

## Vertex similarity in networks

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We consider methods for quantifying the similarity of vertices in networks. We propose a measure of similarity based on the concept that two vertices are similar if their immediate neighbors in the network are themselves similar. This leads to a self-consistent matrix formulation of similarity that can be evaluated iteratively using only a knowledge of the adjacency matrix of the network. We test our similarity measure on computer-generated networks for which the expected results are known, and on a number of real-world networks.

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### I. INTRODUCTION AND BACKGROUND

The study of networked systems, including computer networks, social networks, biological networks, and others, has attracted considerable attention in the recent physics literature [1–3]. A number of structural properties of networks have been the subject of particularly intense scrutiny, including the lengths of paths between vertices [4–7], degree distributions [8–10], community structure [11–14], and various measures of vertex centrality [4,15–17].

Another important network concept that has received comparatively little attention is vertex similarity. There are many situations in which it would be useful to be able to answer questions such as “How similar are these two vertices?” or “Which other vertices are most similar to this vertex?” Of course, there are many senses in which two vertices can be similar. In the network of the World Wide Web, for instance, in which vertices represent Web pages, two pages might be considered similar if the text appearing on them contains many of the same words. In a social network representing friendship between individuals, two people might be considered similar if they have similar professions, interests, or backgrounds. In this paper we consider ways of determining vertex similarity based solely on the structure of a network. Given only the pattern of edges between vertices in a network, we ask, can we define useful measures that tell us when two vertices are similar? Similarity of this type is sometimes called *structural similarity*, to distinguish it from social similarity, textual similarity, or other similarity types. It is a basic premise of research on networks that the structure of a network reflects real information about the vertices the network connects, so it seems reasonable that meaningful structural similarity measures might exist. Here we show that indeed they do and that they can return useful information about networks.

The problem of quantifying similarity between vertices in a network is not a new one. The most common approach taken in previous work has been to focus on so-called *structural equivalence* [18]. Two vertices are considered structurally equivalent if they share many of the same network neighbors. For instance, it may be reasonable to conclude that two individuals in a social network have something in common if they share many of the same friends. Let  $\Gamma_i$  be the neighborhood of vertex  $i$  in a network, i.e., the set of

vertices that are directly connected to  $i$  via an edge. Then the number of common friends of  $i$  and  $j$  is

$$\sigma_{\text{unnorm}} = |\Gamma_i \cap \Gamma_j|, \quad (1)$$

where  $|x|$  indicates the cardinality (i.e., number of elements in) the set  $x$ , so that  $|\Gamma_i|$ , for instance, is simply equal to the degree of vertex  $i$ .

The quantity  $\sigma_{\text{unnorm}}$  can be regarded as a rudimentary measure of similarity between  $i$  and  $j$ . It is, however, not entirely satisfactory. It can take large values for vertices with high degree even if only a small fraction of their neighbors are the same, and in many cases this runs contrary to our intuition about what constitutes similarity. Commonly therefore one normalizes in some way—for instance, so that the similarity is one when  $\Gamma_i = \Gamma_j$ . We are aware of at least three previously proposed ways of doing this [19–21]:

$$\sigma_{\text{Jaccard}} = \frac{|\Gamma_i \cap \Gamma_j|}{|\Gamma_i \cup \Gamma_j|}, \quad (2a)$$

$$\sigma_{\text{cosine}} = \frac{|\Gamma_i \cap \Gamma_j|}{\sqrt{|\Gamma_i| |\Gamma_j|}}, \quad (2b)$$

$$\sigma_{\text{min}} = \frac{|\Gamma_i \cap \Gamma_j|}{\min(|\Gamma_i|, |\Gamma_j|)}. \quad (2c)$$

The first of these, commonly called the *Jaccard index*, was proposed by Jaccard over a hundred years ago [19]; the second, called the *cosine similarity*, was proposed by Salton in 1983 and has a long history of study in the literature on citation networks [20,22,23]. (Measures nonlinear in  $\sigma_{\text{unnorm}}$  are also possible. For example, Refs. [24,25] propose measures involving  $\sqrt{\sigma_{\text{unnorm}}}$  and  $\sigma_{\text{unnorm}}^2$ , respectively.)

There are, however, many cases in which vertices occupy similar structural positions in networks without having common neighbors. For instance, two store clerks in different towns occupy similar social positions by virtue of their numerous professional interactions with customers, although it is quite likely that they have none of those customers in common. Two CEOs of companies occupy similar positions by virtue of their contacts with other high-ranking officers of the companies for which they work, although again none of

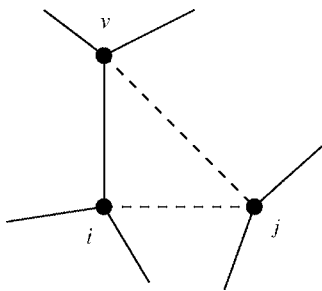


FIG. 1. A vertex  $j$  is similar to vertex  $i$  (dashed line) if  $i$  has a network neighbor  $v$  (solid line) that is itself similar to  $j$ .

the individual officers need be common to both companies. Considerations of this kind lead us to an extended definition of network similarity known as *regular equivalence*. In this case vertices are said to be similar if they are connected to other vertices *that are themselves similar*. It is upon this idea that the measures developed in this paper are based.

Regular equivalence is clearly a self-referential concept: one needs to know the similarity of the neighbors of two vertices before one can compute the similarity of the two vertices themselves. It comes as no surprise to learn therefore that traditional algorithms for computing regular equivalence have an iterative or recursive nature. Two of the best known such algorithms, REGE and CATREGE [26], proceed by searching for optimal matching between the neighbors of the two vertices, while other authors have formulated the calculation as an optimization problem [27].

In this paper, we take a different approach, constructing measures of similarity using the methods of linear algebra. The fundamental statement of our approach is that vertices  $i$  and  $j$  are similar if either of them has a neighbor  $v$  that is similar to the other—see Fig. 1. Coupled with the additional assumption that vertices are trivially similar to themselves, this gives, as we will see, a sensible and straightforward formulation of the concept of regular equivalence for undirected networks. The method has substantial advantages over other similarity measures: it is global—unlike the Jaccard index and related measures, it depends on the whole graph and allows vertices to be similar without sharing neighbors; it has a transparent theoretical rationale, which more complex methods like REGE and CATREGE lack [26]; it avoids the convergence problems that have plagued optimization methods; and it is comparatively fast, since its implementation can take advantage of standard, hardware optimized, linear algebra software.

Some previous authors have also considered similarity measures based on matrix methods [28,29]. We discuss the differences between our measure and these previous ones in Sec. II C.

This paper is organized as follows: In Sec. II we present the derivation of our structural similarity measure. Section III we test the measure on a number of networks, including computer-generated graphs (Secs. III A and III B) and real-world examples (Secs. III C and III D). In Sec. IV we give our conclusions.

