



Global average: 1.1296

User average: 1.0651

Movie average: 1.0533

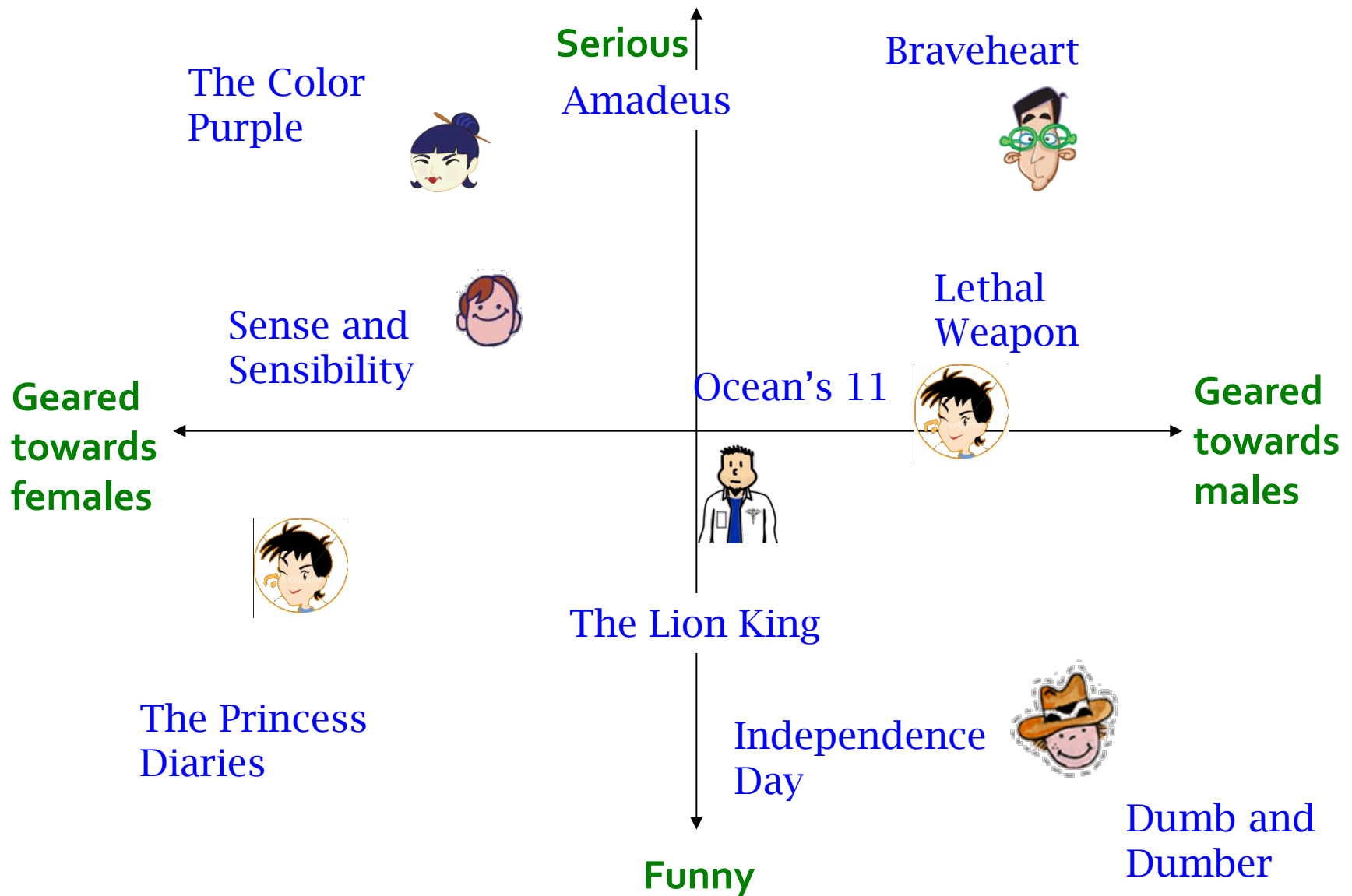
Netflix: 0.9514

Basic Collaborative filtering: 0.94

CF+Biases+learned weights: 0.91

Grand Prize: 0.8563

Latent Factor Models (e.g., SVD)



Latent Factor Models

$$\text{SVD: } A = U \Sigma V^T$$

- “SVD” on Netflix data: $R \approx Q \cdot P^T$

users

items

1		3			5			5		4	
		5	4			4			2	1	3
2	4		1	2		3		4	3	5	
	2	4		5			4			2	
		4	3	4	2					2	5
1		3		3			2			4	

R

≈

items

factors

.1	-.4	.2
-.5	.6	.5
-.2	.3	.5
1.1	2.1	.3
-.7	2.1	-2
-1	.7	.3

Q

users

1.1	-.2	.3	.5	-2	-.5	.8	-.4	.3	1.4	2.4
-.8	.7	.5	1.4	.3	-1	1.4	2.9	-.7	1.2	-.1
2.1	-.4	.6	1.7	2.4	.9	-.3	.4	.8	.7	-.6

P^T

factors

- For now let's assume we can approximate the rating matrix R as a product of “thin” $Q \cdot P^T$
 - R has missing entries but let's ignore that for now!
 - Basically, we will want the reconstruction error to be small on known ratings and we don't care about the values on the missing ones

Ratings as Products of Factors

- How to estimate the missing rating of user x for item i ?

users

items

1		3			5			5		4	
		5	4	?	4			2	1	3	
2	4		1	2		3		4	3	5	
	2	4		5			4			2	
		4	3	4	2					2	5
1		3		3			2			4	

≈

items

.1	-.4	.2
-.5	.6	.5
-.2	.3	.5
1.1	2.1	.3
-.7	2.1	-2
-1	.7	.3

factors

Q

factors

users

1.1	-.2	.3	.5	-2	-.5	.8	-.4	.3	1.4	2.4	-.9
-.8	.7	.5	1.4	.3	-1	1.4	2.9	-.7	1.2	-.1	1.3
2.1	-.4	.6	1.7	2.4	.9	-.3	.4	.8	.7	-.6	.1

P^T

$$\hat{r}_{xi} = q_i \cdot p_x$$

$$= \sum_f q_{if} \cdot p_{xf}$$

q_i = row i of Q
 p_x = column x of P^T

Ratings as Products of Factors

- How to estimate the missing rating of user x for item i ?

users

items

1		3			5			5		4	
		5	4	?	4			2	1	3	
2	4		1	2		3		4	3	5	
	2	4		5			4			2	
		4	3	4	2					2	5
1		3		3			2			4	

≈

$$\hat{r}_{xi} = q_i \cdot p_x$$

$$= \sum_f q_{if} \cdot p_{xf}$$

q_i = row i of Q
 p_x = column x of P^T

items

factors

Q

.1	-.4	.2
-.5	.6	.5
-.2	.3	.5
1.1	2.1	.3
-.7	2.1	-2
-1	.7	.3

factors

users

P^T

1.1	-.2	.3	.5	-2	-.5	.8	-.4	.3	1.4	2.4	-.9
-.8	.7	.5	1.4	.3	-1	1.4	2.9	-.7	1.2	-.1	1.3
2.1	-.4	.6	1.7	2.4	.9	-.3	.4	.8	.7	-.6	.1

Ratings as Products of Factors

- How to estimate the missing rating of user x for item i ?

users

items

1		3			5			5		4	
		5	4	2.4	4			2	1	3	
2	4		1	2		3		4	3	5	
	2	4		5			4			2	
		4	3	4	2					2	5
1		3		3			2			4	

