Definitions

We find the best model for y over function class $\mathcal{G}$. Presume $g^* \in \mathcal{G}$ is the true model and

$$y = g^*(X) + \epsilon$$

Given a training set T , We define the fitted models

$$\hat{g}_\lambda = \|y - g\|_T^2 + \lambda^2 I^v(g)$$

Given a validation set T , let the CV-fitted model be

$$\hat{g}_{\hat{\lambda}} = \arg\min_\lambda \|y - \hat{g}_\lambda\|_V^2$$

We will suppose $I(g^*) > 0$.

Assumptions

Suppose we have sub-Gaussian errors ϵ for constants K and σ_0^2:

$$\max_{i=1:n} K^2 \left(E \left[\exp(|\epsilon_i|^2 K^2) - 1 \right] \right) \le \sigma_0^2$$

Suppose $v > 2\alpha/(2+\alpha)$.

Suppose that the entropy of the class $\mathcal{G}'$ is

$$H\left(\delta, \mathcal{G}' = \left\{ \frac{g - g^*}{I(g) + I(g^*)} : g \in \mathcal{G}, I(g) + I(g^*) > 0 \right\}, P_n \right) \quad \le \quad \tilde{A}\delta^{-\alpha}$$

Suppose for all $\lambda \in \Lambda$, $I^v(\hat{g}_\lambda)$ is upper bounded by $\|\hat{g}_\lambda\|_n^2 = \frac{1}{n}\sum_{i=1}^n \hat{g}_\lambda(x_i)$. See Lemma 1 below for the specific assumption. This assumption includes Ridge, Lasso, Generalized Lasso, and the Group Lasso.

Result 1: Single λ, Single Penalty, cross-validation over $X_T = X_V$

For now, we will suppose $P_n = \{X_i\}_{i=1}^n$ are the same between the validation and training set.

Also, suppose the penalty normalizes the empirical norm such that:

$$\sup_{g \in \mathcal{G}} \frac{\|g - g^*\|_n}{I(g) + I(g^*)} \le R < \infty$$

Suppose for all $\lambda \in \Lambda$, $I^v(\hat{g}_\lambda)$ is upper bounded by its L_2-norm with some constant M and M_0 such that

$$I^v(\hat{g}_\lambda) \le M\|\hat{g}_\lambda\|_n^2 + M_0$$

Then

$$\|\hat{g}_{\hat{\lambda}} - g^*\|_n = O_p(n^{-1/(2+\alpha)}) \left(M^{\alpha/v(2+\alpha)} \|g^*\|_n^{\alpha/2v(2+\alpha)} \vee I^{2\alpha/(2+\alpha)}(g^*) \right)$$

Proof

Let $\tilde{\lambda}$ be the optimal λ under the given assumptions, as specified by Van de geer. From the definition of $\hat{\lambda}$, we get the following basic inequality

$$\begin{aligned}
\|g^* - \hat{g}_{\hat{\lambda}}\|_V^2 &\leq \|g^* - \hat{g}_{\tilde{\lambda}}\|_V^2 + 2(\epsilon, \hat{g}_{\hat{\lambda}} - \hat{g}_{\tilde{\lambda}})_V \\
&\leq \|g^* - \hat{g}_{\tilde{\lambda}}\|_V^2 + 2(\epsilon, \hat{g}_{\hat{\lambda}} - g^*)_V + 2(\epsilon, g^* - \hat{g}_{\tilde{\lambda}})_V \\
&\leq \|g^* - \hat{g}_{\tilde{\lambda}}\|_V^2 + 2\left|(\epsilon, \hat{g}_{\hat{\lambda}} - g^*)_V\right| + 2\left|(\epsilon, g^* - \hat{g}_{\tilde{\lambda}})_V\right|
\end{aligned}$$

By considering the largest term on the RHS, we have following three cases.

Case 1: $\|g^* - \hat{g}_{\tilde{\lambda}}\|_V^2$ is the largest

Since we have assumed that the validation and training set are equal, then $\|g^* - \hat{g}_{\tilde{\lambda}}\|_V$ converges at the optimal rate $O_p(n^{-1/(2+\alpha)})$.

Case 2: $\left|(\epsilon, g^* - \hat{g}_{\tilde{\lambda}})_V\right|$ is the largest

In this case, since ϵ_V is independent of $\hat{g}_{\tilde{\lambda}}$, then by Cauchy Schwarz,

$$\begin{aligned}
\left|(\epsilon, g^* - \hat{g}_{\tilde{\lambda}})_V\right| &\leq \|\epsilon_V\| \|g^* - \hat{g}_{\tilde{\lambda}}\|_V \\
&\leq O_p\left(n^{-1/2}\right) \|g^* - \hat{g}_{\tilde{\lambda}}\|_V
\end{aligned}$$

Hence $\left|(\epsilon, g^* - \hat{g}_{\tilde{\lambda}})_V\right|$ will shrink a bit faster than the optimal rate at a rate of $O_p(n^{-(\frac{1}{2+\alpha}+\frac{1}{2})})$.

Case 3: $\left|(\epsilon, g^* - \hat{g}_{\hat{\lambda}})_V\right|$ is the largest.

