# 5. Integer Programming

Many practical problems require integer solutions. A linear programming problem in which *some* or *all* of the variables are constrained to be *integer-valued* is known as an **integer linear program (IP or ILP)**. If all the variables are integer valued the problem is a *pure integer* LP. A problem that contains both integer and continuous valued variables is a *mixed integer* LP (often abbreviated to **MIP**).

In pure IP problems, the feasible region is contained in the *integer lattice* of points $\mathbb{Z}^n$. IP's are generally harder to solve than LP's. However IP and MIP problems frequently arise in practice. We will examine two solution techniques:

- Method of cutting planes (Gomory)
- Method of branch and bound (B&B)

## 5.1 Gomory's cut (for pure IP)

The standard form of the pure IP problem (minimization case) is

$$\begin{array}{ll} \text{Minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \text{ and integer} \end{array}$$

where $\boldsymbol{c}\,(n \times 1)\,\boldsymbol{,}\,\boldsymbol{A}\,(m \times n)\,,\boldsymbol{b}\,(m \times 1)$ are integer-valued.coefficients.

We seek $\boldsymbol{x} \in \mathbb{Z}^n$. Geometrically, we might expect the optimal solution to a **ILP** to be "not very far" from the optimal solution to the **LP relaxation (**defined as the LP problem obtained by ignoring or *relaxing* the integer restriction on the variables). This is sometimes true, but often not the case. However it seems a good idea, first to solve the relaxation by the simplex method.

**Proposition 1 (Optimality of relaxation)** *Let $x_R$ be the solution to the LP relaxation. If $x_R$ is integer (i.e.* feasible *for IP) then $x_R$ is optimal for the IP.*

**Proof.** (minimization case)

Let $\mathfrak{F}_{IP}$, $\mathfrak{F}_R$ be the feasible regions of the IP and its relaxation, respectively. Relaxing a LP problem removes the integer constraint, so enlarges the feasible region. Hence $\mathfrak{F}_{IP} \subset \mathfrak{F}_R$. As the objective functions for both problems (the IP and its relaxation) are the same, we have $z_R \leq z_{IP}$. We have shown that the optimal value of the LP relaxation is a lower bound for the optimal value of the IP.

If, however, $x_R$ is all integer then $x_R \in \mathfrak{F}_{IP}$ so $z_{IP} \leq z_R$ since $z_{IP} = \min\{z(x) : x \in \mathfrak{F}_{IP}\}$. Therefore $z_R = z_{IP}$ and $x_R$ is optimal for IP. ■

## Properties of a cutting plane

CP1 The cut removes the current LP solution from $\mathfrak{F}_R$

CP2 It doesn't cut away any integer feasible solutions. $\mathfrak{F}_{IP}$ is unchanged.

The equation of the cutting plane is derived from a **source row** in the final *optimal* tableau for the LP relaxation. A new equation, involving an extra slack variable and having the required two properties stated above, is added to the tableau making the tableau *infeasible*. Dual simplex iterations are carried out to find a new LP solution. If this LP solution is integer then we stop having solved the IP. Otherwise we generate a new cut and solve the resulting problem by dual simplex iterations. This process can be shown to converge.

## Algorithm

1. Solve the LP relaxation giving $\boldsymbol{x}^*$ (the current optimum) and $z^*$ (the current optimal value)
2. If $\boldsymbol{x}^* \in \mathbb{Z}^n$ then stop; $\boldsymbol{x}^*$ solves the ILP. Otherwise
3. (At least one variable in the basis is fractional valued). Corresponding rows of the tableau are candidates for the *source row*

   Choose a *source row* using some criterion e.g. $\min_i \left\{|f_{i0} - \frac{1}{2}|\right\}$ (a heuristic) where $f_{i0}$ denotes the fractional part of the tableau right hand side $y_0$ $\left(\bar{b}\right)$. Construct a cut and add a row to the tableau. Go to Step 1.

