Panagiotis Christoforidis Gruppe 2 Übung 13

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A1

$$\int_0^{\pi} x\cdot\cos(x)\,dx = -\int_0^{\pi} 1\cdot\sin(x)\,dx + [x\cdot\sin(x)]_{x=0}^{x=\pi}$$

$f(x) = x \quad f'(x) = 1$

$g(x) = \sin(x) \quad g'(x) = \cos(x)$

$$= -[-\cos(x)]_{x=0}^{x=\pi} + [x\cdot\sin(x)]_{x=0}^{x=\pi}$$

$$= (\cos(\pi) - \cos(0)) + \pi\cdot\sin(\pi) - 0\cdot\sin(0)$$

$$= (-1 - 1) + 0\cdot(\pi + 0)$$

$$= \underline{\underline{-2}}$$ ✓

$$\int_1^e \frac{\ln(x^3)^2}{3x}\,dx = \int_1^e \frac{9\cdot\ln(x)^2}{3x}\,dx = \int_1^e \frac{3\cdot\ln(x)^2}{x}\,dx$$

$$= \int_1^e \frac{3\cdot u^2}{\cancel{x}\cdot\frac{1}{\cancel{x}}}\,du = \int_{e^e}^{e^e} 3u^2\,du$$

$u = \ln(x)$

$u' = \frac{1}{x}$

$\varphi^{-1} = e^x$

$dx = \frac{du}{u'}$

$$= 3\cdot\int_{e^e}^{e^e} u^2\,du$$

$$= 3\cdot\left[\tfrac{1}{3}u^3\right]_{e^e}^{e^e}$$

$$= 3\cdot\tfrac{1}{3}\left[u^3\right]_{e^1}^{e^e}$$

$$= [\ln(x)^3]_1^e$$

$$= \ln(e)^3 - \ln(1)^3$$ ✓

$$= 1^3 - 0^3 = 1$$

$$\int_{-\pi}^{\pi} e^x\cdot\cos(2x)\,dx = -\int_{-\pi}^{\pi} e^x\cdot -2\sin(2x)\,dx + [e^x\cdot\cos(2x)]_{-\pi}^{\pi}$$

$$= \int_{-\pi}^{\pi} e^x\cdot 2\cdot\sin(2x)\,dx + [e^x\cdot\cos(2x)]_{-\pi}^{\pi}$$

$$\int e^x\cos(2x)\,dx = -\int_{-\pi}^{\pi} e^x\cdot 4\cdot\cos(2x)\,dx + [e^x\cos(2x)]_{-\pi}^{\pi} + [e^x 2\sin(2x)]$$

$$5\cdot\int_{-\pi}^{\pi} e^x\cos(2x)\,dx = [e^x\cos(2x)]_{-\pi}^{\pi} + [e^x 2\sin(2x)]_{-\pi}^{\pi}$$

$$\int_{-\pi}^{\pi} e^x\cdot\cos(2x)\,dx = \left[\frac{e^x\cdot(\cos(2x) + 2\sin(2x))}{5}\right]_{-\pi}^{\pi}$$

$$= \frac{e^{\pi}\cdot(\cos(2\pi) + 2\sin(2\pi))}{5} - \frac{e^{-\pi}(\cos(2\pi) + 2\sin(2\pi))}{5}$$

$(= 4{,}6195)$ $$= \frac{e^{\pi} - e^{-\pi}}{5}$$ ✓

$$\int_0^{\pi/2} \frac{\cos(x)}{\sqrt{\sin(x)^2+3}} = \int_0^{\pi/2} \frac{\cancel{\cos(x)}}{\sqrt{u^2+3}\cdot\cancel{\cos(x)}}\,du$$

$u = \sin(x)$

$u' = \cos(x)$

$du = u'\cdot dx$

$dx = \frac{du}{u'}$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{u^2+3}}\,du$$

$$= \sqrt{3}\cdot\int \frac{\sec(s)}{\sqrt{3}}\,ds = \int \sec(s)\,ds$$

$u = \sqrt{3}\tan(s)$

$u' = \sqrt{3}\cdot\sec^2(s)$

$\sqrt{u^2+3} = \sqrt{3\cdot\tan(s)+3}$

$s = \tan^{-1}\left(\frac{u}{\sqrt{3}}\right)$

$$= \left[\log(\tan(s)+\sec(s))\right]$$

$$= \left[\log\left(\tan\left(\tan^{-1}\left(\tfrac{u}{\sqrt{3}}\right)\right) + \sec\left(\tan^{-1}\left(\tfrac{u}{\sqrt{3}}\right)\right)\right)\right]$$

$$= \left[\log\left(\frac{\sqrt{u^2+3}+u}{\sqrt{3}}\right)\right]$$

$$= \left[\log\left(\frac{\sqrt{\sin^2(x)+3}+\sin(x)}{\sqrt{3}}\right)\right]$$

$$= \left[\sinh^{-1}\left(\frac{\sin(x)}{\sqrt{3}}\right)\right]_0^{\pi/2} \checkmark$$

