Algorithmic aspects of upper edge domination

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ABSTRACT

We study the problem of finding a minimal edge dominating set of maximum size in a given graph $G = (V, E)$, called UPPER EDS. We show that this problem is not approximable within a ratio of $n^{\varepsilon - \frac{1}{2}}$, for any $\varepsilon \in (0, \frac{1}{2})$, assuming P $\neq$ NP, where $n = |V|$. On the other hand, for graphs of minimum degree at least 2, we give an approximation algorithm with ratio $\frac{1}{\sqrt{n}}$, matching this lower bound. We further show that UPPER EDS is APX-complete in bipartite graphs of maximum degree 4, and NP-hard in planar bipartite graphs of maximum degree 4.

1. Introduction

Dominating sets have been extensively studied in undirected graphs. Typically, algorithmicists have considered this concept in terms of the minimisation problem MINIMUM DOMINATING SET, which we shorten to MIN DS: find a smallest set of vertices that dominate all vertices of the graph [24,23]. However, researchers have also considered the max-min variant, usually called UPPER DOMINATING SET, which we abbreviate to UPPER DS: find an inclusion-wise minimal dominating set of largest size [13,10,26,16,1,2,7,8,3,9,6]. Both MIN DS and UPPER DS are NP-hard for general graphs; see [19, problem GT2] and [10], respectively.

From the standpoint of polynomial-time approximation, MIN DS is easier than UPPER DS: while the former problem can be approximated within a ratio of $\log n$ on general graphs of order n [27] (but is not approximable within a ratio of $c \log n$, for some constant $c > 0$, unless P $=$ NP [35]), the latter problem is not approximable within a ratio of $n^{\varepsilon - 1}$ for any $\varepsilon \in (0, 1)$, unless P $=$ NP [9,6]. In other words, any (trivial) greedy algorithm for UPPER DS that computes a minimal dominating set is essentially optimal.

Due to the rather discouraging approximability properties of domination problems, and also due to the importance of dominating set graph parameters (especially when using graphs to model practical applications), domination has been studied in several restricted graph classes. In fact, both MIN DS and UPPER DS remain NP-complete in quite restricted settings, such as planar cubic graphs (see [18,30] and [9] respectively). These problems have also been considered in the line graph of a given graph G; alternatively we can study the edge variant of (vertex) domination directly in G.

Given a graph $G = (V, E)$, a set of edges $S \subseteq E$ is an *edge dominating set* in G if every edge in E is either in S or is adjacent to an edge in S. An edge dominating set S in G is *minimal* if no proper subset of S is an edge dominating set in G.

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Again both the minimum and maximum minimal variations exist, referred to as Minimum EDS and Upper EDS respectively, in which we seek a minimal edge dominating set of minimum (respectively maximum) size.

Whilst Min EDS has received considerable attention in the literature, the same is not true for Upper EDS. Min EDS is NP-hard in planar or bipartite graphs of maximum degree 3 [38] and in planar cubic graphs [25], whilst solvable in polynomial time in several graph classes (see [11] for a brief survey). On the other hand Upper EDS has been largely neglected: the only complexity result for the problem that we are aware of is NP-hardness in bipartite graphs [33]. We remedy this situation in this paper.

To highlight some of the differences between Min EDS and Upper EDS, we give the following observation. It is known that the size of a minimum (minimal) edge dominating set is equal to the size of a minimum maximal matching (i.e., a minimum independent edge dominating set) [38]. It is thus tempting to conjecture that the size of a maximum minimal edge dominating set might be related in some way to the size of a maximum (maximal) matching. In fact this is not the case. Although a maximal matching is indeed a minimal edge dominating set, it turns out that a maximum minimal edge dominating set may be arbitrarily larger than a maximum matching. This is illustrated by the following example.

Example 1.1. Let G be the complete bipartite graph $G = K_{2,n-2}$ on n vertices, for any $n \geq 5$. Suppose that the vertices in the two colour classes of G are $\{u_1, u_2\}$ and $\{w_1, w_2, \ldots, w_{n-2}\}$. Then clearly $S_1 = \{u_1 w_1, u_2 w_2\}$ is a maximum matching in G, whereas $S_2 = \{u_1 w_i : 1 \leq i \leq n - 2\}$ is a minimal edge dominating set in G of size $n - 2$.

An explanation of sorts for this phenomenon can be given by observing that, whilst it is well known that Maximum Matching can be solved in polynomial time [14], Upper EDS is NP-hard, and we prove in this paper that NP-hardness holds even in planar bipartite graphs of maximum degree 4.

The main focus of our paper is the approximability of Upper EDS. We show that a similar dichotomy, in terms of approximability properties, holds between Min EDS and Upper EDS, that does between Min DS and Upper DS. However it is the case that the edge domination problems have better approximability properties than the vertex domination problems. These observations are described in more detail as follows.

- As previously mentioned, Min DS is approximable within a ratio of $\log n$ on general graphs of order n [27], but not approximable within a ratio of $c \log n$, for some constant $c > 0$, unless P = NP [35]). On the other hand Min EDS is approximable within a ratio of 2 (any maximal matching serves as a 2-approximation to a minimum edge dominating set [31]) but not approximable within a ratio of $\frac{3}{2} - \varepsilon$, for any $\varepsilon > 0$, assuming the Unique Games Conjecture [36,15].
- Recall that Upper DS is not approximable within a ratio of $n^{\varepsilon-1}$ for any $\varepsilon \in (0, 1)$, unless P = NP [9,6]. We show in this paper that Upper EDS is approximable within a ratio of $\frac{1}{\sqrt{n}}$ in graphs of minimum degree at least 2. On the other hand, for a general graph, we show that the problem is not approximable within a ratio of $n^{\varepsilon - \frac{1}{2}}$, for any $\varepsilon \in (0, \frac{1}{2})$, unless P = NP.

Organisation of the paper. In Section 2, we formally define the key notation, terminology and problems considered in this paper. In Section 3, we discuss some combinatorial properties of minimal edge dominating sets and establish some lower and upper bounds on the upper edge domination number in terms of the order and maximum degree of the given graph. These results are utilised in Sections 4 and 5 where we prove our main (in-)approximability results for Upper EDS in general graphs and in graphs without leaves, respectively, as announced above.

In Section 6, we prove APX-completeness of Upper EDS in bipartite graphs of bounded maximum degree, in particular, for bipartite graphs of maximum degree 4. The construction also yields NP-hardness of Upper EDS for planar bipartite graphs of maximum degree 4. Finally, we give some concluding remarks in Section 7.

A remark on the collaboration. This work was initiated by Jérôme, who had a long-standing interest in the algorithmic complexity of min-max/max-min variants of graph parameters, and had published many nice papers on this topic [1,2,7,8,3,9,28,22,29]. He kindly invited the second and third authors to join him in a new collaboration on upper edge domination; some of Jérôme's papers on "upper" graph parameters were co-authored with the second author [7–9], and the third author's PhD thesis also dealt with this topic [32]. Jérôme put together a preprint containing his observations, which were insightful and elegantly written as always. His untimely death sadly came before the second and third authors could contribute much to the project. This special issue therefore presented an ideal opportunity for the second and third authors to develop Jérôme's ideas into a full paper; his major contribution is reflected in the order of the author names.

