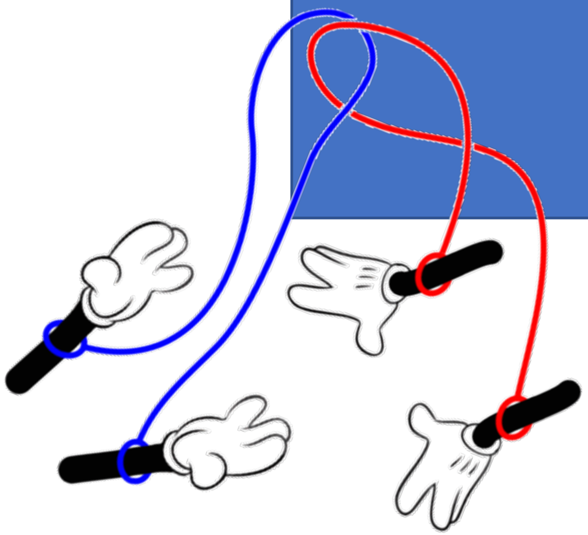
4. Multiple qubits & Entanglement

Quantum Computing



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Composite systems: tensor products

The state of two independent qubits can be written as a tensor (Kronecker) product:

$$|a\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \qquad |b\rangle = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \qquad |\psi\rangle = |ab\rangle \equiv |a\rangle \otimes |b\rangle = \begin{pmatrix} a_0 \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \\ a_1 \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{pmatrix}$$

The state of n independent qubits can be written as $|\psi\rangle = |q_0\rangle \otimes \cdots \otimes |q_n\rangle$. The resulting vector belongs to an Hilbert space of dimension $\boldsymbol{d = 2^n}$.

Single qubit gates can be expressed as tensor product

$$A_p = \mathbb{I} \otimes \cdots \mathbb{I} \otimes \underset{\substack{\uparrow \\ p^{\text{th}}\ \text{qubit}}}{A} \otimes \mathbb{I} \otimes \cdots \mathbb{I}$$

product state

$$A_p(|q_0\rangle \otimes \cdots |q_p\rangle \cdots \otimes |q_n\rangle) = |q_0\rangle \otimes \cdots \big(A\overset{\substack{p^{\text{th}}\ \text{qubit} \\ \downarrow}}{|q_p\rangle}\big) \cdots \otimes |q_n\rangle$$

Entangling gates

$$X_A(\,|0\rangle \otimes |0\rangle) = (X\,|0\rangle) \otimes |0\rangle = |1\rangle \otimes |0\rangle \equiv |10\rangle$$

$$X_A Y_B(\,|0\rangle \otimes |0\rangle) = (X\,|0\rangle) \otimes (Y\,|0\rangle) = |1\rangle \otimes i|1\rangle \equiv i|11\rangle$$

$$H_A(\,|0\rangle \otimes |0\rangle) = (H\,|0\rangle) \otimes |0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle \equiv \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$ Product state

CNOT	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
$\langle 00\|$	1	0	0	0
$\langle 01\|$	0	1	0	0
$\langle 10\|$	0	0	0	1
$\langle 11\|$	0	0	1	0

$$\frac{|00\rangle + |10\rangle}{\sqrt{2}} \xrightarrow{\text{CNOT}} \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

product entangled

It is still a 2-qubit state but cannot be written as product. Instead, it is a superposition of product states

Entangled states

$$|\psi\rangle = \sum_k a_k |\psi_0^k\rangle \otimes |\psi_1^k\rangle \otimes \cdots \otimes |\psi_n^k\rangle$$

Not all multi-qubit states can be written as product states. For the superposition principle, this is also a possible n-qubits state

$$\langle\psi|A_p|\psi\rangle = \sum_k \sum_j a_k\, a_j^* \left\langle \psi_0^j \cdots \psi_p^j \cdots \psi_d^j \left| A_p \right| \psi_0^k \cdots \psi_p^k \cdots \psi_n^k \right\rangle = \sum_k |a_k|^2 \langle \psi_p^k | A | \psi_p^k \rangle$$

2-qubit states which cannot be written as tensor product are called **entangled**.
If $|\psi\rangle$ is entangled, qubit j cannot be in a definite quantum state $|\psi_j\rangle$. For instance

$$|\psi\rangle = \frac{|0_A 1_B\rangle + |1_A 0_B\rangle}{\sqrt{2}} \qquad \langle M_A\rangle_\psi = \langle\psi|M\otimes\mathbb{I}|\psi\rangle = \frac{1}{2}(\langle 0|M|0\rangle + \langle 1|M|1\rangle)$$

