A substitution game in competitive markets

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Outlines of the work

- Starting from [Dieci and Westerhoff, 2010], we consider two interacting cobweb markets;
- two producers in place of a population;
- decisions on the level of substitutability of a product with that of the opponent;
- the aim is to characterize strategically stable configurations of substitution levels;
- some observations on dynamic consequences, such as stability of equilibrium prices and emergence of cyclic behaviors (in a linear environment);

Market structure

- We consider two competitive markets that we label by 1 and 2, each one associated to a type of non storable homogeneous goods;
- the demand and the price of good i are denoted respectively by D_i and P_i respectively;
- We assume two producers, each one producing one type of good. We denote by S_i the supply of good i by producer i.
- Clearing condition occurs in each market, that is the equality

$$D_i = S_i \tag{1}$$

  is always satisfied.
- Profits earned by firm i are of the form

$$\pi_i = P_i S_i - C_i(S_i) \tag{2}$$

  where C_i is the cost functions of i.

Market structure

- With quadratic consumers' utility as considered in [Dixit, 1979] (see also [Singh and Vives, 1984]) and modified as in [Choné and Linnemer, 2020] to account for possible asymmetries in consumers' preferences, linear demands in each market result

$$D_i = \frac{a_i - (P_i - \gamma_{ij} P_j)}{b_i}$$

  where $a_i, b_i > 0$ are positive parameters.
- Effects of price of good j on the demand for i is described by the derivative

$$\frac{\partial D_i}{\partial P_j} = \frac{\gamma_{ij}}{b_i} \tag{3}$$

- We are interested in substitutes or, at least, independent goods. Then we set $\gamma_{ij} \geq 0$ throughout the paper. We will refer to coefficient γ_{ij} as the degree of substitutability of good j with good i.

Market structure

- Optimal supplies are obtained for risk averse producers with exponential utility functions and static expectations

$$\mathbb{E}(P_i') = P_i$$

facing quadratic costs

$$C_i(S) = k_i S + \frac{e_i}{2} S^2 \tag{4}$$

where $k_i, e_i \geq 0$ are parameters (see [Boussard, 1996]);

- The next period optimal output S_i' is given by

$$S_i' = \frac{P_i - k_i}{\delta_i} \tag{5}$$

where

$$\delta_i = e_i + \theta_i \mathbb{V}(P_i')$$

is an aggregate parameter, and $\theta_i > 0$ describes the risk aversion of producer i.

Producer's problem

- With static expectations, i's expected profits are

$$\mathbb{E}(\pi_i') = \mathbb{E}(P_i' S_i - C_i(S_i)) = P_i S_i - C_i(S_i)$$

- and profit variance results

$$\mathbb{V}(\pi_i') = S_i^2 \mathbb{V}(P_i') \tag{6}$$

- With quadratic costs (4), the producer problem reads as

$$\arg\max_{S_i} \left\{ \mathbb{E}(\pi_i') - \frac{\theta_i}{2} \mathbb{V}(\pi_i') \right\} = \arg\max_{S_i} \Big\{ (P_i - k_i) S_i + \\ - \frac{1}{2} \left(e_i + \theta_i \mathbb{V}(P_i') \right) S_i^2 \Big\}$$

- By setting

$$\delta_i = e_i + \theta_i \mathbb{V}(P_i')$$

optimal production of i results

$$S_i = \frac{P_i - k_i}{\delta_i}$$

Equilibrium prices

- With expression of optimal supply, clearing condition (1) takes the form
$$\frac{a_i - (P_i' - \gamma_{ij} P_j')}{b_i} = \frac{P_i - k_i}{\delta_i} \tag{7}$$
- With cost function (4), expected profits results
$$\mathbb{E}(\pi_i') = \frac{(P_i - k_i)^2}{2\delta_i}\left(2 - \frac{e_i}{\delta_i}\right) = S_i^2\left(\delta_i - \frac{e_i}{2}\right)$$
- Equilibrium prices $(\bar{P}_1, \bar{P}_2)$ can be obtained imposing stationary conditions $P_i' = P_i$ in equations (7), giving
$$\bar{P}_i = \frac{(a_i\delta_i + b_i k_i)(\delta_j + b_j) + \gamma_{ij}\delta_i(\delta_j a_j + b_j k_j)}{(\delta_1 + b_1)(\delta_2 + b_2) - \gamma_{12}\gamma_{21}\delta_1\delta_2} \tag{8}$$

Modeling prices' volatility

- Along the line marked by [Boussard, 1996] (see also [Chiarella et al., 2006, Dieci and Westerhoff, 2010]) we assume constant volatility in each market.
- That is, we assume constant variance of adjusted price $P_i - \gamma_{ij} P_j$ that determines the demand D_i of good i by setting

$$\mathbb{V}(P_i' - \gamma_{ij} P_j') = \sigma_i^2 \tag{9}$$

- With this, price variance can be expressed as

$$\mathbb{V}(P_i') = \sigma_i^2 - \gamma_{ij} \sigma_j^2 \tag{10}$$

- According with expression (10), producers expect a reduction in prices' volatility as markets' superposition is enhanced. This reflects the expectation that an undivided consumer base reduces price fluctuations.

