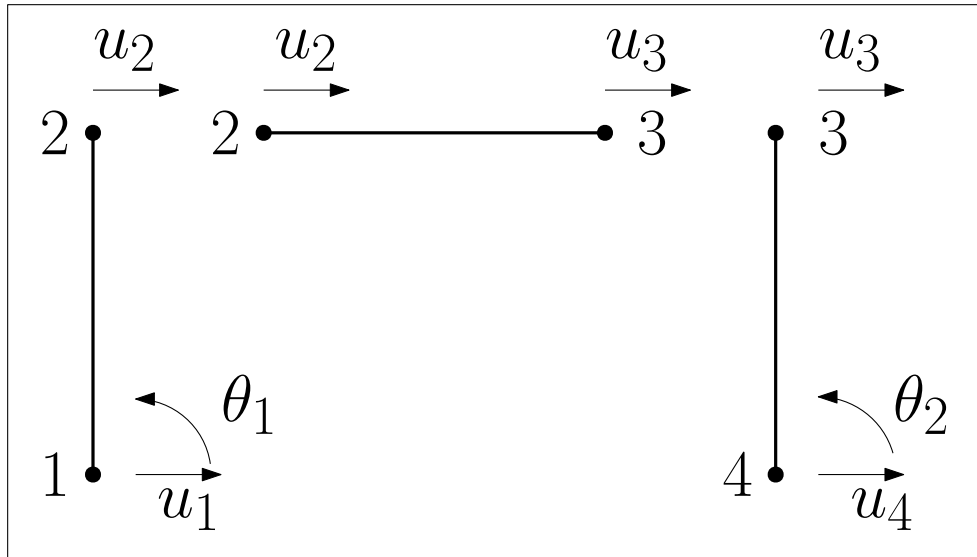


# Stiffness matrix free-body structure

Given the hypothesis of the problem, a finite element model is provided below



Therefore the vector of degrees of freedom and generalized loads read

$$\mathbf{q} = \begin{pmatrix} u_1 \\ \theta_1 \\ u_2 \\ u_3 \\ u_4 \\ \theta_2 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} R_{1x} \\ M_{1z} \\ F \\ 0 \\ R_{2x} \\ M_{2z} \end{pmatrix}$$

where  $R_{ix}$ ,  $M_{iz}$  are the reaction forces at node  $i$ .

This means that the poles are modelled as Euler Bernoulli FE and the floor is discretized using first order Lagrange polynomials. The coordinates of the nodes are the following

Global node number	$x$	$y$
1	0	0
2	0	$L$
3	$L$	0
4	$L$	0

The table of connectivity reads

Element	Left node (1)	Right Node (2)
1	1	2
2	2	3
4	3	4

The stiffness matrix is obtained by construct the local stiffness matrix for each element, converting to the global basis and assembling in the overall stiffness matrix.

For the first beam element, the local stiffness matrix is

$$\mathbf{K}_1^l = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{sym} & & & 4L^2 \end{bmatrix}$$

$v_1^l$  local vertical displacement at node 1  
 $\theta_1^l$  local rotation at node 1  
 $v_2^l$  local vertical displacement at node 2  
 $\theta_2^l$  local rotation at node 2

The rotation at the second node is suppressed, there the fourth row and column of this matrix can be removed.

$$\mathbf{K}_1^l = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 \\ & 4L^2 & -6L \\ \text{sym} & & 12 \end{bmatrix}$$

The local vertical displacement is related to the global degrees of freedom via

$$v_1^l = -u_1^g, \quad v_2^l = -u_2^g$$

So the stiffness matrix in the global basis is obtain via the equation

$$\mathbf{K}_1^g = \mathbf{T}_1 \mathbf{K}_1^l \mathbf{T}_1^\top, \quad \mathbf{T}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

leading to

$$\mathbf{K}_1^g = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 \\ & 4L^2 & 6L \\ \text{sym} & & 12 \end{bmatrix}$$

For the second element (the bar), the local and the global degrees of freedom are aligned, i.e.  $u_1^l = u_2^g$ ,  $u_2^l = u_3^g$ . Therefore the stiffness matrix stays the same when moving from the local to the global basis.

$$\mathbf{K}_2^g = \mathbf{K}_2^l = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For the third element, the first rotational degree of freedom is removed as a consequence of the assumptions of the problem.

$$\mathbf{K}_3^l = \frac{EI}{L^3} \begin{bmatrix} 12 & -12 & 6L \\ & 12 & -6L \\ \text{sym} & & 4L^2 \end{bmatrix} \quad \begin{array}{ll} v_1^l & \text{local vertical displacement at node 1} \\ v_2^l & \text{local vertical displacement at node 2} \\ \theta_2^l & \text{local rotation at node 2} \end{array}$$

The local vertical displacement coincide with the global horizontal displacement of node 3 and 4, i.e.  $v_1^l = u_3^g$  and  $v_2^l = u_4^g$ , therefore  $\mathbf{K}_3^g = \mathbf{K}_3^l$ .