## II. A MEASURE OF SIMILARITY

The fundamental starting point for our measure of similarity is the assumption that the edges in a network them-

selves indicate a similarity between the vertices they connect. Thus, for instance, we assume that two people in a social network are more likely to be connected if they are similar, in some social sense, than if they are dissimilar. The edges of the network provide the raw data from which we will deduce more subtle similarity values, including values for pairs of vertices that are not directly connected.

It is worth noticing that it is not always the case that the edges in a network fall between similar vertices. Some networks are said to be “disassortative” [30], meaning that edges preferentially connect vertices that are different in some way. Although the measures derived in this paper may convey useful information even in these cases, we will for the purposes of argument assume that the networks we are looking at are not disassortative; rather they are “assortative” and edges tend to connect vertices that are fundamentally similar.

This then leads us immediately to the idea of regular equivalence: a pair of vertices  $i, j$  are similar to one another if any pair  $u, v$  of their neighbors are similar. This follows by simple repeated application of the principle above:  $i$  is similar to  $u$ , which is similar to  $v$ , which is similar to  $j$ .

In fact, an even simpler one-step expression of the principle is possible: vertex  $i$  is similar to  $j$  if  $i$  has any network neighbor  $v$  that is itself similar to  $j$ . This idea is illustrated in Fig. 1 and it forms the basis for the measure of similarity developed here. At first glance this definition might appear less satisfactory than the two-step version, having an asymmetry between  $i$  and  $j$  that the two-step definition lacks. As we will see, however, it makes no difference to the results if we swap vertices  $i$  and  $j$ : the mathematical expression for the similarity turns out to be the same and hence the definition is in fact symmetric.

This definition of similarity is clearly recursive and hence we need to provide some starting point for the recursion in order to make the results converge to a useful limit. The starting point we choose is to make each vertex similar to itself, which is natural in most situations. Our definition of similarity will thus have two components: the neighbor term of the previous paragraph and the self-similarity.

Thus our first guess at the form of the similarity (we will improve it later) is to write the similarity  $S_{ij}$  of vertex  $i$  to vertex  $j$  as

$$S_{ij} = \phi \sum_v A_{iv} S_{vj} + \psi \delta_{ij}, \quad (3)$$

where  $\delta_{ij}$  is Kronecker’s function and  $A_{ij}$  is an element of the adjacency matrix of the network, taking the value

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge between } i \text{ and } j, \\ 0 & \text{otherwise,} \end{cases} \quad (4)$$

and  $\phi$  and  $\psi$  are free parameters whose values control the balance between the two components of the similarity.

Considering  $S_{ij}$  to be the  $ij$  element of a similarity matrix  $S$ , we can write Eq. (3) in matrix form as

$$\mathbf{S} = \phi \mathbf{A} \mathbf{S} + \psi \mathbf{I}, \quad (5)$$

where  $\mathbf{I}$  is the identity matrix. Rearranging, this can also be written  $\mathbf{S} = \psi [\mathbf{I} - \phi \mathbf{A}]^{-1}$ . As we see, the parameter  $\psi$  merely contributes an overall multiplicative factor to our similarity. Since in essentially all cases we will be concerned not with the absolute magnitude of the similarity but only with the relative similarity of different pairs of vertices, we can safely set  $\psi=1$ , eliminating one of our free parameters, and giving

$$\mathbf{S} = [\mathbf{I} - \phi \mathbf{A}]^{-1}. \quad (6)$$

This expression for similarity bears a close relation to the matrix-based centrality measure of Katz [31]. In fact, the Katz centrality of a vertex is equal simply to the sum of that vertex's similarities to every other vertex. This is a natural concept: a vertex is prominent in a network if it is closely allied with many other vertices.

We can also consider the similarity of  $i$  and  $j$  when  $j$  has a neighbor  $v$  that is similar to  $i$ . In that case,

$$S_{ij} = \phi \sum_v S_{iv} A_{vj} + \psi \delta_{ij}. \quad (7)$$

It is trivial to show, however, that this leads to precisely the same expression, Eq. (6), for the similarity in the end. Thus, as we claimed above, our definition provides only one similarity value for any pair of vertices, given by the symmetric matrix  $\mathbf{S}$  of Eq. (6).

The remaining parameter  $\phi$  in Eq. (6) is still free. To shed light on the appropriate value for this parameter, let us expand the similarity as a power series thus:

$$\mathbf{S} = \mathbf{I} + \phi \mathbf{A} + \phi^2 \mathbf{A}^2 + \cdots. \quad (8)$$

Noting that the element  $[\mathbf{A}^l]_{ij}$  is equal to the number of (possibly self-intersecting) network paths of length  $l$  from  $i$  to  $j$ , this equation gives us an alternative, term-by-term interpretation of our similarity measure. The first term says that a vertex is identically similar to itself. The second term says that vertices that are immediate neighbors of one another have similarity  $\phi$ . The third term says that vertices that are distance two apart on the network have similarity  $\phi^2$ , and so forth.

But notice also that vertices that have many paths of a given length are considered more similar than those that have few. The similarity of vertices  $i$  and  $j$  acquires a contribution  $\phi^2$  for *every* path of length 2 from  $i$  to  $j$ . We note, however, that some pairs of vertices are *expected* to have one or even many such paths between them: vertices with very high degree, for instance, will almost certainly have one or several paths of length two connecting them, even if connections between vertices are just made at random. So simple counts of number of paths are not enough to establish similarity. We need to know when a pair of vertices has more paths of a given length between than we would expect by chance.

This suggests a strategy for choosing  $\phi$ . We will normalize each term in our series by dividing the number of paths of length  $l$  (given by the power of the adjacency matrix) by the *expected* number of such paths, were vertices in the network connected at random. Then each term will be greater or less than unity by a factor representing the extent to which

the corresponding vertices have more or fewer paths of the appropriate length than would be expected by chance. In fact, there is no single choice of the parameter  $\phi$  that will simultaneously achieve this normalization for every term in the series but, as we will show, there is a choice that achieves it approximately for every term and exactly in the asymptotic limit of high terms in the series, if we allow a slight (and with hindsight sensible) modification of Eq. (6).

### A. Expected number of paths

Let us generalize the series, Eq. (8), to allow an independent coefficient for each term and for each vertex pair  $i, j$ :

$$S_{ij} = \sum_{l=0}^{\infty} C_{ij}^l [\mathbf{A}^l]_{ij}. \quad (9)$$

And let us choose (for the moment) each coefficient to be equal to 1 over the expected number of paths of the corresponding length between the same pair of vertices on a network with the same degree sequence as the network under consideration, but in which the vertices are otherwise randomly connected. Such a network is called a *configuration model*, and the configuration model has been widely studied in the networks literature [32–34].

