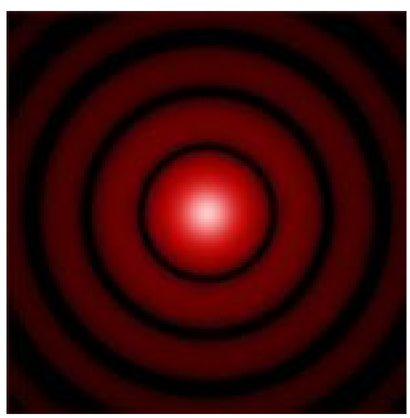
Hiroyuki Tsuda's Laboratory - 2019

Report of exercice 2-Diffraction through apertures



Academic year 2019-2020
Hiroyuki Tsuda's Laboratory

Santacatterina Simon - 81923479

1 Fraunhofer region diffraction

1.1 introduction

The Fraunhofer diffraction formalism is used to study the diffraction pattern of waves through different kind of apertures at a distance far from the diffracting element. It is also called far-field formalism. This report study the effect of the shape of the aperture in the diffracting plane on two quantities of interest that are the the projected amplitude and the projected intensity pattern. The spatial representation of the study is displayed in figure 1.

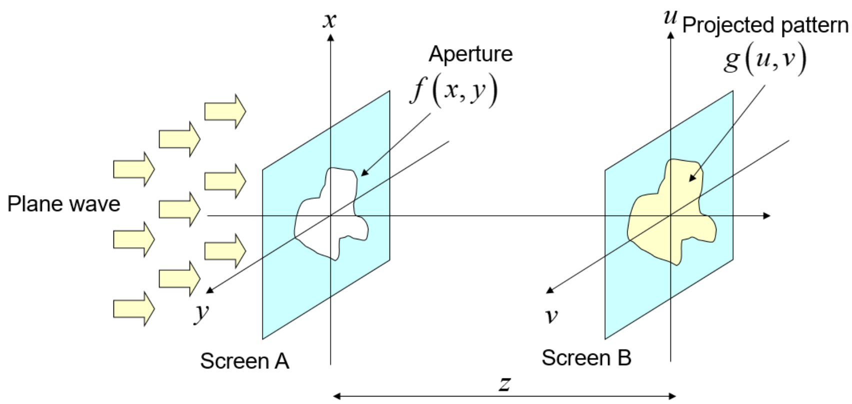


FIGURE 1 – Spatial representation of the study

In this study we consider the incident wave as a plane wave of wavelength λ, the associated wave number is k while the amplitude of the incident wave, propagating in z direction, is constant over the 2-D plane formed by the aperture. In other words A(x,y) = A at z = 0.

1.2 General formalism

The equations are developed using figure 2. The figure is particularly important because it shows the direction angles made by the direction vector

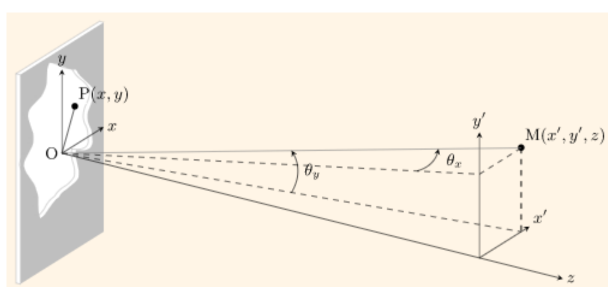


FIGURE 2 – Spatial representation of the study

The general equation associated with Fraunhofer diffraction is given by

$$E(x', y', z) = \frac{j}{\lambda} \iint_S A(x, y) \frac{e^{-jkr}}{r} \, dx \, dy \tag{1}$$

Where r = PM on figure 2.

It is therefore computed, through this equation, the amplitude of the field in the plane defined by x' and y', at a distance d. Given figure 2 we can directly define OM = d and, using trigonometry,

$$\sin\theta_x = \frac{x'}{d} \tag{2}$$

and

$$\sin\theta_y = \frac{y'}{d} \tag{3}$$

Now, additional hypothesis can be used. As we are in far field, for the amplitude term, it can be considered that the distance from the diffracting element is constant over the entire plane of observation as small changes in the value of r are not significant for the amplitude. The $\frac{1}{r}$ term of the amplitude become $\frac{1}{d}$. For the phase term, it is more complicated as a small change in the argument can induce big changes in the value of the phase. It is necessary to use geometry to rewrite the value of r in the argument of the phase. While rewriting the expression we obtain :

$$r = d - x\sin\theta_x - y\sin\theta_y = d - x\frac{x'}{d} - y\frac{y'}{d} \tag{4}$$

Equation 1 is rewriten, using far field approximations, as

$$E(x', y', z) = \frac{j}{\lambda}\frac{e^{-jkd}}{d}\iint_S A(x,y)e^{-\frac{jk}{d}(xx'+yy')}\,dx\,dy \tag{5}$$

While the intensity is obtain by

$$I(x', y', z) = |E(x', y', z)|^2 \tag{6}$$

And equation 5 can be seen as the 2 D Fourier Transform of the function A, on the aperture, but evaluated at frequencies given by $f_x = \frac{x'}{\lambda d}$ and $f_y = \frac{y'}{\lambda d}$.

While it could be interesting mathematically to demonstrate, for each aperture shape, the mathematical formula resulting from the integration, this work is not necessary for this exercise. Instead, knowing the domain of integration, the fact that the incident wave is a plane wave and that is application is a Fourier transform we can comment on the shape of the resulting electric field or intensity pattern. In order to do this, simulation will be undergone.

