Rechenaufgaben

1

$$Z: \sum_{k=1}^{n} \frac{1}{k^2+k} = \frac{n}{n+1} \quad \forall n \geq 1$$

IA $\displaystyle\sum_{k=1}^{1} \frac{1}{k^2+k} = \frac{1}{1^2+1} = \frac{1}{2} = \frac{n}{n+1}$

IV $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2+k} = \frac{n}{n+1}$ gilt für ein $n \in \mathbb{N}$

IS $n \mapsto n+1$

$$\sum_{k=1}^{n+1} \frac{1}{k^2+k} = \sum_{k=1}^{n} \frac{1}{k^2+k} + \frac{1}{(n+1)^2+(n+1)}$$

$$\overset{IV}{=} \frac{n}{n+1} + \frac{1}{n^2+2n+1+n+1}$$

$$= \frac{n \cdot (n^2+3n+2) + (n+1) - 1}{n^3+4n^2+2+5n}$$

$$= \frac{n^3+3n^2+3n+1}{n^3+4n^2+5n+2} = \frac{(n+1)^3}{(n+1)^2(n+2)}$$

$$= \frac{n+1}{n+2} \quad \square$$

NR

$$(n^3+4n^2+5n+2) : (n+2) = n^2+2n+1 = (n+1)^2$$

$$\begin{array}{l} -(n^3+2n^2) \\ \hline \quad 2n^2+5n \\ \quad -(2n^2+4n) \\ \hline \qquad n+2 \\ \qquad -(n+2) \\ \hline \qquad\quad 0 \end{array}$$

② a) $z^2 - 10z + 26 = 0$

<u>p-q-Formel</u>

$$z_{1,2} = \frac{10}{2} \pm \sqrt{5^2 - 26}$$
$$= 5 \pm i$$

b) $z^6 = 64$

$$\Leftrightarrow (r_z e^{i\varphi_z})^6 = 64 \cdot e^{ik2\pi}, k \in \mathbb{Z} \qquad |\sqrt[6]{}$$
$$\Leftrightarrow r_z \cdot e^{i\varphi_z} = (64)^{\frac{1}{6}} \cdot (e^{ik2\pi})^{\frac{1}{6}}$$
$$= 2 \cdot e^{i\frac{k\pi}{3}}$$

Da der Grad 6 ist gibt es nach dem Fundamentalsatz der Algebra 6 Nullstellen

$$z_{1,\dots,6} = 2 \cdot e^{i\frac{k\pi}{3}}, k = 0, \dots, 5$$

⑤

a) $\int x^2 \cdot \sin(x) =$

$$[x^2 \cdot (-\cos(x))] - \int 2x \cdot (-\cos(x)) dx$$
$$= [x^2 \cdot (-\cos(x))] - \left([2x \cdot (-\sin(x))] - \int 2 \cdot (-\sin(x)) dx\right)$$
$$= [x^2 \cdot (-\cos(x)) + 2x \cdot \sin(x) + 2\cos(x)] + C \qquad C \in \mathbb{R}$$

b)

$$\int \frac{1}{1 + \sqrt[3]{x}} dx$$

$$\boxed{\begin{array}{l} \text{I} \quad u = x^{\frac{1}{3}} \qquad u' = \frac{x^{\frac{2}{3}}}{3} = \frac{1}{3(x^{\frac{1}{3}})^2} \\ \frac{du}{dx} = u' \Rightarrow dx = \frac{du}{u'} \end{array}}$$

$$\int \frac{1}{1+\sqrt[3]{x}} \overset{\text{sub I}}{=} \int \frac{1}{1+u} \cdot 3\left(x^{\frac{1}{3}}\right)^2 du$$

