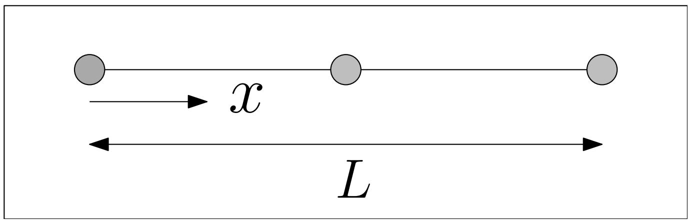
1. Shape Functions Quadratic Lagrange polynomials

The quadratic Lagrange polynomial add a nodal degree at the midpoint of the element



The local coordinate x is expressed in terls of the isoparametric $\xi \in [1, 1]$ coordinate via The function is then approximated via quadratic polynomials

$$u(\xi) = a_0 + a_1\xi + a_2\xi^2$$

Expressing the polynomial coefficients via the degrees of freedom

To obtain the Lagrange shape functions, the coefficients a_i need to be expressed in terms of the degree of freedom $u_i, \ i \in \{1, 2, 3\}$, corresponding to the nodal values at the extremities $\xi = \pm 1$ and the midpoint $\xi = 0$. The following linear system is therefore obtained

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

The inverse of the matrix can be found via this script

In [3]:
```
import numpy as np

mat = np.array([[1, -1, 1],
                [1, 0, 0],
                [1, 1, 1]])

inv_mat = np.linalg.inv(mat)

inv_mat
```

Out[3]:
```
array([[ 0. ,  1. ,  0. ],
       [-0.5,  0. ,  0.5],
       [ 0.5, -1. ,  0.5]])
```

So the shape function are expressed via

$$\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}$$

The following Lagrange basis are obtained

$$\begin{aligned} N_1 &= \frac{\xi(\xi-1)}{2}, \\ N_2 &= 1 - \xi^2, \\ N_3 &= \frac{\xi(\xi+1)}{2}. \end{aligned}$$

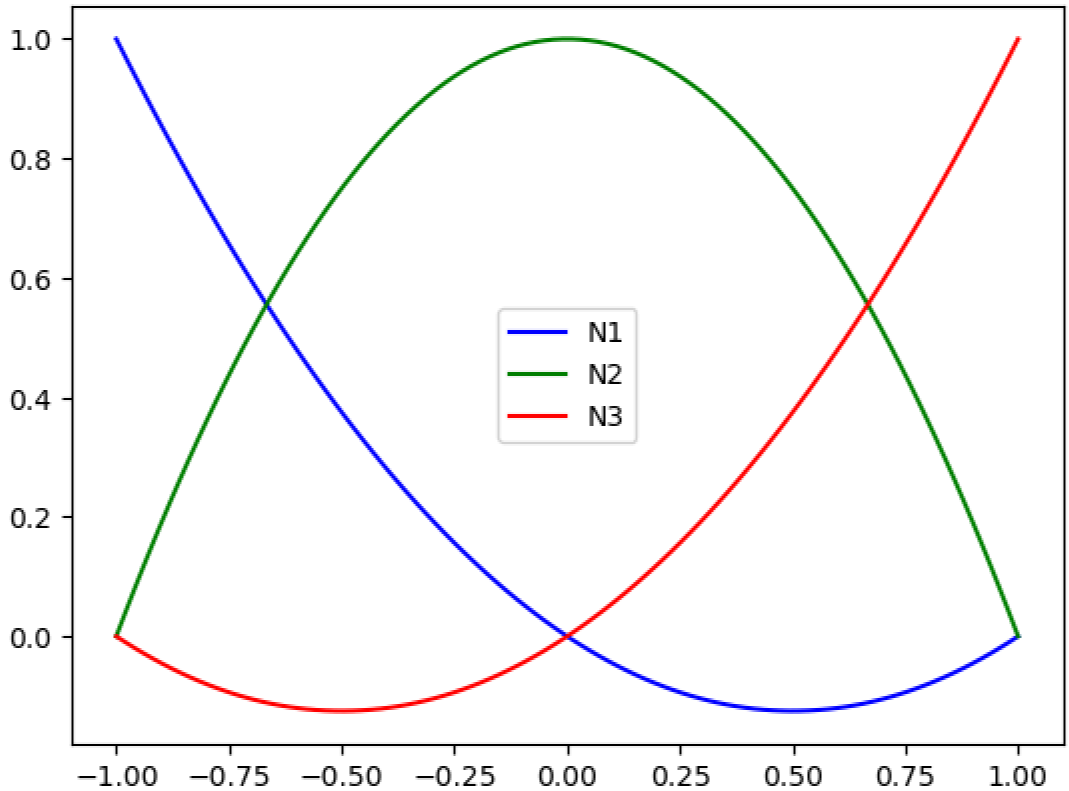
Here's the plot of the shape functions

In [9]:
```
import numpy as np
import matplotlib.pyplot as plt

xi = np.linspace(-1, 1, 100)

N_1 = xi*(xi - 1)/2
N_2 = 1 - xi**2
N_3 = xi*(xi + 1)/2

plt.plot(xi, N_1, 'b', label='N1')
plt.plot(xi, N_2, 'g', label='N2')
plt.plot(xi, N_3, 'r', label='N3')
plt.legend()
plt.show()
```