1.3 Simulation

In this section, Matlab simulation are used to represent the resulting field and intensity pattern. The Matlab method consist in :

1. Sampling the aperture in the 2 dimensions
2. Defining the aperture's domain through the value of the field at different point of the aperture
3. Using Matlab's built in **fft2** function
4. Normalizing the result
5. Scaling the indexes into coordinates

Therefore, the same codes is used for all the aperture geometries. It is only necessary to modify the functions defining the shape of the aperture.

1.4 Results

1.4.1 Circular aperture

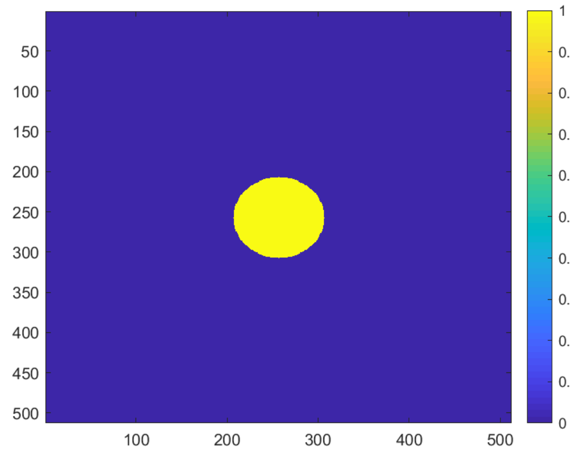


FIGURE 3 – Circular aperture shape

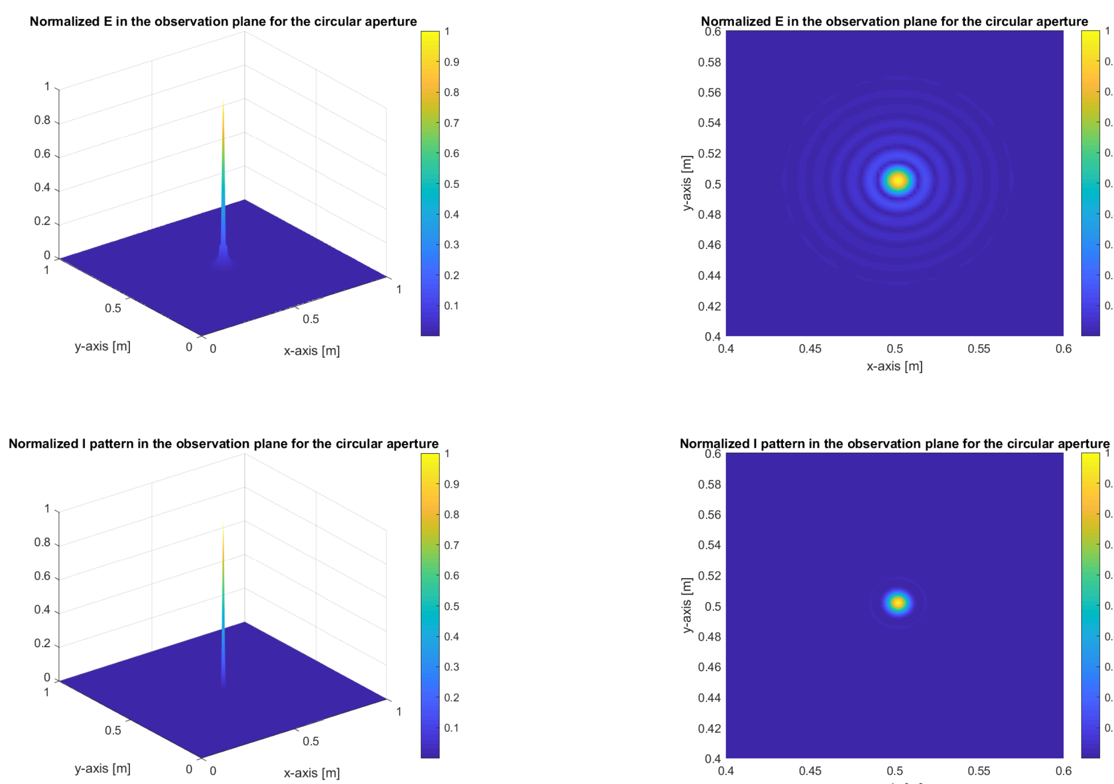


FIGURE 4 – Resulting E and I patterns and their projections in the observation plane for the circular aperture