II $w = 1+u$	$u = w-1$
$w' = 1$	
$du = \frac{dw}{w'}$	

$$\overset{\text{sub I}}{=} 3 \cdot \int \frac{u^2}{1+u} du$$

$$\overset{\text{sub II}}{=} 3 \int \frac{u^2}{w} dw$$

$$\overset{\text{sub II}}{=} 3 \int \frac{(w-1)^2}{w} dw$$

$$= 3 \int \frac{w^2-2w+1}{w} dw$$

$$= 3\left(\int w\, dw - \int 2\, dw + \int \frac{1}{w} dw\right.$$

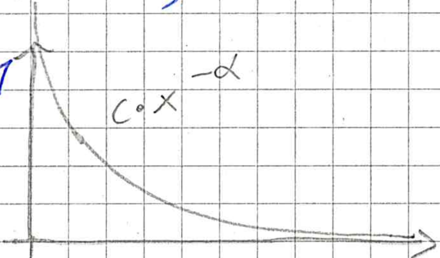
$$= \frac{3}{2} w^2 - 6w + 3\ln(w)$$

$$= \frac{3}{2}(1+u)^2 - 6(1+u) + 3\ln(1+u)$$

$$= \frac{3}{2}\left(1+x^{\frac{1}{3}}\right) - 6 - 6x^{\frac{1}{3}} + 3\ln\left(1+x^{\frac{1}{3}}\right) + C \quad C \in \mathbb{R}$$

6 a) Z: $\int_1^\infty \frac{1}{x \cdot e^x} dx \le \int_1^\infty c \cdot x^{-\alpha} dx$

$\alpha > 1$



$\Leftrightarrow \frac{1}{xe^x} < \frac{c}{x^\alpha}$ ab einem x_0

$\Leftrightarrow \frac{1}{e^x} < \frac{c}{x^{\alpha-1}}$

Wähle $\alpha = 2, c = 1$

$\frac{1}{e^x} < \frac{1}{x}$ dies gilt, da $e^x > x \;\forall x > 1$ $\square$

a) $\int_0^\infty \frac{3x+5}{x^2+4x+3}\,dx$

$$x^2+4x+3=0$$
$$\Leftrightarrow x_{1,2} = -2 \pm \sqrt{4-3}$$
$$\Leftrightarrow x_1 = -1 \quad x_2 = -3$$

Partialbruchzerlegung

$$\frac{3x+5}{x^2+4x+3} = \frac{3x+5}{(x+1)(x+3)} \overset{!}{=} \frac{A}{(x+1)} + \frac{B}{(x+3)}$$
$$= \frac{A(x+3)+B(x+1)}{(x+1)(x+3)}$$

$$\Rightarrow A(x+3)+B(x+1) = 3x+5$$
$$\Rightarrow A=1 \wedge B=2$$

$$\int_0^\infty \frac{3x+5}{x^2+4x+3}\,dx = \int_0^\infty \frac{1}{x+1} + \frac{2}{x+3}\,dx$$
$$= \int_0^\infty \frac{1}{x+1}\,dx + \int_0^\infty \frac{2}{x+3}\,dx$$
$$= \int_0^\infty \frac{1}{x+1}\,dx + 2\int_0^\infty \frac{1}{x+3}\,dx$$
$$= \lim_{\beta\to\infty}\Big(\underbrace{\underbrace{\big[\ln(x+1)\big]_0^\beta}_{\to\infty} + 2\cdot\underbrace{\big[\ln(x+3)\big]_0^\beta}_{\to\infty}}_{\to\infty}\Big)$$

$\Rightarrow$ uneigentl. Integral existiert nicht

④ $T_3(x,1) \quad \alpha = 1$

$$f(x) = \ln^2(x) = \ln(x) \circ \ln(x)$$

$$f'(x) = \frac{1}{x} \cdot \ln(x) + \frac{\ln(x)}{x} = \frac{2\ln(x)}{x}$$

$$f''(x) = 2 \cdot \left(\frac{\frac{1}{x} \cdot x - \ln(x)}{x^2}\right) = 2 \cdot \frac{1-\ln(x)}{x^2}$$

$$f'''(x) = \left(2 \cdot \frac{1-\ln(x)}{x^2}\right)' = 2 \cdot \left(\frac{-\frac{1}{x} \cdot x^2 - (1-\ln(x)) \cdot 2x}{x^4}\right)$$

$$= 2 \cdot \left(\frac{-x-(1-\ln(x))+2x}{x^4}\right)$$

$$= 2 \cdot \left(\frac{-x-2x+\ln(x) \cdot 2x}{x^4}\right)$$

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} \cdot (x-x_0)^j$$

$$T_3(x,1) = \frac{2}{2}(x-1)^2 - \frac{6}{6} \cdot (x-1)^3$$

③

a) $\lim\limits_{x\to 0} \frac{1-x-e^{-x}}{1-\cos^2(x)}$ $\quad \lim(f) = \frac{0}{0}$ stetige Fkt, diffbarer Nenner & Zähler

