Maxwell-Boltzmann Mechanical Proof

We want to prove that the case of random elastic collisions between “spheres” with equal mass moving in every direction (isotropic) without gravity and with velocity distributed according to Maxwell-Boltzmann, without considering the boundary conditions, is representing an equilibrium (stable) condition.

Rewriting the pdf

Starting from the probability density function of the Maxwell-Boltzmann distribution

$$f(v) = \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} 4\pi v^2 e^{-\frac{mv^2}{2k_B T}} \tag{1}$$

And considering the mean (speed) of the distribution, given that m and T are constant

$$\langle v \rangle = \sqrt{\frac{8k_B T}{\pi m}} \tag{2}$$

We can rewrite the probability density function as

$$f(v) = \frac{32}{\pi^2} \frac{v^2}{\langle v \rangle^3} e^{-\frac{4}{\pi}\left(\frac{v}{\langle v \rangle}\right)^2} \tag{3}$$

Calculating the probability of a collision

Let’s now indicate with D the dimension of the spheres (we pick the diameter).

A collision occurs if the center of 2 spheres is within a distance of 2 times their radius (or equivalently, 1 time their diameter). Let’s therefore consider the region of space within 1 diameter from the center of each sphere, and call it the “collision region” (which is also spherical).

During a small interval of time dt , it is easy to show that a sphere traveling with speed v would have covered with its “collision region” a volume in space (excluding the initial “collision region” itself) equal to

$$\pi D^2 v\, dt \tag{4}$$

Let’s for a moment restrict only to spheres having a velocity with magnitude between $\bar{v}_1$ and $\bar{v}_1 + d\bar{v}_1$ and a specific direction within a solid angle $d\Omega$. Due to the isotropy of the distribution and calling N the total number of spheres, the number of spheres satisfying the above conditions is

$$Nf(\bar{v}_1)\, d\bar{v}_1 \frac{d\Omega}{4\pi} \tag{5}$$

If we consider then another group of spheres having velocity with magnitude between $\bar{v}_2$ and $\bar{v}_2 + d\bar{v}_2$ and forming an angle between α and $\alpha + d\alpha$ with the velocity of the spheres of the first group, we can say the following.

Firstly, again because of isotropy, it is easy to show that the angle α will be distributed according to

$$f_\alpha(\alpha) = \frac{1}{2}\sin\alpha\, d\alpha \tag{6}$$

And therefore the number of spheres of the second group is

$$Nf(\bar{v}_2)\, d\bar{v}_2 \frac{1}{2}\sin\alpha\, d\alpha \tag{7}$$

Secondly, sitting in a frame of reference moving with the same velocity of the spheres of the first group, so that those spheres are in rest in it, the spheres of the second group will be seen moving according to the relative velocity, which has magnitude equal to

$$v_0 = \sqrt{\bar{v}_1^2 + \bar{v}_2^2 - 2\bar{v}_1\bar{v}_2\cos\alpha} \tag{8}$$

During a time dt, is it easy to show that the spheres of the second group will have covered with their “collision regions” (again excluding the initial “collision regions” themselves) a volume in space (in their travel relative to the spheres of the first group) of:

$$Nf(\bar{v}_2)\, d\bar{v}_2 \frac{1}{2}\sin\alpha\, d\alpha\pi D^2 v_0\, dt \tag{9}$$

(Clearly, if dt is sufficiently small, there is no overlap in this volume, as the spheres are all moving with the same velocity)

Defining δ as the density of the spheres in the total volume (which we will call V), specifically for the spheres of the first group we have

$$\delta_{v_1} = \frac{1}{V}\, Nf(\bar{v}_1)\, d\bar{v}_1 \frac{d\Omega}{4\pi} \tag{10}$$

We then expect a number of collisions (cases in which spheres from the first group are located within the volume in space covered by the “collision regions” of the spheres from the second group, as saw before) of

$$\delta_{v_1} Nf(\bar{v}_2)\, d\bar{v}_2 \frac{1}{2}\sin\alpha\, d\alpha\pi D^2 v_0\, dt = \frac{1}{V}\, Nf(\bar{v}_1)\, d\bar{v}_1 \frac{d\Omega}{4\pi} Nf(\bar{v}_2)\, d\bar{v}_2 \frac{1}{2}\sin\alpha\, d\alpha\pi D^2 v_0\, dt \tag{11}$$

Integrating over the solid angle $d\Omega$ and rearranging a bit the terms, it's immediate that

$$dcollision_{\bar{v}_1,\bar{v}_2,\alpha} = \frac{N^2}{V}\pi D^2\, f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha\, v_0 d\bar{v}_1 d\bar{v}_2 d\alpha\, dt \tag{12}$$

So that the "rate" of collisions over time, considering that N, V and D are fixed, is proportional to

$$\frac{dcollision_{\bar{v}_1,\bar{v}_2,\alpha}}{dt} \propto\, f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha\, v_0 d\bar{v}_1 d\bar{v}_2 d\alpha \tag{13}$$

If v_0 = 0, for instance, collisions are not possible because the 2 group of spheres are not moving relatively to each other!

We can therefore define a joint pdf for the distribution of the frequency of the collisions as

$$f_{collision|\bar{v}_1,\bar{v}_2,\alpha} = K\, f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha v_0 \tag{14}$$

Where K ensures that the joint pdf has an overall integral equal to 1

$$K = \frac{1}{\int_0^{\pi}\int_0^{\infty}\int_0^{\infty} f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha v_0 d\bar{v}_1 d\bar{v}_2 d\alpha} \tag{15}$$

Calculating the speed distribution of the collisions

Clearly $\bar{v}_1$ and $\bar{v}_2$ are completely interchangeable in the reasoning above, and it is easy to see that $f_{collisions|\bar{v}_1,\bar{v}_2,\alpha} = f_{collisions|\bar{v}_2,\bar{v}_1,\alpha}$.

If we therefore integrate $f_{collisions|\bar{v}_1,\bar{v}_2,\alpha}$ over all the possible values of $\bar{v}_2$ and α, we will get the $\bar{v}_1$ speed distribution of the colliding spheres, which is also equal to the $\bar{v}_2$ speed distribution of the colliding spheres, so that for simplicity we can just call it the speed distribution of all the colliding spheres, naming that speed $\bar{v}$

$$f_{collision|\bar{v}}(\bar{v}) = \int_0^{\infty}\int_0^{\pi} f_{collision|\bar{v}_1,\bar{v}_2,\alpha}\, d\alpha d\bar{v}_2 \tag{16}$$

$$f_{collision|\bar{v}}(\bar{v}) = \int_0^{\infty}\int_0^{\pi} K\, f(\bar{v}_1)\, f(\bar{v}_2)\frac{1}{2}\sin\alpha v_0\, d\alpha d\bar{v}_2 \tag{17}$$

Note that the order of integrals has been rearranged.

Calculating the speed distribution after the collisions

Let's call $\tilde{v}_1$ and $\tilde{v}_2$ the speed of the spheres after the collision.

Considering all the possible configurations of the impact, and the facts that the collisions are elastic (momentum and kinetic energy are conserved) it is possible to show [1] that, for given $\bar{v}_1, \bar{v}_2$ and α, both $\tilde{v}_1$ and $\tilde{v}_2$ are equally distributed, and their probability distribution function (within the boundary of values that are possible to be reached starting from a collision of spheres of speed $\bar{v}_1, \bar{v}_2$ with an $angle\ \alpha$, due to momentum and kinetic energy conservation considerations. [1] provides also such boundaries) is

$$f_{\tilde{v}_1|\bar{v}_1,\bar{v}_2,\alpha} = \frac{\tilde{v}_1}{2r_1r_2} \tag{18}$$

$$f_{\tilde{v}_2|\bar{v}_1,\bar{v}_2,\alpha} = \frac{\tilde{v}_2}{2r_1r_2} \tag{19}$$

Where

$$r_1 = \frac{1}{2}\sqrt{\bar{v}_1^2 + \bar{v}_2^2 \ + \ 2\bar{v}_1\bar{v}_2\cos\alpha} \tag{20}$$

$$r_2 = \frac{1}{2}\sqrt{\bar{v}_1^2 + \bar{v}_2^2 \ - \ 2\bar{v}_1\bar{v}_2\cos\alpha} \tag{21}$$

As $\tilde{v}_1$ and $\tilde{v}_2$ are equally distributed, this represents also the distribution of the speed $\tilde{v}$ of all the spheres after collision, so that

$$f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} = \begin{cases} \frac{\tilde{v}}{2r_1r_2} & within\ the\ boundaries \\ 0 & elsewhere \end{cases} \tag{22}$$

As we saw before, each type of collision has a different probability, dependent not only on the probability density function of $\bar{v}_1, \bar{v}_2, \alpha$ but also on the relative speed v_0.

If we consider the product

$$f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} f_{collision|\bar{v}_1,\bar{v}_2,\alpha} = f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} K\, f(\bar{v}_1)\, f(\bar{v}_2)\, \frac{1}{2}\sin\alpha v_0 \tag{23}$$

This will give us the joint distribution of the speed of the spheres are the collision.

If we fix $\tilde{v}$ and integrate it over all the possible values of $\bar{v}_1, \bar{v}_2, \alpha$, we will get the speed distribution of the spheres after the collision

$$f_{new|\tilde{v}}(\tilde{v}) = \int_0^\infty \int_0^\infty \int_0^\pi f_{\tilde{v}|\bar{v}_1,\bar{v}_2,\alpha} K\, f(\bar{v}_1)\, f(\bar{v}_2)\, \frac{1}{2}\sin\alpha v_0\, d\alpha d\bar{v}_1 d\bar{v}_2 \tag{24}$$

<u>Mechanical proof</u>

Both the speed distribution of the colliding spheres ($f_{collision|\bar{v}}(\bar{v})$) and the speed distribution of the spheres after the colisions ($f_{new|\tilde{v}}(\tilde{v})$) depend on the speed distribution of the population of spheres ($f(v)$, which is assumed to be according to Maxwell-Boltzmann).

Every collision, 2 speeds are "disappearing" from the distribution $f(v)$ (the ones of the spheres colliding), and 2 new speeds are "created" and are "joining" the distribution $f(v)$ (the ones of the spheres after the collision).

If the speeds that are "created" / "joining" $f(v)$ exactly compensate the ones that are "disappearing" from $f(v)$, then the speed distribution of the population of spheres $f(v)$ will be stable.

After some steps (attaching below the calculations) it is possible to show that this is indeed the case. Given

$$F(v) = \int_0^v f(x)\, dx \tag{25}$$

We get

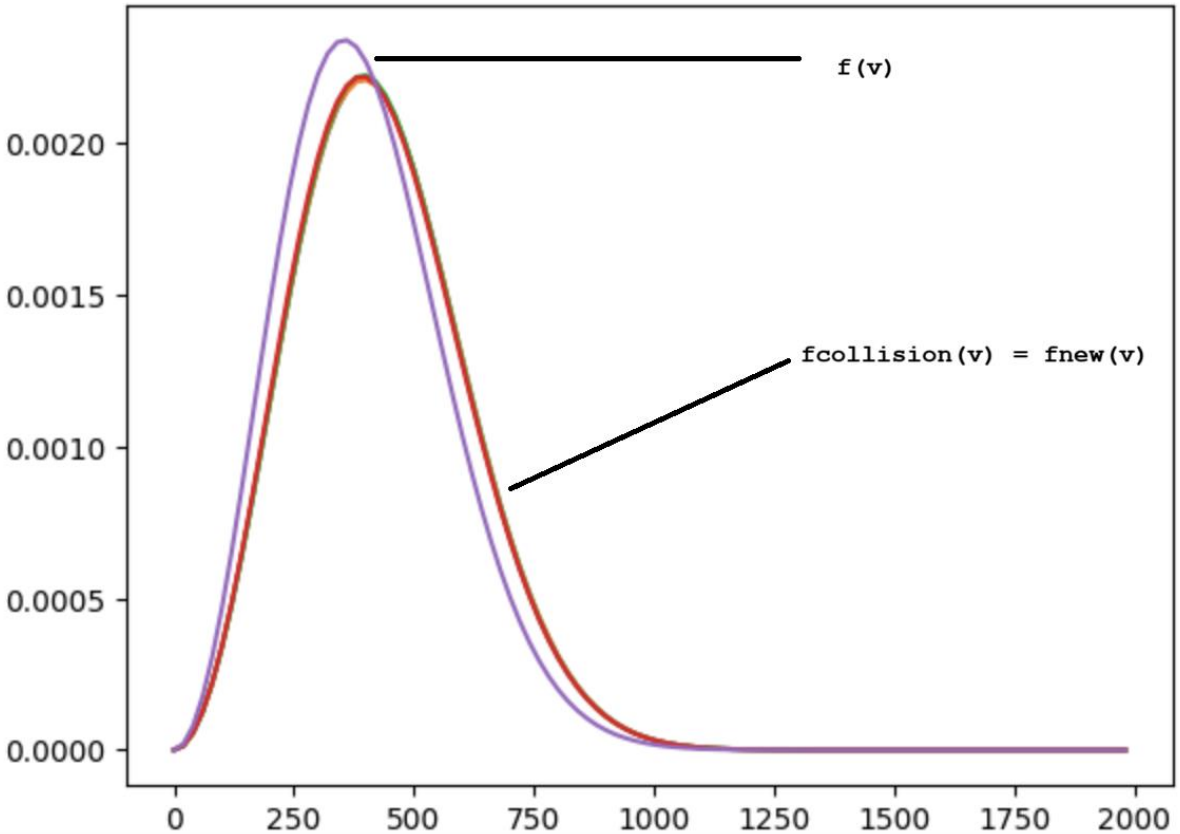
$$f_{collision|\bar{v}}(\bar{v}) \; = \; Kf(\bar{v})\left[\bar{v}F(\bar{v}) + \frac{\pi\langle v\rangle^2}{8}\frac{F(\bar{v})}{\bar{v}} + \frac{\pi^2\langle v\rangle^4}{32}\frac{f(\bar{v})}{\bar{v}^2} + \frac{\pi\langle v\rangle^2}{8}f(\bar{v})\right] \tag{26}$$

$$f_{new|\tilde{v}}(\tilde{v}) \; = \; Kf(\tilde{v})\left[\tilde{v}F(\tilde{v}) + \frac{\pi\langle v\rangle^2}{8}\frac{F(\tilde{v})}{\tilde{v}} + \frac{\pi^2\langle v\rangle^4}{32}\frac{f(\tilde{v})}{\tilde{v}^2} + \frac{\pi\langle v\rangle^2}{8}f(\tilde{v})\right] \tag{27}$$

