Lemma: Additive Models and Additive Penalties

Consider the problem

$$\frac{1}{2}\|y - \sum_{j=1}^{J} g_j\|_T^2 + \sum_{j=1}^{J} \lambda_j \left(P_j(g_j) + \frac{w}{2}\|g_j\|_D^2 \right)$$

We suppose the penalty functions P_j are convex and twice-differentiable. (We do not need the semi-norm assumption.)

Suppose that $\sup_{g \in \mathcal{G}} \|g\|_D \leq G$.

For all $d > 0$, any $\lambda^{(1)}, \lambda^{(2)}$ that satisfy

$$\|\lambda^{(1)} - \lambda^{(2)}\| \leq \frac{dw}{2J} \left(\sqrt{\frac{n}{n_T}} n^{\tau_{min}} \left(2G + \|\epsilon\|_T\right) + 2wG \right)^{-1} n^{-\tau_{\min}}$$

we have

$$\|\hat{g}_j(\cdot|\lambda^{(1)}) - \hat{g}_j(\cdot|\lambda^{(2)})\|_D \leq d/J$$

Hence

$$\|\sum_{j=1}^{J} \hat{g}_j(\cdot|\lambda^{(1)}) - \hat{g}_j(\cdot|\lambda^{(2)})\|_D \leq d$$

Proof

Let $h_j = \hat{g}_j(\cdot|\lambda^{(1)}) - \hat{g}_j(\cdot|\lambda^{(2)})$. Suppose for contradiction that for $\tilde{k}$, we have $\|h_{\tilde{k}}\|_D > d/J$.

Let

$$Z = \{j : \|h_j\| > 0\}$$

Consider the optimization problem

$$\{\hat{m}_j(\lambda)\}_{j \in Z} = \arg\min_{m} \frac{1}{2}\|y - \sum_{j=1}^{J} \left(g_j + m_j h_j\right)\|_T^2 + \sum_{j=1}^{J} \lambda_j \left(P_j(g_j + m_j h_j) + \frac{w}{2}\|g_j + m_j h_j\|_D^2 \right)$$

Note that if $\|h_j\| = 0$, then we just set $m_j = 0$ as a constant.

Now by the KKT conditions, for all $\ell \in Z$, we have

$$\langle y - \sum_{j=1}^{J} \left(g_j + m_j h_j\right), h_\ell \rangle_T + \lambda_\ell \frac{\partial}{\partial m_\ell} P_\ell(g_\ell + m_\ell h_\ell) + \lambda_\ell w \langle h_\ell, g_\ell + m_\ell h_\ell \rangle_D = 0$$

It's implicit derivative with respect to λ_k is

$$\begin{aligned} \langle \sum_{j=1}^{J} \frac{\partial \hat{m}_j(\lambda)}{\partial \lambda_k} h_j, h_\ell \rangle_T + \lambda_\ell \frac{\partial^2}{\partial m_\ell^2} P_\ell(g_\ell + m_\ell h_\ell) \frac{\partial \hat{m}_\ell(\lambda)}{\partial \lambda_k} + \lambda_\ell w \|h_\ell\|_D^2 \frac{\partial \hat{m}_\ell(\lambda)}{\partial \lambda_k} & \\ + 1\left[\ell = k\right] \left(\frac{\partial}{\partial m_\ell} P_\ell(g_\ell + m_\ell h_\ell) + w \langle h_\ell, g_\ell + m_\ell h_\ell \rangle_D \right) &= 0 \end{aligned}$$

Define the following matrices

$$S : S_{ij} = \langle h_j, h_\ell \rangle_T$$

$$D_1 = diag\left(\lambda_\ell \frac{\partial^2}{\partial m_\ell^2} P_\ell(g_\ell + m_\ell h_\ell) \right)$$

$$D_2 = diag\left(\lambda_\ell w \|h_\ell\|_D^2 \right)$$

$$D_3 = diag\left(\frac{\partial}{\partial m_\ell} P_\ell(g_\ell + m_\ell h_\ell) + w\langle h_\ell, g_\ell + m_\ell h_\ell\rangle_D\right)$$

$$M = \begin{pmatrix} \frac{\partial \hat{m}_1(\lambda)}{\partial \lambda} & \frac{\partial \hat{m}_2(\lambda)}{\partial \lambda} & ... & \frac{\partial \hat{m}_J(\lambda)}{\partial \lambda} \end{pmatrix}$$

(You will have to omit certain columns/rows of the matrices if $m_j = 0$ is constant.)

From the implicit differentiation equations, we have the following system of equations:

$$M = D_3 \left(S + D_1 + D_2\right)^{-1}$$

We know that S is a PSD matrix (since it can be written as $S = HH^T$ where $H_j = h_j$ evaluated at covariates T).

