Sobolev penalty:univariate

Given a function h, the Sobolev penalty for h is

$$P(h) = \int (h^{(r)}(x))^2 dx$$

Suppose $\sup_g \|g\|_\infty \leq G$.
We shall suppose for simplicity that the domain is $[0, 1]$.
Suppose we have the function class (so no additional ridge penalty)

$$\hat{\mathcal{G}}(T) = \left\{ \hat{g}(\cdot|\lambda) = \arg\min_{g\in\mathcal{G}} \frac{1}{2}\|y-g\|_T^2 + \lambda P(g) : \lambda \in \Lambda \right\}$$

Using the logic in Example 9.3.2 in Vandegeer, we can express any function in $\mathcal{G}$ as

$$f + g$$

where

$$g = \sum_{k=1}^{r} \alpha_k \psi_k, f = \int_0^1 \beta_u \tilde{\phi}_u$$

where $\langle \psi_k, \tilde{\phi}_u \rangle_T = 0$ and $P(\psi_k) = 0$.
Suppose the observations were drawn from $y = f^*(x) + g^*(x) + \epsilon$ where ϵ are independent sub-gaussian random variables.
Now we have the function class

$$\hat{\mathcal{G}}(T) = \left\{ \hat{g}(\cdot|\lambda), \hat{f}(\cdot|\lambda) = \arg\min_{g\in\mathcal{G}} \frac{1}{2}\|y-(f+g)\|_T^2 + \lambda P(f) : \lambda \in \Lambda, g = \sum_{k=1}^{r} \alpha_k \psi_k, f = \int_0^1 \beta_u \tilde{\phi}_u \right\}$$

We will show that

$$\left\| \left(\hat{g}(\cdot|\lambda^{(1)}) + \hat{f}(\cdot|\lambda^{(1)})\right) - \left(\hat{g}(\cdot|\lambda^{(2)}) + \hat{f}(\cdot|\lambda^{(2)})\right) \right\|_\infty \leq |\lambda^{(1)} - \lambda^{(2)}| n^{\tau_{min}} \sqrt{\frac{n^{\tau_{min}}}{2}\|\epsilon\|_T^2 + P(f^*)G}$$

Proof

First by Vandegeer Example 9.3.2, we know that

$$\hat{g}(\cdot|\lambda) = \arg \min_{g=\sum \alpha_k \psi_k} -2\langle \epsilon, g - g^* \rangle_T + \|g - g^*\|_T^2$$

$$\hat{f}(\cdot|\lambda) = \arg \min_{f=\int_0^1 \beta_u \tilde{\phi}_u} -2\langle \epsilon, f - f^* \rangle_T + \|f - f^*\|_T^2 + \lambda P(f)$$

So $\hat{g}(\cdot|\lambda)$ is actually independent of λ and is therefore constant. We will just denote it $\hat{g}$ from now on.
Now consider

$$h = c\left(\hat{f}(\cdot|\lambda^{(1)}) - \hat{f}(\cdot|\lambda^{(2)})\right)$$

where c is some constant s.t. $P(h) = 1$.
We can assume that $P(h) \neq 0$. Otherwise, if

$$P\left(\hat{f}(\cdot|\lambda^{(1)}) - \hat{f}(\cdot|\lambda^{(2)})\right) = 0$$

then we know that

$$\hat{f}(\cdot|\lambda^{(1)}) - \hat{f}(\cdot|\lambda^{(2)}) \in span\left\{\psi_k\right\}_{k=1}^{r}$$

This is true if and only if $\hat{f}(\cdot|\lambda^{(1)}) \equiv \hat{f}(\cdot|\lambda^{(2)})$ (by the fact that the function spaces are orthogonal).
Consider the optimization problem

$$\hat{m}_h(\lambda) = \arg\min_m \frac{1}{2}\|y - (\hat{g} + \hat{f}(\cdot|\lambda^{(1)}) + mh)\|_T^2 + \lambda P\left(\hat{f}(\cdot|\lambda^{(1)}) + mh\right)$$

By implicit differentiation of the KKT conditions, we get

$$\left.\frac{\partial}{\partial\lambda}\hat{m}_h(\lambda)\right|_{\lambda=\lambda} = -\left(\|h\|_T^2 + \lambda\frac{\partial^2}{\partial m^2}P\left(\hat{f}(\cdot|\lambda^{(1)}) + mh\right)\right)^{-1}\left.\frac{\partial}{\partial m}P\left(\hat{f}(\cdot|\lambda^{(1)}) + mh\right)\right|_{m=\hat{m}_h(\lambda)}$$