It is possible to derive exact values for the expected number of paths between vertices of given degree in the configuration model, and for the interested reader we give some results in the Appendix. But the expressions become more complicated as path length increases and the numbers of distinct topologies of the paths multiply. This makes such calculations useful only for rather short paths and not applicable for the infinite series of Eq. (9). Instead therefore we here proceed by deriving an approximate expression for the expected number of paths that can be evaluated at any order.

The expected number of paths of length  $l$  from  $i$  to  $j$  can be written as the  $j$ th element of the vector  $\mathbf{p}_l$  given by

$$\mathbf{p}_l = \mathbf{A}^l \mathbf{v}, \quad (10)$$

where the vector  $\mathbf{v}$  has all elements zero except for  $v_i=1$ . In the limit of large  $l$ , the vector  $\mathbf{p}_l$  tends toward (a multiple of) the leading eigenvector of the adjacency matrix, and hence in this limit we have  $\mathbf{p}_{l+1} = \lambda_1 \mathbf{p}_l$ , where  $\lambda_1$  is the largest eigenvalue of  $\mathbf{A}$ . Thus the number of paths from  $i$  to  $j$  increases by a factor of  $\lambda_1$  each time we add one extra step to the path length. The first step of the path violates this rule: we know the number of paths increases by exactly a factor of  $k_i$  on the first step. Furthermore, since our paths are constrained to end at vertex  $j$ , the last step must end at one of the  $k_j$  edges emanating from  $j$ , out of a total of  $2m$  possible places that it could end. This introduces a factor of  $k_j/2m$  into the expected number of paths. Thus, to within a multiplicative constant, the number of paths of length  $l$  from  $i$  to  $j$ , for large  $l$ , should be  $(k_i k_j / 2m) \lambda_1^{l-1}$ .

This expression is not in general correct for small  $l$ . It is, however, correct for the particular case  $l=1$  of paths of length one [see Eq. (A1)] and we expect it to be approximately correct for other intermediate values of  $l > 1$ . Guided by these results, we therefore choose the constants  $C_{ij}^l$  appearing in Eq. (9) to take the values

$$C_l^{ij} = \frac{2m}{k_i k_j} \lambda_1^{-l+1}, \quad (11)$$

for  $l \geq 1$ , with  $C_0^{ij} = \delta_{ij}$ . These values approximate the desired values based on expected numbers of paths and are asymptotically correct in the limit of large  $l$ .

### B. Derivation of the similarity

There is one more issue we need to deal with before we arrive at a final expression for our similarity. If we simply substitute  $C_l^{ij}$  from Eq. (11) into Eq. (9) we produce a series that unfortunately does not converge. To see this, note that in the limit of very long path lengths most networks will have roughly the same number of paths between vertices as would be expected by chance. (The local structure around the vertices in question is unimportant in this case, because the length of the path means that correlation with its start and end points is weak for most of its length.) This means that the terms in the series (9) will tend to unity for large  $l$  and, since there are an infinite number of them, the sum will therefore diverge. On the other hand, if, rather than being constant, consecutive terms were to decrease by even the tiniest factor at each order, the sum would converge, as do all series with exponentially decreasing terms. Thus we can ensure convergence by introducing an extra numerical factor  $\alpha$ , giving a series thus:

$$\begin{aligned} S_{ij} &= \delta_{ij} + \frac{2m}{k_i k_j} \sum_{l=1}^{\infty} \alpha^l \lambda_1^{-l+1} [A^l]_{ij} \\ &= \left[ 1 - \frac{2m\lambda_1}{k_i k_j} \right] \delta_{ij} + \frac{2m\lambda_1}{k_i k_j} \left[ \left( \mathbf{I} - \frac{\alpha}{\lambda_1} \mathbf{A} \right)^{-1} \right]_{ij}. \end{aligned} \quad (12)$$

In physical terms, the effect of the parameter  $\alpha$  is to reduce the contribution of long paths relative to short ones. That is, for  $0 < \alpha < 1$ , our similarity measure considers vertices to be more similar if they have a greater than expected number of short paths between them, than if they have a greater than expected number of long ones. While this is a natural route to take, it does mean we have introduced a new free parameter into our calculations. This seems a fair exchange: we have traded the infinite number of free parameters in the expansion of Eq. (9) for just a single such parameter. We discuss the appropriate choice of value for  $\alpha$  in Sec. III B.

The first term in Eq. (12) is diagonal in the vertices  $i$  and  $j$  and hence affects only the similarity of vertices to themselves, which we are not usually interested in, so we will henceforth drop it. Thus our final expression for the similarity is

$$S_{ij} = \frac{2m\lambda_1}{k_i k_j} \left[ \left( \mathbf{I} - \frac{\alpha}{\lambda_1} \mathbf{A} \right)^{-1} \right]_{ij}. \quad (13)$$

Equivalently, we could write this in matrix form thus:

$$\mathbf{S} = 2m\lambda_1 \mathbf{D}^{-1} \left( \mathbf{I} - \frac{\alpha}{\lambda_1} \mathbf{A} \right)^{-1} \mathbf{D}^{-1}, \quad (14)$$

where  $\mathbf{D}$  is the diagonal matrix having the degrees of the vertices in its diagonal elements:  $D_{ij} = k_i \delta_{ij}$ .

This similarity measure takes exactly the form we postulated in Eq. (6) with  $\phi = \alpha/\lambda_1$ , except for an overall multiplier, which is trivial, and the leading factor of  $1/k_i k_j$ , which is not. This factor compensates for the fact that we expect there to be more paths between pairs of vertices with high degree simply because there are more ways of entering and leaving such vertices. Its presence is crucial if we wish to compare the similarities of vertex pairs having very different degrees.

In practical terms, the calculation of the similarity matrix is most simply achieved by direct multiplication. Dropping the constant factor  $2m\lambda_1$  for convenience, we can rewrite Eq. (14) in the form of Eq. (3) thus:

$$\mathbf{DSD} = \frac{\alpha}{\lambda_1} \mathbf{A}(\mathbf{DSD}) + \mathbf{I}. \quad (15)$$

Making any guess we like for an initial value of  $\mathbf{DSD}$ , such as  $\mathbf{DSD} = 0$ , we iterate this equation repeatedly until it converges. In practice, for the networks studied here, we have found good convergence after 100 iterations or less.

### C. Comparison with previous similarity measures

Several other authors have proposed vertex similarity measures based on matrix methods similar to ours [28,29].