So that for a given speed $v = \bar{v} = \tilde{v}$ indeed

$$f_{collision|\bar{v}}(v) = f_{new|\tilde{v}}(v) \tag{28}$$

Below a plot of $f(v)$ and $f_{collision|\bar{v}}(v) = f_{new|\tilde{v}}(v)$ for $\langle v\rangle = 400\ m/s$



<u>References</u>

[1] Mechanical Proof of the Maxwell Speed Distribution Tsung-Wu Lin & Hejie Lin

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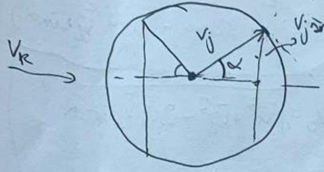
$V_K \rightarrow$



$\pi\left[2(v_j \sin\alpha)\, v_j\, d\alpha\right] =$

$= 2\pi v_j^2 \sin\alpha\, d\alpha$

$\int_0^{\pi} 2\pi v_j^2 \sin\alpha\, d\alpha = -2\cos\alpha \Big|_0^{\pi} \pi v_j^2$

$= 4\pi v_j^2$

$V_{rel} = \sqrt{(v_j\cos\alpha - V_K)^2 + (v_j \sin\alpha)^2} =$

$= \sqrt{v_j^2 + V_K^2 - 2 v_j V_K \cos\alpha}$

$\int_0^{\pi} |v_j \cos\alpha - V_K| \cdot 2\pi V_j^2 \sin\alpha\, d\alpha =$

$= 2\pi v_j^3 \int_0^{\pi} \cos\alpha \sin\alpha\, d\alpha - 2\pi V_K v_j^2 \int_0^{\pi} \sin\alpha\, d\alpha =$ ($\frac{1}{2}\sin 2\alpha$)

$= \pi v_j^3 \left(-\frac{1}{2}\cos 2\alpha\right) - 2\pi V_K$

$\cos 2\pi - \cos 0$

$= 1 \quad 1$

$v_j \cos\alpha - V_K = \varnothing$

$\cos\alpha = \frac{V_K}{V_j}$

$V_j < V_K$

$v_j \cos\alpha - V_K$
 < 0

$V_j = [\quad\quad]$ $100 = 10^2$

$V_K = [\quad\quad]$ 100

$\alpha = [\quad\quad]$ 100

$f[v_j, v_K, \alpha] =$ 10^6

v_j dist, v_K dist $\} \longrightarrow V_{dist} = \frac{v_j\,dist + v_K\,dist}{2}$

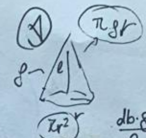
V_j' count

V_j'

V_K' count

V_j' dist, V_K' dist $\} \longrightarrow V'_{dist} = \frac{V_j'\,dist + V_K'\,dist}{2}$

(1)



$\frac{db \cdot g}{2}$

$\frac{2\pi r}{2} g = \pi r g$

$\frac{g - v_j d\alpha}{r - dr} = \frac{g}{r}$

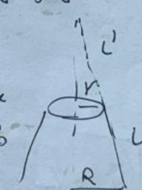
$\cancel{gr} - v_j r\, d\alpha = \cancel{gr} - g\, dr$

$g\, dr = v_j r\, d\alpha$

$\boxed{dr = \frac{v_j r}{g} d\alpha} = \frac{v_j r}{v_j \operatorname{tg}\alpha} d\alpha = \boxed{\frac{r}{\operatorname{tg}\alpha} d\alpha}$

$g \cos\alpha = v_j \sin\alpha$

$g = v_j \operatorname{tg}\alpha$



$\pi R(L+L') - \pi r L'$

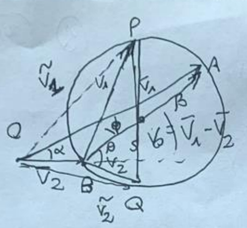
$\frac{L'}{r} = \frac{L+L'}{R} \rightarrow \frac{L'/L}{r} = \frac{1 + L'/L}{R}$

$\pi L\left(R\left(1 + \frac{L'}{L}\right) - r\frac{L'}{L}\right)$

$R\frac{L'}{L} = r + r\frac{L'}{L} \rightarrow (R-r)\frac{L'}{L} = r$

$\pi L\left(R\left(1 + \frac{r}{R-r}\right) - r\frac{r}{R-r}\right)$; $\frac{L'}{L} = \frac{r}{R-r}$

$= \pi L \frac{R^2 - Rr + Rr - r^2}{R-r} = \pi L(R+r)$



$$\vec{V_0} = \vec{\bar{V}_1} - \vec{\bar{V}_2}$$

$$V_2 = V_0 \cos\vartheta$$
$$V_1 = V_0 \sin\vartheta$$

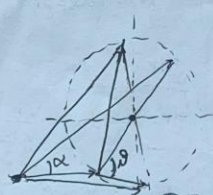
$$\vec{\tilde{V}_2} = \vec{\bar{V}_2} + \vec{V_2}$$

$$\vec{\tilde{V}_1} = \vec{\bar{V}_2} + \vec{V_1}$$

$$\cos\beta = \frac{\bar{V}_1 \cos\alpha - \bar{V}_2}{V_0} \quad \sin\beta = \frac{\bar{V}_1 \sin\alpha}{V_0}$$ (13)

$$V_0 = \sqrt{(\bar{V}_1 \cos\alpha - \bar{V}_2)^2 + (\bar{V}_1 \sin\alpha)^2}$$

$$\boxed{\begin{aligned} V_0 &= \sqrt{\bar{V}_1^2 + \bar{V}_2^2 - 2\bar{V}_1 \bar{V}_2 \cos\alpha} \\ V_2 &= V_0 \cos\vartheta \\ V_1 &= V_0 \sin\vartheta \end{aligned}}$$



$$V_{2 /\!/ V_0} = V_2 \cos\vartheta$$
$$V_{2 \perp V_0} = V_2 \sin\vartheta$$

$$V_{2 /\!/ V_0 /\!/ \bar{V}_2} = V_{2 /\!/ V_0} \cos\beta$$
$$V_{2 /\!/ V_0 \perp \bar{V}_2} = V_{2 /\!/ V_0} \sin\beta$$
$$V_{2 \perp V_0 /\!/ \bar{V}_2} = V_{2 \perp V_0} \sin\beta$$
$$V_{2 \perp V_0 \perp \bar{V}_2} = V_{2 \perp V_0} \cos\beta$$

$$\boxed{\begin{aligned} V_{2 /\!/ \bar{V}_2} &= V_2 \cos\vartheta \cos\beta + V_2 \sin\vartheta \sin\beta \\ V_{2 \perp \bar{V}_2} &= V_2 \cos\vartheta \sin\beta - V_2 \sin\vartheta \cos\beta \end{aligned}}$$ $\to \sin(\beta - \vartheta)$

$-\sin(\vartheta - \beta)$

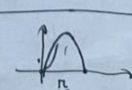
$$\tilde{V}_2 = \sqrt{(\bar{V}_2 + V_{2 /\!/ \bar{V}_2})^2 + V_{2 \perp \bar{V}_2}^2} = \sqrt{\bar{V}_2^2 + V_2^2 \cos^2\vartheta \cos^2\beta + V_2^2 \sin^2\vartheta \sin^2\beta + 2V_2^2 \cos\vartheta \sin\vartheta \cos\beta \sin\beta + V_2^2 \cos^2\vartheta \sin^2\beta + V_2^2 \sin^2\vartheta \cos^2\beta - 2V_2^2 \cos\vartheta \sin\vartheta \cos\beta \sin\beta}$$

$$V_2^2 \cos^2\vartheta + V_2^2 \sin^2\vartheta = V_2^2$$

$$+ 2\bar{V}_2 V_2 \cos\vartheta \cos\beta + 2\bar{V}_2 \sin\vartheta \sin\beta$$

$$\boxed{\tilde{V}_2 = \sqrt{\bar{V}_2^2 + V_2^2 + 2\bar{V}_2 V_2 (\cos\vartheta \cos\beta + \sin\vartheta \sin\beta)}}$$

$$\boxed{P(\alpha) = \frac{1}{2} \sin\alpha}$$



$$P_\vartheta(\vartheta)\, d\vartheta = \frac{2\pi V_0 \cos\vartheta \sin\vartheta\, V_0\, d\vartheta}{\pi V_0^2} = \sin(2\vartheta)\, d\vartheta$$

$1 - 2F(\vartheta)$

$$\vartheta = \frac{1}{2} a\cos(1 - 2F(\vartheta)) \quad F(\vartheta) = \int_0^\vartheta \sin(2\vartheta)\, d\vartheta = -\frac{1}{2}\cos 2\vartheta + \frac{1}{2} \quad \boxed{\frac{1}{2}(1 - \cos 2\vartheta)}$$

$$\int_0^{\frac{\pi}{2}} \sin(2\vartheta)\, d\vartheta = -\frac{1}{2}\cos 2\vartheta \Big|_0^{\frac{\pi}{2}} = -\frac{1}{2}(-1 - 1) = 1$$

$$f(v)=\left[\frac{m}{2\pi K_B T}\right]^{3/2} 4\pi v^2 e^{-\frac{mv^2}{2K_BT}} = \left[\frac{4\pi m}{\pi^2\, 8K_BT}\right]^{3/2} 4\pi v^2 e^{-\frac{\pi m}{8K_BT}\frac{4}{\pi}v^2} =$$

$$= \frac{8}{\pi^{3/2}}\cdot\frac{1}{\langle v\rangle^3}\, 4\pi v^2 e^{-\frac{4}{\pi}\left(\frac{v}{\langle v\rangle}\right)^2}$$

(C)

mean speed $\langle v\rangle = \sqrt{\frac{8K_BT}{\pi m}}$

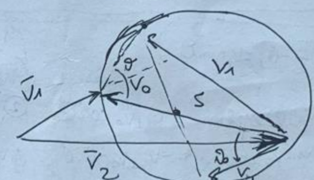
$$\boxed{f(v)= \frac{32\cdot v^2}{\pi^2\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{v}{\langle v\rangle}\right)^2}}$$

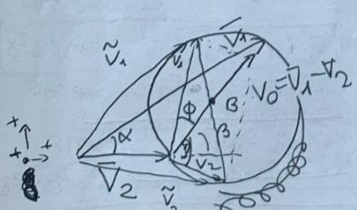
$\langle V\rangle = 400 m/s$

$100\cdot 100\cdot 100\cdot 100$

$100\cdot 10\cdot 10\cdot 10 = 100000$ $\quad 10^3$

$\frac{1}{1000}$





$$\boxed{V_{2\parallel V_0}=V_2\cos\theta}$$

$V_{2\perp V_0}=V_2\sin\theta$

$V_{2\perp V_0 \parallel plane} = V_{2\perp V_0}\cos\gamma$

(5) $V_{2\perp V_0 \perp plane} = V_{2\perp V_0}\sin\gamma$

(1) $V_{2\parallel V_0 \parallel \bar{V}_2} = V_{2\parallel V_0}\cos\beta$

(3) $V_{2\parallel V_0 \perp \bar{V}_2 \parallel plane} = V_{2\parallel V_0}\sin\beta$

(2) $V_{2\perp V_0 \parallel plane \parallel \bar{V}_2} = V_{2\perp V_0 \parallel plane}\sin\beta$

(4) $V_{2\perp V_0 \parallel plane \perp \bar{V}_2 \parallel plane} = V_{2\perp V_0 \parallel plane}\cos\beta$

$\tilde{V}_{2\parallel\bar{V}_2} = \overset{\bar{V}_2+(1)}{V_2\cos\theta\cos\beta} + \overset{(2)}{V_2\sin\theta\sin\beta\cos\gamma}$

$\tilde{V}_{2\perp\bar{V}_2 \parallel plane} = \overset{(3)}{V_2\cos\theta\sin\beta} - \overset{(4)}{V_2\sin\theta\cos\beta\cos\gamma}$

$\tilde{V}_{2\perp\bar{V}_2 \perp plane} = V_2\sin\theta\sin\gamma$

$\tilde{V}_2 = \sqrt{V_2^2\cos^2\theta\cos^2\beta + V_2^2\sin^2\theta\sin^2\beta\cos^2\gamma + 2V_2^2\cos\theta\sin\theta\cos\beta\sin\beta\cos\gamma + V_2^2\cos^2\theta\sin^2\beta + V_2^2\sin^2\theta\cos^2\beta\cos^2\gamma - 2V_2^2\cos\theta\sin\theta\cos\beta\sin\beta\cos\gamma + V_2^2\sin^2\theta\sin^2\gamma + \bar{V}_2^2 + 2\bar{V}_2V_2\cos\theta\cos\beta + 2\bar{V}_2V_2\sin\theta\sin\beta\cos\gamma}$

