Maxwell-Boltzmann Mechanical Proof

and correct Collision Model

This document has two main objectives:

1) Prove that the case of random elastic collisions between “spheres” with equal mass (and zero moment of inertia) moving in every direction isotropically in absence of gravity and with velocity distributed according to Maxwell-Boltzmann, without considering the boundary conditions, is representing an equilibrium (stable) condition
2) Derive the modified Arrhenius equation through a correct collision model

<u>Rewriting the Maxwell-Boltzmann pdf</u>

Starting from the probability density function (pdf) of the Maxwell-Boltzmann distribution

$$f(v) = \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} 4\pi v^2 e^{-\frac{mv^2}{2k_B T}} \qquad (1)$$

And considering the mean (speed) of the distribution, for given m and T

$$\langle v \rangle = \sqrt{\frac{8k_B T}{\pi m}} \qquad (2)$$

We can rewrite the probability density function as

$$f(v) = \frac{32}{\pi^2} \frac{v^2}{\langle v \rangle^3} e^{-\frac{4}{\pi} \frac{v^2}{\langle v \rangle^2}} \qquad (3)$$

<u>Calculating the probability of a collision</u>

Let’s now indicate with $D = 2r$ the dimension of the spheres (we pick the diameter).

A collision occurs if the center of two spheres is within two times their radius (or equivalently, one time their diameter). Let’s therefore consider the region of space within one diameter from the center of each sphere and call it the “collision region” (which is also spherical).

During a small interval of time dt , it is easy to show that a sphere traveling with speed v would have covered with its “collision region” a volume in space (excluding the initial “collision region” itself) equal to

$$\pi D^2 v\, dt$$

Let’s for a moment restrict only to spheres having a velocity with magnitude between $\bar{v}_1$ and $\bar{v}_1 + d\bar{v}_1$ and a specific direction within a solid angle $d\Omega$. Due to the isotropy of the distribution and calling N the total number of spheres, the number of spheres satisfying the above conditions is

$$N f(\bar{v}_1)\, d\bar{v}_1 \frac{d\Omega}{4\pi}$$

If we consider then another group of spheres having velocity with magnitude between $\bar{v}_2$ and $\bar{v}_2 + d\bar{v}_2$ and forming an angle between α and $\alpha + d\alpha$ with the velocity of the spheres of the first group, we can say the following.

Firstly, again because of isotropy, it is easy to show that the angle α will be distributed according to

$$f_\alpha(\alpha) = \frac{1}{2}\sin\alpha\, d\alpha \tag{4}$$

And therefore, the number of spheres of the second group is

$$N f(\bar{v}_2)\, d\bar{v}_2 \frac{1}{2}\sin\alpha\, d\alpha$$

Secondly, sitting in a frame of reference moving with the same velocity of the spheres of the first group, so that those spheres are in rest in it (except for infinitesimal differences), the spheres of the second group will be seen moving according to the relative velocity, which has magnitude equal to

$$v_0 = \sqrt{\bar{v}_1^2 + \bar{v}_2^2 - 2\bar{v}_1\bar{v}_2\cos\alpha} \tag{5}$$

During a time dt, is it easy to show that the spheres of the second group will have covered with their “collision regions” (again excluding the initial “collision regions” themselves) a volume in space (in their travel relative to the spheres of the first group) of:

$$N f(\bar{v}_2)\, d\bar{v}_2 \frac{1}{2}\sin\alpha\, d\alpha \pi D^2 v_0\, dt$$

(Clearly, if dt is sufficiently small, there is no overlap in this volume, as the spheres are all moving with the same velocity)

Defining δ as the density of the spheres in the total volume (which we will call V), specifically for the spheres of the first group we have

$$\delta_{v_1} = \frac{1}{V}\ Nf(\bar{v}_1)\, d\bar{v}_1 \frac{d\Omega}{4\pi} \qquad (6)$$

We then expect the number of collisions (cases in which spheres from the first group are located within the volume in space covered by the "collision regions" of the spheres from the second group, as saw before) to be equal to

$$\delta_{v_1} Nf(\bar{v}_2)\, d\bar{v}_2 \frac{1}{2}\sin\alpha\, d\alpha\pi D^2 v_0\, dt = \frac{1}{V}\ Nf(\bar{v}_1)\, d\bar{v}_1 \frac{d\Omega}{4\pi} Nf(\bar{v}_2)\, d\bar{v}_2 \frac{1}{2}\sin\alpha\, d\alpha\pi D^2 v_0\, dt$$

Integrating over the solid angle $d\Omega$ and rearranging a bit the terms, it's immediate that

$$dcollision_{\bar{v}_1,\bar{v}_2,\alpha} = \frac{N^2}{V}\pi D^2\ f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha\, v_0 d\bar{v}_1 d\bar{v}_2 d\alpha\, dt$$

So that the "rate" of collisions over time, considering that N, V and D are fixed, is proportional to

$$\frac{dcollision_{\bar{v}_1,\bar{v}_2,\alpha}}{dt} \propto\ f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha\, v_0 d\bar{v}_1 d\bar{v}_2 d\alpha \qquad (7)$$

If v_0 = 0, for instance, collisions are not possible because the 2 group of spheres are not moving relatively to each other!

We can therefore define a joint pdf for the distribution of the frequency of the collisions as

$$f_{hit|\bar{v}_1,\bar{v}_2,\alpha} = K\ f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha v_0 \qquad (8)$$

Where K ensures that the joint pdf has an overall integral equal to 1

$$K = \frac{1}{\int_0^\pi \int_0^\infty \int_0^\infty f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha v_0 d\bar{v}_1 d\bar{v}_2 d\alpha} \qquad (9)$$

<u>Expressing the speed distribution of the colliding spheres</u>

Clearly $\bar{v}_1$ and $\bar{v}_2$ are completely interchangeable in the reasoning above, and it is easy to see that $f_{hit|\bar{v}_1,\bar{v}_2,\alpha} = f_{hit|\bar{v}_2,\bar{v}_1,\alpha}$.

If we therefore integrate $f_{hit|\bar{v}_1,\bar{v}_2,\alpha}$ over all the possible values of $\bar{v}_2$ and α, we will get the $\bar{v}_1$ speed distribution of the colliding spheres

$$f_{hit|\bar{v}_1}(\bar{v}_1) = \int_0^\infty \int_0^\pi f_{hit|\bar{v}_1,\bar{v}_2,\alpha}\, d\alpha d\bar{v}_2$$

$$f_{hit|\bar{v}_1}(\bar{v}_1) = \int_0^\infty \int_0^\pi K\ f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha v_0\, d\alpha d\bar{v}_2 \qquad (10)$$

Note that the order of integrals has been rearranged.

For the reason mentioned above (interchangeability of $\bar{v}_1$ and $\bar{v}_2$), this function will represent also the $\bar{v}_2$ speed distribution of the colliding spheres, so that for simplicity we can just call it the speed distribution of all the colliding spheres, naming that speed $\bar{v}$

$$f_{hit|\bar{v}}(\bar{v}) = f_{hit|\bar{v}_1}(\bar{v}) \tag{11}$$

Expressing the speed distribution after the collisions

Let's call $\tilde{v}_1$ and $\tilde{v}_2$ the speed of the spheres after the collision.

Considering all the possible configurations of the impact, and the facts that the collisions are elastic (momentum and kinetic energy are conserved) it is possible to show [1] that, for given $\bar{v}_1$, $\bar{v}_2$ and α, both $\tilde{v}_1$ and $\tilde{v}_2$ are equally distributed, and their probability density function (within the boundary of values that are possible to be reached starting from a collision of spheres of speed $\bar{v}_1$, $\bar{v}_2$ with an angle α due to momentum and kinetic energy conservation considerations. [1] provides also such boundaries) is

$$f_{\tilde{v}_1|\bar{v}_1,\bar{v}_2,\alpha} = \frac{\tilde{v}_1}{2r_1r_2} \tag{12}$$

$$f_{\tilde{v}_2|\bar{v}_1,\bar{v}_2,\alpha} = \frac{\tilde{v}_2}{2r_1r_2} \tag{13}$$

Where

$$r_1 = \frac{1}{2}\sqrt{\bar{v}_1^2 + \bar{v}_2^2 + 2\bar{v}_1\bar{v}_2\cos\alpha} \tag{14}$$

$$r_2 = \frac{1}{2}\sqrt{\bar{v}_1^2 + \bar{v}_2^2 - 2\bar{v}_1\bar{v}_2\cos\alpha} \tag{15}$$

As $\tilde{v}_1$ and $\tilde{v}_2$ are equally distributed, this represents also the distribution of the speed $\tilde{v}$ of all the spheres after collision, so that

$$f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} = \begin{cases} \frac{\tilde{v}}{2r_1r_2} & \textit{within the boundaries} \\ 0 & \textit{elsewhere} \end{cases} \tag{16}$$

As we saw before, each type of collision has a different probability, dependent not only on the probability density function of $\bar{v}_1, \bar{v}_2, \alpha$ but also on the relative speed v_0.

If we consider the product

$$f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} f_{hit|\bar{v}_1,\bar{v}_2,\alpha} = f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} K\, f(\bar{v}_1)\, f(\bar{v}_2) \frac{1}{2}\sin\alpha v_0$$

This will give us the joint probability density function of the speed of the spheres after the collision.

If we fix $\tilde{v}$ and integrate it over all the possible values of $\bar{v}_1, \bar{v}_2, \alpha$, we will get the speed distribution of the spheres after the collision

$$f_{new|\tilde{v}}(\tilde{v}) = \int_0^\infty \int_0^\infty \int_0^\pi f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} K\, f(\bar{v}_1)\, f(\bar{v}_2)\, \frac{1}{2} \sin\alpha\, v_0\, d\alpha d\bar{v}_1 d\bar{v}_2 \qquad (17)$$

Mechanical proof

Both the speed distribution of the colliding spheres ($f_{hit|\bar{v}}(\bar{v})$) and the speed distribution of the spheres after the colisions ($f_{new|\tilde{v}}(\tilde{v})$) depend on the speed distribution of the population of spheres ($f(v)$, which is assumed to be according to Maxwell-Boltzmann).

Every collision, 2 speeds are "disappearing" from the distribution $f(v)$ (the ones of the spheres colliding), and 2 new speeds are "created" and are "joining" the distribution $f(v)$ (the ones of the spheres after the collision).

If the speeds that are "created" / "joining" $f(v)$ exactly compensate the ones that are "disappearing" from $f(v)$, then the speed distribution of the population of spheres $f(v)$ will be in equilibrium.

This is expressed by the condition that for any given speed $v_{coll} = \bar{v} = \tilde{v}$

$$f_{hit|\bar{v}}(v_{coll}) = f_{new|\tilde{v}}(v_{coll}) \qquad (18)$$

After some steps (detailed below) it is possible to show that this is indeed the case.

We can then call the probability density function of the speed of the spheres colliding (which is the same before and after the collision) with a single name $f_{coll}(v_{coll})$ such that

$$f_{coll}(v_{coll}) = f_{hit|\bar{v}}(v_{coll}) = f_{new|\tilde{v}}(v_{coll}) \qquad (19)$$

And we can also find the value of the constant K so that finally we can write

$$f_{coll}(v_{coll}) = \frac{1}{\sqrt{2}\langle v\rangle} f(v_{coll}) \left[v_{coll} F(v_{coll}) + \frac{\pi\langle v\rangle^2}{8} \frac{F(v_{coll})}{v_{coll}} + \frac{\pi^2\langle v\rangle^4}{32} \frac{f(v_{coll})}{v_{coll}^2} + \frac{\pi\langle v\rangle^2}{8} f(v_{coll}) \right] \qquad (20)$$

Which after few steps can be rewritten as

$$f_{coll}(v_{coll}) = \frac{1}{\sqrt{2}\langle v\rangle} f(v_{coll}) \left[\left(v_{coll} + \frac{\pi\langle v\rangle^2}{8 v_{coll}} \right) erf\left(\frac{2}{\sqrt{\pi}} \frac{v_{coll}}{\langle v\rangle}\right) + \frac{\langle v\rangle}{2} e^{-\frac{4 v_{coll}^2}{\pi\, \langle v\rangle^2}} \right] \qquad (21)$$

Where $erf(x)$ is the error function.