We are interested in bounding the gradient of $\hat{m}_{\tilde{k}}(\lambda)$ wrt λ, which is the $\tilde{k}$-th column of M has norm. By Lemma PSD_Matrix_Inverse, we know that

$$\begin{aligned}
\|\nabla_\lambda \hat{m}_{\tilde{k}}(\lambda)\| &= \|Me_{\tilde{k}}\| \\
&= \|D_3 \left(S + D_1 + D_2\right)^{-1} e_{\tilde{k}}\| \\
&\leq \|D_3 \left(D_1 + D_2\right)^{-1} e_{\tilde{k}}\| \\
&\leq \left|\frac{\partial}{\partial m_{\tilde{k}}} P_{\tilde{k}}(g_{\tilde{k}} + m_{\tilde{k}} h_{\tilde{k}}) + w\langle h_{\tilde{k}}, g_{\tilde{k}} + m_{\tilde{k}} h_{\tilde{k}}\rangle_D\right| \lambda_{\tilde{k}}^{-1} w^{-1} \|h_{\tilde{k}}\|_D^{-2}
\end{aligned}$$

where the last inequality is derived by plugging in the $\tilde{k}$th entry in the diagonal matrices.

Note that from the KKT conditions, we have that

$$\begin{aligned}
\left|\frac{\partial}{\partial m_{\tilde{k}}} P_{\tilde{k}}(g_{\tilde{k}} + m_{\tilde{k}} h_{\tilde{k}})\right| &= \left|\frac{1}{\lambda_{\tilde{k}}}\langle y - \sum_{j=1}^{J}\left(g_j + m_j h_j\right), h_{\tilde{k}}\rangle_T + w\langle h_{\tilde{k}}, g_{\tilde{k}} + m_{\tilde{k}} h_{\tilde{k}}\rangle_D\right| \\
&\leq n^{\tau_{min}} \|y - \sum_{j=1}^{J}\left(g_j + m_j h_j\right)\|_T \|h_{\tilde{k}}\|_T + w\|h_{\tilde{k}}\|_D \|g_{\tilde{k}} + m_{\tilde{k}} h_{\tilde{k}}\|_D \\
&\leq \left(\sqrt{\frac{n}{n_T}} n^{\tau_{min}}\left(2G + \|\epsilon\|_T\right) + wG\right)\|h_{\tilde{k}}\|_D
\end{aligned}$$

Also

$$w\langle h_{\tilde{k}}, g_{\tilde{k}} + m_{\tilde{k}} h_{\tilde{k}}\rangle_D \leq w\|h_{\tilde{k}}\|_D G$$

Hence

$$\|\nabla_\lambda \hat{m}_{\tilde{k}}(\lambda)\| \leq \left(\sqrt{\frac{n}{n_T}} n^{\tau_{min}}\left(2G + \|\epsilon\|_T\right) + 2wG\right) n^{\tau_{\min}} w^{-1} \|h_{\tilde{k}}\|_D^{-1}$$

By the MVT, there is some $\alpha \in [0, 1]$ such that

$$\begin{aligned}
\left|\hat{m}_{\tilde{k}}(\lambda^{(2)}) - \hat{m}_{\tilde{k}}(\lambda^{(1)})\right| &= \left|\left\langle \lambda^{(2)} - \lambda^{(1)}, \nabla_\lambda \hat{m}_{\tilde{k}}(\lambda)\right\rangle_{\lambda = \alpha\lambda^{(1)} + (1-\alpha)\lambda^{(2)}}\right| \\
&\leq \|\lambda^{(2)} - \lambda^{(1)}\| \left(\sqrt{\frac{n}{n_T}} n^{\tau_{min}}\left(2G + \|\epsilon\|_T\right) + 2wG\right) n^{\tau_{\min}} \frac{J}{dw} \\
&= 1/2
\end{aligned}$$

But this is a contradiction since we know that $\hat{m}_{\tilde{k}}(\lambda^{(2)}) = 1$ and $\hat{m}_{\tilde{k}}(\lambda^{(1)}) = 0$.

Lemma: Additive Models and Additive Penalties, Nonsmooth

Same assumptions as above, but we allow the penalties to be nonsmooth.

Suppose for almost every λ, the differentiable space $\Omega^{L_T(\cdot,\lambda)}(\hat{g}(\cdot|\lambda))$ is a local optimality space.

Suppose for almost every λ, the penalty function is twice differentaible in the differentiable space.

The conclusions are the same as before.

For all $d > 0$, any $\lambda^{(1)}, \lambda^{(2)}$ that satisfy

$$\|\lambda^{(1)} - \lambda^{(2)}\| \le \frac{dw}{2J}\left(\frac{n}{n_T} n^{\tau_{min}}\left(2G + \|\epsilon\|_T\right) + wG + G\right)^{-1} n^{-\tau_{\min}}$$

we have

$$\|\hat{g}_j(\cdot|\lambda^{(1)}) - \hat{g}_j(\cdot|\lambda^{(2)})\|_D \le d/J$$

Hence

$$\|\sum_{j=1}^{J} \hat{g}_j(\cdot|\lambda^{(1)}) - \hat{g}_j(\cdot|\lambda^{(2)})\|_D \le d$$