2. Definitions of notation and terminology

Throughout this paper, we consider simple undirected graphs. A *graph* $G = (V, E)$ can be specified by the set V of vertices and the set E of edges; every edge has two endpoints and these two endpoints are called *adjacent*; if v is an endpoint of e, we also say that e and v are *incident* and two edges e and e' are *adjacent* if they share a common endpoint.

Given a set $S \subseteq V$, the *neighbourhood* of S in G, denoted by $N_G(S)$, is defined as follows: $N_G(S) = \{v \in V : \exists u \in S, \ uv \in E\}$. The *closed neighbourhood* is defined to be $N_G[S] = S \cup N_G(S)$. Similarly, given a set $S \subseteq E$, the *neighbourhood* of S in G, denoted by $N_G(S)$, is defined as follows:

$$N_G(S) = \{e = uv \in E : \exists e' = vw \in S \text{ for some } w \in V\}.$$

The *closed neighbourhood* is defined to be $N_G[S] = S \cup N_G(S)$. For singleton sets $S = \{s\}$, where $s \in V \cup E$, we simply write $N_G(s)$ or $N_G[s]$, even omitting G if clear from the context. If $v \in V$, the cardinality of $N_G(v)$ is called *degree* of v, denoted $d_G(v)$. A graph where all vertices have degree r is called *r-regular*. The minimum (respectively maximum) degree taken over all vertices in G is denoted by $\delta(G)$ (respectively $\Delta(G)$).

For a subset of edges S, $V(S)$ denotes the vertices incident to S. A vertex set S induces the graph $G[S]$ with vertex set S and $e \in E$ being an edge in $G[S]$ iff both endpoints of e are in S. If $S \subseteq E$ is an edge set, then we define $\overline{S} = E \setminus S$. Edge set S induces the graph $G[V(S)]$, while $G_S = (V, S)$ denotes the partial graph induced by S. In particular, $G_{\overline{S}} = (V, \overline{S})$.

A *star* $S \subseteq E$ of a graph $G = (V, E)$ is a tree of G where at most one vertex has a degree greater than 1, or, equivalently, it is isomorphic to $K_{1,\ell}$ for some $\ell \geq 0$. The vertices of degree 1 (in the case that $\ell > 1$) are called *leaves* of the star while the remaining vertices are called *centres* of the star; usually in a star, there is only one centre except for the case $\ell = 1$ where we consider that the star (indeed the edge $K_2 = K_{1,1}$) has two centres and no leaves. An ℓ-star is a star of ℓ leaves (hence, by assumption $\ell \neq 1$). If $\ell = 0$, the star is called *trivial* and it is reduced to a single vertex (the centre); otherwise, the star is said to be *non-trivial*. Finally, $G_S = (V, S)$ with $S \subseteq E$ is a *spanning star forest* of G if each connected component of G_S is a star. For a spanning star forest G_S of G, leaf(S) denotes the whole set of vertices that are leaves and triv(S) denotes the vertices of the trivial stars of G_S.

A vertex set $S \subseteq V$ is an *independent set* if the induced graph $G[S]$ comprises isolated vertices, or equivalently, S is a set of pairwise non adjacent vertices. The *independence number* of a graph G, denoted $\alpha(G)$ is the maximum size of an independent set of G. We let MAX IS denote the problem of computing $\alpha(G)$, given a graph G. An edge set $S \subseteq E$ is a *matching* if S is a set of pairwise non adjacent edges. The *matching number*, denoted $\alpha'(G)$, is the maximum size of a matching of G.[2]

A vertex set $S \subseteq V$ is a *dominating set* if every vertex not in S is adjacent to at least one vertex of S, i.e., $N_G[S] = V$. An edge set $S \subseteq E$ is an *edge dominating set* if every edge $e \in E \setminus S$ is adjacent to some edge of S. The *edge domination number*, denoted $\gamma'(G)$, is the minimum size of an edge dominating set of G.

A set S is *minimal* with respect to a graph property π if S satisfies π and any proper subset $S' \subset S$ of S does not satisfy π. Let $S \subseteq E$ be an edge dominating set. Define an edge $e \in E$ to be *private* if e is dominated by exactly one edge of S, i.e., $|N[S] \cap N[e]| = 1$. We will demonstrate a connection between minimal edge dominating sets and private edges in Section 3. Define $\Gamma'(G)$ to be the maximum size of a minimal edge dominating set in G, referred to as the *upper edge domination number* of G. The central optimisation problem that we consider in this paper is as follows:

UPPER EDS
Input: A graph $G = (V, E)$.
Solution: A minimal edge dominating set $S \subseteq E$ of maximum size.

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3. Lower and upper bounds for $\Gamma'(G)$

In this section we establish lower and upper bounds for $\Gamma'(G)$ that will be used in later sections of the paper, particularly in relation to approximability results. We begin by giving a necessary and sufficient condition for an edge dominating set to be minimal in terms of private edges.

Proposition 3.1. *Let $G = (V, E)$ be a graph and let $S \subseteq E$ be an edge dominating set. Then $S \subseteq E$ is a minimal edge dominating set if and only if every edge $e \in S$ has a private edge in $N[e]$.*

Proof. Suppose firstly that S is a minimal edge dominating set. Suppose that there exists some $e \in S$ that has no private edge in $N[e]$. Let $S' = S \setminus \{e\}$. As e is not a private edge, e is dominated by a neighbour in S'. Similarly, for every $e' \in N(e)$, e' is dominated by some neighbour in S'. Hence S' is an edge dominating set, contradicting the minimality of S.

Conversely suppose that every edge $e \in S$ has a private edge in $N[e]$. Suppose that S is not minimal. Then there exists some $e \in S$ such that $S' = S \setminus \{e\}$ is an edge dominating set of G. Thus e is dominated by some neighbour, and thus e cannot be a private edge relative to S. Similarly every edge e' in $N(e)$ is dominated by some neighbour in S', and hence e' cannot be a private edge relative to S. Hence there is no private edge in $N[e]$ relative to S. This is a contradiction, and hence S is minimal.   □

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[2] Here and in the following, we follow the convention that if $\rho(G)$ is a graph parameter indicating the maximum or minimum number of vertices with a certain property (e.g., being independent or dominating) of a graph G, then $\rho'(G)$ denotes the corresponding maximum or minimum number of edges with a similar property for edge sets.

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Our next result builds on Proposition 3.1 and gives a characterisation of minimal edge dominating sets in terms of spanning stars and vertex dominating sets of a graph.