Lagrange basis canonical construction

Given a set of $k+1$ distinct nodes $\{\xi_0, \xi_1, \ldots, \xi_k\}$ the Lagrange basis is the set $\{l_0(\xi),\, l_1(\xi),\, \ldots,\, l_k(\xi)\}$ of polynomials (each of degree k) that satisfy the property

$$l_j(\xi_m) = \delta_{jm}, \qquad \delta_{jm} = \begin{cases} 0, & j \neq m, \\ 1, & j = m \end{cases}$$

Given this simple rule, the Lagrange basis can be explicity written as follows

$$l_j(\xi) = \prod_{\substack{0 \le m \le k \\ m \neq j}} \frac{\xi - \xi_m}{\xi_j - \xi_m}.$$

For the quadratic Lagrange polynomial the points are given by $\xi_0 = -1, \xi_1 = 0, \xi_2 = 1$, leading to

$$\begin{aligned} l_0(\xi) &= \frac{\xi(\xi-1)}{2}, \\ l_1(\xi) &= 1 - \xi^2, \\ l_2(\xi) &= \frac{\xi(\xi+1)}{2}. \end{aligned}$$

For a generic degree, the Lagrange basis functions can be much more effinctly computed in this way, rather that by the degrees of freedom definition.

Integration via quadrature rules of the Stiffness and mass matrix

The stiffness matrix is computed via the elastic energy as

$$E_{\text{el}}(u) = \int_{-1}^{1} \frac{EA}{2}\left(\frac{du}{dx}\right)^2 J\, d\xi = \frac{1}{2}\mathbf{u}^\top \mathbf{K}\mathbf{u}, \qquad \mathbf{K} = \int_{-1}^{1} EA \frac{d\mathbf{N}^\top}{d\xi}\frac{d\mathbf{N}}{d\xi} J^{-1} d\xi$$

Since $x = L(\xi+1)/2$ the jacobian is $J = L/2$

The mass matrix is computed using the kinetic energy as

$$E_{\text{cin}}(u) = \int_{-1}^{1} \frac{\rho A}{2}\dot{u}^2\, J\, d\xi = \frac{1}{2}\mathbf{u}^\top \mathbf{M}\mathbf{u}, \qquad \mathbf{M} := \int_{-1}^{1} \rho A \mathbf{N}^\top \mathbf{N}\, J\, d\xi$$

We start by considering the Stiffness matrix. The derivative of the shape functions is given by

$$\frac{d\mathbf{N}^\top}{d\xi} = \begin{pmatrix} \xi - \frac{1}{2} \\ -2\xi \\ \xi + \frac{1}{2} \end{pmatrix}$$

Suppose the cross section varies linearly $A = A_1\frac{1-\xi}{2} + A_2\frac{1+\xi}{2}$, then the stiffness matrix is given by

$$\mathbf{K} = \int_{-1}^{1} EA\left(A_1\frac{1-\xi}{2} + A_2\frac{1+\xi}{2}\right)\begin{bmatrix} \xi - \frac{1}{2} \\ -2\xi \\ \xi + \frac{1}{2} \end{bmatrix}\begin{bmatrix} \xi - \frac{1}{2} & -2\xi & \xi + \frac{1}{2} \end{bmatrix}\frac{2}{L} d\xi$$

The integrand is a cubic polynomial (multiplication of a linear with a matrix valued quadratic polynomial). Therefore, for the integration to be exact, it must hold $2n - 1 \geq 3$. This means that at least 2 quadrature points are needed.

The mass matrix is instead given by

$$\mathbf{K} = \int_{-1}^{1} EA\left(A_1\frac{1-\xi}{2} + A_2\frac{1+\xi}{2}\right)\begin{bmatrix} \xi(\xi-1)/2 \\ 1-\xi^2 \\ \xi(\xi+1)/2 \end{bmatrix}\begin{bmatrix} \xi(\xi-1)/2 & 1-\xi^2 & \xi(\xi+1

The integrand is a quintic polynomial (multiplication of a linear with a matrix valued quartuc polynomial). Therefore, for the integration to be exact, it must hold $2n-1 \geq 5$. This means that at least 3 quadrature points are needed.

If the cross section is constant then the integrand of the stiffness matrix is a quadratic polynomial. For the integration to be exact, two points again needed ($2n-1 \geq 2$). The integrand of the mass matrix is a quartic polynomial and at least 3 points are needed ($2n-1 \geq 4$).