Markov Chain Hitting Times and Penney's Game

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1 Introduction

We will collect here several interesting facts about hitting times of finite Markov chains, which do not appear to be as well known as they should be. We give an interesting application to the famous Penney game.

Consider a finite Markov Chain with K states and transition matrix $T_{ij} = P(X_n = j | X_{n-1} = i)$. T is assumed to be irreducible; the stationary distribution is denoted by π. The initial state X_1 is sampled according to some fixed distribution μ and the state thereafter evolves according to the transition matrix T.

2 Hitting time distribution

Fix some finite subset S of states. It is required that $1 \leq |S| \leq K-1$. Define τ_S to be the first time *after t=1* that the walk visits S. Thus if the initial state X_1 is in S, then τ_S is a kind of return time; otherwise τ_S is the first passage time to S.

We will characterize the full distribution of τ_S.

Let $\phi_S(t) = \sum_{n\geq 1} P(\tau_S = n)t^n$ be the standard probability generating function and set $\overline{\phi}_S(t) := \sum_{n\geq 1} P(\tau_S > n)t^n$. Knowing ϕ_S is of course equivalent to knowing the full distribution of τ_S.

It turns out that there is a simple expression for $\overline{\phi}$. To derive this, we will consider the values $p_{n,k} := P_\mu(\tau_S > n, X_n = k)$. Since we do not count the initial X_1 in τ_S, it follows $p_{1,k} = \mu_k$. On the other hand, for $n > 1$, evidently $p_{n,k} = 0$ if $k \in S$. Thus for $n > 1$:

$$\begin{aligned} p_{n,k} &= P(\tau_S > n, X_n = k) && (1) \\ &= \sum_{k'} P(\tau_S > n, X_n = k, X_{n-1} = k') && (2) \\ &= \delta_{k\notin S} \sum_{k'} p_{n-1,k'} T_{k',k} && (3) \end{aligned}$$

Introducing the partial generating function $\overline{\phi}_k(t) := \sum_{n\geq 1} p_{n,k}t^n$ the above can be written

$$\begin{aligned}
\overline{\phi}_k(t) &= t\mu_k + \sum_{n\geq 2}\sum k' p_{n-1,k'}T_{k',k}t^n && (4)\\
&= t\mu_k + t\delta_{k\notin S}\sum_{k'}\overline{\phi}_{k'}(t)T_{k',k} && (5)\\
\overline{\phi}.(t)^T &= t\mu^t + t\overline{\phi}.(t)^T T_{\notin S} && (6)\\
\overline{\phi}.(t)^T &= t\mu^T(I - tT_{\notin S})^{-1} && (7)\\
& && (8)
\end{aligned}$$

And since $\overline{\phi}_S(t) = \sum_k \overline{\phi}_k(t)$, it follows that

$$\overline{\phi}_S(t) = t\mu^T(I - tT_{\notin S})^{-1}\mathbf{1} \tag{9}$$

To relate back to ϕ_S we note that

$$\begin{aligned}
P(\tau_S = n) &= P(\tau_S > n-1) - P(\tau_S > n) && (10)\\
\phi_S(t) &= \sum_{n\geq 1} P(\tau_S > n-1)t^n - \sum_{n\geq 1} P(\tau_S > n)t^n && (11)\\
&= t\sum_{n\geq 0} P(\tau_S > n)t^n - \overline{\phi}_S(t) && (12)\\
&= t(1 + \overline{\phi}_S(t)) + \overline{\phi}_S(t) && (13)\\
&= t + \overline{\phi}_S(t)(t-1) && (14)
\end{aligned}$$

We have proved:

Proposition 2.0.1 *The probability generating function $\phi_S(t) = \sum_{n\geq 1} P_\mu(\tau_S = n)t^n$ is given by*

$$\phi_S(t) = t + (t^2 - t)\mu^T(I - tT_{\notin S})^{-1}\mathbf{1} \tag{15}$$

where $(T_{\notin S})_{ij} = T_{ij}\delta_{j\notin S}$

An immediate consequence is that the factorial moments of τ_S are given by the derivatives of ϕ_S evaluated at $t = 1$. We can work these out by simple calculus; the result is (for $n > 1$):

$$\begin{aligned}
fm_n = \phi_S^{(n)}(1) &= n!\mu^T T^n_{\notin S}(I - T_{\notin S})^{-n}\mathbf{1} && (16)\\
&= n!\mu^T(-I + (I - T_{\notin S})^{-1})^n\mathbf{1} && (17)
\end{aligned}$$

while

$$fm_1 = \phi_S^{(1)}(1) = \mu^T(I - T_{\notin S})^{-1}\mathbf{1} \tag{18}$$

Thus we see that the matrix $(I - T_{\notin S})^{-1}$ completely controls the moments of the hitting time τ_S- the moments are given explicitly by $fm_n = \mu^T p_n((I - T_{\notin S})^{-1})\mathbf{1}$ where $p_n(x) = n!(x-1)^n$ for $n > 1$ and $p_1(x) = x$.

In general, the raw moments can be expressed in terms of the factorial moments

$$m_k = \sum_{j \le k} S(k,j) fm_j$$

where S are Stirling numbers of the second kind. So again the raw moments are polynoials in the matrix $(I - T_{\notin S})^{-1}$.

There is also a convenient formula for this matrix in terms of a group inverse, which we turn to next.

3 Low rank correction

We saw before that the matrix $(I - T_{\notin S})^{-1}$ plays a central role in characterizing the distribution of τ_S. This matrix also has an alternate expression which is sometimes more convenient, in terms of a *group inverse* [1].

Proposition 3.0.1 *Let T be a $K \times K$ irreducible stochastic matrix with stationary distribution π, S a proper subset of $\{1, \dots, K\}$ and let $T_{\notin S}$ be the matrix given by zeroing out every column of T that is in S. Then*

$$(I - T_{\notin S})^{-1} = Q + (I-Q)_{\cdot,S} Q_{S,S}^{-1} Q_{S,\cdot} + \frac{(\mathbf{1}_K + (I-Q)_{\cdot,S} Q_{S,S}^{-1} \mathbf{1}_{|S|})(\pi^T - \pi_S^T Q_{S,S}^{-1} Q_{S,\cdot})}{\pi_S^T Q_{S,S}^{-1} \mathbf{1}_{|S|}} \tag{19}$$

where $Q = (I - T)^g$.

At first glance, it might appear that we can prove this by writing $I - T_{\notin S}$ as a rank $|S|$ correction to $I - T$, and then using the Woodbury formula. However, this doesn't work because $I - T$ is not invertible. The trick is to instead consider $I - tT_{\notin S}$ for some $t < 1$, and write this as a rank $|S|$ correction to the invertible matrix $I - tT$. Thus for $t < 1$ the Woodbury formula is valid, and we get a formula for $(I - tT_{\notin S})^{-1}$ that involves $(I - tT)^{-1}$. The next step is to take the limit $t \to 1$ of the resulting expression, so we need to know how $(I - tT)^{-1}$ behaves in this limit. For this, we need the following lemma:

Lemma 3.1 *For an irreducible T with stationary distribution π, we have*

$$\lim_{t \to 1} (I - tT)^{-1} - \frac{\mathbf{1}\pi^T}{1 - t} = (I - T)^g \tag{20}$$

This lemma is proved in the Appendix (Section 7). Given this, the proposition follows from a bunch of tedious algebra.

[1] note that this is not the same as the more well-known *pseudoinverse*, although they happen to coincide in some cases

4 Asymptotic questions via Markov Chain CLT

Now consider problems of the following sort: what is the probability that the walk stays entirely outisde of S for the first n steps? This is an interesting application of the central limit theorem developed in the coin-tossing notes. The probability turns out to die off exponentially with a particular rate that can easily be explicitly determined in terms of the pseudoinverse Q. See other notes for details.

Similarly we can handle questions like "what is the probability that the walk spends between 10 and 20 percent of the time in S"? (again, just apply the CLT from the other set of notes)

5 First-Hit State Distribution

Another interesting question is what is the distribution over the first encountered element of S? That is, what is $P_\mu(X_{\tau_S} = s)$ for $s \in S$?