Modeling prices' volatility

- More precisely, we have
$$\sigma_i^2 = \mathbb{V}(P_i' - \gamma_{ij} P_j') = \mathbb{V}(P_i') + \gamma_{ij}^2 \mathbb{V}(P_j') - 2\gamma_{ij}\mathbb{C}(P_i', P_j') \tag{11}$$
- Assuming constant covariance
$$\mathbb{C}(P_i', P_j') = c$$
prices' volatility have the following expressions
$$\mathbb{V}(P_i') = \left(\sigma_i^2 - \gamma_{ij}^2\sigma_j^2 - c\gamma_{ij}^2\gamma_{ji}\right)\frac{1}{1-\gamma_{12}^2\gamma_{21}^2} \tag{12}$$
where we assume $\gamma_{12}^2\gamma_{21}^2 \neq 1$.
- Second order approx. of $\mathbb{V}(P_i')$ around $(\gamma_{21}, \gamma_{12}) = (0, 0)$ gives
$$\mathbb{V}(P_i') \simeq I_i^2(P_i) := \sigma_i^2 - \gamma_{ij}^2\sigma_j^2$$
- We further simplify as
$$\mathbb{V}(P_i') \simeq I_i^1(P_i) := \sigma_i^2 - \gamma_{ij}\sigma_j^2$$
the linear interpolation of $I_i^2(P_i)$ in $[0, 1]$.

Technical assumptions

- We consider the parameter restrictions

$$a_i - k_i > 0 \tag{A1}$$

and

$$\frac{e_i + \theta_i \sigma_i^2}{\theta_i \sigma_j^2} \geq 1 \tag{A2}$$

- Under Assumption (A1), the maximum price exceeds marginal cost k_i in each market. As a consequence, relation $\bar{P}_i - k_i > 0$ is satisfied, ensuring positiveness of equilibrium production $\bar{S}_i$.
- Assumption (A2) will be considered to ensure positiveness of parameter $\delta_i = e_i + \theta_i(\sigma_i^2 - \gamma_{ij}\sigma_j^2)$. Indeed, by (A2), the relation $\delta_i > 0$ is satisfied for all $\gamma_{ij} \in [0, 1)$.

Noncooperative substitution game

- We assume that each producer can estimate the consumers' demand through market analysis and can regulate the degree of substitutability of its product with that of the opponent.
- Specifically, producer i can strategically select the degree of substitution γ_{ji} of good i with the j one (and analogously for producer j). Such individual choices reflect on producers' profits.
- This defines a strategic environment that we call *substitution game*, where $X_i = [0, 1]$ is the strategy space for producer i and strategy tuples are pairs $(\gamma_{21}, \gamma_{12}) \in X_1 \times X_2$.
- We specify this substitution game at the market equilibrium, where

$$\bar{\pi}_i = \bar{P}_i(\gamma_{21}, \gamma_{12}) S_i(P_i(\gamma_{21}, \gamma_{12})) - C_i(S_i(\bar{P}_i(\gamma_{21}, \gamma_{12}))$$

- In this setting, the best reply of i is

$$BR_i(\gamma_{ij}) = \arg \max_{\gamma_{ji} \in [0,1]} \bar{\pi}_i(\gamma_{ji}, \gamma_{ij}) \tag{13}$$

Noncooperative substitution game

- A Nash equilibrium of the substitution game we will denoted by $(\gamma_{21}^*, \gamma_{12}^*)$.
- Under appropriate hypotheses, we will show that an internal Nash equilibrium of the substitution game exists, where both producers select intermediate values of substitution degrees.
- In order to lighten the notation, the following aggregate parameters will be used.

$$A_i = a_i\delta_i + \gamma_{ij}\delta_i a_j + b_i k_i$$
$$B_i = a_i\delta_i b_j + \gamma_{ij}\delta_i b_j k_j + b_i b_j k_i$$

- The following preliminary Lemma is needed.