Jeh and Widom [28] have proposed a method that they call ‘‘SimRank,’’ predicated, as ours is, on the idea that vertices are similar if their neighbors are similar. In our notation, their measure is

$$S_{ij} = \frac{C}{k_i k_j} \sum_{u,v} A_{iu} A_{vj} S_{uv}, \quad (16)$$

where  $C$  is a constant. While this expression bears some similarity to ours, Eq. (3), it has an important difference also. Starting from an initial guess for  $S_{ij}$ , one can iterate to converge on a complete expression for the similarity, and this final expression contains terms representing path counts between vertex pairs, as in our case. However, since the adjacency matrix appears twice on the right-hand side of Eq. (16), the expression includes *only paths of even length*. This can make a substantial difference to the resulting figures for similarity. An extreme example would be a bipartite network, such as a tree or a square lattice, in which vertices are separated either only by paths of even length or only by paths of odd length. In such cases, those vertices that are separated only by paths of odd length will have similarity *zero*. Even vertices that are directly connected to one another by an edge will have similarity zero. Most people would consider this result counterintuitive, and our measure, which counts paths of all lengths, seems clearly preferable.

Blondel *et al.* [29] considered similarity measures for directed networks, i.e., based on asymmetric adjacency matrices, which is a more complex situation than the one we consider. However, for the special case of a symmetric matrix, the measure of Blondel *et al.* can be written as



$$S_{ij} = C \sum_{u,v} A_{iu} A_{vj} S_{uv}, \quad (17)$$

where  $C$  is again a constant. This is very similar to the measure of Jeh and Widom, differing only in the omission of the factor  $1/k_i k_j$ . Like the measure of Jeh and Widom, it can be written in terms of paths between vertices, but counts only paths of even lengths, so that again vertices separated only by odd paths (such as adjacent vertices on bipartite graphs) have similarity zero.

#### D. A measure of structural equivalence

An interesting corollary of the theory developed in the previous sections is an alternative measure of structural equivalence. The structural equivalence measures of Eq. (2) can be viewed as similarity measures that count only the paths of length two between vertex pairs; the number of common neighbors of two vertices is exactly equal to the number of paths of length two. Thus structural equivalence can be thought of as just one term—the second-order term—in the infinite series that defines our measure of regular equivalence.

The measures of Eq. (2) differ from one another in their normalization. The developments outlined in this paper suggest another possible normalization, one in which we divide the number of paths of length 2 by its expected value in the configuration model. An exact expression for this expected value is derived in the appendix, Eq. (A2), and the resulting normalized equivalence measure is then

$$\sigma = \frac{2m}{k_i k_j} \left( \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle} \right) |\Gamma_i \cap \Gamma_j|. \quad (18)$$

If we are concerned only with the comparative similarities of different pairs of vertices within a given graph, then we can neglect multiplicative constants and write

$$\sigma = \frac{|\Gamma_i \cap \Gamma_j|}{k_i k_j} = \frac{|\Gamma_i \cap \Gamma_j|}{|\Gamma_i| |\Gamma_j|}. \quad (19)$$

This is, we feel, in many ways a more sensible measure of structural equivalence than those of Eq. (2). It gives high similarity to vertex pairs that have many common neighbors compared not to the maximum number possible but to the *expected* number of such neighbors, and therefore highlights vertices that have a statistically improbable coincidence of neighborhoods. Of course, one could define similar measures for paths of length 1 or 3 or any other length. Or one could combine all such lengths, which is precisely what our overall similarity measure does.

### III. TESTS OF THE METHOD

In this section we test our method on a number of different networks. Our first example is a set of computer-generated networks designed to have known similarities between vertices. In the following sections we also test the method against some real-world examples.

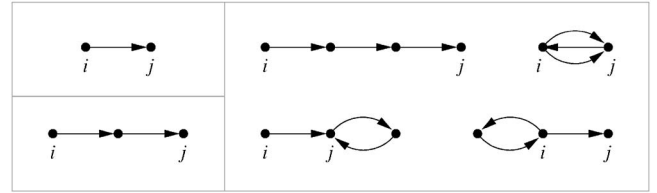


FIG. 2. (a) A density plot of the similarities of all vertex pairs not directly connected by an edge in our stratified network model. The points give the average similarity as a function of age difference and the line is a least-squares fit to a straight line. (b) A density plot of the cosine similarity values for the same network.

#### A. Stratified model network

In many social networks, individuals make connections with others preferentially according to some perceived similarity, such as age or income. Such networks are said to be *stratified*, and stratified networks present a perfect opportunity to test our similarity measure: ideally we would like to see that given only the network structure our measure can correctly identify vertices that are similar in age (or whatever the corresponding variable is) even when the vertices are not directly connected to one another.

As a first test of our measure, we have created artificial stratified networks on a computer. Such networks offer a controlled structure for which we believe we know the “correct” answers for vertex similarity.

In our model networks, each of  $n=1000$  vertices was given one of ten integer “ages.” Then edges were created between vertices with probability

$$P(\Delta t) = p_0 e^{-a \Delta t}, \quad (20)$$

where  $\Delta t$  is the difference in ages of the vertices and  $p_0$  and  $a$  are constants, whose values in our calculations were chosen to be  $p_0=0.12$  and  $a=2.0$ . Thus the probability of “acquaintance” between two individuals drops by a factor of  $e^2$  for every additional year separating their ages.

In order to calculate our similarity measure for this or any network we need first to choose a value for the single parameter  $\alpha$  appearing in Eq. (13). In the present calculations we used a value of  $\alpha=0.97$ , which, as we will see, is fairly typical. Since  $\alpha$  must be strictly less than one if Eq. (13) is to converge,  $\alpha=0.97$  is quite close to the maximum possible value. We discuss in the following section why values close to the maximum are usually desirable.

Figure 2(a) shows a density plot of the similarity values for all vertex pairs in the model network not directly connected by an edge, on semilog scales as a function of the age difference between the vertices. The average similarity as a function of age difference is also plotted, along with a fit to the data. We exclude directly connected pairs in the figure because it is trivial that such pairs will have high similarity and most of the interest in our method is in its ability to detect similarity in nontrivial cases.

For comparison, we also show in Fig. 2(b) a density plot of the cosine similarity, Eq. (2b), for the same network. As the plots show, cosine similarity is in this case a much less revealing measure of similarity. It is only possible for cosine

similarity to be nonzero for a pair of vertices if there exists a path of length two between them. Vertices with an age difference of three or more rarely have such a path in this network and, as Fig. 2(b) shows, such vertices therefore nearly all have a cosine similarity of zero. Thus cosine similarity finds only highly similar vertices in this case and entirely fails to distinguish between vertices with age differences between 3 and 9. Our similarity measure by contrast distinguishes these cases comfortably.