The assembled stiffness matrix for the free structure is there given by

$$\mathbf{K}_{\text{free}}^g = \begin{bmatrix} 12 & -6L\alpha & -12\alpha & 0 & 0 & 0 \\ & 4L^2\alpha & -6L\alpha & 0 & 0 & 0 \\ & & 12\alpha + \beta & -\beta & 0 & 0 \\ & & & 12\alpha + \beta & -12\alpha & 6L\alpha \\ & & & & 12\alpha & -6L\alpha \\ \text{sym} & & & & & 4L^2\alpha \end{bmatrix} \quad \begin{array}{l} u_1 \\ \theta_1 \\ u_2 \\ u_3 \\ u_4 \\ \theta_4 \end{array}$$

where  $\alpha := \frac{EI}{L^3}$  and  $\beta := \frac{EA}{L}$

## Boundary conditions

Since node 1 and 2 are fixed, the boundary conditions for the problem read

$$u_1 = 0, \quad u_4 = 0, \quad \theta_1 = 0, \quad \theta_4 = 0.$$

The FE problem to be solved includes  $\mathbf{q} = [u_2, u_3]^\top$  and the system reads

$$\begin{bmatrix} 12\alpha + \beta & -\beta \\ -\beta & 12\alpha + \beta \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

Once the displacement have been found, the reactions are then computed via

$$\begin{pmatrix} R_{1x} \\ M_{1z} \\ R_{2x} \\ M_{2x} \end{pmatrix} = \begin{bmatrix} -12\alpha & 0 \\ -6L\alpha & 0 \\ 0 & -12L\alpha \\ 0 & 6L\alpha \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$$

## III conditioning of the resulting linear system

The stiffness matrix for the clamped structure can be written as follows

$$\mathbf{K} = \beta \begin{bmatrix} 1 + \varepsilon & -1 \\ -1 & 1 + \varepsilon \end{bmatrix}, \quad \varepsilon = \frac{12\alpha}{\beta} = \frac{12I}{AL^2}$$

So the system to be solved reads

$$\begin{bmatrix} 1 + \varepsilon & -1 \\ -1 & 1 + \varepsilon \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} F/\beta \\ 0 \end{pmatrix}$$

Solving the system analytically yields

$$\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \frac{1}{\varepsilon(2 + \varepsilon)} \begin{bmatrix} 1 + \varepsilon & -1 \\ -1 & 1 + \varepsilon \end{bmatrix} \begin{pmatrix} F/\beta \\ 0 \end{pmatrix}$$

The numerical value of the varepsilon is  $\varepsilon = \frac{12I}{AL^2} = 1.836 \cdot 10^{-4}$ .

- If four significant digits are kept, the system becomes singular as  $\varepsilon \approx 0$ .
- If six significant digitd are kept, the system becomes solvable  $\varepsilon \approx 0.00018$ .

The reason why the system becomes unsolvable when only 4 significant digits are kept is the condition number of the stiffness matrix. In fact the condition number for this case is

$$\text{cond}(\mathbf{K}) = \|\mathbf{K}\| \cdot \|\mathbf{K}^{-1}\| = \frac{\lambda_{\max}}{\lambda_{\min}} = 1 + \frac{1}{\varepsilon} \approx 10^4,$$

where the eigenvalues are obtained as the roots of the characteristic polynomial

$$(1 + \varepsilon - \lambda)^2 - 1 = 0, \quad \lambda = 1 + \varepsilon \pm \sqrt{1 + \varepsilon^2} \approx 1 + \varepsilon \pm 1, \\ \lambda_{\max} = 2 + \varepsilon, \quad \lambda_{\min} = \varepsilon.$$

So small error in the stiffness matrix or in the load vector (in this case the source of error is the truncation of the value of  $\varepsilon$ ).

# Refresher: condition number of a matrix (cf. Chapter 3 course notes)

Given a matrix  $A \in \mathbb{R}^{n \times n}$ , its norm is given by

$$M = \|A\| := \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

where  $\|\cdot\|$  can be any vector norm.