$\sqrt{V_2^2\cos^2\theta + V_2^2\sin^2\theta\cos^2\gamma + V_2^2\sin^2\theta\sin^2\gamma}$

$\sqrt{V_2^2+\bar{V}_2^2+2\bar{V}_2V_2\left(\cos\theta\cos\beta+\frac{\sin\theta}{\sin\beta}\cos\gamma\right)}$

(D)

$$\tilde{V}_{2 /\!/ \bar{V}_2} = \bar{V}_2 + V_2 \cos\vartheta \cos\beta + V_2 \sin\vartheta \sin\beta \cos\gamma$$

$$\tilde{V}_{2 \perp \bar{V}_2 /\!/ \text{plane}} = V_2 \cos\vartheta \sin\beta - V_2 \sin\vartheta \cos\beta \cos\gamma$$

$$\tilde{V}_{2 \perp \bar{V}_2 \perp \text{plane}} = V_2 \sin\vartheta \sin\gamma$$

$$\tilde{V}_2 = \sqrt{\bar{V}_2^2 + V_2^2 \cos^2\vartheta \cos^2\beta + V_2^2 \sin^2\vartheta \sin^2\beta \cos^2\gamma + 2\bar{V}_2 V_2 \cos\vartheta \cos\beta + 2\bar{V}_2 V_2 \sin\vartheta \sin\beta \cos\gamma \cancel{+ 2V_2^2 \cos\vartheta \sin\vartheta \cos\beta \sin\beta \cos\gamma} + V_2^2 \cos^2\vartheta \sin^2\beta + V_2^2 \sin^2\vartheta \cos^2\beta \cos^2\gamma \cancel{- 2V_2^2 \cos\vartheta \sin\vartheta \cos\beta \sin\beta \cos\gamma} + V_2^2 \sin^2\vartheta \sin^2\gamma} =$$

$$= \sqrt{\bar{V}_2^2 + V_2^2 \cos^2\vartheta + V_2^2 \sin^2\vartheta \cos^2\gamma + 2\bar{V}_2 V_2 (\cos\vartheta \cos\beta + \sin\vartheta \sin\beta \cos\gamma) + V_2^2 \sin^2\vartheta \sin^2\gamma} =$$

$$\boxed{\tilde{V}_2 = \sqrt{\bar{V}_2^2 + V_2^2 + 2\bar{V}_2 V_2 (\cos\vartheta \cos\beta + \sin\vartheta \sin\beta \cos\gamma)}}$$

$$V_{1 /\!/ V_0} = V_1 \cos\phi = V_1 \cos\left(\frac{\pi}{2} - \vartheta\right) = V_1 \sin\vartheta$$

$$V_{1 \perp V_0} = V_1 \cos\vartheta \quad \begin{cases} V_{1 \perp V_0 /\!/ \text{plane}} = V_{1 \perp V_0} \cos\gamma \\ V_{1 \perp V_0 \perp \text{plane}} = -V_{1 \perp V_0} \sin\gamma \end{cases}$$

① $V_{1 /\!/ V_0 /\!/ V_2} = V_{1 /\!/ V_0} \cos\beta$

③ $V_{1 /\!/ V_0 \perp V_2 /\!/ \text{plane}} = V_{1 /\!/ V_0} \sin\beta$

② $V_{1 \perp V_0 /\!/ \text{plane} /\!/ \bar{V}_2} = -V_{1 \perp V_0 /\!/ \text{plane}} \cdot \sin\beta$

④ $V_{1 \perp V_0 /\!/ \text{plane} \perp \bar{V}_2 /\!/ \text{plane}} = V_{1 \perp V_0 /\!/ \text{plane}} \cos\beta$

$$\tilde{V}_{1 /\!/ \bar{V}_2} = \bar{V}_2 \overset{①}{+} V_1 \sin\vartheta \cos\beta \overset{②}{-} V_1 \cos\vartheta \sin\beta \cos\gamma$$

$$\tilde{V}_{1 \perp \bar{V}_2 /\!/ \text{plane}} = \overset{③}{V_1 \sin\vartheta \sin\beta} \overset{④}{+} V_1 \cos\vartheta \cos\beta \cos\gamma$$

$$\tilde{V}_{1 \perp \bar{V}_2 \perp \text{plane}} = -V_1 \cos\vartheta \sin\gamma$$

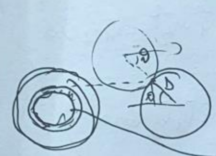
$$\tilde{V}_1 = \sqrt{\bar{V}_2^2 + V_1^2 \sin^2\vartheta \cos^2\beta + V_1^2 \cos^2\vartheta \sin^2\beta \cos^2\gamma + 2\bar{V}_2 V_1 \sin\vartheta \cos\beta - 2\bar{V}_2 V_1 \cos\vartheta \sin\beta \cos\gamma \cancel{- 2V_1^2 \sin\vartheta \cos\vartheta \sin\beta \cos\beta \cos\gamma} + V_1^2 \sin^2\vartheta \sin^2\beta + V_1^2 \cos^2\vartheta \cos^2\beta \cos^2\gamma \cancel{+ 2V_1^2 \sin\vartheta \cos\vartheta \sin\beta \cos\beta \cos\gamma} + V_1^2 \cos^2\vartheta \sin^2\gamma}$$

$$\boxed{\tilde{V}_1 = \sqrt{\bar{V}_2^2 + V_1^2 + 2\bar{V}_2 V_1 (\sin\vartheta \cos\beta - \cos\vartheta \sin\beta \cos\gamma)}}$$

cons. momentum

$/\!/ \bar{V}_2$: $\cancel{\bar{V}_2 + \bar{V}_1 \cos\alpha = \bar{V}_2 + V_2 (\cos\vartheta \cos\beta + \sin\vartheta \sin\beta \cos\gamma)}$

E



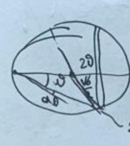
$\Delta\sin\vartheta$ $\qquad 2\pi\Delta\sin\vartheta\,\Delta\cos\vartheta\,d\vartheta$ $\qquad \Delta\sin(\vartheta+d\vartheta)$

$\Delta\sin\vartheta\cos d\vartheta + \cancel{\cos\vartheta\sin d\vartheta}$

$\Delta\cos\vartheta\, d\vartheta$

$$\boxed{2\pi\Delta^2\sin\vartheta\cos\vartheta\,d\vartheta}$$

$1-\frac{d\vartheta^2}{2}$



$2d\vartheta$

$$2\pi\left(\frac{V_0}{2}\sin 2\vartheta\right)\left(\frac{V_0}{2}2d\vartheta\right) = \pi V_0^2\sin(2\vartheta)\,d\vartheta =$$

$$\boxed{= 2\pi V_0^2\sin\vartheta\cos\vartheta\,d\vartheta}$$

$$\text{ratio} = \left(\frac{V_0}{D}\right)^2 \text{ not } f(\vartheta)$$

$$\frac{\pi V_0^2\sin(2\vartheta)\,d\vartheta}{4\pi\left(\frac{V_0}{2}\right)^2 \rightarrow \pi V_0^2} =$$

$f_\vartheta = \sin(2\vartheta)\,d\vartheta$

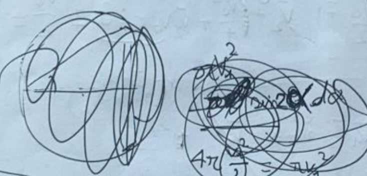
$F = \int_0^\vartheta \sin(2\vartheta)\,d\vartheta = -\frac{1}{2}\cos(2\vartheta)\Big|_0^\vartheta =$

$= -\frac{1}{2}\cos 2\vartheta + \frac{1}{2}\cos 0 =$

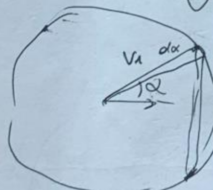
$$\boxed{F = \frac{1}{2}(1-\cos 2\vartheta)}$$

$2F = 1-\cos 2\vartheta \qquad \cos 2\vartheta = 1-2F$

$\vartheta = \frac{1}{2}\arccos(1-2F)$



$\sin 2\alpha\, d\alpha$



$2\pi V_1\sin\alpha\, V_1\, d\alpha =$

$$\boxed{= 2\pi V_1^2\sin\alpha\,d\alpha}$$

$$\rightarrow \frac{2\pi V_1^2\sin\alpha\,d\alpha}{4\pi V_1^2} \qquad \boxed{f_\alpha = \frac{1}{2}\sin\alpha\,d\alpha}$$

---

so that $\int_{d\bar{V}_1, d\bar{V}_2, d\alpha} = 1$

$$\boxed{f_{\text{collision}}(\bar{V}_1,\bar{V}_2,\alpha) = K\cdot f(\bar{V}_1)*f(\bar{V}_2)*\frac{1}{2}\sin(\alpha)*V_0*d\bar{V}_1*d\bar{V}_2*d\alpha}$$

$$\boxed{\begin{aligned} f_\vartheta(\vartheta) &= \sin(2\vartheta)\,d\vartheta \\ f_\gamma(\gamma) &= \frac{1}{2\pi}\,d\gamma \end{aligned}} \Longrightarrow \boxed{f_\omega(\omega) = \frac{1}{2}\sin\omega\,d\omega} \qquad \boxed{f_{\tilde{V}_1}(\tilde{V})\Big/_{\alpha,\bar{V}_1,\bar{V}_2} = \frac{\tilde{V}_1}{2V_1V_2}}$$

$$\boxed{\tilde{V}_1 = \tilde{V}_1(\bar{V}_1, \bar{V}_2, \alpha, \omega)}$$

$$\boxed{V_0 = \sqrt{\bar{V}_1^2 + \bar{V}_2^2 - 2\bar{V}_1\bar{V}_2\cos\alpha}}$$

assuming

$$r_1(\alpha,\bar v_1,\bar v_2) = \frac{1}{2}\sqrt{\bar v_1^2+\bar v_2^2+2\bar v_1\bar v_2\cos\alpha}$$

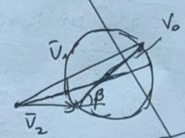
F

$$P_{V1\alpha}(V) = \frac{V}{2r_1r_2} =$$

$$r_2 = \frac{1}{2}\sqrt{\bar v_1^2+\bar v_2^2-2\bar v_1\bar v_2\cos\alpha}$$

$$\boxed{P_{V1\alpha}(V) = \frac{2V}{\sqrt{(\bar v_1^2+\bar v_2^2)^2-4\bar v_1^2\bar v_2^2\cos^2\alpha}}}$$

assume it's correct!



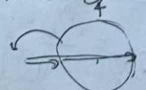
$$V_{max/min} = \sqrt{\left(\bar v_2+\frac{v_0}{2}\cos\beta\right)^2+\left(\frac{v_0}{2}\sin\beta\right)^2} \pm \frac{v_0}{2} = \sqrt{\bar v_2^2+\frac{v_0^2}{4}+\bar v_2\underbrace{v_0\cos\beta}_{\bar v_1\cos\alpha-\bar v_2}} \pm \frac{v_0}{2} =$$

$$= \sqrt{\cancel{\bar v_2^2}+\frac{\bar v_1^2}{4}+\frac{\bar v_2^2}{4}-\frac{\bar v_1\bar v_2\cos\alpha}{2}+\bar v_1\bar v_2\cos\alpha-\cancel{\bar v_2^2}} \pm \frac{v_0}{2}$$

$$V_{max/min} = \sqrt{\frac{\bar v_1^2}{4}+\frac{\bar v_2^2}{4}-\frac{\bar v_1\bar v_2\cos\alpha}{2}}$$

$$\boxed{V_{max/min} = \sqrt{\frac{\bar v_1^2}{4}+\frac{\bar v_2^2}{4}+\frac{\bar v_1\bar v_2\cos\alpha}{2}} \pm \sqrt{\frac{\bar v_1^2}{4}+\frac{\bar v_2^2}{4}-\frac{\bar v_1\bar v_2\cos\alpha}{2}}}$$