Noncooperative substitution game

Lemma

If $\gamma_{ij} = 0$*, then*

$$BR_i(0) = [0, 1] \tag{14}$$

Moreover, let Assumptions (A1) and (A2) be fulfilled. If $\gamma_{ij} \in (0, 1)$*, then*

$$BR_i(\gamma_{ij}) = \begin{cases} 0 & \textit{if } (\gamma_{ji}^*)_- \leq 0 \\ (\gamma_{ji}^*)_- & \textit{if } (\gamma_{ji}^*)_- \in (0, 1) \\ 1 & \textit{if } (\gamma_{ji}^*)_- \geq 1 \end{cases} \tag{15}$$

$$(\gamma_{ji}^*)_- = \frac{1}{\theta_j \sigma_i^2} \left(\frac{B_i}{A_i} + e_j + \theta_j \sigma_j^2 + \right. \tag{16}$$

$$\left. - \sqrt{\left(\frac{B_i}{A_i}\right)^2 + \frac{B_i}{A_i}\left(e_j + \theta_j \sigma_j^2\right) + \frac{\theta_j \sigma_i^2}{\gamma_{ij} \delta_i}(b_i + \delta_i)\left(b_j - \frac{B_i}{A_i}\right)} \right)$$

Noncooperative strategies

Best reply functions are shown in Figures (1) and (2) for an illustrative purpose.

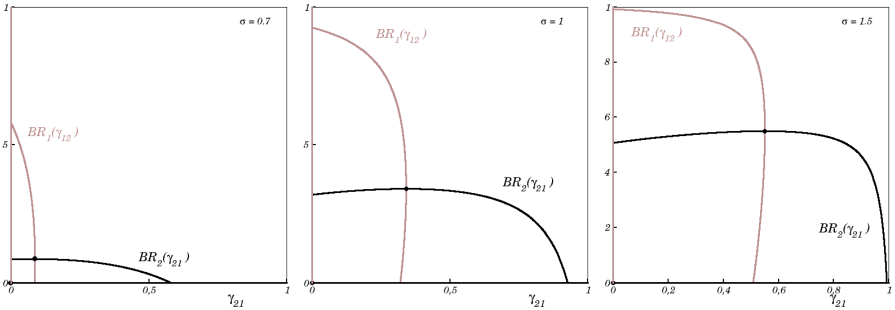


Figure: Best replies in the symmetric case varying $\sigma := \sigma_1 = \sigma_2$. Nash equilibria are highlighted by black dots. Other parameters are $a_i = 10$, $b_i = 1$, $k_i = 1$, $e_i = 0$, and $\theta_i = 2$.

Noncooperative strategies

Best reply functions are shown in Figures (1) and (2) for an illustrative purpose.

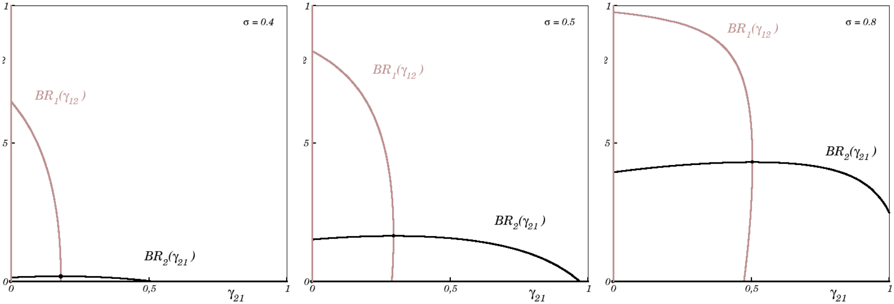


Figure: Best replies in varying $\sigma := \sigma_1 = \sigma_2$. Nash equilibria are highlighted by black dots. Parameters are $a_i = 10$, $b_i = 1$, $k_1 = 1$, $k_2 = 2$, $e_1 = 0$, $e_2 = 0.2$ and $\theta_i = 4$.

- The following Proposition shows the existence of an internal Nash equilibrium, under the hypothesis of symmetry between producers and markets and under the following parameters' restrictions

$$a_i = a,\ b_i = b,\ k_i = 0,\ e_i = 0,\ \theta_i = \theta,\ \sigma_i = \sigma \qquad \text{(A3)}$$

Proposition

Point $(0,0)$ is a Nash equilibrium of the substitution game. Moreover, let Assumptions (A1) and (A2) be fulfilled, symmetry between producers and markets be given with the parameter restrictions (A3). If $\theta\sigma^2 > b$, a Nash equilibrium (γ^*, γ^*) exists such that

$$\gamma^* = 1 - \sqrt{\frac{b}{\theta\sigma^2}} \in (0,1) \qquad (17)$$

Internal Nash equilibrium

- The existence of an internal Nash equilibrium describes strategically stable degrees of substitution between goods.
- This fact justifies the endogenous emergence of substitute goods as a result of strategic behavior;
- The existence of an internal Nash equilibrium attests for a trade-off between full substitutability and complete differentiation.
- We explain this trade-off assuming ideal perturbations of i's strategy with respect to a certain intermediate level.
- In the explanation, we consider that, under Assumption (A1), profits π_i and price P_i share the same monotonicity properties with respect to strategy γ_{ji}, indeed

$$\frac{\partial \bar{\pi}_i}{\partial \gamma_{ji}} = \frac{\partial}{\partial \gamma_{ji}} \frac{(\bar{P}_i - k_i)^2}{2\delta_i} \left(2 - \frac{e_i}{\delta_i} \right) = \frac{\bar{P}_i - k_i}{\delta_i} \left(2 - \frac{e_i}{\delta_i} \right) \frac{\partial \bar{P}_i}{\partial \gamma_{ji}} \tag{18}$$

