

1 Simple model

Definitions

We find the best model for y over function class \mathcal{G} . Presume $g^* \in \mathcal{G}$ is the true model and

$$y = g^*(X) + \epsilon$$

where ϵ are sub-Gaussian errors for constants K and σ_0^2

$$\max_{i=1:n} K^2 (E [\exp(|\epsilon_i|^2 K^2) - 1]) \leq \sigma_0^2$$

Given a training set T , We define the fitted models

$$\hat{g}_\lambda = \|y - g\|_T^2 + \lambda^2 I^v(g)$$

Given a validation set V , let the CV-fitted model be

$$\hat{g}_{\hat{\lambda}} = \arg \min_{\lambda} \|y - \hat{g}_\lambda\|_V^2$$

We will suppose $I(g^*) > 0$.

Assumptions

Suppose the entropy of the class \mathcal{G}' is

$$H \left(\delta, \mathcal{G}' = \left\{ \frac{g - g^*}{I(g) + I(g^*)} : g \in \mathcal{G}, I(g) + I(g^*) > 0 \right\}, P_T \right) \leq \tilde{A} \delta^{-\alpha} \quad (1)$$

Suppose $v > 2\alpha/(2 + \alpha)$.

Suppose for all $\lambda \in \Lambda$, $I^v(\hat{g}_\lambda)$ is upper bounded by $\|\hat{g}_\lambda\|_n^2 = \frac{1}{n} \sum_{i=1}^n \hat{g}_\lambda(x_i)$. See Lemma 1 below for the specific assumption. This assumption includes Ridge, Lasso, Generalized Lasso, and the Group Lasso.

Result 1: Single λ , Single Penalty, cross-validation over general X_T, X_V

Suppose that the training and validation set are independently sampled, so the values X_i are not necessarily the same. Suppose the training and validation sets are both of size n . Suppose X is bounded s.t. $|X| \leq R_X$ and the domain of $g \in \mathcal{G}$ is over $(-R_X, R_X)$.

Suppose the same entropy bound (2) for both the training set P_T and validation set P_V .

Suppose for all $\lambda \in \Lambda$, $I^v(\hat{g}_\lambda)$ is upper bounded by its L_2 -norm with some constant M and M_0 such that

$$I^v(\hat{g}_\lambda) \leq M \|\hat{g}_\lambda\|_n^2 + M_0$$

Suppose the entropy bound for both training set P_T and validation set P_V .

Suppose that

$$\sup_{g \in \mathcal{G}} \frac{\|g - g^*\|_\infty}{I(g) + I(g^*)} \leq K < \infty$$

Let $\tilde{\lambda}$ be the optimal λ by Vandegeer. Then

$$\|\hat{g}_{\tilde{\lambda}} - \hat{g}_{\hat{\lambda}}\|_V = O_p \left(n^{-1/(2+\alpha)} \right) \left(I^{\alpha/(2+\alpha)}(g^*) + I(g^*) \right)$$

and $\|\hat{g}_{\bar{\lambda}} - g^*\|_V$ is of the same order (differs by some constant).

Proof:

By the triangle inequality,

$$\|\hat{g}_{\bar{\lambda}} - g^*\|_V \leq \|\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}\|_V + \|\hat{g}_{\hat{\lambda}} - g^*\|_V$$

We bound each component on the RHS separately.

First bound $\|\hat{g}_{\bar{\lambda}} - g^*\|_V$. By Vandegeer Thrm 10.2 and Lemma 2,

$$\begin{aligned} \|\hat{g}_{\bar{\lambda}} - g^*\|_V &\leq \|\hat{g}_{\bar{\lambda}} - g^*\|_T + \left| \|\hat{g}_{\bar{\lambda}} - g^*\|_V - \|\hat{g}_{\bar{\lambda}} - g^*\|_T \right| \\ &\leq O_p\left(n^{-1/(2+\alpha)}\right) I^{\alpha/(2+\alpha)}(g^*) + O_p\left(n^{-1/(2+\alpha)}\right) (I(g^*) + I(\hat{g}_{\bar{\lambda}})) \\ &\leq O_p\left(n^{-1/(2+\alpha)}\right) \left(I^{\alpha/(2+\alpha)}(g^*) + I(g^*)\right) \end{aligned}$$

Next bound $\|\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}\|_V$. The basic inequality gives us

$$\|\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}\|_V^2 \leq 2 \left| \langle \epsilon, \hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}} \rangle_V \right| + 2 \left| \langle g^* - \hat{g}_{\bar{\lambda}}, \hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}} \rangle_V \right|$$

Case a: $\left| \langle \epsilon, \hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}} \rangle_T \right|$ is the bigger term on the RHS

By Vandegeer (10.6),

$$\|\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}\|_V^2 \leq O_P(n^{-1/2}) \|\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}\|^{1-\alpha/2} (I(\hat{g}_{\bar{\lambda}}) + I(\hat{g}_{\hat{\lambda}}))^{\alpha/2}$$

If $I(\hat{g}_{\bar{\lambda}}) > I(\hat{g}_{\hat{\lambda}})$, then

$$\|\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}\|_V \leq O_P(n^{-1/(2+\alpha)}) I(g^*)^{\alpha/(2+\alpha)}$$

Otherwise, suppose $I(\hat{g}_{\bar{\lambda}}) < I(\hat{g}_{\hat{\lambda}})$. Since I is a pseudo-norm,

$$\begin{aligned} \|\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}\|_V &\leq O_P(n^{-1/(2+\alpha)}) I(\hat{g}_{\hat{\lambda}})^{\alpha/(2+\alpha)} \\ &\leq O_P(n^{-1/(2+\alpha)}) (I(\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}) + I(\hat{g}_{\hat{\lambda}}))^{\alpha/(2+\alpha)} \end{aligned}$$