$\alpha=0$ $\quad V_{max/min} = \sqrt{\frac{\bar v_1^2}{4}+\frac{\bar v_2^2}{4}+\frac{\bar v_1\bar v_2}{2}} \pm \sqrt{\ }$



$$\frac{1}{2}\sqrt{(\bar v_1+\bar v_2)^2} \pm \frac{1}{2}\sqrt{(\bar v_1-\bar v_2)^2} = \frac{1}{2}(\bar v_1+\bar v_2) \pm \frac{1}{2}|\bar v_1-\bar v_2|$$

$$\boxed{V_{max/min} = \frac{1}{2}\sqrt{\bar v_1^2+\bar v_2^2+2\bar v_1\bar v_2\cos\alpha} \pm \frac{1}{2}\sqrt{\bar v_1^2+\bar v_2^2-2\bar v_1\bar v_2\cos\alpha}}$$

$$f(v) = \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} 4\pi v^2 e^{-\frac{mv^2}{2k_BT}} \qquad \langle v\rangle = \sqrt{\frac{8k_BT}{\pi m}}$$

$$f(v) = \left(\frac{4}{\pi^2}\frac{\pi m}{8k_BT}\right)^{\frac{3}{2}} 4\pi v^2 e^{-\frac{\pi m}{8k_BT}\frac{4v^2}{\pi}}$$

$$\boxed{f(v) = \frac{32}{\pi^2}\frac{v^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{v}{\langle v\rangle}\right)^2}}$$

$$\lim_{\alpha\to\pi}\frac{\sin\alpha}{\sqrt{2}\bar{v}_1\sqrt{1+\cos\alpha}}$$

$$\sqrt{\lim_{\alpha\to 0}\frac{\sin^2\alpha}{1-\cos\alpha}} \qquad \lim_{\alpha\to 0}\frac{\sin\alpha}{\sqrt{2}\bar{v}_1\sqrt{1-\cos\alpha}}$$

$$\sqrt{\lim_{\alpha\to 0}\frac{2\sin\alpha\cos\alpha}{\sin\alpha}} = \sqrt{2}$$

$(a-b)(a^2+ab+b^2)$

$a^3 - \cancel{a^2b} + \cancel{a^2b} - \cancel{ab^2} + \cancel{ab^2} - b^3$

$$\boxed{A = B\cdot\bar{v}_2^2 e^{-c\bar{v}_2^2}}$$

$$\frac{\sqrt{2}}{\sqrt{2}\bar{v}_1} = \frac{1}{\bar{v}_1}$$

$$K f(\bar{v}_1) f(\bar{v}_2)\cdot\frac{1}{2}\sin(\alpha)\cdot v_0 =$$

$$= \boxed{K\frac{32}{\pi^2}\frac{\bar{v}_1^2\bar{v}_2^2}{\langle v\rangle^6} e^{-\frac{4}{\pi}\frac{\bar{v}_1^2+\bar{v}_2^2}{\langle v\rangle^2}}\cdot\frac{1}{2}}\sin(\alpha)\cdot\sqrt{\bar{v}_1^2+\bar{v}_2^2-2\bar{v}_1\bar{v}_2\cos\alpha}$$

A

$$\int_0^\infty\int_0^\pi A\sin(\alpha)\sqrt{\bar{v}_1^2+\bar{v}_2^2-2\bar{v}_1\bar{v}_2\cos\alpha}\,d\alpha\,d\bar{v}_2 =$$

$(\bar{v}_1+\bar{v}_2 - |\bar{v}_1-\bar{v}_2|)\left((\bar{v}_1+\bar{v}_2)^2 + (\bar{v}_1+\bar{v}_2)|\bar{v}_1-\bar{v}_2| + |\bar{v}_1-\bar{v}_2|^2\right)$

$(\bar{v}_1+\bar{v}_2)^3 - |\bar{v}_1-\bar{v}_2|^3$

$$= \int_0^\infty A\left[\frac{2}{3\cdot 2\bar{v}_1\bar{v}_2}\left(\bar{v}_1^2+\bar{v}_2^2-2\bar{v}_1\bar{v}_2\cos\alpha\right)^{\frac{3}{2}}\right]_0^\pi d\bar{v}_2 = \int_0^\infty\frac{A}{3\bar{v}_1\bar{v}_2}\left[\left(\bar{v}_1^2+\bar{v}_2^2+2\bar{v}_1\bar{v}_2\right)^{\frac{3}{2}} - \left(\bar{v}_1^2+\bar{v}_2^2-2\bar{v}_1\bar{v}_2\right)^{\frac{3}{2}}\right]d\bar{v}_2$$

$\bar{v}_1 > \bar{v}_2$ $\quad(\cancel{\bar{v}_1}+\bar{v}_2-\cancel{\bar{v}_1}+\bar{v}_2)(\bar{v}_1^2+\bar{v}_2^2+2\bar{v}_1\bar{v}_2$

$+\bar{v}_1^2-\bar{v}_2^2$

$+\bar{v}_1^2+\bar{v}_2^2-2\bar{v}_1\bar{v}_2)$

$(2\bar{v}_2)$ $\quad(3\bar{v}_1^2+\bar{v}_2^2)$

$\bar{v}_2 > \bar{v}_1$ $\quad(\bar{v}_1+\cancel{\bar{v}_2}-\cancel{\bar{v}_2}+\bar{v}_1)(\bar{v}_1^2+2\bar{v}_1\bar{v}_2+\bar{v}_2^2+$

$+\bar{v}_2^2-\bar{v}_1^2$

$+\bar{v}_1^2+\bar{v}_2^2-2\bar{v}_1\bar{v}_2$

$(2\bar{v}_1)$ $\quad(3\bar{v}_2^2+\bar{v}_1^2)$

$$\int_0^{\bar{v}_1}\frac{A}{3\bar{v}_1\cancel{\bar{v}_2}}2\cancel{\bar{v}_2}(3\bar{v}_1^2+\bar{v}_2^2)\,dV_2 +$$

$$+\int_{\bar{v}_1}^{\infty}\frac{A}{3\cancel{\bar{v}_1}\bar{v}_2}2\cancel{\bar{v}_1}(3\bar{v}_2^2+\bar{v}_1^2)\,dV_2$$

$$\boxed{\int f'g = fg - \int fg'}$$

M $\quad\bar{v}_1^2+\bar{v}_2^2+2\bar{v}_1\bar{v}_2\cos\alpha = \emptyset$

$\cos\alpha < 0$

$\bar{v}_1^2+\bar{v}_2^2 = -2\bar{v}_1\bar{v}_2\cos\alpha$

$$\cos\alpha = -\frac{\frac{\bar{v}_1}{\bar{v}_2}+\frac{\bar{v}_2}{\bar{v}_1}}{2} \qquad -\frac{x+\frac{1}{x}}{2}$$

$$\frac{x+\frac{1}{x}}{2}\leq 1 \quad x>0$$

$x^2-2x+1\leq 0$

$(x-1)^2\leq 0$ $\quad\boxed{x=1}$

$\bar{v}_1=\bar{v}_2$

$\cos\alpha=-1$

only pos

$$\frac{\sin(\alpha)}{\sqrt{2}\bar{v}_1\sqrt{1+\cos\alpha}}$$

$$\int_0^{\bar{v}_1} \frac{B\bar{v}_2^{\cancel{2}} e^{-c\bar{v}_2^2}}{3\bar{v}_1 \cancel{\bar{v}_2}} \cdot 2\bar{v}_2 (3\bar{v}_1^2+\bar{v}_2^2)\, dV_2 = \int_0^{\bar{v}_1} \frac{2B\bar{v}_1^{\cancel{2}}\bar{v}_2^{-2}}{\cancel{3\bar{v}_1}} e^{-c\bar{v}_2^2}\, dV_2 + \quad \text{H}$$

$$+ \int_0^{\bar{v}_1} \frac{2B\bar{v}_2^4}{3\bar{v}_1} e^{-c\bar{v}_2^2}\, dV_2$$

$$\int_0^{\bar{v}_1} 2A\bar{v}_1\, dV_2 = \int_0^{\bar{v}_1} \cancel{2}\cdot K \frac{32}{\pi^2} \cdot \frac{\bar{v}_1^2\bar{v}_2^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{\bar{v}_1^2+\bar{v}_2^2}{\langle v\rangle^2}} \cdot \frac{1}{\cancel{2}} \cdot \bar{v}_1\, d\bar{v}_2 = \int_0^{\bar{v}_1} \frac{32}{\pi^2}\frac{\bar{v}_2^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar{v}_2}{\langle v\rangle}\right)^2} \cdot \frac{K\bar{v}_1^3}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar{v}_1}{\langle v\rangle}\right)^2} d\bar{v}_2 =$$

$$\frac{K\bar{v}_1^3}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar{v}_1}{\langle v\rangle}\right)^2} \int_0^{\bar{v}_1} f(\bar{v}_2)\, dV_2 + \frac{K\bar{v}_1}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar{v}_1}{\langle v\rangle}\right)^2} \int_0^{\bar{v}_1} \bar{v}_2^2 f(\bar{v}_2)\, d\bar{v}_2$$

$$\frac{\bar{v}_1}{\bar{v}_2} \cdot \frac{\bar{v}_2^2}{\bar{v}_1^2} = \frac{\bar{v}_2}{\bar{v}_1} \qquad \frac{\bar{v}_1}{\bar{v}_2} \quad \frac{\bar{v}_1}{\bar{v}_2^2}$$

$$\frac{K\bar{v}_1^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar{v}_1}{\langle v\rangle}\right)^2} \int_{\bar{v}_1}^{\infty} \bar{v}_2 f(\bar{v}_2)\, dV_2 + K\frac{\bar{v}_1^4}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar{v}_1}{\langle v\rangle}\right)^2} \int_{\bar{v}_1}^{\infty} \frac{1}{\bar{v}_2} f(\bar{v}_2)\, d\bar{v}_2$$

$$\bar{v}_1^2+\bar{v}_2^2 - 2\tilde{v}_1\sqrt{\bar{v}_1^2+\bar{v}_2^2-\tilde{v}_1^2} \quad + \bar{v}_1^2+\bar{v}_2^2+2\cancel{\tilde{v}_1}\sqrt{} = -2\sqrt{(\bar{v}_1^2+\bar{v}_2^2)^2 - 4\tilde{v}_1^2(\bar{v}_1^2+\bar{v}_2^2-\tilde{v}_1^2)}$$

$$= [illegible] = 2\bar{v}_1^2+2\bar{v}_2^2-2\sqrt{}$$

$$v < r_1 - r_2 \;\vee\; v < r_2 - v_1$$

$$v > r_1 + r_2 \qquad v > r_1 + r_2$$

$$v > \tfrac{1}{2}\sqrt{-+} + \tfrac{1}{2}\sqrt{} -$$

$$\tilde{v}_1^2 = r_1^2 + r_2^2 + 2r_1r_2\cos\omega$$

$$\tilde{v}_1^2 = \frac{1}{4}(\bar{v}_1^2+\bar{v}_2^2+2\bar{v}_1\bar{v}_2\cos\alpha) + \frac{1}{4}(\bar{v}_1^2+\bar{v}_2^2-2\bar{v}_1\bar{v}_2\cos\alpha) + \frac{1}{2}\sqrt{(\bar{v}_1^2+\bar{v}_2^2)^2 - 4\bar{v}_1^2\bar{v}_2^2\cos^2\alpha}\,\cos\omega$$

$$\tilde{v}_1^2 = \frac{1}{2}(\bar{v}_1^2+\bar{v}_2^2) + \frac{1}{2}\sqrt{(\bar{v}_1^2+\bar{v}_2^2)^2 - 4\bar{v}_1^2\bar{v}_2^2\cos^2\alpha}\,\cos\omega$$

$$\tilde{v}_1^4 + \frac{1}{4}(\bar{v}_1^2+\bar{v}_2^2)^2 - \tilde{v}_1^2(\bar{v}_1^2+\bar{v}_2^2) = \frac{1}{4}(\bar{v}_1^2+\bar{v}_2^2)^2\cos^2\omega - \bar{v}_1^2\bar{v}_2^2\cos^2\alpha\cos^2\omega$$

$$\bar{v}_1^2\bar{v}_2^2\cos^2\alpha\cos^2\omega = \tilde{v}_1^2(\bar{v}_1^2+\bar{v}_2^2) - \tilde{v}_1^4 - \frac{1}{4}(\bar{v}_1^2+\bar{v}_2^2)^2\sin^2\omega$$

$$\frac{-(b+c)\pm\sqrt{b^2+c^2-2bc}}{-2} = \frac{+(b+c)+|b-c|}{2}$$

$$\frac{\bar{v}_1^2+\bar{v}_2^2+|\bar{v}_1^2-\bar{v}_2^2|}{2}$$

$$\bar{v}_1 > \bar{v}_2$$

$$= \bar{v}_1^2 \quad v_{max}$$

$$\sin^2\omega = \emptyset$$

$$\cos^2\omega = 1$$

$$\omega = 0$$

$$\omega = \pi$$

$$\cos^2\alpha = \frac{\tilde{v}_1^2(\bar{v}_1^2+\bar{v}_2^2-\tilde{v}_1^2)}{\bar{v}_1^2\bar{v}_2^2}$$

$$\emptyset \div 1$$

$$\cos\alpha = \frac{\tilde{v}_1}{\bar{v}_1\bar{v}_2}\sqrt{\bar{v}_1^2+\bar{v}_2^2-\tilde{v}_1^2}$$

$$x_{1,2} = \frac{-(b+c)\pm\sqrt{(b+c)^2-4bc}}{-2}$$

$$-1 \le \frac{\tilde{v}_1}{\bar{v}_1\bar{v}_2}\sqrt{\bar{v}_1^2+\bar{v}_2^2-\tilde{v}_1^2} \le 1 \Rightarrow \frac{\tilde{v}_1^2}{\bar{v}_1^2\bar{v}_2^2}(\bar{v}_1^2+\bar{v}_2^2-\tilde{v}_1^2) \le 1$$

$$\frac{a}{bc}(b+c-a) \le 1$$

$$\frac{a}{bc}(b+c-a) = 1 \qquad -a^2+(b+c)a-bc = \emptyset$$

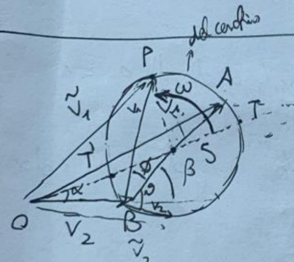
$\boxed{\int_0^x \frac{e^x}{x}\,dx = E_i(x)+c}$ $\boxed{\int_0^x \frac{e^{-x^2}}{x} = \frac{1}{2}E_i(-x^2)+c}$

$$\int_0^\infty \underbrace{x^m e^{-ax^2}}_{x^{m-1}\cdot xe^{-ax^2}}\,dx = \Big[\underbrace{x^{m-1}\left(-\tfrac{1}{2a}\right)e^{-ax^2}}_{f\cdot g}\Big]_0^\infty - \int_0^\infty \underbrace{(m-1)x^{m-2}}_{f'}\underbrace{\left(-\tfrac{1}{2a}\right)e^{-ax^2}}_{g}\,dx =$$