There is no state $|\psi_A\rangle = \alpha|0\rangle + \beta|1\rangle$ such that $\langle M_A\rangle_\psi = \langle\psi_A|M|\psi_A\rangle$. Indeed,

$$\langle\psi_A|M|\psi_A\rangle = |\alpha|^2\langle 0|M|0\rangle + \alpha\beta^*\langle 1|M|0\rangle + \beta\alpha^*\langle 0|M|1\rangle + |\beta|^2\langle 1|M|1\rangle \quad \neq \quad \langle M_A\rangle_\psi$$

The state of qubit A is an incoherent mixture, not a linear superposition.

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$$|\psi\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \qquad \langle\psi| M \otimes \mathbb{1} |\psi\rangle =$$

$$= \frac{1}{2}\left(\langle 01| + \langle 10|\right)\left(M \otimes \mathbb{1}\right)\left(|01\rangle + |10\rangle\right) =$$

$$= \frac{1}{2}\Big(\langle 01| M \otimes \mathbb{1} |01\rangle + \langle 01| M \otimes \mathbb{1} |10\rangle + \langle 10| M \otimes \mathbb{1} |01\rangle + \langle 10| M \otimes \mathbb{1} |10\rangle\Big) =$$

$$= \frac{1}{2}\Big(\langle 0|M|0\rangle\langle 1|1\rangle + \cancel{\langle 0|M|1\rangle\langle 1|0\rangle} + \cancel{\langle 1|M|0\rangle\langle 0|1\rangle} + \langle 1|M|1\rangle\langle 0|0\rangle\Big)$$

Exercise: exchange interaction

$X_1 \otimes X_2 + Y_1 \otimes Y_2$

$$H = J(s_{1x}\, s_{2x} + s_{1y}\, s_{2y})$$

$S_x = \frac{X}{2}$ $S_y = \frac{Y}{2}$

autostati $J > 0$ ←

$$H = J\mathbf{s}_1 \cdot \mathbf{s}_2$$

$|0\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$

$|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$ degeneri

- Determine eigenvalues and eigenvectors.
- Compute the time evolution for a system initialized in $|\uparrow\uparrow\rangle$. Autostato
- Compute the time evolution for a system initialized in $|\uparrow\downarrow\rangle$.

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Quantum no-cloning theorem

Goal: copy the **unknown** pure state $|\chi\rangle$ into slot $|s\rangle$ by means of a unitary operator U_{copy} :

IMPOSSIBLE

$$|\chi\rangle\otimes|s\rangle \xrightarrow{U_{\text{copy}}} U_{\text{copy}}(|\chi\rangle\otimes|s\rangle) = |\chi\rangle\otimes|\chi\rangle$$

$$|\chi\rangle\otimes|0\rangle = (\alpha|0\rangle + \beta|1\rangle)\otimes|0\rangle = \alpha|00\rangle + \beta|10\rangle \xrightarrow{U_{\text{CNOT}}} \alpha|00\rangle + \beta|11\rangle$$ **ENTANGLED**

$$\neq |\chi\rangle\otimes|\chi\rangle = \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$$ **NON-ENTANGLED**

PROOF: Suppose there exist an universal operator able to copy two unknown quantum states

$$U_{\text{copy}}|\chi_1\otimes s\rangle = |\chi_1\otimes\chi_1\rangle$$

$$U_{\text{copy}}|\chi_2\otimes s\rangle = |\chi_2\otimes\chi_2\rangle$$

We now evaluate the scalar product $a = \langle\chi_1\otimes\phi|U_{\text{copy}}^\dagger U_{\text{copy}}|\chi_2\otimes\phi\rangle$ in two different ways:

1. $a = \langle\chi_1\otimes\phi|\chi_2\otimes\phi\rangle = \langle\chi_1|\chi_2\rangle$
2. $a = \langle\chi_1\otimes\chi_1|\chi_2\otimes\chi_2\rangle = (\langle\chi_1|\chi_2\rangle)^2$

As a result, either $|\chi_1\rangle \equiv |\chi_2\rangle$ or $\langle\chi_1|\chi_2\rangle = 0 \Longrightarrow$ we cannot clone a superposition of the two.