Internal Nash equilibrium

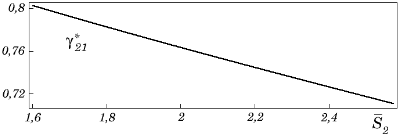
- suppose that, on one hand, producer i revises her decision increasing the substitutability γ_{ji} of good i with the other j.
- As a result, the optimal production of j increases because of the reduced price volatility of good j
- Therefore, there may be so much quantity to satisfy the demand for i so that a collapse in price P_i follows, being good j a substitute of i.
- This effect is captured by expressing price $\bar{P}_i$ in terms of individual productions using clearing condition in market i as

$$\bar{P}_i(1-\gamma_{12}\gamma_{21}) = a_i + \gamma_{ij}a_j - b_i\bar{S}_i - \gamma_{ij}b_j\bar{S}_j \tag{19}$$

  showing a negative (and linear) relation between $\bar{P}_i$ and $\bar{S}_j$.
- As a result, an ideal increase of γ_{ji} may entail profits' disadvantages for firm i, as follows by equation (18).

Internal Nash equilibrium

- We observe that this mechanism implies that an increase in the production of a player, let say j, forces its opponent i to weaken the superposition γ_{ji} of its product.
- Indeed, an increase of $\bar{S}_j$ implies greater amounts of the aggregate $\widetilde{S}_i := \bar{S}_i + \gamma_{ij} b_j \bar{S}_j / b_i$ (see (19)), with detrimental consequences on price $\bar{P}_i$ and, in turn, on profits $\bar{\pi}_i$.
- In order to limit this negative effect, player i has incentive to enhance the diversification of good i by reducing substitutability γ_{ji}. In this way, risk averse producer j will reduce its output.



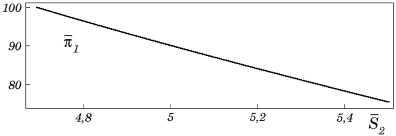


Figure: Equilibrium strategy γ^*_{21} (left) and profits $\bar{\pi}_1$ of player 1 varying 2's production $\bar{S}_2$, obtained by variations of k_2 from 0 to 10, given $a_i = 10$, $b_i = 1$, $k_1 = 1$, $e_i = 0.5$, $\sigma_i = 2$, and $\theta_i = 2$, with $i = 1, 2$.

Internal Nash equilibrium

- Now suppose that producer i revises her decision decreasing substitutability of i with j.
- Then, the influence of price P_i on j's demand is reduced. However, the adjusted price $\widetilde{P}_j := \bar{P}_j - \gamma_{ji}\bar{P}_i$ must not increase to support the same (or weakly reduced) optimal production of producer j.
- Hence, price $\bar{P}_j$ needs to adapt in order to compensate the diminished impact of price $\bar{P}_i$. When such an impact is substantial, price $\bar{P}_j$ will considerably decrease to compensate it.
- This, in turn, implies a diminished price $\bar{P}_i$ to sustain the unperturbed demand for i. Indeed, the adjusted price $\widetilde{P}_i := \bar{P}_i - \gamma_{ij}\bar{P}_j$ must remain constant. This effect is captured by clearing condition in market i, giving

$$\bar{P}_i = \frac{1}{b_i^{-1} + \delta_i^{-1}} \left(\frac{a_i}{b_i} + \frac{k_i}{\delta_i} + \frac{\gamma_{ij}}{b_i} \bar{P}_j \right)$$

- By (18), player i may undergo profits disadvantages from choosing lower values of γ_{ji}.

Internal Nash equilibrium

- This mechanism implies that an increase of the demand D_j in market j incentives player i to enhance the superposition of good i with the j one.
- In this way, player i takes advantage from higher prices in market j.