items

f factors

Q

.1	-.4	.2
-.5	.6	.5
-.2	.3	.5
1.1	2.1	.3
-.7	2.1	-2
-1	.7	.3

f factors

users

P^T

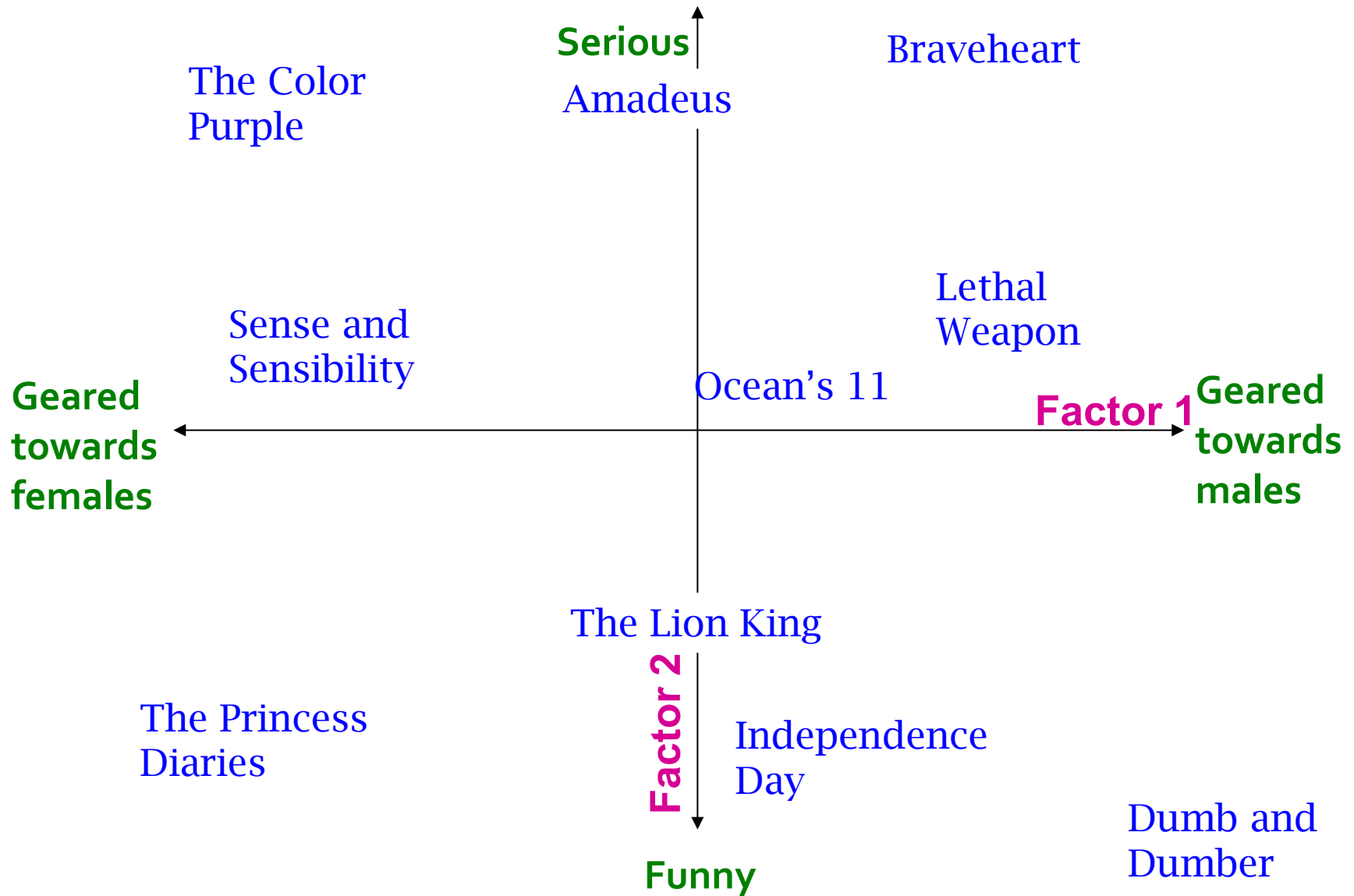
1.1	-.2	.3	.5	-2	-.5	.8	-.4	.3	1.4	2.4	-.9
-.8	.7	.5	1.4	.3	-1	1.4	2.9	-.7	1.2	-.1	1.3
2.1	-.4	.6	1.7	2.4	.9	-.3	.4	.8	.7	-.6	.1

$$\hat{r}_{xi} = q_i \cdot p_x$$

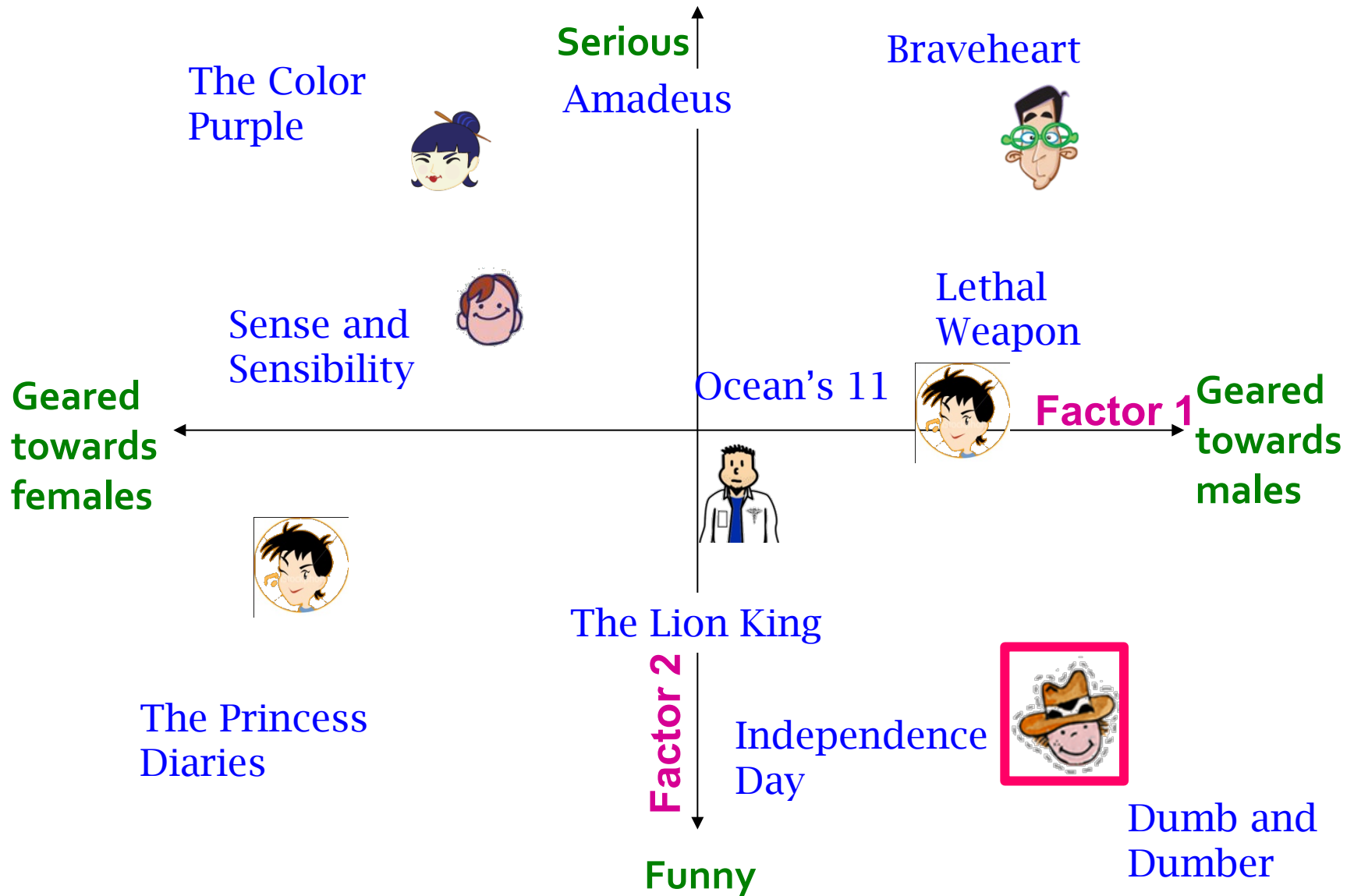
$$= \sum_f q_{if} \cdot p_{xf}$$

q_i = row i of Q
 p_x = column x of P^T

Latent Factor Models



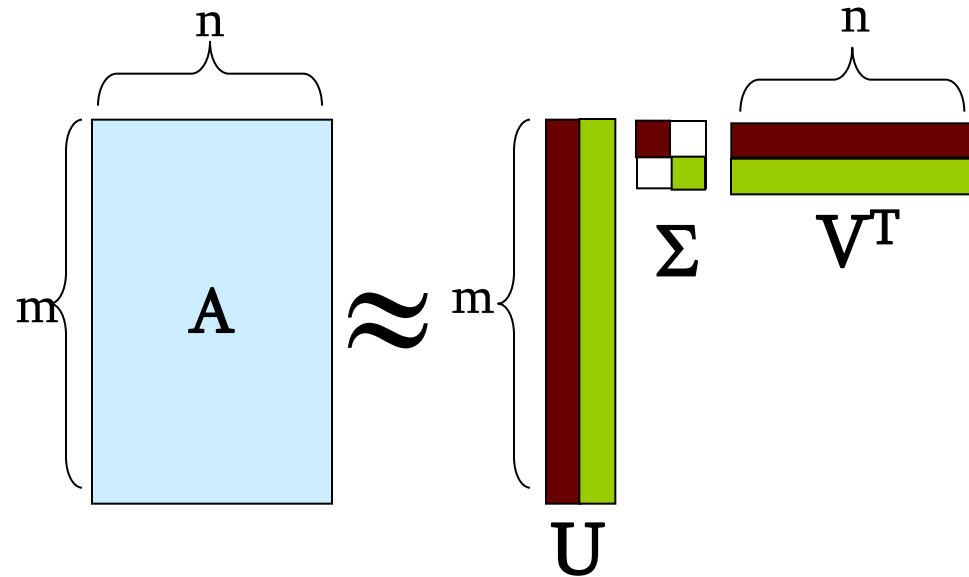
Latent Factor Models



Recap: SVD

■ Remember SVD:

- **A**: Input data matrix
- **U**: Left singular vecs
- **V**: Right singular vecs
- Σ : Singular values



■ So in our case:

“SVD” on Netflix data: $R \approx Q \cdot P^T$

$$A = R, \quad Q = U, \quad P^T = \Sigma V^T$$

$$\hat{r}_{xi} = q_i \cdot p_x$$

SVD: More good stuff

- We already know that SVD gives minimum reconstruction error (Sum of Squared Errors):

$$\min_{U, V, \Sigma} \sum_{ij \in A} \left(A_{ij} - [U \Sigma V^T]_{ij} \right)^2$$

- Note two things:
 - SSE and RMSE are monotonically related:
 - $RMSE = \frac{1}{c} \sqrt{SSE}$ Great news: SVD is minimizing RMSE
 - **Complication:** The sum in SVD error term is over all entries (no-rating is interpreted as zero-rating). But our R has missing entries!