1.4.2 Rectangular aperture

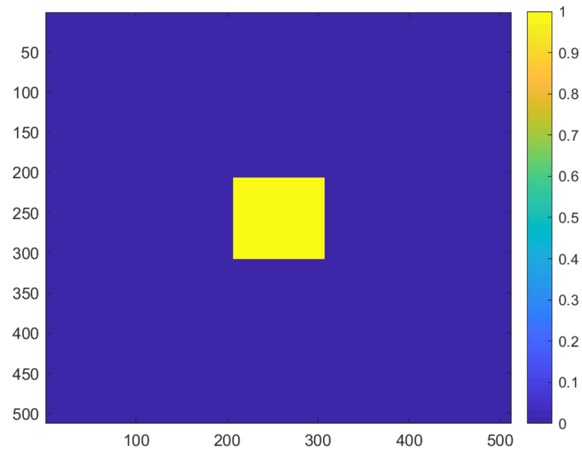


Figure 5 – Rectangular aperture shape

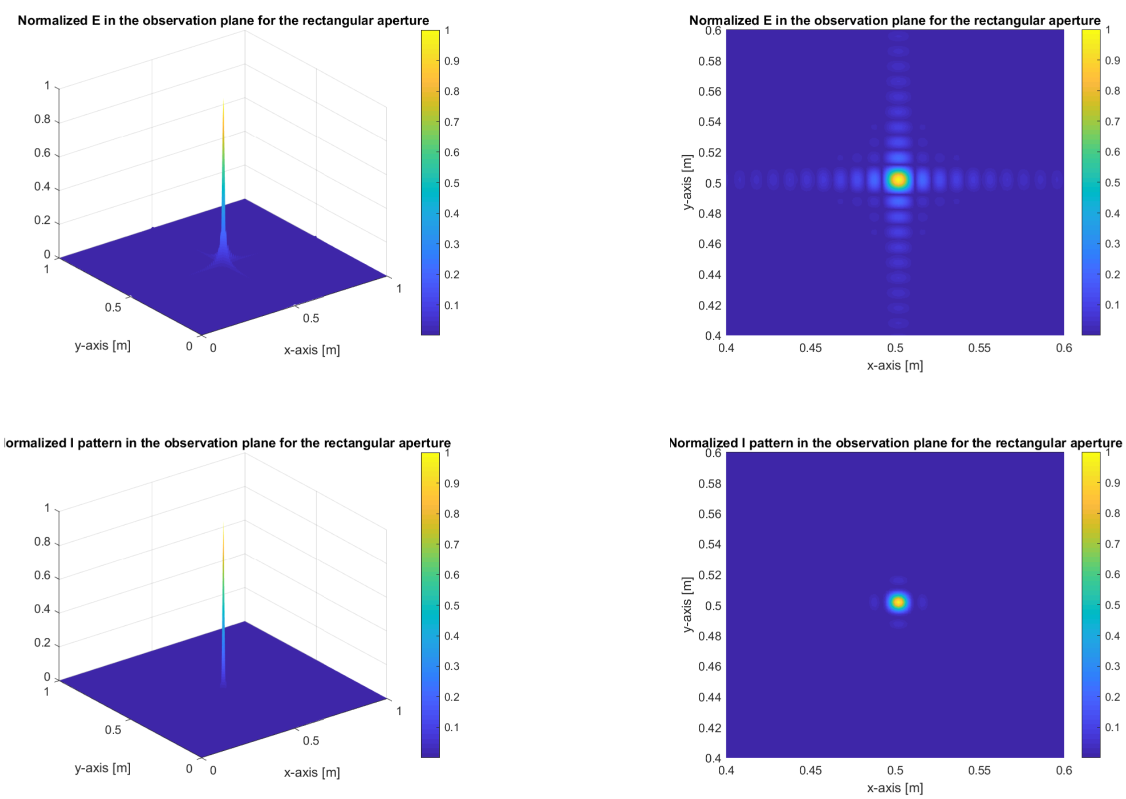


Figure 6 – Resulting E and I patterns and their projections in the observation plane for the rectangular aperture