## The cut equation (pure IP)

For pure IP problems, Gomory's cut, sometimes termed a *fractional* cut, takes the form

$$\sum_{j \in J} f_{ij} x_j \geq f_{i0} \tag{5.1}$$

or equivalently, introducing a slack variable $s$

$$\boxed{s - \sum_{j \in J} f_{ij} x_j = -f_{i0}} \tag{5.2}$$

where

- row $i$ of the optimal tableau for the LP relaxation is the chosen *source row*, corresponding to some basic variable $x_i$ having a fractional (non integer) value $y_{i0}$ at the current optimum.
- $f_{ij}$ is the fractional part of the tableau entry $y_{ij}$, $j \in J \cup \{0\}$

  ($J$ = index set of non-basic variables at the current optimum)

## Result (Validity of cut equation)

The cut Eq.(5.2) can be shown to satisfy the required properties

**Proof.** Let $x_i$ be a fractional valued basic variable in the final tableau (for the LP relaxation). The tableau equation has the canonical form

$$x_i + \sum_{j \in J} y_{ij} x_j = y_{i0}$$

which may be written

$$x_i + \sum_{j \in J} (n_{ij} + f_{ij}) x_j = n_{i0} + f_{i0}$$

where $n_{ij}$ is the integer part of $y_{ij}$ and $f_{ij}$ is the fractional part.

e.g. if $y_{ij} = 3.6$ then $n_{ij} = 3$ and $f_{ij} = 0.6$

if $y_{ij} = -3.6$ then $n_{ij} = -4$ and $f_{ij} = 0.4$

$$x_i + \sum_{j \in J} n_{ij} x_j - n_{i0} = f_{i0} - \sum_{j \in J} f_{ij} x_j \tag{5.3}$$

- collecting all integer terms to the lhs.

At the current LP optimum $\boldsymbol{x}^*$we have $x_j = 0$ for all $j \in J$ so $f_{i0} > \sum_{j \in J} f_{ij} x_j$. Thus $s < 0$ in Eq.(5.2) and $\boldsymbol{x}^*$ is infeasible for the problem with the cut added **proving property P1.**

At any feasible *integer solution* the lhs. of (5.3) is integer-valued. Therefore the rhs. must also be integer-valued. Hence rhs.$\in \{0, -1, -2, ..\}$ so rhs. of (5.3)$\leq 0$, hence Eq.(5.2) is satisfied at every *integer* feasible solution, **proving property P2.** ■

## Example

$$\begin{array}{ll} \text{Maximize} & 7x_1 + 9x_2 \\ \text{s.t.} & -x_1 + 3x_2 \leq 6 \\ & 7x_1 + x_2 \leq 35 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{array}$$

The optimal tableau to the LP relaxation is

| | $s_2$ | $s_1$ | |
|---|---|---|---|
| $x_2$ | $\frac{1}{22}$ | $\frac{7}{22}$ | $\frac{7}{2}$ |
| $x_1$ | $\frac{3}{22}$ | $-\frac{1}{22}$ | $\frac{9}{2}$ |
| | $\frac{30}{22}$ | $\frac{56}{22}$ | 63 |

Using the $x_2-$row as the source row $\left(\text{criterion } |f_{i0} - \frac{1}{2}| \text{ is tied}\right)$ leads to

| | $s_2$ | $s_1$ | |
|---|---|---|---|
| $x_2$ | $\frac{1}{22}$ | $\frac{7}{22}$ | $\frac{7}{2}$ |
| $x_1$ | $\frac{3}{22}$ | $-\frac{1}{22}$ | $\frac{9}{2}$ |
| $s_3$ | $-\frac{1}{22}$ | $\boxed{-\frac{7}{22}}$ | $-\frac{1}{2}$ |
| | $\frac{30}{22}$ | $\frac{56}{22}$ | 63 |

After a dual simplex iteration we find rhs.$\geq 0$, therefore have found a new optimal LP solution

| | $s_2$ | $s_3$ | |
|---|---|---|---|
| $x_2$ | 0 | 1 | 3 |
| $x_1$ | $\frac{1}{7}$ | $-\frac{1}{7}$ | $\frac{32}{7}$ |
| $s_1$ | $\frac{1}{7}$ | $-\frac{22}{7}$ | $\frac{11}{7}$ |
| | 1 | 8 | 59 |

which however is still fractional. We obtain a new cut using $x_1$ as the source row.