By the assumptions given, Vandegeer (10.6) gives us that

$$\sup_{g \in \mathcal{G}} \frac{|(\epsilon, g - g*)_n|}{\|g - g*\|_n^{1-\alpha/2} (I(g^*) + I(g))^{\alpha/2}} = O_p(n^{-1/2})$$

Hence

$$\left|(\epsilon, g^* - \hat{g}_{\hat{\lambda}})_V\right| \leq O_p(n^{-1/2}) \|\hat{g}_{\hat{\lambda}} - g*\|_n^{1-\alpha/2} (I(g^*) + I(\hat{g}_{\hat{\lambda}}))^{\alpha/2}$$

If $I(g^*) \geq I(g_{\hat{\lambda}})$, then

$$\|g^* - \hat{g}_{\hat{\lambda}}\|_V \leq O_p(n^{-1/(2+\alpha)}) I(g^*)^{\alpha/(2+\alpha)}$$

Otherwise, we have

$$\|\hat{g}_{\hat{\lambda}} - g*\|_n^{1+\alpha/2} \leq O_p(n^{-1/2}) I(\hat{g}_{\hat{\lambda}})^{\alpha/2}$$

By Lemma 1 below, using the assumption that the penalty of $\hat{g}_\lambda$ is bounded above by its $L_2(P_n)$ norm, we have that

$$\|g^* - \hat{g}_{\hat{\lambda}}\|_n \leq O_p(n^{-1/(2+\alpha)}) M^{\alpha/v(2+\alpha)} \|g^*\|_n^{\alpha/2v(2+\alpha)}$$

Result 2: Single λ, Single Penalty, cross-validation over general X_T, X_V

Now suppose that the training and validation set are independently sampled, so the values X_i are not necessarily the same. Suppose X is bounded s.t. $|X| \leq R_X$ and the domain of $g \in \mathcal{G}$ is over $(-R_X, R_X)$.

We suppose the training and validation sets are both of size n.

Suppose the penalty normalizes the empirical norm as follows:

$$\sup_{g \in \mathcal{G}} \frac{\|g - g^*\|_T}{I(g) + I(g^*)} \leq R < \infty, \;\; \sup_{g \in \mathcal{G}} \frac{\|g - g^*\|_V}{I(g) + I(g^*)} \leq R < \infty$$

Suppose that

$$\sup_{g\in\mathcal{G}} \frac{\|g-g^*\|_\infty}{I(g)+I(g^*)} \quad \leq \quad K < \infty$$

Suppose for all $\lambda \in \Lambda$, $I^v(\hat{g}_\lambda)$ is upper bounded by its L_2-norm with constants M and M_0:

$$I^v(\hat{g}_\lambda) \leq M\left(\|\hat{g}_\lambda\|_T^2 + \|\hat{g}_\lambda\|_V^2\right) + M_0 = M\|\hat{g}_\lambda\|_{2n}^2 + M_0$$

Then for any $\xi > 0$,

$$\|\hat{g}_{\hat{\lambda}} - g^*\|_V = O_p(n^{-1/(2+\alpha+\xi)})I(g^*)$$

Proof: We follow the same proof structure of going thru the three cases, modifying the proofs as appropriate:

Case 1: $\|g^* - \hat{g}_{\tilde{\lambda}}\|_V^2$ is the largest

By Lemma 2, we have

$$Pr\left(\sup_{g\in\mathcal{G}} \frac{\left|\|g^*-g\|_{P_n} - \|g^*-g\|_{P_{n''}}\right|}{I(g^*)+I(g)} \geq 6\delta\right) \leq 2\exp\left(2\tilde{A}\delta^{-\alpha} - \frac{4\delta^2 n}{K^2}\right)$$

Hence for any $\xi > 0$,

$$\frac{\left|\|g^*-\hat{g}_{\tilde{\lambda}}\|_T - \|g^*-\hat{g}_{\tilde{\lambda}}\|_V\right|}{I(g^*)+I(\hat{g}_{\tilde{\lambda}})} \leq O_p(n^{-1/(2+\alpha+\xi)})$$

Therefore

$$\begin{aligned}
\|g^*-\hat{g}_{\tilde{\lambda}}\|_V &\leq \|g^*-\hat{g}_{\tilde{\lambda}}\|_T + O_p(n^{-1/(2+\alpha+\xi)})\left(I(g^*)+I(\hat{g}_{\tilde{\lambda}})\right) \\
&\leq \|g^*-\hat{g}_{\tilde{\lambda}}\|_T + O_p(n^{-1/(2+\alpha+\xi)})I(g^*)
\end{aligned}$$

Hence we can attain a rate that is infinitely close to the optimal rate.

Case 2: $\left|(\epsilon, g^*-\hat{g}_{\tilde{\lambda}})_V\right|$ is the largest

The same proof still holds.

Case 3: $\left|(\epsilon, g^*-\hat{g}_{\hat{\lambda}})_V\right|$ is the largest.

Again, we have by Van de geer (10.6),

$$\left|(\epsilon, g^*-\hat{g}_{\hat{\lambda}})_V\right| \leq O_p(n^{-1/2})\|\hat{g}_{\hat{\lambda}} - g*\|_V^{1-\alpha/2}(I(g^*)+I(\hat{g}_{\hat{\lambda}}))^{\alpha/2}$$

If $I(g^*) \geq I(g_{\hat{\lambda}})$ is true, then result is clearly attained.