Latent Factor Models

The diagram illustrates the matrix factorization process. It shows three matrices:

- Matrix Q (User-Item Matrix):** A 6x12 grid representing user ratings. The top row is highlighted in yellow. The labels "users" and "items" are on the top and left, and "factors" is on the right.
- Matrix U (User Factors):** A 6x3 grid representing latent factors for each user. The label "users" is on the top, and "factors" is on the right.
- Matrix V (Item Factors):** A 12x3 grid representing latent factors for each item. The label "items" is on the left, and "factors" is on the right.

The matrices are related by the equation $Q = UV^T$, where Q is the product of U and V^T . The green "Q" and "PT" labels at the bottom indicate the matrices involved in the factorization.

- **SVD isn't defined when entries are missing!**
- **Use specialized methods to find P, Q**

- $$\min_{P,Q} \sum_{(i,x) \in R} (r_{xi} - q_i \cdot p_x)^2 \quad \hat{r}_{xi} = q_i \cdot p_x$$

■ **Note:**

- We don't require cols of P, Q to be orthogonal/unit length
- P, Q map users/movies to a latent space
- The most popular model among Netflix contestants

Finding the Latent Factors

The diagram illustrates the matrix factorization process. It shows three matrices:

- Matrix Q (users × items):** A 5x12 matrix with yellow cells containing integers. The values are:


1		3			5			5		4	
		5	4			4			2	1	3
2	4		1	2		3		4	3	5	
	2	4		5			4			2	
		4	3	4	2					2	5
1		3		3			2			4	
- Matrix \hat{Q} (users × factors):** A 5x3 matrix with white cells containing decimal values. The values are:

.1	-.4	.2
-.5	.6	.5
-.2	.3	.5
1.1	2.1	.3
-.7	2.1	-2
-1	.7	.3
- Matrix P^T (users × factors):** A 12x12 matrix with white cells containing decimal values. The values are:

1.1	-.2	.3	.5	-.2	-.5	.8	-.4	.3	1.4	2.4	-.9
-.8	.7	.5	1.4	.3	-1	1.4	2.9	-.7	1.2	-.1	1.3
2.1	-.4	.6	1.7	2.4	.9	-.3	.4	.8	.7	-.6	.1

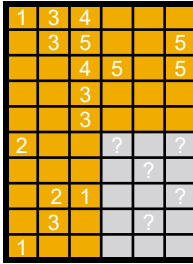
A tilde symbol (\approx) is placed between Q and \hat{Q} , indicating that \hat{Q} is an approximation of Q . A large Q is also shown at the bottom right.

Back to Our Problem

- **Want to minimize SSE for unseen test data**
 - **Idea: Minimize SSE on training data**
 - Want large k (# of factors) to capture all the signals
 - But, **SSE** on test data begins to rise for $k > 2$
 - This is a classical example of **overfitting**:
 - With too much freedom (too many free parameters) the model starts fitting noise
 - That is it fits too well the training data and thus **not generalizing** well to unseen test data
- 

1	3	4			
	3	5			5
		4	5		5
		3			
		3			
2			?	?	?
				?	
	2	1			?
	3			?	
1					

Dealing with Missing Entries



1	3	4							
	3	5						5	
		4	5					5	
			3						
			3						
2									
	2								
	3								
1									

- To solve overfitting we introduce **regularization:**

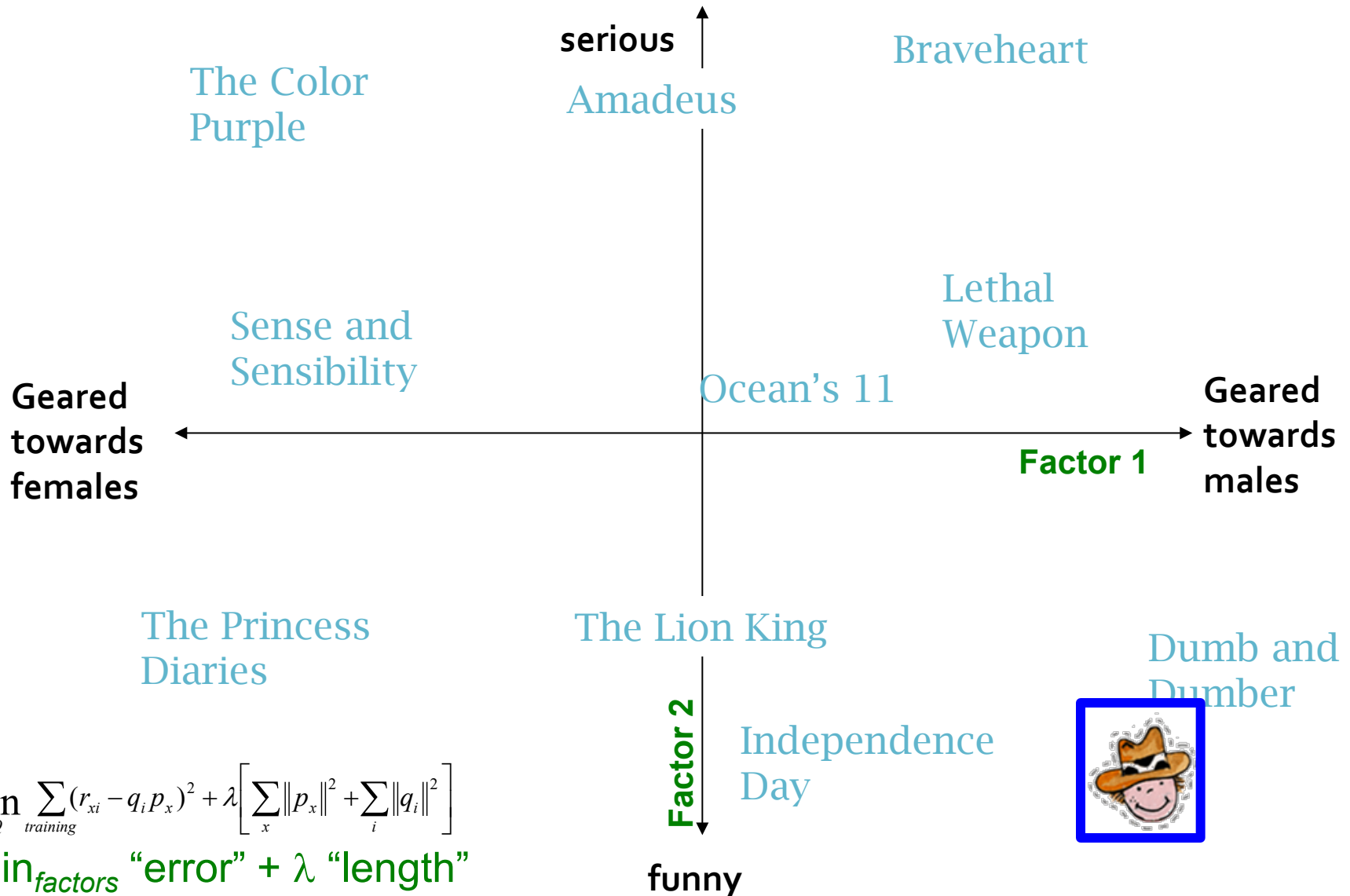
- Allow rich model where there are sufficient data
- Shrink aggressively where data are scarce

$$\min_{P,Q} \underbrace{\sum_{training} (r_{xi} - q_i p_x)^2}_{\text{"error"}} + \underbrace{\left[\lambda_1 \sum_x \|p_x\|^2 + \lambda_2 \sum_i \|q_i\|^2 \right]}_{\text{"length"}}$$