Analyzing the speed distribution of the colliding spheres in comparison with Maxwell-Boltzmann

Let's now compare the initial Maxwell-Boltzmann distribution, which has mode v_p (where $f'(v) = 0$), mean speed $\langle v \rangle$ and mean squared speed $\langle v^2 \rangle$, where v_p and $\langle v^2 \rangle$ can be easily found in the literature

$$v_p = \frac{\sqrt{\pi}}{2}\langle v \rangle \quad , \quad \langle v^2 \rangle = \frac{3\pi}{8}\langle v \rangle^2 \tag{22}$$

With the distribution f_{coll} which has mean speed $\langle v_{coll} \rangle$ and mean squared speed $\langle v_{coll}^2 \rangle$, which can be derived through integration over all the possible values of the speed v_{coll}

$$\langle v_{coll} \rangle = \frac{\sqrt{2}(\pi+3)}{8}\langle v \rangle \quad , \quad \langle v_{coll}^2 \rangle = \frac{7\pi}{16}\langle v \rangle^2 \tag{23}$$

$\langle v_{coll} \rangle$ is approximately 8.57 % bigger than $\langle v \rangle$, while $\langle v_{coll}^2 \rangle$ is approximately 16.67 % bigger than $\langle v^2 \rangle$. This makes sense, considering that faster spheres have higher probability to collide, and thus the distribution f_{coll} is skewed towards the right.

Analyzing the distribution of relative speeds

First let's analyze how the relative speed between 2 spheres is distributed across the population of all the spheres.

We can start from the probability of having a sphere with speed between v_1 and $v_1 + dv_1$, a second sphere with speed between v_2 and $v_2 + dv_2$, and with an angle between their velocity vectors between α and $\alpha + d\alpha$.

$$f(v_1)\, f(v_2)\frac{1}{2}\sin\alpha \, dv_1 dv_2 d\alpha$$

We can then differentiate their relative speed v_0 with respect to α to get

$$\frac{dv_0}{d\alpha} = \frac{2v_1v_2\sin\alpha}{2\sqrt{v_1^2+v_2^2-2v_1v_2\cos\alpha}} = \frac{v_1v_2\sin\alpha}{v_0} \rightarrow \sin\alpha\, d\alpha = \frac{v_0}{v_1v_2}dv_0 \tag{24}$$

Substituting in the equation above, the probability becomes

$$f(v_1)\, f(v_2)\frac{1}{2}\frac{v_0}{v_1v_2}dv_1dv_2dv_0 \tag{25}$$

Keeping v_0 constant, and integrating over all the values of v_1 and v_2 allowing that specific value of v_0 (the basic constraint is that $|v_1 - v_2| \leq v_0 \leq v_1 + v_2$) we can obtain the probability density function of the relative speed v_0

$$f_{rel}(v_0) = \frac{8\sqrt{2}}{\pi^2}\frac{v_0^2}{\langle v\rangle^3}e^{-\frac{2}{\pi}\frac{v_0^2}{\langle v\rangle^2}} = \frac{1}{\sqrt{2}}f(\frac{v_0}{\sqrt{2}}) \tag{26}$$

This distribution, however, doesn't consider the probability of collisions.

If we want to know instead how the relative velocity is distributed among the colliding spheres, we can start from the elementary probability of a collision

$$Kf(v_1)\,f(v_2)\frac{1}{2}\sin\alpha v_0\,dv_1dv_2d\alpha$$

And again, apply the substitution which we obtained by differentiating v_0 with respect to α

$$Kf(v_1)\,f(v_2)\frac{1}{2}\frac{v_0^2}{v_1v_2}dv_1dv_2d\alpha$$

This is result is similar to the previous one, but with an extra Kv_0 factor, so that if we integrate as before over all the values of v_1 and v_2 allowing that specific value of v_0, the result we obtain will also be multiplied by a factor Kv_0. The probability density function of the relative speed among the colliding spheres will therefore be (where we use v_{0_coll} in place of v_0)

$$f_{rel\_coll}(v_{0\_coll}) = Kv_{0\_coll}f_{rel}(v_{0\_coll}) = \frac{1}{\sqrt{2}\langle v\rangle}v_{0\_coll}\frac{8\sqrt{2}}{\pi^2}\frac{v_{0\_coll}^2}{\langle v\rangle^3}e^{-\frac{2}{\pi}\frac{v_{0\_coll}^2}{\langle v\rangle^2}} =$$

$$= \frac{8}{\pi^2}\frac{v_{0\_coll}^3}{\langle v\rangle^4}e^{-\frac{2}{\pi}\frac{v_{0\_coll}^2}{\langle v\rangle^2}} \tag{27}$$

We can then compare as done before the mode v_{0p}, mean speed $\langle v_0\rangle$ and mean squared speed $\langle v_0^2\rangle$ of the relative speed distribution of the population of spheres (by themselves respectively $\sqrt{2}$, $\sqrt{2}$ and 2 times bigger than the respective values of the Maxwell-Boltzmann distribution)

$$v_{0p} = \frac{\sqrt{\pi}}{\sqrt{2}}\langle v\rangle \quad , \quad \langle v_0\rangle = \sqrt{2}\langle v\rangle \quad , \quad \langle v_0^2\rangle = \frac{3\pi}{4}\langle v\rangle^2 \tag{28}$$

With the mode v_{0p_coll}, mean speed $\langle v_{0\_coll}\rangle$ and mean squared speed $\langle v_{0\_coll}^2\rangle$ of the distribution of the relative speed among the colliding spheres

$$v_{0p\_coll} = \frac{\sqrt{3\pi}}{2}\langle v\rangle \quad , \quad \langle v_{0\_coll}\rangle = \frac{3\sqrt{2}\pi}{8}\langle v\rangle \quad , \quad \langle v_{0\_coll}^2\rangle = \pi\langle v\rangle^2 \tag{29}$$

v_{0p_coll} is approximately 22.47 % bigger than v_{0p}, $\langle v_{0\_coll}\rangle$ is approximately 17.81 % bigger than $\langle v_0\rangle$, while $\langle v_{0\_coll}^2\rangle$ is approximately 33.33 % bigger than $\langle v_0^2\rangle$. This makes sense as well, considering that faster spheres have higher probability to collide, but also higher probability to have high relative speed with the other spheres, and thus the distribution f_{rel_coll} is skewed towards the right.

We can take one final step, which is to consider the distribution of the component of the relative speed normal to the contact surface of the colliding spheres ($v_{0\_coll\_norm}$).

To do this, let's first calculate the probability, for a given relative speed v_{0_coll} during a collision, that the component normal to the contact surface is at least equal to v_{min}.

By looking at Fig. 1a (showing the plane containing the vector v_{0_coll} and the center of the two spheres) we can see how

$$v_{0\_coll\_norm} = v_{0\_coll} \cos\theta \tag{30}$$

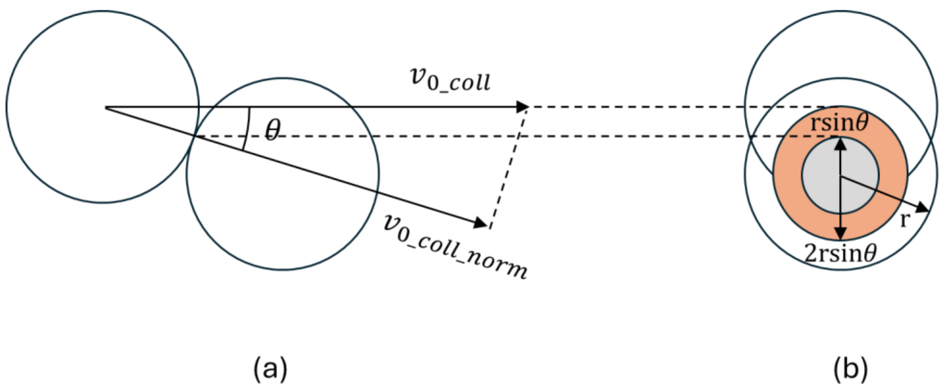


Fig. 1

So if we want

$$v_{0\_coll\_norm} > v_{min} \rightarrow v_{0\_coll} \cos\theta > v_{min} \rightarrow \cos\theta > \frac{v_{min}}{v_{0\_collision}} \rightarrow \cos^2\theta > \frac{v_{min}^2}{v_{0\_coll}^2} \rightarrow$$

$$\rightarrow \sin^2\theta < 1 - \frac{v_{min}^2}{v_{0\_coll}^2} \rightarrow \sin^2\theta < \sin^2\theta_{min} = 1 - \frac{v_{min}^2}{v_{0\_coll}^2} \tag{31}$$

And, by looking at Fig. 1b (which shows the plane perpendicular to v_{0_coll}, and thus perpendicular to the plane shown in Fig. 1a), we can see how the distance between the projection of the center of the first sphere (with speed v_{0_coll}) and the projection of the center of the second sphere is $2\mathrm{r}\sin\theta$. The distance between the projection of the contact point between the two spheres and the projection of the center of the second sphere is instead $\mathrm{r}\sin\theta$.

The projection of the center of the first sphere can fall in any point of the plane with equal likelihood (given the boundaries of a collision), and as we saw the projection of the contact point is halfway between the projection of the two centers and therefore can also be in any point of the plane (within the boundaries of collision) with equal likelihood.

Therefore, the probability of the projection of the contact point to be inside the grey circle shown in Fig. 1b for a generic collision is proportional to the ratio between the area of the grey circle and the area of the projection of the sphere (representing the boundaries, as the contact point must be on both spheres)

$$p(contact\ point\ inside\ grey\ circle) = \frac{\pi(\mathrm{r}\sin\theta)^2}{\pi \mathrm{r}^2} = \sin^2\theta \tag{32}$$

Let's also note that every point inside the grey circle represents a condition of smaller θ and thus smaller $\sin\theta$.

We can then conclude that the probability of $v_{0\_coll\_norm}$ to be at least equal to v_{min} (for a given v_{0_coll}), given the condition on $\sin\theta$, is equal to

$$p(v_{0\_coll\_norm} > v_{min} | v_{0\_coll}) = \sin^2\theta_{min} = 1 - \frac{v_{min}^2}{v_{0\_coll}^2} \tag{33}$$

And by integrating over all the possible values of v_{0_coll}, considering the distribution f_{rel_coll} and the fact that v_{0_coll} clearly must be at least equal to v_{min}, we can derive the probability of $v_{0\_coll\_norm}$ to be at least equal to v_{min} for a generic collision

$$p(v_{0\_coll\_norm} > v_{min}) = \int_{v_{min}}^{\infty} p(v_{0\_coll\_norm} > v_{min} | v_{0\_coll}) f_{rel\_coll}(v_{0\_coll})\, dv_{0\_coll} =$$

$$= e^{-\frac{2v_{min}^2}{\pi\langle v\rangle^2}} \tag{34}$$

If we name $f_{rel\_coll\_norm}$ the probability density function of $v_{0\_coll\_norm}$ and $F_{rel\_coll\_norm}$ its cumulative distribution function, it's then true that

$$F_{rel\_coll\_norm}(v_{min}) = p(v_{0\_coll\_norm} \leq v_{min}) = 1 - p(v_{0\_coll\_norm} > v_{min}) =$$

$$= 1 - e^{-\frac{2v_{min}^2}{\pi\langle v\rangle^2}}$$

So

$$F_{rel\_coll\_norm}(v_{0\_coll\_norm}) = 1 - e^{-\frac{2v_{0\_coll\_norm}^2}{\pi\langle v\rangle^2}} \tag{35}$$

$$f_{rel\_coll\_norm}(v_{0\_coll\_norm}) = F'_{rel\_coll\_norm} = \frac{4v_{0\_coll\_norm}}{\pi\langle v\rangle^2} e^{-\frac{2v_{0\_coll\_norm}^2}{\pi\langle v\rangle^2}} \tag{36}$$

We can then calculate mode, mean, and mean squared speed as done before

$$v_{0p\_coll\_norm} = \frac{\sqrt{\pi}}{2}\langle v\rangle \quad , \quad \langle v_{0\_coll\_norm}\rangle = \frac{\sqrt{2}\pi}{4}\langle v\rangle \quad , \quad \langle v_{0\_coll\_norm}^2\rangle = \frac{\pi}{2}\langle v\rangle^2 \tag{37}$$

$v_{0p\_coll\_norm}$ is approximately 42.26 % smaller than v_{0p_coll}, $\langle v_{0\_coll\_norm} \rangle$ is approximately 33.33 % smaller than $\langle v_{0\_coll} \rangle$, while $\langle v_{0\_coll\_norm}^2 \rangle$ is 50.00 % smaller than $\langle v_{0\_coll}^2 \rangle$ due to the fact that only normal component is counted.

We can plot in Fig. 2 all the probability density functions derived so far, where a specific value of $\langle v \rangle = 400\ m/s$ has been chosen. The position of the curves reflects what we previously found for their modes and means.