Universality

Digital computer

- Converts sets of input bit strings to sets of output bit strings
- *Universality* = ability to realize any Boolean function on an arbitrary bit string
- *Universal gate set*: combination of NOT and any two-bit gate

Quantum Computer

- Converts sets of orthogonal input states to orthogonal output states
- *Universality* = capability to obtain any unitary on an arbitrary number of qubits
- *Universal gate set*: combination of single-qubit gates and a universal 2-qubit gate

Universality

- Special interesting case: input and output states represent real physical systems, described by a given Hamiltonian. H
- In that case the unitary would correspond to the time evolution of the system under investigation. $U = e^{-iHt}$ $\quad U|\psi_0\rangle \longrightarrow |\psi(t)\rangle$
- Implementing any unitary would mean **SIMULATE ANY QUANTUM TIME EVOLUTION** and would have crucial applications in the study and design of new materials (we will come back to this point in the Chapter 6 on Applications, *Quantum Simulation* section).

Clifford gates

Clifford gates: transform Paulis into Paulis

$U: P \longrightarrow UPU^\dagger$

Conjugation by unitary

$H = |+\rangle\langle 0| + |-\rangle\langle 1| = |0\rangle\langle +| + |1\rangle\langle -|$

$|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$

$X = |0\rangle\langle 1| + |1\rangle\langle 0|$

$R_z(\pi/2) = S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$HXH = Z$
$HZH = X$
$HYH = -Y$

$SXS^\dagger = Y$
$SYS^\dagger = -X$
$SZS^\dagger = Z$

It transforms the eigenstates of P but not the eigenvalues (all Paulis share the same set of eigenvalues)

$XZX = -Z$
$XYX = -Y$
$XXX = X$

Paulis are Clifford

Multi-qubit Clifford gates: transform tensor products of Paulis into other tensor products of Paulis

$CX(X \otimes \mathbb{I})CX = X \otimes X$

$CX = (Z \otimes \mathbb{I} - Z \otimes X + \mathbb{I} \otimes X + \mathbb{I})/2$

Non-Clifford gates

$H \, R_z(\vartheta) H = R_x(\vartheta)$

$$R_x(\vartheta) = e^{-i\vartheta X/2} \xrightarrow{\text{Conjugation}} U R_x(\vartheta) U^\dagger = e^{-i\vartheta U X U^\dagger/2}$$

Clifford gates can be used to expand the power of non-Clifford gates: here we have changed the rotation axis. Similarly:

$$CX(R_x(\vartheta)\otimes\mathbb{I})CX = CX\left(e^{-i\vartheta X\otimes\mathbb{I}/2}\right)CX = e^{-i\vartheta CX(X\otimes\mathbb{I})CX/2} = e^{-i\vartheta(X\otimes X)/2}$$

Conjugation by CNOT

$$(\mathbb{I}\otimes H)e^{-i\vartheta(X\otimes X)/2}(\mathbb{I}\otimes H) = e^{-i\vartheta(X\otimes HXH)/2} = e^{-i\vartheta(X\otimes Z)/2}$$

Conjugation by single-qubit gates

Combining 1 and 2-qubits Cliffords with R_x we can implement a large set of operations
The technique can be extended to many-qubit interactions (see notebook)

Proving Universality

We split the problem. First, suppose we wish to implement

$$U = e^{i(aX+bZ)}$$

But we are only able to implement $\quad R_x(\theta) = e^{iX\theta/2} \quad\quad R_z(\theta) = e^{iZ\theta/2}$

Unfortunately $\quad U = \underline{e^{i(aX+bZ)} \neq e^{iaX} e^{ibZ}} = R_x(2a) R_z(2b)$

However, $\quad U = \lim_{N\to\infty} \left(e^{iaX/N} e^{ibZ/N}\right)^N \quad \Longrightarrow \quad e^{iaX/N} e^{ibZ/N} \approx e^{i(aX+bZ)/N}$

with the error scaling as $1/N^2$

$\Rightarrow$ U can be approximated arbitrary well by using a sufficient number of slices N

Universality

The same method can be applied to multi-qubit unitaries. For instance

$$U = e^{i(aX\otimes X\otimes X+bZ\otimes Z)}$$

We have shown that we can implement both $e^{iaX\otimes X\otimes X}$ and $e^{ibZ\otimes Z}$ (by decomposing them in terms of elementary gates). However, since $[XXX, ZZ] \neq 0$, we need to resort to the "**slice**" technique (a.k.a. *Suzuki-Trotter* decomposition) introduced before

$$\left(e^{iaXXX/N}e^{ibZZ/N}\right)^N \approx U$$

By increasing N we get an arbitrarily accurate decomposition

The same method works on an arbitrary number of qubits and of terms in the exponential, provided they can be decomposed into Pauli matrices. Since all matrices can be expressed in this way, this proves that **we can implement any unitary using 1 and 2 qubit gates**.