Then the first multiplicand is bounded by

$$\begin{aligned}\left|\|h\|_T^2 + \lambda\frac{\partial^2}{\partial m^2}P\left(\hat{f}(\cdot|\lambda^{(1)}) + mh\right)\right|^{-1} &\leq n^{\tau_{min}}\frac{\partial^2}{\partial m^2}P\left(\hat{f}(\cdot|\lambda^{(1)}) + mh\right)^{-1} \\ &= \frac{n^{\tau_{min}}}{2P(h)}\end{aligned}$$

The equality follows from the Lemma Sobolev Facts (see below).
From the Lemma Sobolev Facts and by the fact that $P(h) = 1$, we have

$$\begin{aligned}\left|\frac{\partial}{\partial\lambda}\hat{m}_h(\lambda)\right|_{\lambda=\lambda} &\leq \frac{n^{\tau_{min}}}{P(h)}\sqrt{P\left(\hat{f}(\cdot|\lambda^{(1)}) + \hat{m}_h(\lambda)h\right)P(h)} \\ &= n^{\tau_{min}}\sqrt{P\left(\hat{f}(\cdot|\lambda^{(1)}) + \hat{m}_h(\lambda)h\right)}\end{aligned}$$

By the definition of $\hat{m}_h(\lambda)$ and $\hat{f}(\cdot|\lambda^{(1)})$, we have that

$$\begin{aligned}\lambda P\left(\hat{f}(\cdot|\lambda^{(1)}) + \hat{m}_h(\lambda)h\right) &\leq \frac{1}{2}\|y - (\hat{g} + \hat{f}(\cdot|\lambda^{(1)}))\|_T^2 + \lambda P\left(\hat{f}(\cdot|\lambda^{(1)})\right) \\ &= \frac{1}{2}\|y - (\hat{g} + \hat{f}(\cdot|\lambda^{(1)}))\|_T^2 + \lambda^{(1)}P\left(\hat{f}(\cdot|\lambda^{(1)})\right) + \left(\lambda - \lambda^{(1)}\right)P\left(\hat{f}(\cdot|\lambda^{(1)})\right) \\ &\leq \frac{1}{2}\|y - (g^* + f^*)\|_T^2 + \lambda^{(1)}P\left(f^*\right) + \left(\lambda - \lambda^{(1)}\right)P\left(\hat{f}(\cdot|\lambda^{(1)})\right)\end{aligned}$$

In addition, by definition of $\hat{f}(\cdot|\lambda^{(1)})$, we have

$$P\left(\hat{f}(\cdot|\lambda^{(1)})\right) \leq \frac{1}{2\lambda^{(1)}}\|y - (g^* + f^*)\|_T^2 + P\left(f^*\right)$$

Combining the two inequalities above, we have

$$\begin{aligned}\lambda P\left(\hat{f}(\cdot|\lambda^{(1)}) + \hat{m}_h(\lambda)h\right) &\leq \frac{1}{2}\|\epsilon\|_T^2 + \lambda^{(1)}P\left(f^*\right) + \left(\lambda - \lambda^{(1)}\right)\left(\frac{1}{2\lambda^{(1)}}\|\epsilon\|_T^2 + P\left(f^*\right)\right) \\ &\leq \frac{\lambda}{2\lambda^{(1)}}\|\epsilon\|_T^2 + \lambda P\left(f^*\right)\end{aligned}$$

Therefore

$$P\left(\hat{f}(\cdot|\lambda^{(1)}) + \hat{m}_h(\lambda)h\right) \leq \frac{n^{\tau_{min}}}{2}\|\epsilon\|_T^2 + P(f^*)$$

Then by the MVT, we have

$$\begin{aligned}\|\hat{f}(\cdot|\lambda^{(1)}) - \hat{f}(\cdot|\lambda^{(2)})\|_\infty &= \|m_h(\lambda^{(2)})h\|_\infty \\ &\leq \left|\lambda^{(1)} - \lambda^{(2)}\right|\left(\sup_{\lambda\in[\lambda^{(1)},\lambda^{(2)}]}\left|\frac{\partial}{\partial\lambda}\hat{m}_h(\lambda)\right|_{\lambda=\lambda}\right)G \\ &\leq \left|\lambda^{(1)} - \lambda^{(2)}\right|Gn^{\tau_{min}}\sqrt{\frac{n^{\tau_{min}}}{2}\|\epsilon\|_T^2 + P(f^*)}\end{aligned}$$