Otherwise, we have

$$\|\hat{g}_{\hat{\lambda}} - g*\|_V^{1+\alpha/2} \leq O_p(n^{-1/2})I(\hat{g}_{\hat{\lambda}})^{\alpha/2}$$

By Lemma 1 below, since the penalty is bounded above by the $L_2(P_n)$ norm, it follows that

$$\|g^*-\hat{g}_{\hat{\lambda}}\|_V \leq O_p(n^{-1/(2+\alpha)})M^{\alpha/v(2+\alpha)}\|g^*\|_{2n}^{\alpha/2v(2+\alpha)}$$

Result 3: Single λ, Multiple Penalties, cross-validation over general X_T, X_V

Consider an additive model:

$$y = \sum_{j=1}^{J} g_j^* + \epsilon$$

We fit the model by least squares with separate penalties for each function g_j:

$$\{\hat{g}_j\}_{j=1}^{J} = \arg \min_{g_j \in \mathcal{G}_j} \|y - \sum_{j=1}^{J} g_j\|_T^2 + \frac{\lambda^2}{J} \sum_{j=1}^{J} I_j^{v_j}(g_j)$$

Suppose for all j, there is some $0 < \alpha_j < 2$ s.t. for all $\delta > 0$,

$$H\left(\delta, \left\{\frac{g_j - g_j^*}{I(g_j) + I(g_j^*)} : g_j \in \mathcal{G}_j, I(g_j) + I(g_j^*) > 0\right\}, \|\cdot\|_T\right) \leq A\delta^{-\alpha_j}$$

If

$$\tilde{\lambda}_T^{-1} = O_p\left(n^{1/(2+\alpha_{max})}\right) I_{(j)}^{(2v_{(j)} - 2\alpha_{max} + v_{(j)}\alpha_{max})/2(2+\alpha_{max})}(g_{(j)}^*)$$

then

$$\|\sum_{j=1}^{J} g_j - g_j^*\|_T^2 = O_p\left(\tilde{\lambda}_T\right)\left(1 \vee J^{\frac{1-\alpha_{max}}{2+\alpha_{max}}} \vee \max_{j \in 1:J}\left\{J^{\frac{v_j - v_j\alpha_{max} + \alpha_{max}}{2v_j + v_j\alpha_{max} - 2\alpha_{max}}}\right\}\right) \max_{j \in 1:J}\left(\left(I_j^{v_j}(g_j^*)\right)^{1/2}\right)$$

Proof:

We have the basic inequality

$$\|\sum_{j=1}^{J} \hat{g}_j - g_j^*\|_T^2 + \frac{\lambda^2}{J}\sum_{j=1}^{J} I_j^{v_j}(\hat{g}_j) \leq 2\left|\left(\epsilon_T, \sum_{j=1}^{J} \hat{g}_j - g_j^*\right)\right| + \frac{\lambda^2}{J}\sum_{j=1}^{J} I_j^{v_j}\left(g_j^*\right)$$

Case 1:
Suppose the RHS is dominated by the penalty term:

$$\left|\left(\epsilon_T, \sum_{j=1}^{J} \hat{g}_j - g_j^*\right)\right| \leq \frac{\lambda^2}{J}\sum_{j=1}^{J} I_j^{v_j}\left(g_j^*\right)$$

It follows that

$$\|\sum_{j=1}^{J} \hat{g}_j - g_j^*\|_T^2 + \lambda^2 \sum_{j=1}^{J} I_j^{v_j}(\hat{g}_j) \leq O_p(1)\frac{\lambda^2}{J}\sum_{j=1}^{J} I_j^{v_j}\left(g_j^*\right)$$

Obviously,

$$\|\sum_{j=1}^{J} \hat{g}_j - g_j^*\|_T^2 \leq O_p(1)\frac{\lambda^2}{J}\sum_{j=1}^{J} I_j^{v_j}\left(g_j^*\right) \leq O_p(1)\lambda^2 \max_{j \in 1:J} I_j^{v_j}\left(g_j^*\right)$$

Therefore

$$\|\sum_{j=1}^{J} \hat{g}_j - g_j^*\|_T \leq O_p(\lambda)\left(\sup_{j \in 1:J} I_j^{v_j}\left(g_j^*\right)\right)^{1/2}$$

Case 2:
Suppose the RHS is dominated by the empirical process

$$\left|\left(\epsilon_T, \sum_{j=1}^{J} \hat{g}_j - g_j^*\right)\right| \geq \frac{\lambda^2}{J} \sum_{j=1}^{J} I_j^{v_j}\left(g_j^*\right)$$

We bound the empirical process as follows. By Lemma 5, we know for sufficiently small $\delta > 0$,

$$H\left(\delta, \left\{\frac{\sum_{j=1}^{J} g_j - g_j^*}{\max_{j\in 1:J}\left(I(g_j) + I(g_j^*)\right)} : g_j \in \mathcal{G}_j, I(g_j) + I(g_j^*) > 0\right\}, \|\cdot\|_T\right) \leq 2AJ\left(\frac{\delta}{2J\left(1+R\right)}\right)^{-\alpha_{max}}$$

Hence by Lemma 6,

$$\sup_{g_j\in\mathcal{G}_j} \frac{\left|\left(\epsilon_T, \sum_{j=1}^{J} g_j - g_j^*\right)\right|}{\|\sum_{j=1}^{J} g_j - g_j^*\|^{1-\alpha_{max}/2} \max_{j\in 1:J}\left(I(g_j) + I(g_j^*)\right)^{\alpha_{max}/2}} = O_p\left(n^{-1/2} J^{(1-\alpha_{max})/2}\right)$$

Consequently, in this case, the basic inequality becomes

$$\|\sum_{j=1}^{J} \hat{g}_j - g_j^*\|_T^2 + \frac{\lambda^2}{J} \sum_{j=1}^{J} I_j^{v_j}(\hat{g}_j) \leq O_p\left(n^{-1/2} J^{(1-\alpha_{max})/2}\right) \|\sum_{j=1}^{J} \hat{g}_j - g_j^*\|_T^{1-\alpha_{max}/2} \max_{j\in 1:J}\left(I(\hat{g}_j) + I(g_j^*)\right)^{\alpha_{max}/2}$$