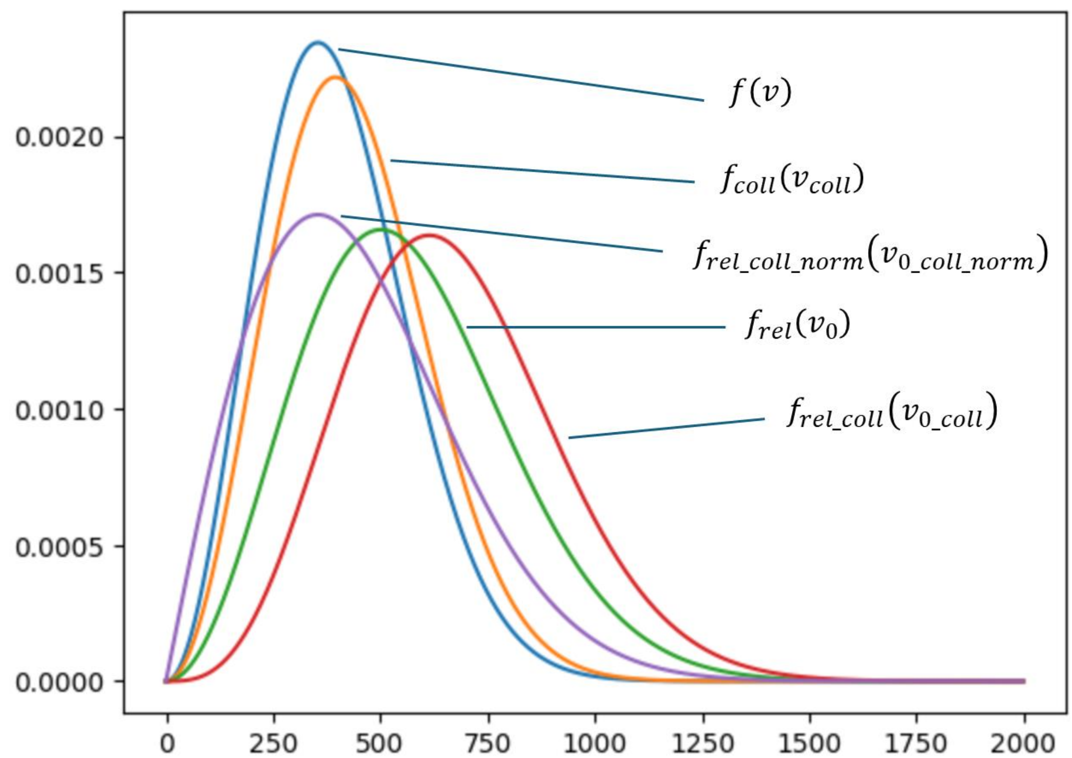


Fig. 2

Deriving the modified Arrhenius equation through a correct collision model

We calculated before the rate of collisions $dcollision_{\bar{v}_1,\bar{v}_2,\alpha}$, which we can express as a rate over time and per unit of volume

$$\frac{dcollision_{\bar{v}_1,\bar{v}_2,\alpha}}{dt}\frac{1}{V} = \left(\frac{N}{V}\right)^2 \pi D^2\, f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha\, v_0 d\bar{v}_1 d\bar{v}_2 d\alpha$$

By integrating over all the possible values of $\bar{v}_1$, $\bar{v}_2$ and α (rearranging the order of the integrals) we find

$$\frac{dcollision}{dt}\frac{1}{V} = \left(\frac{N}{V}\right)^2 \pi D^2 \int_0^\infty \int_0^\infty \int_0^\pi f(\bar{v}_1)\, f(\bar{v}_2)\, \frac{1}{2}\sin\alpha\, v_0\, d\alpha d\bar{v}_1 d\bar{v}_2 = \left(\frac{N}{V}\right)^2 \pi D^2 \frac{1}{K} =$$

$$= \pi D^2 \sqrt{2}\langle v\rangle \left(\frac{N}{V}\right)^2 \qquad (38)$$

If we then assume that a chemical reaction between two colling spheres can happen only if the speed $v_{0\_coll\_norm}$ is greater than a minimum threshold v_{min}, and if that condition is met we introduce also a probability for the collision to happen ρ (called the steric factor) the number of reactions per unit of time and volume will be

$$\frac{dreaction}{dt}\frac{1}{V} = \frac{dcollision}{dt}\frac{1}{V} p\big(v_{0\_coll\_norm} > v_{min}\big)\rho = \pi D^2 \rho \sqrt{2}\langle v\rangle e^{-\frac{2v_{min}^2}{\pi\langle v\rangle^2}} \left(\frac{N}{V}\right)^2 \qquad (39)$$

Let's now express $\langle v\rangle$ as a function of the temperature, as saw at the beginning

$$\frac{dreaction}{dt}\frac{1}{V} = \pi D^2 \rho \sqrt{2}\sqrt{\frac{8k_BT}{\pi m}} e^{-\frac{mv_{min}^2}{4k_BT}} \left(\frac{N}{V}\right)^2 = 4\pi D^2 \rho \sqrt{\frac{k_BT}{\pi m}} e^{-\frac{mv_{min}^2}{4k_BT}} \left(\frac{N}{V}\right)^2 \qquad (40)$$

Now let's examine Fig 3., where the springs are supposed to be ideal and without mass.

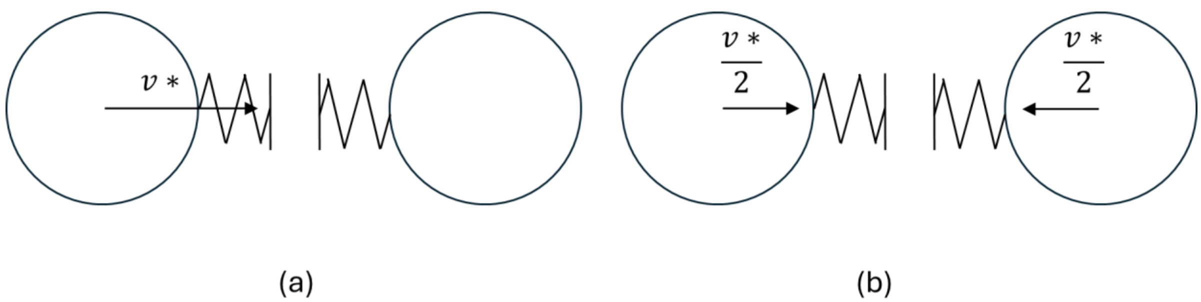


Fig. 3

During an elastic collision, part of the kinetic energy of the sphere with speed $v*$ will become potential energy stored in the springs, and then released again to accelerate the other sphere (initially at rest).

We can study the same collision in a frame of reference with moves with half the speed of $v*$, in the same direction. In this frame of reference, both spheres will have a speed equal to half of $v*$, and will be moving towards each other with that speed.

For symmetry, a point in time will be reached where both spheres will have zero speed, before they start accelerating in the opposite direction. This will help us determine the total energy stored in the springs at its maximum level

$$\varepsilon = \frac{1}{2}m\left(\frac{v*}{2}\right)^2 + \frac{1}{2}m\left(\frac{v*}{2}\right)^2 = \frac{1}{4}mv*^2 \qquad (41)$$

Let’s now assume that a similar situation happens during the elastic collision of the spheres in our model (where the “energy storing” mechanism doesn’t necessarily have to be a spring) and specifically that a chemical reaction happens if the kinetic energy transferred to the “energy storing” mechanism is at least equal or higher than a certain threshold. Here the component of the velocity to consider is $v_{0\_coll\_norm}$, along which direction forces are exchanged.

So v_{min} (here, the minimum value that the component of the relative speed normal to the contact surface of the colliding spheres $v_{0\_coll\_norm}$ needs to have for the chemical reaction to happen) is related to the activation energy of the chemical reaction by the formula

$$\varepsilon_a = \frac{1}{4} m v_{min}^2 \qquad (42)$$

We can then substitute and get

$$\frac{dreaction}{dt}\frac{1}{V} = 4\pi D^2 \rho \sqrt{\frac{k_B T}{\pi m}} e^{-\frac{\varepsilon_a}{k_B T}} \left(\frac{N}{V}\right)^2 \qquad (43)$$

As a last step, if we want to express this relationship introducing the concentration $[C]$ expressed in mol/l, the molar mass M, the activation energy per mole E_a and expressing all the volumes in l and numbers in moles, considering the relationships

$$[C] = \frac{N/N_a}{10^3 V} \qquad (44)$$

$$M = N_A m \qquad (45)$$

$$E_a = N_A \varepsilon_a \qquad (46)$$

$$R = N_A k_B \qquad (47)$$

Where R is the molar gas constant and N_A is the Avogadro constant

$$\frac{dreaction}{dt}\frac{1}{10^3 V N_A} = \frac{4\pi D^2 \rho}{10^3 N_A} \sqrt{\frac{N_A k_B T}{\pi N_A m}} e^{-\frac{N_A \varepsilon_a}{N_A k_B T}} \left(\frac{N/N_a}{10^3 V} 10^3 N_A\right)^2 = \frac{4\pi D^2 \rho}{10^3 N_A} \sqrt{\frac{RT}{\pi M}} e^{-\frac{E_a}{RT}} 10^6 N_A^2 [C]^2 =$$

$$= 10^3 N_A 4\pi D^2 \rho \sqrt{\frac{RT}{\pi M}} e^{-\frac{E_a}{RT}} [C]^2 \qquad (48)$$

Appendix

<u>Solving the integrals for the speed distribution of the colliding spheres</u>

First, we solve the inner integral

$$f_{hit|\bar{v}_1}(\bar{v}_1) = \int_0^\infty \int_0^\pi Kf(\bar{v}_1)\, f(\bar{v}_2) \frac{1}{2} \sin \alpha v_0 \, d\alpha d\bar{v}_2 =$$

$$= \int_0^\infty Kf(\bar{v}_1) f(\bar{v}_2) \left(\int_0^\pi \frac{1}{2} \sin \alpha v_0 \, d\alpha \right) d\bar{v}_2 =$$

$$= \int_0^\infty Kf(\bar{v}_1) f(\bar{v}_2) \left(\int_0^\pi \frac{1}{2} \sin \alpha \sqrt{\bar{v}_1^2 + \bar{v}_2^2 \, - \, 2\bar{v}_1 \bar{v}_2 \cos \alpha} \, d\alpha \right) d\bar{v}_2 =$$

$$= \int_0^\infty Kf(\bar{v}_1) f(\bar{v}_2) \left(\frac{1}{2} \frac{2}{3} \frac{1}{2\bar{v}_1 \bar{v}_2} (\bar{v}_1^2 + \bar{v}_2^2 \, - \, 2\bar{v}_1 \bar{v}_2 \cos \alpha)^{\frac{3}{2}} \right) \Big|_0^\pi d\bar{v}_2 =$$

$$= \int_0^\infty Kf(\bar{v}_1) f(\bar{v}_2) \frac{1}{6\bar{v}_1 \bar{v}_2} \left((\bar{v}_1^2 + \bar{v}_2^2 + 2\bar{v}_1 \bar{v}_2)^{\frac{3}{2}} - (\bar{v}_1^2 + \bar{v}_2^2 + 2\bar{v}_1 \bar{v}_2)^{\frac{3}{2}} \right) d\bar{v}_2 =$$

$$= \int_0^\infty Kf(\bar{v}_1) f(\bar{v}_2) \frac{1}{6\bar{v}_1 \bar{v}_2} \left(((\bar{v}_1 + \bar{v}_2)^2)^{\frac{3}{2}} - ((\bar{v}_1 - \bar{v}_2)^2)^{\frac{3}{2}} \right) d\bar{v}_2 =$$

$$= \int_0^\infty Kf(\bar{v}_1) f(\bar{v}_2) \frac{1}{6\bar{v}_1 \bar{v}_2} ((\bar{v}_1 + \bar{v}_2)^3 - |\bar{v}_1 - \bar{v}_2|^3) d\bar{v}_2$$

