Lemma PSD_Matrix_Inverse

Suppose A is a PSD matrix and D is a diagonal matrix with positive entries. Then for any vector x, we have

$$\|D^{-1}x\| \geq \|(A+D)^{-1}x\|$$

Proof

Notation: For matrix B, define $B^2 = BB$.

It suffices to show that for all x,

$$x^T \left(D^{-2} - (A+D)^{-2}\right) x \geq 0$$

That is, we are interested in showing that $D^{-2} - (A+D)^{-2}$ is PSD. This can be shown by noting that

$$(A+D)^2 \succeq D^2 \implies D^{-2} \succeq (A+D)^{-2}$$

Lemma: Lipschitz Definition Equivalence

The following two conditions are equivalent:

1. For all $u > 0$ and any $\lambda^{(1)}, \lambda^{(2)}$ that satisfy

$$\|\lambda^{(1)} - \lambda^{(2)}\| \leq Cu$$

then

$$\|\hat{g}(\cdot|\lambda^{(1)}) - \hat{g}(\cdot|\lambda^{(2)})\|_D \leq u$$

1. $\hat{g}(\cdot|\lambda)$ is $1/C$-Lipschitz in λ:

$$\|\hat{g}(\cdot|\lambda^{(1)}) - \hat{g}(\cdot|\lambda^{(2)})\|_D \leq \frac{1}{C}\|\lambda^{(1)} - \lambda^{(2)}\|$$

Proof

It is clear that Condition 2 implies Condition 1.

To show Condition 1 implies Condition 2, suppose for any $\lambda^{(1)}, \lambda^{(2)}$, we have

$$\|\lambda^{(1)} - \lambda^{(2)}\| = d = C\frac{d}{C}$$

Then

$$\|\hat{g}(\cdot|\lambda^{(1)}) - \hat{g}(\cdot|\lambda^{(2)})\|_D \leq \frac{d}{C} = \frac{1}{C}\|\lambda^{(1)} - \lambda^{(2)}\|$$

Lemma: Application of Bernstein's inequality

For n independent sub-gaussian RVs ϵ with constants σ, K, the norm is bounded as follows

$$Pr\left(\|\epsilon\|_n \geq 2\sigma\right) \leq \exp\left(-n\frac{\sigma^2}{K}\right)$$