41 4P

IA $n=1$

$$\sum_{k=1}^{1} \frac{1}{k^2+k} = \frac{1}{1^2+1} = \frac{1}{1+1} = \frac{1}{2} = \frac{n}{n+1}$$

IV $\sum_{k=1}^{n} \frac{1}{k^2+k} = \frac{n}{n+1}$ gilt für ein $n \in \mathbb{N}$

IS: $n \mapsto n+1$

Z: $$\sum_{k=1}^{n+1} \frac{1}{k^2+k} = \frac{n+1}{n+1+1} = \frac{n+1}{n+2}$$

$$\sum_{n=1}^{n+1} \frac{1}{k^2+k} = \sum_{k=1}^{n} \frac{1}{k^2+k} + \frac{1}{(n+1)^2+n+1}$$

$$\overset{IV}{=} \frac{n}{n+1} + \frac{1}{n^2+2n+1+n+1}$$

$$= \frac{n\cdot(n^2+2n+1+n+1) + n+1}{(n+1)(n^2+2n+1+n+1)}$$

$$= \frac{n^3+2n^2+n+n^2+n+n+1}{n^3+2n^2+n+n^2+n+n^2+2n+1+n+1}$$

$$n^3+4n^2+5n+2 : (n+2) = n^2+2n+1 = (n+1)^2$$
$$-(n^3+2n^2)$$
$$2n^2+5n$$
$$-(2n^2+4n)$$
$$n+2$$

$$= \frac{n^3+3n^2+3n+1}{n^3+4n^2+5n+2} = \frac{(n+1)^3}{(n+2)\cancel{(n+1)^2}}$$

$$= \frac{n+1}{n+2}$$ $\square$

falsch

~~A1~~

~~IA: $n=1$~~

~~$$\sum_{n=1}^{1} \frac{1}{k^2+k} = \frac{1}{1^2+1} = \frac{1}{2} = \frac{1}{1+1} = \frac{n}{n+1} \quad \square$$~~

~~IV gilt für ein $n \in \mathbb{N}$~~

~~IS $n \to n+1$~~

~~$$\sum_{n=1}^{n+1} \frac{1}{k^2+k} = \frac{n+1}{n+1+1} = \frac{n+1}{n+2}$$~~

~~$$\sum_{n=1}^{n+1} \frac{1}{k^2+k} = \sum_{n=1}^{n} \frac{1}{k^2+k} + \frac{n+1}{(n+1)^2+n+1}$$~~

~~$$\overset{IV}{=} \frac{n}{n+1} + \frac{n+1}{(n+1)^2+(n+1)}$$~~

~~$$= \frac{n+1}{n^2+n2} \quad \square \quad (n+1)((n+1)^2+n+1)$$~~

~~$n+1 \quad (n^2+2n+1) \quad +n^2+n+n+1)$~~

~~$n^3+2n^2+n+n^2+2n+1+n^2+n+n+1$~~

~~n^3+4n^2+5n+n~~

~~$\frac{1}{2}+\frac{1}{6}=$~~

~~$\frac{3}{6}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$~~

~~$\frac{1}{2}+\frac{1}{3}$~~

~~$+\frac{2}{3}+\frac{1}{12}$~~

~~$\frac{6}{12}+\frac{8}{12}+\frac{1}{12}=\frac{15}{12}$~~

~~$\frac{1}{3^2+3}=\frac{1}{9+3}=\frac{1}{12}$~~

~~$\frac{1}{2}+\frac{2}{3}+\frac{1}{12}$~~

~~$\frac{6}{12}+\frac{8}{12}+\frac{1}{12}=\frac{15}{12}$~~ X

A2

(a) $z^2 - 10z + 26 = 0$

$$z_{1,2} = \frac{10}{2} \pm \sqrt{\left(\frac{10}{2}\right)^2 - 26}$$

$$z_{1,2} = 5 \pm \sqrt{25-26}$$

$$z_{1,2} = 5 \pm \sqrt{-1}$$

$$z_1 = 5 + i$$

$$z_2 = 5 - i$$

2P | 1

(b) $z^6 = 64$ eine Lösung ist $2^6 = 64$

also

2P

$z_1 = 2 \cdot e^{i \cdot 0}$ da $\frac{2\pi}{6} = \frac{1}{3}\pi$ rotiere um $\frac{1}{3}\pi$, um alle Lösungen zu erhalten (Potenzierung v. komplexen Zahlen)