Increasing the number of terms increases the complexity of the method polynomially.

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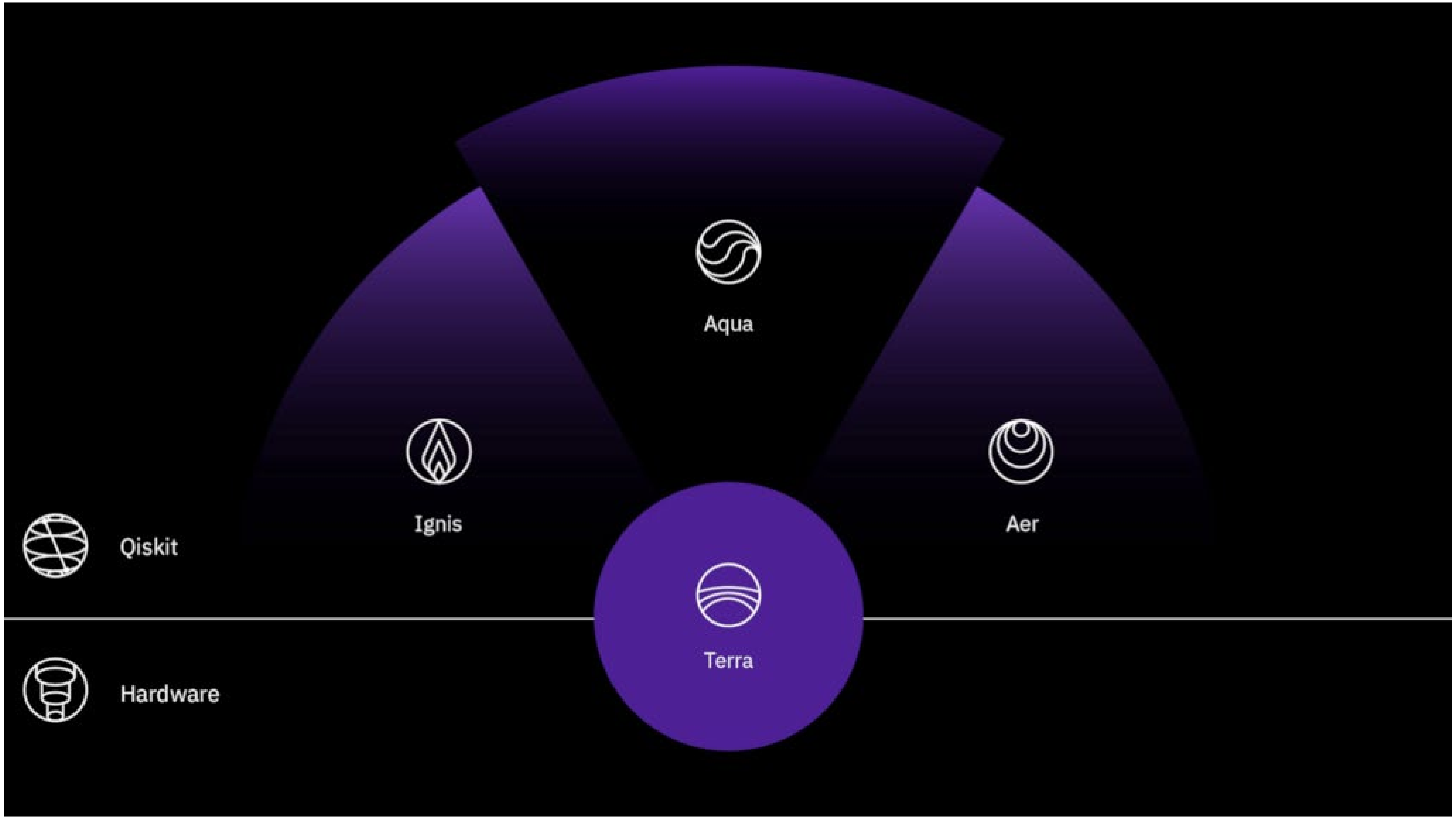

https://qiskit.org/documentation/