1.4.3 Double slit aperture

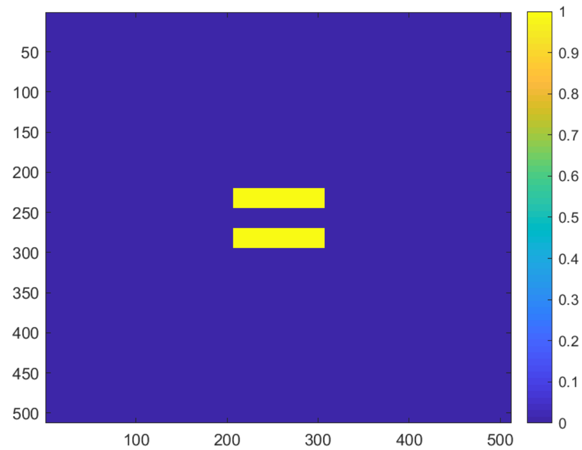


FIGURE 7 – Double slit aperture shape

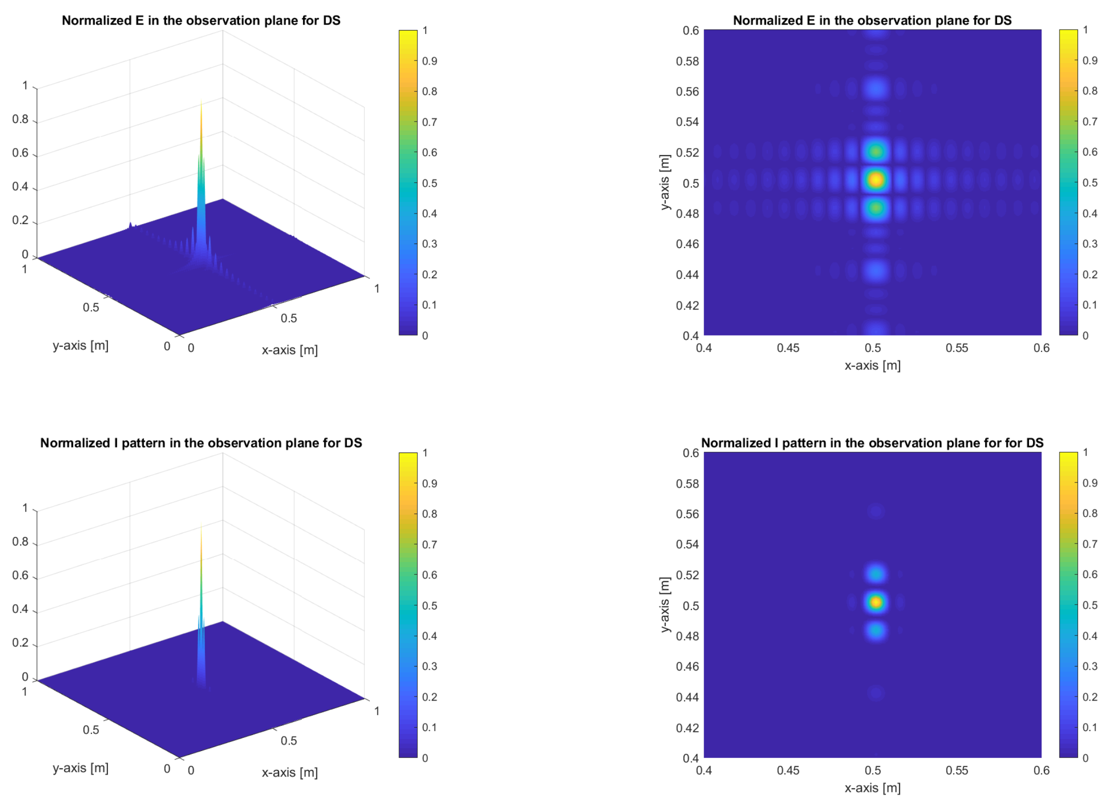


FIGURE 8 – Resulting E and I patterns and their projections in the observation plane for the double slit aperture

1.4.4 Periodic aperture

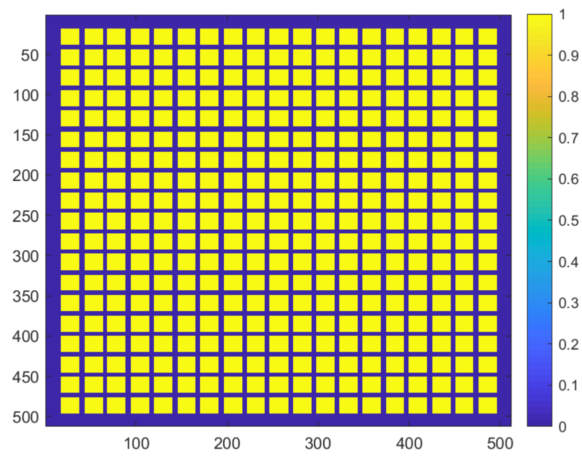


FIGURE 9 – Periodic aperture shape

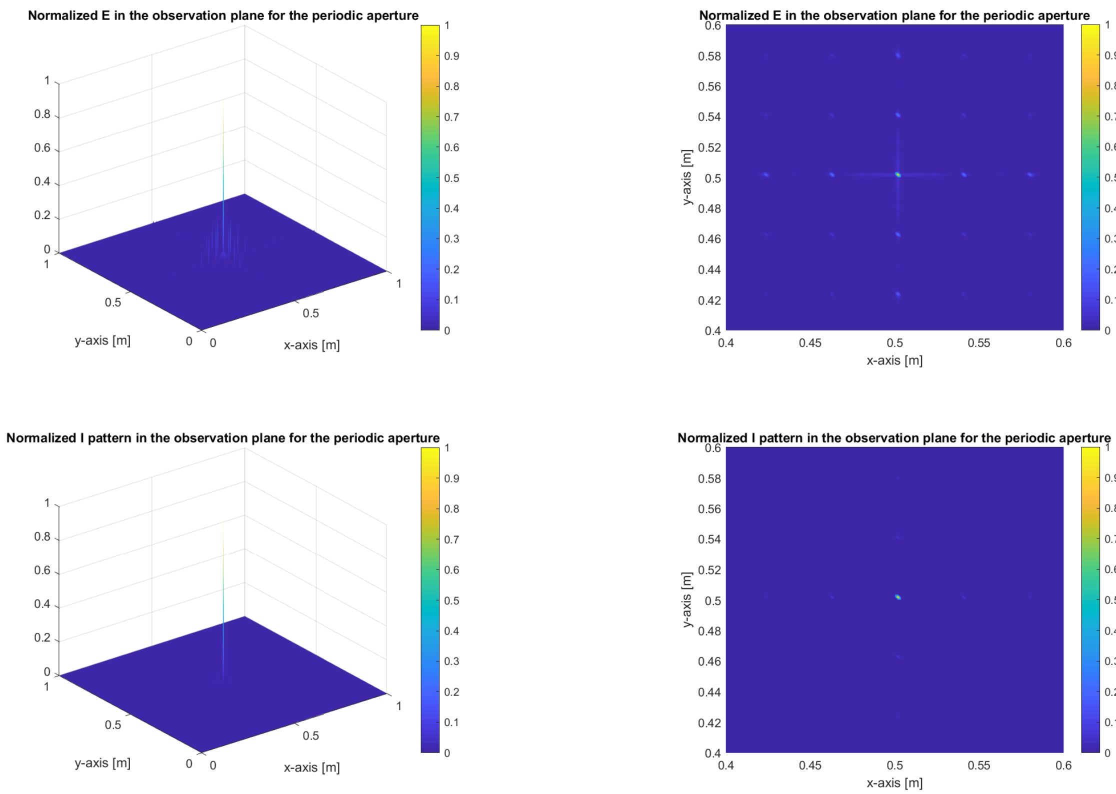


FIGURE 10 – Resulting E and I patterns and their projections in the observation plane for the periodic aperture