If $I(\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}) \leq I(\hat{g}_{\hat{\lambda}})$, then we're done. Otherwise if $I(\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}) \geq I(\hat{g}_{\hat{\lambda}})$, by the assumption that $I^V(\cdot)$ is bounded by the L2 norm,

$$\|\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}\|_V \leq O_P(n^{-1/(2+\alpha)}) (M \|\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}\|_V^2 + M_0)^{\alpha/v(2+\alpha)}$$

If M_0 is bigger, we're done. Otherwise,

$$\|\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}\|_V \leq O_P(n^{-v/(2v-2\alpha+\alpha v)}) < O_P(n^{-1/(2+\alpha)})$$

Case b: $\left| \langle g^* - \hat{g}_{\bar{\lambda}}, \hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}} \rangle_V \right|$ is the bigger term on the RHS

By Cauchy Schwarz,

$$\|\hat{g}_{\bar{\lambda}} - \hat{g}_{\hat{\lambda}}\|_V \leq O_P(1) \|g^* - \hat{g}_{\bar{\lambda}}\|_V$$

2 General Additive Model

Definitions

We find the best model for y over function classes $\mathcal{G} = \left\{ \sum_{j=1}^J g_j : g_j \in \mathcal{G}_j \right\}$. Suppose we observe:

$$y = \sum_{j=1}^J g_j^* + \epsilon$$

where $\sum_{j=1}^J g_j^* \in \mathcal{G}$. Suppose ϵ are sub-Gaussian errors for constants K and σ_0^2 :

$$\max_{i=1:n} K^2 (E [\exp(|\epsilon_i|^2 K^2) - 1]) \leq \sigma_0^2$$

Given a training set T , we fit models by least squares with multiple penalties

$$\{\hat{g}_{\lambda,j}\}_{j=1}^J = \arg \min_{\sum g_j \in \mathcal{G}} \|y - \sum_{j=1}^J g_j\|_T^2 + \lambda^2 \sum_{j=1}^J I_j^{v_j}(g_j)$$

Given a validation set V , let the CV-fitted model be

$$\{\hat{g}_{\hat{\lambda},j}\}_{j=1}^J = \arg \min_{\lambda} \|y - \sum_{j=1}^J \hat{g}_{\lambda,j}\|_V^2$$

Reasonable assumption:

- The entropy bound (2) in result 2 comes from the assumptions in Lemma 3. The α below is $\alpha = \max_{j=1:J} \{\alpha_j\}$, so convergence is only as fast as fitting the highest-entropy function class. The constant A must be appropriately inflated such that the entropy bound holds for all $\delta \in (0, R]$.

“Special” assumptions:

- We assume exponents $v_j = 1$, whereas Vandegeer Thrm 10.2 only assumes $v > 2\alpha/(2 + \alpha)$. Without this assumption, I wasn’t able to form inequalities between $\sum_{j=1}^J I_j(g_j) \leq \text{something} + \sum_{j=1}^J I_j^{v_j}(g_j)$. Indeed, Remark 1 in “High-dimensional Additive Modeling” (Vandegeer 2009) notes the importance of using the semi-norm instead of the square of the semi-norm.
- We suppose the following incoherence condition, in the spirit of Vandegeer 2014 “The additive model with different smoothness for the components”: Let $p_V(\vec{x})$ be the empirical density over the validation set. Let p_{Vj} be the marginal density of x_j for the empirical distribution of the validation set. Let

$$r_V(\vec{x}) = \frac{p_V(\vec{x})}{\prod_{j=1}^J p_{Vj}(x_j)}, \quad \gamma_V^2 = \int r_V(\vec{x}) \prod_{j=1}^J p_{Vj}(x_j) d\mu$$

Suppose that $\gamma_V < 1/(J - 1)$. Furthermore, we will suppose that $\int g_j p_{Vj} d\mu = 0$ for $j = 2, \dots, J$.

Result 2: Additive Model with multiple penalties, Single oracle λ over X_T

Suppose there is some $0 < \alpha < 2$ s.t. for all $\delta \in (0, R]$,

$$H \left(\delta, \left\{ \frac{\sum_{j=1}^J g_j - g_j^*}{\sum_{j=1}^J I_j(g_j) + I_j(g_j^*)} : g_j \in \mathcal{G}_j, \sum_{j=1}^J I_j(g_j) + I_j(g_j^*) > 0 \right\}, \|\cdot\|_T \right) \leq A\delta^{-\alpha} \quad (2)$$

If λ is chosen s.t.

$$\tilde{\lambda}_T^{-1} = O_p \left(n^{1/(2+\alpha)} \right) \left(\sum_{j=1}^J I_j(g_j^*) \right)^{(2-\alpha)/2(2+\alpha)}$$

then

$$\left\| \sum_{j=1}^J \hat{g}_j - g_j^* \right\|_T = O_p \left(\tilde{\lambda}_T \right) \left(\sum_{j=1}^J I_j(g_j^*) \right)^{1/2}$$

and

$$\sum_{j=1}^J I_j(\hat{g}_j) = O_p(1) \sum_{j=1}^J I_j(g_j^*)$$

Proof:

The basic inequality gives us:

$$\left\| \sum_{j=1}^J \hat{g}_j - g_j^* \right\|_T^2 + \lambda^2 \sum_{j=1}^J I_j(\hat{g}_j) \leq 2 \left| \left(\epsilon_T, \sum_{j=1}^J \hat{g}_j - g_j^* \right) \right| + \lambda^2 \sum_{j=1}^J I_j(g_j^*)$$

Case 1: $\left| \left(\epsilon_T, \sum_{j=1}^J \hat{g}_j - g_j^* \right) \right| \leq \lambda^2 \sum_{j=1}^J I_j(g_j^*)$

$$\left\| \sum_{j=1}^J \hat{g}_j - g_j^* \right\|_T \leq O_p(\lambda) \left(\sum_{j=1}^J I_j(g_j^*) \right)^{1/2}$$

Case 2: $\left| \left(\epsilon_T, \sum_{j=1}^J \hat{g}_j - g_j^* \right) \right| \geq \lambda^2 \sum_{j=1}^J I_j(g_j^*)$