We then consider separately the cases where $\bar{v}_1 - \bar{v}_2 \geq 0$ (in other words when $\bar{v}_2$ is between 0 and $\bar{v}_1$) and where $\bar{v}_1 - \bar{v}_2 < 0$ (when $\bar{v}_2$ is greater than $\bar{v}_1$)

$$f_{hit|\bar{v}_1}(\bar{v}_1) = \int_0^{\bar{v}_1} Kf(\bar{v}_1) f(\bar{v}_2) \frac{1}{6\bar{v}_1 \bar{v}_2} ((\bar{v}_1 + \bar{v}_2)^3 - (\bar{v}_1 - \bar{v}_2)^3) d\bar{v}_2 \; +$$
$$\int_{\bar{v}_1}^\infty Kf(\bar{v}_1) f(\bar{v}_2) \frac{1}{6\bar{v}_1 \bar{v}_2} ((\bar{v}_1 + \bar{v}_2)^3 - (\bar{v}_2 - \bar{v}_1)^3) d\bar{v}_2 =$$

$$= \int_0^{\bar{v}_1} Kf(\bar{v}_1) f(\bar{v}_2) \frac{\left((2\bar{v}_2) \left((\bar{v}_1^2 + \bar{v}_2^2 + 2\bar{v}_1 \bar{v}_2) + (\bar{v}_1^2 - \bar{v}_2^2) + (\bar{v}_1^2 + \bar{v}_2^2 - 2\bar{v}_1 \bar{v}_2) \right) \right)}{6\bar{v}_1 \bar{v}_2} d\bar{v}_2 \; +$$
$$\int_{\bar{v}_1}^\infty Kf(\bar{v}_1) f(\bar{v}_2) \frac{\left((2\bar{v}_1) \left((\bar{v}_1^2 + \bar{v}_2^2 + 2\bar{v}_1 \bar{v}_2) + (\bar{v}_2^2 - \bar{v}_1^2) + (\bar{v}_2^2 + \bar{v}_1^2 - 2\bar{v}_1 \bar{v}_2) \right) \right)}{6\bar{v}_1 \bar{v}_2} d\bar{v}_2 =$$

$$= \int_0^{\bar{v}_1} Kf(\bar{v}_1) f(\bar{v}_2) \frac{2\bar{v}_2 (3\bar{v}_1^2 + \bar{v}_2^2)}{6\bar{v}_1 \bar{v}_2} d\bar{v}_2 \; + \int_{\bar{v}_1}^\infty Kf(\bar{v}_1) f(\bar{v}_2) \frac{2\bar{v}_1 (3\bar{v}_2^2 + \bar{v}_1^2)}{6\bar{v}_1 \bar{v}_2} d\bar{v}_2 =$$

$$= \int_0^{\bar{v}_1} Kf(\bar{v}_1) f(\bar{v}_2) \frac{3\bar{v}_1^2 + \bar{v}_2^2}{3\bar{v}_1} d\bar{v}_2 \; + \int_{\bar{v}_1}^\infty Kf(\bar{v}_1) f(\bar{v}_2) \frac{3\bar{v}_2^2 + \bar{v}_1^2}{3\bar{v}_2} d\bar{v}_2 =$$

$$= \int_0^{\bar{v}_1} Kf(\bar{v}_1) f(\bar{v}_2) \bar{v}_1 d\bar{v}_2 \; + \int_0^{\bar{v}_1} Kf(\bar{v}_1) f(\bar{v}_2) \frac{\bar{v}_2^2}{3\bar{v}_1} d\bar{v}_2 \; + \int_{\bar{v}_1}^\infty Kf(\bar{v}_1) f(\bar{v}_2) \bar{v}_2 d\bar{v}_2 \, +$$
$$\int_{\bar{v}_1}^\infty Kf(\bar{v}_1) f(\bar{v}_2) \frac{\bar{v}_1^2}{3\bar{v}_2} d\bar{v}_2$$

Now let's define and solve separately

$$I_1 = \int_0^{\bar{v}_1} Kf(\bar{v}_1)f(\bar{v}_2)\bar{v}_1 d\bar{v}_2 \tag{49}$$

$$I_2 = \int_0^{\bar{v}_1} Kf(\bar{v}_1)f(\bar{v}_2)\frac{\bar{v}_2^2}{3\bar{v}_1} d\bar{v}_2 \tag{50}$$

$$I_3 = \int_{\bar{v}_1}^{\infty} Kf(\bar{v}_1)f(\bar{v}_2)\bar{v}_2 d\bar{v}_2 \tag{51}$$

$$I_4 = \int_{\bar{v}_1}^{\infty} Kf(\bar{v}_1)f(\bar{v}_2)\frac{\bar{v}_1^2}{3\bar{v}_2} d\bar{v}_2 \tag{52}$$

So that

$$f_{hit|\bar{v}_1}(\bar{v}_1) = I_1 + I_2 + I_3 + I_4 \tag{53}$$

Next

$$I_1 = \int_0^{\bar{v}_1} Kf(\bar{v}_1)f(\bar{v}_2)\bar{v}_1 d\bar{v}_2 = Kf(\bar{v}_1)\bar{v}_1 \int_0^{\bar{v}_1} f(\bar{v}_2) d\bar{v}_2$$

If we define

$$F(v) = \int_0^{v} f(x)\, dx \tag{54}$$

Then

$$I_1 = Kf(\bar{v}_1)\bar{v}_1 F(\bar{v}_1) \tag{55}$$

Furthermore

$$I_2 = \frac{Kf(\bar{v}_1)}{3\bar{v}_1}\int_0^{\bar{v}_1} \bar{v}_2^2 f(\bar{v}_2) d\bar{v}_2 = \frac{Kf(\bar{v}_1)}{3\bar{v}_1}\int_0^{\bar{v}_1} \bar{v}_2^2 \frac{32}{\pi^2}\frac{\bar{v}_2^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{\bar{v}_2^2}{\langle v\rangle^2}} d\bar{v}_2 = \frac{Kf(\bar{v}_1)}{3\bar{v}_1}\int_0^{\bar{v}_1} \frac{-4\bar{v}_2^3}{\pi\langle v\rangle}\frac{-8\bar{v}_2}{\pi\langle v\rangle^2} e^{-\frac{4}{\pi}\frac{\bar{v}_2^2}{\langle v\rangle^2}} d\bar{v}_2$$

We can then use integration by parts

$$I_2 = \frac{Kf(\bar{v}_1)}{3\bar{v}_1}\left(\frac{-4\bar{v}_2^3}{\pi\langle v\rangle} e^{-\frac{4}{\pi}\frac{\bar{v}_2^2}{\langle v\rangle^2}}\right)\Bigg|_0^{\bar{v}_1} - \frac{Kf(\bar{v}_1)}{3\bar{v}_1}\int_0^{\bar{v}_1} \frac{-12\bar{v}_2^2}{\pi\langle v\rangle} e^{-\frac{4}{\pi}\frac{\bar{v}_2^2}{\langle v\rangle^2}} d\bar{v}_2 =$$

$$= \frac{Kf(\bar{v}_1)}{3\bar{v}_1}\left(\frac{-4\bar{v}_1^3}{\pi\langle v\rangle} e^{-\frac{4}{\pi}\frac{\bar{v}_1^2}{\langle v\rangle^2}} - 0\right) - \frac{Kf(\bar{v}_1)}{3\bar{v}_1}\int_0^{\bar{v}_1} \frac{-12\bar{v}_2^2}{\pi\langle v\rangle} e^{-\frac{4}{\pi}\frac{\bar{v}_2^2}{\langle v\rangle^2}} d\bar{v}_2 =$$

$$= \frac{Kf(\bar{v}_1)}{3}\frac{-\pi\langle v\rangle^2}{8}\frac{32}{\pi^2}\frac{\bar{v}_1^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{\bar{v}_1^2}{\langle v\rangle^2}} - \frac{Kf(\bar{v}_1)}{3\bar{v}_1}\int_0^{\bar{v}_1} \frac{-3\pi\langle v\rangle^2}{8}\frac{32}{\pi^2}\frac{\bar{v}_2^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{\bar{v}_2^2}{\langle v\rangle^2}} d\bar{v}_2 =$$

$$= \frac{Kf(\bar{v}_1)}{3}\frac{-\pi\langle v\rangle^2}{8} f(\bar{v}_1) - \frac{Kf(\bar{v}_1)}{3\bar{v}_1}\frac{-3\pi\langle v\rangle^2}{8} F(\bar{v}_1) = \frac{Kf(\bar{v}_1)\pi\langle v\rangle^2}{24\bar{v}_1}\left(3F(\bar{v}_1) - \bar{v}_1 f(\bar{v}_1)\right) \tag{56}$$

And using similar techniques

$$I_3 = Kf(\bar{v}_1)\int_{\bar{v}_1}^{\infty} \bar{v}_2 \frac{32}{\pi^2}\frac{\bar{v}_2^2}{\langle v\rangle^3} e^{-\frac{4}{\pi\langle v\rangle^2}\bar{v}_2^2} d\bar{v}_2 = Kf(\bar{v}_1)\int_{\bar{v}_1}^{\infty} \frac{-4\bar{v}_2^2}{\pi\langle v\rangle}\frac{-8\bar{v}_2}{\pi\langle v\rangle^2} e^{-\frac{4}{\pi\langle v\rangle^2}\bar{v}_2^2} d\bar{v}_2 =$$

$$= Kf(\bar{v}_1)\left(\frac{-4\bar{v}_2^2}{\pi\langle v\rangle} e^{-\frac{4}{\pi\langle v\rangle^2}\bar{v}_2^2}\right)\Bigg|_{\bar{v}_1}^{\infty} - Kf(\bar{v}_1)\int_{\bar{v}_1}^{\infty} \frac{-8\bar{v}_2}{\pi\langle v\rangle} e^{-\frac{4}{\pi\langle v\rangle^2}\bar{v}_2^2} d\bar{v}_2 =$$

$$= Kf(\bar{v}_1)\left(0 - \frac{-4\bar{v}_1^2}{\pi\langle v\rangle} e^{-\frac{4}{\pi\langle v\rangle^2}\bar{v}_1^2}\right) - Kf(\bar{v}_1)\int_{\bar{v}_1}^{\infty} \langle v\rangle \frac{-8\bar{v}_2}{\pi\langle v\rangle^2} e^{-\frac{4}{\pi\langle v\rangle^2}\bar{v}_2^2} d\bar{v}_2 =$$

$$= Kf(\bar{v}_1)\frac{\pi\langle v\rangle^2}{8}\frac{32}{\pi^2}\frac{\bar{v}_1^2}{\langle v\rangle^3} e^{-\frac{4}{\pi\langle v\rangle^2}\bar{v}_1^2} - Kf(\bar{v}_1)\int_{\bar{v}_1}^{\infty} \langle v\rangle \frac{-8\bar{v}_2}{\pi\langle v\rangle^2} e^{-\frac{4}{\pi\langle v\rangle^2}\bar{v}_2^2} d\bar{v}_2 =$$

$$= Kf(\bar{v}_1)\frac{\pi\langle v\rangle^2}{8} f(\bar{v}_1) - Kf(\bar{v}_1)\langle v\rangle \left(e^{-\frac{4}{\pi\langle v\rangle^2}\bar{v}_2^2}\right)\Bigg|_{\bar{v}_1}^{\infty} =$$

$$= K\frac{\pi\langle v\rangle^2}{8} f^2(\bar{v}_1) - Kf(\bar{v}_1)\langle v\rangle \left(0 - e^{\frac{4}{\pi\langle v\rangle^2}\bar{v}_1^2}\right) =$$

$$= K\frac{\pi\langle v\rangle^2}{8} f^2(\bar{v}_1) + Kf(\bar{v}_1)\frac{\pi^2\langle v\rangle^4}{32\bar{v}_1^2}\frac{32}{\pi^2}\frac{\bar{v}_1^2}{\langle v\rangle^3} e^{\frac{4}{\pi\langle v\rangle^2}\bar{v}_1^2} = K\frac{\pi\langle v\rangle^2}{8} f^2(\bar{v}_1) + Kf(\bar{v}_1)\frac{\pi^2\langle v\rangle^4}{32\bar{v}_1^2} f(\bar{v}_1) =$$

$$= K\frac{\pi\langle v\rangle^2}{8} f^2(\bar{v}_1) + K\frac{\pi^2\langle v\rangle^4}{32\bar{v}_1^2} f^2(\bar{v}_1) \tag{57}$$

Finally

$$I_4 = \frac{Kf(\bar{v}_1)\bar{v}_1^2}{3}\int_{\bar{v}_1}^{\infty} \frac{32}{\pi^2}\frac{\bar{v}_2^2}{\langle v\rangle^3} e^{-\frac{4}{\pi\langle v\rangle^2}\bar{v}_2^2}\frac{1}{\bar{v}_2} d\bar{v}_2 = \frac{Kf(\bar{v}_1)\bar{v}_1^2}{3}\int_{\bar{v}_1}^{\infty} \frac{-4}{\pi\langle v\rangle}\frac{-8\bar{v}_2}{\pi\langle v\rangle^2} e^{-\frac{4}{\pi\langle v\rangle^2}\bar{v}_2^2} d\bar{v}_2 =$$

$$= \frac{Kf(\bar{v}_1)\bar{v}_1^2}{3}\frac{-4}{\pi\langle v\rangle}\left(e^{-\frac{4}{\pi\langle v\rangle^2}\bar{v}_2^2}\right)\Bigg|_{\bar{v}_1}^{\infty} = \frac{Kf(\bar{v}_1)\bar{v}_1^2}{3}\frac{-4}{\pi\langle v\rangle}\left(0 - e^{\frac{4}{\pi\langle v\rangle^2}\bar{v}_1^2}\right) = \frac{K\pi\langle v\rangle^2 f(\bar{v}_1)}{24}\frac{32}{\pi^2}\frac{\bar{v}_1^2}{\langle v\rangle^3} e^{\frac{4}{\pi\langle v\rangle^2}\bar{v}_1^2} =$$

$$= \frac{K\pi\langle v\rangle^2 f(\bar{v}_1)}{24} f(\bar{v}_1) = \frac{K\pi\langle v\rangle^2}{24} f^2(\bar{v}_1) \tag{58}$$

