1A) $P(E=1)=.9 \Rightarrow P(E=0)=.1$

$P(N=1)=.25 \Rightarrow P(N=0)=.75$

Since independent + no condition its equal to $P(N=0)=\boxed{.75}=P(N=0|R)$

1B) $P(R=1|E=1, N=1)=1$ $\qquad P(R=0|E=1, N=1)=0$

$P(R=1|E=1, N=0)=0$ $\Rightarrow$ $P(R=0|E=1, N=0)=1$

$P(R=1|E=0, N=1)=.1$ $\qquad P(R=0|E=0, N=1)=.9$

$P(R=1|E=0, N=0)=.1$ $\qquad P(R=0|E=0, N=0)=.9$

$$P(N=0|R=0)=\frac{P(R=0|N=0)\cdot P(N=0)}{P(R=0)} = \frac{.99\ (.75)}{(.765)}$$

$$P(R=0)= P(R=0|E=1,N=1)\cdot P(E=1,N=1) + P(R=0|E=0,N=1)\cdot P(E=0,N=1)$$
$$+ P(R=0|E=1,N=0)\cdot P(E=1,N=0) + P(R=0|E=0,N=0)\cdot P(E=0,N=1)$$

$$= (0)\cdot(.9)(.25) + (1)(.9)(.75) + (.9)(.1)(.25) + (.9)(.1)(.75)$$

$$0 + .675 + .0225 + .0675 =$$

$$\underline{P(R=0)=.765^{*}} \Rightarrow P(R=1)=.235$$

$$P(R=0|N=0)=\frac{1}{P(N=0)}\cdot\left(P(R=0|N=0,E=1)(P(N=0,E=1)) + P(R=0|N=0,E=0)P(N=0,E=0)\right)$$

$$\frac{1}{.75}\cdot\left(1\cdot(.75)(.9) + (.9)(.1)(.75)\right) = .9 + .09 = \underline{.99} = P(R=0|N=0)$$

$$P(N=0|R=0)=\frac{.99\ (.75)}{.765} = \boxed{.97 = P(N=0|R=0)}$$

This makes sense because it should be much higher than a since we are given information about if a spike was recorded, which in this case wasn't and lead to higher chance no spike occured.

1c) $P(N=0 \mid R=0, E=0)$

$$= \frac{P(R=0, N=0, E=0)}{P(R=0, E=0)} = \frac{\overset{.9}{P(R=0 \mid E=0, N=0)} \cdot \overset{(.1)\,(.75)}{P(E=0, N=0)}}{P(R=0, E=0)}$$

$$P(R=0, E=0) = P(E=0) \cdot P(R=0 \mid E=0) \quad = .1$$

$$P(R=0 \mid E=0) = P(R=0 \mid E=0, N=1)P(E=0, N=1) + P(R=0 \mid E=0, N=0) \,/\, P(E=0)$$

$$= ((.9)(.1)(.25) + (.9)(.1)(.75))/.1 = .9$$

$$P(R=0, E=0) = (.1)(.9) = .09$$

$$P(N=0 \mid R=0, E=0) = \frac{(.9)(.1).75}{.09} = \boxed{.75}$$

This makes sense because when you are given additional information that the equipment was broken, it increases the chances that no spike was recorded due to the equipment being broken and lowers the chances of it being from no spike occurring, and we see that the probability of No spike occuring was less in this scenario.

1d) $P(E, N \mid R) \overset{?}{=} P(E \mid R) P(N \mid R)$

Case $R=1, E=1, N=1$

$$\frac{P(R=1, E=1, N=1)}{(R=1)} = \frac{P(R=1 \mid E=1) P(E=1)}{P(R=1)} \cdot \frac{P(R=1 \mid N=1) P(N=1)}{P(R=1)}$$

$$\frac{P(E=1)P(N=1)P(R=1 \mid E=1, N=1)}{P(R=1)} = \frac{P(E=1)\left[P(R=1 \mid E=1, N=0) P(N=0) + P(R=1 \mid E=1, N=1) P(N=1)\right]}{P(R=1)}$$

$$\frac{(.9)(.25)(1)}{.235} \overset{?}{=} \frac{P(N=1)\left[P(R=1 \mid N=1, E=0) \cdot P(E=0) + P(R=1 \mid N=1, E=1) P(E=1)\right]}{P(R=1)}$$

$$.957 \overset{?}{=} \frac{.9\,[0 + 1(.25)]}{(.235)} \cdot \frac{(.25)\,[(.1)(.1) + .9]}{(.235)}$$

from earlier

$$.957 \neq .927$$

Since $P(E, N \mid R) \neq P(E \mid R) P(N \mid R)$

for $R=1, E=1, N=1$ they are $\boxed{\text{not independent}}$

Intuitively it makes sense because knowing a spike recorded and the machine is working will tell you with 100% guarantee that a spike must have occurred.