### B. Choice of $\alpha$

Our similarity measure, Eq. (13), contains one free parameter  $\alpha$ , which controls the relative weight placed on short and long paths. This parameter lies strictly in the range  $0 < \alpha < 1$ , with low values placing most weight on short paths between vertices and high values placing weight more equally both on short and long paths. (Values  $\alpha > 1$  would place more weight on long paths than on short, but for such values the series defining our similarity does not converge.)

There is in general no single value of  $\alpha$  that works perfectly for every network, but experience suggests some reliable rules of thumb. Our stratified network model, for instance, provides a good guide. Consider Fig. 3. In this figure we have calculated the correlation coefficient of the similarity values for vertex pairs determined using our method against the probabilities, Eq. (20), of connections between the vertices, which, following the ideas outlined at the beginning of Sec. II, we consider to be a fundamental measure of vertices' *a priori* similarity. As the figure shows, the correlation is quite low for low values of  $\alpha$ , but becomes strong as  $\alpha$  approaches 1. Only as  $\alpha$  gets very close to 1 does the correlation fall off again. This appears to imply that a value of  $\alpha = 0.9$  or greater should give the best results in this case. Furthermore, it appears that, for values of  $\alpha$  in this range, the precise value does not matter greatly, all values around the maximum in the correlation coefficient giving roughly comparable performance.

This we have found to be a good general rule: values of  $\alpha$  close to the maximum value of 1 perform the best, with values in the range 0.90–0.99 being typical. Within this range the results are not highly sensitive to the exact value. We give another example to reinforce this conclusion below. (We also note that the measures of Jeh and Widom [28] and Blondel *et al.* [29], Sec. II C, give poorer correlations than our measure—0.4890(4) and 0.0069(1), respectively.)

The large typical values of  $\alpha$  mean that paths of different lengths are weighted almost equally in our similarity measure. In other words, it appears that our measure works best when long paths are accorded almost as much consideration as short ones. This contrasts strongly with structural equivalence measures like the Jaccard index and the cosine similarity, which are based exclusively on short paths—those of length 2. Indeed, these measures can be considered analogous to measures such as ours in the limit of small  $\alpha$ , where all the weight is placed on the shortest paths, which effectively means paths of length two when we are considering vertex pairs that are not directly connected. Thus, in a sense, our measure, with its near-maximal value of  $\alpha$ , can be con-

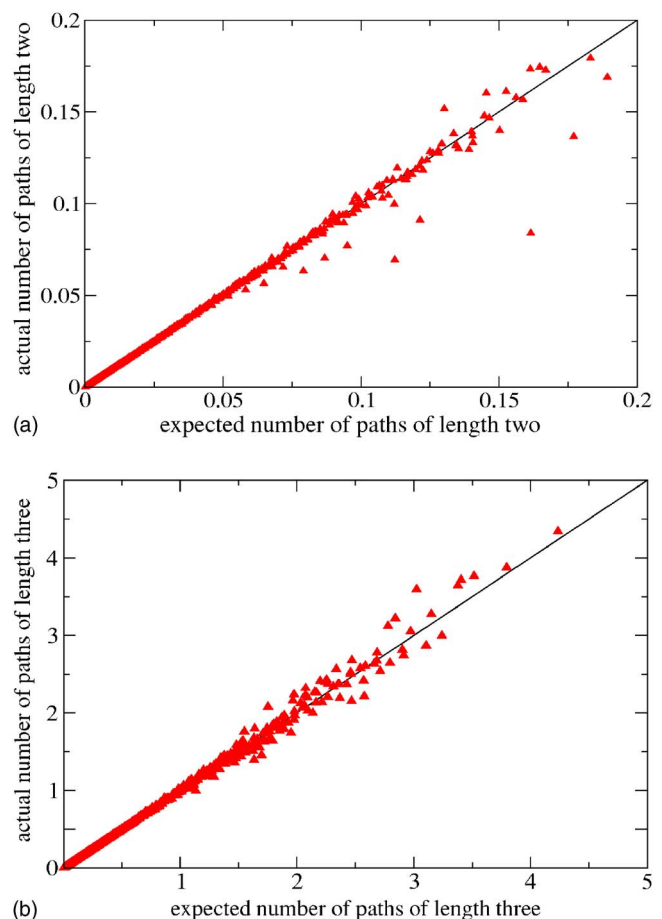


FIG. 3. (Color online) The correlation coefficient  $r(\sigma, \sigma_{\text{age}})$  for correlation between our similarity measure and the probability of connection, Eq. (20), in our stratified model, for a range of values of  $\alpha$ . The values given are averaged over an ensemble of graphs generated from the model. The maximum value is found to occur for  $\alpha \approx 0.97$ .

sidered at the farthest possible extreme from the traditional structural equivalence measures.

### C. Thesaurus network

We now consider two applications of our method to real-world networks. The first is to a network of words extracted from a supplemented version of the 1911 U.S. edition of *Roget's Thesaurus* [35]. The thesaurus consists of a five-level hierarchical categorization of English words. For example, the word “paradise” (level 5) is cataloged under “heaven” (level 4), “superhuman beings and regions” (level 3), “religious affections” (level 2), and “words relating to the sentient and moral powers” (level 1). Here we study the network composed of the 1000 level-4 words, in which two such words are linked if one or more of the level-5 words cataloged below them are common to both. For instance, the level-4 words “book” and “knowledge” are connected because the entries for both in the thesaurus contain the level-5 terms “book learning” and “encyclopedia.”

In Table I we show the words most similar to the words “alarm,” “hell,” “mean,” and “water,” as ranked first by our

TABLE I. The words most similar to “alarm,” “heaven,” “mean,” and “water,” in the word network of the 1911 edition of *Roget’s Thesaurus*, as quantified by our similarity measure and by the more rudimentary cosine similarity of Eq. (2b). We used a value of 0.98 for the parameter  $\alpha$ .

Word	Our measure		Cosine similarity	
alarm	warning	32.014	omen	0.51640
	danger	25.769	threat	0.47141
	omen	18.806	prediction	0.34816
hell	heaven	63.382	pleasure	0.40825
	pain	28.927	discontent	0.28868
	discontent	7.034	weariness	0.26726
mean	compromise	20.027	gravity	0.23570
	generality	19.811	inferiority	0.22222
	middle	17.084	littleness	0.20101
water	plunge	33.593	dryness	0.44721
	air	25.267	wind	0.31623
	moisture	25.267	ocean	0.31623

similarity measure and second by cosine similarity. We used a value of  $\alpha=0.98$  in this case, on the grounds that this value gave the best performance in other test cases (see below).

Since cosine similarity can be regarded as a measure of the number of paths of length 2 between vertices, it tends in this example to give high similarity scores for words at distance two in the thesaurus—synonyms of synonyms, antonyms of synonyms, and so forth. For example, cosine similarity ranks “pleasure” as the word most similar to “hell,” probably because it is closely associated with hell’s antonym “heaven.” By contrast, our measure ranks “heaven” itself first, which appears to be a more sensible association. Similarly, cosine similarity links “water” with “dryness,” whereas our measure links “water” with “plunge.”