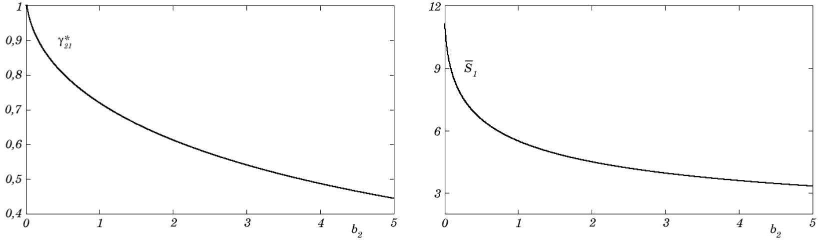


Figure: Nash level γ^*_{21} (left) and production $\bar{S}_1$ varying 2's demand slope b_2, given $a_i = 10$, $b_1 = 1$, $k_i = 1$, $e_i = 0.5$, $\sigma_i = 2$, and $\theta_i = 2$.

Risk aversion's effects

- The risk aversion of a producer induces a monotonic response on its own strategy.
- Indeed, increasing θ_i determines the decrease of i's production $\bar{S}_i$ and, in turn, increasing prices $\bar{P}_i$. Hence, player j enhances the superposition γ_{ij} of its output to take advantage of high price of good i.
- In response to this tendency, producer i increases substitutability γ_{ji} to lower the production $\bar{S}_j$, preserving price $\bar{P}_i$.
- not surprising since risk averse producers are expected to benefit from reduced prices' volatility.

Risk aversion's effects

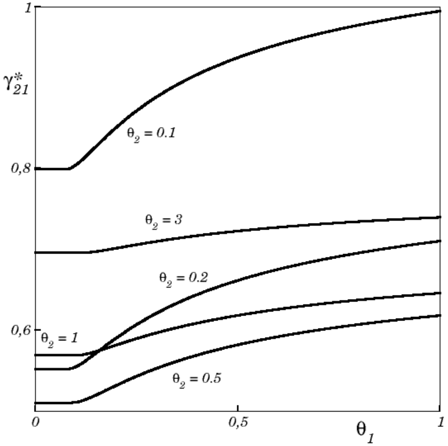


Figure: Equilibrium strategies γ^*_{21} as θ_1 varies. Parameters are $a_i = 10$, $b_i = 1$, $k_i = 1$, $e_i = 0.5$, $\sigma_i = 2$, and $\theta_2 = 3$.

Risk aversion's effects

- The risk aversion of a producer induces two distinct strategic responses from its opponent.
- Specifically, depending on the magnitude of, let say, player 1's aversion θ_1, player 2 acts decreasing substitutability γ_{12} in order to hedge price $\bar{P}_2$ of its production by controlling the opponent's output $\bar{S}_2$ and, at the same time, by enhancing γ_{12} in order to take advantages of high prices in market 1.

Risk aversion's effects

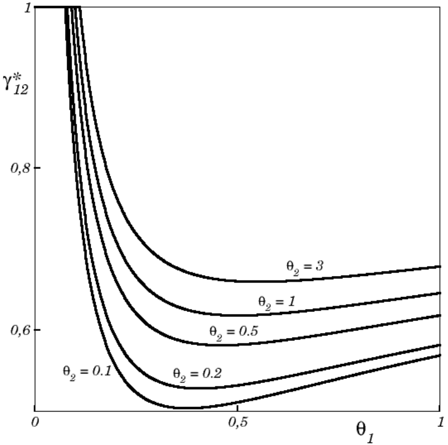


Figure: Equilibrium strategies γ^*_{12} (left) and γ^*_{21} (right) as θ_1 varies. Parameters are $a_i = 10$, $b_i = 1$, $k_i = 1$, $e_i = 0.5$, $\sigma_i = 2$, and $\theta_2 = 3$.

Cooperative strategies

- In the hypothesis of cooperative attitudes of producers, each agent maximizes the aggregate payoff of the industry. In this occurrence, player i chooses

$$\gamma_{ji}^{c} \in \mathrm{C}_i(\gamma_{ij}) := \arg \max_{\gamma_{ji} \in [0,1]} \left(\overline{\pi}_i(\gamma_{ji}, \gamma_{ij}) + \overline{\pi}_j(\gamma_{ij}, \gamma_{ji})\right) \tag{20}$$

- A cooperative equilibrium is a pair $(\gamma_{ji}^c, \gamma_{ij}^c)$ such that $\gamma_{ji}^c \in C_i(\gamma_{ij}^c)$ and $\gamma_{ij}^c \in C_j(\gamma_{ji}^c)$ hold.
- We remark that such a cooperative equilibrium is Pareto efficient.

Proposition

Let Assumptions (A1) be fulfilled, symmetry between producers and markets be given with the parameter restrictions (A3). If $\theta\sigma^2 > b/3$, then an internal cooperative equilibrium (γ^c, γ^c) exists such that

$$\gamma^c = 1 - \sqrt{\frac{b}{3\theta\sigma^2}} \in (0,1) \tag{21}$$

Cooperative strategies

- Previous proposition establishes the existence of an internal cooperative equilibrium, where aggregate profits are maximized.
- The degree of substitutability $\gamma^c < 1$ avoids prices' falls due to excess supply.
- This fact prevents the two market from merging one with the other under cooperative attitudes.

