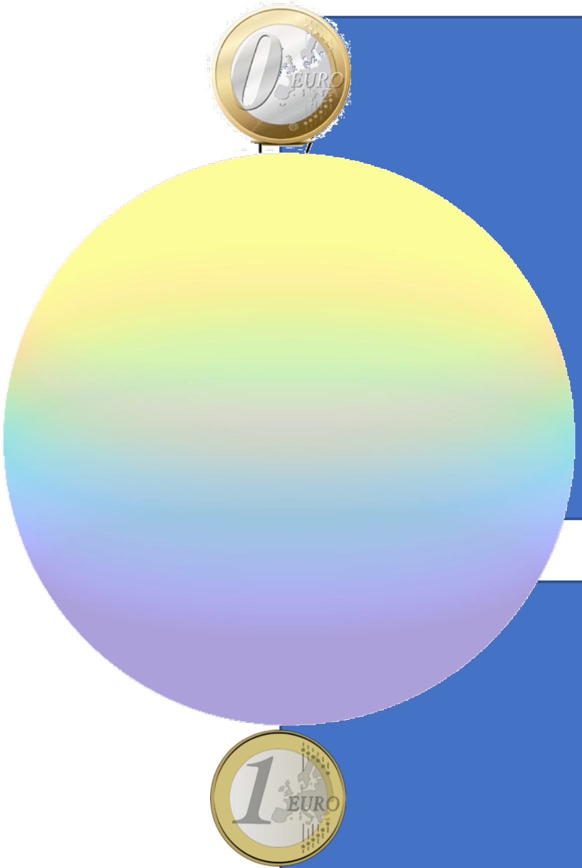


3. Qubits

Quantum Computing

UNIVERSITÀ DI PARMA

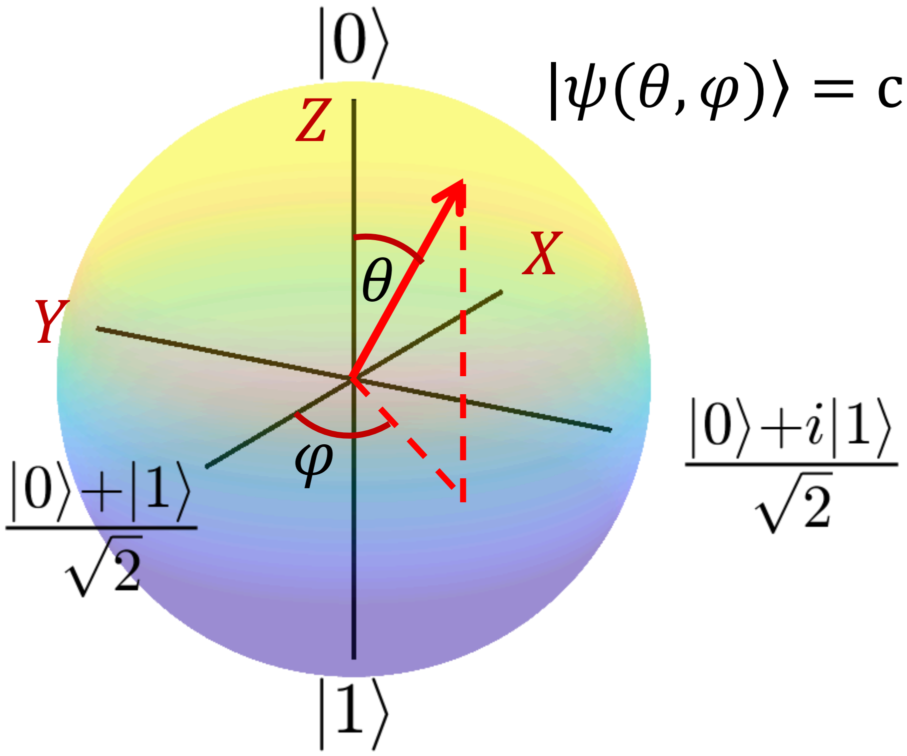
Qubit states

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \quad \text{BASIS}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