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-1+e^{-x}}{-(-2\cos(x) \cdot \sin(x))} = \lim_{x\to 0} \frac{-1+e^{-x}}{2\cos(x)\sin(x)} \overset{L'H}{=} \lim_{x\to 0} \frac{-e^{-x}}{2 \cdot \cos^2(x) + 2\sin^2(x)}$$

$$= \lim_{x\to 0} \frac{-e^{-x}}{2} = \frac{-1}{2}$$

b) $\lim\limits_{x\to\infty} \frac{\cos^2(x)}{\ln(x)}$ $\qquad \cos : \mathbb{R} \to [-1,1]$

$\qquad\qquad \lim\limits_{x\to\infty} \ln(x) \to \infty$

$$\Rightarrow \lim_{x\to\infty} \frac{\cos(x)}{\ln(x)} = 0$$

c) $\lim\limits_{n\to\infty} \dfrac{n^4+5n^2+n\cdot e^{-n}}{(2n^2-1)^2}$ $\qquad \lim\limits_{n\to\infty} n\cdot e^{-n}=0$

$= \lim\limits_{n\to\infty} \dfrac{n^4+5n^2}{4n^4-4n^2+1}$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{n^4}(1+5n^{-2})}{\cancel{n^4}\,\underbrace{(4+n^{-2}+n^{-4})}_{\to 0}}$ $\qquad \lim\limits_{n\to\infty} 5n^{-2}=0$

$= \dfrac{1}{4}$

Beweisteil

Kettenregel

1 $$(f(g(x_0)))' = \lim_{x\to x_0} \frac{f(g(x))-f(g(x_0))}{x-x_0}$$

$$= \lim_{x\to x_0} \frac{f(g(x))-f(g(x_0))}{g(x)-g(x_0)} \cdot \frac{g(x)-g(x_0)}{x-x_0} = f'(g(x_0))\cdot g'(x_0)$$

Umkehrfunktion

Z: $f'(x_0) = \frac{1}{f^{-1\prime}(f(x_0))}$

Bew: $$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(f(x))-g(f(x_0))}$$

$$= \lim_{x\to x_0} \frac{1}{\frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)}} = \frac{1}{g'(f(x_0))}$$

2

a) $f_1(x) = a^x \qquad f_1'(x) = \ln(a)\cdot a^x$

$$(f_1(x))' = (a^x)' = (\exp(\ln(a)\cdot x))' = \ln(a)\cdot\exp(x\cdot\ln(a))$$

$$= \ln(a)\cdot a^x \quad \square$$

b) $f_2(x) = \ln(x) \qquad f_2'(x) = \frac{1}{x}$

$$(f_2(x))' = \frac{1}{\exp'(\ln(x))} = \frac{1}{\exp(\ln(x))} = \frac{1}{x} \quad \square$$

c) $f_3(x) = x^\alpha \qquad f_3'(x) = \alpha\cdot x^{\alpha-1}$

$$(f_3(x))' = (\exp(\alpha\cdot\ln(x)))' = \alpha\cdot\frac{1}{x}\cdot\exp(\alpha\cdot\ln(x))$$

$$= \alpha\cdot\exp(\ln(\tfrac{1}{x}))\cdot\exp(\alpha\cdot\ln(x))$$

$$= \alpha\cdot\exp(-\ln(x))\cdot\exp(\alpha\cdot\ln(x))$$

$$= \alpha\cdot\exp(\alpha\cdot\ln(x)-\ln(x)) = \alpha\cdot\exp(\ln(x)\cdot(\alpha-1))$$

$$= \alpha x^{\alpha-1} \quad \square$$

(3) z.z. $\int_a^b f(x)\,dx = \int_0^b f(x)\,dx - \int_0^a f(x)\,dx$

$$\int_0^b f(x)\,dx - \int_0^a f(x)\,dx = \sum_{k=0}^{b} \Delta x_k \cdot f(z_k) - \sum_{k=0}^{a} \Delta x_k \cdot f(z_k)$$