Remark

Under the hypotheses of Propositions 2, the degree of substitution in the cooperative case is always greater than the one in the Nash case. That is, $\theta\sigma^2 > b/3$ *implies*

$$0 \leq \gamma^* < \gamma^c < 1$$

Prices' stability

- Non equilibrium prices' adaptation processes are driven by clearing conditions (1). Under Assumption (A0), these conditions fix the following linear recurrences among incoming and current prices

$$P_i' = \frac{1}{1-\gamma_{12}\gamma_{21}}\left(a_i + \gamma_{ij}a_j - \frac{b_i}{\delta_i}(P_i - k_i) - \gamma_{ij}\frac{b_j}{\delta_j}(P_j - k_j)\right) \quad (22)$$

- This defines a two dimensional discrete time linear dynamical system, with fixed point $(\bar{P}_1, \bar{P}_2)$ determined by (8). The associated matrix of coefficients results

$$J = \frac{-1}{1-\gamma_{12}\gamma_{21}}\begin{pmatrix} b_1/\delta_1 & \gamma_{12}b_2/\delta_2 \\ \gamma_{21}b_1/\delta_1 & b_2/\delta_2 \end{pmatrix} \quad (23)$$

Prices' stability

- The stability of equilibrium point $(\bar{P}_1, \bar{P}_2)$ is ensured provided that inequalities $p(1) > 0$, $p(-1) > 0$, and $\det J < 1$ are satisfied, where $p(\lambda) = \lambda^2 - \mathrm{tr}J\lambda + \det J$ is the characteristic polynomial of matrix J, $\mathrm{tr}J$ is its trace, and $\det J$ its determinant.
- The first condition $p(1) > 0$ is always satisfied, while the latter two conditions $p(-1) > 0$ and $\det J < 1$ correspond, respectively, to the following inequalities

$$\frac{b_1}{\delta_1} + \frac{b_2}{\delta_2} - \frac{b_1 b_2}{\delta_1 \delta_2} < 1 - \gamma_{12}\gamma_{21} \tag{24a}$$

$$\frac{b_1 b_2}{\delta_1 \delta_2} < 1 - \gamma_{12}\gamma_{21} \tag{24b}$$

Prices' stability

Remark

If $\gamma_{12} = \gamma_{21} = 0$*, the two goods are independent and the two markets are disjoint. In this event, stability in each market is ensured when*

$$\left| -\frac{b_i}{\delta_i} \right| < 1 \tag{25}$$

holds. This relation corresponds to the classical single-market stability conditions relating the slopes of demand and supply curves (see e.g. [Ezekiel, 1938]).

Prices' stability

- On one hand, the stability of two partially overlapped unstable markets can be recovered by reduction of goods' superpositions, provided that the two markets are stable when independent. This is an immediate consequence of Remark (5), taking into account the continuity of stability conditions (24) with respect to degrees of substitution.
- On the other hand, the stability of two partially overlapped unstable markets can not be recovered through enhancing product substitutability. Also, enhancing product substitutability can destabilize otherwise stable markets.

Proposition

If the conditions (24) are not satisfied given the pair $(\gamma_{21}, \gamma_{12})$ *with* $\gamma_{21}, \gamma_{12} < 1$, *then they are not satisfied for any pair* $(\gamma'_{21}, \gamma'_{12})$ *with* $\gamma_{21} \leq \gamma'_{21} < 1$ *and* $\gamma_{12} \leq \gamma'_{12} < 1$. *Moreover, conditions (24) are never satisfied as* $\gamma_{21}\gamma_{12} \to 1$.

Prices' stability

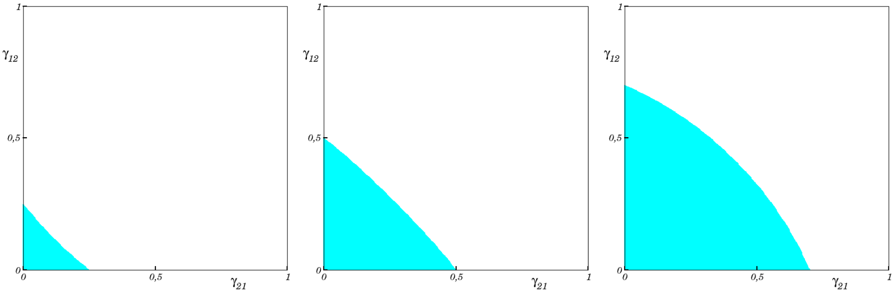


Figure: Stability region in the $(\gamma_{21}, \gamma_{12})$ space varying risk aversion. Blue points represent pairs strategy pairs for which market equilibrium $(\bar{P}_1, \bar{P}_2)$ is locally stable. Parameters are $a_i = 10$, $b_i = 1$, $k_i = 1$, $e_i = 0.5$, $\sigma_i = 1$, $\theta_i = \theta$ with $\theta = 2$ (left), $\theta = 3$ (center), $\theta = 5$ (right).