By Vandegeer (10.6), the basic inequality becomes

$$\left\| \sum_{j=1}^J \hat{g}_j - g_j^* \right\|_T^2 + \lambda^2 \sum_{j=1}^J I_j(\hat{g}_j) \leq O_p \left(n^{-1/2} \right) \left\| \sum_{j=1}^J \hat{g}_j - g_j^* \right\|_T^{1-\alpha/2} \left(\sum_{j=1}^J I_j(\hat{g}_j) + I_j(g_j^*) \right)^{\alpha/2}$$

Case 2a: $\sum_{j=1}^J I_j(\hat{g}_j) \leq \sum_{j=1}^J I_j(g_j^*)$

Then

$$\left\| \sum_{j=1}^J \hat{g}_j - g_j^* \right\|_T \leq O_p \left(n^{-1/(2+\alpha)} \right) \left(\sum_{j=1}^J I_j(g_j^*) \right)^{\alpha/(2+\alpha)}$$

Case 2b: $\sum_{j=1}^J I_j(\hat{g}_j) \geq \sum_{j=1}^J I_j(g_j^*)$

Then

$$\sum_{j=1}^J I_j(\hat{g}_j) \leq O_p \left(n^{-1/(2-\alpha)} \right) \lambda^{-4/(2-\alpha)} \left\| \sum_{j=1}^J \hat{g}_j - g_j^* \right\|_T$$

Hence

$$\left\| \sum_{j=1}^J \hat{g}_j - g_j^* \right\|_T \leq O_p \left(n^{-1/(2-\alpha)} \right) \lambda^{-2\alpha/(2-\alpha)}$$

Result 3: Additive Model with multiple penalties, Single cross-validation λ over general X_T, X_V

Suppose that the training and validation set are independently sampled, so the values X_i are not necessarily the same. Suppose the training and validation sets are both of size n . Suppose X is bounded s.t. $|X| \leq R_X$ and the domain of $g \in \mathcal{G}$ is over $(-R_X, R_X)$.

Suppose the same entropy bound (2) for both the training set P_T and validation set P_V .

In addition to the assumptions in Result 4, suppose the infinity norm is also bounded

$$\sup_{g_j \in \mathcal{G}_j} \frac{\|\sum_{j=1}^J g_j - g_j^*\|_\infty}{\sum_{j=1}^J I_j(g_j) + I_j(g_j^*)} \leq K < \infty$$

Suppose there exist constants M, M_0 s.t. for all j and all $\lambda \in \Lambda$

$$I_j(\hat{g}_{\lambda,j}) \leq M \|\hat{g}_{\lambda,j}\|_V^2 + M_0$$

Special assumption: Suppose the incoherence condition $\gamma_V < 1/(J-1)$. We will also suppose $\int g_j p_{V,j} d\mu = 0$ for $j = 2, \dots, J$.

Let $\hat{\lambda}$ be the optimal λ as specified in Result 2. Then

$$\left\| \sum_{j=1}^J \hat{g}_{\hat{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right\|_V = O_p \left(n^{-1/(2+\alpha)} \right) (1 - \gamma(J-1))^{\alpha/(2+\alpha)} \left(\left(\sum_{j=1}^J I_j(g_j^*) \right)^{\alpha/(2+\alpha)} + \sum_{j=1}^J I_j(g_j^*) + \left\| \sum_{j=1}^J g_j^* \right\|_V^{\alpha/2(2+\alpha)} \right)$$

and $\left\| \sum_{j=1}^J g_j^* - \hat{g}_{\hat{\lambda},j} \right\|_V$ is on the same order (differs by a constant).

Proof:

By the triangle inequality,

$$\left\| \sum_{j=1}^J g_j^* - \hat{g}_{\hat{\lambda},j} \right\|_V \leq \left\| \sum_{j=1}^J \hat{g}_{\hat{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right\|_V + \left\| \sum_{j=1}^J g_j^* - \hat{g}_{\hat{\lambda},j} \right\|_V$$

By Lemma 2 and Result 2, we can easily bound $\left\| \sum_{j=1}^J g_j^* - \hat{g}_{\hat{\lambda},j} \right\|_V$.

$$\begin{aligned} \left\| \sum_{j=1}^J g_j^* - \hat{g}_{\hat{\lambda},j} \right\|_V &\leq \left\| \sum_{j=1}^J g_j^* - \hat{g}_{\hat{\lambda},j} \right\|_T + \left| \left\| \sum_{j=1}^J g_j^* - \hat{g}_{\hat{\lambda},j} \right\|_T - \left\| \sum_{j=1}^J g_j^* - \hat{g}_{\hat{\lambda},j} \right\|_V \right| \\ &\leq O_p \left(n^{-1/(2+\alpha)} \right) \left(\left(\sum_{j=1}^J I_j(g_j^*) \right)^{\alpha/(2+\alpha)} + \sum_{j=1}^J I_j(g_j^*) \right) \end{aligned}$$

Next bound $\left\| \sum_{j=1}^J \hat{g}_{\hat{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right\|_V$. By definition of $\hat{\lambda}$, we have the basic inequality

$$\left\| \sum_{j=1}^J \hat{g}_{\hat{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right\|_V^2 \leq 2 \left| \left(\epsilon, \sum_{j=1}^J \hat{g}_{\hat{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right)_V \right| + 2 \left| \left(\sum_{j=1}^J g_j^* - \hat{g}_{\hat{\lambda},j}, \sum_{j=1}^J \hat{g}_{\hat{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right)_V \right|$$

Case 1: $\left| \left(\epsilon, \sum_{j=1}^J \hat{g}_{\hat{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right)_V \right|$ is bigger

By Vandegeer (10.6),

$$\left\| \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right\|_V^{1+\alpha/2} \leq O_p(n^{-1/2}) \left(\sum_{j=1}^J I_j(\hat{g}_{\tilde{\lambda},j}) + I_j(\hat{g}_{\hat{\lambda},j}) \right)^{\alpha/2}$$

If $\sum_{j=1}^J I_j(\hat{g}_{\tilde{\lambda},j}) \geq \sum_{j=1}^J I_j(\hat{g}_{\hat{\lambda},j})$, we're done.