Lemma 3.2. *Let $G = (V, E)$ be a graph. $S \subseteq E$ is a minimal edge dominating set if and only if the two following conditions hold:*

(i) G_S is a spanning star forest and $\mathrm{triv}(S)$ is an independent set of G.
(ii) $\mathrm{triv}(S)$ is a dominating set of $G[\mathrm{triv}(S) \cup \mathrm{leaf}(S)]$.[3]

Proof. Suppose that Conditions (*i*) and (*ii*) hold. Suppose firstly that S is not an edge dominating set. Then there exists some $e = uv \in E \setminus N[S]$. Thus $\{u, v\} \subseteq \mathrm{triv}(S)$, a contradiction to Condition (*i*). Now suppose that S is not minimal. Then by Proposition 3.1, there exists some edge $e = vw \in S$ that does not have a private edge in $N[e]$. Hence there exists some $e' = uv \in N(e) \cap S$. Hence $v \notin \mathrm{leaf}(S)$. Moreover $w \in \mathrm{leaf}(S)$, by Condition (i). If $N(w) = \{v\}$ then clearly Condition (*ii*) is not satisfied, a contradiction. Hence pick some edge $e'' = wx$ where $x \neq v$. Then by Condition (*i*), $e'' \notin S$. Also as e'' is not a private edge, x is incident to some edge of S, which implies that $x \notin \mathrm{triv}(S)$. Hence again Condition (*ii*) is not satisfied, a contradiction.

Conversely suppose that S is a minimal edge dominating set. We claim that the partial graph $G_S = (V, S)$ is a spanning star forest and $\mathrm{triv}(S)$ is an independent set of G. Firstly, G_S is acyclic and P_4-free (i.e., without 3 consecutive edges), as otherwise S will be not a minimal edge dominating set; hence, G_S is a spanning star forest. Moreover $\mathrm{triv}(S)$ is an independent set since S needs to dominate all edges. Hence Condition (*i*) is satisfied. Suppose for a contradiction that Condition (*ii*) is not satisfied. Then there exists some $v \in \mathrm{leaf}(S)$ that is not adjacent in $H = G[\mathrm{triv}(S) \cup \mathrm{leaf}(S)]$ to any vertex in $\mathrm{triv}(S)$. By Condition (*i*), $uv \in S$ for some $u \in N_H(v)$ that is the centre of a star $K_{1,\ell}$ in G_S for some $\ell \geq 2$. Any $w \in N_G(v)$ is either a leaf or the centre of a star in G_S. Thus $e = uv \in S$ does not have a private edge in $N[e]$. Hence by Proposition 3.1, S is not minimal, a contradiction. □

We now give an upper bound for $\Gamma'(G)$ in terms of the order of G, together with a characterisation of when the bound is tight.

Corollary 3.3. *For any connected graph $G = (V, E)$ with $n \geq 3$ vertices,*

$$\Gamma'(G) \leq n - 2. \tag{1}$$

Inequality (1) is tight if and only if (i) $n = 3$, (ii) $n = 4$ and G contains $2K_2$ as a subgraph, or (iii) $n \geq 5$ and G contains $K_{2,n-2}$ as a subgraph.

Proof. Let S be any minimal edge dominating set of G. To prove the inequality, suppose firstly that G_S contains a trivial star. Then the non-trivial stars of G_S form a spanning star forest on at most $n - 1$ vertices, and hence $|S| \leq n - 2$. Now suppose that G_S contains no trivial stars. Then every star in G_S is a $K_{1,1}$, for otherwise G_S contains a vertex $v \in \mathrm{leaf}(S)$, which is impossible, since there is no vertex in $\mathrm{triv}(S)$ to dominate v in $G[\mathrm{triv}(S) \cup \mathrm{leaf}(S)]$. This is a contradiction to Condition (*ii*) in Lemma 3.2. Hence $|S| \leq n/2$, and as $n \geq 3$, the conclusion is immediate.

To show the characterisation regarding the tightness of the inequality, suppose firstly that $n = 3$. Then $G = K_{1,2}$ or $G = K_3$ as G is connected, and hence $\Gamma'(G) = 1$. Now suppose that $n = 4$. If G contains $2K_2$ as a subgraph then $\Gamma'(G) = 2$, otherwise $G = K_{1,3}$, in which case $\Gamma'(G) = 1$. Finally suppose that $n \geq 5$. Suppose firstly that G contains $K_{2,n-2}$ as a subgraph. As in Example 1.1, we may form a minimal edge dominating set of G of size $n - 2$. Now suppose that G does not contain $K_{2,n-2}$ as a subgraph.
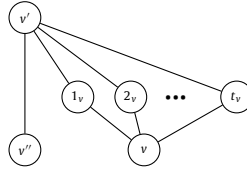
Suppose that G_S contains no trivial stars. Then as in the first paragraph of this proof, every star in G_S is a $K_{1,1}$. Hence $|S| \leq n/2$ and thus $|S| \leq n - 3$ as $n \geq 5$. If G_S contains two trivial stars, then the non-trivial stars of G_S form a spanning star forest on at most $n - 2$ vertices, and hence $|S| \leq n - 3$. Hence G_S contains one trivial star. The non-trivial stars of G_S form a spanning star forest on at most $n - 1$ vertices. If $\Gamma'(G) = n - 2$ then G_S contains one trivial star on a vertex v and one star $K_{1,n-1}$. By Condition (*ii*) of Lemma 3.2, v must be adjacent to every leaf in the star $K_{1,n-1}$. Thus, G contains $K_{2,n-2}$ as a subgraph after all, a contradiction. □

Our next result establishes lower and upper bounds for $\Gamma'(G)$ in terms of both the order of G and the maximum degree of G.

Theorem 3.4. *For any connected graph $G = (V, E)$ where $n = |V|$ and $\Delta = \Delta(G) \geq 2$,*

$$\frac{n}{\Delta + 1} \leq \Gamma'(G) \leq \frac{\Delta n}{\Delta + 2}.$$

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[3] By assumption the empty set is a dominating set of the empty graph.

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Fig. 1. Gadget $H_t(v)$ for $t \geq 2$.

Proof. For the left part of the inequality, it is sufficient to show that $\alpha'(G) \geq \frac{n}{\Delta+1}$, since a maximum matching M^* is a minimal edge dominating set. Let x be the number of exposed vertices relative to M^*, i.e., the number of vertices that are not endpoints of edges from the matching. Then $x \leq \alpha'(G)(\Delta - 1)$, since each edge of M^* is adjacent to at most $\Delta - 1$ edges with an exposed endpoint vertex, and each exposed vertex is incident to at least one edge that is adjacent to an edge in M^*. Since $x = n - 2\alpha'(G)$, the inequality follows.

For the right part of the inequality, let S be a maximum minimal edge dominating set in G. With respect to G_S, let n_1 be the number of trivial stars, let n_2 be the number of centres of stars of the form $K_{1,1}$, let n_3 be the number of centres of stars of the form $K_{1,r}$, for any $r \geq 2$, and let n_4 be the number of leaves. Then

$$n = n_1 + n_2 + n_3 + n_4. \tag{2}$$

By Lemma 3.2, each vertex in leaf(S) must be adjacent in G to some vertex in triv(S). However each vertex in triv(S) can dominate at most Δ vertices in leaf(S). Hence $n_4 \leq \Delta n_1$. Also each non-trivial star of the form $K_{1,r}$, for some $r \geq 2$, can have at most Δ leaves, and thus $n_4 \leq \Delta n_3$. Thus we obtain the following inequality from Equation (2):

$$\Delta n \geq \Delta n_2 + (\Delta + 2)n_4. \tag{3}$$

By Lemma 3.2, S forms a spanning star forest, and hence $\Gamma'(G) = |S| = \frac{n_2}{2} + n_4$. This implies that

$$(\Delta + 2)\Gamma'(G) = \left(1 + \frac{\Delta}{2}\right)n_2 + (\Delta + 2)n_4$$
$$\leq \Delta n_2 + (\Delta + 2)n_4 \tag{4}$$

since $\Delta \geq 2$. Thus, putting Inequalities (4) and (3) together, the result follows. $\square$

4. Approximability in general graphs

In this section we study the approximability of Upper EDS, given a general graph G, establishing lower and upper bounds in terms of both the order and maximum degree G. We begin with a lower bound involving the order of G.