We can argue similarly to the derivation of the pgf of τ_S.

Consider $q_{n,k} := P_\mu(\tau_S = n, X_n = k)$. Clearly $q_{1,k} = 0$ for any k. On the other hand, for $n > 1$, it follows that if $\tau_S = n$ and $X_n = k$, then necessarily $X_n \in S$ and $X_{n-1} \notin S$. Thus for $n > 1$,

$$\begin{aligned}
q_{n,k} &= \sum_{k'} P(\tau_S = n, X_n = k, X_{n-1} = k') && (21)\\
&= \delta_{k\in S} \sum_{k'} P(\tau_S > n-1, X_{n-1} = k') T_{k',k} && (22)\\
&= \delta_{k\in S} \sum_{k'} p_{n-1,k'} T_{k',k} && (23)
\end{aligned}$$

where $p_{n,k}$ is as in the previous section. Using also the notation $\overline{\phi}_k(t)$ as in the previous section, and noting that $P_\mu(X_{\tau_S} = k) = \sum_{n\geq 1} q_{n,k}$, we see that

$$\begin{aligned}
P_\mu(X_{\tau_S} = \cdot) &= \sum_{k'} \delta_{k\in S} T_{k',k} \overline{\phi}_{k'}(1) && (24)\\
&= \overline{\phi}.(t)^T T_{\in S} && (25)\\
&= \overline{\phi}.(1)^T (T - T_{\notin S}) && (26)\\
&= \mu^T (I - T_{\notin S})^{-1} (T - T_{\notin S}) && (27)\\
&= \mu^T (I - T_{\notin S})^{-1} ((I - T_{\notin S}) + (T - I)) && (28)\\
&= \mu^T (I - (I - T_{\notin S})^{-1} (I - T)) && (29)
\end{aligned}$$

where we used equation 7.

This expression combines nicely with the low rank correction formula (Proposition 3.0.1). Indeed, when evaluating $(I - T_{\notin S})^{-1}(I - T)$,we know any term like $(\ldots)\pi^T(I - T)$ will evaluate to zero, and moreover $Q(I - T) = I - \mathbf{1}\pi^T$, so at the end of the day we get

$$I-(I-T_{\notin S})^{-1}(I-T) = -(I-Q)_{\cdot,S}Q_{S,S}^{-1}I_{S,\cdot} + \frac{(\mathbf{1}+(I-Q)_{\cdot,S}Q_{S,S}^{-1}\mathbf{1})\pi_S^T Q_{S,S}^{-1}I_{S,\cdot}}{\pi_S^T Q_{S,S}^{-1}\mathbf{1}} \tag{30}$$

For example, the i,j entry of this matrix is equal to the probability that j is the first-encountered element of S, given that the random walk starts at i.

The expression simplifies considerably in the interesting case where the initial distribution μ over states is taken to be $\mu = \pi$. In this case $\mu^T Q = \pi^T Q = 0$ so multiplying on left by π^T and simplifying we get the following beautifully simple formula:

$$P_\pi(X_{\tau_S} = \cdot) = \frac{\pi_S^T Q_{S,S}^{-1} I_{S,\cdot}}{\pi_S^T Q_{S,S}^{-1}\mathbf{1}} \tag{31}$$

6 string counting games

We now consider the famous Penney game and some generalizations. In this game, player 1 and player 2 select binary strings of some fixed length L. We then flip a coin- whichever player has their string appear first in the sequence of flips wins the game.

Here the transition structure is given by the substring transition matrix as described in the other set of notes. There it is shown that this transition matrix has $\pi \propto \mathbf{1}$ and is irreducible, so the above results apply.

One slight detail is that in the Penney game, the appeareance of one of the special strings at $t = 1$ "counts". That is, if the first L flips happen to be s_1, then player 1 wins. This is a bit different than the definition of τ_S, in which the initial state does not "count" if it happens to lie in S.