Otherwise, suppose $\sum_{j=1}^J I_j(\hat{g}_{\tilde{\lambda},j}) < \sum_{j=1}^J I_j(\hat{g}_{\hat{\lambda},j})$.

By the incoherence assumption, we can apply Lemma 4

$$\begin{aligned} \sum_{j=1}^J I_j(\hat{g}_{\tilde{\lambda},j}) &\leq M \sum_{j=1}^J \|\hat{g}_{\tilde{\lambda},j}\|_V^2 + M_0 J \\ &\leq M(1 - \gamma(J-1)) \left\| \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} \right\|_V^2 + M_0 J \end{aligned}$$

Then

$$\left\| \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right\|_V^{1+\alpha/2} \leq O_p(n^{-1/2}) \left(M(1 - \gamma(J-1)) \left\| \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} \right\|_V^2 + M_0 J \right)^{\alpha/2}$$

If $M_0 J$ is the biggest, we're done. Otherwise,

$$\begin{aligned} \left\| \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right\|_V^{1+\alpha/2} &\leq O_p(n^{-1/2}) (1 - \gamma(J-1))^{\alpha/2} \left\| \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} \right\|_V^\alpha \\ &\leq O_p(n^{-1/2}) (1 - \gamma(J-1))^{\alpha/2} \left(\left\| \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right\|_V + \left\| \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} - g_j^* \right\|_V + \left\| \sum_{j=1}^J g_j^* \right\|_V \right)^\alpha \end{aligned}$$

If $\left\| \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right\|_V$ or $\left\| \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} - g_j^* \right\|_V$ is the biggest on the RHS, then the rate is faster than $O_p(n^{-1/(2+\alpha)})$. If $\left\| \sum_{j=1}^J g_j^* \right\|_V$ is the biggest, then

$$\left\| \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right\|_V \leq O_p(n^{-1/(2+\alpha)}) \left\| \sum_{j=1}^J g_j^* \right\|_V^{\alpha/2(2+\alpha)}$$

Case 2: $\left| \left(\sum_{j=1}^J g_j^* - \hat{g}_{\hat{\lambda},j}, \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right)_V \right|$ is bigger

By Cauchy Schwarz,

$$\left\| \sum_{j=1}^J \hat{g}_{\tilde{\lambda},j} - \hat{g}_{\hat{\lambda},j} \right\|_V \leq O_p(1) \left\| \sum_{j=1}^J g_j^* - \hat{g}_{\hat{\lambda},j} \right\|_V$$

3 General Additive Model: Multiple Lambdas

Definitions

We find the best model for y over function classes $\mathcal{G} = \left\{ \sum_{j=1}^J g_j : g_j \in \mathcal{G}_j \right\}$. Suppose we observe:

$$y = \sum_{j=1}^J g_j^* + \epsilon$$

where $\sum_{j=1}^J g_j^* \in \mathcal{G}$. Suppose ϵ are sub-Gaussian errors for constants K and σ_0^2 :

$$\max_{i=1:n} K^2 \left(E \left[\exp(|\epsilon_i|^2 K^2) - 1 \right] \right) \leq \sigma_0^2$$

Given a training set T , we fit models by least squares with multiple penalties and tuning parameters

$$\{\hat{g}_{\lambda,j}\}_{j=1}^J = \arg \min_{\sum g_j \in \mathcal{G}} \|y - \sum_{j=1}^J g_j\|_T^2 + \sum_{j=1}^J \lambda_j^2 I_j^{v_j}(g_j)$$

Suppose $1 \leq v_j \leq 2$.

Given a validation set V , let the CV-fitted model be

$$\{\hat{g}_{\hat{\lambda},j}\}_{j=1}^J = \arg \min_{\lambda} \|y - \sum_{j=1}^J \hat{g}_{\lambda,j}\|_V^2$$

Result 4: Additive Model, Oracle $\{\lambda_i\}$ given X_T

These results are implied by Vandegeer's paper "The additive model with different smoothness for the components."

Suppose for all $j = 1 : J$

$$\mathcal{H} \left(\delta, \left\{ \frac{g_j - g_j^*}{I(g_j) + I(g_j^*)} \right\}, \|\cdot\|_n \right) \leq A_j \delta^{-\alpha_j} \forall \delta > 0$$

Let

$$\lambda_j = O_p(n^{-1/(2+\alpha_j)})$$

and

$$\left(\sum_{j=1}^J I_j^{q_j}(g_j^*) \right)^{1/2} \lambda_{\max} = O_P(1)R$$

There are some constants c_1, c_2 s.t. for $\lambda_j = O_p(n^{-1/(2+\alpha_j)})$, we have

$$\left\| \sum g_j^* - \sum \hat{g}_{\hat{\lambda},j} \right\| \leq c_2 \lambda_{(j)}$$

where $(j) = \arg \max \alpha_j$. That is, the convergence rate depends on the highest-entropy function class (with respect to the penalty)

$$\left\| \sum_{j=1}^J g_j^* - \sum_{j=1}^J \hat{g}_j \right\|_T = O_p(n^{-1/(2+\alpha_{(j)})})$$

Jean's version of the Proof for Vandegeer Thrm 3.1:

Suppose for some constant R , we define the function class

$$\mathcal{M}(R) = \left\{ \{g_j\} : (\lambda_j/R)^{(1-q_j)/q_j} \lambda_j I_j(g_j - g_j^*) \leq R, \left\| \sum_{j=1}^J g_j - g_j^* \right\|_T \leq R \right\}$$

Recall that

$$\sup_{g_j \in \mathcal{G}_j} \frac{|(\epsilon^T, g_j - g_j^*)|}{(I_j(g_j) + I_j(g_j^*))^{\alpha_j/2} \|g_j - g_j^*\|^{1-\alpha_j/2}} = O_p(n^{-1/2})$$