#### D. Friendship network of high school students

As a second real-world test of our similarity measure, we apply it to a set of networks of friendships between school children. The network data were collected as part of the National Longitudinal Study of Adolescent Health (AddHealth) [36], and describe 90 118 students at 168 schools, including their school grade (i.e., year), race, and gender, as well as their recent patterns of friendship. It is well known that people with similar social traits tend to associate with one another [37], so we expect there to be a correlation between similarity in terms of personal traits and similarity based on network position. This gives us another method for checking the efficacy of our similarity measure.

The AddHealth data were gathered through questionnaires handed out to students at 84 pairs of American schools, a school pair typically consisting of one junior high school (grades 7 and 8, ages 12–14) and one high school (grades 9–12, ages 14–18). Here we consider the data as a set of networks in which each network contains students from one

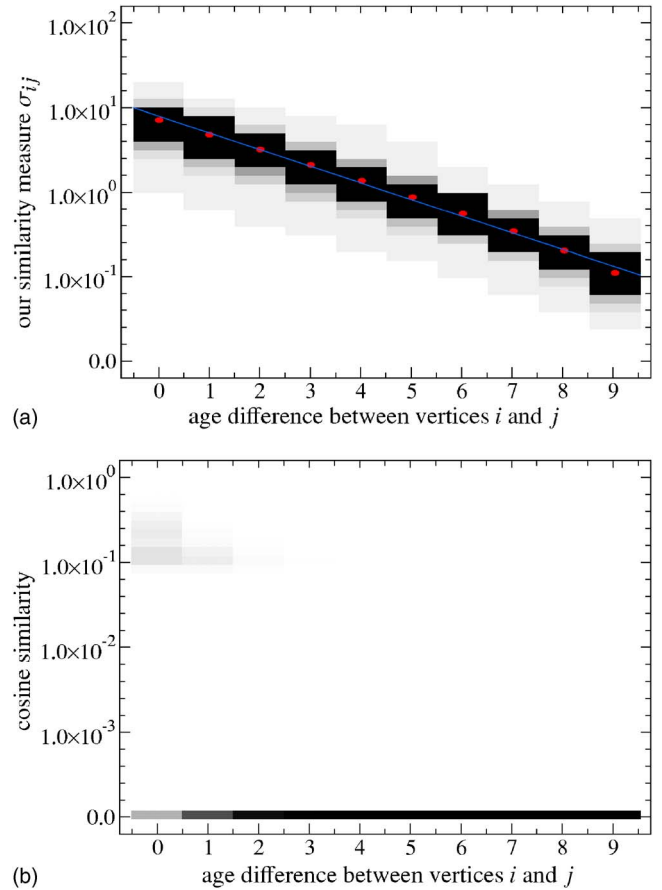


FIG. 4. (Color online) The correlation coefficient for correlation between our similarity measure and the age difference of all vertex pairs in a single network, as a function of  $\alpha$ . This plot is typical for the school networks studied.

pair of schools, with students in all six grades.

The questionnaires asked respondents, among other things, to “List your closest (male/female) friends. List your best (male/female) friend first, then your next best friend, and so on. (Girls/Boys) may include (boys/girls) who are friends and (boy/girl) friends.” For each of the friends listed, the student was asked to state in which of five particular activities they had participated recently with that friend, such as “you spent time with (him/her) last weekend.” From these answers a weight  $w(i, j)$  was assigned to every ordered pair of students  $(i, j)$  such that  $w(i, j)$  is 0 if  $i$  has not listed  $j$  as a friend, or 1 plus the number of activities conducted otherwise. (The additional 1 is necessary because some students list another as a friend but have not participated in any of the listed activities with them recently.) From these weights we construct an unweighted, undirected friendship network by adding a link between vertices  $i$  and  $j$  if  $w(i, j)$  and  $w(j, i)$  are both greater than or equal to a specified threshold value  $W$ . As it turns out, our conclusions are not very sensitive to the choice of  $W$ ; the results described here use  $W=2$ .

The networks derived in this way are not necessarily connected; they may, and often do, consist of more than one component for each school studied. To simplify matters we here consider only on the largest component of each network. The largest component in some of the networks is

TABLE II. Network size  $n$  and ratios of average similarity values for school networks in the AddHealth data set. The column labeled SG:DG gives the ratio of average similarity for students in the same grade (SG) to average similarity for students in different grades (DG). The column labeled SR:DR gives the ratio of average similarity for students of the same race (SR) to average similarity for students of different races (DR). Columns marked with asterisks (\*) give values of the same ratios but omitting vertex pairs connected directly by an edge.

School	$n$	Similarity ratios			
		SG:DG	SG:DG*	SR:DR	SR:DR*
A	1090	8.0	6.1	1.1	1.1
B	1302	6.2	4.4	2.6	2.6
C	1996	2.2	1.9	5.0	5.0
D	1530	3.3	2.6	4.0	3.6

quite small, however, so to avoid finite size effects we have focused on networks whose largest component contains more than 1000 students.

We first test our similarity measure using the method we used for the stratified network of Sec. III A: we determine the linear correlation coefficient between age difference (measured as difference in grade) and our network similarity measure, for all vertex pairs in a network. We have calculated this correlation coefficient for a range of values of  $\alpha$ , the free parameter in our measure, and for a selection of different networks. The results for one particular network are shown in Fig. 4. In this case the correlation coefficient is maximized for  $\alpha \approx 0.99$ , which is again close to the maximum possible value of 1. For other networks we find maxima in the range from 0.96 to 0.99, which is in accord with the results of Sec. III B. (We also calculated correlation coefficients for the similarity measures of Jeh and Widom [28] and Blondel *et al.* [29] discussed in Sec. II C. These values were routinely lower than ones found with our measure.)

These correlations between age difference and network similarity appear to indicate that our similarity measure is able to detect some aspects of the social structure of these networks. To investigate this further, we have also calculated the average similarity of vertex pairs that have a known common characteristic, either grade or race, comparing that average with the average similarity for vertex pairs that differ with respect to the same characteristic. The values of  $\alpha$  used were those corresponding to the peak in the correlation, as above. The results are given in Table II.

For school A, for example, the average similarity for pairs of students in the same grade is a factor of 8 greater than that for pairs in different grades. It is possible, however, that this impressive difference could result purely from the contribution to the similarity from vertex pairs that are directly connected by an edge. It would come as no surprise that such pairs tend to be in the same grade. To guard against this, we give in the fourth column of Table II results for calculations in which all directly connected vertex pairs were removed. Even with these pairs removed we see that same-grade vertex pairs are on average significantly more similar than pairs from different grades.