Let $(j) = \arg\max_{j\in 1:J} I(\hat{g}_j) + I(g_j^*)$.
Case 2a: Suppose $I(\hat{g}_{(j)}) \leq I(g_{(j)}^*)$.
Then

$$\begin{aligned}\|\sum_{j=1}^{J} \hat{g}_j - g_j^*\|_T &\leq O_p\left(n^{-1/(2+\alpha_{max})} J^{(1-\alpha_{max})/(2+\alpha_{max})}\right) I_{(j)}^{\alpha_{max}/(2+\alpha_{max})}(g_{(j)}^*) \\ &\leq O_p(\lambda) J^{(1-\alpha_{max})/(2+\alpha_{max})} \sup_{j\in 1:J}\left(I_j^{v_j}(g_j^*)\right)^{1/2}\end{aligned}$$

Case 2b: Suppose $I(\hat{g}_{(j)}) \geq I(g_{(j)}^*)$.
Then

$$\|\sum_{j=1}^{J} \hat{g}_j - g_j^*\|_T \leq O_p\left(n^{-1/(2+\alpha_{max})} J^{(1-\alpha_{max})/(2+\alpha_{max})}\right) I_{(j)}^{\alpha_{max}/(2+\alpha_{max})}(\hat{g}_{(j)})$$

and

$$\lambda^2 I_{(j)}^{v_{(j)}}(\hat{g}_{(j)}) \leq \lambda^2 \sum_{j=1}^{J} I_j^{v_j}(\hat{g}_j) \leq O_p\left(n^{-1/2} J^{(3-\alpha_{max})/2}\right) \|\sum_{j=1}^{J} \hat{g}_j - g_j^*\|_T^{1-\alpha_{max}/2} I_{(j)}^{\alpha_{max}/2}(\hat{g}_{(j)})$$

Hence

$$I_{(j)}^{v_{(j)}-\alpha_{max}/2}(\hat{g}_{(j)}) \leq O_p\left(n^{-1/2} J^{(3-\alpha_{max})/2}\right) \lambda^{-2} \|\sum_{j=1}^{J} \hat{g}_j - g_j^*\|_T^{1-\alpha_{max}/2}$$

Simplifying, we get

$$\|\sum_{j=1}^{J} \hat{g}_j - g_j^*\|_T \leq O_p\left(n^{-v_{(j)}/(2v_{(j)}-2\alpha_{max}+v_{(j)}\alpha_{max})} J^{\frac{v_{(j)}-v_{(j)}\alpha_{max}+\alpha_{max}}{2v_{(j)}+v_{(j)}\alpha_{max}-2\alpha_{max}}}\right) \lambda^{-2\alpha_{max}/(2v_{(j)}-2\alpha_{max}+v_{(j)}\alpha_{max})}$$

By our choice of $\tilde{\lambda}$, we have

$$\begin{aligned}
\|\sum_{j=1}^{J}\hat{g}_j - g_j^*\|_T &\leq O_p(\lambda) J^{\frac{v_{(j)}-v_{(j)}\alpha_{max}+\alpha_{max}}{2v_{(j)}+v_{(j)}\alpha_{max}-2\alpha_{max}}} I_{(j)}^{\alpha_{max}/(2+\alpha_{max})}(g_{(j)}^*) \\
&\leq O_p(\lambda) \max_{j\in 1:J}\left(J^{\frac{v_j-v_j\alpha_{max}+\alpha_{max}}{2v_j+v_j\alpha_{max}-2\alpha_{max}}} \left(I_j^{v_j}(g_j^*)\right)^{1/2}\right)
\end{aligned}$$

Lemmas

Lemma 1:

Suppose for all $\lambda \in \Lambda$, the penalty function $I^v(g_\lambda)$ is upper-bounded by $\|g_\lambda\|_n^2 = \frac{1}{n}\sum_{i=1}^n g_\lambda^2(x_i)$ with constants M_0 and M:

$$I^v(g_\lambda) \leq M\|g_\lambda\|_n^2 + M_0$$

Suppose there is some function $g^* \in \mathcal{G}$ such that

$$\|g^* - g_\lambda\|_n^{1+\alpha/2} \leq O_p(n^{-1/2}) I^{\alpha/2}(g_\lambda)$$

then for sufficiently large n,

$$\|g^* - g_\lambda\|_n \leq O_p(n^{-1/(2+\alpha)}) M^{\alpha/v(2+\alpha)} \|g^*\|_n^{\alpha/2v(2+\alpha)}$$

Proof:
From the assumption that $I^v(g_\lambda)$ is upper-bounded by $\|g_\lambda\|_n^2$,

$$\|g^* - g_\lambda\|_n^{1+\alpha/2} \leq O_p(n^{-1/2})\left(M\|g_\lambda\|_n^2 + M_0\right)^{\alpha/2v}$$

If $M_0 > \|g_\lambda\|_n^2$, then the result immediately follows.
Otherwise, if $M_0 \leq \|g_\lambda\|_n^2$, then

$$\begin{aligned}
\|g^* - g_\lambda\|_n^{1+\alpha/2} &\leq O_p(n^{-1/2}) M^{\alpha/2v} \|g_\lambda\|_n^{\alpha/v} \\
&\leq O_p(n^{-1/2}) M^{\alpha/2v} \left(\|g_\lambda - g^*\|_n + \|g^*\|_n\right)^{\alpha/v}
\end{aligned}$$

Case 1: $\|g_\lambda - g^*\|_n \leq \|g^*\|_n$
The result immediately follows.
Case 2: $\|g_\lambda - g^*\|_n > \|g^*\|_n$
We show for sufficiently large n, this case will not occur. Suppose this case occurs. Then

$$\|g^* - g_\lambda\|_n^{1+\alpha/2} \leq O_p(n^{-1/2}) M^{\alpha/v(2+\alpha)} \|g_\lambda - g^*\|_n^{\alpha/v}$$

Rearranging, we have that

$$\|g^* - g_\lambda\|_n^{1+\alpha/2-\alpha/v} \leq O_p(n^{-1/2}) M^{\alpha/v(2+\alpha)}$$

Since the LHS exponent is $1 + \alpha/2 - \alpha/v > 0$, $\|g^* - g_\lambda\|_n$ decreases with n. With sufficiently large n, we can ensure that only Case 1 occurs.