Putting everything together

$$f_{hit|\bar{v}_1}(\bar{v}_1) = Kf(\bar{v}_1)\bar{v}_1 F(\bar{v}_1) + \frac{Kf(\bar{v}_1)\pi\langle v\rangle^2}{24\bar{v}_1}\left(3F(\bar{v}_1) - \bar{v}_1 f(\bar{v}_1)\right) + K\frac{\pi\langle v\rangle^2}{8} f^2(\bar{v}_1) +$$
$$K\frac{\pi^2\langle v\rangle^4}{32\bar{v}_1^2} f^2(\bar{v}_1) + \frac{K\pi\langle v\rangle^2}{24} f^2(\bar{v}_1) =$$

$$= Kf(\bar{v}_1)\bar{v}_1 F(\bar{v}_1) + \frac{Kf(\bar{v}_1)\pi\langle v\rangle^2}{8\bar{v}_1} F(\bar{v}_1) - \frac{K\pi\langle v\rangle^2}{24} f^2(\bar{v}_1) + K\frac{\pi\langle v\rangle^2}{8} f^2(\bar{v}_1) + K\frac{\pi^2\langle v\rangle^4}{32\bar{v}_1^2} f^2(\bar{v}_1) +$$
$$\frac{K\pi\langle v\rangle^2}{24} f^2(\bar{v}_1) =$$

$$= Kf(\bar{v}_1)\bar{v}_1 F(\bar{v}_1) + \frac{Kf(\bar{v}_1)\pi\langle v\rangle^2}{8\bar{v}_1} F(\bar{v}_1) + K\frac{\pi^2\langle v\rangle^4}{32\bar{v}_1^2} f^2(\bar{v}_1) \ + \frac{K\pi\langle v\rangle^2}{8} f^2(\bar{v}_1) =$$

$$= Kf(\bar{v}_1)\left[\bar{v}_1 F(\bar{v}_1) + \frac{\pi\langle v\rangle^2}{8}\frac{F(\bar{v}_1)}{\bar{v}_1} + \frac{\pi^2\langle v\rangle^4}{32}\frac{f(\bar{v}_1)}{\bar{v}_1^2} \ + \frac{\pi\langle v\rangle^2}{24} f(\bar{v}_1)\right] \quad (59)$$

And for the reason discussed before

$$f_{hit|\bar{v}}(\bar{v}) = f_{hit|\bar{v}_1}(\bar{v}) =$$

$$= Kf(\bar{v})\left[\bar{v}F(\bar{v}) + \frac{\pi\langle v\rangle^2}{8}\frac{F(\bar{v})}{\bar{v}} + \frac{\pi^2\langle v\rangle^4}{32}\frac{f(\bar{v})}{\bar{v}^2} + \frac{\pi\langle v\rangle^2}{8} f(\bar{v})\right] \quad (60)$$

<u>Solving the integrals for the speed distribution after the collisions</u>

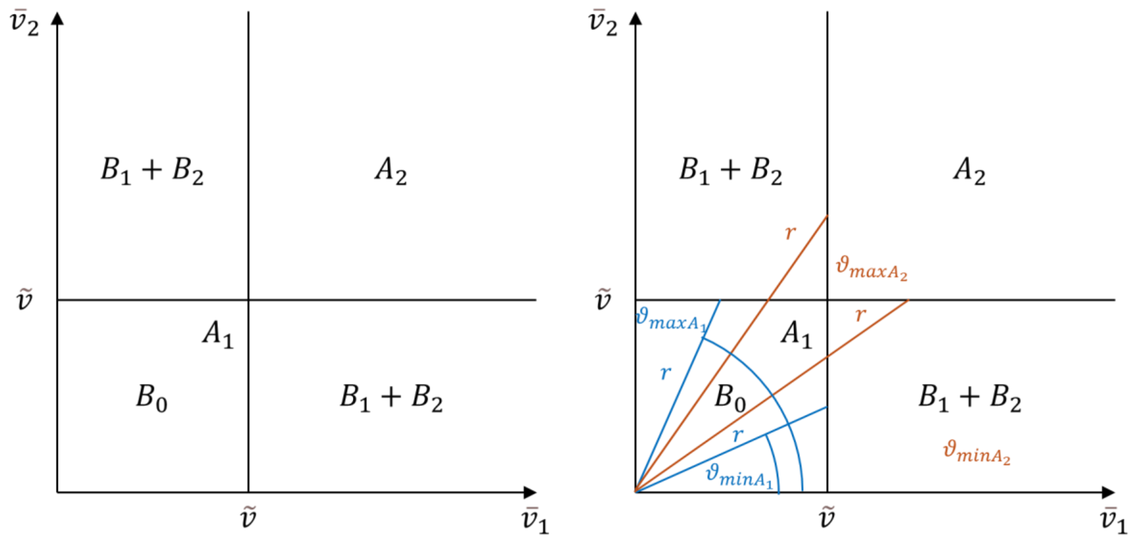
As mentioned before, [1] provide the boundaries for which $f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} = \frac{\tilde{v}}{2r_1r_2}$.

More specifically, referring to Fig. 4a (which represent the integration space for $\bar{v}_1$ and $\bar{v}_2$, for a given value of $\tilde{v}$) the paper identifies different regions:

- B_0 : where $\bar{v}_1^2 + \bar{v}_2^2 < \tilde{v}^2$
  Here $f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} = 0$ for every α (it is easy to see that $\tilde{v}$ is impossible to obtain due to conservation of kinetic energy considerations)
- B_1 and B_2: where $min\,\{\bar{v}_1, \bar{v}_2\} < \tilde{v} < max\,\{\bar{v}_1, \bar{v}_2\}$
  Here $f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} = \frac{\tilde{v}}{2r_1r_2}$ for every α
- A_1 and A_2: where $\tilde{v} < min\,\{\bar{v}_1, \bar{v}_2\}$ (A_1) or $\tilde{v} > max\,\{\bar{v}_1, \bar{v}_2\}$ (A_2)
  Here $f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} = \frac{\tilde{v}}{2r_1r_2}$ for $\alpha_{min} < \alpha < \alpha_{max}$ and $f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} = 0$ otherwise, where:

  $$\cos\alpha_{min} = \frac{\tilde{v}}{\bar{v}_1\bar{v}_2}\sqrt{\bar{v}_1^2 + \bar{v}_2^2 - \tilde{v}^2} \quad (61)$$

  $$\cos\alpha_{max} = -\frac{\tilde{v}}{\bar{v}_1\bar{v}_2}\sqrt{\bar{v}_1^2 + \bar{v}_2^2 - \tilde{v}^2} \quad (62)$$



(a) (b)

Fig. 4

Skipping the values for which $f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} = 0$, and considering that for regions B_1 and B_2 we can equivalently say that $\alpha_{min} = 0$ and $\alpha_{max} = \pi$

$$f_{new|\tilde{v}}(\tilde{v}) = \iint_{A_1+A_2+B_1+B_2} \int_{\alpha_{min}}^{\alpha_{max}} \frac{\tilde{v}}{2r_1r_2} K\, f(\bar{v}_1)\, f(\bar{v}_2) \frac{1}{2} \sin\alpha\, v_0\, d\alpha\, d\bar{v}_1 d\bar{v}_2 =$$

$$= \iint_{A_1+A_2+B_1+B_2} \int_{\alpha_{min}}^{\alpha_{max}} \frac{\tilde{v}K\, f(\bar{v}_1)\, f(\bar{v}_2) \frac{1}{2} \sin\alpha \sqrt{\bar{v}_1^2+\bar{v}_2^2 - 2\bar{v}_1\bar{v}_2\cos\alpha}}{2\frac{1}{2}\sqrt{\bar{v}_1^2+\bar{v}_2^2 + 2\bar{v}_1\bar{v}_2\cos\alpha}\frac{1}{2}\sqrt{\bar{v}_1^2+\bar{v}_2^2 - 2\bar{v}_1\bar{v}_2\cos\alpha}} d\alpha\, d\bar{v}_1 d\bar{v}_2 =$$

$$= \iint_{A_1+A_2+B_1+B_2} \int_{\alpha_{min}}^{\alpha_{max}} \frac{\tilde{v}K\, f(\bar{v}_1)\, f(\bar{v}_2) \sin\alpha}{\sqrt{\bar{v}_1^2+\bar{v}_2^2 + 2\bar{v}_1\bar{v}_2\cos\alpha}} d\alpha\, d\bar{v}_1 d\bar{v}_2 =$$

$$= \iint_{A_1+A_2+B_1+B_2} \int_{\alpha_{min}}^{\alpha_{max}} \tilde{v}K\, f(\bar{v}_1) f(\bar{v}_2) \left(-\frac{1}{\bar{v}_1\bar{v}_2}\right) \frac{-\bar{v}_1\bar{v}_2 \sin\alpha}{\sqrt{\bar{v}_1^2+\bar{v}_2^2 + 2\bar{v}_1\bar{v}_2\cos\alpha}} d\alpha\, d\bar{v}_1 d\bar{v}_2 =$$

$$= \iint_{A_1+A_2+B_1+B_2} -\frac{\tilde{v}K\, f(\bar{v}_1) f(\bar{v}_2)}{\bar{v}_1\bar{v}_2} \left(\sqrt{\bar{v}_1^2 + \bar{v}_2^2 + 2\bar{v}_1\bar{v}_2\cos\alpha}\right)\Big|_{\alpha_{min}}^{\alpha_{max}} d\bar{v}_1 d\bar{v}_2 =$$

$$= \iint_{A_1+A_2} -\frac{\tilde{v}K\, f(\bar{v}_1) f(\bar{v}_2)}{\bar{v}_1\bar{v}_2} \left(\sqrt{\bar{v}_1^2 + \bar{v}_2^2 + 2\bar{v}_1\bar{v}_2\left(-\frac{\tilde{v}}{\bar{v}_1\bar{v}_2}\sqrt{\bar{v}_1^2 + \bar{v}_2^2 - \tilde{v}^2}\right)} - \sqrt{\bar{v}_1^2 + \bar{v}_2^2 + 2\bar{v}_1\bar{v}_2\left(\frac{\tilde{v}}{\bar{v}_1\bar{v}_2}\sqrt{\bar{v}_1^2 + \bar{v}_2^2 - \tilde{v}^2}\right)}\right) d\bar{v}_1 d\bar{v}_2 +$$
$$\iint_{B_1+B_2} -\frac{\tilde{v}K\, f(\bar{v}_1) f(\bar{v}_2)}{\bar{v}_1\bar{v}_2} \left(\sqrt{\bar{v}_1^2 + \bar{v}_2^2 - 2\bar{v}_1\bar{v}_2} - \sqrt{\bar{v}_1^2 + \bar{v}_2^2 + 2\bar{v}_1\bar{v}_2}\right) d\bar{v}_1 d\bar{v}_2$$

Let’s define and solve separately

$$I_5 = \iint_{A_1+A_2} -\frac{\tilde{v}K\, f(\bar{v}_1)f(\bar{v}_2)}{\bar{v}_1\bar{v}_2}\left(\sqrt{\bar{v}_1^2 + \bar{v}_2^2 \ + \ 2\bar{v}_1\bar{v}_2\left(-\frac{\tilde{v}}{\bar{v}_1\bar{v}_2}\sqrt{\bar{v}_1^2 + \bar{v}_2^2 - \tilde{v}^2}\right)} - \sqrt{\bar{v}_1^2 + \bar{v}_2^2 \ + \ 2\bar{v}_1\bar{v}_2\left(\frac{\tilde{v}}{\bar{v}_1\bar{v}_2}\sqrt{\bar{v}_1^2 + \bar{v}_2^2 - \tilde{v}^2}\right)}\right)d\bar{v}_1 d\bar{v}_2 \quad (63)$$

$$I_6 = \iint_{B_1+B_2} -\frac{\tilde{v}K\, f(\bar{v}_1)f(\bar{v}_2)}{\bar{v}_1\bar{v}_2}\left(\sqrt{\bar{v}_1^2 + \bar{v}_2^2 \ - \ 2\bar{v}_1\bar{v}_2} - \sqrt{\bar{v}_1^2 + \bar{v}_2^2 \ + \ 2\bar{v}_1\bar{v}_2}\right)d\bar{v}_1 d\bar{v}_2 \quad (64)$$