$z_2 = 2 \cdot e^{i \cdot \frac{1}{3}\pi}$

$z_3 = 2 \cdot e^{i \cdot \frac{2}{3}\pi}$

$z_4 = 2 \cdot e^{i \cdot \pi}$ $\quad z_5 = 2 \cdot e^{i \cdot \frac{4}{3}\pi}$ $\quad z_6 = 2 \cdot e^{i \cdot \frac{5}{3}\pi}$

A3

(a) $\lim\limits_{x\to 0} \dfrac{1-x-e^{-x}}{1-\cos^2(x)}$ $\frac{0}{0}$ stetige Fkt l'Hospital $= \lim\limits_{x\to 0} \dfrac{-1+e^{-x}}{2\cos(x)\cdot -\sin(x)}$ $0/0$ $= \lim\limits_{x\to 0} \dfrac{-e^{-x}}{-(\sin^2(x)+\cos^2(x))}$ l'Hospital

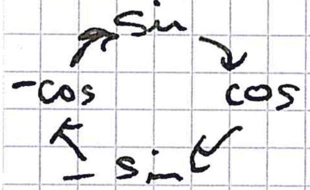
$(1-x-e^{-x})' = -1+e^{-x}$

$(1-\cos^2(x))'$

$(\cos(x)\cdot\cos(x))' = -\sin\cdot\cos(x)$

$= \lim\limits_{x\to 0} \dfrac{-e^{-x}}{-1} = \lim\limits_{x\to 0} \dfrac{e^{-x}}{1}$

$\Rightarrow = 1$ f.



$(x^2)' = 2x$

$(\cos)' = -\sin(x)$

$\cos^2(x) = 2\cdot\cos(x)\cdot -\sin(x)$

$(\cos(x)\cdot -\sin(x))' = -\sin^2(x) - \cos^2(x)$

(b) $\lim\limits_{x\to\infty} \dfrac{\cos^2(x)}{\ln(x)}$ $= 0.$ 2P

$\cos(x): \mathbb{R} \to [-1,1]$

$\cancel{-1}\,0 \le \cos^2 \le 1 \quad \forall x\in\mathbb{R}$

$\ln(x) \xrightarrow{x\to\infty} \infty$

(c) $\lim\limits_{n\to\infty} \dfrac{n^4+5n^2+n\cdot e^{-n}}{(2n^2-1)^2}$ $\to 0$ für $n\to\infty$ $= \lim\limits_{n\to\infty} \dfrac{n^4+5n^2}{(2n^2-1)^2}$

$= \lim\limits_{n\to\infty} \dfrac{n^4+5n^2}{4n^4-4n^2+1}$

$= \lim\limits_{n\to\infty} \dfrac{n^4(1+5\cdot n^{-2})}{n^4(4\underbrace{-4n^{-2}+n^{-4}}_{\to 0})}$

$= \dfrac{1}{4}$! 1P

A4 $T_3(x,1)$ $a = 1$

$f(x) = \ln^2(x) = \ln(x) \cdot \ln(x)$ $f'(1) = 2 \cdot e$

$f'(x) = \frac{1}{x} \cdot \ln(x) + \frac{1}{x} \cdot \ln(x) = \frac{2}{x} \cdot \ln(x)$

$= 2 \cdot x^{-1} \cdot \ln(x)$

$f''(x) = 2 \cdot -1 \cdot x^{-2} \cdot \ln(x) + 2 \cdot x^{-1} \cdot \frac{1}{x}$

$= -2x^{-2} \ln(x) + 2$ $(2)' = 0$

$f'''(x) = (-2) \cdot (-2) \cdot x^{-3} \cdot \ln(x) + (-2) \cdot x^{-2} \cdot \frac{1}{x}$