Note: I believe we can often provide a good estimate of M for the entire class $\mathcal{G}$, which means that we can always estimate the sample size needed to ensure this case never occurs. That is, I believe we can often estimate M s.t.

$$I^v(g) \leq M\|g\|_n^2 + M_0 \forall g \in \mathcal{G}$$

Lemma 2:

Let $P_{n'}$ and $P_{n''}$ be empirical distributions over $\{X'_i\}_{i=1}^n, \{X''_i\}_{i=1}^n$. Let $P_{2n} = \frac{1}{2}(P_{n'} + P_{n''})$. Suppose X is bounded s.t. $|X| < R_X$.

Let $\mathcal{G}' = \left\{ \frac{g-g^*}{I(g)+I(g^*)} : g \in \mathcal{G}, I(g) + I(g^*) > 0 \right\}$. Suppose g is defined over the domain over X (and zero otherwise). Suppose

$$\sup_{f \in \mathcal{G}'} \|f\|_{P_{2n}} \leq R < \infty, \ \sup_{f \in \mathcal{G}'} \|f\|_\infty \leq K < \infty$$

and

$$H(\delta, \mathcal{G}', P_{n'}) \leq \tilde{A}\delta^{-\alpha}, \ H(\delta, \mathcal{G}', P_{n''}) \leq \tilde{A}\delta^{-\alpha}$$

Then

$$Pr\left(\sup_{g \in \mathcal{G}} \frac{\left| \|g^* - g\|_{P_n} - \|g^* - g\|_{P_{n''}} \right|}{I(g^*) + I(g)} \geq 6\delta \right) \leq 2\exp\left(2\tilde{A}\delta^{-\alpha} - \frac{4\delta^2 n}{K^2} \right)$$

Proof: The proof is very similar to that in Pollard 1984 (page 32), so some details below are omitted.

First note that for any function f and h, we have

$$\|f\|_{P_{n'}} - \|h\|_{P_{n'}} \leq \|f - h\|_{P_{n'}} \leq \sqrt{2}\|f - h\|_{P_{2n}}$$

Similarly for $P_{n''}$.

Let $\{h_j\}_{j=1}^N$ be the $\sqrt{2}\delta$-cover for $\mathcal{G}'$ (where $N = N(\sqrt{2}\delta, \mathcal{G}', P_{2n})$). Let h_j be the closest function (in terms of $\|\cdot\|_{P_{2n}}$) to some $f \in \mathcal{G}'$. Then

$$\begin{aligned}
\|f\|_{P_{n'}} - \|f\|_{P_{n''}} &\leq \|f - h_j\|_{P_{n'}} + \left| \|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}} \right| + \|f - h_j\|_{P_{n''}} \\
&\leq 4\delta + \left| \|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}} \right|
\end{aligned}$$

Therefore for $f = \frac{g^* - g}{I(g^*) + I(g)}$, we have

$$\begin{aligned}
Pr\left(\sup_{g \in \mathcal{G}} \frac{\left| \|g^* - g\|_{P_n} - \|g^* - g\|_{P_{n''}} \right|}{I(g^*) + I(g)} \geq 6\delta \right) &\leq Pr\left(\sup_{j \in 1:N} \left| \|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}} \right| \geq 2\delta \right) \\
&\leq N \max_{j \in 1:N} Pr\left(\left| \|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}} \right| \geq 2\delta \right)
\end{aligned}$$

Now note that

$$\begin{aligned}
\left| \|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}} \right| &= \frac{\left| \|h_j\|_{P_{n'}}^2 - \|h_j\|_{P_{n''}}^2 \right|}{\|h_j\|_{P_{n'}} + \|h_j\|_{P_{n''}}} \\
&\leq \frac{\left| \|h_j\|_{P_{n'}}^2 - \|h_j\|_{P_{n''}}^2 \right|}{\sqrt{2}\|h_j\|_{P_{2n}}}
\end{aligned}$$

By Hoeffding's inequality,

$$\begin{aligned}
Pr\left(\left| \|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}} \right| \geq 2\delta \right) &\leq Pr\left(\left| \|h_j\|_{P_{n'}}^2 - \|h_j\|_{P_{n''}}^2 \right| \geq 2\sqrt{2}\delta\|h_j\|_{P_{2n}} \right) \\
&= Pr\left(\left| \sum_{i=1}^n W_i \left(h_j^2(x'_i) - h_j^2(x''_i) \right) \right| \geq 2\sqrt{2}n\delta\|h_j\|_{P_{2n}} \right) \\
&\leq 2\exp\left(-\frac{16\delta^2 n^2 \|h_j\|_{P_{2n}}^2}{4\sum_{i=1}^n \left(h_j^2(x'_i) - h_j^2(x''_i) \right)^2} \right)
\end{aligned}$$

Since $\|h_j\|_\infty < K$, then

$$\begin{aligned}\sum_{i=1}^{n}\left(h_j^2(x_i') - h_j^2(x_i'')\right)^2 &\leq \sum_{i=1}^{n} h_j^4(x_i') + h_j^4(x_i'') \\ &\leq nK^2\|h_j\|_{P_{2n}}^2\end{aligned}$$

Hence

$$Pr\left(\left|\|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}}\right| \geq 2\delta\right) \leq 2\exp\left(-\frac{4\delta^2 n}{K^2}\right)$$