Prices' stability and risk aversion

- The widening of the stability region increasing risk aversion does not depend on the specific parameters' selection.

Proposition

Let stability conditions (24) be not satisfied for some pair $(\theta_1, \theta_2) \neq (0,0)$ *and let condition*

$$\left| -\frac{b_i}{\delta_i} \right| < 1$$

be satisfied for at least one value of index i*. Then a threshold* θ_j^* *exists with* $j \neq i$ *such that* $\theta_j > \theta_j^*$ *implies that conditions (24) are fulfilled.*

Prices' stability and equilibrium strategies

- Assuming the parameters' restrictions (A3) and assuming symmetry between producers and between markets, equilibrium $(\bar{P}_1, \bar{P}_2)$ is Lyapunov stable when the Nash equilibrium of the substitution game is played, namely when the pair $(\gamma_{12}, \gamma_{21}) = (\gamma^*, \gamma^*)$ is chosen;
- Indeed, stability conditions (24) are not simultaneously satisfied, since relation (24a) reduces to $1 < 1$. Matrix (23) becomes

$$J = \frac{-1}{1+\gamma^*} \begin{pmatrix} 1 & \gamma^* \\ \gamma^* & 1 \end{pmatrix}$$

- eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = (\gamma^* - 1)/(\gamma^* + 1)$ with eigenvectors $\boldsymbol{v}_1 = (1, 1)$ and $\boldsymbol{v}_2 = (1, -1)$ respectively
- the generic pice path is

$$\begin{pmatrix} P_1(t) \\ P_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} (-1)^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left(\frac{1-\gamma^*}{1+\gamma^*}\right)^t$$

Prices' stability and equilibrium strategies

- Under the same hypotheses, the point $(\bar{P}_1, \bar{P}_2)$ is unstable when the cooperative equilibrium of the substitution game is played, namely when the pair $(\gamma_{12}, \gamma_{21}) = (\gamma^c, \gamma^c)$ is chosen.
- This fact directly follows from stability conditions (24), which are not satisfied with strict inequalities, thus attesting for eigenvalues outside the unit circle in the complex plane.

Prices' stability and equilibrium strategies

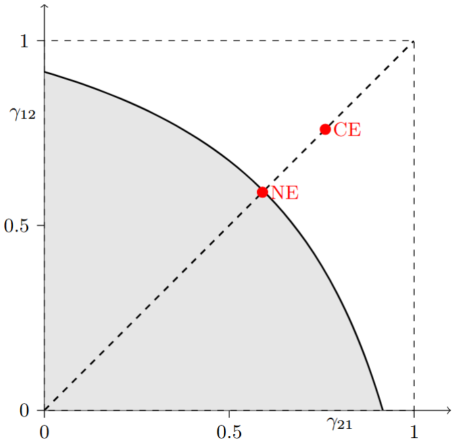
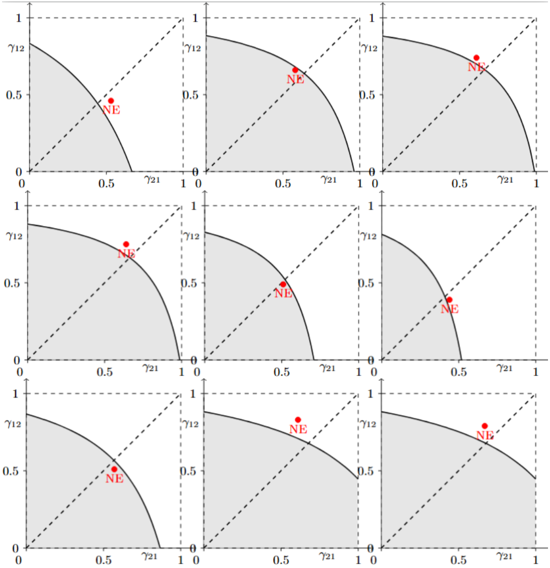


Figure: Stability Region in the $(\gamma_{12}, \gamma_{21})$-parameter space, Nash equilibrium, and cooperative equilibrium with $a_i = 10$, $\sigma_i^2 = 3$, $\theta_i = 4$, $b_i = 2$, $k_i = 0$, $e_i = 0$; $\gamma^* = 0.59$; $\gamma^c = 0.76$.