So that

$$f_{new|\tilde{v}}(\tilde{v}) = I_5 + I_6 \quad (65)$$

For I_5 we can change to polar coordinates

$$\bar{v}_1 = r\cos\vartheta \quad (66)$$

$$\bar{v}_2 = r\sin\vartheta \quad (67)$$

$$d\bar{v}_1 d\bar{v}_2 = r dr d\vartheta \quad (68)$$

And

$$I_5 = \iint_{A_1+A_2} -\frac{\tilde{v}K\frac{32\,r^2\cos^2\vartheta}{\pi^2\ \langle v\rangle^3}e^{-\frac{4r^2\cos^2\vartheta}{\pi\ \ \langle v\rangle^2}}\frac{32\,r^2\sin^2\vartheta}{\pi^2\ \langle v\rangle^3}e^{-\frac{4r^2\sin^2\vartheta}{\pi\ \ \langle v\rangle^2}}}{r\cos\vartheta\, r\sin\vartheta}\cdot$$

$$\cdot\left(\sqrt{r^2\cos^2\vartheta + r^2\sin^2\vartheta \ + \ 2r\cos\vartheta\, r\sin\vartheta\left(-\frac{\tilde{v}}{r\cos\vartheta r\sin\vartheta}\sqrt{r^2\cos^2\vartheta + r^2\sin^2\vartheta - \tilde{v}^2}\right)} - \sqrt{r^2\cos^2\vartheta + r^2\sin^2\vartheta \ + \ 2r\cos\vartheta\, r\sin\vartheta\left(\frac{\tilde{v}}{r\cos\vartheta r\sin\vartheta}\sqrt{r^2\cos^2\vartheta + r^2\sin^2\vartheta - \tilde{v}^2}\right)}\right) r dr d\vartheta =$$

$$= \iint_{A_1+A_2} -\tilde{v}K\,\frac{1024r^2\cos\vartheta\sin\vartheta}{\pi^4\langle v\rangle^6}e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}}\left(\sqrt{r^2 - 2\tilde{v}\sqrt{r^2-\tilde{v}^2}} - \sqrt{r^2 + 2\tilde{v}\sqrt{r^2-\tilde{v}^2}}\right) r dr d\vartheta =$$

$$= \iint_{A_1+A_2} \tilde{v}K\,\frac{1024r^3\cos\vartheta\sin\vartheta}{\pi^4\langle v\rangle^6}e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}}\left(\sqrt{r^2 + 2\tilde{v}\sqrt{r^2-\tilde{v}^2}} - \sqrt{r^2 - 2\tilde{v}\sqrt{r^2-\tilde{v}^2}}\right) dr d\vartheta =$$

$$= \iint_{A_1+A_2} \tilde{v}K\,\frac{1024r^3\cos\vartheta\sin\vartheta}{\pi^4\langle v\rangle^6}e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}}\left(\sqrt{\left(\sqrt{r^2-\tilde{v}^2}+\tilde{v}\right)^2} - \sqrt{\left(\sqrt{r^2-\tilde{v}^2}-\tilde{v}\right)^2}\right) dr d\vartheta$$

Considering that in $A_1 + A_2$ we have $r > \tilde{v}$

$$I_5 = \iint_{A_1+A_2} \tilde{v}K\,\frac{1024r^3\cos\vartheta\sin\vartheta}{\pi^4\langle v\rangle^6}e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}}\left(\sqrt{r^2-\tilde{v}^2}+\tilde{v} - \left|\sqrt{r^2-\tilde{v}^2}-\tilde{v}\right|\right) dr d\vartheta =$$

$$= \iint_{A_1+A_2} \tilde{v}K\,\frac{256r^3 2\sin 2\vartheta}{\pi^4\langle v\rangle^6}e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}}\left(\sqrt{r^2-\tilde{v}^2}+\tilde{v} - \left|\sqrt{r^2-\tilde{v}^2}-\tilde{v}\right|\right) d\vartheta dr$$

Where in the last step note that the order of the integrals has been inverted.

We can see from Fig. 4b that in the region A_1 the angle ϑ is delimited by

$$\cos\vartheta_{minA_1} = \frac{\tilde{v}}{r} \rightarrow \sin\vartheta_{minA_1} = \sqrt{1-\left(\frac{\tilde{v}}{r}\right)^2} \quad (69)$$

$$\sin\vartheta_{maxA_1} = \frac{\tilde{v}}{r} \rightarrow \cos\vartheta_{maxA_1} = \sqrt{1-\left(\frac{\tilde{v}}{r}\right)^2} \quad (70)$$

While in the region A_2 it is delimited by

$$\sin\vartheta_{minA_2} = \frac{\tilde{v}}{r} \rightarrow \cos\vartheta_{minA_2} = \sqrt{1-\left(\frac{\tilde{v}}{r}\right)^2} \quad (71)$$

$$\cos\vartheta_{maxA_2} = \frac{\tilde{v}}{r} \rightarrow \sin\vartheta_{maxA_2} = \sqrt{1-\left(\frac{\tilde{v}}{r}\right)^2} \quad (72)$$

Furthermore, in A_1, r is between $\tilde{v}$ and $\sqrt{2}\tilde{v}$, therefore $\left|\sqrt{r^2-\tilde{v}^2}-\tilde{v}\right|$ is less than or equal to 0, while in A_2, r is greater than $\sqrt{2}\tilde{v}$, so that $\left|\sqrt{r^2-\tilde{v}^2}-\tilde{v}\right|$ is greater than 0.

We can then split the integral between the 2 regions

$$I_5 = \int_{\tilde{v}}^{\sqrt{2}\tilde{v}}\int_{\vartheta_{minA_1}}^{\vartheta_{maxA_1}} \tilde{v}K\frac{256r^3 2\sin 2\vartheta}{\pi^4\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}\left(\sqrt{r^2-\tilde{v}^2}+\tilde{v}+\sqrt{r^2-\tilde{v}^2}-\tilde{v}\right)d\vartheta dr +$$

$$+\int_{\sqrt{2}\tilde{v}}^{\infty}\int_{\vartheta_{minA_2}}^{\vartheta_{maxA_2}} \tilde{v}K\frac{256r^3 2\sin 2\vartheta}{\pi^4\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}\left(\sqrt{r^2-\tilde{v}^2}+\tilde{v}-\sqrt{r^2-\tilde{v}^2}+\tilde{v}\right)d\vartheta dr =$$

$$= \int_{\tilde{v}}^{\sqrt{2}\tilde{v}} \tilde{v}K\frac{256r^3}{\pi^4\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}\left(2\sqrt{r^2-\tilde{v}^2}\right)(-\cos 2\vartheta)\Big|_{\vartheta_{minA_1}}^{\vartheta_{maxA_1}} dr +$$

$$+\int_{\sqrt{2}\tilde{v}}^{\infty} \tilde{v}K\frac{256r^3}{\pi^4\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}(2\tilde{v})(-\cos 2\vartheta)\Big|_{\vartheta_{minA_2}}^{\vartheta_{maxA_2}} dr =$$

$$= \int_{\tilde{v}}^{\sqrt{2}\tilde{v}} \tilde{v}K\frac{512r^3\left(\sqrt{r^2-\tilde{v}^2}\right)}{\pi^4\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}\left(-\cos^2\vartheta_{maxA_1}+\sin^2\vartheta_{maxA_1}+\cos^2\vartheta_{minA_1}-\sin^2\vartheta_{minA_1}\right)dr +$$

$$+\int_{\sqrt{2}\tilde{v}}^{\infty} \tilde{v}^2K\frac{512r^3}{\pi^4\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}\left(-\cos^2\vartheta_{maxA_2}+\sin^2\vartheta_{maxA_2}+\cos^2\vartheta_{minA_2}-\sin^2\vartheta_{minA_2}\right)dr =$$

$$= \int_{\tilde{v}}^{\sqrt{2}\tilde{v}} \tilde{v}K\frac{512r^3\left(\sqrt{r^2-\tilde{v}^2}\right)}{\pi^4\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}2\left(2\left(\frac{\tilde{v}}{r}\right)^2-1\right)dr + \int_{\sqrt{2}\tilde{v}}^{\infty} \tilde{v}^2K\frac{512r^3}{\pi^4\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}2\left(1-2\left(\frac{\tilde{v}}{r}\right)^2\right)dr =$$

$$= \int_{\tilde{v}}^{\sqrt{2}\tilde{v}} \tilde{v}K\frac{1024r\left(2\tilde{v}^2-r^2\right)\left(\sqrt{r^2-\tilde{v}^2}\right)}{\pi^4\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}dr + \int_{\sqrt{2}\tilde{v}}^{\infty} \tilde{v}^2K\frac{1024r\left(r^2-2\tilde{v}^2\right)}{\pi^4\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}dr$$

We can then define and solve separately

$$I_7 = \int_{\tilde{v}}^{\sqrt{2}\tilde{v}} \tilde{v}K\frac{1024r\left(2\tilde{v}^2-r^2\right)\left(\sqrt{r^2-\tilde{v}^2}\right)}{\pi^4\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}dr \quad (73)$$

$$I_8 = \int_{\sqrt{2}\tilde{v}}^{\infty} \tilde{v}^2 K \frac{1024r\left(r^2-2\tilde{v}^2\right)}{\pi^4\langle v\rangle^6} e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}} dr \tag{74}$$

So that

$$I_5 = I_7 + I_8 \tag{75}$$

For I_7 we can perform a change of coordinates taking $u = \sqrt{r^2 - \tilde{v}^2}$ so that $r^2 = u^2 + \tilde{v}^2$ and $2udu = 2rdr$ or $udu = rdr$. Concerning the limits of integration, when $r = \tilde{v}$ then $u = 0$ and when $r = \sqrt{2}\tilde{v}$ then $u = \tilde{v}$. By substituting these relationships in the equation of I_7

$$I_7 = \int_0^{\tilde{v}} \tilde{v}K \frac{1024u\left(2\tilde{v}^2-u^2-\tilde{v}^2\right)u}{\pi^4\langle v\rangle^6} e^{-\frac{4u^2+\tilde{v}^2}{\pi\ \langle v\rangle^2}} du = \int_0^{\tilde{v}} \tilde{v}K \frac{1024\left(\tilde{v}^2-u^2\right)u^2}{\pi^4\langle v\rangle^6} e^{-\frac{4\ \tilde{v}^2}{\pi\langle v\rangle^2}} e^{-\frac{4\ u^2}{\pi\langle v\rangle^2}} du =$$

$$= \int_0^{\tilde{v}} K \frac{32\left(\tilde{v}^2-u^2\right)u^2}{\tilde{v}\pi^2\langle v\rangle^3} \frac{32\tilde{v}^2}{\pi^2\langle v\rangle^3} e^{-\frac{4\ \tilde{v}^2}{\pi\langle v\rangle^2}} e^{-\frac{4\ u^2}{\pi\langle v\rangle^2}} du = \int_0^{\tilde{v}} K \frac{32\left(\tilde{v}^2-u^2\right)u^2}{\tilde{v}\pi^2\langle v\rangle^3}\ f(\tilde{v}) e^{-\frac{4\ u^2}{\pi\langle v\rangle^2}} du =$$

$$= Kf(\tilde{v}) \int_0^{\tilde{v}} \frac{32\tilde{v}u^2}{\pi^2\langle v\rangle^3}\ e^{-\frac{4\ u^2}{\pi\langle v\rangle^2}} du + Kf(\tilde{v}) \int_0^{\tilde{v}} -\frac{32u^4}{\tilde{v}\pi^2\langle v\rangle^3}\ e^{-\frac{4\ u^2}{\pi\langle v\rangle^2}} du =$$

$$= Kf(\tilde{v})\tilde{v} \int_0^{\tilde{v}} f(u)du + Kf(\tilde{v}) \int_0^{\tilde{v}} \frac{4u^3}{\tilde{v}\pi\langle v\rangle} \left(-\frac{8u}{\pi\langle v\rangle^2}\right)\ e^{-\frac{4\ u^2}{\pi\langle v\rangle^2}} du =$$

$$= Kf(\tilde{v})\tilde{v}F(\tilde{v}) + Kf(\tilde{v}) \left(\frac{4u^3}{\tilde{v}\pi\langle v\rangle}\ e^{-\frac{4\ u^2}{\pi\langle v\rangle^2}}\right)\Bigg|_0^{\tilde{v}} - Kf(\tilde{v}) \int_0^{\tilde{v}} \frac{12u^2}{\tilde{v}\pi\langle v\rangle}\ e^{-\frac{4\ u^2}{\pi\langle v\rangle^2}} du =$$

$$= Kf(\tilde{v})\tilde{v}F(\tilde{v}) + Kf(\tilde{v}) \left(\frac{4\tilde{v}^3}{\tilde{v}\pi\langle v\rangle}\ e^{-\frac{4\ \tilde{v}^2}{\pi\langle v\rangle^2}} - 0\right) - Kf(\tilde{v}) \int_0^{\tilde{v}} \frac{3\pi\langle v\rangle^2}{8\tilde{v}} \frac{32u^2}{\pi^2\langle v\rangle^3}\ e^{-\frac{4\ u^2}{\pi\langle v\rangle^2}} du =$$

$$= Kf(\tilde{v})\tilde{v}F(\tilde{v}) + Kf(\tilde{v}) \frac{\pi\langle v\rangle^2}{8} \frac{32\tilde{v}^2}{\pi^2\langle v\rangle^3}\ e^{-\frac{4\ \tilde{v}^2}{\pi\langle v\rangle^2}} - Kf(\tilde{v}) \frac{3\pi\langle v\rangle^2}{8\tilde{v}} \int_0^{\tilde{v}} f(u)du =$$

$$= Kf(\tilde{v})\tilde{v}F(\tilde{v}) + Kf(\tilde{v}) \frac{\pi\langle v\rangle^2}{8} f(\tilde{v}) - Kf(\tilde{v}) \frac{3\pi\langle v\rangle^2}{8\tilde{v}} F(\tilde{v}) =$$

$$= Kf(\tilde{v})\tilde{v}F(\tilde{v}) + K \frac{\pi\langle v\rangle^2}{8} f^2(\tilde{v}) - K \frac{3\pi\langle v\rangle^2}{8} f(\tilde{v}) \frac{F(\tilde{v})}{\tilde{v}} \tag{76}$$