Theorem 4.1. *Given a graph G with n vertices,* Upper EDS *is not approximable within ratio $n^{\varepsilon - \frac{1}{2}}$, for any $\varepsilon \in (0, \frac{1}{2})$, unless* P $=$ NP.

Proof. Let $G = (V, E)$ be a connected graph where $n = |V|$ and $m = |E|$, given as an instance of Max IS. We build a graph $H = (V', E')$ as follows:

- H contains G as induced subgraph.
- For each vertex $v \in V$, we add a copy of the gadget $H_t(v)$ for $t = n + 2$, where the vertex v of $H_t(v)$ is identified to vertex $v \in V$. An illustration of the gadget is given in Fig. 1.

This construction is clearly computable within polynomial time. Moreover, the graph H has $(t + 3)n$ vertices and $m + (2t + 1)n$ edges.

Let S be a maximum independent set of G. Construct the following set of edges:

$$S' = \{v'i_v : \ 1 \leq i \leq t \text{ and } v \in S\} \cup \{v1_v, v'v'' : \ v \in V \setminus S\}.$$

Clearly, using Lemma 3.2 we deduce that S' is a minimal edge dominating set of H with size $t|S| + 2(n - |S|)$. Hence, we deduce:

$$\Gamma'(H) \geq |S'| = 2n + (t - 2)|S| = 2n + (t - 2)\alpha(G). \tag{5}$$

Conversely, let S' be a maximum minimal edge dominating set of H. We firstly establish some properties of S' relating to gadget $H_t(v)$, for any $v \in V$.

Property 4.2. *The following properties hold for S' relative to $H_t(v)$, for any $v \in V$:*

(i) *If v is incident to an edge of S', then $|S' \cap E(H_t(v))| \leq 2$.*

(ii) *If v is not incident to any edge of S', then $|S' \cap E(H_t(v))| = t$.*

Proof. Assume v is incident to an edge e of S'. Firstly, suppose that e is inside $E(H_t(v))$ (i.e., $e = vi_v$ for some i where $1 \leq i \leq t$). Then $|S' \cap E(H_t(v))| = 2$ because on the one hand $v'v''$ must be dominated and on other hand S' is minimal and it has to satisfy Lemma 3.2. Secondly, suppose that e is outside $E(H_t(v))$ (i.e., $e = vu$ for some $u \in V$). As before, we deduce $|S' \cap E(H_t(v))| = 1$ and $S' \cap \{vi_v : 1 \leq i \leq t\} = \emptyset$. Finally, assume that v is not incident to any edge of S'. The t edges vi_v ($1 \leq i \leq t$) need to be dominated, and hence $|S' \cap E(H_t(v))| = t$. □

Now, we will polynomially transform S' into a minimal edge dominating set S'' satisfying $S'' \cap E = \emptyset$ and $|S''| \geq |S'|$ by applying an inductive procedure. Assume $uv \in S' \cap E$. We replace S' by the following set:

$$S'' = \{u1_u, u'u'', v1_v, v'v''\} \cup S' \setminus \big(E(H_t(u)) \cup \{uv\} \cup E(H_t(v))\big).$$

The new solution S'' remains a minimal edge dominating set of H, and using Case (ii) of Property 4.2 we deduce that $|S''| = |S'| + 1$.

At the end of the procedure, we obtain a minimal edge dominating set S'' satisfying $S'' \cap E = \emptyset$ and $|S''| \geq |S'|$. Let $S = \{v \in V : |S'' \cap E(H_t(v))| = t\}$. From Property 4.2 and the edge dominating set property of S'', we deduce that S is an independent set of G, and that $|S''| = t|S| + 2(n - |S|)$. Thus, we obtain:

$$(t - 2)\alpha(G) + 2n \geq (t - 2)|S| + 2n = |S''| \geq |S'| = \Gamma'(H). \tag{6}$$

Using Inequality (5) and the choice of $t = n + 2$, we deduce that:

$$\Gamma'(H) = 2n + (t - 2)\alpha(G) = n(\alpha(G) + 2). \tag{7}$$

Suppose that, for some $\varepsilon \in (0, \frac{1}{2})$, UPPER EDS is approximable within a ratio of $N^{\varepsilon - \frac{1}{2}}$, where $N = |V(H)|$. Using the supposed approximation algorithm in H for UPPER EDS with ratio $N^{\varepsilon - \frac{1}{2}}$, we may build a minimal edge dominating set S' in H such that $|S'| \geq N^{\varepsilon - \frac{1}{2}} \Gamma'(H)$.

From the arguments above, starting from S', we can in polynomial time build an independent set S of G satisfying $|S| = \frac{|S'| - 2n}{t - 2}$; recall also that $\alpha(G) = \frac{\Gamma'(H) - 2n}{t - 2}$. It follows that

$$\alpha(G) - |S| \leq \frac{\Gamma'(H) - |S'|}{t - 2} = \frac{\Gamma'(H) - |S'|}{n}. \tag{8}$$

Moreover, by Equation (7), we obtain $\Gamma'(H) \leq 2n\alpha(G)$, as we may assume without loss of generality that $\alpha(G) \geq 2$. It follows from this inequality, and from Inequality (8) that

$$N^{\varepsilon - \frac{1}{2}} \leq \frac{|S'|}{\Gamma'(H)} \leq 1 - n\left(\frac{\alpha(G) - |S|}{\Gamma'(H)}\right) \leq 1 - n\left(\frac{\alpha(G) - |S|}{2n\alpha(G)}\right) = \frac{1}{2} + \frac{|S|}{2\alpha(G)}.$$

Hence $\frac{|S|}{\alpha(G)} \geq 2N^{\varepsilon - \frac{1}{2}} - 1$. Now $N = (t + 3)n = n(n + 5) > n^2$. It follows that

$$\frac{|S|}{\alpha(G)} > n^{2\varepsilon - 1} + n^{2\varepsilon - 1} - 1 \geq n^{2\varepsilon - 1}. \tag{9}$$

It is known that MAX IS is not approximable within ratio $n^{\delta - 1}$, for any $\delta \in (0, 1)$, unless P = NP [37,39]. Setting $\delta = 2\varepsilon$, Inequality (9), gives a contradiction, establishing the desired result. □

We next show that it is straightforward to modify the argument given in the proof of Theorem 4.1 to provide a strong lower bound for the approximability of UPPER EDS in terms of the maximum degree of the given graph.