Since (Pollard and Vandegeer say that)

$$N(\sqrt{2}\delta, \mathcal{G}', P_{2n}) \leq N(\delta, \mathcal{G}', P_{n''}) + N(\delta, \mathcal{G}', P_{n''})$$

then

$$Pr\left(\sup_{g\in\mathcal{G}}\frac{\left|\|g^* - g\|_{P_n} - \|g^* - g\|_{P_{n''}}\right|}{I(g^*) + I(g)} \geq 6\delta\right) \leq 2\exp\left(2\tilde{A}\delta^{-\alpha} - \frac{4\delta^2 n}{K^2}\right)$$

Using shorthand, we can write that for any $\xi > 0$,

$$\sup_{g\in\mathcal{G}}\frac{\left|\|g^* - g\|_{P_n} - \|g^* - g\|_{P_{n''}}\right|}{I(g^*) + I(g)} = O_p(n^{-1/(2+\alpha+\xi)})$$

Lemma 3:

Suppose the function class $\mathcal{F}$ is bounded s.t. $\sup_{f\in\mathcal{F}}\|f\|_n \leq R < \infty$. Let

$$\tilde{\mathcal{F}} = \{\gamma f \ : \ f \in \mathcal{F}, \gamma \in (0, 1]\}$$

$$H\left(\delta(1 + R + \delta), \tilde{\mathcal{F}}, \|\cdot\|_n\right) \leq \log(1 + \lfloor\frac{1}{\delta}\rfloor) + H\left(\delta, \mathcal{F}, \|\cdot\|_n\right)$$

Proof: Let $\{h_i\}_{i=1}^N$ be the δ-cover for $\mathcal{F}$. Consider any $f \in \mathcal{F}$ and let $h_{(f)}$ be the closest function in δ-cover for $\mathcal{F}$. Choose $j \in \mathbb{Z}^+$ such that $|\gamma - \delta j| < \delta$.

$$\begin{aligned}\|\gamma f - \delta j h_{(f)}\|_n &\leq \|\gamma f - \gamma h_{(f)}\|_n + \|\gamma h_{(f)} - \delta j h_{(f)}\|_n \\ &\leq \gamma\|f - h_{(f)}\|_n + |\gamma - \delta j|\|h_{(f)}\|_n \\ &\leq \gamma\delta + \delta\left(\|f - h_{(f)}\|_n + \|f\|_n\right) \\ &\leq \gamma\delta + \delta\left(\delta + R\right) \\ &\leq \delta\left(1 + R + \delta\right)\end{aligned}$$

Hence we have found that the following $N(1 + \lfloor\frac{1}{\delta}\rfloor)$ functions form a $\delta(1 + R + \delta)$-cover for $\tilde{\mathcal{F}}$:

$$\{h_i\}_{i=1}^N \cup \left\{j\delta h_i \ : \ j \in 1 : \lfloor\frac{1}{\delta}\rfloor, i \in 1 : N\right\}$$

Lemma 4:

Define function classes $\{\mathcal{F}_j\}_{j=1}^J$ and

$$\tilde{\mathcal{F}} = \left\{ \sum_{j=1}^J f_j \ : \ f_j \in \mathcal{F}_j \right\}$$

Then

$$H\left(J\delta, \tilde{\mathcal{F}}, \|\cdot\|_n\right) \leq \sum_{j=1}^J H\left(\delta, \mathcal{F}_j, \|\cdot\|_n\right)$$

Proof: For every $j = 1 : J$, consider any $f_j \in \mathcal{F}_j$ and let $h_{(j)}$ be the closest function in the δ-cover for $\mathcal{F}_j$.

$$\|\sum_{j=1}^J f_j - \sum_{j=1}^J h_{(j)}\| \quad \leq \quad \sum_{j=1}^J \|f_j - h_{(j)}\| \leq J\delta$$

Henceexp $\left(\sum_{j=1}^J H\left(\delta, \mathcal{F}_j, \|\cdot\|_n\right)\right)$ functions form a $J\delta$-cover for $\tilde{\mathcal{F}}$.

Lemma 5:

Suppose for all $j = 1, ..., J$, there is some $\alpha_j > 0$ and $A_j > 0$ s.t. the following entropy bound holds for all $\delta > 0$

$$H\left(\delta, \left\{\frac{g_j - g_j^*}{I(g_j) + I(g_j^*)} : g_j \in \mathcal{G}_j, I(g_j) + I(g_j^*) > 0\right\}, \|\cdot\|_T\right) \leq A\delta^{-\alpha_j}$$

Then for sufficiently small$\delta > 0$, we have

$$H\left(\delta, \left\{\frac{\sum_{j=1}^J g_j - g_j^*}{\sup_{j\in 1:J}\left(I(g_j) + I(g_j^*)\right)} : g_j \in \mathcal{G}_j, I(g_j) + I(g_j^*) > 0\right\}, \|\cdot\|_T\right) \leq 2JA\left(\frac{\delta}{2J\left(1+R\right)}\right)^{-\alpha_{max}}$$

where $\alpha_{max} = \max_{j\in 1:J}\alpha_j$.