$\boxed{2a)}$ $P(a,b,c,d) =$

$$\begin{tikzcd} a \arrow[r] & b \arrow[d] \\ c \arrow[u] & d \end{tikzcd}$$

$$\boxed{P(c)\cdot P(d)\, P(a|c)\cdot P(b|d,a)}$$

$\boxed{2b}$ $P(c,d) \stackrel{?}{=} P(c)\cdot P(d)$

$$= \sum_a \sum_b P(a,b,c,d)$$

$$= P(c)\cdot P(d)\cdot \underbrace{\sum_a \sum_b P(a|c)\cdot P(b|a,d)}_{=1}$$

$$= P(c)\cdot P(d)$$

$c \perp\!\!\!\perp d$

$\boxed{\text{Yes } c+d \text{ are independent}}$

$\boxed{2c)}$ $P(c,d|a,b) \stackrel{?}{=} P(c|a,b)\cdot P(d|a,b)$

$$= \frac{P(a,b,c,d)}{P(a,b)} = \frac{P(c)\cdot P(d)\, P(a|c)\cdot P(b|a,d)}{P(a,b)}$$

$$P(a|c) = \frac{P(a)\, P(c|a)}{P(c)}$$

$$= \frac{P(c)\cdot P(d)\, P(a|c)\cdot P(b|a)\, P(d|b,a)}{P(a,b)\cdot P(d|a)} \frac{P(b|a,c)}{P(b|a,c)}$$

$$\frac{P(c|a,b)\cdot P(d|a,b)}{P(b|a)}$$

$$= \frac{P(c)\cdot P(a|c)\cdot P(b|a,c)}{P(a,b)} \cdot \frac{P(d)\, P(b|a,d)}{P(b|a,c)}$$

$$\frac{P(c|a,b)\cdot P(d)\cdot P(b|a)\, P(d|a,b)}{P(b|a)\, \underbrace{P(d|a)}_{=P(d)}}$$

$$= P(c|a,b)\cdot \frac{P(d)\, P(b|a,d)}{P(b|a,c)} \longrightarrow = P(b|a) \underbrace{\sum_d P(d|a,b)}_{=1}$$

$$= P(c|a,b)\cdot P(d|a,b)$$

$\boxed{c \perp\!\!\!\perp d \text{ Yes } c+d \text{ independent given } a+b}$

2d) $P(a,d) \overset{?}{=} P(a) \cdot P(d)$

$$\sum_b \sum_c P(a,b,c,d)$$

$$\sum_b \sum_c P(C) P(d) P(a|C) P(b|a,d)$$

$$= P(d) \sum_C P(c) \frac{P(a) P(c|a)}{P(c)} \cdot \underbrace{\sum_b P(b|a,d)}_{=1}$$

$$= P(d) \cdot P(a) \cdot \underbrace{\sum_c P(c|a)}_{=1}$$

$$= P(C) \cdot P(d)$$ $\boxed{a \perp\!\!\!\perp d \quad \text{Yes a+d are independent}}$

2E) $P(a,d|b) \overset{?}{=} P(a|b) \cdot P(d|b)$

$$= \sum_c \frac{P(a,b,c,d)}{P(b)} = \sum_c \frac{P(c) P(d) P(a|c) P(b|a,d)}{P(b)}$$

$$= \frac{P(d) P(b|a,d)}{P(b)} \sum_c \frac{\cancel{P(c)} P(a) P(c|a)}{\cancel{P(c)}}$$

$$= \frac{P(a) \cancel{P(d)} \cdot P(b|a) P(d|a,b)}{P(b) \cancel{P(d|a)}}$$

$$= \frac{\cancel{P(a)} \cancel{P(b)} \cdot P(a|b) P(d|a,b)}{\cancel{P(a)} \cancel{P(b)}}$$

$$= P(a|b) P(d|a,b)$$

$a \not\perp\!\!\!\perp d | b$ $\boxed{\text{No c and d are not independent}}$

2f) $P(b,c) \stackrel{?}{=} P(b) \cdot P(c)$

$$= \sum_a \sum_d P(a,b,c,d)$$

$$\sum_a \sum_d P(c)\, P(d)\, P(a|c)\, P(b|a,d)$$

$$= P(c) \sum_a \sum_d \cancel{P(d)}\, P(a|C) \cdot \frac{P(b|a)\, P(d|a,b)}{\cancel{P(d|a)}}$$

$$= P(c)\, P(b|a) \underbrace{\sum_a P(a|C)}_{1} \underbrace{\sum_d P(d|a,b)}_{1}$$

$$= P(b|a) \cdot P(c)$$

$\cancel{b \perp\!\!\!\perp} C$ $\boxed{\text{No } b + c \text{ are Not independent}}$