Sobolev penalty: multivariate

The function class of interest

$$\hat{\mathcal{G}}(T) = \left\{ \left\{ \hat{g}_j(\cdot|\lambda), \hat{f}_j(\cdot|\lambda) \right\} = \arg\min_{g \in \mathcal{G}} \frac{1}{2} \|y - \sum_{j=1}^J g_j(x_j)\|_T^2 + \sum_{j=1}^J \lambda_j P(g_j) : \lambda \in \Lambda \right\}$$

We can show that

$$\|\hat{f}_\ell(\cdot|\lambda^{(1)}) - \hat{f}_\ell(\cdot|\lambda^{(2)})\|_\infty \quad \leq \quad G \left\| \lambda^{(1)} - \lambda^{(2)} \right\| \frac{n^{\tau_{max}/2 + 2\tau_{min}}}{2} \sqrt{\frac{n^{\tau_{min}} + J}{2} \|\epsilon\|_T^2 + \sum_{j=1}^J P\left(f_j^*\right)}$$

A second approach gives a different bound:

$$\|\hat{f}_\ell(\cdot|\lambda^{(1)}) - \hat{f}_\ell(\cdot|\lambda^{(2)})\|_\infty \quad \leq \quad \left\| \lambda^{(1)} - \lambda^{(2)} \right\| \frac{G^2(2G + \|\epsilon\|_T) n^{3\tau_{min}}}{4}$$

Proof

First by Vandegeer Example 9.3.2, we know that

$$\{\hat{g}_j(\cdot|\lambda)\}_{j=1}^J = \arg \min_{g_j = \sum \alpha_k \psi_k} -2\langle \epsilon, \sum_{j=1}^J g_j - g_j^* \rangle_T + \| \sum_{j=1}^J g_j - g_j^* \|_T^2$$

$$\left\{ \hat{f}_j(\cdot|\lambda) \right\}_{j=1}^J = \arg \min_{f_j = \int_0^1 \beta_u \tilde{\phi}_u} -2\langle \epsilon, \sum_{j=1}^J f_j - f_j^* \rangle_T + \| \sum_{j=1}^J f_j - f_j^* \|_T^2 + \sum_{j=1}^J \lambda_j P(f_j)$$

For every $j = 1 : J$, we again notice that $\hat{g}_j(\cdot|\lambda)$ is independent of λ. We will just denote it $\hat{g}_j$ from now on.

For every $j = 1 : J$, define functions

$$h_j = c\left(\hat{f}_j(\cdot|\lambda^{(1)}) - \hat{f}_j(\cdot|\lambda^{(2)}) \right)$$

where c is some constant s.t. $P(h_j) = 1$.

We can assume that $P(h_j) \neq 0$. Otherwise, if

$$P\left(\hat{f}_j(\cdot|\lambda^{(1)}) - \hat{f}_j(\cdot|\lambda^{(2)}) \right) \quad = \quad 0$$

then we know that

$$\hat{f}_j(\cdot|\lambda^{(1)}) - \hat{f}_j(\cdot|\lambda^{(2)}) \in span\, \{\psi_k\}_{k=1}^r$$

This is true if and only if $\hat{f}_j(\cdot|\lambda^{(1)}) \equiv \hat{f}_j(\cdot|\lambda^{(2)})$ (by the fact that the function spaces are orthogonal).

Now consider the optimization problem

$$\{\hat{m}_j(\lambda, h)\}_{j=1}^J = \arg\min_{m_j} \frac{1}{2} \|y - \sum_{j=1}^J (\hat{g}_j + \hat{f}_j(\cdot|\lambda^{(1)}) + m_j h_j)\|_T^2 + \sum_{j=1}^J \lambda_j P\left(\hat{f}_j(\cdot|\lambda^{(1)}) + m_j h_j \right)$$

(If $h_j \equiv 0$, then set $m_j = 0$ as a constant.) For simplicity, we will assume $h_j \neq 0$.