Proof: By Lemma 3,

$$H\left(\delta(1+R+\delta), \left\{\gamma\frac{g_j - g_j^*}{I(g_j) + I(g_j^*)} \ : \ g_j \in \mathcal{G}_j, I(g_j) + I(g_j^*) > 0, \gamma \in (0,1]\right\}, \|\cdot\|_T\right) \leq \log(1+\lfloor\frac{1}{\delta}\rfloor) + A\delta^{-\alpha_j}$$

Note that

$$\frac{\sum_{j=1}^J g_j - g_j^*}{\sup_{j\in 1:J}\left(I(g_j) + I(g_j^*)\right)} = \sum_{j=1}^J \left(\frac{I(g_j) + I(g_j^*)}{\sup_{\ell\in 1:J} I(g_\ell) + I(g_\ell^*)}\right) \frac{g_j - g_j^*}{I(g_j) + I(g_j^*)}$$

By Lemma 4,

$$H\left(J\delta(1+R+\delta), \left\{\frac{\sum_{j=1}^J g_j - g_j^*}{\sup_{j\in 1:J}\left(I(g_j) + I(g_j^*)\right)} \ : g_j \in \mathcal{G}_j, I(g_j) + I(g_j^*) > 0\right\}, \|\cdot\|_T\right) \leq J\log(1+\lfloor\frac{1}{\delta}\rfloor) + JA\delta^{-\alpha_j}$$

Hence for sufficiently small δ,

$$H\left(J\delta(1+R+\delta), \left\{\frac{\sum_{j=1}^J g_j - g_j^*}{\sup_{j\in 1:J}\left(I(g_j)+I(g_j^*)\right)} : g_j \in \mathcal{G}_j, I(g_j)+I(g_j^*) > 0\right\}, \|\cdot\|_T\right) \le 2JA\delta^{-\alpha_{max}}$$

Rearranging, we get

$$\begin{aligned} H\left(\delta, \left\{\frac{\sum_{j=1}^J g_j - g_j^*}{\sup_{j\in 1:J}\left(I(g_j)+I(g_j^*)\right)} : g_j \in \mathcal{G}_j, I(g_j)+I(g_j^*) > 0\right\}, \|\cdot\|_T\right) &\le 2AJ\left(\sqrt{\left(\frac{1+R}{2}\right)^2 + \frac{\delta}{J}} - \frac{1+R}{2}\right)^{-\alpha_{max}} \\ &\le 2AJ\left(\frac{\delta}{2J(1+R)}\right)^{-\alpha_{max}} \end{aligned}$$

(Used the fact that for $b > 0$ small enough, $\sqrt{a^2+b} - a \ge \sqrt{(a+\frac{b}{4a})^2} - a = \frac{b}{4a}$)

Lemma 6:

Suppose ϵ_i are sub-gaussian errors and for the function class $\mathcal{F}$, we have that for some $0 < \alpha < 2$, $A' > 0$, and $J > 0$

$$H\left(\delta, \mathcal{F}, \|\cdot\|_T\right) \le A' J^\tau \delta^{-\alpha} \ \forall \delta > 0$$

Then for $T = 2C_1CA'^{1/2}J^{\tau/2}2^{1-\alpha/2}$

$$Pr\left(\sup_{f\in\mathcal{F}} \frac{\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_i f(z_i)\right|}{\|f\|_n^{1-\alpha/2}} \ge T\right) \le c\exp(-T^2/c^2)$$

Proof: Follow proof for Lemma 8.4 in Vandegeer, but with $A = A'J^{-\alpha}$. Note that we then have $A_0 = A'^{1/2}J^{\tau/2}$. We then get

$$Pr\left(\sup_{f\in\mathcal{F}} \frac{\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_i f(z_i)\right|}{\|f\|_n^{1-\alpha/2}} \ge 2C_1CA'^{1/2}J^{\tau/2}2^{1-\alpha/2}\right) \le c\exp(-T^2/c^2)$$

Note that we can write via shorthand that

$$\sup_{f\in\mathcal{F}} \frac{\left|\frac{1}{n}\sum_{i=1}^n \epsilon_i f(z_i)\right|}{\|f\|_n^{1-\alpha/2}} = O_p(J^{\tau/2}n^{-1/2})$$

Example 1: Sobelov norm (NOT DONE)

Consider the functions

$$\mathcal{G} = \left\{g : [0,1] \mapsto \mathbb{R} : \int_0^1 g^{(m)}(z)^2 dz < \infty\right\}$$

Suppose x_i are all unique. Then the Sobelov norm for the class $\{\hat{g}_\lambda \in \mathcal{G} : \lambda \in \Lambda\}$ is bounded above by its $L_2(P_n)$ norm.

$$I^2(\hat{g}_\lambda) = \int_0^1 \left(\hat{g}_\lambda^{(m)}(z)\right)^2 dz \le 2\|\hat{g}_\lambda\|_n^2 + 4I^2(\tilde{g}) + 4\|y\|_n^2 \ \forall \lambda \in \Lambda$$

PROBLEM: as defined, it is possible that $I^2(\tilde{g})$ grows with n, which is not okay!

Proof:

Let $\tilde{g}$ satisfy $\tilde{g}(x_i) = y_i$ and have the smallest value for $\int_0^1 \left(\tilde{g}^{(m)}(z)\right)^2 dz$. This function $\tilde{g}$ should always exist.

Case 1: $\lambda \leq 1/2$

By definition of $\hat{g}_\lambda$

$$\|y - \hat{g}_\lambda\|_n^2 + \lambda^2 I^2(\hat{g}_\lambda) \leq \|y - (\tilde{g} - \lambda\hat{g}_\lambda)\|_n^2 + \lambda^2 I^2(\tilde{g} - \lambda\hat{g}_\lambda)$$

Note that

$$\begin{aligned}
I^2(\tilde{g} - \lambda\hat{g}_\lambda) &= \int_0^1 \left(\tilde{g}^{(m)} - \lambda\hat{g}_\lambda^{(m)}\right)^2 dz \\
&= 2\int_0^1 \max\left(\left|\tilde{g}^{(m)}\right|^2, \left|\lambda\hat{g}_\lambda^{(m)}\right|^2\right) dz \\
&= 2\left(\int_0^1 \left|\tilde{g}^{(m)}\right|^2 dz + \int_0^1 \left|\lambda\hat{g}_\lambda^{(m)}\right|^2 dz\right)
\end{aligned}$$