A bit more formally, we can define $\overline{\tau}_S$ to be the first time a state in S is encountered, *including* $t = 1$ *if applicable*. By the above reuslts, we know the distribution of X_{τ_S}, but for the standard Penney game we really want $X_{\overline{\tau_S}}$. *It turns out these two random variables have the same distribution*! Why? It follows from the fact that (a) the initial distribution over states μ in this case coincides with the stationary distribution π and (b), the mapping $(X_1, X_2, \ldots) \to (X_2, X_3, \ldots)$ is measure preserving assuming $X_1 \sim \pi$.

Thus the formula 31 for the distribution over the first-hit state is exactly applicable.

In our case, of course $|S| = 2$. The matrix inverse simplifies a bit and we see that the odds in favor of player 2 are

$$P(X_\tau = s_2)/P(X_\tau = s_1) \quad = \quad \frac{Q_{s_1,s_1} - Q_{s_1,s_2}}{Q_{s_2,s_2} - Q_{s_2,s_1}} \tag{32}$$

In particular, we only need 4 specific entries of Q to determine these probabilities. In the other set of notes, I explained how to compute Q_{ij} in terms of overlaps of i and j, so this gives an explicit way to compute these odds, which works for any pair of substrings i and j, and any length L.

This is basically equivalent to *Conway's algorithm*.

But the formula 31 applies to any set of states, not just $|S| = 2$. In particular, what if there are k players, each picks their own substring, what is the rank ordering of the players? This can easily be determined by looking at $\pi_S^T Q_{S,S}^{-1} \propto \mathbf{1}^T Q_{S,S}^{-1}$ where S is the set of strings.

There can also be weird higher-order inconsistencies. For example suppose there are three players who select strings A, B and C. It could happen that A beats B when all three players are playing, but that B beats A when only these two are playing.

7 Appendix: Limit expression for group inverse

Here we prove Lemma 3.1.

The first observation is that $T^n = \mathbf{1}\pi^T + O(c^n)$ where $c < 1$. Why? We know by Perron-Frobenius that the dominant eigenvector is *simple*, i.e. it has multiplicitly 1 in the characteristic polynomial. So if we consider the Jordan form of T, it will have a 1×1 Jordan Block with eigenvalue 1, and every other block will have an eigenvalue < 1. Now it is easy to see that we can take c to be the (magnitude of the) next-largest eigenvalue of T.

The next step is to show that the limit $(I - tT)^{-1} - \mathbf{1}\pi^T/(1-t)$ exists. To see this, write it as

$$\sum_n t^n (T^n - \mathbf{1}\pi^t) \tag{33}$$

By fact 1, the term in parenthesis is $O(c^n)$ where c is *strictly* less than 1. So the sum is $O(1/(1 - tc))$ and thus stays finite in the limit.

Since the limit exists, we can set $Q := \lim_{t\to1}(I - tT)^{-1} - \mathbf{1}\pi^T/(1 - t)$. To show that $Q = (I - T)^g$, we just need to check a few algebraic facts.

First, the fact that Q is finite implies that $(1 - t)(I - tT)^{-1} \to \mathbf{1}\pi^T$.

Furthermore,for $t < 1$,

$$\begin{aligned}
(I - tT)^{-1}T &= t^{-1}((I - tT)^{-1}(tT - I) + (I - tT)^{-1}) && (34)\\
&= t^{-1}(-I + (I - tT)^{-1}) && (35)\\
& && (36)
\end{aligned}$$

Combining the two observations,

$$\begin{aligned}
((I - tT)^{-1} - \mathbf{1}\pi^t/(1 - t))(I - T) &= (I - tT)^{-1}(I - T) && (37)\\
&= (I - tT)^{-1} - t^{-1}(-I + (I - tT)^{-1}) && (38)\\
&= t^{-1}I + (1 - t^{-1})(I - tT)^{-1} && (39)\\
&= t^{-1}I - t^{-1}(1 - t)(I - tT)^{-1} && (40)\\
&\Rightarrow && (41)\\
Q(I - T) &= I - \mathbf{1}\pi^T && (42)
\end{aligned}$$

By an analogous argument, $(I - T)Q = I - \mathbf{1}\pi^T$

At this point, we can check $Q(I-T)Q = Q$ and $(I-T)Q(I-T) = I-T$, which is a simple matter of algebra. It follows that $Q = (I-T)^g$ by the uniqueness property of the group inverse.