2G) $P(b,c|a) \stackrel{?}{=} P(b|a) \cdot P(c|a)$

$$\sum_d \frac{P(a,b,c,d)}{P(a)}$$

$$= \sum_d \frac{P(c)P(d)\, P(a|c)\, P(b|a,d)}{P(a)}$$

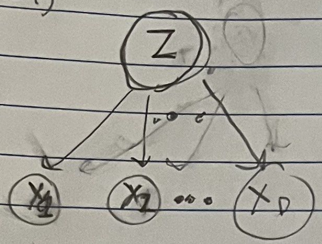
$$\sum_d \frac{\cancel{P(c)}\, \cancel{P(d)}\, \cancel{P(a)}\, P(c|a)\, P(b|a)\, P(d|a,b)}{\cancel{P(a)}\, \cancel{P(c)}\, \cancel{P(d|a)}}$$

$$= P(b|a)\, P(c|a) \sum_d \underbrace{P(d|a,b)}_{1}$$

$$= P(b|a)\, P(c|a)$$

$b \perp\!\!\!\perp c | a$ $\boxed{\text{Yes } b + c \text{ are independent given } a}$

3a)



3b) $P(X_0, X_1 \ldots X_D, Z) = \boxed{P(Z)\,P(X_1|Z)\,P(X_2|Z)\ldots P(X_D|Z)}$

3c) $P(X_1, X_2) \stackrel{?}{=} P(X_1, X_2)$

$= X_1 \not\perp X_2$

Intuitively No, knowing information about any X_i gives you some information about Z which gives you information for all X

4a) $P(X_1, X_2, X_3, X_4) = P(X_1)\,P(X_2|X_1)\,(X_3|X_2)\,P(X_4|X_3)$

4b) $X_1 \sim N(0, \sigma^2)$ $\quad X_t | X_{t-1} \sim (X_{t-1}, \sigma^2)$

$t = 2, 34$

$COV(X,Y) = E[XY] - E[X]E[Y]$ $\quad Var[X] = E[X^2] - E[X]^2$

let $N(0,\sigma^2)$ $1234 = i$ be iid gaussian

$x_1 = y_1$ $\sim N(0, \sigma^2)$

$x_2 = y_1 + y_2$ $\sim N(0, 2\sigma^3)$

$x_3 = y_1 + y_2 + y_3$ $\sim N(0, 3\sigma^2)$

$x_4 = y_1 + y_2 + y_3 + y_4$ $\sim N(0, 4\sigma^2)$

$N(\mu_1, \sigma_1^2) + N(\mu_2, \sigma_2^2) =$

$N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

4b Contined...

$cov(x_1, x_1) = E[x_1^2] - 2E[x_1]$ $\quad$ $var(x_1) = E[x_1]^2 - E[x_1]^2$

$\sigma^2 - 2(0)$ $\quad$ $\sigma^2 = E[x_1^2] - 0$

$= \sigma^2$

$cov(x_1, x_2) = E[x_1 x_2] \cdot E[x_1] \cdot E[x_2]$

$= E[y_1(y_1 + y_2)] - 0$

$= E[y_1^2] + E[y_1 y_2]$

$= \sigma^2$

$cov(x_2, x_2) = E[x_2^2] - 2E[x_2]$ $\quad$ $var(x_2) = E[x_2^2] - E[x_2]^2$

$= 2\sigma^2 - 0$ $\quad$ $2\sigma^2 = E[x_2^2] - 0$

$= 2\sigma^2$

...From this the pattern we get

$cov(x_1, x_2) = cov(x_2, x_1) \rightarrow$
$$\begin{bmatrix} \sigma^2 & \sigma^2 & \sigma^2 & \sigma^2 \\ \sigma^2 & 2\sigma^2 & 2\sigma^2 & 2\sigma^2 \\ \sigma^2 & 2\sigma^2 & 3\sigma^2 & 3\sigma^2 \\ \sigma^2 & 2\sigma^2 & 3\sigma^2 & 4\sigma^2 \end{bmatrix}$$

In terms of σ^2

4b

$$\boxed{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \cdot \sigma^2 = cov}$$

4c

$$\boxed{\sigma^2 \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}}$$

4d Zeroes correspond to places in the graph where there are no connections present