Proof

Let $\lambda^{(1)}, \lambda^{(2)}$ be the penalty parameters satisfying the distance constraint above. Let C be the constant defined in the assumption

$$\|\lambda^{(1)} - \lambda^{(2)}\| \le dC$$

Under the assumptions about the differentiable space and the local optimality space, we know that for almost every pair $\lambda^{(1)}, \lambda^{(2)}$, there is a line

$$\mathcal{L} = \left\{\alpha\lambda^{(1)} + (1-\alpha)\lambda^{(2)} : \alpha \in [0,1]\right\}$$

containing a finite set of points $\{\ell_i\}_{i=0}^{N+1} \subset \mathcal{L}$ where $\ell_0 = \lambda^{(1)}$ and $\ell_{N+1} = \lambda^{(2)}$ such that:

1. The differentiable spaces $\Omega^{L_T(\cdot,\ell_i)}(\hat{g}(\cdot|\ell_i))$ satisfy the condition that the differentiable space is a local optimality differentiable space conditions and
2. The union of the differentiable spaces contains the entire line $\mathcal{L}$:

$$\mathcal{L} \subset \cup_{i=0}^{N+1} \Omega^{L_T(\cdot,\ell_i)}(\hat{g}(\cdot|\ell_i))$$

Now we partition $\mathcal{L}$ according to the differentiable spaces. We will partition with the centers of each differentiable space and points in the intersection of all the differentiable spaces. Let $\{\ell_{(i)}\}_{i=0}^{N} \subset \mathcal{L}$ be the points such that $\ell_{(i)}$ is in the differentiable space $\Omega^{L_T(\cdot,\ell_i)}(\hat{g}(\cdot|\ell_i))$ and $\Omega^{L_T(\cdot,\ell_{i+1})}(\hat{g}(\cdot|\ell_{i+1}))$. That is, we choose

$$\ell_{(i)} \in \Omega^{L_T(\cdot,\ell_i)}(\hat{g}(\cdot|\ell_i)) \cap \Omega^{L_T(\cdot,\ell_{i+1})}(\hat{g}(\cdot|\ell_{i+1}))$$

Hence the following points form a partition of $\mathcal{L}$

$$\left(\ell_0, \ell_{(0)}\right), \left(\ell_{(0)}, \ell_1\right), ..., \left(\ell_N, \ell_{(N)}\right), \left(\ell_{(N)}, \ell_{N+1}\right)$$

Note that

$$\|\ell_i - \ell_{(i)}\| \le \frac{\|\ell_i - \ell_{(i)}\|}{\|\lambda^{(1)} - \lambda^{(2)}\|} dC$$

Applying the smooth lemma to the pairs of points above, we have that

$$\|g(\cdot|\ell_i) - g(\cdot|\ell_{(i)})\|_D \le \frac{\|\ell_i - \ell_{(i)}\|}{\|\lambda^{(1)} - \lambda^{(2)}\|} d$$

Similarly,

$$\|g(\cdot|\ell_{i+1}) - g(\cdot|\ell_{(i)})\|_D \leq \frac{\|\ell_{i+1} - \ell_{(i)}\|}{\|\lambda^{(1)} - \lambda^{(2)}\|} d$$

Hence

$$\begin{aligned}
\|g(\cdot|\lambda^{(1)}) - g(\cdot|\lambda^{(2)})\|_D &\leq \sum_{i=0}^{N} \|g(\cdot|\ell_i) - g(\cdot|\ell_{(i)})\|_D + \|g(\cdot|\ell_{i+1}) - g(\cdot|\ell_{(i)})\|_D \\
&\leq d \left(\sum_{i=0}^{N} \frac{\|\ell_{i+1} - \ell_{(i)}\|}{\|\lambda^{(1)} - \lambda^{(2)}\|} + \frac{\|\ell_i - \ell_{(i)}\|}{\|\lambda^{(1)} - \lambda^{(2)}\|} \right) \\
&= d
\end{aligned}$$

Lemma PSD_Matrix_Inverse

Suppose A is a PSD matrix and D is a diagonal matrix with positive entries. Then for any vector x, we have

$$\|D^{-1}x\| \geq \|(A+D)^{-1}x\|$$

Moreover, for any diagonal matrix D_1 with positive entries, we have

$$\|D_1 D^{-1}x\| \geq \|D_1(A+D)^{-1}x\|$$

Proof

Notation: For matrix B, define $B^2 = BB$.

It suffices to show that for all x,

$$x^T \left(D^{-2} - (A+D)^{-2}\right) x \geq 0$$

That is, we are interested in showing that $D^{-2} - (A+D)^{-2}$ is PSD. This can be shown by noting that

$$(A+D)^2 \succeq D^2 \implies D^{-2} \succeq (A+D)^{-2}$$

To show the "moreover" part, note that it suffices to show that $D_1^2\left(D^{-2} - (A+D)^{-2}\right)$ is PSD. Since $D^{-2} - (A+D)^{-2}$ is PSD, we have

$$\|D_1 D^{-1}x\|^2 \geq \|D_1(A+D)^{-1}x\|^2$$

(Note that if D_1 were a PSD matrix, this would also hold.)