Corollary 4.3. *Given a graph G where $\Delta = \Delta(G)$, UPPER EDS is not approximable within ratio $\Delta^{\varepsilon - 1}$, for any $\varepsilon \in (0, 1)$, unless P = NP.*

Proof. We modify the argument in the latter stages of the proof of Theorem 4.1 as follows. Suppose that, for some $\varepsilon \in (0, 1)$, UPPER EDS is approximable within a ratio of $\Delta^{\varepsilon - 1}$, where $\Delta = \Delta(H)$. As in the proof of Theorem 4.1, we obtain $\frac{|S|}{\alpha(G)} \geq 2\Delta^{\varepsilon - 1} - 1$. Clearly $\Delta \geq t \geq n$. It follows that $\frac{|S|}{\alpha(G)} \geq n^{\varepsilon - 1}$, so again we contradict the lower bound for the approximability of MAX IS. □

It is straightforward to give a matching upper bound for the lower bound given in Corollary 4.3.

Proposition 4.4. *UPPER EDS is approximable within a ratio of $\frac{1}{\Delta}$, given a graph G of maximum degree $\Delta = \Delta(G)$.*

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Proof. The result follows immediately by constructing a maximum matching, and then using the lower and upper bounds given by Theorem 3.4. □

5. Approximability in graphs without leaves

In this section we give an approximation algorithm for UPPER EDS with performance ratio $\frac{1}{\sqrt{n}}$, given a connected graph G on n vertices such that G contains no leaves, i.e., G has minimum degree at least 2.

The approximation algorithm is described in Algorithm 1. The set S that is returned is either a maximum matching in G, computed in line 1, or is a set T of edges computed between lines 2-15, whichever is the larger.

Algorithm 1: Approximation algorithm for Upper Eds.

Input: A connected graph $G = (V, E)$ where $n = |V|$ and $\delta(G) \geq 2$
Output: A minimal edge dominating set S where $|S| \geq \Gamma'(G)/\sqrt{n}$

1 $S :=$ maximum matching in G
2 $v :=$ vertex in G of maximum degree
3 $W = \{w_1, \ldots, w_r\} := N_G(v)$
4 $W_1 := \{w_i \in W : N_G(w_i) \cap W = \emptyset\}$
5 $W_2 := W \setminus W_1$
6 $T_1 := \{w_i x_i : w_i \in W_1 \wedge x_i \in N_G(w_i) \setminus \{v\}\}$ // the x_i need not be distinct
7 $T_2 :=$ maximum matching in $G[W_2]$
8 $T_3 := T_2$
9 **while** some vertex $w_i \in W_2$ is unmatched by T_3 **do**
10 | $w_j :=$ any vertex in $N_G(w_i) \cap W_2$ // w_j is matched by T_2
11 | $T_3 := T_3 \cup \{w_i w_j\}$
12 $N := N_G[v] \cup \{x_i : w_i \in W_1\}$
13 $G' := G[V \setminus N]$
14 $T_4 :=$ maximum matching in G'
15 $T := T_1 \cup T_3 \cup T_4$
16 **if** $|T| > |S|$ **then** $S := T$
17 **return** S

If S is a maximum matching in G, then clearly S is a minimal edge dominating set. Otherwise, lines 2-15 compute T, which is then returned as S, as follows. We firstly identify a vertex v with maximum degree (i.e., $d_G(v) = \Delta(G)$). The set W comprising the neighbours of v is computed, and we then compute the set W_1 comprising those members of W that have no neighbours in W. The set W_2 is assigned to be those members of W not in W_1.

As each vertex has degree at least 2 in G, each vertex $w_i \in W_1$ has a neighbour x_i that is not equal to v. The algorithm adds each edge $w_i x_i$ to the set of edges T_1, noting that $x_i \notin W$ by definition of W_1; also the x_i need not be distinct. Each such edge in T_1 has $v w_i$ as a private edge.

Next we compute a maximum matching T_2 in the subgraph of G induced by W_2. Each vertex in $w_i \in W_2$ has at least one neighbour in W_2, by definition of W_2. Each edge $w_i w_j$ added to T_2 has $v w_i$ and $v w_j$ as private edges. Some vertices in W_2 might be unmatched in T_2, so T_2 might not yet dominate all edges between v and a member of W_2.

Lines 8-11 extend T_2 to a set of edges T_3 that does dominate all edges between v and a member of W_2. Initially we set $T_3 = T_2$. As long as some vertex $w_i \in W_2$ is unmatched by T_3, we pick a neighbour w_j of w_i in W_2. Then, vertex w_j must be matched by T_2, or else T_2 is not maximal. We add $w_i w_j$ to T_3. Then this edge has $v w_i$ as a private edge. Note that no P_4 in T_3 is created by this procedure, otherwise T_2 admits an augmenting path, a contradiction. Thus any edge $w_i w_j$ in T_2 will retain either $v w_i$ or $v w_j$ as a private edge. Once this procedure terminates, all vertices in W_2 are matched in T_3. Moreover every edge in $T_1 \cup T_3$ has a private edge incident to v.

The next step is to ensure that the rest of the graph is dominated. We compute the set N of vertices that are matched by $T_1 \cup T_3$, plus v. Then N comprises v, $N_G(v)$ and the vertices x_i that are incident to edges in T_1. We let G' be the subgraph of G induced by $\bar{N}$, where $\bar{N} = V \setminus N$. We compute a maximum matching T_4 in G' and then assign to T the edges in $T_1 \cup T_3 \cup T_4$.

Clearly T_4 is an edge dominating set of G', and as previously argued, $T_1 \cup T_3$ is an edge dominating set of the subgraph of G induced by N. It remains to show that T is an edge dominating set of G. To show this, it is sufficient to show that each edge xy is dominated, where $x \in N$ and $y \in \bar{N}$. This is clearly the case, as x must be incident to an edge in $T_1 \cup T_3$.

Finally, T is minimal, since each edge in $T_1 \cup T_3$ has a private edge incident to v as previously argued, and each edge in T_4 is a private edge for itself.

We now show that Algorithm 1 achieves a performance ratio of $\frac{1}{\sqrt{n}}$.

Theorem 5.1. UPPER EDS *is approximable within a ratio of* $\frac{1}{\sqrt{n}}$, *given a connected graph G on n vertices without any leaves (i.e.,* $\delta(G) \geq 2$).

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Proof. By the above argument, Algorithm 1 returns a minimal edge dominating set T. To establish the performance ratio, we consider three main cases depending on the value of $\Delta(G)$.