| | $s_2$ | $s_3$ | |
|---|---|---|---|
| $x_2$ | 0 | 1 | 3 |
| $x_1$ | $\frac{1}{7}$ | $-\frac{1}{7}$ | $\frac{32}{7}$ |
| $s_1$ | $\frac{1}{7}$ | $-\frac{22}{7}$ | $\frac{11}{7}$ |
| $s_4$ | $\boxed{-\frac{1}{7}}$ | $-\frac{6}{7}$ | $-\frac{4}{7}$ |
| | 1 | 8 | 59 |

| | $s_4$ | $s_3$ | |
|---|---|---|---|
| $x_2$ | | | 3 |
| $x_1$ | | | 4 |
| $s_1$ | | | 1 |
| $s_2$ | | | 4 |
| | | | 55 |

and find an optimal integer feasible solution after one iteration. Therefore this solves the ILP. We can retrace the steps of the above general proof for the 2nd cut in this example. The source row

$$x_1 + \tfrac{1}{7}s_2 - \tfrac{1}{7}s_3 = 4\tfrac{4}{7}$$

may be rewritten as

$$x_1 - s_3 - 4 = \tfrac{4}{7} - \tfrac{1}{7}s_2 - \tfrac{6}{7}s_3$$

At any feasible *integer* solution therefore, we have

$$\tfrac{4}{7} - \tfrac{1}{7}s_2 - \tfrac{6}{7}s_3 \leq 0$$

which is the constraint represented by the $s_4$ row in the penultimate tableau.

## 5.2 The mixed integer cut

Consider $x_i$ to be an *integer* variable of a MIP which, as before, is not integer valued in the optimal tableau of the LP relaxation There must be such an $x_i$ otherwise the optimal LP solution is feasible for MIP and we are done.

The equation corresponding to the $i$ th tableau row is

$$x_i + \sum_{j \in J} y_{ij} x_j = y_{i0} = n_{i0} + f_{i0} \tag{5.4}$$

$$x_i - n_{i0} = f_{i0} - \sum_{j \in J} y_{ij} x_j \tag{5.5}$$

At any feasible solution to MIP, $x_i$ *is integer valued* and the lhs. of Eq. 5.5 is integer.

Therefore either

**Case A: rhs.** $\leq 0$, or

**Case B: rhs.** $\geq 1$ must hold.

Define $J^+ = \{j : y_{ij} \geq 0\}$ and $J^- = \{j : y_{ij} < 0\}$. Using the property that *all* $x_j \geq 0$ *at a feasible solution to MIP* (we no longer have the integer property holding), we **either** have for **Case A:**

$$
\begin{aligned}
-\sum_{j \in J} y_{ij} x_j &\leq -f_{i0} \\
-\sum_{j \in J^+} y_{ij} x_j - \sum_{j \in J^-} y_{ij} x_j &\leq -f_{i0}
\end{aligned}
$$

so, as the second term $\geq 0$ ($x_j \geq 0$ and $-y_{ij} \geq 0$ for $j \in J^-$) we have

$$-\sum_{j \in J^+} y_{ij} x_j \leq -f_{i0} \tag{5.6}$$

**or**, for **Case B**

$$
\begin{aligned}
\sum_{j \in J} y_{ij} x_j &\leq -(1 - f_{i0}) \\
\sum_{j \in J^+} y_{ij} x_j + \sum_{j \in J^-} y_{ij} x_j &\leq -(1 - f_{i0}) \\
\sum_{j \in J^-} y_{ij} x_j &\leq -(1 - f_{i0})
\end{aligned}
$$

since $x_j \geq 0$ at any feasible solution to MIP. Therefore

$$\left( \frac{f_{i0}}{1 - f_{i0}} \right) \sum_{j \in J^-} y_{ij} x_j \leq -f_{i0} \tag{5.7}$$

Combining the mutually exclusive inequalities (5.6) and (5.7) we obtain Gomory's **mixed integer cut**

$$-\sum_{j \in J^+} y_{ij} x_j + \left( \frac{f_{i0}}{1 - f_{i0}} \right) \sum_{j \in J^-} y_{ij} x_j \leq -f_{i0}. \tag{5.8}$$