By our choice of λ , we have that for function sets $\{g_j - g_j^*\} \in \mathcal{M}(R)$, the empirical process term decreases with n :

$$\begin{aligned} |(\epsilon^T, g_j - g_j^*)| &\leq O_P(n^{-1/2}) (I_j(g_j) + I_j(g_j^*))^{\alpha_j/2} \|g_j - g_j^*\|^{1-\alpha_j/2} \\ &\leq O_P(n^{-1/2}) \left(\lambda_j^{-1/q_j} R^{1/q_j} \right)^{\alpha_j/2} R^{1-\alpha_j/2} \\ &\leq O_P(n^{-1/(2+\alpha_j)}) R^2 \end{aligned}$$

Hence for sufficiently large n , Vandegeer Lemma's 5.4 (Jean's version below) states that the fitted functions \hat{g}_j are also within R of the truth:

$$\{\hat{g}_j - g_j^*\} \in \mathcal{M}(R) \implies \left\| \sum_{j=1}^J g_j^* - \hat{g}_j \right\|_T \leq R$$

Now we just need to determine the right value for R . Choose n sufficiently large s.t. the penalty term for function (j) is the highest (for the truth)

$$\lambda_j^2 I_j^{q_j}(g_j^*) \leq \lambda_{(j)}^2 I_{(j)}^{q_{(j)}}(g_{(j)}^*) \quad \forall j$$

Then choose R s.t.

$$\left(\lambda_{(j)}^2 I_{(j)}^{q_{(j)}}(g_{(j)}^*) \right)^{1/2} J^{1/2} = O_P(1) R$$

Hence

$$\left\| \sum_{j=1}^J g_j^* - \hat{g}_j \right\|_T \leq n^{-1/(2+\alpha_{(j)})} J^{1/2} I_{(j)}^{q_{(j)}/2}(g_{(j)}^*)$$

Result 5: Additive Model, Cross-validated $\{\lambda_i\}$ over general X_T, X_V

Assume the same conditions as result 4, but also for the validation set.

Condition 2.4: Incoherence condition on the validation set. Let $p_V(\vec{x})$ be the empirical density over the validation set. Let p_{Vj} be the marginal density of x_j for the empirical distribution of the validation set. Let

$$r_V(\vec{x}) = \frac{p_V(\vec{x})}{\prod_{j=1}^J p_{Vj}(x_j)}, \quad \gamma_V^2 = \int r_V(\vec{x}) \prod_{j=1}^J p_{Vj}(x_j) d\mu$$

Suppose that $\gamma_V < 1/(J-1)$. Furthermore, we will suppose that $\int g_j p_{Vj} d\mu = 0$ for $j = 2, \dots, J$.

Additionally, suppose there exist constants M, M_0 s.t. for all j and all $\lambda \in \Lambda$

$$I_j(\hat{g}_{\lambda,j}) \leq M \|\hat{g}_{\lambda,j}\|_V^2 + M_0$$

Let $\tilde{\lambda}$ be the optimal $\{\lambda_i\}$ as specified in Result 4. Then

$$\left\| \sum_{j=1}^J \hat{g}_{\hat{\lambda},j} - \hat{g}_{\tilde{\lambda},j} \right\|_V = O_p \left(n^{-1/(2+\alpha_{(j)})} \right) (1 - \gamma(J-1))^{\alpha_{(j)}/(2+\alpha_{(j)})} \left(\left(\sum_{j=1}^J I_j(g_j^*) \right)^{\alpha_{(j)}/(2+\alpha_{(j)})} + \sum_{j=1}^J I_j(g_j^*) + \left\| \sum_{j=1}^J g_j^* \right\|_V^{\alpha_{(j)}/2} \right)$$

and $\left\| \sum_{j=1}^J g_j^* - \hat{g}_{\hat{\lambda},j} \right\|_V = O_p(1) \left\| \sum_{j=1}^J g_j^* - \hat{g}_{\tilde{\lambda},j} \right\|_T$.

Proof:

Exactly the same as Result 3

Lemmas

Lemma 1:

Suppose for all $\lambda \in \Lambda$, the penalty function $I^v(g_\lambda)$ is upper-bounded by $\|g_\lambda\|_n^2 = \frac{1}{n} \sum_{i=1}^n g_\lambda^2(x_i)$ with constants M_0 and M :

$$I^v(g_\lambda) \leq M\|g_\lambda\|_n^2 + M_0$$

Suppose there is some function $g \in \mathcal{G}$ such that

$$\|g - g_\lambda\|_n^{1+\alpha/2} \leq O_p(n^{-1/2})I^{\alpha/2}(g_\lambda)$$

Then

$$\|g - g_\lambda\|_n \leq O_p(n^{-1/(2+\alpha)})M^{\alpha v/(2+\alpha)}\|g\|_n^{2\alpha/v(2+\alpha)}$$

Proof:

From the assumptions, we have

$$\|g - g_\lambda\|_n^{1+\alpha/2} \leq O_p(n^{-1/2}) (M\|g_\lambda\|_n^2 + M_0)^{\alpha/2v}$$

If $M_0 > \|g_\lambda\|_n^2$, we're done. Otherwise,

$$\begin{aligned} \|g - g_\lambda\|_n^{1+\alpha/2} &\leq O_p(n^{-1/2})M^{\alpha/2v}\|g_\lambda\|_n^{\alpha/v} \\ &\leq O_p(n^{-1/2})M^{\alpha/2v}(\|g_\lambda - g\|_n + \|g\|_n)^{\alpha/v} \end{aligned}$$

Case 1: $\|g_\lambda - g\|_n \geq \|g\|_n$

Then

$$\|g - g_\lambda\|_n \leq O_p(n^{-v/(2v+\alpha v-2\alpha)})M^{\alpha v^2/(2v+\alpha v-2\alpha)}$$

Note that $\sup_v -\frac{v}{2v+\alpha v-2\alpha} = -\frac{1}{2+\alpha}$, so this rate is faster than $O_p(n^{-\frac{1}{2+\alpha}})$.