Case 1: suppose that $\Delta(G) \leq \sqrt{n}$. Recall from the proof of Proposition 4.4 that computing a maximum matching gives an approximation ratio of $\frac{1}{\Delta(G)}$ for UPPER EDS. It follows that

$$|S| \geq \alpha'(G) \geq \frac{\Gamma'(G)}{\Delta(G)} \geq \frac{\Gamma'(G)}{\sqrt{n}}.$$

Case 2: suppose that $\Delta(G) \geq 2\sqrt{n}$. Considering the execution of Algorithm 1, we may deduce that $|S| \geq |T_1| + |T_3| + |T_4|$. Using the notation of Algorithm 1, it is straightforward to verify that $|T_1| = |W_1|$, $|T_3| \geq \left\lceil \frac{|W_2|}{2} \right\rceil$, and $|W_1| + |W_2| = \Delta(G)$. It follows that

$$|S| \geq |T_1| + |T_3| + |T_4| \geq \frac{|W_1| + \Delta(G)}{2} + |T_4|. \tag{10}$$

As $\Delta(G) \geq 2\sqrt{n}$, it follows from Inequality (10) that $|S| \geq \sqrt{n}$. It follows by Inequality (1) that

$$|S| \geq \frac{n}{\sqrt{n}} \geq \frac{\Gamma'(G)}{\sqrt{n}}.$$

Case 3: suppose that $\sqrt{n} < \Delta(G) < 2\sqrt{n}$. Observe that $|N| \leq \Delta(G) + 1 + |W_1|$. Moreover, as in the proof of Theorem 3.4,

$$|T_4| = \alpha'(G') \geq \frac{|V(G')|}{\Delta(G') + 1} \geq \frac{n - |N|}{\Delta(G) + 1} \geq \frac{n - (\Delta(G) + 1 + |W_1|)}{\Delta(G) + 1}. \tag{11}$$

Let $\Delta = \Delta(G)$ and assume that $\Delta \geq 1$ (if $\Delta = 0$ then G is an isolated vertex and $\Gamma'(G) = 0$). It follows from Inequalities (10) and (11) that

$$\begin{aligned}
|S| &\geq \frac{|W_1|(\Delta - 1) + \Delta^2 - \Delta + 2n - 2}{2(\Delta + 1)} \\
&\geq \frac{\Delta^2 - \Delta + 2n - 2}{2(\Delta + 1)} \\
&= \left(\frac{\Delta\sqrt{n}}{n+2}\right)\left(\frac{n+2}{\Delta\sqrt{n}}\right)\left(\frac{\Delta^2 - \Delta + 2n - 2}{2(\Delta + 1)}\right) \\
&= \left(\frac{\Delta\sqrt{n}}{n+2}\right)\left(\frac{\Delta^2 n - \Delta n + 2n^2 - 2n + 2\Delta^2 - 2\Delta + 4n - 4}{2\Delta^2\sqrt{n} + 2\Delta\sqrt{n}}\right) \\
&= \left(\frac{\Delta\sqrt{n}}{n+2}\right)\left(1 + \frac{f(n, \Delta)}{2\Delta^2\sqrt{n} + 2\Delta\sqrt{n}}\right)
\end{aligned} \tag{12}$$

where $f(n, \Delta) = \Delta^2 n - \Delta n + 2n^2 - 2n + 2\Delta^2 - 2\Delta + 4n - 4 - 2\Delta^2\sqrt{n} - 2\Delta\sqrt{n}$. We will show that $f(n, \Delta) \geq 0$ provided that $n \geq 4$ (if $n \leq 3$ then either $G = K_n$ or $G = P_3$; recall that $\Delta(G) \geq 1$, so $n \geq 2$. In any of the cases that $G = K_2$, $G = K_3$ or $G = P_3$, $\Gamma'(G) = 1$ and the approximation algorithm produces an optimal solution). Using the fact that $\sqrt{n} < \Delta(G) < 2\sqrt{n}$, it follows that
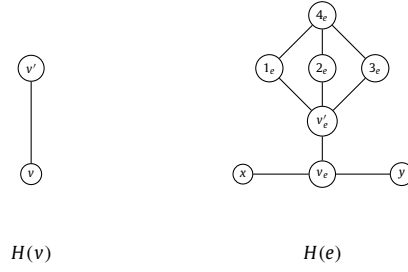
$$\begin{aligned}
f(n, \Delta) &= \Delta^2(n - 2\sqrt{n} + 2) - \Delta(n + 2\sqrt{n} + 2) + 2n^2 + 2n - 4 \\
&> n(n - 2\sqrt{n} + 2) - 2\sqrt{n}(n + 2\sqrt{n} + 2) + 2n^2 + 2n - 4 \\
&= 3n^2 - 4n\sqrt{n} - 4\sqrt{n} - 4 \\
&= (2n^2 - 4n\sqrt{n}) + \left(\frac{1}{2}n^2 - 4\sqrt{n}\right) + \left(\frac{1}{2}n^2 - 4\right).
\end{aligned} \tag{13}$$

It is easy to see that each bracketed term in Inequality (13) is greater than or equal to 0, given that $n \geq 4$. It follows from Inequality (12) and Theorem 3.4 that
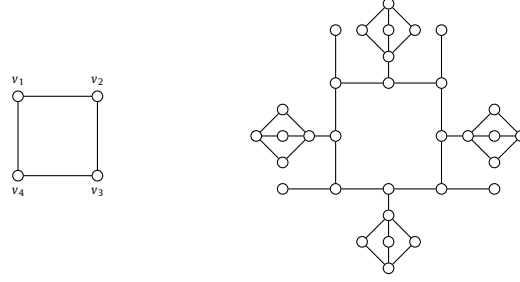
$$|S| \geq \frac{\Delta\sqrt{n}}{n+2} = \left(\frac{1}{\sqrt{n}}\right)\left(\frac{\Delta n}{n+2}\right) \geq \frac{\Gamma'(G)}{\sqrt{n}}.$$

This completes the proof. $\quad \square$

Fig. 2. On the left side, vertex gadget $H(v)$ and on the right side, edge gadget $H(e)$ for $e = xy$.

Fig. 3. On the left side, an instance of MAX IS, and on the right side, the corresponding bipartite instance of UPPER EDS.

6. Approximability in bipartite graphs

In this section we improve the result given in [33] by showing that UPPER EDS is APX-complete and NP-complete in bounded degree bipartite graphs and planar bipartite graphs, respectively.

Theorem 6.1. UPPER EDS *is* APX-*complete in bipartite graphs of maximum degree 4, and not approximable within a ratio of* $\frac{2374}{2375} + \varepsilon$, *for any* $\varepsilon > 0$, *unless* $\mathsf{P} = \mathsf{NP}$.

Proof. By Proposition 4.4, UPPER EDS belongs to APX in graphs of bounded degree. To show APX-hardness, we give an L-reduction [34,5] from MAX IS in connected graphs of maximum degree 3, which is APX-complete [4]. Let $G = (V, E)$ be a connected graph of maximum degree 3 given as an instance of MAX IS, where $n = |V|$ and $m = |E|$. We build a bipartite graph H of maximum degree 4 using a vertex gadget $H(v)$ for every $v \in V$ as described on the left side of Fig. 2 and an edge gadget $H(e)$ for every edge $e = uv \in E$ as described on the right side of Fig. 2. More formally:

- $V(H) = V \cup V' \cup V'_E$ contains V; moreover, it contains n new vertices $V' = \{v' : v \in V\}$ and $6m$ new vertices $V'_E = \{v_e, v'_e, 1_e, 2_e, 3_e, 4_e : e \in E\}$. Also, H has $n + 9m$ edges.
- For each vertex $v \in V$, we add a pendant edge, that is copy of the vertex gadget $H(v)$;
- Each edge $e = uv$ of G is replaced by the edge gadget $H(e)$ between vertices u and v.