1.5 Results

Such shapes of the resulting electric field are not trivial and need to be commented extensively to be understood. However, those shapes can be understood using the Fourier tranform mathematical properties. Let's first remember that the Fourier Transform of a constant over a limited window in one domain results in a sinus cardinal which as for typical equation $sinc(x) = \frac{sin(x)}{x}$, as seen in figure 11, in the other domain. More precisely, while integrating a constant over a limited domain the shape of the domain will strongly affect the resulting Fourier Transform. Here we can transpose the mathematical properties discussed above in our study case. Considering performing the Fourier tranform of an constant incident field in the spatial domain of the aperture we therefore obtain, in the resulting domain, sinus cardinal shapes for the resulting amplitude.

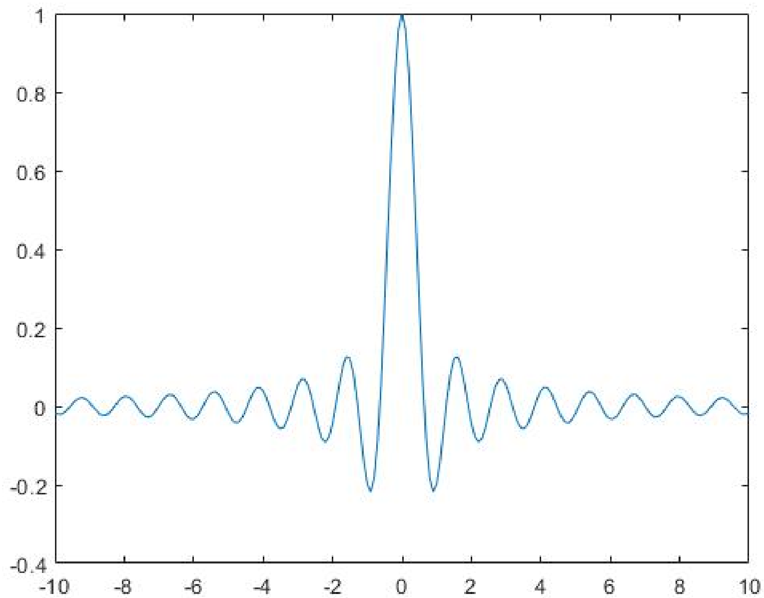


FIGURE 11 – Illustrative example of the sinc shape

For the circular aperture, the amplitude of the sinus cardinal lobes decrease as ρ' increases, in other word, as we get further from the center of the spot the intensity is smaller. Also, for $\rho' = constant$ and making θ' vary from 0 to 2π the amplitude of the lobes is constant. This is due to the spatial invariance of the situation.

For the rectangular aperture, the two components x and y of the intensity are independent from each other. In both case, the amplitude is maximum at 0 and the amplitude of the sinus cardinal lobes decrease as we move away from the origin. The width of the aperture in each direction influence the shape of the intensity such that the longer the width in each direction the narrower the central peak and the smaller the secondary lobes.

In the case of the double slit, the statement for the rectangular aperture holds but interference can be observed in the resulting pattern. The fields from each slit interfere constructively along an axis passing by the middle of the two slits. This explains why the peak of intensity is maximum at this point in figure 8, while in the aperture domain light is blocked at the same place as seen on figure 7. High secondary lobes can also be observed, those are mainly due to the each rectangular shape independently.

Finally, the case of the periodic aperture is a bit more complex. Each aperture is emitting a field independently, the effect is such that the contribution of each of them has to be taken into account. This is done by summing the field of each of them in both direction. The resulting field is not anymore a $sinc(x) = \frac{sin(x)}{x}$ shape but approximately a $\frac{sin^2(Nx)}{sin^2(x)}$, as observed on figure 10. The peaks are much more narrower and the secondary lobes smaller. The big peaks get higher and there are more and smaller secondary lobes as the number of elements in the grid, seen on figure 9, increases.

2 Fresnel region diffraction

While the Fraunhofer approximation was a far field approximation we might also need to compute field that are closer. In this case, the equation of the resulting field cannot be simplified. A scheme for this situation in the case of a rectangular aperture can be seen on figure 12.

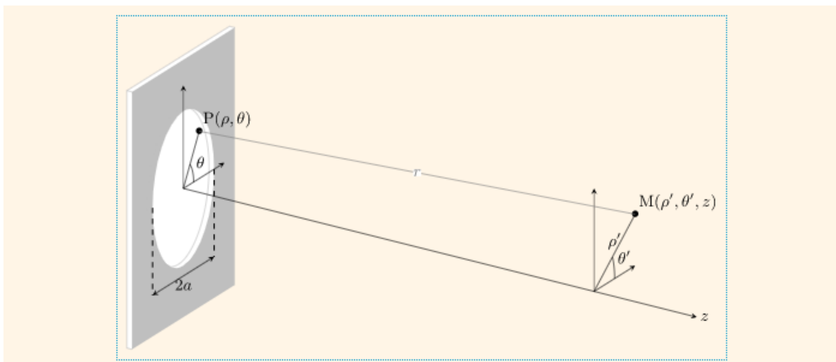


FIGURE 12 – Fresnel diffraction case

In this study we consider the incident wave as a plane wave of wavelength λ, the associated wave number is k while the amplitude of the incident wave, propagating in z direction, is constant over the 2-D plane formed by the aperture. In other words A(x,y) = A at z = 0.