Breaking the symmetry



Adaptive adjustments

- At the beginning of period $t+1$, player i observes the new price P_i'. Profits at this stage are

$$\pi_i' = P_i' S_i' - C_i(S_i') = P_i' \cdot \left(\frac{P_i - k_i}{\delta_i} \right) - C_i \left(\frac{P_i - k_i}{\delta_i} \right)$$

- Player i selects the substitution level γ_{ji}' for the period $t+1$ maximizing π_i', assuming static expectations on the level γ_{ij}' chosen by her opponent for period $t+1$. That is, player i evaluates

$$\begin{aligned} \gamma_{ji}' &= \arg\max_{\gamma_{ji}} \pi_i' = \arg\max_{\gamma_{ji}} \left(P_i' \cdot \left(\frac{P_i - k_i}{\delta_i} \right) - C_i \left(\frac{P_i - k_i}{\delta_i} \right) \right) \\ &= \arg\max_{\gamma_{ji}} P_i' \end{aligned}$$

Adaptive adjustments

By (22), it follows that the maximum of P_i' is attained at

$$\gamma_{ji}' = \begin{cases} 0 & \text{if } \widetilde{\gamma}_{ji} < 0 \\ \widetilde{\gamma}_{ji} & \text{if } \widetilde{\gamma}_{ji} \in (0,1) \\ 1 & \text{if } \widetilde{\gamma}_{ji} \geq 1 \end{cases}$$

where

$$\widetilde{\gamma}_{ji} = \frac{\sigma_j^2}{\sigma_i^2} - \frac{\widetilde{\delta}_j - e_j}{\theta_j \sigma_i^2}$$

and

$$\widetilde{\delta}_j = \left(a_i + \gamma_{ij} a_j - b_i \frac{P_i - k_i}{\delta_i} \right)^{-1} \left\{ \gamma_{ij} b_j (P_j - k_j) + \left(\gamma_{ij}^2 b_j^2 (P_j - k_j)^2 + \right. \right.$$

$$\left. \left. + \theta_j b_j (P_j - k_j) \left(\sigma_i^2 - \gamma_{ij} \left(\sigma_j^2 + \frac{e_j}{\theta_j} \right) \right) \left(a_i + \gamma_{ij} a_j - b_i \frac{P_i - k_i}{\delta_i} \right) \right)^{1/2} \right\}$$

Adaptive adjustments

- We assume adaptive adjustments towards best reply

$$\gamma_{ji}' = \alpha\gamma_{ji} + (1-\alpha)\arg\max_{\gamma_{ji}} P_i'$$

- price dynamics

$$P_i' = \frac{1}{1-\gamma_{12}\gamma_{21}}\left(a_i + \gamma_{ij}a_j - \frac{b_i}{\delta_i}(P_i - k_i) - \gamma_{ij}\frac{b_j}{\delta_j}(P_j - k_j)\right)$$

- 4 dimensional dynamical system, with fixed point $(\bar{P}_1, \bar{P}_2, \gamma_{21}^*, \gamma_{12}^*)$.

Adaptive adjustments

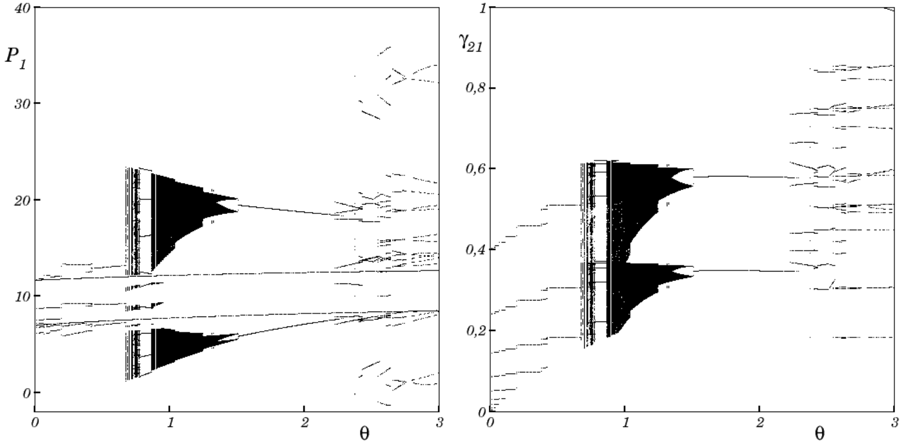


Figure: Bifurcation diagrams of P_1 (left) and γ_{21} (right) varying $\theta_1 = \theta_2 = \theta$. Parameters are $a_i = 10$, $b_i = 1$, $k_i = 1$, $e_i = 2$, $\sigma_i = 1$.

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