For I_8 instead we proceed directly with integration

$$I_8 = \int_{\sqrt{2}\tilde{v}}^{\infty} \tilde{v}^2 K \frac{1024r\left(r^2-2\tilde{v}^2\right)}{\pi^4\langle v\rangle^6} e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}} dr = K \int_{\sqrt{2}\tilde{v}}^{\infty} \frac{1024\tilde{v}^2 r^3}{\pi^4\langle v\rangle^6} e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}} dr - K \int_{\sqrt{2}\tilde{v}}^{\infty} \frac{2048r\tilde{v}^4}{\pi^4\langle v\rangle^6} e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}} dr =$$

$$= K \int_{\sqrt{2}\tilde{v}}^{\infty} \frac{-128\tilde{v}^2 r^2}{\pi^3\langle v\rangle^4} \left(-\frac{8r}{\pi\langle v\rangle^2}\right) e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}} dr - K \int_{\sqrt{2}\tilde{v}}^{\infty} \frac{-256\tilde{v}^4}{\pi^3\langle v\rangle^4} \left(-\frac{8r}{\pi\langle v\rangle^2}\right) e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}} dr =$$

$$= K \left(\frac{-128\tilde{v}^2 r^2}{\pi^3\langle v\rangle^4} e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}}\right)\Bigg|_{\sqrt{2}\tilde{v}}^{\infty} - K \int_{\sqrt{2}\tilde{v}}^{\infty} \frac{-256\tilde{v}^2 r}{\pi^3\langle v\rangle^4} e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}} dr - K \frac{-256\tilde{v}^4}{\pi^3\langle v\rangle^4} \left(e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}}\right)\Bigg|_{\sqrt{2}\tilde{v}}^{\infty} =$$

$$= K\left(0 - \frac{-128\tilde{v}^2 2\tilde{v}^2}{\pi^3\langle v\rangle^4} e^{-\frac{4\,2\tilde{v}^2}{\pi\langle v\rangle^2}}\right) - K\int_{\sqrt{2}\tilde{v}}^{\infty} \frac{32\tilde{v}^2}{\pi^2\langle v\rangle^2}\left(-\frac{8r}{\pi\langle v\rangle^2}\right) e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}} dr - K\frac{-256\tilde{v}^4}{\pi^3\langle v\rangle^4}\left(0 - e^{-\frac{4\,2\tilde{v}^2}{\pi\langle v\rangle^2}}\right) =$$

$$= K\frac{256\tilde{v}^4}{\pi^3\langle v\rangle^4} e^{-\frac{4\,2\tilde{v}^2}{\pi\langle v\rangle^2}} - K\frac{32\tilde{v}^2}{\pi^2\langle v\rangle^2}\left(e^{-\frac{4\ r^2}{\pi\langle v\rangle^2}}\right)\Bigg|_{\sqrt{2}\tilde{v}}^{\infty} - K\frac{256\tilde{v}^4}{\pi^3\langle v\rangle^4} e^{-\frac{4\,2\tilde{v}^2}{\pi\langle v\rangle^2}} = -K\frac{32\tilde{v}^2}{\pi^2\langle v\rangle^2}\left(0 - e^{-\frac{4\,2\tilde{v}^2}{\pi\langle v\rangle^2}}\right)\Bigg|_{\sqrt{2}\tilde{v}}^{\infty} =$$

$$= K\frac{32\tilde{v}^2}{\pi^2\langle v\rangle^2} e^{-\frac{4\,2\tilde{v}^2}{\pi\langle v\rangle^2}} = K\frac{\pi^2\langle v\rangle^4}{32\tilde{v}^2}\left(\frac{32\tilde{v}^2}{\pi^2\langle v\rangle^3} e^{-\frac{4\ \tilde{v}^2}{\pi\langle v\rangle^2}}\right)^2 = K\frac{\pi^2\langle v\rangle^4}{32\tilde{v}^2} f^2(\tilde{v}) \tag{77}$$

Therefore

$$I_5 = I_7 + I_8 = Kf(\tilde{v})\tilde{v}F(\tilde{v}) + K\frac{\pi\langle v\rangle^2}{8} f^2(\tilde{v}) - K\frac{3\pi\langle v\rangle^2}{8} f(\tilde{v})\frac{F(\tilde{v})}{\tilde{v}} + K\frac{\pi^2\langle v\rangle^4}{32\tilde{v}^2} f^2(\tilde{v}) \tag{78}$$

Finally, we can integrate I_6, considering that the region (and the integrand) is symmetrically split between cases where $\bar{v}_1^2$ is greater than $\bar{v}_2^2$, or vice versa, we can choose the case of $\bar{v}_1^2 \geq \bar{v}_2^2$ (lower right in Fig. 4a) and say that

$$I_6 = \iint_{B_1+B_2} -\frac{\tilde{v}K\ f(\bar{v}_1)f(\bar{v}_2)}{\bar{v}_1\bar{v}_2}\left(\sqrt{\bar{v}_1^2 + \bar{v}_2^2\ -\ 2\bar{v}_1\bar{v}_2} - \sqrt{\bar{v}_1^2 + \bar{v}_2^2\ +\ 2\bar{v}_1\bar{v}_2}\right) d\bar{v}_1 d\bar{v}_2 =$$

$$= 2\int_0^{\tilde{v}}\int_{\tilde{v}}^{\infty} \frac{\tilde{v}K\ f(\bar{v}_1)f(\bar{v}_2)}{\bar{v}_1\bar{v}_2}\left(-\sqrt{(\ \bar{v}_1 -\ \bar{v}_2)^2} + \sqrt{(\ \bar{v}_1 +\ \bar{v}_2)^2}\right) d\bar{v}_1 d\bar{v}_2 =$$

$$= 2\int_0^{\tilde{v}}\int_{\tilde{v}}^{\infty} \frac{\tilde{v}K\frac{32\ \bar{v}_1^2}{\pi^2\langle v\rangle^3} e^{-\frac{4\ \bar{v}_1^2}{\pi\langle v\rangle^2}} \frac{32\ \bar{v}_2^2}{\pi^2\langle v\rangle^3} e^{-\frac{4\ \bar{v}_2^2}{\pi\langle v\rangle^2}}}{\bar{v}_1\bar{v}_2}(-\bar{v}_1 + \bar{v}_2 +\ \bar{v}_1 +\ \bar{v}_2) d\bar{v}_1 d\bar{v}_2 =$$

$$= 2\int_0^{\tilde{v}}\int_{\tilde{v}}^{\infty} \frac{\tilde{v}K\frac{32\ \bar{v}_1^2}{\pi^2\langle v\rangle^3} e^{-\frac{4\ \bar{v}_1^2}{\pi\langle v\rangle^2}} \frac{32\ \bar{v}_2^2}{\pi^2\langle v\rangle^3} e^{-\frac{4\ \bar{v}_2^2}{\pi\langle v\rangle^2}}}{\bar{v}_1\bar{v}_2} 2\bar{v}_2 d\bar{v}_1 d\bar{v}_2 = 2\int_0^{\tilde{v}}\int_{\tilde{v}}^{\infty} \tilde{v}K\ \frac{2048}{\pi^4}\frac{\bar{v}_1\bar{v}_2^2}{\langle v\rangle^6} e^{-\frac{4\ \bar{v}_1^2}{\pi\langle v\rangle^2}} e^{-\frac{4\ \bar{v}_2^2}{\pi\langle v\rangle^2}} d\bar{v}_1 d\bar{v}_2 =$$

$$= 2\int_0^{\tilde{v}}\int_{\tilde{v}}^{\infty} \tilde{v}K\ \frac{-256}{\pi^3}\frac{\bar{v}_2^2}{\langle v\rangle^4} e^{-\frac{4\ \bar{v}_2^2}{\pi\langle v\rangle^2}}\left(\frac{-8\bar{v}_1}{\pi\langle v\rangle^2}\right) e^{-\frac{4\ \bar{v}_1^2}{\pi\langle v\rangle^2}} d\bar{v}_1 d\bar{v}_2 =$$

$$= 2\int_0^{\tilde{v}} \tilde{v}K\ \frac{-256}{\pi^3}\frac{\bar{v}_2^2}{\langle v\rangle^4} e^{-\frac{4\ \bar{v}_2^2}{\pi\langle v\rangle^2}}\left(e^{-\frac{4\ \bar{v}_1^2}{\pi\langle v\rangle^2}}\right)\Bigg|_{\tilde{v}}^{\infty} d\bar{v}_2 =$$

$$= 2\int_0^{\tilde{v}} \tilde{v}K\ \frac{-256}{\pi^3}\frac{\bar{v}_2^2}{\langle v\rangle^4} e^{-\frac{4\ \bar{v}_2^2}{\pi\langle v\rangle^2}}\left(0 - e^{-\frac{4\ \tilde{v}^2}{\pi\langle v\rangle^2}}\right) d\bar{v}_2 = 2\tilde{v}K\ \frac{8}{\pi\langle v\rangle} e^{-\frac{4\ \tilde{v}^2}{\pi\langle v\rangle^2}} \int_0^{\tilde{v}} \frac{32}{\pi^2}\frac{\bar{v}_2^2}{\langle v\rangle^3} e^{-\frac{4\ \bar{v}_2^2}{\pi\langle v\rangle^2}} d\bar{v}_2 =$$

$$= K\frac{\pi\langle v\rangle^2}{2\tilde{v}}\frac{32}{\pi^2}\frac{\tilde{v}^2}{\langle v\rangle^3} e^{-\frac{4\ \tilde{v}^2}{\pi\langle v\rangle^2}} F(\tilde{v}) = K\frac{\pi\langle v\rangle^2}{2\tilde{v}} f(\tilde{v})F(\tilde{v}) \tag{79}$$

We can now obtain

$$f_{new|\tilde{v}}(\tilde{v}) = I_5 + I_6 = Kf(\tilde{v})\tilde{v}F(\tilde{v}) + K\frac{\pi\langle v\rangle^2}{8}f^2(\tilde{v}) - K\frac{3\pi\langle v\rangle^2}{8}f(\tilde{v})\frac{F(\tilde{v})}{\tilde{v}} + K\frac{\pi^2\langle v\rangle^4}{32\tilde{v}^2}f^2(\tilde{v}) +$$
$$+ K\frac{\pi\langle v\rangle^2}{2\tilde{v}}f(\tilde{v})F(\tilde{v}) =$$
$$= Kf(\tilde{v})\left[\tilde{v}F(\tilde{v}) + \frac{\pi\langle v\rangle^2}{8}f(\tilde{v}) - \frac{3\pi\langle v\rangle^2}{8}\frac{F(\tilde{v})}{\tilde{v}} + \frac{\pi^2\langle v\rangle^4}{32\tilde{v}^2}f(\tilde{v}) + \frac{\pi\langle v\rangle^2}{2\tilde{v}}F(\tilde{v})\right] =$$
$$= Kf(\tilde{v})\left[\tilde{v}F(\tilde{v}) + \frac{\pi\langle v\rangle^2}{8}\frac{F(\tilde{v})}{\tilde{v}} + \frac{\pi^2\langle v\rangle^4}{32}\frac{f(\tilde{v})}{\tilde{v}^2} + \frac{\pi\langle v\rangle^2}{8}f(\tilde{v})\right] \tag{80}$$

<u>Calculating the value of the constant K</u>

As saw before

$$K = \frac{1}{\int_0^\pi\int_0^\infty\int_0^\infty f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha v_0 d\bar{v}_1 d\bar{v}_2 d\alpha}$$

But we already calculated the value of

$$f_{hit|\bar{v}_1}(\bar{v}_1) = \int_0^\infty\int_0^\pi K\, f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha v_0\, d\alpha d\bar{v}_2$$

So we can rearrange the order of the integrals

$$K = \frac{1}{\int_0^\infty\int_0^\infty\int_0^\pi f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha v_0 d\alpha d\bar{v}_2 d\bar{v}_1} = \frac{1}{\int_0^\infty\frac{f_{hit|\bar{v}_1}(\bar{v}_1)}{K}d\bar{v}_1}$$