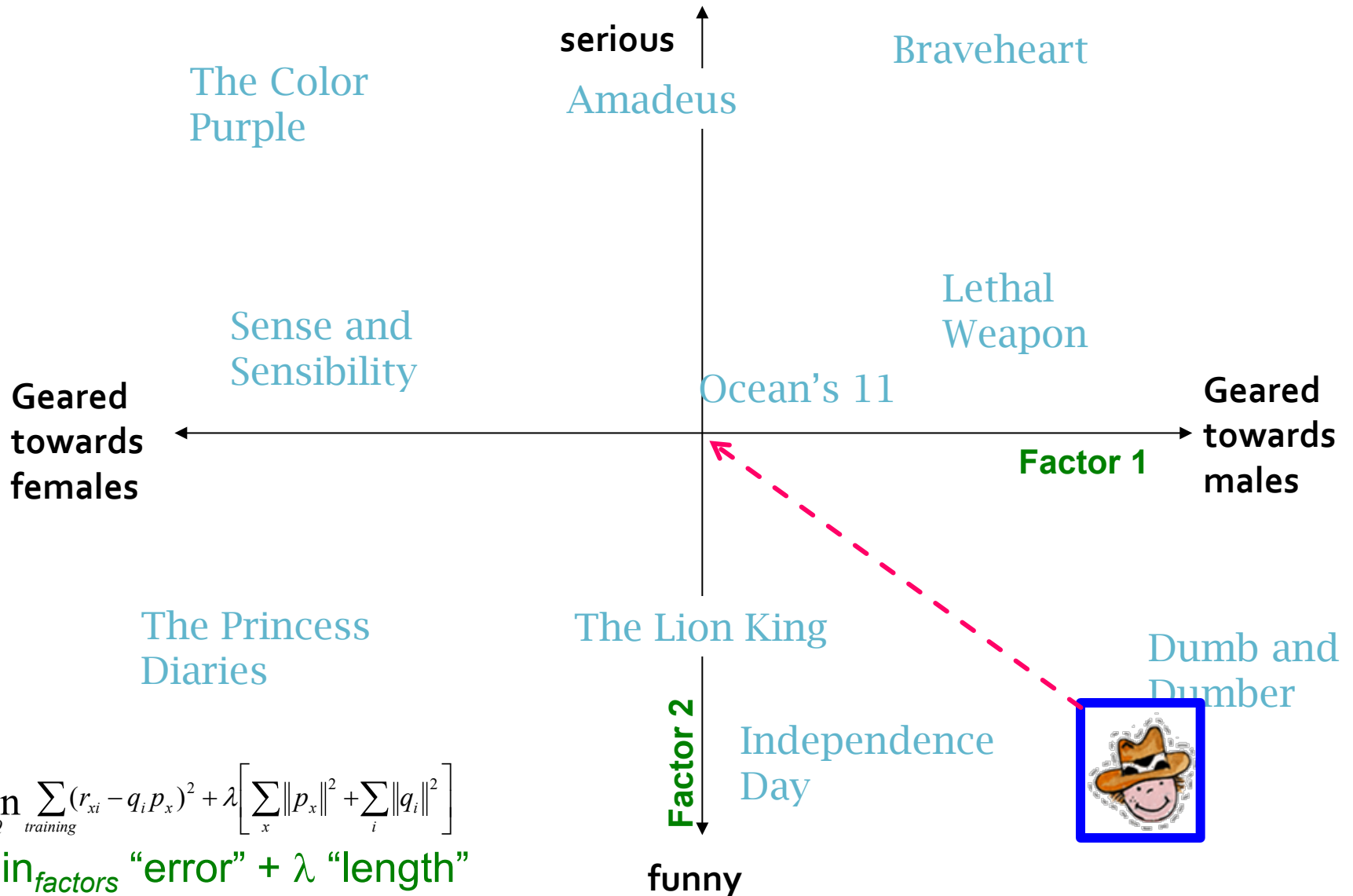
$\lambda_1, \lambda_2 \dots$ user set regularization parameters

Note: We do not care about the “raw” value of the objective function, but we care in P,Q that achieve the minimum of the objective