$= -4 x^{-3} \cdot \ln(x) - 2x^{-1}$

$\cancel{T_3(x,1) = \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!} \cdot (x-1)^n}$

$e^x = 1$

$x \cdot \ln(e) = 1$

$e^x = e$

$x \cdot \ln(e) = \ln(e)$

$e^1 = e$

$e^0 = 1$

$\cancel{T_3(x,1) = 1 \cdot (1-1) +}$

$= 1 +$

$f(1) = \ln^2(x) = \ln^2(1) = 0$ ✓

$f'(1) = \frac{1}{1} \cdot \ln(1) = 0$ ✓

$f''(1) = -2 \cdot \frac{1}{1^2} \cdot \ln(1) + 2 = 2$ ✓

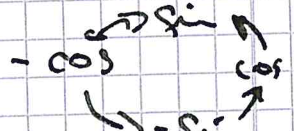
$f'''(1) = -4 \cdot 1 \cdot 0 - 2 \cdot \frac{1}{1} = -2$ X

$T_3(x) = 2 \cdot x^2 + -2x^3$ X

A5 $\int f'g = -\int g'f + [fg]$

(a) $\int x^2 \cdot \sin(x)\,dx = -\int 2 \cdot x \cdot -\cos(x) + [-\cos(x) \cdot x^2]$

$g = x^2$ $g'(x) = 2x$ $= \int 2x \cdot \cos(x) + [-\cos(x) \cdot x^2]$



$f' = \sin(x)$ $= -\int 2 \cdot \sin(x) + [-\cos(x) \cdot x^2]$

$+ [2x \cdot -\sin(x)]$

$f = -\cos(x)$

$g = 2x$ $g' = 2$ $= -2 \cdot \int \sin(x) - [\cos(x) \cdot x^2]$

$f' = -\cos$ $f = -\sin$ $- [2 \cdot x \cdot \sin(x)]$

2

$= [-2 \cdot -\cos(x) - \cos(x) \cdot x^2$

$-2x\sin(x)]$

$= [2\cos(x) - \cos(x)x^2 - 2x\sin(x)]$

(b) $\int \frac{1}{1+\sqrt[3]{x}}\,dx$

$u = x^{1/3} = \sqrt[3]{x}$

$u' = \frac{1}{3}\cdot x^{-\frac{2}{3}}$

$\frac{du}{dx} = u'$

$dx = \frac{\cancel{x'}\,du}{u'}$

$\int \frac{1}{1+u}\cdot\frac{1}{\frac{1}{3}\cdot x^{-\frac{2}{3}}}$

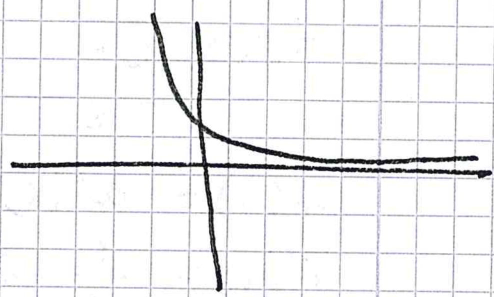
A6

a) $\int_0^\infty \frac{3x+5}{x^2+4x+3}\,dx = \lim_{\beta\to\infty}\int_0^\beta \frac{3x+5}{x^2+4x+3}$

b) $\int_1^\infty \frac{1}{x\cdot e^x}$ existiert

$\int_1^\infty \frac{1}{x\cdot e^x} \leq \int_1^\infty c\cdot x^{-1}$

$\leq \int_1^\infty \frac{c}{x}$



$\int_1^\infty \frac{1}{x}\cdot e^{-x} \leq c\cdot\int_1^\infty \frac{1}{x}$

$\int_1^\infty e^{-x} \leq c$ $\square$

1P

Beweise

A1

$f: (a,b) \to \mathbb{R}$ streng monoton + stetig

$$f'(x_0) = \frac{1}{f^{\prime -1}(f(x_0))}$$

sei $f, g: (a,b) \to \mathbb{R}$ stetig, dann

$$(f \circ g)' = \cancel{1}\, g'(f(x)) \cdot f'(x)$$

$$(f \circ g)' = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

$$= g'(f(x)) \cdot f'(x) \quad \square$$

$f^{-1}(f(x)) = x$, dann gilt

$(f^{-1}(f(x)))' = x' = 1$

und somit folgt:

$$f^{-1\prime}(f(x)) \cdot f'(x) = 1$$

$$f'(x) = \frac{1}{f^{\prime -1}(f(x))} \quad \square$$

A2

a) $f_1(t) = a^x$ $\qquad (e^{x \cdot \ln(a)})' = \ln(a) \cdot e^{x \cdot \ln(a)}$

$a^x = \exp(\ln(a^x)) = \exp(x \cdot \ln(a))$

$(a^x)' = (\exp(x \cdot \ln(a)))' = \cancel{x} \cdot \exp(\ln(a) \cdot x) \cdot \ln(a)$

$= \ln(a) \cdot a^x$

b) $f_2(x) = \ln(x)$

$f_2'(x)$ über Umkehrfkt. e^x

$$f_2'(x) = \frac{1}{f_2^{-1'}(f_2(x))} = \frac{1}{\exp'(\ln(x))} = \frac{1}{\exp(\ln(x))}$$

$$= \frac{1}{x}$$

c) $f_3(x) = x^\alpha = \exp(\ln(x^\alpha))$

$$= \exp(\alpha \cdot \ln(x))$$

$$f_3(x)' = (\exp(\alpha \cdot \ln(x)))'$$

$$= \exp(\alpha \cdot \ln(x)) \cdot \frac{1}{x}$$

$$= \alpha \cdot \exp(\alpha \cdot \ln(x)) \cdot \exp(\ln(\tfrac{1}{x}))$$

$$= \exp(\alpha \cdot \ln(x)) - \ln(\tfrac{1}{x})) \cdot \alpha$$

$$= \exp(\ln(x)(\alpha - 1)) \cdot \alpha$$

$$= \alpha \cdot x^{\alpha - 1}$$

<u>A4</u> $f'g + g'f = (f \cdot g)'$

$$(f \cdot g)' = \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x)}{x - x_0} + \frac{f(x_0)g(x) - f(x_0)g(}{x - x_0}$$

$$= \lim_{x \to x_0} g(x) \cdot \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} f(x_0) \cdot \frac{g(x) - g(x_0)}{x - x_0}$$

$$= f'(x)g(x_0) + g'(x) \cdot f(x)$$

$$\int f' g = -\int f g' + [fg]$$

$$\cancel{\int (f \cdot g)'}$$

$$\int f \cdot g = \int (f \cdot g)' = \int f'g + g'f \qquad F' = f, \; G' = G$$

$$\int f'g = -\int g'f + [f \cdot g]$$

$$\int f'g = -\int g'f + [f \cdot g] \qquad \square$$

A5

a) $g = 1 + \frac{1}{g} \qquad g > 0$

$\Leftrightarrow g^2 = g + 1$

$g^2 - g - 1 = 0$

$$g_{1,2} = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 1}$$

$$g_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1}$$

$$g_{1,2} = \frac{1}{2} \pm \sqrt{\frac{5}{4}} = \frac{1 \pm \sqrt{5}}{2}$$

da $g > 0$: $g = \frac{1 + \sqrt{5}}{2}$ $(\approx 1{,}618)$

b) $x_{n+1} = 1 + \frac{1}{x_n} \qquad x_0 = 1$

$$|x_n - g| \leq \frac{1}{g^{n+1}}$$

Induktion:

IA $\quad n = 0$

$$|x_0 - g| \leq \frac{1}{g^{n+1}}$$

3/

$$|1 - (1 + \frac{1}{g})| \leq \frac{1}{g^{n+1}}$$

$$\left|1-1-\frac{1}{g}\right| \leq \frac{1}{g^{\cancel{n+1}}1}$$

$$\frac{1}{g} \leq \frac{1}{g}$$

<u>IV</u>: gilt für ein $n \in \mathbb{N}$

$$\left|x_n - g\cancel{\tfrac{1}{g}}\right| \leq \frac{1}{g^{n+1}}$$

<u>IS</u>: $n \mapsto n+1$

$$\left|x_{n+1} - g\cancel{\tfrac{1}{g}}\right| \leq \frac{1}{g^{n+2}}$$

$$\left|1+\frac{1}{x_n}-\left(1+\frac{1}{g}\right)\right| \leq \frac{1}{g^2 \cdot g^n}$$

$$\left|1+\frac{1}{x_n}-1-\frac{1}{g}\right| \leq \frac{1}{g^2 \cdot g^n}$$

$$\left|\frac{1}{x_n}-\frac{1}{g}\right| \leq \frac{1}{g^2 g^n}$$

$$\left|\frac{g-x_n}{x_n \cdot g}\right| \leq \frac{1}{g^2 g^n}$$

$$\frac{1}{|x_n| \cdot g} \cdot |g-x_n| \leq \frac{1}{g^2 g^n}$$

<u>Nach IV</u>

$$\leq \frac{1}{|x_n| g} \cdot \frac{1}{g^{n+1}} = \frac{1}{|x_n| \cdot g^{n+2}} \overset{x_n > 0}{\leq} \frac{1}{g^{n+2}} \quad \square$$

c) Mit $\varepsilon = \frac{1}{g^{n+1}}$ und $n > N$ folgt

$$|x_n - g| \leq \frac{1}{g^{n+1}} = \varepsilon \quad \square$$

A3

$f: [0,b] \to \mathbb{R}$ stetig $a \in [0,b]$

$$\int_a^b f(x) = \int_0^b f(x)\,dx - \int_0^a f(x)\,dx$$

$$\int_0^b f(x)\,dx - \int_0^a f(x)\,dx = \sum_{k=0}^{b} \Delta x_k \cdot f(z_k) - \sum_{k=0}^{a} \Delta x_k \cdot f(z_k)$$

da a in $[0,b]$ werden alle Elemente vor a durch $\sum_{k=0}^{a} \Delta x_k \cdot f(z_k)$ eliminiert, hieraus folgt:

$$\sum_{k=0}^{b} \Delta x_k \cdot f(z_k) - \sum_{k=0}^{a} \Delta x_k \cdot f(z_k) = \sum_{k=a}^{b} \Delta x_k \cdot f(z_k)$$

$$= \int_a^b f(x) \quad \square$$

5

A6

a)

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

$\forall \delta > 0 \quad \forall \varepsilon > 0$

b) $y_n \to x_0$

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \text{ geben}$$

mit jeder Folge $y_n \to x_0$:

$$|x - \cancel{x_0}\, y_n| < \delta \Rightarrow |f(x) - f(y_n)| < \varepsilon,$$

sodass $\forall n > N$ gilt, dass

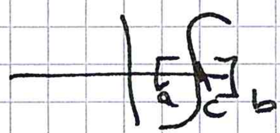
$$|f(x) - f(y_n)| < \varepsilon \text{ und } |x - y_n| < \delta$$

und

$|y_n - x_0| < \varepsilon$, weshalb Folgenstetigkeit gegeben ist.

A7

Annahme $f(a)\, f(b) \le 0$



~~Für~~ Mit Intervallschachtelung

$I_N = [a, b]$ und $c_n = \frac{a_n + b_n}{2}$

$$I_{n+1} = \begin{cases} [a_n, c_n], & a_n c_n \le 0 \\ [c_n, b_n], & c_n b_n \le 0 \end{cases}$$

Sofern $n = 0$ gilt, ist klar, dass es ein x geben muss mit $f(x) = 0$, da $f(a)\, f(b) \le 0$.

Weiterhin muss es auch für alle n gelten, da I_{n+1} immer ~~paarweise~~ Vorzeichenverschiedene Paare als Grenzen eine hat, sodass bei stetiger Ausfüllung das Intervall gegen den x-Wert konvergiert, der $f(x) = 0$ entspricht.

Hieraus folgt: $\forall y \in [a,b]\ \exists x_0 : f(x_0) = y$ $\square$

6/6