2.1 General formalism

The equations are developed using figure 12. The figure is particularly important because it shows the system of coordinates which are used. It is important to notice that the figure display a circular aperture case using polar coordinates. In the two next cases, which are the knife-edge aperture and the double slit aperture, it is more appropriate to work in a Cartesian coordinates planes (x, y) at z = 0 and (x', y') at z = k where k is usually taken as a multiple of the wavelength, k = nλ.

$$E(x', y', z) = \iint_S KA(x, y)\frac{e^{-jkr}}{r}\, dS \tag{7}$$

Where r = PM = $\sqrt{\rho^2 + z^2 + \rho'^2 - \rho\rho' \cos(\theta' - \theta)}$ on figure 12. Also as there is invariance in the (x',y',z) plane the wave (field) is independent from $\theta^{'}$. It is therefore set to 0. This integration is realized by varying θ from 0 to 2π and ρ from 0 to a, a being the aperture radius.

The intensity is obtained by

$$I(x', y', z) = |E(x', y', z)|^2 \tag{8}$$

For the knife-edge aperture and the double slit aperture we use the same equation but the value of r is computed as a function of z, x, y, x' and y'. In this case, $r = \sqrt{z^2 + x^2 - x'^2 + y^2 - y'^2}$.

2.2 Simulation

In the following, the FFT cannot be used, the simulation just consist in calculating and integral in the spatial domain by iterating on all the spatial coordinates in x, y, x' and y'. The computational time (s) is several order of magnitude higher.

2.3 Results

2.3.1 Circular aperture

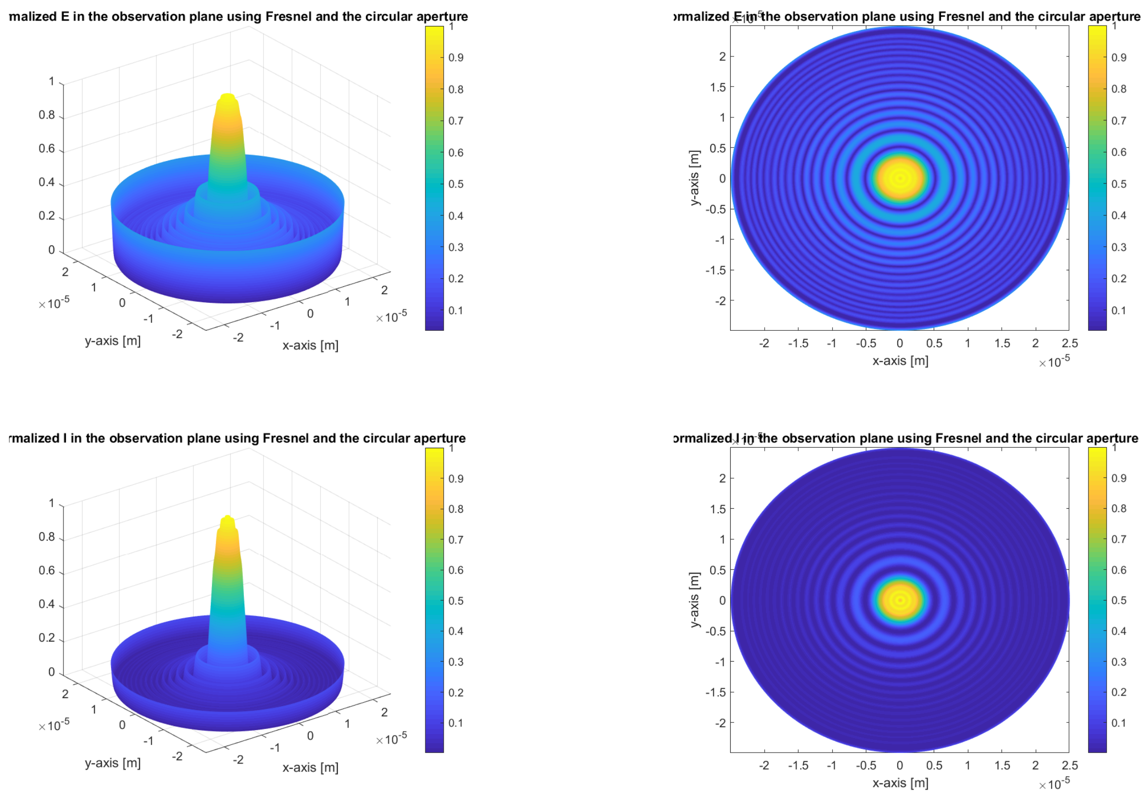


FIGURE 13 – Resulting E and I patterns and their projections in the observation plane for the circular aperture