Case 2: $\|g_\lambda - g\|_n \leq \|g\|_n$

Then

$$\|g - g_\lambda\|_n \leq O_p(n^{-1/(2+\alpha)})M^{\alpha v/(2+\alpha)}\|g\|_n^{2\alpha/v(2+\alpha)}$$

I believe we can often provide a good estimate of M for the entire class \mathcal{G} , which means that we can always estimate the sample size needed to ensure this case never occurs. That is, I believe we can often estimate M s.t.

$$I^v(g) \leq M\|g\|_n^2 + M_0 \forall g \in \mathcal{G}$$

Lemma 2:

Let $P_{n'}$ and $P_{n''}$ be empirical distributions over $\{X_i'\}_{i=1}^n, \{X_i''\}_{i=1}^n$. Let $P_{2n} = \frac{1}{2}(P_{n'} + P_{n''})$. Suppose X is bounded s.t. $|X| < R_X$.

Let $\mathcal{G}' = \left\{ \frac{g-g^*}{I(g)+I(g^*)} : g \in \mathcal{G}, I(g) + I(g^*) > 0 \right\}$. Suppose g is defined over the domain over X (and zero otherwise). Suppose

$$\sup_{f \in \mathcal{G}'} \|f\|_{P_{2n}} \leq R < \infty, \quad \sup_{f \in \mathcal{G}'} \|f\|_\infty \leq K < \infty$$

and

$$H(\delta, \mathcal{G}', P_{n'}) \leq \tilde{A}\delta^{-\alpha}, \quad H(\delta, \mathcal{G}', P_{n''}) \leq \tilde{A}\delta^{-\alpha}$$

Then

$$Pr \left(\sup_{g \in \mathcal{G}} \frac{|\|g^* - g\|_{P_{n'}} - \|g^* - g\|_{P_{n''}}|}{I(g^*) + I(g)} \geq 6\delta \right) \leq 2 \exp \left(2\tilde{A}\delta^{-\alpha} - \frac{4\delta^2 n}{K^2} \right)$$

Proof: The proof is very similar to that in Pollard 1984 (page 32), so some details below are omitted. First note that for any function f and h , we have

$$\|f\|_{P_{n'}} - \|h\|_{P_{n'}} \leq \|f - h\|_{P_{n'}} \leq \sqrt{2}\|f - h\|_{P_{2n}}$$

Similarly for $P_{n''}$.

Let $\{h_j\}_{j=1}^N$ be the $\sqrt{2}\delta$ -cover for \mathcal{G}' (where $N = N(\sqrt{2}\delta, \mathcal{G}', P_{2n})$). Let h_j be the closest function (in terms of $\|\cdot\|_{P_{2n}}$) to some $f \in \mathcal{G}'$. Then

$$\begin{aligned} \|f\|_{P_{n'}} - \|f\|_{P_{n''}} &\leq \|f - h_j\|_{P_{n'}} + |\|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}}| + \|f - h_j\|_{P_{n''}} \\ &\leq 4\delta + |\|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}}| \end{aligned}$$

Therefore for $f = \frac{g^* - g}{I(g^*) + I(g)}$, we have

$$\begin{aligned} Pr \left(\sup_{g \in \mathcal{G}} \frac{|\|g^* - g\|_{P_n} - \|g^* - g\|_{P_{n''}}|}{I(g^*) + I(g)} \geq 6\delta \right) &\leq Pr \left(\sup_{j \in 1:N} |\|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}}| \geq 2\delta \right) \\ &\leq N \max_{j \in 1:N} Pr \left(|\|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}}| \geq 2\delta \right) \end{aligned}$$

Now note that

$$\begin{aligned} |\|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}}| &= \frac{|\|h_j\|_{P_{n'}}^2 - \|h_j\|_{P_{n''}}^2|}{\|h_j\|_{P_{n'}} + \|h_j\|_{P_{n''}}} \\ &\leq \frac{|\|h_j\|_{P_{n'}}^2 - \|h_j\|_{P_{n''}}^2|}{\sqrt{2}\|h_j\|_{P_{2n}}} \end{aligned}$$

By Hoeffding's inequality,

$$\begin{aligned} Pr \left(|\|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}}| \geq 2\delta \right) &\leq Pr \left(|\|h_j\|_{P_{n'}}^2 - \|h_j\|_{P_{n''}}^2| \geq 2\sqrt{2}\delta\|h_j\|_{P_{2n}} \right) \\ &= Pr \left(\left| \sum_{i=1}^n W_i (h_j^2(x'_i) - h_j^2(x''_i)) \right| \geq 2\sqrt{2}n\delta\|h_j\|_{P_{2n}} \right) \\ &\leq 2 \exp \left(- \frac{16\delta^2 n^2 \|h_j\|_{P_{2n}}^2}{4 \sum_{i=1}^n (h_j^2(x'_i) - h_j^2(x''_i))^2} \right) \end{aligned}$$

Since $\|h_j\|_\infty < K$, then

$$\begin{aligned} \sum_{i=1}^n (h_j^2(x'_i) - h_j^2(x''_i))^2 &\leq \sum_{i=1}^n h_j^4(x'_i) + h_j^4(x''_i) \\ &\leq nK^2 \|h_j\|_{P_{2n}}^2 \end{aligned}$$

Hence

$$Pr \left(|\|h_j\|_{P_{n'}} - \|h_j\|_{P_{n''}}| \geq 2\delta \right) \leq 2 \exp \left(- \frac{4\delta^2 n}{K^2} \right)$$

Since (Pollard and Vandegeer say that)

$$N(\sqrt{2}\delta, \mathcal{G}', P_{2n}) \leq N(\delta, \mathcal{G}', P_{n''}) + N(\delta, \mathcal{G}', P_{n''})$$

then

$$Pr \left(\sup_{g \in \mathcal{G}} \frac{|\|g^* - g\|_{P_n} - \|g^* - g\|_{P_{n''}}|}{I(g^*) + I(g)} \geq 6\delta \right) \leq 2 \exp \left(2\tilde{A}\delta^{-\alpha} - \frac{4\delta^2 n}{K^2} \right)$$