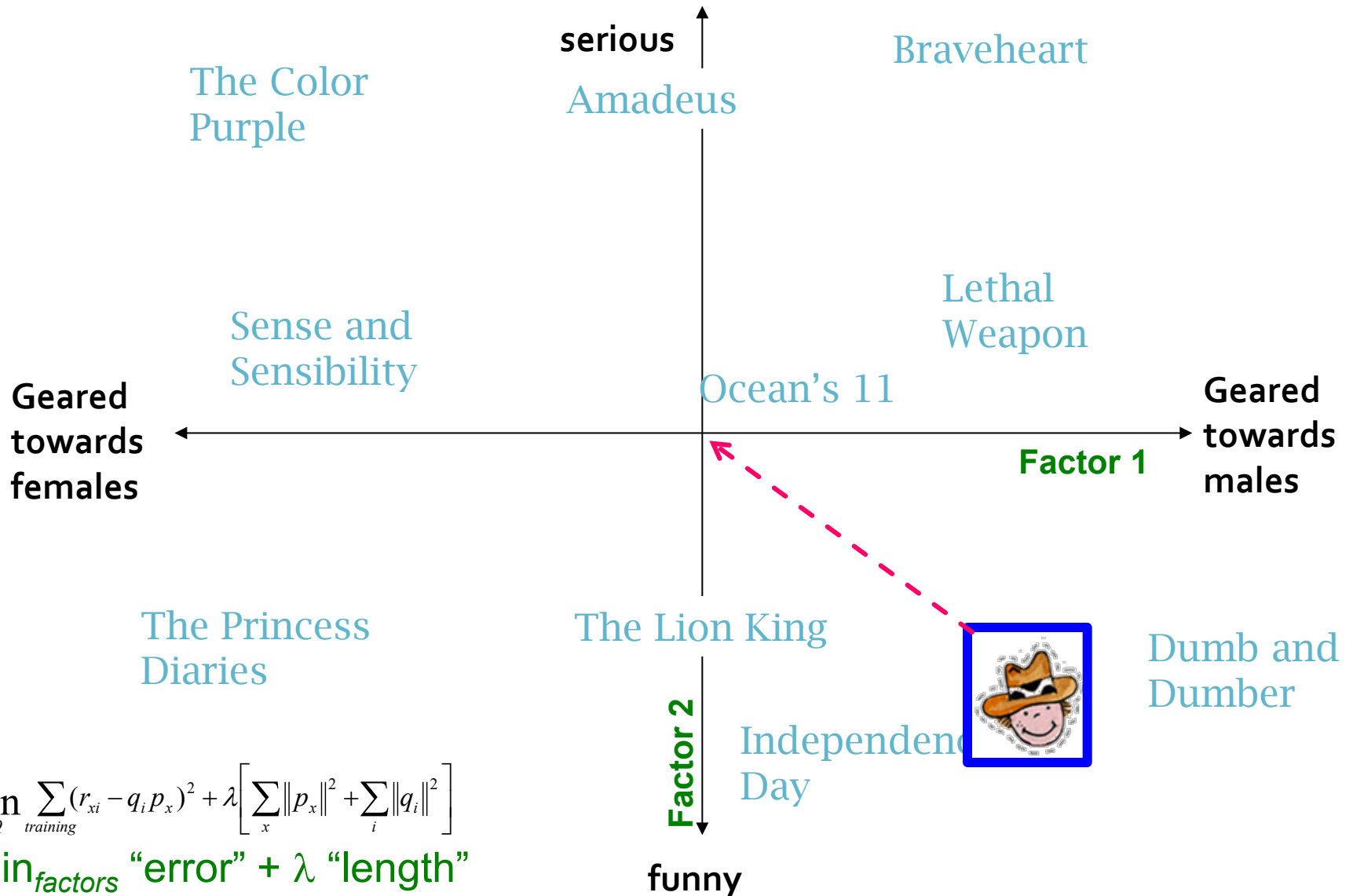
The Effect of Regularization



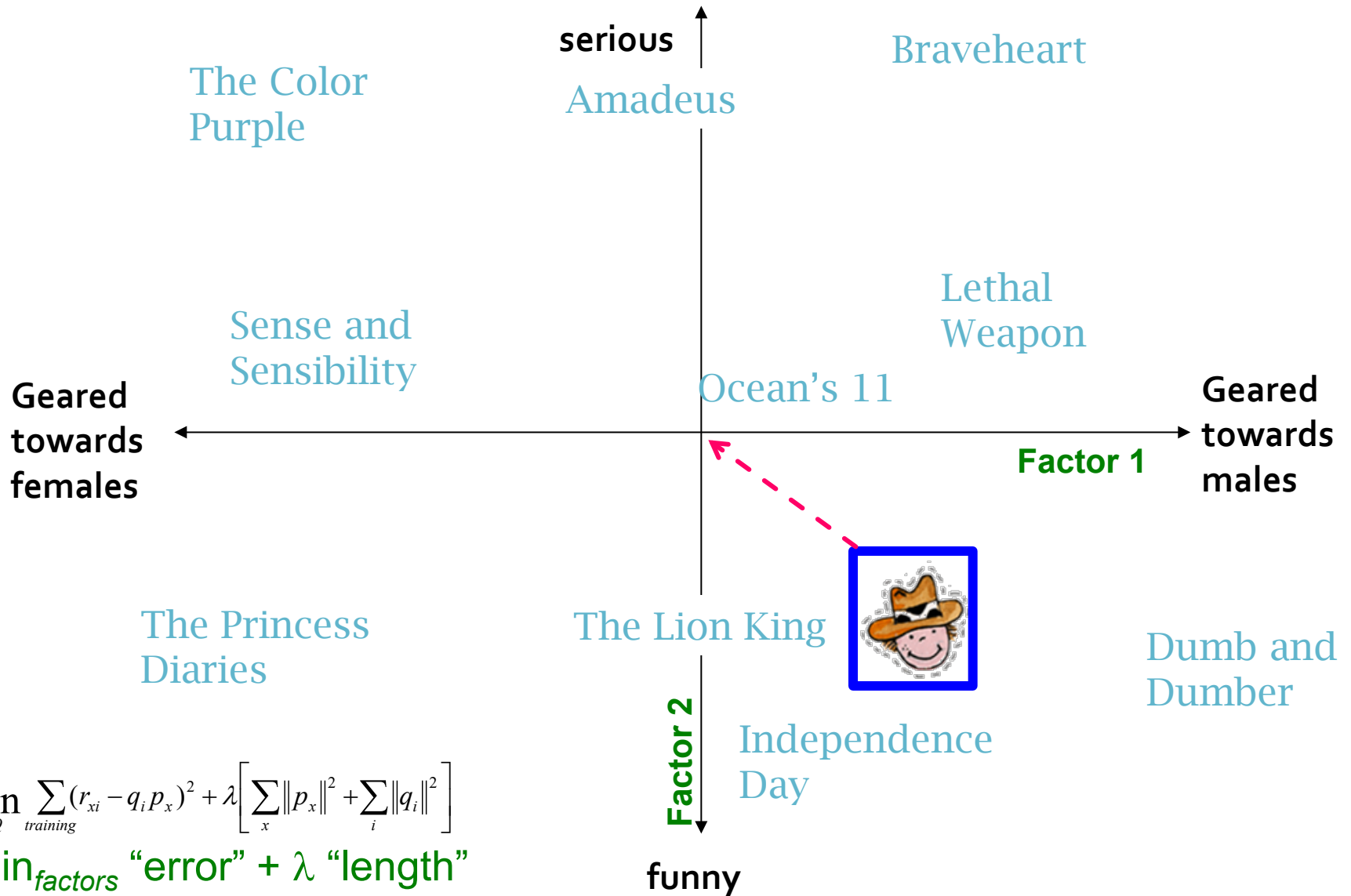
The Effect of Regularization



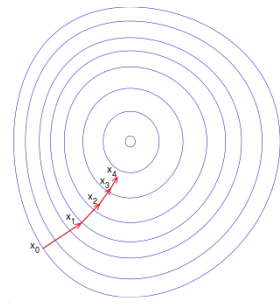
The Effect of Regularization



The Effect of Regularization



Stochastic Gradient Descent



- Want to find matrices **P** and **Q** :

$$\min_{P, Q} \sum_{training} (r_{xi} - q_i p_x)^2 + \left[\lambda_1 \sum_x \|p_x\|^2 + \lambda_2 \sum_i \|q_i\|^2 \right]$$

- **Gradient decent:**

- Initialize **P** and **Q** (using SVD, pretend missing ratings are 0)

- Do gradient descent:

- $P \leftarrow P - \eta \cdot \nabla P$

- $Q \leftarrow Q - \eta \cdot \nabla Q$

- where ∇Q is gradient/derivative of matrix **Q** :

- $\nabla Q = [\nabla q_{if}]$ and $\nabla q_{if} = \sum_{x,i} -2(r_{xi} - q_i p_x) p_{xf} + 2\lambda_2 q_{if}$

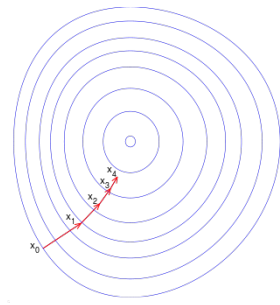
- Here q_{if} is entry **f** of row **q_i** of matrix **Q**

- **Observation: Computing gradients is slow!**

How to compute gradient
of a matrix?

Compute gradient of every
element independently!

Stochastic Gradient Descent



■ Gradient Descent (GD) vs. Stochastic GD

- **Observation:** $\nabla Q = [\nabla q_{if}]$ where

$$\nabla q_{if} = \sum_{x,i} -2(r_{xi} - q_{if}p_{xf})p_{xf} + 2\lambda q_{if} = \sum_{x,i} \nabla Q(r_{xi})$$

- Here q_{if} is entry f of row q_i of matrix Q

- $Q = Q - \square$ $Q = Q - \eta[\sum_{x,i} \nabla Q(r_{xi})]$

- **Idea:** Instead of evaluating gradient over all ratings evaluate it for each individual rating and make a step

- **GD:** $Q \leftarrow Q - \eta[\sum_{r_{xi}} \nabla Q(r_{xi})]$

- **SGD:** $Q \leftarrow Q - \mu \nabla Q(r_{xi})$

- **Faster convergence!**

- Need more steps but each step is computed much faster

GD vs SGD on linear regression

$$h_{\theta}(x) = \sum_{j=0}^n \theta_j x_j$$

GD

$$J_{train}(\theta) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$$

Repeat{

$$\theta_j := \theta_j - \eta \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

(for every $j = 0, \dots, n$)

}

SGD

New cost function

$$\text{cost}(\theta, (x^{(i)}, y^{(i)})) = \frac{1}{2} (h_{\theta}(x^{(i)}) - y^{(i)})^2$$

$$J_{train}(\theta) = \frac{1}{m} \sum_{i=1}^m \text{cost}(\theta, (x^{(i)}, y^{(i)}))$$

1. Randomly shuffle dataset;

2. Repeat{

for $i = 1, \dots, m$ {

$$\theta_j := \theta_j - \eta (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

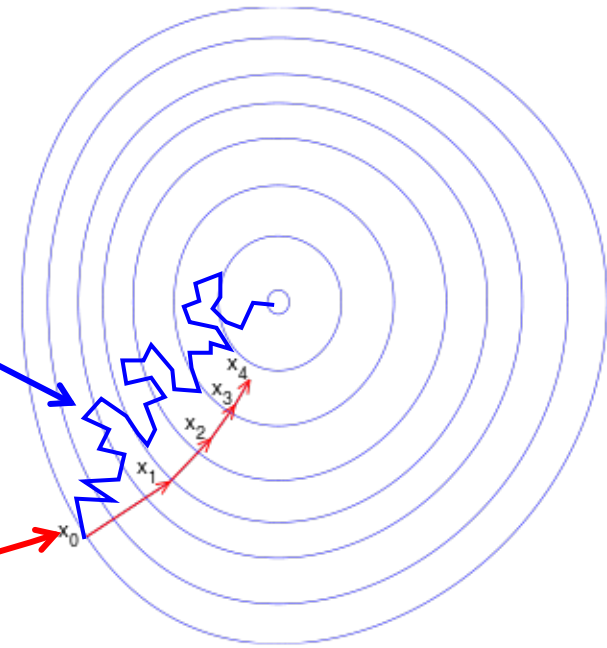
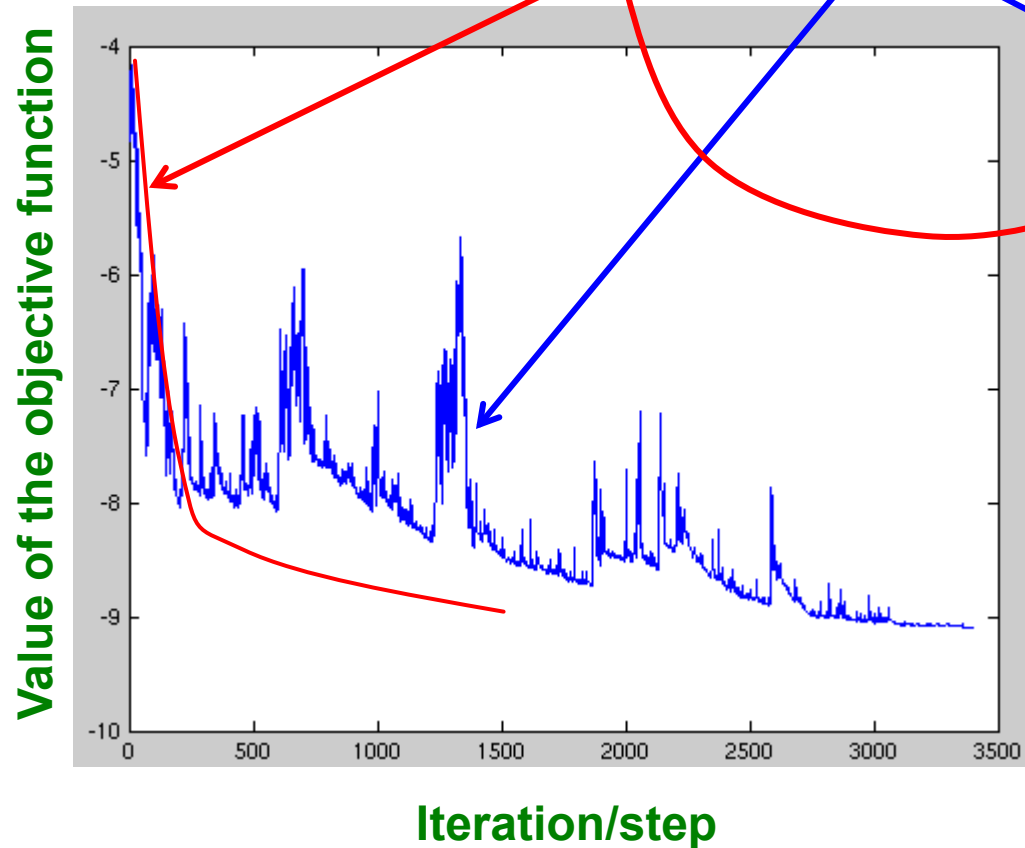
(for every $j = 0, \dots, n$)