2.3.2 Knife-edge aperture

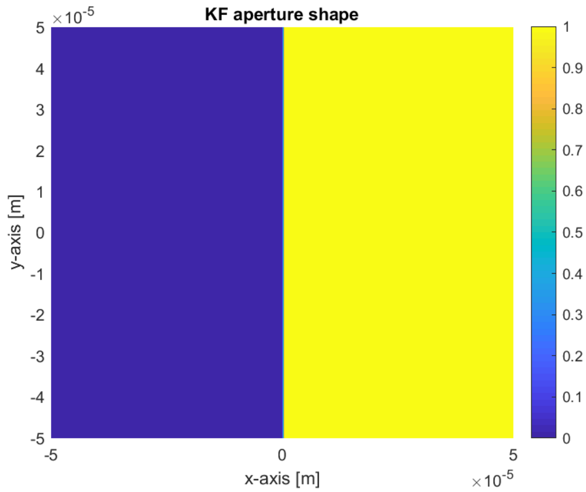


FIGURE 14 – Knife-edge shape

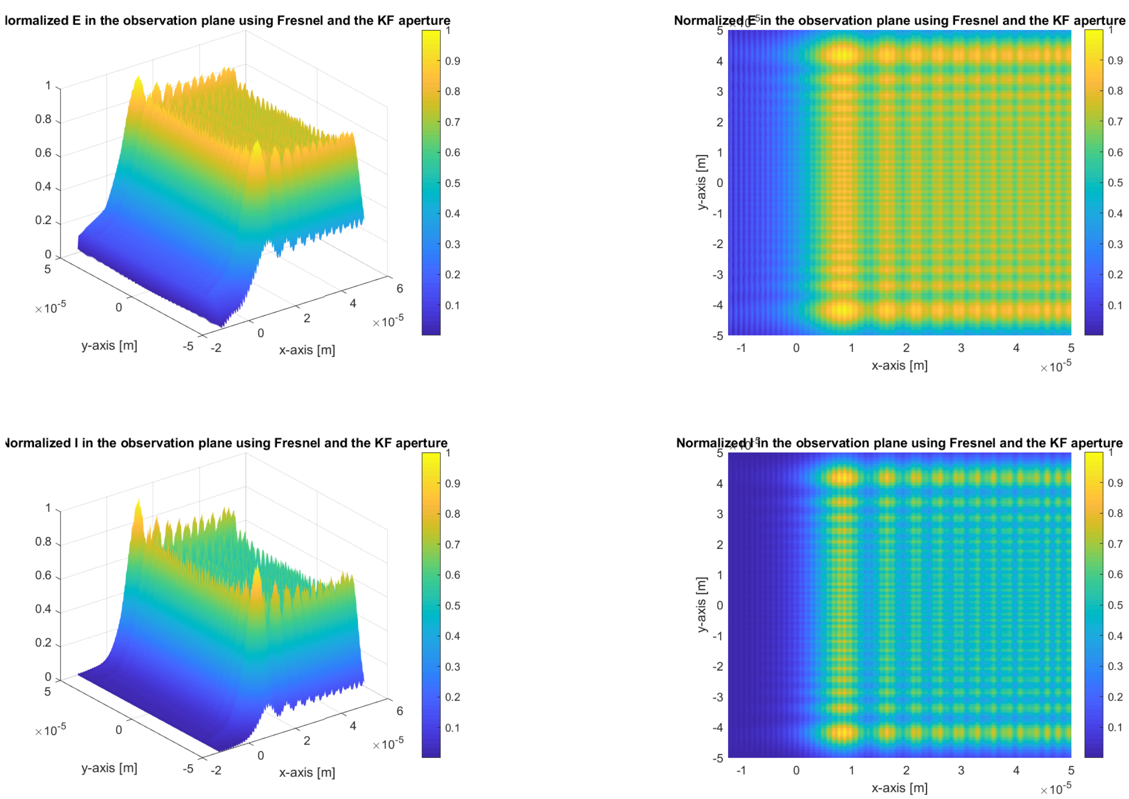


FIGURE 15 – Resulting E and I patterns and their projections in the observation plane for the knife-edge aperture