Hence

$$\lambda^2 I^2(\hat{g}_\lambda) \leq \lambda^2\|\hat{g}_\lambda\|_n^2 + 2\lambda^2 I^2(\tilde{g}) + 2\lambda^4 I^2(\hat{g}_\lambda)$$

The following ineq follows, where the RHS is maximized when $\lambda = 1/2$

$$I^2(\hat{g}_\lambda) \leq \frac{\lambda^2}{\lambda^2 - 2\lambda^4}\left(\|\hat{g}_\lambda\|_n^2 + 2I^2(\tilde{g})\right) \leq 2\|\hat{g}_\lambda\|_n^2 + 4I^2(\tilde{g})$$

Case 2: $\lambda > 1/2$

By definition of $\hat{g}_\lambda$

$$\|y - \hat{g}_\lambda\|_n^2 + \lambda^2 I^2(\hat{g}_\lambda) \leq \|y\|_n^2$$

The RHS is maximized when $\lambda = 1/2$, so

$$I^2(\hat{g}_\lambda) \leq 4\|y\|_n^2$$

Hence we have an upper bound for the Sobelov norm

$$I^2(\hat{g}_\lambda) \leq 2\|\hat{g}_\lambda\|_n^2 + 4I^2(\tilde{g}) + 4\|y\|_n^2$$

Appendix

A cute lemma I found but never used: Supposing that $I^v(\hat{g}_\lambda)$ is continuous in λ, then given training data T,

$$\frac{\partial}{\partial\lambda} L_T(\hat{g}_\lambda, \lambda) \quad = \quad 2\lambda I^v(\hat{g}_\lambda)$$

Also, L_T is convex in λ.

Proof:

By definition,

$$L_T(\hat{g}_\lambda, \lambda) = \|y - \hat{g}_\lambda\|_T^2 + \lambda^2 I^v(\hat{g}_\lambda) \leq \|y - \hat{g}_{\lambda'}\|_T^2 + \lambda^2 I^v(\hat{g}_{\lambda'}) = L_T(\hat{g}_{\lambda'}, \lambda)$$

Then we can provide upper and lower bounds for $L_T(\hat{g}_{\lambda_2}, \lambda_2) - L_T(\hat{g}_{\lambda_1}, \lambda_1)$:

$$\begin{aligned}
L_T(\hat{g}_{\lambda_2}, \lambda_2) - L_T(\hat{g}_{\lambda_1}, \lambda_1) &\leq L_T(\hat{g}_{\lambda_1}, \lambda_2) - L_T(\hat{g}_{\lambda_1}, \lambda_1) \\
&= \|y - \hat{g}_{\lambda_1}\|_T^2 + \lambda_2^2 I^v(\hat{g}_{\lambda_1}) - \|y - \hat{g}_{\lambda_1}\|_T^2 - \lambda_1^2 I^v(\hat{g}_{\lambda_1}) \\
&= (\lambda_2^2 - \lambda_1^2) I^v(\hat{g}_{\lambda_1})
\end{aligned}$$

$$
\begin{aligned}
L_T(\hat{g}_{\lambda_2}, \lambda_2) - L_T(\hat{g}_{\lambda_1}, \lambda_1) &\geq L_T(\hat{g}_{\lambda_2}, \lambda_2) - L_T(\hat{g}_{\lambda_2}, \lambda_1) \\
&= \|y - \hat{g}_{\lambda_2}\|_T^2 + \lambda_2^2 I^v(\hat{g}_{\lambda_2}) - \|y - \hat{g}_{\lambda_2}\|_T^2 - \lambda_1^2 I^v(\hat{g}_{\lambda_2}) \\
&= (\lambda_2^2 - \lambda_1^2) I^v(\hat{g}_{\lambda_2})
\end{aligned}
$$

So suppose WLOG $\lambda_2 > \lambda_1$:

$$
(\lambda_2 + \lambda_1) I^v(\hat{g}_{\lambda_2}) \leq \frac{L_T(\hat{g}_{\lambda_2}, \lambda_2) - L_T(\hat{g}_{\lambda_1}, \lambda_1)}{\lambda_2 - \lambda_1} \leq (\lambda_2 + \lambda_1) I^v(\hat{g}_{\lambda_1})
$$

So as $\lambda_1 \to \lambda_2 = \lambda$, we have by the sandwich theorem,

$$
\frac{\partial}{\partial \lambda} L_T(\hat{g}_\lambda, \lambda) = 2\lambda I^v(\hat{g}_\lambda)
$$

Furthermore, given training data T

$$
\frac{\partial}{\partial \lambda} L_T(\hat{g}_\lambda, \lambda) = \frac{\partial}{\partial \lambda} \|y - \hat{g}_\lambda\|_T^2 + 2\lambda I^v(\hat{g}_\lambda) + \lambda^2 \frac{\partial}{\partial \lambda} I^v(\hat{g}_\lambda)
$$

then, combining this with the lemma, we have that

$$
\frac{\partial}{\partial \lambda} \|y - \hat{g}_\lambda\|_T^2 = -\lambda^2 \frac{\partial}{\partial \lambda} I^v(\hat{g}_\lambda)
$$

Finally, to see that L_T is convex in λ, note that

$$
\frac{\partial^2}{\partial \lambda^2} L_T(\hat{g}_\lambda, \lambda) = 2I^v(\hat{g}_\lambda) + 2\lambda v I^{v-1}(\hat{g}_\lambda) \frac{\partial}{\partial \lambda} I(\hat{g}_\lambda) > 0
$$

since $\frac{\partial}{\partial \lambda} I(\hat{g}_\lambda) > 0$.