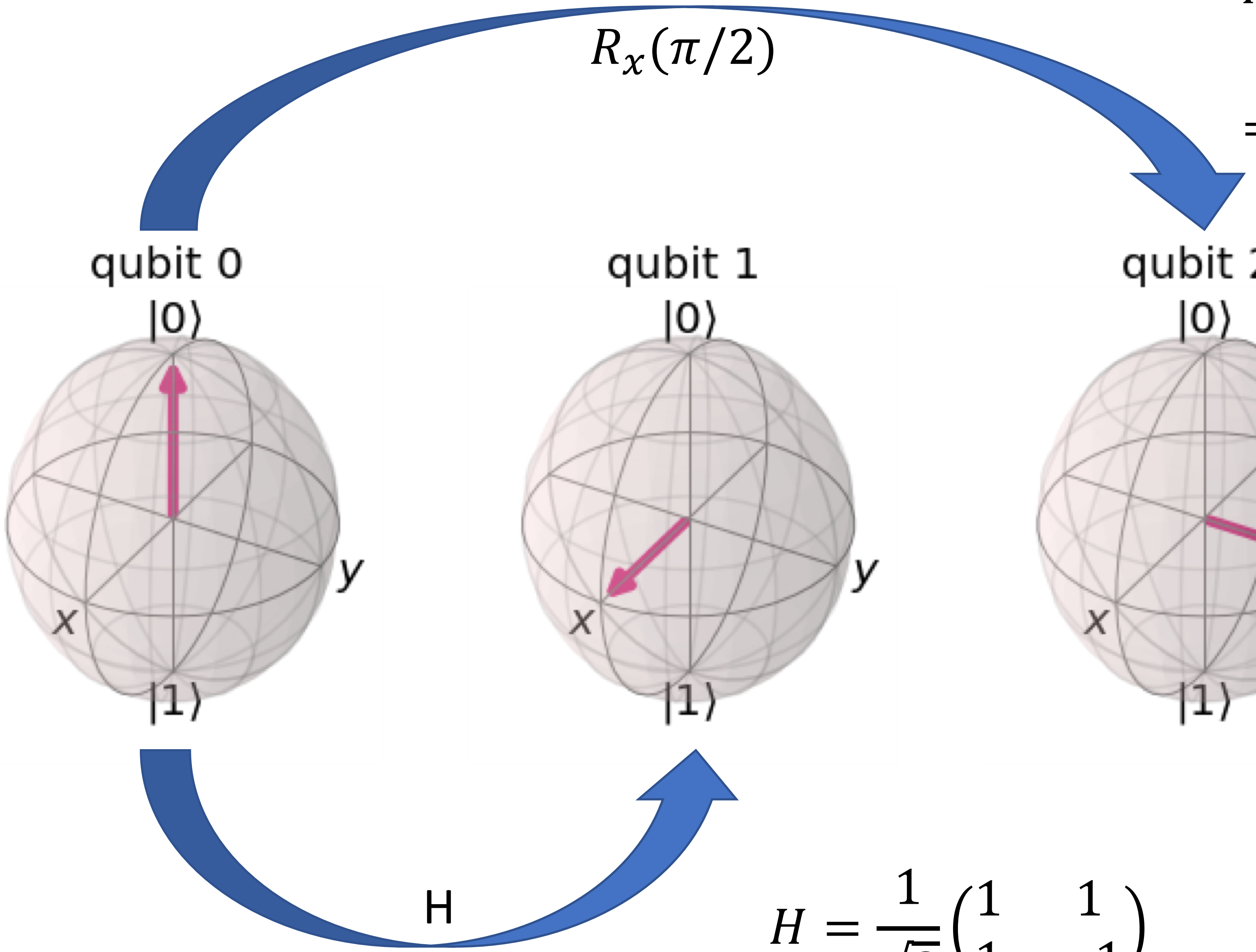
$$\alpha,\beta \in \mathbb{C}$$
$$|\alpha|^2 + |\beta|^2 = 1$$

$$|\psi(\theta,\varphi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$



- The surface of this sphere is a valid Hilbert space
- Unitary operators (preserving the norm) induce rotations on the Bloch sphere

$$R_x(\vartheta) = e^{-i\vartheta X/2} = \begin{pmatrix} \cos\frac{\vartheta}{2} & -i\sin\frac{\vartheta}{2} \\ -i\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix}$$