{If either $\alpha \leq \gamma$ or $\beta \leq \gamma$ with $\alpha, \beta \leq 0$ then $\alpha + \beta \leq \gamma$}

Introducing the slack variable $s$, we obtain

$$\boxed{s - \sum_{j \in J^+} y_{ij} x_j + \left( \frac{f_{i0}}{1 - f_{i0}} \right) \sum_{j \in J^-} y_{ij} x_j = -f_{i0}} \tag{5.9}$$

Notice that all the coefficients of terms on the lhs. are in fact $\leq 0$ when the signs of $y_{ij}$ are taken into account.

### Example (MIP cut)

Previous example with only $x_1$ integer

$$\begin{array}{ll} \text{Maximize} & 7x_1 + 9x_2 \\ \text{s.t.} & -x_1 + 3x_2 \leq 6 \\ & 7x_1 + x_2 \leq 35 \\ & x_1, x_2 \geq 0 \text{ and } x_1 \text{ integer} \end{array}$$

The final tableau for the LP relaxation was

| | $s_2$ | $s_1$ | |
|---|---|---|---|
| $x_2$ | $\frac{1}{22}$ | $\frac{7}{22}$ | $\frac{7}{2}$ |
| $x_1$ | $\frac{3}{22}$ | $-\frac{1}{22}$ | $\frac{9}{2}$ |
| | $\frac{30}{22}$ | $\frac{56}{22}$ | 63 |

The only integer variable is $x_1$, so becomes the source row for the MIP cut. Formula (5.9) with gives

$$s_3 - \tfrac{3}{22}s_2 - \tfrac{1}{22}s_1 = -\tfrac{1}{2}$$

which we insert into the tableau

| | $s_2$ | $s_1$ | |
|---|---|---|---|
| $x_2$ | $\frac{1}{22}$ | $\frac{7}{22}$ | $\frac{7}{2}$ |
| $x_1$ | $\frac{3}{22}$ | $-\frac{1}{22}$ | $\frac{9}{2}$ |
| $s_3$ | $\boxed{-\frac{3}{22}}$ | $-\frac{1}{22}$ | $-\frac{1}{2}$ |
| | $\frac{30}{22}$ | $\frac{56}{22}$ | 63 |

Here $f_{i0} = \frac{1}{2}$ and $\dfrac{f_{i0}}{f_{i0} - 1} = -1$. Carry out a dual simplex iteration

| | $s_3$ | $s_1$ | |
|---|---|---|---|
| $x_2$ | | | $\frac{10}{3}$ |
| $x_1$ | | | 4 |
| $s_2$ | | | $\frac{11}{3}$ |
| | | | 58 |

The rhs.$\geq 0$, therefore is feasible for the relaxation. Stop dual simplex iterations. Check whether the solution is feasible for MILP. $x_1$ is integer, therefore it is. We conclude that the solution is optimal for MILP. Stop.#

We can retrace the steps of the above general proof for this example. The $x_1$-row represents

$$x_1 + \tfrac{3}{22}s_2 - \tfrac{1}{22}s_1 = \tfrac{9}{2}$$

$$x_1 - 4 = \tfrac{1}{2} - \tfrac{3}{22}s_2 + \tfrac{1}{22}s_1$$

Case A: $rhs \leq 0$

$$\begin{aligned} \tfrac{1}{2} - \tfrac{3}{22}s_2 + \tfrac{1}{22}s_1 &\leq 0 \\ -\tfrac{3}{22}s_2 &\leq -\tfrac{1}{2} \end{aligned}$$

Case B: $rhs \geq 1$

$$\begin{aligned} \tfrac{1}{2} - \tfrac{3}{22}s_2 + \tfrac{1}{22}s_1 &\geq 1 \\ -\tfrac{1}{22}s_1 &\leq -\tfrac{1}{2} \end{aligned}$$

Hence

$$-\tfrac{3}{22}s_2 - \tfrac{1}{22}s_1 \leq -\tfrac{1}{2}$$

confirming the cut.