We have also made similar calculations with respect to the race of students. Students in school A did not appear to have

any significant division along racial lines (columns 5 and 6 of Table II), but this school was almost entirely composed of students of a single race anyway, so this result is not very surprising; it seems likely that the numbers were just too small to show a significant effect. School B was similar. Schools C and D, however, show a marked contrast. In school C, the average similarity for students of the same race is a factor of 5 greater than the average similarity for students of different races. School C had a population split 2:1 between two racial groups, in marked contrast with schools A and B. School D similarly appears to be divided by race, although a little less strongly. In this case there is a three-way split within the population between different racial groups. Possibly this more even split with no majority group was a factor in the formation of friendships between students from different groups.

These results indicate that our measure of similarity is able to identify real social similarity between vertices in these networks. That is, using only the structure of the network, our similarity measure identifies students of the same race and in the same grade to be more similar to each other than students of different grades or different races. For comparison, we performed similar calculations using the similarity measures proposed by Jeh and Widom [28] and Blondel *et al.* [29], finding again that the average similarity of pairs of vertices sharing characteristics was higher than for pairs of vertices that differed by the same characteristic. However, the factors by which these methods differentiated between vertices with similar characteristics and vertices with different characteristics was consistently less than with our measure.

#### IV. CONCLUSIONS

In this paper we have proposed a measure of structural similarity for pairs of vertices in networks. The method is fundamentally iterative, with the similarity of a vertex pair being given in terms of the similarity of the vertices' neighbors. Alternatively, our measure can be viewed as a weighted count of the number of paths of all lengths between the vertices in question. We expect the measure to be applicable to any network where the vertices do not have a tendency to attach to dissimilar vertices. The weights appearing in this count are asymptotically equal to the expected numbers of



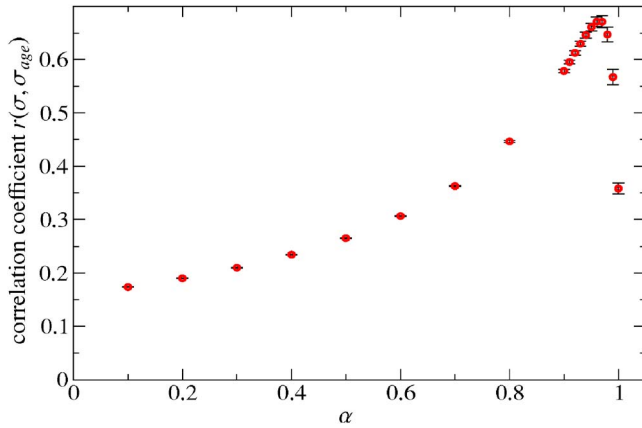


FIG. 5. (Color online) There is only one possible topology for paths of length 1 between distinct vertices, and only one for paths of length 2, but there are four possible topologies for paths of length 3.

network paths between the vertices, which we express in terms of the leading eigenvalue of the adjacency matrix of the network and the degrees of the vertices of interest. The resulting expression for our similarity measure is given in Eq. (14).

We have tested our measure against computer-generated and real-world networks, with promising results. In tests on computer-generated networks the measure is particularly good at discerning similarity between vertices connected by relatively long paths, an area in which more traditional similarity measures such as cosine similarity perform poorly. In tests on real-world networks the method was able to extract sensible synonyms to words from a network representing the structure of Roget's Thesaurus, and showed strong correlations with similarity of age and race in a number of networks of friendship among school children. Taken together, these results seem to indicate that the measure is capable of extracting useful information about vertex similarity based on network topology.

The strength of similarity measures such as ours is their generality—in any network where the function or role of a

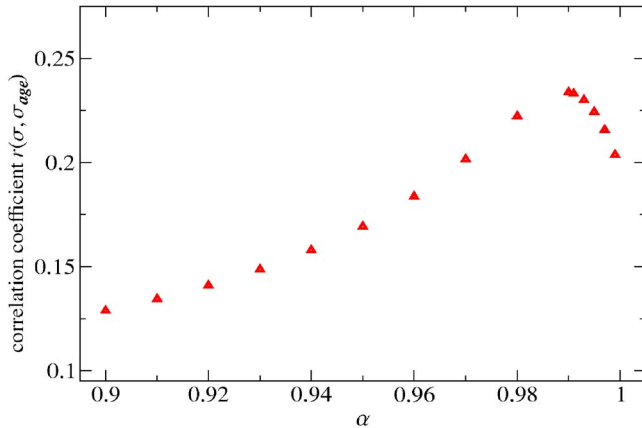


FIG. 6. (Color online) (a) The actual number of paths of length 2 between vertex pairs in a configuration model versus the expected number of paths, as determined from Eq. (A2). (b) Same as in plot (a), but for paths of length 3.

vertex is related in some way to its structural surroundings, structural similarity measures can be used to find vertices with similar functions. For instance, similarity measures can be used to divide vertices into functional categories [21,38,39] or for functional prediction in cases where the functionality of vertices is partly known ahead of time [40]. We believe that the application of similarity measures to problems such as these will prove a fruitful topic for future work.

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## APPENDIX: EXPECTED NUMBER OF SHORT PATHS

In this Appendix we derive exact values for the first few coefficients  $C_1^{ij}$  in Eq. (9). These coefficients are equal to the inverse of the expected number of paths between vertices of given degree in a configuration model having a specified degree sequence.

To begin with, consider the first-order coefficient  $C_1^{ij}$ . If vertices  $i$  and  $j$  have degrees  $k_i$  and  $k_j$  respectively, then we can calculate the expected number of paths of length 1 between them as follows. For any of the  $k_i$  edges emerging from vertex  $i$ , there are  $2m$  places where it could terminate, where  $m$  is the total number of edges in the network. Of these,  $k_j$  end at vertex  $j$  and hence result in a direct path of length 1 from  $i$  to  $j$ . Thus for each edge emerging from  $i$  there is a probability  $k_j/2m$  of a length-1 path to  $j$ , and overall the expected number of such paths is  $k_i k_j / 2m$ . Thus

$$C_1^{ij} = \frac{2m}{k_i k_j}. \quad (\text{A1})$$

Now consider the second-order term in the series. A path of length 2 between  $i$  and  $j$  must go through a single intermediate vertex  $v$ , whose degree we denote  $k_v$ . Using the argument of the preceding paragraph, the expected number of paths of length 1 from  $i$  to  $v$  is  $k_i k_v / 2m$ . This uses up one of the edges emerging from  $v$ , leaving  $k_v - 1$  remaining edges and thus the expected number of paths of length 1 from  $v$  to  $j$ , given that there is already a path from  $i$  to  $v$ , is  $(k_v - 1) k_j / 2m$ . The expected number of paths of length 2 from  $i$  to  $j$  via  $v$  is then the product  $k_i k_v (k_v - 1) k_j / (2m)^2$ . Summing over all  $v$ , the total expected number of paths of length 2 is

$$\frac{k_i k_j}{(2m)^2} \sum_v k_v (k_v - 1) = \frac{k_i k_j}{2m} \left( \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} \right), \quad (\text{A2})$$

where  $\langle k \rangle$  and  $\langle k^2 \rangle$  are the mean degree and mean-square degree of the network, respectively, and we have made use of the result  $2m = n\langle k \rangle$ , where  $n$  is the total number of vertices in the network.  $C_2^{ij}$  is then the reciprocal of this quantity.