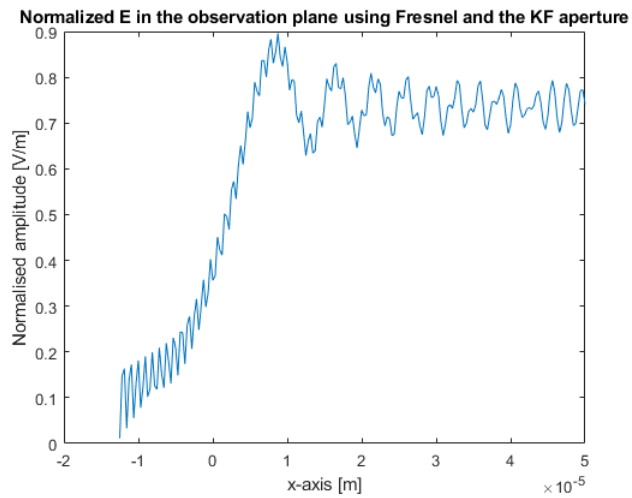


FIGURE 16 – Knife-edge resulting field amplitude in x direction

2.3.3 Double slit aperture

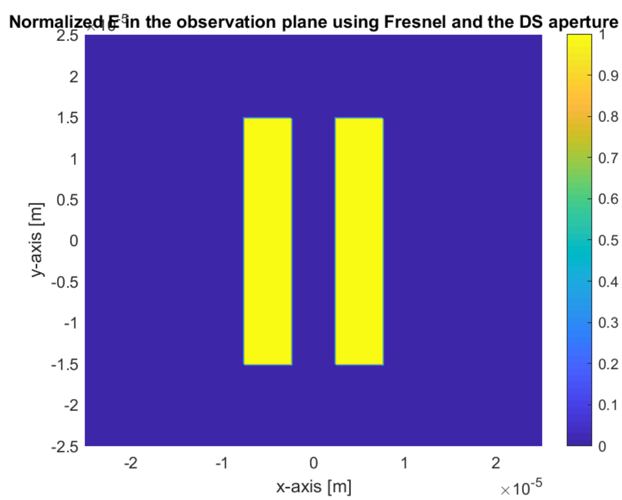


FIGURE 17 – Double slit shape

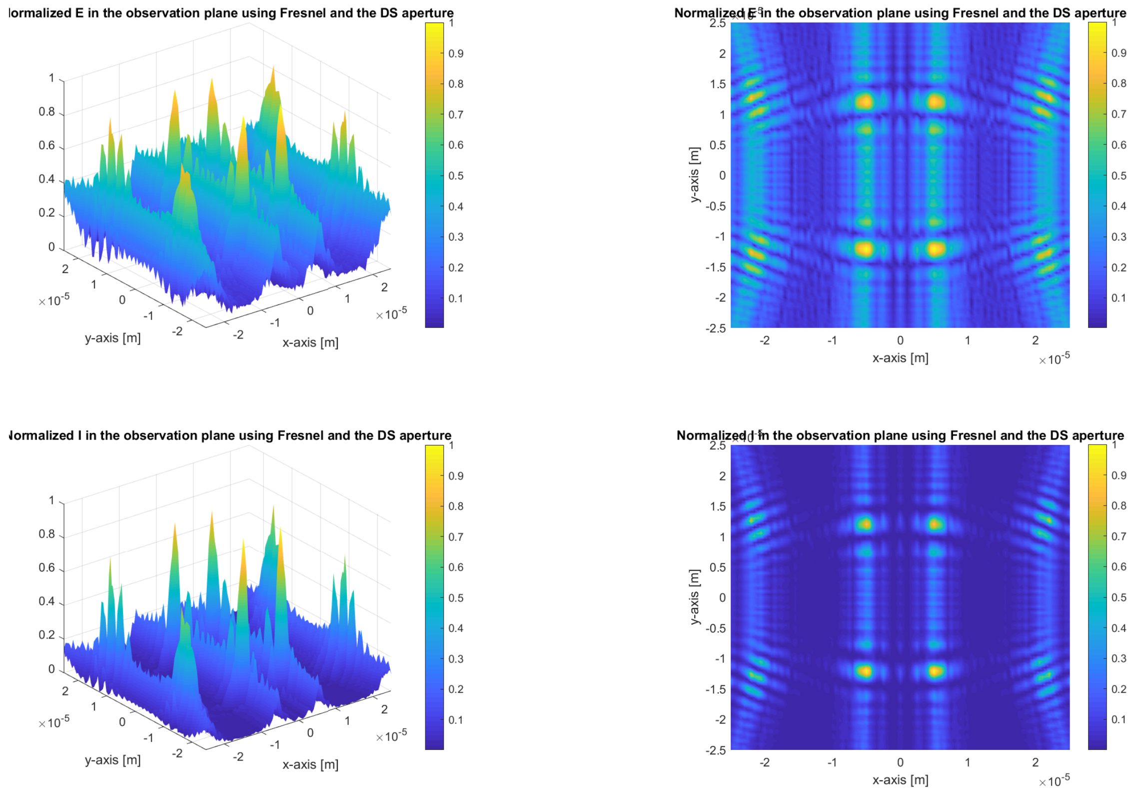


FIGURE 18 – Resulting E and I patterns and their projections in the observation plane for the double slit aperture

2.4 Results

In this case of near field approximation the result are different from what we obtained using the Fraunhofer approach. An important thing is to observe that the shape of the beams will change as we get further from the diffracting plane.

Concerning the circular aperture we can see on figure 13, calculated at a distance z = 70λ, that the central beam is larger than on figure 4. However, as the diffracting plane and the observation plane get further apart we obtain as resulting pattern similar than the one for the Fraunhofer approximation.

The knife-edge aperture exhibits a particular pattern of parallel rays as seen on figure 15. This is what is expected from a typical knife edge aperture. It as to be taken into account that here the plane, illustrated by figure 14, was not really semi infinite and then the pattern is not perfect but is really similar to the expected pattern from a knife-edge aperture. It can be seen from figure 15 that the phenomena of parallel rays happens in both direction because this plane is finite in both directions x and y. Still as one direction (the x direction) is the one which is half obstructed, the phenomena in this direction is more prominent. It can be observed on figure 16.

Finally the last aperture studied was the double slit in case of Fresnel diffraction. What can be seen is that the closer it is from the diffracting plane, the more the two slits seem to emit independently, as seen on figure 18, and the further we get the more a situation similar to the Fraunhofer approximation is obtained, as in figure 8.

3 Conclusion

The Fraunhofer approximation is a very useful and easy way to compute the resulting fields of diffraction patterns. It permits to obtain independent integrals in both directions (x and y) and therefore to use the FFT, a powerful and very fast algorithm to compute Fourier Transforms. The computation time is then small. On the opposite, the Fresnel formalism for the computation of close fields does not allow us to simplify the equations and we end up with complicated formula. These formula have to be solved numerically by large iterations, which induces a long computational time.