$f\cdot g'$

$$= \emptyset - \emptyset + \frac{m-1}{2a}\int_0^\infty x^{m-2}e^{-ax^2}\,dx$$

$$\boxed{\int_0^\infty x^m e^{-ax^2}\,dx = \frac{m-1}{2a}\int_0^\infty x^{m-2}e^{-ax^2}\,dx}$$



$$\boxed{\begin{aligned} r_1 &= \overline{OS} = \frac{1}{2}\sqrt{(\bar v_1\cos\alpha+\bar v_2)^2+(\bar v_1\sin\alpha)^2} \\ &= \frac{1}{2}\sqrt{\bar v_1^2+\bar v_2^2+2\bar v_1\bar v_2\cos\alpha}\end{aligned}}$$

$(\alpha,\bar v_1,\bar v_2)$

$$\boxed{r_2 = \overline{SB} = \frac{1}{2}\sqrt{\bar v_1^2+\bar v_2^2-2\bar v_1\bar v_2\cos\alpha}}$$

$(\alpha,\bar v_1,\bar v_2)$

$\overline{SP} = \overline{SB}$

↓

!!! anche nel paper

$$\tilde v_1(\omega,r_1,r_2) = \overline{OP} = \sqrt{(\overline{SP}\cos\omega+\overline{OS})^2+(\overline{SP}\sin\omega)^2} = \sqrt{(r_2\cos\omega+r_1)^2+(r_2\sin\omega)^2} =$$

$(\omega,\alpha,\bar v_1,\bar v_2)$ $\boxed{\tilde v_1 = \sqrt{r_1^2+r_2^2+2r_1r_2\cos\omega}}$

$$\boxed{P_\omega = P_{\omega|\alpha}(\omega) = \frac{1}{2}\sin\omega\,d\omega}$$

$$d\tilde v_1 = \frac{-\cancel{2}r_1r_2\sin\omega}{\cancel{2}\sqrt{\dots}}\,d\omega \longrightarrow \frac{d\tilde v_1}{d\omega} = \frac{-r_1r_2\sin\omega}{\tilde v_1}$$

monotona decrescente con ω

$$P_{\tilde v_1|\alpha}(v)\,|d\tilde v_1| = P_{\omega|\alpha}(\omega)\,|d\omega| \rightarrow P_{\tilde v_1|\alpha}(v) = P_{\omega|\alpha}(\omega)\left|\frac{d\omega}{d\tilde v_1}\right| = \frac{1}{2}\sin\omega\left|\frac{\tilde v_1}{-r_1r_2\sin\omega}\right|$$

$$\boxed{P_{\tilde v_1|\alpha}(\tilde v_1) = \frac{\tilde v_1}{2r_1r_2}}$$

$$\frac{\tilde{V}_1}{\bar{V}_1\bar{V}_2}\sqrt{\bar{V}_1^2+\bar{V}_2^2-\tilde{V}_1^2}=1$$

$$\tilde{V}_1^2(\bar{V}_1^2+\bar{V}_2^2)-\tilde{V}_1^4=\bar{V}_1^2\bar{V}_2^2$$

$$\tilde{V}_1^4-\tilde{V}_1^2(\bar{V}_1^2+\bar{V}_2^2)+\bar{V}_1^2\bar{V}_2^2=\emptyset$$

$$\tilde{V}_1^2=\frac{(\bar{V}_1^2+\bar{V}_2^2)\pm\sqrt{(\bar{V}_1^2+\bar{V}_2^2)^2-4\bar{V}_1^2\bar{V}_2^2}}{2}=$$

$$=\frac{(\bar{V}_1^2+\bar{V}_2^2)\pm\sqrt{(\bar{V}_1^2-\bar{V}_2^2)^2}}{2}=$$

$$=\frac{(\bar{V}_1^2+\bar{V}_2^2)\pm|\bar{V}_1^2-\bar{V}_2^2|}{2}$$

$\bar{V}_1>\bar{V}_2$ $\quad$ $\bar{V}_2>\bar{V}_1$

$\tilde{V}_1$ $\quad$ //

$\tilde{V}_2$ $\quad$ //

ok every α $\quad$ $\boxed{\bar{V}_1<\tilde{V}_1<\bar{V}_2}$

$\cos\alpha_{min}=\frac{\tilde{V}_1}{\bar{V}_1\bar{V}_2}\sqrt{\bar{V}_1^2+\bar{V}_2^2-\tilde{V}_1^2}$ $\quad$ $\boxed{\text{otherwise}}$

$\cos\alpha_{max}=-\frac{\tilde{V}_1}{\bar{V}_1\bar{V}_2}\sqrt{\bar{V}_1^2+\bar{V}_2^2-\tilde{V}_1^2}$

$$\int_0^\infty\int_0^\infty\int_0^\pi f(\bar{V}_1,\bar{V}_2,\alpha)\,d\bar{V}_1\,d\bar{V}_2\,d\alpha=1$$

$$f(\bar{V}_1,\bar{V}_2,\alpha)=K\,f(\bar{V}_1)*f(\bar{V}_2)*\frac{1}{2}\sin(\alpha)*V_0 \qquad (d\bar{V}_1*d\bar{V}_2*d\alpha)$$

$$f_{hit}(\bar{V})=\int_\emptyset^\infty\int_\emptyset^\pi K\,f(\bar{V}_1)f(\bar{V}_2)\,\frac{1}{2}\sin(\alpha)*V_0\,d\alpha\,d\bar{V}_2=$$

$$=\int_\emptyset^\infty K f(\bar{V}_1)f(\bar{V}_2)\frac{1}{2}\left(\int_0^\pi \sin(\alpha)\cdot\sqrt{\bar{V}_1^2+\bar{V}_2^2-2\bar{V}_1\bar{V}_2\cos\alpha}\,d\alpha\right)d\bar{V}_2=$$

$$=\int_0^\infty K f(\bar{V}_1)f(\bar{V}_2)\,\frac{1}{2}\,\frac{1}{3\bar{V}_1\bar{V}_2}(\bar{V}_1^2+\bar{V}_2^2-2\bar{V}_1\bar{V}_2\cos\alpha)^{3/2}\Big|_0^\pi d\bar{V}_2=$$

$$=\int_0^\infty K f(\bar{V}_1)f(\bar{V}_2)\frac{1}{6\bar{V}_1\bar{V}_2}\left(\left((\bar{V}_1+\bar{V}_2)^2\right)^{3/2}-\left((\bar{V}_1-\bar{V}_2)^2\right)^{3/2}\right)d\bar{V}_2=$$

$$=\int_0^\infty K f(\bar{V}_1)f(\bar{V}_2)\frac{1}{6\bar{V}_1\bar{V}_2}\left((\bar{V}_1+\bar{V}_2)^3-|\bar{V}_1-\bar{V}_2|^3\right)d\bar{V}_2=$$

$$=\int_0^{\bar{V}_1} K f(\bar{V}_1)f(\bar{V}_2)\frac{1}{6\bar{V}_1\bar{V}_2}\left((\bar{V}_1+\bar{V}_2)^3-(\bar{V}_1-\bar{V}_2)^3\right)d\bar{V}_2+\int_{\bar{V}_1}^\infty K f(\bar{V}_1)f(\bar{V}_2)\frac{1}{6\bar{V}_1\bar{V}_2}\left((\bar{V}_2+\bar{V}_1)^3-(\bar{V}_2-\bar{V}_1)^3\right)d\bar{V}_2$$

$$\int_0^{\bar v_1} K f(\bar v_1) f(\bar v_2) \frac{1}{6\bar v_1 \bar v_2}\left((2\bar v_2)\left((\bar v_1^2+\bar v_2^2+2\bar v_1\bar v_2)+(\bar v_1^2-\bar v_2^2)+(\bar v_1^2+\bar v_2^2-2\bar v_1\bar v_2)\right)\right)d\bar v_2 + K$$

$$+\int_{\bar v_1}^{\infty} K f(\bar v_1) f(\bar v_2) \frac{1}{6\bar v_1 \bar v_2}\left((2\bar v_1)\left((\bar v_2^2+\bar v_1^2+2\bar v_2\bar v_1)+(\bar v_2^2-\bar v_1^2)+(\bar v_2^2+\bar v_1^2-2\bar v_2\bar v_1)\right)\right)d\bar v_2 =$$

$$= \int_0^{\bar v_1} K f(\bar v_1) f(\bar v_2) \frac{\cancel{2}\cancel{\bar v_2}(3\bar v_1^2+\bar v_2^2)}{\underset{3}{\cancel{6}}\bar v_1 \cancel{\bar v_2}} d\bar v_2 + \int_{\bar v_1}^{\infty} K f(\bar v_1) f(\bar v_2) \frac{\cancel{2}\cancel{\bar v_1}(3\bar v_2^2+\bar v_1^2)}{\underset{3}{\cancel{6}}\cancel{\bar v_1}\bar v_2} d\bar v_2 =$$

$$= \int_0^{\bar v_1} K f(\bar v_1) f(\bar v_2) \frac{3\bar v_1^2+\bar v_2^2}{3\bar v_1} d\bar v_2 + \int_{\bar v_1}^{\infty} K f(\bar v_1) f(\bar v_2) \frac{3\bar v_2^2+\bar v_1^2}{3\bar v_2} d\bar v_2 =$$

$$= \underbrace{\int_0^{\bar v_1} K f(\bar v_1) f(\bar v_2)\cdot \bar v_1\, d\bar v_2}_{I_1} + \underbrace{\int_0^{\bar v_1} K f(\bar v_1) f(\bar v_2)\cdot \frac{\bar v_2^2}{3\bar v_1} d\bar v_2}_{I_2} + \underbrace{\int_{\bar v_1}^{\infty} K f(\bar v_1) f(\bar v_2)\, \bar v_2\, d\bar v_2}_{I_3} + \underbrace{\int_{\bar v_1}^{\infty} K f(\bar v_1) f(\bar v_2) \frac{\bar v_1^2}{3\bar v_2} d\bar v_2}_{I_4}$$

$$I_1 = K f(\bar v_1)\bar v_1 \int_0^{\bar v_1} f(\bar v_2)\, d\bar v_2 = \boxed{K f(\bar v_1)\bar v_1 F(\bar v_1)}$$

$$I_2 = \frac{K f(\bar v_1)}{3\bar v_1}\int_0^{\bar v_1} \bar v_2^2 f(\bar v_2)\, d\bar v_2 = \frac{K f(\bar v_1)}{3\bar v_1}\int_0^{\bar v_1} \bar v_2^2 \cdot \frac{32}{\pi^2}\frac{\bar v_2^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar v_2}{\langle v\rangle}\right)^2} d\bar v_2 =$$

$\cdots \frac{K f(\bar v_1)}{3\bar v_1}$

$$= \int_0^{\bar v_1} \underbrace{\bar v_2^3 \cdot \left(\frac{-4}{\pi\langle v\rangle}\right)}_{f} \underbrace{\left(-\frac{8\bar v_2}{\pi\langle v\rangle^2}\right) e^{-\frac{4}{\pi}\left(\frac{\bar v_2}{\langle v\rangle}\right)^2}}_{g'} d\bar v_2 = \underbrace{\left[\bar v_2^3\left(\frac{-4}{\pi\langle v\rangle}\right) e^{-\frac{4}{\pi}\left(\frac{\bar v_2}{\langle v\rangle}\right)^2}\right]_0^{\bar v_1}}_{fg} - \int_0^{\bar v_1} \underbrace{\frac{-12}{\pi\langle v\rangle}\bar v_2^2}_{f'} \underbrace{e^{-\frac{4}{\pi}\left(\frac{\bar v_2}{\langle v\rangle}\right)^2}}_{g} d\bar v_2 =$$

$$= -\frac{4\bar v_1^3}{\pi\langle v\rangle} e^{-\frac{4}{\pi}\left(\frac{\bar v_1}{\langle v\rangle}\right)^2} - \cancel{0} - \int_0^{\bar v_1} \frac{\pi\langle v\rangle^2}{8}\cdot 3\ \frac{32}{\pi^2}\frac{\bar v_2^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar v_2}{\langle v\rangle}\right)^2} d\bar v_2 =$$

$$= \left(-\frac{\pi\langle v\rangle^2 \bar v_1}{8}\right)\left(\frac{32}{\pi^2}\frac{\bar v_1^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar v_1}{\langle v\rangle}\right)^2}\right) + \frac{3\pi\langle v\rangle^2}{8}\int_0^{\bar v_1} \frac{32}{\pi^2}\frac{\bar v_2^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar v_2}{\langle v\rangle}\right)^2} d\bar v_2 =$$

$$= -\frac{\pi\langle v\rangle^2 \bar v_1}{8} f(\bar v_1) + \frac{3\pi\langle v\rangle^2}{8} F(\bar v_1) \quad \times \frac{K f(\bar v_1)}{3\bar v_1} \qquad \boxed{\frac{K f(\bar v_1)\pi\langle v\rangle^2}{24\bar v_1}\left(3F(\bar v_1) - \bar v_1 f(\bar v_1)\right)}$$

$$I_3=\int_{\bar V_1}^{\infty} k f(\bar V_1)\, f(\bar V_2)\, \bar V_2\, d\bar V_2 = k f(\bar V_1) \int_{\bar V_1}^{\infty} \bar V_2\, \frac{32}{\pi^2}\frac{\bar V_2^2}{\langle v\rangle^3}\, e^{-\frac{4}{\pi}\left(\frac{\bar V_2}{\langle v\rangle}\right)^2} d\bar V_2 =$$