For paths of length 3 and greater, the calculations become more complicated. Since paths can be self-intersecting, we have to consider topologies for those paths that include loops or that traverse the same edge more than once. While there is only one topology for paths of length 1 or 2 between a specified pair of vertices, there are four distinct topologies for

paths of length 3 (Fig. 5), and we must calculate and sum the expected numbers of each of them to get the total expected number of paths. The end result for paths of length 3 is

$$\frac{k_i k_j}{2m} \left[ \left( \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} \right)^2 + k_i + k_j - 1 \right]. \quad (\text{A3})$$

As a check on our calculations, we compare our analytic expressions for the numbers of paths of length 2 and 3 to actual path counts for randomly generated networks in Fig. 6. There is increased scatter in the numerical data at longer path lengths because the graphs studied are finite in size, but overall the agreement between analytic and numerical calculations is good.

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- [1] R. Albert and A.-L. Barabási, *Rev. Mod. Phys.* **74**, 47 (2002).
  - [2] S. N. Dorogovtsev and J. F. F. Mendes, *Adv. Phys.* **51**, 1079 (2002).
  - [3] M. E. J. Newman, *SIAM Rev.* **45**, 167 (2003).
  - [4] D. J. Watts and S. H. Strogatz, *Nature (London)* **393**, 440 (1998).
  - [5] F. Chung and L. Lu, *Proc. Natl. Acad. Sci. U.S.A.* **99**, 15879 (2002).
  - [6] R. Cohen and S. Havlin, *Phys. Rev. Lett.* **90**, 058701 (2003).
  - [7] A. Fronczak, P. Fronczak, and J. A. Hołyst, *Phys. Rev. E* **70**, 056110 (2004).
  - [8] A.-L. Barabási and R. Albert, *Science* **286**, 509 (1999).
  - [9] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, *Phys. Rev. Lett.* **85**, 4633 (2000).
  - [10] P. L. Krapivsky, S. Redner, and F. Leyvraz, *Phys. Rev. Lett.* **85**, 4629 (2000).
  - [11] M. Girvan and M. E. J. Newman, *Proc. Natl. Acad. Sci. U.S.A.* **99**, 7821 (2002).
  - [12] F. Radicchi, C. Castellano, F. Cecconi, V. Loreto, and D. Parisi, *Proc. Natl. Acad. Sci. U.S.A.* **101**, 2658 (2004).
  - [13] R. Guimerà, M. Sales-Pardo, and L. A. N. Amaral, *Phys. Rev. E* **70**, 025101(R) (2004).
  - [14] L. Donetti and M. A. Muñoz, *J. Stat. Mech.: Theory Exp.* 2004, P10012.
  - [15] M. E. J. Newman, *Phys. Rev. E* **64**, 016132 (2001).
  - [16] K.-I. Goh, B. Kahng, and D. Kim, *Phys. Rev. Lett.* **87**, 278701 (2001).
  - [17] P. Holme, B. J. Kim, C. N. Yoon, and S. K. Han, *Phys. Rev. E* **65**, 056109 (2002).
  - [18] F. Lorrain and H. C. White, *J. Math. Sociol.* **1**, 49 (1971).
  - [19] P. Jaccard, *Bulletin de la Société Vaudoise des Sciences Naturelles* **37**, 547 (1901).
  - [20] G. Salton, *Automatic Text Processing: The Transformation, Analysis, and Retrieval of Information by Computer* (Addison-Wesley, Reading, MA, 1989).
  - [21] E. Ravasz, A. L. Somera, D. A. Mongru, Z. N. Oltvai, and A.-L. Barabási, *Science* **297**, 1553 (2002).
  - [22] G. Salton and M. J. McGill, *Introduction to Modern Information Retrieval* (McGraw-Hill, Auckland, 1983).
  - [23] L. Hamers, Y. Hemeryck, G. Herweyers, M. Janssen, H. Ketters, R. Rousseau, and A. Vanhoutte, *Inf. Process. Manage.* **25**, 315 (1989).
  - [24] R. S. Burt, *Social Forces* **55**, 93 (1976).
  - [25] D. S. Goldberg and F. P. Roth, *Proc. Natl. Acad. Sci. U.S.A.* **100**, 4372 (2003).
  - [26] S. P. Borgatti and M. G. Everett, *Soc. Networks* **15**, 361 (1993).
  - [27] V. Batagelj, P. Doreian, and A. Ferligoj, *Soc. Networks* **14**, 121 (1992).
  - [28] G. Jeh and J. Widom, in *Proceedings of the 8th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining* (Association of Computing Machinery, New York, 2002), pp. 538-543.
  - [29] V. D. Blondel, A. Gajardo, M. Heymans, P. Senellart, and P. van Dooren, *SIAM Rev.* **46**, 647 (2004).
  - [30] M. E. J. Newman, *Phys. Rev. E* **67**, 026126 (2003).
  - [31] L. Katz, *Psychometrika* **18**, 39 (1953).
  - [32] T. Łuczak, in *Proceedings of the Symposium on Random Graphs, Poznań 1989*, edited by A. M. Frieze and T. Łuczak (John Wiley, New York, 1992), pp. 165-182.
  - [33] M. Molloy and B. Reed, *Random Struct. Algorithms* **6**, 161 (1995).
  - [34] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, *Phys. Rev. E* **64**, 026118 (2001).
  - [35] *Thesaurus of English Words and Phrases*, edited by S. Mawson (T. Y. Crowell Co., New York, 1911).
  - [36] P. Bearman, J. Moody, and K. Stovel, *Am. J. Sociol.* **110**, 44 (2004).
  - [37] J. M. McPherson, L. Smith-Lovin, and J. Cook, *Annu. Rev. Sociol.* **27**, 415 (2001).
  - [38] J. J. Luczkovich, S. P. Borgatti, J. C. Johnson, and M. G. Everett, *J. Theor. Biol.* **220**, 303 (2003).
  - [39] A. W. Wolfe, *Connections* **26**, 105 (2005).
  - [40] P. Holme and M. Huss, *J. R. Soc., Interface* **2**, 327 (2005).