}

}

SGD vs. GD

■ Convergence of **GD** vs. **SGD**



GD improves the value of the objective function at every step.

SGD improves the value but in a “noisy” way.

GD takes fewer steps to converge but each step takes much longer to compute.

In practice, **SGD** is much faster!

Stochastic Gradient Descent

- **Stochastic gradient decent:**
 - Initialize \mathbf{P} and \mathbf{Q} (using SVD, pretend missing ratings are 0)
 - Then iterate over the ratings (multiple times if necessary) and update factors:
 - For each r_{xi} :
 - $\varepsilon_{xi} = 2(r_{xi} - q_i \cdot p_x)$ (derivative of the "error")
 - $q_i \leftarrow q_i - \mu_1 (\varepsilon_{xi} p_x - \lambda_2 q_i)$ (update equation)
 - $p_x \leftarrow p_x - \mu_2 (\varepsilon_{xi} q_i - \lambda_1 p_x)$ (update equation)
 - **2 for loops:**
 - For until convergence:
 - For each r_{xi}
 - Compute gradient, do a "step"

■ Stochastic gradient decent:

- Initialize \mathbf{P} and \mathbf{Q} (using SVD, pretend missing ratings are 0)
- Then iterate over the ratings (multiple times if necessary) and update factors:

For each r_{xi} :

- $\varepsilon_{xi} = 2(r_{xi} - q_i \cdot p_x)$ (derivative of the "error")
 - $q_i \leftarrow q_i - \mu_1 (\varepsilon_{xi} p_x - \lambda_2 q_i)$ (update equation)
 - $p_x \leftarrow p_x - \mu_2 (\varepsilon_{xi} q_i - \lambda_1 p_x)$ (update equation)
- μ ... learning rate

■ 2 for loops:

- For until convergence:
 - For each r_{xi}
 - Compute gradient, do a "step"

Extending Latent Factor Model to Include Biases

Modeling Biases and Interactions

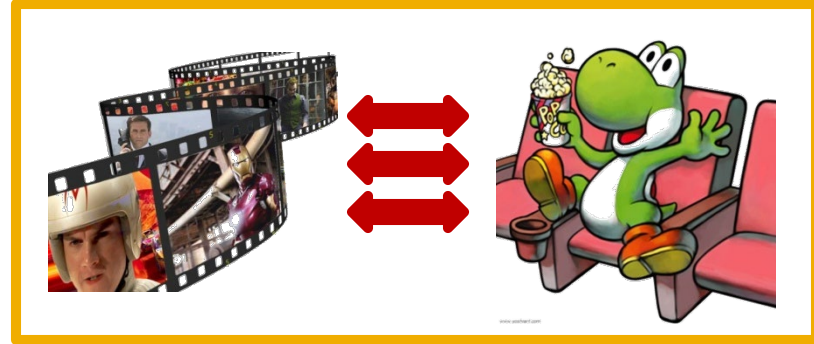
user bias



movie bias



user-movie interaction



Baseline predictor

- Separates users and movies
- Benefits from insights into user's behavior
- Among the main practical contributions of the competition

User-Movie interaction

- Characterizes the matching between users and movies
- Attracts most research in the field
- Benefits from algorithmic and mathematical innovations

- μ = overall mean rating
- b_x = bias of user x
- b_i = bias of movie i

Baseline Predictor

- We have expectations on the rating by user x of movie i , even without estimating x 's attitude towards movies like i



- Rating scale of user x
- Values of other ratings user gave recently (day-specific mood, anchoring, multi-user accounts)

- (Recent) popularity of movie i
- Selection bias; related to number of ratings user gave on the same day ("frequency")

Putting It All Together

$$r_{xi} = \underbrace{\mu}_{\text{Overall mean rating}} + \underbrace{b_x}_{\text{Bias for user } x} + \underbrace{b_i}_{\text{Bias for movie } i} + \underbrace{q_i \cdot p_x}_{\text{User-Movie interaction}}$$

■ Example:

- Mean rating: $\mu = 3.7$
- You are a critical reviewer: your ratings are 1 star lower than the mean: $b_x = -1$
- Star Wars gets a mean rating of 0.5 higher than average movie: $b_i = +0.5$
- Predicted rating for you on Star Wars:
 $= 3.7 - 1 + 0.5 = 3.2$

Fitting the New Model

■ Solve:

$$\min_{Q,P} \sum_{(x,i) \in R} \left(r_{xi} - (\mu + b_x + b_i + q_i p_x) \right)^2$$

goodness of fit

$$+ \left(\lambda_1 \sum_i \|q_i\|^2 + \lambda_2 \sum_x \|p_x\|^2 + \lambda_3 \sum_x \|b_x\|^2 + \lambda_4 \sum_i \|b_i\|^2 \right)$$

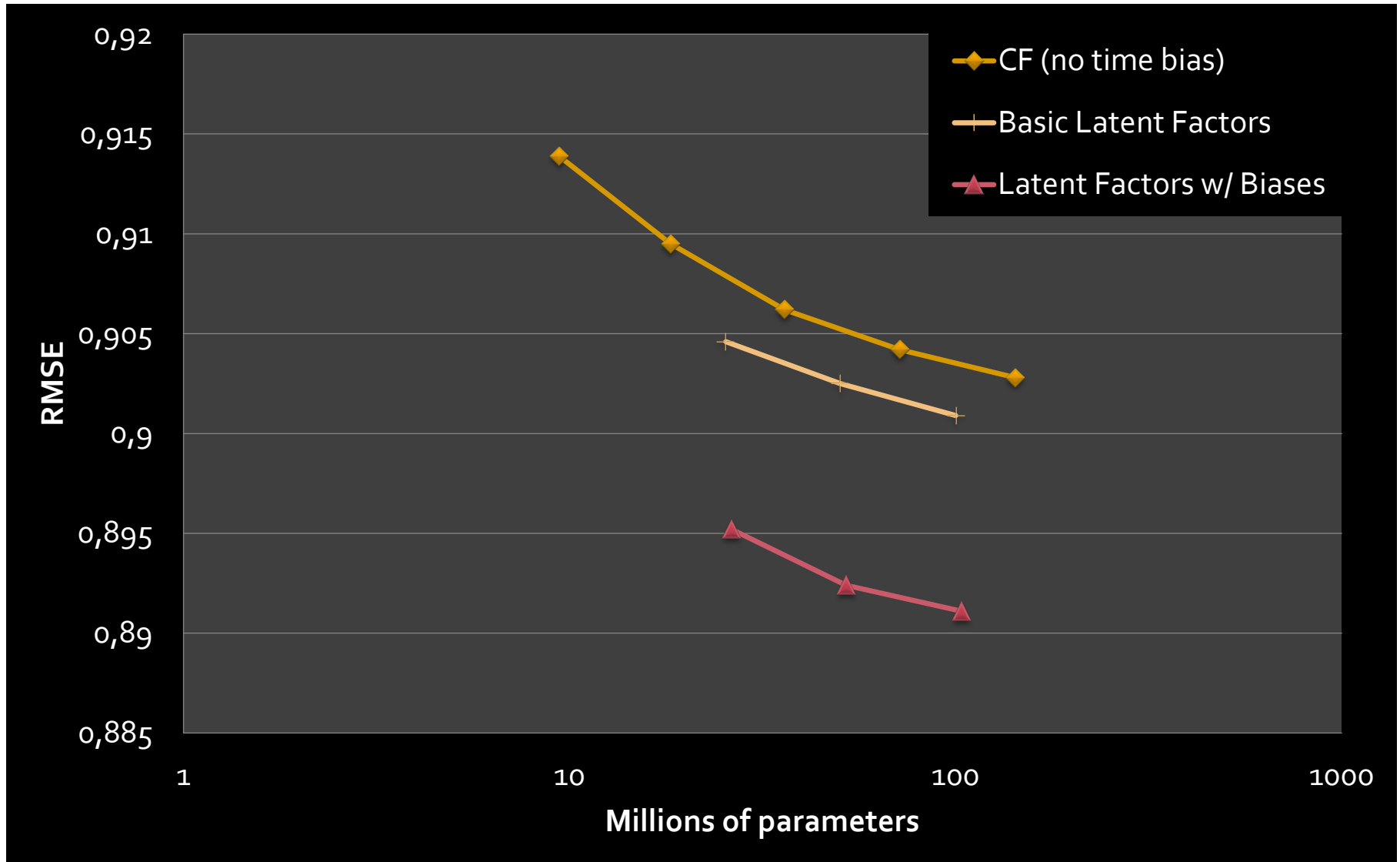
regularization

λ is selected via grid-search on a validation set

■ Stochastic gradient decent to find parameters

- **Note:** Both biases b_x, b_i as well as interactions q_i, p_x are treated as parameters (we estimate them)