Using shorthand, we can write

$$\sup_{g \in \mathcal{G}} \frac{|\|g^* - g\|_{P_n} - \|g^* - g\|_{P_{n''}}|}{I(g^*) + I(g)} = O_p(n^{-1/(2+\alpha)})$$

Lemma 3:

Suppose the function classes \mathcal{F}_j is a cone and $I_j : \mathcal{F}_j \mapsto [0, \infty)$ is a psuedonorm. Furthermore, suppose

$$H(\delta, \{f_j \in \mathcal{F}_j : I_j(f_j) \leq 1\}, \|\cdot\|_n) \leq A_j \delta^{-\alpha_j}$$

Then if $f_j^* \in \mathcal{F}_j$, then

$$H \left(\delta, \left\{ \frac{\sum_{j=1}^J f_j - f_j^*}{\sum_{j=1}^J I_j(f_j) + I_j(f_j^*)} : f_j \in \mathcal{F}_j, I_j(f_j) + I_j(f_j^*) > 0 \right\}, \|\cdot\|_n \right) \leq 2 \sum_{j=1}^J A_j \left(\frac{\delta}{2J} \right)^{-\alpha_j}$$

Proof: Let $\tilde{f}_j = \frac{f_j}{\sum_{j=1}^J I_j(f_j) + I_j(f_j^*)}$. Then $\tilde{f}_j \in \mathcal{F}_j$ and $I_j(\tilde{f}_j) \leq 1$. Let $h_{(j)}$ be the closest function to \tilde{f}_j in the δ cover of \mathcal{F}_j . Similarly, let $h_{(j)}^*$ be the closest function to \tilde{f}_j^* in the δ cover of \mathcal{F}_j . Then

$$\begin{aligned} \left\| \frac{\sum_{j=1}^J f_j - f_j^*}{\sum_{j=1}^J I_j(f_j) + I_j(f_j^*)} - \left(\sum_{j=1}^J h_{(j)} - h_{(j)}^* \right) \right\| &\leq \sum_{j=1}^J \left\| \frac{f_j - f_j^*}{\sum_{j=1}^J I_j(f_j) + I_j(f_j^*)} - (h_{(j)} - h_{(j)}^*) \right\| \\ &\leq \sum_{j=1}^J \left\| \frac{f_j}{\sum_{j=1}^J I_j(f_j) + I_j(f_j^*)} - h_{(j)} \right\| + \left\| \frac{f_j^*}{\sum_{j=1}^J I_j(f_j) + I_j(f_j^*)} - h_{(j)}^* \right\| \\ &\leq 2J\delta \end{aligned}$$

Hence

$$H \left(2J\delta, \left\{ \frac{\sum_{j=1}^J f_j - f_j^*}{\sum_{j=1}^J I_j(f_j) + I_j(f_j^*)} : f_j \in \mathcal{F}_j, I_j(f_j) + I_j(f_j^*) > 0 \right\}, \|\cdot\|_n \right) \leq 2 \sum_{j=1}^J A_j \delta^{-\alpha_j}$$

Lemma 4:

Let $p_n(\vec{x})$ be some empirical density and let p_{nj} be the corresponding empirical marginal density of x_j . Let

$$r(\vec{x}) = \frac{p_n(\vec{x})}{\prod_{j=1}^J p_{nj}(x_j)}, \quad \gamma^2 = \int (r(\vec{x}) - 1)^2 \prod_{j=1}^J p_{nj}(x_j) d\mu$$

Suppose $\gamma < 1/(J-1)$. Furthermore, suppose $\int g_j p_{nj} d\mu = 0$ for $j = 2, \dots, J$. Then

$$\left\| \sum_{j=1}^J g_j \right\|_n^2 \geq (1 - \gamma(J-1)) \left(\sum_{j=1}^J \|g_j\|_n^2 \right)$$

Proof: The proof is very similar to Lemma 5.1 in Vandegeer 2014 “The additive model with different smoothness for the components.”

$$\left\| \sum_{j=1}^J g_j \right\|_n^2 = \sum_{j=1}^J \|g_j\|_n^2 + \sum_{j \neq k} \int g_j g_k p_n(\vec{x}) d\mu$$

We bound the latter term:

$$\begin{aligned} \left| \int g_j g_k p_n(\vec{x}) d\mu \right| &= \left| \int g_j g_k (r(\vec{x}) - 1) \Pi_{j=1}^J p_{n_j}(x_j) d\mu \right| \\ &\leq \gamma \left| \int g_j^2 g_k^2 \Pi_{j=1}^J p_{n_j}(x_j) d\mu \right|^{1/2} \\ &= \gamma \|g_j\|_n \|g_k\|_n \end{aligned}$$

Hence

$$\begin{aligned} \left\| \sum_{j=1}^J g_j \right\|_n^2 &\geq \sum_{j=1}^J \|g_j\|_n^2 - \gamma \sum_{j \neq k} \|g_j\|_n \|g_k\|_n \\ &\geq (1 - \gamma(J-1)) \sum_{j=1}^J \|g_j\|_n^2 + \gamma \sum_{j < k} (\|g_j\|_n - \|g_k\|_n)^2 \\ &\geq (1 - \gamma(J-1)) \sum_{j=1}^J \|g_j\|_n^2 \end{aligned}$$

Vandegeer’s Lemma 5.4 (Jean’s version)

Let

$$\tau_R(\{f_j\}) = \left\| \sum f_j \right\|_T + \sum_{j=1}^J (\lambda_j/R)^{(1-q_j)/q_j} \lambda_j I_j(f_j)$$

Suppose

$$\sum_{j=1}^J \lambda_j^2 I_j^{q_j}(f_j^*) \leq \delta_0^2 R^2$$

and for all function sets $\{f_j\}$ s.t. $\tau_R(\{f_j\}) \leq R$, suppose

$$\sup_{f_j} |(\epsilon_T, f_j)| \leq \delta_0^2 R^2$$

Let

$$\hat{f}_j = \arg \min \|y - \sum_{j=1}^J f_j\|_T^2 + \sum_{j=1}^J \lambda_j^2 I_j^{q_j}(f_j)$$

Then $\tau_R(\{\hat{f}_{\lambda,j} - f_j^*\}) \leq R$.