This construction is computable within polynomial-time, $\Delta(H) = 4$, H is connected, bipartite with $2n + 6m$ vertices and $9m + n$ edges. An illustration of this construction is given in Fig. 3 for $G = C_4$. We claim that there exists a maximal independent set S in G of size k if and only if there exists a minimal edge dominating set S' in H of size $4m + k$.

For, let S be a maximal independent set in G. As $V \setminus S$ is a vertex cover in G, for each edge $e \in E$, choose $v^e \in V \setminus S$ to be a vertex of G that covers edge e. Define the set S' of edges in H as follows:

$$S' = \{vv' : v \in S\} \cup \{v^e v_e, 1_e 4_e, 2_e 4_e, 3_e 4_e : e \in E\}.$$

By construction, S' is a minimal edge dominating set of H of size $4m + |S|$. Hence, we deduce:

$$\Gamma'(H) \geq 4m + \alpha(G) \tag{14}$$

Conversely, let S' be a minimal edge dominating set of H. We will transform S' into another minimal edge dominating set S'', where $|S''| \geq |S'|$, using properties of H.

Claim 6.2. *Given a minimal edge dominating set S' of H, there exists a minimal edge dominating set S'' of H, where (i) $|S''| \geq |S'|$, (ii) for every $e \in E$, $v_e v'_e \notin S''$, (iii) for every $e = xy \in E$, $|S'' \cap \{xv_e, v_e y\}| = 1$, and (iv) for every $e \in E$, $\{1_e 4_e, 2_e 4_e, 3_e 4_e\} \subseteq S''$.*

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Proof. We prove this claim by modifying S' according to several cases; in general we require to carry out the process iteratively for each case until the constructed solution S'' satisfies the required properties. It can be assumed that, at the beginning of the second or later iterations, the latest constructed set S'' plays the role of S' in what follows.

1. Assume $|S' \cap \{xv_e, v_e y, v_e v'_e\}| = 3$ for some edge $e = xy \in E$. Then $|S' \cap \big(E(H(e) \setminus \{xv_e, v_e y, v_e v'_e\})\big)| = 1$ using a similar argument to that given in the proof of Property 4.2. We replace S' by $S'' = (S' \setminus E(H(e))) \cup \{xv_e, v_e y, 1_e 4_e, 2_e 4_e, 3_e 4_e\}$; solution S'' remains a minimal edge dominating set of H and $|S''| = |S'| + 1$.

2. Assume $|S' \cap \{xv_e, v_e y, v_e v'_e\}| \leq 2$ for every edge $e \in E$, and the equality is tight for some $e = xy \in E$. We distinguish two cases depending on $v_e v'_e \in S'$ or not.
   (2.1). If $v_e v'_e \in S'$, then we have $|S' \cap \big(E(H(e) \setminus \{xv_e, v_e y, v_e v'_e\})\big)| = 1$. We replace S' by $S'' = (S' \setminus \{v_e v'_e, 1_e v'_e, 2_e v'_e, 3_e v'_e\}) \cup \{1_e 4_e, 2_e 4_e, 3_e 4_e\}$. As before, S'' remains a minimal edge dominating set of H and $|S''| = |S'| + 1$.
   (2.2). Assume $|S' \cap \{xv_e, v_e y\}| = 2$. Then, $xx' \notin S'$ and $yy' \notin S'$. We replace S' by $S'' = (S' \setminus \{xv_e\}) \cup \{xx'\}$. As before, S'' remains a minimal edge dominating set of H and $|S''| = |S'|$.

3. Assume $|S' \cap \{xv_e, v_e y, v_e v'_e\}| \leq 1$ for every edge $e \in E$, and $v_e v'_e \in S'$ for some $e = xy \in E$.
   (3.1). If $xx' \in S'$ or $yy' \in S'$ (without loss of generality, we assume that $xx' \in S'$), then we replace S' by

   $$S'' = (S' \setminus \{xx', v_e v'_e, 1_e v'_e, 2_e v'_e, 3_e v'_e\}) \cup \{xv_e, 1_e 4_e, 2_e 4_e, 3_e 4_e\}.$$

   As before, S'' remains a minimal edge dominating set of H (because $\forall e \in E, |S' \cap \{xv_e, v_e y, v_e v'_e\}| \leq 1$) and $|S''| = |S'| + 1$.
   (3.2). Assume now that $\{xx', yy'\} \cap S' = \emptyset$. Then to dominate xx', there must exist an edge $f = xz \in E$ such that $xv_f \in S'$. As $|S' \cap \{xv_f, v_f z, v_f v'_f\}| \leq 1$, it follows that $v_f v'_f \notin S'$. Thus we replace S' by

   $$S'' = \left(S' \setminus \{v_e v'_e, 1_e v'_e, 2_e v'_e, 3_e v'_e, 1_f v'_f, 2_f v'_f, 3_f v'_f\}\right)$$
   $$\cup \{xv_e, 1_e 4_e, 2_e 4_e, 3_e 4_e, 1_f 4_f, 2_f 4_f, 3_f 4_f\}.$$

   The constructed set S'' is a minimal edge dominating set of H because xv_e has $v_e v'_e$ as a private edge and xv_f has $v_f v'_f$ as a private edge. Moreover, there exists an edge $yv_{f'} \in S' \cap S''$ for some $f' \in E$, where $f' \neq e$, that has yy' as a private edge. Finally $|S''| \geq |S'| + 2$.

4. Finally, assume $|S' \cap \{xv_e, v_e y, v_e v'_e\}| \leq 1$ for every edge $e \in E$, and $|S' \cap \{xv_e, v_e y, v_e v'_e\}| = 0$ for some $e = xy \in E$.
   (4.1). If $xx' \in S'$, then we replace S' by $S'' = (S' \setminus (E(H(e)) \cup \{xx'\})) \cup \{xv_e, 1_e 4_e, 2_e 4_e, 3_e 4_e\}$. As before, S'' remains a minimal edge dominating set of H and $|S''| \geq |S'|$.
   (4.2). Assume that $\{xx', yy'\} \cap S' = \emptyset$. We proceed as in Case (3.2), as follows. To dominate xx', there must exist an edge $f = xz \in E$ such that $xv_f \in S'$. As $|S' \cap \{xv_f, v_f z, v_f v'_f\}| \leq 1$, it follows that $v_f v'_f \notin S'$. Thus we replace S' by

   $$S'' = \left(S' \setminus \{1_e v'_e, 2_e v'_e, 3_e v'_e, 1_f v'_f, 2_f v'_f, 3_f v'_f\}\right)$$
   $$\cup \{xv_e, 1_e 4_e, 2_e 4_e, 3_e 4_e, 1_f 4_f, 2_f 4_f, 3_f 4_f\}.$$

   The constructed set S'' is a minimal edge dominating set of H because xv_e has $v_e v'_e$ as a private edge and xv_f has $v_f v'_f$ as a private edge. Moreover, there exists an edge $yv_{f'} \in S' \cap S''$ for some $f' \in E$, where $f' \neq e$, that has yy' as a private edge. Finally $|S''| \geq |S'| + 2$.