(L)

!!! $k f(\bar V_1)$

$$= \int_{\bar V_1}^{\infty} \underbrace{\bar V_2^2\left(-\frac{4}{\pi\langle v\rangle}\right)}_{f}\ \underbrace{\frac{-8\bar V_2}{\pi\langle v\rangle^2}\, e^{-\frac{4}{\pi}\left(\frac{\bar V_2}{\langle v\rangle}\right)^2}}_{g'} d\bar V_2 = \left[\underbrace{\bar V_2^2\left(-\frac{4}{\pi\langle v\rangle}\right)}_{f}\ \underbrace{e^{-\frac{4}{\pi}\left(\frac{\bar V_2}{\langle v\rangle}\right)^2}}_{g}\right]_{\bar V_1}^{\infty} - \int_{\bar V_1}^{\infty} \underbrace{\frac{-8\bar V_2}{\pi\langle v\rangle}}_{f'}\ \underbrace{e^{-\frac{4}{\pi}\left(\frac{\bar V_2}{\langle v\rangle}\right)^2}}_{g} d\bar V_2 =$$

$$= \emptyset - \bar V_1^2\left(-\frac{4}{\pi\langle v\rangle}\right) e^{-\frac{4}{\pi}\left(\frac{\bar V_1}{\langle v\rangle}\right)^2} - \int_{\bar V_1}^{\infty} \langle v\rangle\left(-\frac{8\bar V_2}{\pi\langle v\rangle^2}\right) e^{-\frac{4}{\pi}\left(\frac{\bar V_2}{\langle v\rangle}\right)^2} d\bar V_2 =$$

$$= \frac{4\bar V_1^2}{\pi\langle v\rangle} e^{-\frac{4}{\pi}\left(\frac{\bar V_1}{\langle v\rangle}\right)^2} - \left[\langle v\rangle\, e^{-\frac{4}{\pi}\left(\frac{\bar V_2}{\langle v\rangle}\right)^2}\right]_{\bar V_1}^{\infty} = \frac{4\bar V_1^2}{\pi\langle v\rangle} e^{-\frac{4}{\pi}\left(\frac{\bar V_1}{\langle v\rangle}\right)^2} - \emptyset + \langle v\rangle\, e^{-\frac{4}{\pi}\left(\frac{\bar V_1}{\langle v\rangle}\right)^2} =$$

$$= \frac{\pi\langle v\rangle^2}{8}\cdot \frac{32}{\pi^2}\frac{\bar V_1^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar V_1}{\langle v\rangle}\right)^2} + \frac{\pi^2\langle v\rangle^4}{32\bar V_1^2}\, \frac{32}{\pi^2}\frac{\bar V_1^2}{\langle v\rangle^3}\, e^{-\frac{4}{\pi}\left(\frac{\bar V_1}{\langle v\rangle}\right)^2} =$$

$$= \frac{\pi\langle v\rangle^2}{8} f(\bar V_1) + \frac{\pi^2\langle v\rangle^4}{32\bar V_1^2} f(\bar V_1) \xrightarrow[\times k f(\bar V_1)]{} \boxed{k\frac{\pi\langle v\rangle^2}{8} f^2(\bar V_1) + k\frac{\pi^2\langle v\rangle^4}{32\bar V_1^2} f^2(\bar V_1)}$$

$$I_4=\int_{\bar V_1}^{\infty} k f(\bar V_1) f(\bar V_2)\frac{\bar V_1^2}{3\bar V_2} d\bar V_2 = \frac{k f(\bar V_1)\bar V_1^2}{3}\int_{\bar V_1}^{\infty} \frac{f(\bar V_2)}{\bar V_2} d\bar V_2 =$$

!!! $\frac{k f(\bar V_1)\bar V_1^2}{3}$

$$= \int_{\bar V_1}^{\infty} \frac{\frac{32}{\pi^2}\frac{\bar V_2^{\cancel{2}}}{\langle v\rangle^3}}{\cancel{\bar V_2}}\, e^{-\frac{4}{\pi}\left(\frac{\bar V_2}{\langle v\rangle}\right)^2} d\bar V_2 = \int_{\bar V_1}^{\infty}\left(\frac{-4}{\pi\langle v\rangle}\right)\left(-\frac{8\bar V_2}{\pi\langle v\rangle^2} e^{-\frac{4}{\pi}\left(\frac{\bar V_2}{\langle v\rangle}\right)^2}\right) d\bar V_2 =$$

$$= -\frac{4}{\pi\langle v\rangle}\left[e^{-\frac{4}{\pi}\left(\frac{\bar V_2}{\langle v\rangle}\right)^2}\right]_{\bar V_1}^{\infty} = -\frac{4}{\pi\langle v\rangle}\left(\emptyset - e^{-\frac{4}{\pi}\left(\frac{\bar V_1}{\langle v\rangle}\right)^2}\right) = \frac{\pi\langle v\rangle^2}{8\bar V_1^2}\cdot \frac{32}{\pi^2}\frac{\bar V_1^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar V_1}{\langle v\rangle}\right)^2} =$$

$$= \frac{\pi\langle v\rangle^2}{8\bar V_1^2} f(\bar V_1) \xrightarrow[\times k\frac{f(\bar V_1)\bar V_1^2}{3}]{} \boxed{\frac{k\pi\langle v\rangle^2}{24} f^2(\bar V_1)}$$

$$f_{hit}(\bar V) = \underset{\bar v \cdot f \cdot F}{k f(\bar V_1)\bar V_1 F(\bar V_1)} + \frac{k f(\bar V_1)\pi\langle v\rangle^2}{24\bar V_1}\left(\underset{\frac{fF}{v}}{3F(\bar V_1)} - \underset{f^2}{\bar V_1 f(\bar V_1)}\right) + \underset{f^2}{\frac{k\pi\langle v\rangle^2}{8} f^2(\bar V_1)} + \underset{\frac{f^2}{v^2}}{k\frac{\pi^2\langle v\rangle^4}{32\bar V_1^2} f^2(\bar V_1)}$$

$$+ \cancel{\frac{k\pi\langle v\rangle^2}{24} f^2(\bar V_1)} \sim f^2$$

$$= k\left[\bar V_1 f(\bar V_1) F(\bar V_1) + \frac{\pi\langle v\rangle^2}{8}\frac{f(\bar V_1)F(\bar V_1)}{\bar V_1} + \frac{\pi^2\langle v\rangle^4}{32}\frac{f^2(\bar V_1)}{\bar V_1^2} + \frac{\pi\langle v\rangle^2}{8} f^2(\bar V_1)\right]$$

$$\psi(V, V_j, V_K) = \int_{\alpha_{min}}^{\pi-\alpha_{min}} \underbrace{P_{V|\alpha}(V)}_{\frac{V}{2r_1r_2}} \underbrace{P_\alpha(\alpha)}_{\frac{1}{2}\sin\alpha}\, d\alpha$$

M

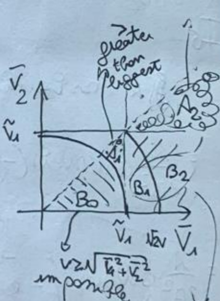
$$P_{new}(V) = \int_0^\infty\int_0^\infty \psi(V, V_j, V_K)\, P_{old}(V_j)\, P_{old}(V_K)\, dV_j\, dV_K$$

$$\rightarrow P_{new}(V) = \int_0^\infty\int_0^\infty\int_{\alpha_{min}}^{\pi-\alpha_{min}} P_{V|\alpha} P_\alpha P_{old} P_{old}\, d\, d\, d$$

$$f_{new}(V) = \int_0^\infty\int_0^\infty\int_{\alpha_{min}}^{\cancel{\pi}-\alpha_{min}} P_{\tilde{V}_1|_{\alpha,\bar{V}_1,\bar{V}_2}}\, K \cdot f(\bar{V}_1)\cdot f(\bar{V}_2)\, \underbrace{\tfrac{1}{2}\sin(\alpha)}_{P_\alpha} V_0\, d\cancel{\alpha}\, d\bar{V}_1\, d\bar{V}_2 =$$

$$= \int_0^\infty\int_0^\infty\int_{\alpha_{min}}^{\pi-\alpha_{min}} \frac{\tilde{V}_1 \cdot K \cdot f(\bar{V}_1) f(\bar{V}_2)\, \frac{1}{2}\sin(\alpha)\, \cancel{\sqrt{\bar{V}_1^2+\bar{V}_2^2-2\bar{V}_1\bar{V}_2\cos\alpha}}}{\underbrace{2\cdot\frac{1}{8}\sqrt{\bar{V}_1^2+\bar{V}_2^2+2\bar{V}_1\bar{V}_2\cos\alpha}}_{r_1}\, \underbrace{\cancel{\sqrt{\bar{V}_1^2+\bar{V}_2^2-2\bar{V}_1\bar{V}_2\cos\alpha}}}_{r_2}}\, d\alpha\, d\bar{V}_1\, d\bar{V}_2 =$$

$$= \int_0^\infty\int_0^\infty\int_{\alpha_{min}}^{\pi-\alpha_{min}} \tilde{V}_1 K f(\bar{V}_1) f(\bar{V}_2) \frac{\sin(\alpha)}{\underbrace{\sqrt{\bar{V}_1^2+\bar{V}_2^2+2\bar{V}_1\bar{V}_2\cos\alpha}}_{-\frac{1}{2}}}\, d\alpha\, d\bar{V}_1\, d\bar{V}_2 =$$



less than smallest

in between

$$= \int_0^\infty\int_0^\infty\int_{\alpha_{min}}^{\pi-\alpha_{min}} \tilde{V}_1 K f(\bar{V}_1) f(\bar{V}_2) \left(-\frac{1}{\bar{V}_1\bar{V}_2}\right)\left(\frac{-\bar{V}_1\bar{V}_2\sin(\alpha)}{\sqrt{\bar{V}_1^2+\bar{V}_2^2+2\bar{V}_1\bar{V}_2\cos\alpha}}\right) d\alpha\, d\bar{V}_1\, d\bar{V}_2 =$$

$$= \int_0^\infty\int_0^\infty -\frac{\tilde{V}_1 K f(\bar{V}_1) f(\bar{V}_2)}{\bar{V}_1\bar{V}_2}\cdot \sqrt{\bar{V}_1^2+\bar{V}_2^2+2\bar{V}_1\bar{V}_2\cos\alpha}\,\Big|_{\alpha_{min}}^{\pi-\alpha_{min}}\, d\bar{V}_1\, d\bar{V}_2 =$$

$$= \iint_{A_1+A_2} -\frac{\tilde{V}_1 K f(\bar{V}_1) f(\bar{V}_2)}{\bar{V}_1\bar{V}_2}\cdot\left(\sqrt{\bar{V}_1^2+\bar{V}_2^2+2\cancel{\bar{V}_1\bar{V}_2}\cdot\underbrace{\left(-\frac{\tilde{V}_1}{\cancel{\bar{V}_1\bar{V}_2}}\sqrt{\bar{V}_1^2+\bar{V}_2^2-\tilde{V}_1^2}\right)}_{\cos(\pi-\alpha_{min})}} - \sqrt{\bar{V}_1^2+\bar{V}_2^2+2\cancel{\bar{V}_1\bar{V}_2}\frac{\tilde{V}_1}{\cancel{\bar{V}_1\bar{V}_2}}\sqrt{\bar{V}_1^2+\bar{V}_2^2-\tilde{V}_1^2}}\right) d\bar{V}_1\, d\bar{V}_2 +$$

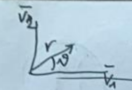
$$+ \iint_{B_1+B_2} -\frac{\tilde{V}_1 K f(\bar{V}_1) f(\bar{V}_2)}{\bar{V}_1\bar{V}_2}\left(\sqrt{\bar{V}_1^2+\bar{V}_2^2-2\bar{V}_1\bar{V}_2} - \sqrt{\bar{V}_1^2+\bar{V}_2^2+2\bar{V}_1\bar{V}_2}\right) d\bar{V}_1\, d\bar{V}_2 =$$

I_5

$$= \iint_{A_1+A_2} \frac{\tilde{V}_1 K f(\bar{V}_1) f(\bar{V}_2)}{\bar{V}_1\bar{V}_2}\left(\sqrt{\bar{V}_1^2+\bar{V}_2^2+2\tilde{V}_1\sqrt{\bar{V}_1^2+\bar{V}_2^2-\tilde{V}_1^2}} - \sqrt{\bar{V}_1^2+\bar{V}_2^2-2\tilde{V}_1\sqrt{\bar{V}_1^2+\bar{V}_2^2-\tilde{V}_1^2}}\right) d\bar{V}_1\, d\bar{V}_2 +$$

I_6

$$+ \iint_{B_1 B_2} \frac{\tilde{V}_1 K f(\bar{V}_1) f(\bar{V}_2)}{\bar{V}_1\bar{V}_2}\left(\sqrt{\bar{V}_1^2+\bar{V}_2^2+2\bar{V}_1\bar{V}_2} - \sqrt{\bar{V}_1^2+\bar{V}_2^2-2\bar{V}_1\bar{V}_2}\right) d\bar{V}_1\, d\bar{V}_2$$



$$\boxed{\begin{aligned}\bar{v}_1 &= r\cos\vartheta \\ \bar{v}_2 &= r\sin\vartheta\end{aligned}} \quad d\bar{v}_1 d\bar{v}_2 = r\,dr\,d\vartheta$$

$$d\bar{v}_1 = dr\cos\vartheta - r\sin\vartheta\, d\vartheta$$
$$d\bar{v}_2 = dr\sin\vartheta + r\cos\vartheta\, d\vartheta$$
$$d\bar{v}_1 d\bar{v}_2 = dr^2 \sin\vartheta\cos\vartheta + r\cos^2\vartheta\, dr\, d\vartheta - r\sin^2\vartheta\, dr\, d\vartheta - r^2\cos\vartheta\sin\vartheta\, d\vartheta^2$$