da $b \geq a$, eleminieren sich alle Elemente von 0 bis a

$$= \sum_{k=a}^{b} \Delta x_k\, f(z_k) = \int_a^b f(x)\,dx \quad \square$$

(4) $$(f(x_0)\cdot g(x_0))' = \lim_{x \to x_0} \left(\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \right)$$

$$= \lim_{x \to x_0} \left(\frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \right)$$

$$= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right)$$

$$= f(x_0)' g(x_0) + f(x_0) g'(x_0)$$

$$(f(x)\cdot g(x))' = f'(x)\,g(x) + f(x)\,g'(x)$$

$$\Leftrightarrow \int (f(x)\cdot g(x))'\,dx = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx$$

$$\Leftrightarrow [f(x)\cdot g(x)] = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx$$

$$\Leftrightarrow \int f'(x)\,g(x)\,dx = [f(x)\,g(x)] - \int f(x)\,g'(x)\,dx$$

5 a) $g = 1 + \frac{1}{g} \quad | -1 - \frac{1}{g}$

$\Leftrightarrow 0 = g - 1 - \frac{1}{g} \quad | \cdot g$

$\Leftrightarrow 0 = g^2 - g - 1$

$\Rightarrow g_{1,2} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$\Rightarrow g = \frac{1+\sqrt{5}}{2}$

b) $x_{n+1} = 1 + \frac{1}{x_n} \quad Z: |x_n - g| \leq \frac{1}{g^{n+1}}$

IA: $n=0 \quad x_0 = 1$

$|1 - (1 + \frac{1}{g})| \leq \frac{1}{g}$

$\Leftrightarrow |1 - 1 - \frac{1}{g}| \leq \frac{1}{g}$

$|-\frac{1}{g}| \leq \frac{1}{g}$

$\frac{1}{g} \leq \frac{1}{g}$

IV: $\exists n \in \mathbb{N}_0 : |x_n - g| \leq \frac{1}{g^{n+1}}$

IS: $n \mapsto n+1$

$|x_{n+1} - g| \leq \frac{1}{g^{n+2}}$

$|1 + \frac{1}{x_n} - 1 - \frac{1}{g}| = |\frac{1}{x_n} - \frac{1}{g}| = |\frac{g}{x_n g} - \frac{x_n}{x_n g}|$

$= \frac{g - x_n}{x_n g} = \frac{1}{|x_n|} \cdot \frac{1}{|g|} \cdot |g - x_n| \leq \frac{1}{|x_n|} \frac{1}{|g|} \frac{1}{g^{n+1}}$ | mit $|x_n| \geq 1$

$\leq \frac{1}{g^{n+2}} \quad \square$

c) $\varepsilon = \frac{1}{g^{n+1}} \; \forall n > N \; |x_n - g| \leq \varepsilon \quad \square$

6

a) $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

b) → siehe GitHub

7

$\forall y_0 \in [a,b] \; \exists x_0 : f(x_0) = y_0$ Zwischenwertsatz

- Man kann annehmen $y_0 = 0$, ansonsten $f(x) - y_0$
- Setze $[a_0, b_0] = [a, b]$ und konstruiere Intervallschachtelung $[a_n, b_n]$, sodass $f(a_n)$ und $f(b_n)$ unterschiedliche Vorzeichen haben, also $f(a_n) f(b_n) < 0$
- Nun hat $f\left(\frac{a_n + b_n}{2}\right)$ höchstens mit einen der beiden Zahlen $f(a_n)$ oder $f(b_n)$ das gleiche Vorzeichen, deshalb Folgeintervall entweder $\left[a_n, \frac{a_n+b_n}{2}\right]$ oder $\left[\frac{a_n+b_n}{2}, b_n\right]$.
- Der durch Intervallschachtelung bestimmte Punkt $x \in [a,b]$ ist ein Nullpunkt somit gilt $f(x)^2 = \lim_{n \to \infty} f(a_n) f(b_n) \leq 0$

$$0 \leq f(x)^2 = \lim_{n \to \infty} f(a_n) f(b_n) \leq 0$$

$\Rightarrow f(x) = 0$

$\Rightarrow \forall y \in [a,b] \; \exists x : f(x) = y$