The norm of a matrix measures the maximum stretching the matrix produces on a vector. To define the condition number it is necessary to the minimum value of the ratio

$$m = \min_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

The reciprocal of the minimum stretching is the norm of the inverse, since

$$m = \min_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \min_{y \neq 0} \frac{\|y\|}{\|A^{-1}y\|} = 1 / \max_{y \neq 0} \frac{\|A^{-1}y\|}{\|y\|} = \frac{1}{\|A^{-1}\|}$$

For a singular matrix  $m = 0$  so we assume the matrix  $A$  to be invertible. The condition number of a matrix is the ratio between the maximum and minimum stretching, i.e.

$$\kappa(A) := \frac{M}{m} = \|A\| \cdot \|A^{-1}\|$$

Consider a linear system

$$Ax = b$$

Imagine that the data  $b$  is not perfectly known, but only up to an additive perturbation  $\delta b$ . The perturbed system reads

$$A(x + \delta x) = b + \delta b$$

The error on the solution is therefore given by  $A\delta x = \delta b$ .

The definitions of  $M$  and  $m$  imply that

$$\begin{aligned} \|b\| &\leq M\|x\|, \\ \|\delta b\| &\geq m\|\delta x\| \end{aligned} \quad \implies \quad \frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}$$

This inequality shows that the condition number is a relative error magnification factor. Changes in the right-hand side can cause changes  $\kappa(A)$  times as large in

the solution.

An analogous reasoning can be performed if the matrix  $A$  is not perfectly known (like in the exercise above). Indeed consider an additive perturbation  $\delta A$  to matrix  $A$ . The perturbed system reads

$$(A + \delta A)(x + \delta x) = b.$$

Again since  $Ax = b$  it holds  $A\delta x = -\delta A(x + \delta x)$ . So

$$\delta x = -A^{-1}(\delta A(x + \delta x))$$

By definition of matrix norm and condition number

$$\|\delta x\| \leq \|A^{-1}\| \cdot \|\delta A\| \cdot \|x + \delta x\| = \kappa(A) \frac{\|\delta A\|}{\|A\|} \|x + \delta x\|$$

So the following estimate holds

$$\frac{\|\delta x\|}{\|x + \delta x\|} \leq \kappa(A) \frac{\|\delta A\|}{\|A\|}.$$

## Using the Euclidian norm as the vector norm

If the Euclidian norm  $\|x\|_2 = (\sum_{i=1}^n x_i^2)^{1/2}$  is employed then the condition number is the ratio between the maximum and minimum singular values of the matrix  $A$ .

There are several equivalent definition of the singular values. The most commonly used is the following

$$\{\sigma_1, \dots, \sigma_n\} = \{\sqrt{\lambda_1(A^\top A)}, \dots, \sqrt{\lambda_n(A^\top A)}\}$$

where  $\lambda_1(A^\top A), \dots, \lambda_n(A^\top A)$  are the ordered eigenvalues (from the biggest to the smallest) of the matrix  $A^\top A$ . This definition implies that

$$\sigma_1 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}, \quad \sigma_n = \min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

so that  $\kappa(A) = \sigma_1/\sigma_n$ .

## The special case of symmetric/positive definite matrices

If the matrix  $A$  is symmetric then the spectral theorem guarantees it can be decomposed as

$$A = Q\Lambda Q^\top, \quad \text{if } A = A^\top$$

where  $Q$  is a orthogonal matrix and  $\lambda$  is a diagonal matrix containing the eigenvalues. Its singular values are equal to the absolute value of its eigenvalues:

$$\sigma_i = \sqrt{\lambda_i(A^\top A)} = \sqrt{\lambda_i(Q\Lambda^2 Q^\top)} = \sqrt{\lambda_i(A)^2} = |\lambda_i(A)|.$$

Then we have  $\kappa(A) = |\lambda_1(A)|/|\lambda_n(A)|$ . If the matrix  $A$  is positive definite then  $\lambda_1(A) > \dots > \lambda_n(A) > 0$  so that

$$\kappa(A) = \frac{\lambda_1(A)}{\lambda_n(A)} \quad \text{if } A \text{ is symmetric and positive definite.}$$

Since the stiffness matrix in linear elasticity is always symmetric and positive definite, the condition number is computed via its eigenvalues.

## References

[Cleve Corner post on Mathworks](#)

[Article by Nick Higham](#)