The KKT conditions give us for all $\ell = 1 : J$

$$\left\langle h_\ell, y - \left(\sum_{j=1}^{J} \hat{g}_j(\cdot|\lambda^{(1)}) + \hat{f}_j(\cdot|\lambda^{(1)}) + \hat{m}_j(\lambda, h) h_j \right) \right\rangle_T + \lambda_\ell \frac{\partial}{\partial m_\ell} P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + m_\ell h \right) \Bigg|_{m_\ell = \hat{m}_\ell(\lambda, h)} = 0$$

For all $k = 1 : J$, by implicit differentiation of the KKT conditions with respect to λ_k, we get

$$\begin{aligned} \left\langle h_\ell, y - \sum_{j=1}^{J} h_j \frac{\partial}{\partial \lambda_k} \hat{m}_j(\lambda, h) \right\rangle_T + \lambda_\ell \frac{\partial^2}{\partial m_\ell^2} P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + m_\ell h \right) \frac{\partial}{\partial \lambda_k} \hat{m}_\ell(\lambda, h) & \\ + 1[\ell = k] \frac{\partial}{\partial m_\ell} P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + m_\ell h \right) &= 0 \end{aligned}$$

Define the following matrices

$$S : S_{ij} = \langle h_j, h_\ell \rangle_T$$

$$D_1 = diag\left(\lambda_\ell \frac{\partial^2}{\partial m_\ell^2} P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + \hat{m}_\ell(\lambda) h_\ell \right) \right)$$

$$D_3 = diag\left(\frac{\partial}{\partial m_\ell} P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + \hat{m}_\ell(\lambda) h_\ell \right) \right)$$

$$M = \begin{pmatrix} \frac{\partial \hat{m}_1(\lambda)}{\partial \lambda} & \frac{\partial \hat{m}_2(\lambda)}{\partial \lambda} & \dots & \frac{\partial \hat{m}_J(\lambda)}{\partial \lambda} \end{pmatrix}$$

From the implicit differentiation equations, we have the following system of equations:

$$M = D_3 \left(S + D_1 \right)^{-1}$$

We know that S is a PSD matrix (since it can be written as $S = HH^T$ where $H_j = h_j$ evaluated at covariates T).

We are interested in bounding $\nabla_\lambda \hat{m}_\ell(\lambda, h)$, which is the ℓ-th column of M has norm. By Lemma PSD_Matrix_Inverse, we know that

$$\begin{aligned} \|\nabla_\lambda \hat{m}_\ell(\lambda, h)\| &= \|M e_\ell\| \\ &= \left\|D_3 \left(S + D_1 \right)^{-1} e_\ell\right\| \\ &\le \|D_3 D_1^{-1} e_\ell\| \\ &= \left| \frac{\partial}{\partial m_\ell} P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + \hat{m}_\ell(\lambda) h_\ell \right) \right| \left| \lambda_\ell \frac{\partial^2}{\partial m_\ell^2} P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + \hat{m}_\ell(\lambda) h_\ell \right) \right|^{-1} \end{aligned}$$

By Lemma Sobolev Facts (below), we have

$$\frac{\partial^2}{\partial m_\ell^2} P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + \hat{m}_\ell(\lambda) h_\ell \right) = 2P(h_\ell) = 2$$

Also by Lemma Sobolev Facts (below), we note that

$$\begin{aligned} \left| \frac{\partial}{\partial m_\ell} P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + \hat{m}_\ell(\lambda) h_\ell \right) \right| &\le 2\sqrt{P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + \hat{m}_\ell(\lambda) h_\ell \right) P(h_\ell)} \\ &= 2\sqrt{P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + \hat{m}_\ell(\lambda) h_\ell \right)} \end{aligned}$$

By the definition of $\hat{m}_\ell(\lambda)$ and $\hat{f}(\cdot|\lambda^{(1)})$, we have

$$\begin{aligned}
\lambda_\ell P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + \hat{m}_\ell(\lambda)h_\ell\right) &\leq \frac{1}{2}\|y - (\hat{g} + \hat{f}(\cdot|\lambda^{(1)}))\|_T^2 + \sum_{j=1}^J \lambda_j P\left(\hat{f}_j(\cdot|\lambda^{(1)})\right) \\
&= \frac{1}{2}\|y - (g^* + f^*)\|_T^2 + \sum_{j=1}^J \lambda_j^{(1)} P\left(f_j^*\right) + \sum_{j=1}^J \left(\lambda_j - \lambda_j^{(1)}\right) P\left(\hat{f}_j(\cdot|\lambda^{(1)})\right) \\
&\leq \frac{1}{2}\|\epsilon\|_T^2 + \sum_{j=1}^J \lambda_j^{(1)} P\left(f_j^*\right) + J\lambda_{max}\left[\max_{j=1:J} P\left(\hat{f}_j(\cdot|\lambda^{(1)})\right)\right]
\end{aligned}$$