We can just rename $\bar{v}_1$ as $\bar{v}$

$$K = \frac{1}{\int_0^\infty\frac{f_{hit|\bar{v}}(\bar{v})}{K}d\bar{v}} = \frac{1}{\int_0^\infty f(\bar{v})\left[\bar{v}F(\bar{v})+\frac{\pi\langle v\rangle^2}{8}\frac{F(\bar{v})}{\bar{v}}+\frac{\pi^2\langle v\rangle^4}{32}\frac{f(\bar{v})}{\bar{v}^2}+\frac{\pi\langle v\rangle^2}{8}f(\bar{v})\right]d\bar{v}} =$$
$$= \frac{1}{\int_0^\infty \bar{v}f(\bar{v})F(\bar{v})d\bar{v}+\int_0^\infty\frac{\pi\langle v\rangle^2}{8}\frac{f(\bar{v})F(\bar{v})}{\bar{v}}d\bar{v}+\int_0^\infty\frac{\pi^2\langle v\rangle^4}{32}\frac{f^2(\bar{v})}{\bar{v}^2}d\bar{v}+\int_0^\infty\frac{\pi\langle v\rangle^2}{8}f^2(\bar{v})d\bar{v}}$$

Let’s as usual solve each integral separately

$$I_9 = \int_0^\infty \bar{v}f(\bar{v})F(\bar{v})d\bar{v} \tag{81}$$

$$I_{10} = \int_0^\infty \frac{\pi\langle v\rangle^2}{8}\frac{f(\bar{v})F(\bar{v})}{\bar{v}}d\bar{v} \tag{82}$$

$$I_{11} = \int_0^\infty \frac{\pi^2\langle v\rangle^4}{32}\frac{f^2(\bar{v})}{\bar{v}^2}d\bar{v} \tag{83}$$

$$I_{12} = \int_0^\infty \frac{\pi\langle v\rangle^2}{8}f^2(\bar{v})d\bar{v} \tag{84}$$

So that

$$K = \frac{1}{I_9 + I_{10} + I_{11} + I_{12}} \tag{85}$$

Next

$$I_9 = \int_0^\infty \bar{v} \frac{32}{\pi^2} \frac{\bar{v}^2}{\langle v \rangle^3} e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v \rangle^2}} F(\bar{v}) d\bar{v} = \int_0^\infty \frac{-4}{\pi} \frac{\bar{v}^2}{\langle v \rangle} F(\bar{v}) \left(-\frac{8\bar{v}}{\pi \langle v \rangle^2}\right) e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v \rangle^2}} d\bar{v} =$$

$$= \left(\frac{-4}{\pi} \frac{\bar{v}^2}{\langle v \rangle} F(\bar{v}) e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v \rangle^2}}\right)\Bigg|_0^\infty - \int_0^\infty \left(-\frac{8}{\pi} \frac{\bar{v}}{\langle v \rangle} F(\bar{v}) - \frac{4}{\pi} \frac{\bar{v}^2}{\langle v \rangle} f(\bar{v})\right) e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v \rangle^2}} d\bar{v} =$$

$$= (0 - 0) + \int_0^\infty \frac{8}{\pi} \frac{\bar{v}}{\langle v \rangle} F(\bar{v}) e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v \rangle^2}} d\bar{v} + \int_0^\infty \frac{4}{\pi} \frac{\bar{v}^2}{\langle v \rangle} f(\bar{v}) e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v \rangle^2}} d\bar{v} =$$

$$= (0 - 0) + \int_0^\infty -\langle v \rangle F(\bar{v}) \left(-\frac{8\bar{v}}{\pi \langle v \rangle^2}\right) e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v \rangle^2}} d\bar{v} + \int_0^\infty \frac{4}{\pi} \frac{\bar{v}^2}{\langle v \rangle} \frac{32}{\pi^2} \frac{\bar{v}^2}{\langle v \rangle^3} e^{-\frac{4}{\pi}\frac{2\bar{v}^2}{\langle v \rangle^2}} d\bar{v} =$$

$$= \left(-\langle v \rangle F(\bar{v}) e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v \rangle^2}}\right)\Bigg|_0^\infty - \int_0^\infty -\langle v \rangle f(\bar{v}) e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v \rangle^2}} d\bar{v} + \int_0^\infty \frac{4}{\pi} \frac{\bar{v}^2}{\langle v \rangle} \frac{32}{\pi^2} \frac{\bar{v}^2}{\langle v \rangle^3} e^{-\frac{4}{\pi}\frac{2\bar{v}^2}{\langle v \rangle^2}} d\bar{v} =$$

$$= (0 - 0) + \int_0^\infty \langle v \rangle \frac{32}{\pi^2} \frac{\bar{v}^2}{\langle v \rangle^3} e^{-\frac{4}{\pi}\frac{2\bar{v}^2}{\langle v \rangle^2}} d\bar{v} + \int_0^\infty \frac{4}{\pi} \frac{\bar{v}^2}{\langle v \rangle} \frac{32}{\pi^2} \frac{\bar{v}^2}{\langle v \rangle^3} e^{-\frac{4}{\pi}\frac{2\bar{v}^2}{\langle v \rangle^2}} d\bar{v} =$$

We can now perform a change of coordinates for the last two integrals taking $u = \sqrt{2}\bar{v}$ so that $u^2 = 2\bar{v}^2$ and $du = \sqrt{2} d\bar{v}$. It is easy to see that the limits of integration remain the same

$$I_9 = \int_0^\infty \langle v \rangle \frac{32}{\pi^2} \frac{\frac{u^2}{2}}{\langle v \rangle^3} e^{-\frac{4}{\pi}\frac{u^2}{\langle v \rangle^2}} \frac{du}{\sqrt{2}} + \int_0^\infty \frac{4}{\pi} \frac{\frac{u^2}{2}}{\langle v \rangle} \frac{32}{\pi^2} \frac{\frac{u^2}{2}}{\langle v \rangle^3} e^{-\frac{4}{\pi}\frac{u^2}{\langle v \rangle^2}} \frac{du}{\sqrt{2}} =$$

$$= \frac{\sqrt{2}\langle v \rangle}{4} (F(u))|_0^\infty + \int_0^\infty \frac{-2\sqrt{2}}{\pi^2} \frac{u^3}{\langle v \rangle^2} \left(-\frac{8u}{\pi \langle v \rangle^2}\right) e^{-\frac{4}{\pi}\frac{u^2}{\langle v \rangle^2}} du =$$

$$= \frac{\sqrt{2}\langle v \rangle}{4} (1 - 0) + \left(\frac{-2\sqrt{2}}{\pi^2} \frac{u^3}{\langle v \rangle^2} e^{-\frac{4}{\pi}\frac{u^2}{\langle v \rangle^2}}\right)\Bigg|_0^\infty - \int_0^\infty \frac{-6\sqrt{2}}{\pi^2} \frac{u^2}{\langle v \rangle^2} e^{-\frac{4}{\pi}\frac{u^2}{\langle v \rangle^2}} du =$$

$$= \frac{\sqrt{2}\langle v \rangle}{4} + (0 - 0) + \int_0^\infty \frac{3\sqrt{2}\langle v \rangle}{16} \frac{32}{\pi^2} \frac{u^2}{\langle v \rangle^3} e^{-\frac{4}{\pi}\frac{u^2}{\langle v \rangle^2}} du = \frac{\sqrt{2}\langle v \rangle}{4} + \frac{3\sqrt{2}\langle v \rangle}{16} (F(u))\Big|_0^\infty =$$

$$= \frac{\sqrt{2}\langle v \rangle}{4} + \frac{3\sqrt{2}\langle v \rangle}{16} (1 - 0) = \frac{\sqrt{2}\langle v \rangle}{4} + \frac{3\sqrt{2}\langle v \rangle}{16} = \frac{7\sqrt{2}}{16} \langle v \rangle \tag{86}$$

Then

$$I_{10} = \int_0^\infty \frac{\pi\langle v\rangle^2}{8} \frac{\frac{32}{\pi^2}\frac{\bar{v}^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v\rangle^2}} F(\bar{v})}{\bar{v}} d\bar{v} = \int_0^\infty \frac{4}{\pi}\frac{\bar{v}}{\langle v\rangle} e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v\rangle^2}} F(\bar{v}) d\bar{v} = \int_0^\infty \frac{-\langle v\rangle}{2} F(\bar{v}) \left(-\frac{8\bar{v}}{\pi\langle v\rangle^2}\right) e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v\rangle^2}} d\bar{v} =$$

$$= \left(\frac{-\langle v\rangle}{2} F(\bar{v}) e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v\rangle^2}}\right)\Bigg|_0^\infty - \int_0^\infty \frac{-\langle v\rangle}{2} f(\bar{v}) e^{-\frac{4}{\pi}\frac{\bar{v}^2}{\langle v\rangle^2}} d\bar{v} = (0-0) - \int_0^\infty \frac{-\langle v\rangle}{2} \frac{32}{\pi^2} \frac{\bar{v}^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{2\bar{v}^2}{\langle v\rangle^2}} d\bar{v}$$

Again by substitution

$$I_{10} = \int_0^\infty \frac{\langle v\rangle}{2} \frac{32}{\pi^2} \frac{\frac{u^2}{2}}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}} \frac{du}{\sqrt{2}} = \int_0^\infty \frac{\sqrt{2}\langle v\rangle}{8} \frac{32}{\pi^2} \frac{u^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}} du = \frac{\sqrt{2}\langle v\rangle}{8} (F(u))\Big|_0^\infty = \frac{\sqrt{2}\langle v\rangle}{8}(1-0) =$$

$$= \frac{\sqrt{2}}{8}\langle v\rangle \tag{87}$$

Further

$$I_{11} = \int_0^\infty \frac{\pi^2\langle v\rangle^4}{32} \frac{\frac{32^2}{\pi^4}\frac{\bar{v}^4}{\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{2\bar{v}^2}{\langle v\rangle^2}}}{\bar{v}^2} d\bar{v} = \int_0^\infty \frac{32}{\pi^2} \frac{\bar{v}^2}{\langle v\rangle^2} e^{-\frac{4}{\pi}\frac{2\bar{v}^2}{\langle v\rangle^2}} d\bar{v}$$

One more time by substitution

$$I_{11} = \int_0^\infty \frac{32}{\pi^2} \frac{\frac{u^2}{2}}{\langle v\rangle^2} e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}} \frac{du}{\sqrt{2}} = \int_0^\infty \frac{\sqrt{2}\langle v\rangle}{4} \frac{32}{\pi^2} \frac{u^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}} du = \frac{\sqrt{2}\langle v\rangle}{4} (F(u))\Big|_0^\infty = \frac{\sqrt{2}\langle v\rangle}{4}(1-0) =$$

$$= \frac{\sqrt{2}}{4}\langle v\rangle \tag{88}$$

Finally

$$I_{12} = \int_0^\infty \frac{\pi\langle v\rangle^2}{8} f^2(\bar{v}) d\bar{v} = \int_0^\infty \frac{\pi\langle v\rangle^2}{8} \frac{32^2}{\pi^4} \frac{\bar{v}^4}{\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{2\bar{v}^2}{\langle v\rangle^2}} d\bar{v} = \int_0^\infty \frac{128}{\pi^3} \frac{\bar{v}^4}{\langle v\rangle^4} e^{-\frac{4}{\pi}\frac{2\bar{v}^2}{\langle v\rangle^2}} d\bar{v}$$

With one last substitution

$$I_{12} = \int_0^\infty \frac{128}{\pi^3} \frac{\frac{u^4}{4}}{\langle v\rangle^4} e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}} \frac{du}{\sqrt{2}} = \int_0^\infty \frac{16\sqrt{2}}{\pi^3} \frac{u^4}{\langle v\rangle^4} e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}} du = \int_0^\infty \frac{-2\sqrt{2}}{\pi^2} \frac{u^3}{\langle v\rangle^2} \left(-\frac{8u}{\pi\langle v\rangle^2}\right) e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}} du =$$

$$= \left(\frac{-2\sqrt{2}}{\pi^2} \frac{u^3}{\langle v\rangle^2} e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}}\right)\Bigg|_0^\infty - \int_0^\infty \frac{-6\sqrt{2}}{\pi^2} \frac{u^2}{\langle v\rangle^2} e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}} = (0-0) + \int_0^\infty \frac{3\sqrt{2}\langle v\rangle}{16} \frac{32}{\pi^2} \frac{u^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}} =$$

$$= \frac{3\sqrt{2}\langle v\rangle}{16} (F(u))|_0^\infty = \frac{3\sqrt{2}\langle v\rangle}{16}(1-0) = \frac{3\sqrt{2}}{16}\langle v\rangle \tag{89}$$

We can now at last determine K

$$K = \frac{1}{I_9+I_{10}+I_{11}+I_{12}} = \frac{1}{\frac{7\sqrt{2}}{16}\langle v\rangle + \frac{\sqrt{2}}{8}\langle v\rangle + \frac{\sqrt{2}}{4}\langle v\rangle + \frac{3\sqrt{2}}{16}\langle v\rangle} = \frac{1}{\frac{16\sqrt{2}}{16}\langle v\rangle} = \frac{1}{\sqrt{2}\langle v\rangle} \tag{90}$$

References

[1] Mechanical Proof of the Maxwell Speed Distribution Tsung-Wu Lin & Hejie Lin

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