$(= 0{,}54931)$ besser: $\operatorname{arcsinh}\left(\frac{1}{\sqrt{3}}\right)$

3/3

A2 a) $|f(x)| \leq g(x)$, $-\infty < a < b < \infty$

Z: $\int_a^b g(x)\,dx < \infty \Rightarrow \int_a^b f(x)\,dx < \infty$

Annahme: $\int_a^b g(x)\,dx < \infty \Rightarrow \int_a^b f(x)\,dx \not< \infty$

$$\Rightarrow \int_a^b f(x)\,dx \to \infty$$

$$\Rightarrow \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \to -\infty$$

$$\Rightarrow \int_a^b g(x) - f(x)\,dx \to -\infty$$

$$\Rightarrow \exists x : \underline{f(x) \geq g(x)}$$ ↯ Widerspruch zur Annahme

Warum?

b) $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt = \lim_{k\to\infty}\left(\int_0^1 e^{-t} t^{x-1}\,dt + \int_1^k e^{-t} t^{x-1}\,dt\right)$

Z: $\Gamma(x)$ konvergiert und $\Gamma(x) < \infty \ \forall x$

$e^{-t}\cdot t^{x-1} < c\cdot t^{-\alpha}$ für $\alpha, t > 1$ $x > 0$ $c < \infty$

$\frac{1}{e^t}\cdot t^{x-1} < c\cdot t^{-\alpha}$

$\Leftrightarrow e^{-t} \cdot t^{x-1} \cdot t^{\alpha} < c$

$\Leftrightarrow e^{-t} \cdot t^{x+\alpha-1} < c$ wähle $z = x + \alpha - 1$

$\Leftrightarrow \underbrace{e^{-t} \cdot t^{z}}_{\Theta(t)} < c$ da $x > 0$, $\alpha > 1$ $z > 0 \ (= 0 + 1 - 1)$

$\Theta'(t) = -t \cdot e^{-t} \cdot z \cdot t^{z-1}$ ↯ $\Theta'(t) < 0 \quad \forall t$ Produktregel verwenden

$\Rightarrow \Theta(t)$ streng monoton fallend für alle t

mit $c = 2e^{-1}$ und für $\Theta(1) = e^{-1}$ folgt

$$\frac{1}{e} < \frac{2}{e}$$

$$\Rightarrow \Theta(t) < \frac{2}{e} \quad \forall t \qquad \square$$

c) fehlt. 1/3

A3 a) $\arctan(x) = (\tan(x))^{-1}$ $\qquad \cos^2 + \sin^2 = 1$

$f(x) = \arctan(x)$ $\qquad \sin = \sqrt{1 - \cos^2}$

$g := f^{-1}$ $\qquad \cos = \sqrt{1 - \sin^2}$

$g(x) = \tan(x) \qquad g'(x) = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos^2(x) - (-\sin^2(x))}{\cos^2(x)}$

$f'(x) = \frac{1}{g'(f(x))}$ $\qquad = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$

$= \frac{1}{\cos^2(\arctan(x))^{-1}} = \cos^2(\arctan(x)) = \frac{1}{\cos^2(x)}$

$$\tan(\arctan(x)) = \frac{\sin(\arctan(x))}{\cos(\arctan(x))}$$

$$x = \frac{\sqrt{1 - \cos^2(\arctan(x))}}{\cos(\arctan(x))}$$

$$x^2 = \frac{1 - \cos^2(\arctan(x))}{\cos^2(\arctan(x))}$$

$$x^2 = \frac{1}{\cos^2(\arctan(x))} - 1$$

$$x^2 + 1 = \frac{1}{\cos^2(\arctan(x))} \qquad \cos^2(\arctan(x)) = \frac{1}{x^2+1}$$

$$\Rightarrow f'(x) = \frac{1}{x^2+1} = (\arctan(x))'$$

$$\Rightarrow \int \frac{1}{x^2+1} = \arctan(x), \quad da$$

$$(\arctan(x))' = \frac{1}{\tan'(\arctan(x))} = \cos^2(\arctan(x)) = \frac{1}{x^2+1}$$

$\square$

b) $$\int_a^b \arctan(x)\,dx = \int_a^b 1 \cdot \arctan(x)\,dx$$

$$= -\int_a^b x \cdot \frac{1}{x^2+1}\,dx + [x \cdot \arctan(x)]_{x=a}^{x=b}$$

$= x^2+1$ $$= -\int_a^b \frac{x}{x^2+1}\,dx + [x \cdot \arctan(x)]_{x=a}^{x=b}$$

$= 2 \cdot x$ $$= -\int_a^b \frac{\cancel{x}}{u \cdot 2 \cdot \cancel{x}}\,du + [x \cdot \arctan(x)]_{x=a}^{x=b}$$

$= \frac{dx}{u'}$ $$= -\int_a^b \frac{1}{2} \cdot \frac{1}{u}\,du + [x \cdot \arctan(x)]_{x=a}^{x=b}$$

$$= -\frac{1}{2} \cdot \int_a^b \frac{1}{u}\,du + [x \cdot \arctan(x)]_{x=a}^{x=b}$$

it $u = x^2+1$ $$= -\frac{1}{2}[\log(x^2+1)]_{x=a}^{x=b} + [x \cdot \arctan(x)]_{x=a}^{x=b}$$

$$\cancel{= [x \cdot \arctan(x)]_{x=a}^{x=b} + [x \cdot \arctan(x)]_{x=a}^{x=b}}$$

$$= \left[x \cdot \arctan(x) - \frac{1}{2} \cdot \log(x^2+1)\right]_{x=a}^{x=b}$$

$\square$

3/3