Proof We use the convexity of the penalties and the least squares function. Consider $\tilde{f}_j = t\hat{f}_j + (1-t)f_j^*$ where

$$t = \frac{R}{R + \tau_R(\{\hat{f}_j - f_j^*\})}$$

First note that by convexity,

$$\tau_R(\{\tilde{f}_j - f_j^*\}) = \frac{R}{R + \tau_R(\{\hat{f}_j - f_j^*\})} \tau_R(\{\hat{f}_j - f_j^*\}) \leq R$$

Hence

$$\sup_{f_j} \left| \left(\epsilon_T, f_j^* - \tilde{f}_j \right) \right| \leq \delta_0^2 R^2$$

So by the basic inequality,

$$\left\| \sum_{j=1}^J f_j^* - \sum_{j=1}^J \tilde{f}_j \right\|_T^2 + \sum_{j=1}^J \lambda_j^2 I_j^{q_j}(\tilde{f}_j) \leq \sum_{j=1}^J \left| \left(\epsilon_T, f_j^* - \tilde{f}_j \right) \right| + \sum_{j=1}^J \lambda_j^2 I_j^{q_j}(f_j^*)$$

and with gross algebra, we can show that

$$(\lambda_j/R)^{(1-q_j)/q_j} \lambda_j I_j(\tilde{f}_j - f_j^*) \leq 4\delta_0 R$$

Then

$$\begin{aligned} \frac{R}{R + \tau_R(\{\hat{f}_j - f_j^*\})} \tau_R(\{\hat{f}_j - f_j^*\}) &= \tau_R(\{\tilde{f}_j - f_j^*\}) \\ &= \left\| \sum_{j=1}^J \tilde{f}_j - f_j^* \right\|_T + \sum_{j=1}^J (\lambda_j/R)^{(1-q_j)/q_j} \lambda_j I_j(\tilde{f}_j - f_j^*) \\ &\leq O_P(1) J \delta_0 R \end{aligned}$$

So for small enough δ_0 , we have

$$\tau_R(\{\hat{f}_j - f_j^*\}) \leq R$$

4 Examples

Our goal here is to show that the assumptions hold for various examples.

4.1 Sobolev Norm

Suppose \mathcal{G} is the class of smooth functions $g : [0, 1] \mapsto \mathbb{R}$ s.t. $I_{(k)}^2(g) = \int_0^1 g^{(k)}(t)^2 dt < \infty$.

$$\arg \min_{g \in \mathcal{G}, f \in \mathcal{F}} \|y - g(x_1) + f(x_2)\|_T^2 + \lambda_g^2 I_{(k)}(g) + \lambda_f^2 I_{(k)}(f)$$

Note that it can be shown that g can always be expressed using natural $2k$ -order B-splines with knots at the training points x_{1T} . (De Boor, Thrm XIII.5) So we can express $g(t) = \sum_{i=1}^n \beta_i(t) \gamma_i = B\gamma$ where B is positive definite. Similarly, $I_{(k)}^2(g) = \gamma^T \Omega \gamma$ where $\Omega_{ij} = \int_0^1 \beta_i^{(k)}(u) \beta_j^{(k)}(u) du$.

Assumption 1: Show for some constant K ,

$$\frac{\|g\|_\infty}{I_{(k)}(g)} \leq K$$

Proof:

From De Boor (p.110), B-splines have the property that $|\beta_i(t)| \leq 1$ and $\beta_i(t) \geq 0$. Hence $\|g\|_\infty \leq \max_i |\gamma_i|$. Then

$$\frac{\|g\|_\infty}{I_{(k)}(g)} \leq \frac{\|\gamma\|_\infty}{\|\Omega^{1/2} \gamma\|}$$

Assuming that the smallest nonzero eigenvalue of $\Omega^{1/2}$ is at least greater than some constant $c > 0$ (in fact, the nonzero eigenvalues of $\Omega^{1/2}$ likely grow with n), then

$$\frac{\|g\|_\infty}{I_{(k)}(g)} \leq \frac{\|\gamma\|_\infty}{c \|\gamma\|_2} \leq \frac{1}{c}$$

Assumption 2: Show that there are constants M, M_0 s.t. the penalty is bounded by the squared L2 norm:

$$I_{(k)}^2(g_\lambda) = M \|g_\lambda\|^2 + M_0$$

Proof:

By Noah's reformulation of the smoothing spline problem in terms of the reproducing Kernel Hilbert space, we have

$$\|g_\lambda\|_n^2 = \|\Omega \alpha\|_2^2, \quad I_{(k)}^2(g_\lambda) = \alpha^T \Omega \alpha$$

Then suppose $\|\Omega \alpha\| = O_p(n^\tau) \|\alpha\|_2$ for some constant τ . Then

$$\begin{aligned} I_{(k)}^2(g_\lambda) &= \alpha^T \Omega \alpha \\ &\leq \|\alpha\| \|\Omega \alpha\| \\ &= O_p(n^\tau) \|\alpha\|_2^2 \\ &\leq O_p(n^{-\tau}) \|\Omega \alpha\|_2^2 \\ &\leq \|\Omega \alpha\|_2^2 + M_0 \\ &= \|g_\lambda\|_2^2 + M_0 \end{aligned}$$

(That is, for sufficiently large n , we bound the penalty with the constant M_0 if $\tau < 0$ and we bound the penalty with $\|g_\lambda\|_2^2$ if $\tau > 0$)