5. Suppose that $\{1_e 4_e, 2_e 4_e, 3_e 4_e\} \nsubseteq S'$ for some $e \in E$. Then we replace S' by $S'' = (S' \setminus \{1_e v'_e, 2_e v'_e, 3_e v'_e\}) \cup \{1_e 4_e, 2_e 4_e, 3_e 4_e\}$. Since $v_e v'_e \notin S''$, it follows that S'' is a minimal edge dominating set of H and $|S''| \geq |S'|$.

To see that this process terminates, we apply Case 1 until there are no edges of G satisfying this property (every time Case 1 is applied we obtain a new instance of Case 2). We then apply Case 2 until there are no edges of G satisfying this property (every time Case 2 is applied we may obtain a new instance of Case 3, but no new instance of no other case). Every time Cases 3 and 4 are applied we obtain no new instances of any case. Once Cases 1-4 no longer apply, Properties (i)-(iii) of the claim are satisfied. We then apply Case 5 until there are no edges of G satisfying this property, and once this occurs, Properties (i)-(iv) of the claim hold. □

Let $S = \{v \in V : vv' \in S''\}$. We claim that S is an independent set in G. For, suppose $x \in S$ and $y \in S$ for some edge $e = xy \in E$. Then $xx' \in S''$ and $yy' \in S''$. By Property (iii) of Claim 6.2, suppose without loss of generality that $xv_e \in S''$. Then xx' does not have a private edge, a contradiction to the minimality of S''. Moreover by Properties (iii) and (iv) of Claim 6.2, it follows that $|S| \geq |S''| - 4m$. Thus, by letting S' be a minimal edge dominating set of maximum size in H, it follows that $|S''| = |S'| = \Gamma'(H)$. Hence, using inequality (14), we obtain:

$$\Gamma'(H) = 4m + \alpha(G) \tag{15}$$

We finally note that, from any minimal edge dominating set S' of H, we can in polynomial time build an independent set S of G satisfying $\alpha(G) - |S| \leq \Gamma'(H) - |S'|$. Moreover, observe that $n \leq \chi(G)\alpha(G)$, where $\chi(G)$ is the chromatic number

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of G. Also $\chi(G) \leq \Delta(G) + 1$ by Brooks' Theorem. Recall that $\Delta(G) \leq 3$. It follows that $2m \leq 3n \leq 12\alpha(G)$, and hence by Equation 15, we obtain $\Gamma'(H) \leq 25\alpha(G)$.

Thus the reduction shown above is an L-reduction with parameters $\alpha = 25$ and $\beta = 1$ (see [5]). It follows that Upper EDS is APX-complete even for bipartite graphs of maximum degree 4.

Let ρ_I and ρ_U denote the best possible approximation ratios for Max IS and Upper EDS, respectively. Given a graph G of maximum degree 3 as an instance of Max IS, we build a bipartite graph H of maximum degree 4 as in the above reduction. Using the supposed approximation algorithm in H for Upper EDS with ratio ρ_U, we may build a minimal edge dominating set S' in H such that $|S'| \geq \rho_U \Gamma'(H)$.

As above, from S', we may build an independent set S in G in polynomial time such that $\alpha(G) - |S| \leq \Gamma'(H) - |S'|$. It follows that

$$\rho_U \leq \frac{|S'|}{\Gamma'(H)} \leq 1 - \left(\frac{\alpha(G) - |S|}{\Gamma'(H)} \right) \leq 1 - \left(\frac{\alpha(G) - |S|}{25\alpha(G)} \right) = \frac{24}{25} + \frac{|S|}{25\alpha(G)}$$

Hence $\frac{|S|}{\alpha(G)} \geq 25\rho_U - 24$. If $25\rho_U - 24 > \rho_I$ then we contradict the choice of ρ_I as the best possible approximation ratio for Max IS. It follows that $25\rho_U - 24 \leq \rho_I$, and hence $\rho_U \leq \frac{24 + \rho_I}{25}$.

Max IS in graphs of maximum degree 3 is not approximable a within ratio $\frac{94}{95} + \varepsilon$, for any $\varepsilon > 0$, unless P = NP [12]. It follows that Upper EDS in bipartite graphs of maximum degree 4 is not approximable a within ratio of $\frac{2374}{2375} + \varepsilon$, for any $\varepsilon > 0$, unless P = NP.   □

Corollary 6.3. Upper EDS *is* NP-*hard in planar bipartite graphs of maximum degree 4.*

Proof. The reduction given in Theorem 6.1 maintains planarity. Hence, using the NP-hardness of Max IS in planar graphs of maximum degree 3 [20,17], we obtain the expected result.   □

7. Concluding remarks

We conclude our paper by giving several open problems concerning the complexity and approximability of Upper EDS.

- Can our lower bound of $n^{\varepsilon - \frac{1}{2}}$ for the approximability of Upper EDS in general graphs be shown to hold for graphs of minimum degree at least 2? If not, can the upper bound of $\frac{1}{\sqrt{n}}$ for the approximability of Upper EDS in graphs without leaves be improved?
- Can we approximate Upper EDS within a ratio of $\frac{1}{\sqrt{n}}$ in general graphs? Or, can our lower bound of $n^{\varepsilon - \frac{1}{2}}$ for the approximability of Upper EDS in general graphs be improved?
- Our NP-hardness result for Upper EDS in planar bipartite graphs of maximum degree 4 leaves open the complexity in (planar) bipartite graphs of maximum degree 3. Is it possible to design a polynomial-time algorithm for such graphs, or is the problem NP-hard?
- Is Upper EDS in APX in bipartite graphs? Or, can we extend our lower bound result for general graphs to the bipartite case? Or, can we observe some approximability behaviour that is in between, such as a ratio of $\Theta(\log n)$?
- Is there a PTAS for approximating Upper EDS in planar graphs?

It also remains open to investigate (exponential-time) exact algorithms for Upper EDS, and to study Upper EDS from the perspective of parameterised complexity. It is also not clear whether one can enumerate all minimal edge dominating sets in time $O^*(c^n)$ for some $c < 2$. The only prior work on this problem that we are aware of is an algorithm to output all minimal edge dominating sets in incremental polynomial time [21].

As an aside let us finally remark that even for Upper DS, there are many open questions concerning parameterised complexity. For instance it is open if Upper DS (with the standard parameterisation, on general graphs) belongs to W[1] or if it is hard for W[2], see [9].

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Highlights

- This is the first paper to focus on Upper EDS, the problem of finding a minimal edge dominating set of maximum size in a given graph $G = (V, E)$.
- We show that Upper EDS is not approximable within a ratio of $n^{\epsilon - \frac{1}{2}}$, for any $\epsilon \in (0, \frac{1}{2})$, unless $P = NP$, where $n = |V|$.
- For graphs of minimum degree at least 2, we show that Upper EDS is approximable within a ratio of $1/\sqrt{n}$.
- We show that Upper EDS is APX-complete in bipartite graphs of maximum degree 4, and NP-hard in planar bipartite graphs of maximum degree 4.