Performance of Various Methods





Global average: 1.1296

User average: 1.0651

Movie average: 1.0533

Netflix: 0.9514

Basic Collaborative filtering: 0.94

Collaborative filtering++: 0.91

Latent factors: 0.90

Latent factors+Biases: 0.89

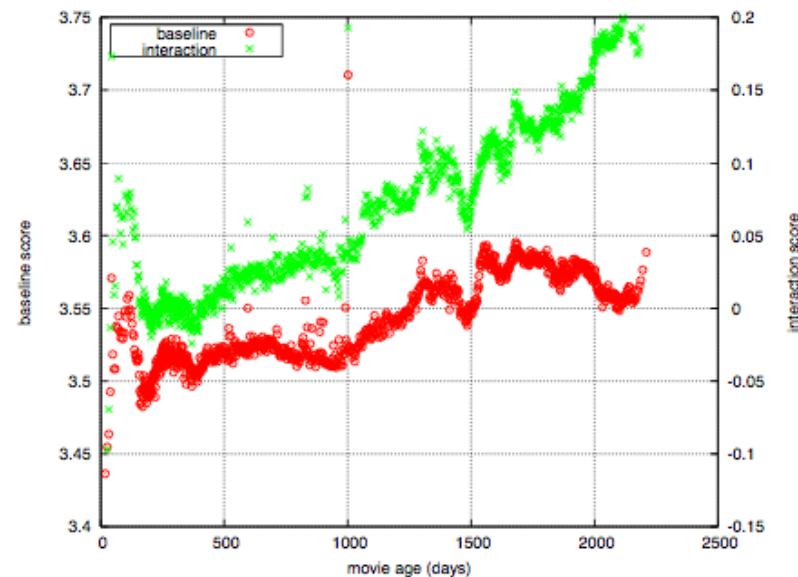
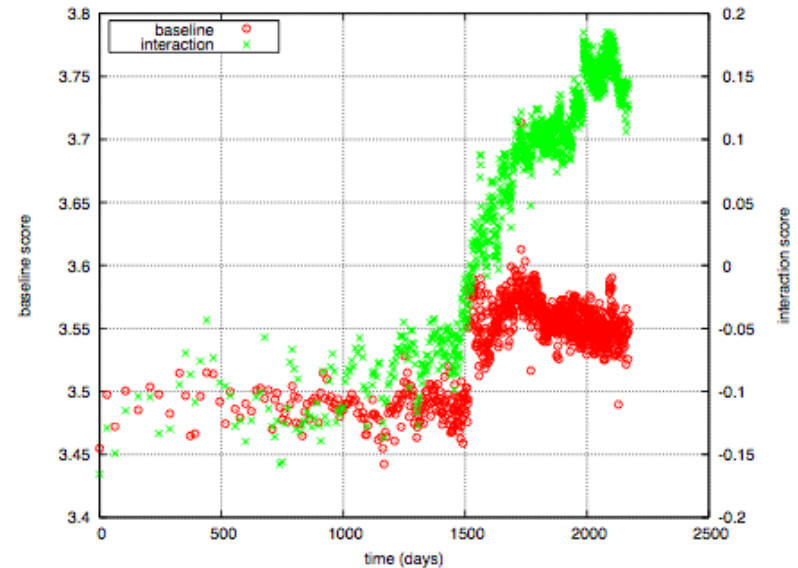
Grand Prize: 0.8563

The Netflix Challenge: 2006-09

Temporal Biases Of Users

- **Sudden rise in the average movie rating (early 2004)**
 - Improvements in Netflix
 - GUI improvements
 - Meaning of rating changed
- **Movie age**
 - Users prefer new movies without any reasons
 - Older movies are just inherently better than newer ones

Y. Koren, Collaborative filtering with temporal dynamics, KDD '09



Temporal Biases & Factors

- **Original model:**

$$r_{xi} = \mu + b_x + b_i + q_i \cdot p_x$$

- **Add time dependence to biases:**

$$r_{xi} = \mu + b_x(t) + b_i(t) + q_i \cdot p_x$$

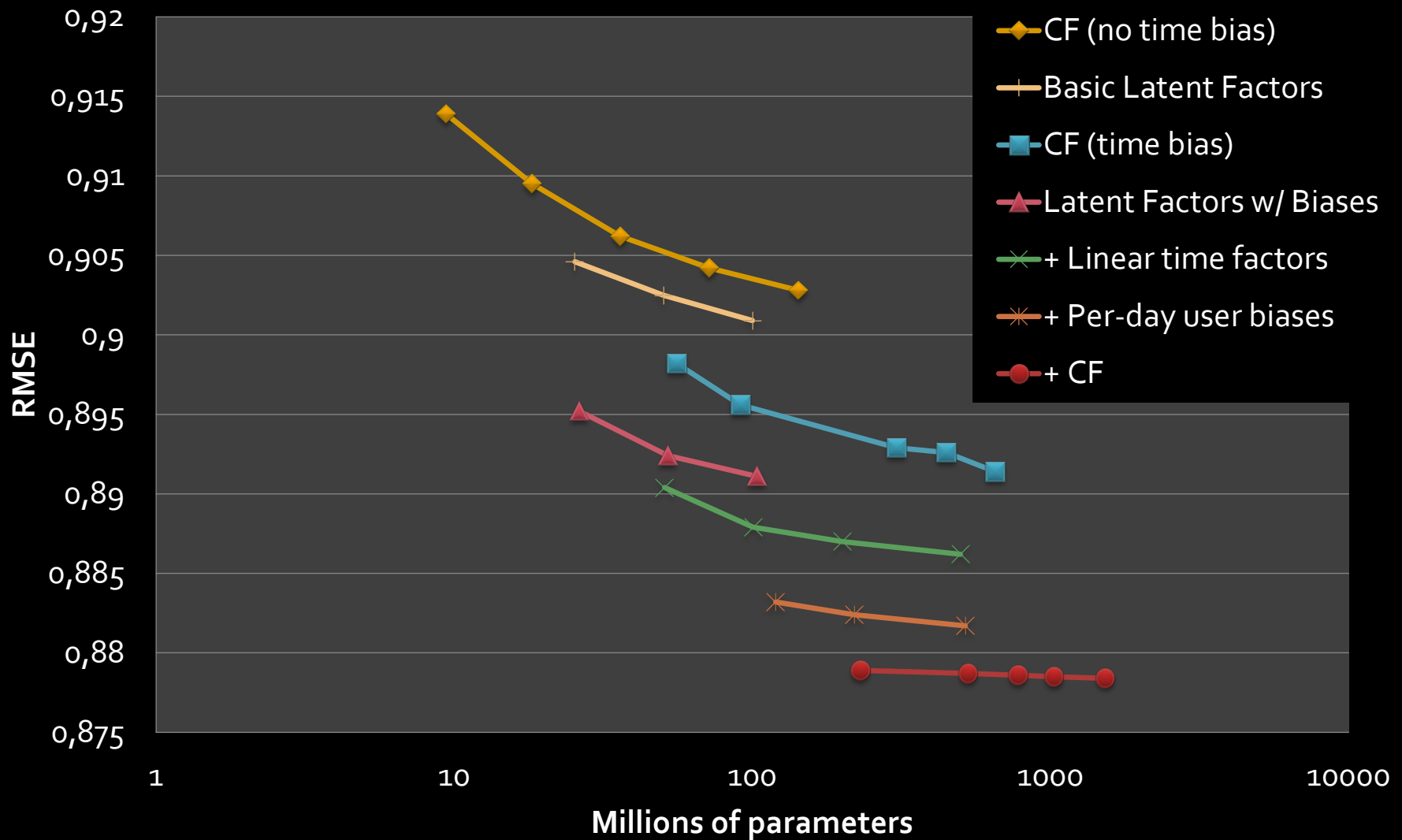
- Make parameters b_x and b_i to depend on time
- (1) Parameterize time-dependence by linear trends
- (2) Each bin corresponds to 10 consecutive weeks