Furthermore, we know by definition of $\hat{f}_j(\cdot|\lambda^{(1)})$ that

$$P\left(\hat{f}_j(\cdot|\lambda^{(1)})\right) \leq \frac{1}{2\lambda_{min}}\|\epsilon\|_T^2 + \frac{1}{\lambda_{min}}\sum_{j=1}^J \lambda_j^{(1)} P\left(f_j^*\right)$$

So

$$P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + m_\ell h_\ell\right) \leq \frac{n^{\tau_{min}} + Jn^{\tau_{max}+2\tau_{min}}}{2}\|\epsilon\|_T^2 + n^{\tau_{max}+2\tau_{min}}\sum_{j=1}^J P\left(f_j^*\right)$$

Then by the MVT, we have

$$\begin{aligned}
\|\hat{f}_\ell(\cdot|\lambda^{(1)}) - \hat{f}_\ell(\cdot|\lambda^{(2)})\|_\infty &= \|\hat{m}_\ell(\lambda, h)h_\ell\|_\infty \\
&\leq G\left\|\lambda^{(1)} - \lambda^{(2)}\right\| \|\nabla_\lambda \hat{m}_\ell(\lambda, h)\| \\
&\leq G\left\|\lambda^{(1)} - \lambda^{(2)}\right\| \frac{n^{\tau_{min}}}{2}\sqrt{\frac{n^{\tau_{min}} + Jn^{\tau_{max}+2\tau_{min}}}{2}\|\epsilon\|_T^2 + n^{\tau_{max}+2\tau_{min}}\sum_{j=1}^J P\left(f_j^*\right)}
\end{aligned}$$

A second approach:
By the KKT conditions, we also know that

$$\begin{aligned}
\left|\frac{\partial}{\partial m_\ell} P\left(\hat{f}_\ell(\cdot|\lambda^{(1)}) + \hat{m}_\ell(\lambda)h_\ell\right)\right| &= \frac{1}{\lambda_\ell}\left|\left\langle h_\ell, y - \left(\sum_{j=1}^J \hat{g}_j(\cdot|\lambda^{(1)}) + \hat{f}_j(\cdot|\lambda^{(1)}) + \hat{m}_j(\lambda, h)h_j\right)\right\rangle_T\right| \\
&\leq \frac{1}{\lambda_{min}}\|h_\ell\|_T\left\|y - \left(\sum_{j=1}^J \hat{g}_j(\cdot|\lambda^{(1)}) + \hat{f}_j(\cdot|\lambda^{(1)}) + \hat{m}_j(\lambda, h)h_j\right)\right\|_T \\
&\leq G(2G + \|\epsilon\|_T)n^{\tau_{min}}
\end{aligned}$$

Hence

$$\|\nabla_\lambda \hat{m}_\ell(\lambda, h)\| \leq G(2G + \|\epsilon\|_T)n^{2\tau_{min}}\frac{1}{2}$$

Then by the MVT, we have

$$\begin{aligned}
\|\hat{f}_\ell(\cdot|\lambda^{(1)}) - \hat{f}_\ell(\cdot|\lambda^{(2)})\|_\infty &= \|\hat{m}_\ell(\lambda, h)h_\ell\|_\infty \\
&\leq G\left\|\lambda^{(1)} - \lambda^{(2)}\right\| \|\nabla_\lambda \hat{m}_\ell(\lambda, h)\| \\
&\leq \left\|\lambda^{(1)} - \lambda^{(2)}\right\| G^2(2G + \|\epsilon\|_T)n^{3\tau_{min}}\frac{1}{4}
\end{aligned}$$

Lemma: Sobolev Facts

For any function h, we have

$$\begin{aligned}\left|\frac{\partial}{\partial m}P(g+mh)\right| &= \left|2\int (g^{(r)}(x)+mh^{(r)}(x))h^{(r)}(x)dx\right| \\ &\leq 2\sqrt{P(g+mh)P(h)}\end{aligned}$$

and

$$\frac{\partial^2}{\partial m^2}P(g+mh) = 2\int (h^{(r)}(x))^2 dx = 2P(h)$$