N

$$= \iint_{A_1+A_2} \frac{\tilde{v}_1 K \frac{32}{\pi^2}\frac{r^2\cos^2\vartheta}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{r^2\cos^2\vartheta}{\langle v\rangle^2}} \cdot \frac{32}{\pi^2}\frac{r^2\sin^2\vartheta}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{r^2\sin^2\vartheta}{\langle v\rangle^2}}}{r\cos\vartheta\, r\sin\vartheta} \left(\sqrt{r^2+2\tilde{v}_1\sqrt{r^2-\tilde{v}_1^2}} - \sqrt{r^2-2\tilde{v}_1\sqrt{r^2-\tilde{v}_1^2}}\right) r\,dr\,d\vartheta +$$

$$\cancel{\iint_{B_1+B_2} \frac{\tilde{v}_1 K \frac{32}{\pi^2}\frac{r^2\cos^2\vartheta}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{r^2\cos^2\vartheta}{\langle v\rangle^2}} \cdot \frac{32}{\pi^2}\frac{r^2\sin^2\vartheta}{\langle v\rangle^3} e^{-\frac{4}{\pi}\frac{r^2\sin^2\vartheta}{\langle v\rangle^2}}}{r\cos\vartheta\, r\sin\vartheta} \left(\sqrt{r^2+2r^2\cos\vartheta\sin\vartheta} - \sqrt{r^2-2r^2\cos\vartheta\sin\vartheta}\right) r\,dr\,d\vartheta}$$

I_5

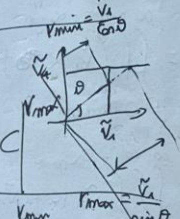
$$= \iint_{A_1+A_2} \tilde{v}_1 K \frac{1024}{\pi^4\langle v\rangle^6} r^2\cos\vartheta\sin\vartheta\, e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}} \left(\sqrt{r^2+2\tilde{v}_1\sqrt{r^2-\tilde{v}_1^2}} - \sqrt{r^2-2\tilde{v}_1\sqrt{r^2-\tilde{v}_1^2}}\right) r\,dr\,d\vartheta +$$

$$\cancel{+\iint_{B_1+B_2} \tilde{v}_1 K \frac{1024}{\pi^4\langle v\rangle^6} r^2\cos\vartheta\sin\vartheta\, e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}} \left[r\left(\sqrt{1+2\cos\vartheta\sin\vartheta} - \sqrt{1-2\cos\vartheta\sin\vartheta}\right) r\,dr\,d\vartheta\right.}$$

I_6

$$\int A r^3 \cos\vartheta\sin\vartheta\, e^{-Br^2}\left(\sqrt{1+2\cos\vartheta\sin\vartheta} - \sqrt{1-2\cos\vartheta\sin\vartheta}\right) dr =$$

$$= \frac{A}{2B^2}\cos\vartheta\sin\vartheta\left(-e^{-Br^2} - Be^{-Br^2} r^2\right)\left(\sqrt{1+\sin(2\vartheta)} - \sqrt{1-\sin(2\vartheta)}\right) + C$$



$$\cancel{= \frac{A}{2B^2}\cos\vartheta\sin\vartheta\left[-e^{-B\frac{\tilde{v}_1^2}{\sin^2\vartheta}} - Be^{-B\frac{\tilde{v}_1^2}{\sin^2\vartheta}} \cdot \frac{\tilde{v}_1^2}{\sin^2\vartheta} + e^{-B\frac{\tilde{v}_1^2}{\cos^2\vartheta}} + Be^{-B\frac{\tilde{v}_1^2}{\cos^2\vartheta}}\frac{\tilde{v}_1^2}{\cos^2\vartheta}\right]\left(\sqrt{1+\sin 2\vartheta} - \sqrt{1-\sin 2\vartheta}\right)}$$

I_6

$$I_6 = 2\times\iint_{B_1+B_2} \frac{\tilde{v}_1 K \frac{32}{\pi^2}\frac{\bar{v}_1^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar{v}_1}{\langle v\rangle}\right)^2} \cdot \frac{32}{\pi^2}\frac{\bar{v}_2^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\bar{v}_2}{\langle v\rangle}\right)^2}}{\bar{v}_1\bar{v}_2}\left(\cancel{\bar{v}_1}+\bar{v}_2 - (\cancel{\bar{v}_1}-\bar{v}_2)\right) d\bar{v}_1 d\bar{v}_2 =$$

arriva $\bar{v}_1 \geq \bar{v}_2$

$$= 2\times\iint_{\substack{B_1+B_2\\ \bar{v}_1\geq\bar{v}_2}} \tilde{v}_1 K \frac{2048}{\pi^4\langle v\rangle^6}\bar{v}_1\bar{v}_2^2\, e^{-\frac{4}{\pi}\left(\frac{\bar{v}_1}{\langle v\rangle}\right)^2} e^{-\frac{4}{\pi}\left(\frac{\bar{v}_2}{\langle v\rangle}\right)^2} d\bar{v}_1 d\bar{v}_2 = 2\times\iint_{\substack{B_1+B_2\\ \bar{v}_1\geq\bar{v}_2}} \tilde{v}_1 K\left(\frac{256\,\bar{v}_2^2}{\pi^3\langle v\rangle^4}\right)\left(-\frac{8\,\bar{v}_1}{\pi\langle v\rangle^2}\right) e^{-\frac{4}{\pi}\left(\frac{\bar{v}_1}{\langle v\rangle}\right)^2} e^{-\frac{4}{\pi}\left(\frac{\bar{v}_2}{\langle v\rangle}\right)^2} d\bar{v}_1 d\bar{v}_2 =$$

(simmetria)

$$= 2\times\int_{\substack{B_1+B_2\\ \bar{v}_1\geq\bar{v}_2}} \tilde{v}_1 K\left(-\frac{256\,\bar{v}_2^2}{\pi^3\langle v\rangle^4}\right) e^{-\frac{4}{\pi}\left(\frac{\bar{v}_2}{\langle v\rangle}\right)^2} e^{-\frac{4}{\pi}\left(\frac{\bar{v}_1}{\langle v\rangle}\right)^2}\Bigg|_{\tilde{v}_1}^{\infty} d\bar{v}_2 = 2\times\int_{\substack{B_1+B_2\\ \bar{v}_1\geq\bar{v}_2}} \tilde{v}_1 K\left(-\frac{256\,\bar{v}_2^2}{\pi^3\langle v\rangle^4}\right) e^{-\frac{4}{\pi}\left(\frac{\bar{v}_2}{\langle v\rangle}\right)^2}\left(\emptyset - e^{-\frac{4}{\pi}\left(\frac{\tilde{v}_1}{\langle v\rangle}\right)^2}\right) d\bar{v}_2 =$$

$$= 2\times\int \cancel{\tilde{v}_1} K \frac{8\,\bar{v}_2^2}{\pi\langle v\rangle \tilde{v}_1^2} e^{-\frac{4}{\pi}\left(\frac{\bar{v}_2}{\langle v\rangle}\right)^2} \cdot \frac{32}{\pi^2}\frac{\tilde{v}_1^2}{\langle v\rangle^3} e^{-\frac{4}{\pi}\left(\frac{\tilde{v}_1}{\langle v\rangle}\right)^2} d\bar{v}_2 =$$

$$= 2\times\int_0^{\tilde v_1} K\,\frac{8\bar v_2^{\,2}}{\tilde v_1\pi\langle v\rangle}\,e^{-\frac{4}{\pi}\left(\frac{\bar v_2}{\langle v\rangle}\right)^2} f(\tilde v_1)\,d\bar v_2 = 2\times\int_0^{\tilde v_1}\underbrace{\left(-K\frac{\langle v\rangle}{\tilde v_1}f(\tilde v_1)\bar v_2\right)}_{f}\underbrace{\left(-\frac{8\bar v_2}{\pi\langle v\rangle^2}\right)e^{-\frac{4}{\pi}\left(\frac{\bar v_2}{\langle v\rangle}\right)^2}}_{g'}d\bar v_2 =$$

$$= 2\times\left[-K\frac{\langle v\rangle}{\tilde v_1}f(\tilde v_1)\bar v_2\, e^{-\frac{4}{\pi}\left(\frac{\bar v_2}{\langle v\rangle}\right)^2}\right]_0^{\tilde v_1} - \int_0^{\tilde v_1}\underbrace{-K\frac{\langle v\rangle}{\tilde v_1}f(\tilde v_1)}_{f'}\underbrace{e^{-\frac{4}{\pi}\left(\frac{\bar v_2}{\langle v\rangle}\right)^2}}_{g}d\bar v_2 =$$

$$= 2\times\left[-K\frac{\langle v\rangle}{\tilde v_1}f(\tilde v_1)\tilde v_1\, e^{-\frac{4}{\pi}\left(\frac{\tilde v_1}{\langle v\rangle}\right)^2} + \emptyset\right] - \int_0^{\tilde v_1} \dots\, d\bar v_2 =$$

$$= 2\times\left[-K\frac{\pi^2\langle v\rangle^4}{32\tilde v_1^2}f(\tilde v_1)\,\frac{32}{\pi^2}\frac{\tilde v_1^2}{\langle v\rangle^3}e^{-\frac{4}{\pi}\left(\frac{\tilde v_1}{\langle v\rangle}\right)^2}\right] - \int_0^{\tilde v_1} K\frac{\pi^2\langle v\rangle^4}{\dots}\left(-\frac{32}{\pi^2}\frac{\bar v_2^{\,2}}{\langle v\rangle^3}\right)e^{-\frac{4}{\pi}\left(\frac{\bar v_2}{\langle v\rangle}\right)^2}$$

$$= 2\times\int_0^{\tilde v_1} K\frac{\pi\langle v\rangle^2}{4\tilde v_1}f(\tilde v_1)\,\frac{32}{\pi^2}\frac{\bar v_2^{\,2}}{\langle v\rangle^3}e^{-\frac{4}{\pi}\left(\frac{\bar v_2}{\langle v\rangle}\right)^2}d\bar v_2 = 2\cdot K\frac{\pi\langle v\rangle^2}{4\tilde v_1}f(\tilde v_1)F(\tilde v_1) = \boxed{K\frac{\pi\langle v\rangle^2}{2\tilde v_1}f(\tilde v_1)F(\tilde v_1)}$$

I_5 polar coordinates $\quad \int \frac{1}{2}\sin 2\theta \rightarrow -\frac{1}{4}\cos(2\theta)$

$$\iint_{A_1+A_2}\left(\tilde v_1 K\frac{1024}{\pi^4\langle v\rangle^6}r^3\right)\cos\theta\sin\theta\, e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}\underbrace{\left(\sqrt{r^2+2\tilde v_1\sqrt{r^2-\tilde v_1^2}}-\sqrt{r^2-2\tilde v_1\sqrt{r^2-\tilde v_1^2}}\right)}_{C}dr\,d\theta =$$

$$= \int_{A_1}\dots\cos(2\theta)\Big|_{\theta_{min}}^{\theta_{max}}(-B)^C dr + \int_{A_2}-\cos(2\theta)\Big|_{\theta_{min}}^{\theta_{max}}(-B)^C dr =$$

$$= \int_{A_1}-\left(\cos^2\theta_{max}-\sin^2\theta_{max}-\cos^2\theta_{min}+\sin^2\theta_{min}\right)B^C dr + \int_{A_2}-\left(\cos^2\theta_{max}-\sin^2\theta_{max}-\cos^2\theta_{min}+\sin^2\theta_{min}\right)B^C dr$$

$\cos\theta_{min}=\frac{\tilde v_1}{r}$ $\quad \cos\theta_{max}=\sin\theta_{min}=\sqrt{1-\left(\frac{\tilde v_1}{r}\right)^2}$ $\quad \cos\theta_{min}=\sqrt{1-\left(\frac{\tilde v_1}{r}\right)^2}$ $\quad \sin\theta_{max}=\cos\theta_{min}=\frac{\tilde v_1}{r}$

$\sin\theta_{min}=\sqrt{1-\left(\frac{\tilde v_1}{r}\right)^2}$ $\quad \sin\theta_{max}=\cos\theta_{max}=\frac{\tilde v_1}{r}$ $\quad \sin\theta_{min}=\frac{\tilde v_1}{r}$ $\quad \cos\theta_{max}=\sin\theta_{min}=\sqrt{1-\left(\frac{\tilde v_1}{r}\right)^2}$

$$= \int_{A_1}-\underbrace{\left(1-\left(\frac{\tilde v_1}{r}\right)^2-\left(\frac{\tilde v_1}{r}\right)^2-\left(\frac{\tilde v_1}{r}\right)^2+1-\left(\frac{\tilde v_1}{r}\right)^2\right)}_{}B^C dr + \int_{A_2}-\underbrace{\left(\left(\frac{\tilde v_1}{r}\right)^2-1+\left(\frac{\tilde v_1}{r}\right)^2-1+\left(\frac{\tilde v_1}{r}\right)^2+\left(\frac{\tilde v_1}{r}\right)^2\right)}_{}B^C dr$$

$$-2\left(1-2\frac{\tilde v_1^2}{r^2}\right) = 2\left(2\frac{\tilde v_1^2}{r^2}-1\right) \qquad\qquad -2\left(2\frac{\tilde v_1^2}{r^2}-1\right) = 2\left(1-2\frac{\tilde v_1^2}{r^2}\right)$$