$$b_i(t) = b_i + b_{i,\text{Bin}(t)}$$

- **Add temporal dependence to factors**

- $p_x(t)$... user preference vector on day t

Adding Temporal Effects



Global average: 1.1296

User average: 1.0651

Movie average: 1.0533

Netflix: 0.9514

Basic Collaborative filtering: 0.94

Collaborative filtering++: 0.91

Latent factors: 0.90

Latent factors+Biases: 0.89

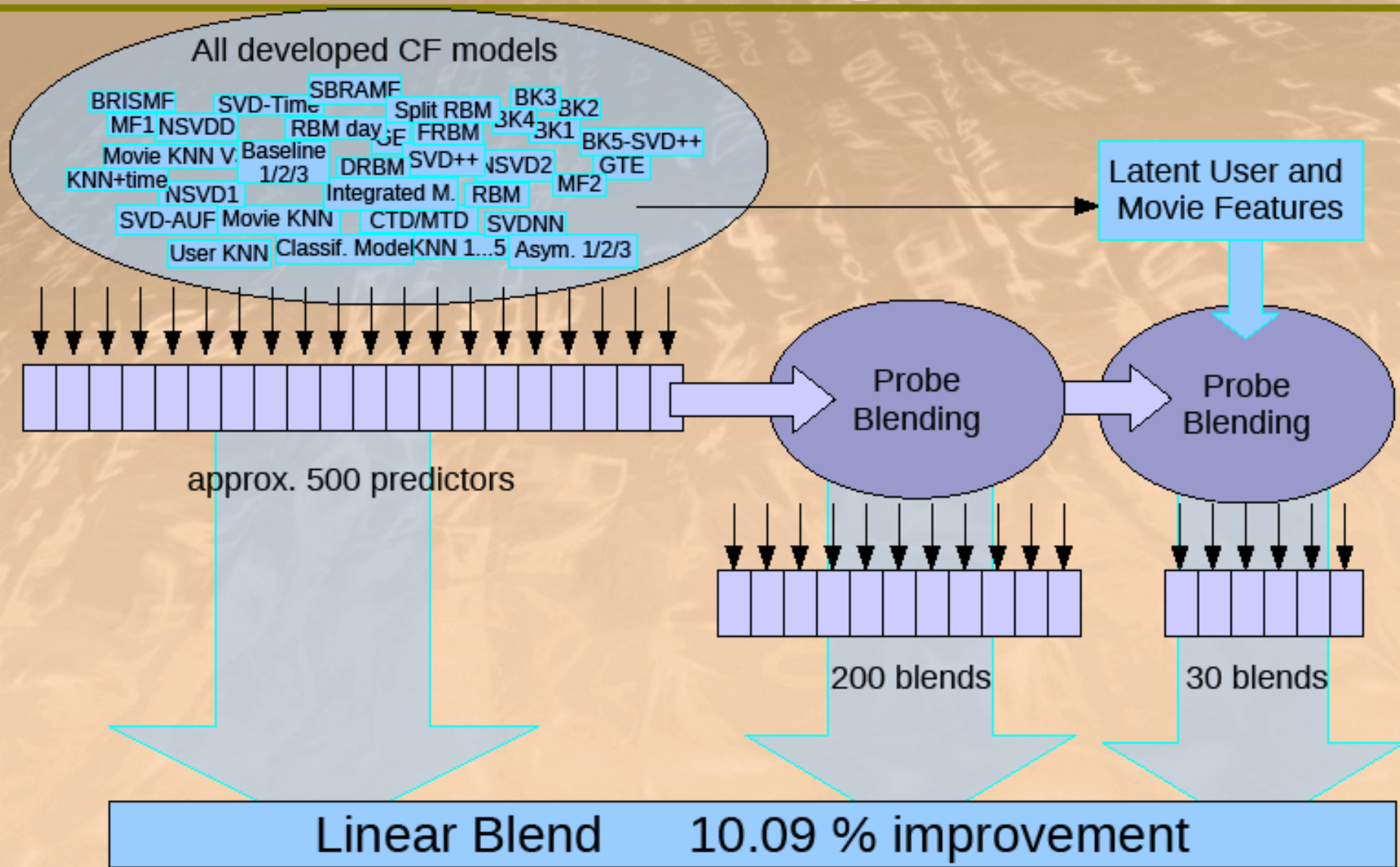
Latent factors+Biases+Time: 0.876

Grand Prize: 0.8563

Still no prize! 😞
Getting desperate.
Try a “kitchen
sink” approach!

The big picture

Solution of BellKor's Pragmatic Chaos



Standing on June 26th 2009

NETFLIX

Netflix Prize

HomeRulesLeaderboardRegisterUpdateSubmitDownload

Leaderboard

Display top 20 leaders.

Rank	Team Name	Best Score	% Improvement	Last Submit Time
1	BellKor's Pragmatic Chaos	0.8558	10.05	2009-06-26 18:42:37
Grand Prize - RMSE \leq 0.8563				
2	PragmaticTheory	0.8582	9.80	2009-06-25 22:15:51
3	BellKor in BigChaos	0.8590	9.71	2009-05-13 08:14:09
4	Grand Prize Team	0.8593	9.68	2009-06-12 08:20:24
5	Dace	0.8604	9.56	2009-04-22 05:57:03
6	BigChaos	0.8613	9.47	2009-06-23 23:06:52
Progress Prize 2008 - RMSE = 0.8616 - Winning Team: BellKor in BigChaos				
7	BellKor	0.8620	9.40	2009-06-24 07:16:02
8	Gravity	0.8634	9.25	2009-04-22 18:31:32
9	Opera Solutions	0.8638	9.21	2009-06-26 23:18:13
10	BruceDengDaoCiYiYou	0.8638	9.21	2009-06-27 00:55:55
11	pengpengzhou	0.8638	9.21	2009-06-27 01:06:43
12	xlvector	0.8639	9.20	2009-06-26 13:49:04
13	xiangliang	0.8639	9.20	2009-06-26 07:47:34

June 26th submission triggers 30-day “last call”

Netflix Prize

COMPLETED

[Home](#) [Rules](#) [Leaderboard](#) [Update](#) [Download](#)

Leaderboard

Showing Test Score. [Click here to show quiz score](#)Display top leaders.

Rank	Team Name	Best Test Score	% Improvement	Best Submit Time
Grand Prize - RMSE = 0.8567 - Winning Team: BellKor in BigChaos				
1	BellKor's Pragmatic Chaos	0.8567	10.06	2009-07-26 18:18:28
2	The Ensemble	0.8567	10.06	2009-07-26 18:38:22
3	Grand Prize Team	0.8582	9.90	2009-07-10 21:24:40
4	Opera Solutions and Vandelay United	0.8588	9.84	2009-07-10 01:12:31
5	Vandelay Industries!	0.8591	9.81	2009-07-10 00:32:20
6	PragmaticTheory	0.8594	9.77	2009-06-24 12:06:56
7	BellKor in BigChaos	0.8601	9.70	2009-05-13 08:14:09
8	Dace	0.8612	9.59	2009-07-24 17:18:43
9	Feeds2	0.8622	9.48	2009-07-12 13:11:51
10	BigChaos	0.8623	9.47	2009-04-07 12:33:59
11	Opera Solutions	0.8623	9.47	2009-07-24 00:34:07
12	BellKor	0.8624	9.46	2009-07-26 17:19:11

Progress Prize 2008 - RMSE = 0.8627 - Winning Team: BellKor in BigChaos

13	xiangliang	0.8642	9.27	2009-07-15 14:53:22
14	Gravity	0.8643	9.26	2009-04-22 18:31:32
15	Ces	0.8651	9.18	2009-06-21 19:24:53
16	Invisible Ideas	0.8653	9.15	2009-07-15 15:53:04
17	Just a guy in a garage	0.8662	9.06	2009-05-24 10:02:54
18	J Dennis Su	0.8666	9.02	2009-03-07 17:16:17
19	Craig Carmichael	0.8666	9.02	2009-07-25 16:00:54
20	acmehill	0.8668	9.00	2009-03-21 16:20:50

Progress Prize 2007 - RMSE = 0.8723 - Winning Team: KorBell

Million \$ Awarded Sept 21st 2009



Acknowledgments

- Some slides and plots borrowed from Yehuda Koren, Robert Bell and Padhraic Smyth
- **Further reading:**
 - Y. Koren, Collaborative filtering with temporal dynamics, KDD '09
 - <http://www2.research.att.com/~volinsky/netflix/bpc.html>
 - <http://www.the-ensemble.com/>