$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

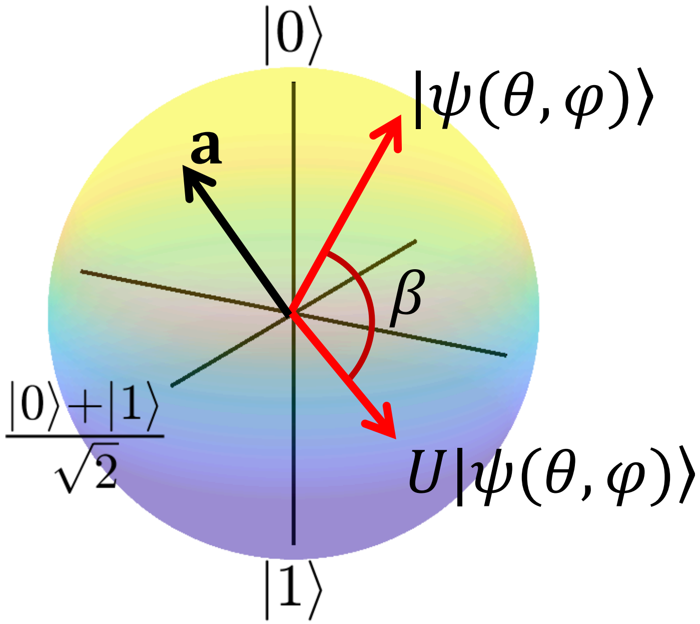
Single-qubit gates

$$U = e^{i(\alpha\mathbb{I}+\beta\,\mathbf{a}\cdot\boldsymbol{\sigma}/2)} = e^{i\alpha}\left(\cos\frac{\beta}{2}\mathbb{I} + i\sin\frac{\beta}{2}\mathbf{a}\cdot\boldsymbol{\sigma}\right)$$

General unitary transformation:
Rotation of β about axis $\mathbf{a}$.

Decomposition in Euler angles:

$$U = e^{i\alpha}e^{i\xi Z/2}e^{i\mu Y/2}e^{i\gamma Z/2}$$



$$R_\alpha(\vartheta) = e^{-i\vartheta\sigma_\alpha/2}\begin{cases} R_x(\vartheta) = \begin{pmatrix} \cos\vartheta/2 & -i\sin\vartheta/2 \\ -i\sin\vartheta/2 & \cos\vartheta/2 \end{pmatrix} \\ R_y(\vartheta) = \begin{pmatrix} \cos\vartheta/2 & -\sin\vartheta/2 \\ \sin\vartheta/2 & \cos\vartheta/2 \end{pmatrix} \\ R_z(\vartheta) = \begin{pmatrix} e^{-i\vartheta/2} & 0 \\ 0 & e^{i\vartheta/2} \end{pmatrix} = e^{-i\vartheta/2}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\vartheta} \end{pmatrix} \end{cases}$$

QISKIT

$$U_1(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{pmatrix} = e^{-\frac{i\lambda}{2}} R_z(\lambda) \propto R_z(\lambda)$$

$$U_2(\phi,\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{i\lambda} \\ e^{i\phi} & e^{i(\phi+\lambda)} \end{pmatrix}$$

$$H = U_2(0,\pi)$$

$$U_3(\theta,\phi,\lambda) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} e^{i\lambda} \\ \sin\frac{\theta}{2} e^{i\phi} & \cos\frac{\theta}{2} e^{i(\phi+\lambda)} \end{pmatrix}$$

$$R_x(\theta) = U_3(\theta, -\pi/2, \pi/2)$$

$$R_y(\theta) = U_3(\theta, 0,0)$$

$$R_z(\lambda) \propto U_3(0,0,\lambda)$$

Measurements

Let's consider the pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- **Z-basis measurement**: $p_0 = |\alpha|^2, \quad p_1 = |\beta|^2 = 1 - |\alpha|^2$. $\langle Z\rangle = p_0 - p_1$
- **X-basis measurement**: first apply H, then measure

$$|\psi_X\rangle = H|\psi\rangle = \alpha\frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$

$$p_0 = \left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2 = \frac{1}{2}(|\alpha|^2 + \alpha\beta^* + \alpha^*\beta + |\beta|^2)$$

$$p_1 = \left|\frac{\alpha-\beta}{\sqrt{2}}\right|^2 = \frac{1}{2}(|\alpha|^2 - \alpha\beta^* - \alpha^*\beta + |\beta|^2)$$

$$p_0 - p_1 = \alpha\beta^* + \alpha^*\beta = \\ = \langle X\rangle = \langle\psi|X|\psi\rangle$$

Hence we call this X-basis measurement.
How can we get Y-basis measurements?

State tomography

Let's consider the pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & \mathrm{Re}[\alpha\beta^*] + i\mathrm{Im}[\alpha\beta^*] \\ \mathrm{Re}[\alpha\beta^*] - i\mathrm{Im}[\alpha\beta^*] & 1 - |\alpha|^2 \end{pmatrix}$$

- 3 Independent elements to be determined for full state tomography
- Diagonal elements from Z-basis measurements
- Real part of the off diagonal element by X-basis measurement: $\langle X\rangle = 2\mathrm{Re}[\alpha\beta^*]$
- Imaginary part of the off diagonal element by Y-basis measurement: $\langle Y\rangle = 2i\mathrm{Im}[\alpha\beta^*]$
- You can try with Qiskit!