$\le\emptyset$ in A_1 $\qquad\qquad$ $\le\emptyset$ in A_2

$$\cancel{\int_{\tilde{v}_1}^{\sqrt{2}\tilde{v}_1} 2\left(2\frac{\tilde{v}_1^2}{r^2}-1\right)\tilde{v}_1 K \frac{256}{\pi^4\langle v\rangle^6} r^3 e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}} \left(\sqrt{r^2+2\tilde{v}_1\sqrt{r^2-\tilde{v}_1^2}} - \sqrt{r^2-2\tilde{v}_1\sqrt{r^2-\tilde{v}_1^2}}\right) dr}$$

P

$$C=\sqrt{r^2+2\tilde{v}_1\sqrt{r^2-\tilde{v}_1^2}}-\sqrt{r^2-2\tilde{v}_1\sqrt{r^2-\tilde{v}_1^2}}=$$

$$=\sqrt{\left(\sqrt{r^2-\tilde{v}_1^2}+\tilde{v}_1\right)^2}-\sqrt{\left(\sqrt{r^2-\tilde{v}_1^2}-\tilde{v}_1\right)^2}=$$

$r>\tilde{v}_1 \Rightarrow$ always >0

$$C=\left(\sqrt{r^2-\tilde{v}_1^2}+\tilde{v}_1-\left|\sqrt{r^2-\tilde{v}_1^2}-\tilde{v}_1\right|\right)$$

$$\sqrt{r^2-\tilde{v}_1^2}-\tilde{v}_1>0 \qquad r^2>2\tilde{v}_1^2$$

$$\sqrt{r^2-\tilde{v}_1^2}>\tilde{v}_1 \qquad \boxed{r>\sqrt{2}\,\tilde{v}_1}$$

$$r^2-\tilde{v}_1^2>\tilde{v}_1^2$$

$A_1:\ \tilde{v}_1\le r\le\sqrt{2}\tilde{v}_1 \rightarrow C_{A_1}=\sqrt{r^2-\tilde{v}_1^2}+\tilde{v}_1-\left(\tilde{v}_1-\sqrt{r^2-\tilde{v}_1^2}\right)=2\sqrt{r^2-\tilde{v}_1^2}$

$A_2:\ \sqrt{2}\tilde{v}_1\le r\le\infty \rightarrow C_{A_2}=\sqrt{r^2-\tilde{v}_1^2}+\tilde{v}_1-\sqrt{r^2-\tilde{v}_1^2}+\tilde{v}_1=2\tilde{v}_1$

$$\boxed{\int_{\tilde{v}_1}^{\sqrt{2}\tilde{v}_1} 2\left(2\frac{\tilde{v}_1^2}{r^2}-1\right)\tilde{v}_1 K\frac{256}{\pi^4\langle v\rangle^6} r^3 e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}\cdot 2\sqrt{r^2-\tilde{v}_1^2}\,dr+} \longrightarrow I_7$$

$$\boxed{+\int_{\sqrt{2}\tilde{v}_1}^{\infty} 2\left(1-2\frac{\tilde{v}_1^2}{r^2}\right)\tilde{v}_1\cdot K\frac{256}{\pi^4\langle v\rangle^6} r^3 e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}\,2\tilde{v}_1\,dr} \longrightarrow I_8$$

$$I_5=I_7+I_8$$

$$I_8\cdot K\int_{\sqrt{2}\tilde{v}_1}^{\infty}\frac{1024\tilde{v}_1^2}{\pi^4\langle v\rangle^6}r\left(r^2-2\tilde{v}_1^2\right)e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}dr=$$

$$=K\int_{\sqrt{2}\tilde{v}_1}^{\infty}\frac{1024\tilde{v}_1^2}{\pi^4\langle v\rangle^6}r^3e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}dr-K\int_{\sqrt{2}\tilde{v}_1}^{\infty}\frac{2048\tilde{v}_1^4}{\pi^4\langle v\rangle^6}re^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}dr=$$

$$=K\int_{\sqrt{2}\tilde{v}_1}^{\infty}\left(-\frac{128\tilde{v}_1^2r^2}{\pi^3\langle v\rangle^4}\right)\left(-\frac{8r}{\pi\langle v\rangle^2}\right)e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}dr+K\int_{\sqrt{2}\tilde{v}_1}^{\infty}\frac{256\tilde{v}_1^4}{\pi^3\langle v\rangle^4}\left(-\frac{8r}{\pi\langle v\rangle^2}\right)e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}dr=$$

$$=K\left[-\frac{128\tilde{v}_1^2r^2}{\pi^3\langle v\rangle^4}e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}\right]_{\sqrt{2}\tilde{v}_1}^{\infty}-K\int_{\sqrt{2}\tilde{v}_1}^{\infty}-\frac{256\tilde{v}_1^2r}{\pi^3\langle v\rangle^4}e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}dr+K\cdot\frac{256\tilde{v}_1^4}{\pi^3\langle v\rangle^4}\left[e^{-\frac{4r^2}{\pi\langle v\rangle^2}}\right]_{\sqrt{2}\tilde{v}_1}^{\infty}=$$

$$=K\left[\cancel{\varnothing+\frac{128\tilde{v}_1^2\cdot2\tilde{v}_1^2}{\pi^3\langle v\rangle^4}e^{-\frac{4}{\pi}\frac{2\tilde{v}_1^2}{\langle v\rangle^2}}}\right]+\ -K\int_{\sqrt{2}\tilde{v}_1}^{\infty}\left(\frac{32\tilde{v}_1^2}{\pi^2\langle v\rangle^2}\right)\left(-\frac{8r}{\pi\langle v\rangle^2}\right)e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}dr+\cancel{K\frac{256\tilde{v}_1^4}{\pi^3\langle v\rangle^4}\left(\varnothing-e^{-\frac{4\cdot2\tilde{v}_1^2}{\pi\langle v\rangle^2}}\right)}$$

$$=-K\frac{32\tilde{v}_1^2}{\pi^2\langle v\rangle^2}\left[e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}\right]_{\sqrt{2}\tilde{v}_1}^{\infty}=-K\frac{32\tilde{v}_1^2}{\pi^2\langle v\rangle^2}\left[\varnothing-e^{-\frac{4}{\pi}\frac{2\tilde{v}_1^2}{\langle v\rangle^2}}\right]=K\frac{\pi^2\langle v\rangle^4}{32\tilde{v}_1^2}\left(\frac{32\tilde{v}_1^2}{\pi^2\langle v\rangle^3}e^{-\frac{4}{\pi}\frac{\tilde{v}_1^2}{\langle v\rangle^2}}\right)^2=\boxed{K\frac{\pi^2\langle v\rangle^4}{32\tilde{v}_1^2}f^2(\tilde{v}_1)}$$

$$I_7 = K\int_{\tilde v_1}^{\sqrt{2}\,\tilde v_1} \frac{1024}{\pi^4\langle v\rangle^6}\,\tilde v_1\, r\,(2\tilde v_1^2 - r^2)\sqrt{r^2-\tilde v_1^2}\; e^{-\frac{4}{\pi}\frac{r^2}{\langle v\rangle^2}}\,dr =$$

Q

$$\boxed{u=\sqrt{r^2-\tilde v_1^2}}$$

$$\boxed{r^2 = u^2+\tilde v_1^2}$$

$$\boxed{u^2 = r^2 - \tilde v_1^2} \qquad r \ge \tilde v_1 \quad u^2>0$$

$$u>0$$

$$r=\tilde v_1 \to u^2 = \tilde v_1^2 - \tilde v_1^2 = 0$$

$$r=\sqrt{2}\,\tilde v_1 \to u^2 = 2\tilde v_1^2 - \tilde v_1^2 \to u^2 = \tilde v_1^2 \to u = \tilde v_1$$

$$\cancel{2}u\,du = \cancel{2}r\,dr$$

$$I_7 = K\int_0^{\tilde v_1} \frac{1024}{\pi^4\langle v\rangle^6}\,\tilde v_1\, u\,(2\tilde v_1^2 - u^2 - \cancel{\tilde v_1^2})\,u\, e^{-\frac{4}{\pi}\frac{u^2+\tilde v_1^2}{\langle v\rangle^2}}\,du =$$

$$= K\int_0^{\tilde v_1} \frac{1024}{\pi^4\langle v\rangle^6}\,\tilde v_1(\tilde v_1^2-u^2)u^2\, e^{-\frac{4}{\pi}\frac{\tilde v_1^2}{\langle v\rangle^2}}\, e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}}\,du =$$

$$= K\int_0^{\tilde v_1} \frac{32}{\tilde v_1\pi^2\langle v\rangle^3}\,\underbrace{\frac{32}{\pi^2}\frac{\tilde v_1^2}{\langle v\rangle^3}e^{-\frac{4}{\pi}\frac{\tilde v_1^2}{\langle v\rangle^2}}}_{f(\tilde v_1)}\,(\tilde v_1^2-u^2)u^2\, e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}}\,du =$$

$$= Kf(\tilde v_1)\int_0^{\tilde v_1} \frac{32\tilde v_1}{\pi^2\langle v\rangle^3}\,u^2 e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}}\,du + Kf(\tilde v_1)\int_0^{\tilde v_1} -\frac{32}{\tilde v_1\pi^2\langle v\rangle^3}\,u^4 e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}}\,du =$$

$$= Kf(\tilde v_1)\int_0^{\tilde v_1} \tilde v_1\,\frac{32}{\pi^2\langle v\rangle^3}\,u^2 e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}}\,du + Kf(\tilde v_1)\int_0^{\tilde v_1} \underbrace{\frac{4u^3}{\tilde v_1\pi\langle v\rangle}}_{g}\cdot\underbrace{\left(-\frac{8u}{\pi\langle v\rangle^2}\right)e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}}}_{g'}\,du =$$

$$= Kf(\tilde v_1)\tilde v_1 F(\tilde v_1) + Kf(\tilde v_1)\left[\frac{4u^3}{\tilde v_1\pi\langle v\rangle}e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}}\right]_0^{\tilde v_1} - Kf(\tilde v_1)\int_0^{\tilde v_1} \frac{12u^2}{\tilde v_1\pi\langle v\rangle}\,e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}}\,du =$$

$$= Kf(\tilde v_1)\tilde v_1 F(\tilde v_1) + Kf(\tilde v_1)\left[\frac{4\tilde v_1^{\cancel{3}\,2}}{\cancel{\tilde v_1}\pi\langle v\rangle}e^{-\frac{4}{\pi}\frac{\tilde v_1^2}{\langle v\rangle^2}} - 0\right]$$

$$-Kf(\tilde v_1)\int_0^{\tilde v_1} \frac{3\pi\langle v\rangle^2}{8\tilde v_1}\cdot\frac{32u^2}{\pi^2\langle v\rangle^3}\,e^{-\frac{4}{\pi}\frac{u^2}{\langle v\rangle^2}}\,du =$$

$$= Kf(\tilde v_1)\tilde v_1 F(\tilde v_1) + Kf(\tilde v_1)\frac{\pi\langle v\rangle^2}{8}\cdot\underbrace{\frac{32}{\pi^2}\frac{\tilde v_1^2}{\langle v\rangle^3}e^{-\frac{4}{\pi}\frac{\tilde v_1^2}{\langle v\rangle^2}}}_{f(\tilde v_1)} - Kf(\tilde v_1)\frac{3}{8}\frac{\pi\langle v\rangle^2}{\tilde v_1}F(\tilde v_1) =$$

$$\boxed{= Kf(\tilde v_1)\tilde v_1 F(\tilde v_1) + K\frac{\pi\langle v\rangle^2}{8}f^2(\tilde v_1) - K\frac{3}{8}\frac{\pi\langle v\rangle^2}{\tilde v_1}f(\tilde v_1)F(\tilde v_1)}$$

$$I_5 + I_6 = I_7 + I_8 + I_6 =$$

$$= \underbrace{K f(\tilde{v}_1)\, \tilde{v}_1 F(\tilde{v}_1) + K\frac{\pi\langle v\rangle^2}{8} f^2(\tilde{v}_1) - K\frac{3}{8}\frac{\pi\langle v\rangle^2}{\tilde{v}_1} f(\tilde{v}_1)F(\tilde{v}_1)}_{I_7} + \underbrace{K\frac{\pi^2\langle v\rangle^4}{32\tilde{v}_1^2} f^2(\tilde{v}_1)}_{I_8}$$

R

$$\underbrace{+K\frac{\pi\langle v\rangle^2}{2\tilde{v}_1} f(\tilde{v}_1)F(\tilde{v}_1)}_{I_6}$$

$$= K f(\tilde{v}_1)\left[\tilde{v}_1 F(\tilde{v}_1) + \left(\frac{1}{2} - \frac{3}{8}\right)\pi\langle v\rangle^2 \frac{F(\tilde{v}_1)}{\tilde{v}_1} + \frac{\pi^2\langle v\rangle^4}{32}\frac{f(\tilde{v}_1)}{\tilde{v}_1^2} + \frac{\pi\langle v\rangle^2}{8} f(\tilde{v}_1)\right] =$$

$$\boxed{= K f(\tilde{v}_1)\left[\tilde{v}_1 F(\tilde{v}_1) + \frac{\pi\langle v\rangle^2}{8}\frac{F(\tilde{v}_1)}{\tilde{v}_1} + \frac{\pi^2\langle v\rangle^4}{32}\frac{f(\tilde{v}_1)}{\tilde{v}_1^2} + \frac{\pi\langle v\rangle^2}{8} f(\tilde{v}_1)\right]}$$