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On a Quantum System with Memory

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Dedicated to Professor Dr. G. Vojta on the Occasion of his 60th Birthday

Abstract. We consider the integro-differential equation for the classical trajectory of an oscillator coupled to another one. On the quantum level the elimination of the coordinate A of the "unvisible" oscillator leads to an effective path integral (X, Ξ, μ) for the associated imaginary time stochastic process $t \in (-\infty, \infty) \to x(t)$. We prove reflection positivity of the measure $d\mu \approx F \cdot d\xi$, where $d\xi$ governes the free oscillator x and F is the counterpart of Feynman's influence functional. Finally, realizing the Hamiltonian semigroup $\exp(-tH)$, $t \geqq 0$, in the physical Hilbert space $\mathscr{H} = L^2(X, \Gamma, \mu)$, where $\Gamma \subseteq \Xi_+$, we try to understand what is memory.

Über ein Quantensystem mit Gedächtnis

Inhaltsübersicht. Wir untersuchen die Integro-Differentialgleichung für die klassische Trajektorie eines Oszillators, welcher an einen zweiten gekoppelt ist. Was passiert in der Quantenmechanik, wenn man die Koordinate des „unsichtbaren" Oszillators eliminiert? In imaginärer Zeit erhalten wir ein effektives Funktionalintegral (X, Ξ, μ) für den assoziierten stochastischen Prozeß $t \in (-\infty, \infty) \to x(t)$. Formal gilt $d\mu \approx F \cdot d\xi$. Hierbei beschreibt das Maß $d\xi$ die Dynamik des freien Oszillators „x" und F entspricht dem Feynmanschen Einflußfunktional. Wir zeigen, daß $d\mu$ reflexionspositiv ist und realisieren die Halbgruppe $\exp(-tH)$, $t \geqq 0$, in $\mathscr{H} = L^2(X, \Xi_+, \mu)$. Dabei versuchen wir zu verstehen, wie in der Quantentheorie Gedächtnis entsteht.

1. Introduction

Excited by the beauty of the $P(\varphi)_2$-theory in terms of a Markov process [1] mathematicians attacked more realistic models for fundamental interactions. The concept of imaginary time path integral for gauge and Fermi fields requires a linking with differential geometry and non-commutative probability. Actually, the main effort is to quantize gravity within a string theory.

A powerfull tool to solve those models is the method of effective action. As example let us look at the classical equations of motion

$$\left.\begin{aligned} m\ddot{x} &= -e\dot{A} \\ \ddot{A} + k^2 A &= e\dot{x} \end{aligned}\right\}, \tag{1}$$

for a system with two degrees of freedom. Following ideas of Feynman and Wheeler [2] one may try to eliminate the variable A. It is possible to describe the above caricature of QED in one space-time dimension purely in terms of a stochastic process $t \in (-\infty, \infty) \to x(t)$ with memory. Unfortunately [3], the Hamiltonian does not admit a normalized ground state Ω (with a bare particle potential the propagator is unknown) and hence the model does not fit the Osterwalder-Schrader like axioms of Klein [4]. To learn something below we analize to identical oscillators "x" and "A" with harmonic instead of "weak" coupling as in (1). Let ω_0 denote their eigenfrequency. Our textbook Lagrangean reads

$$\mathscr{L} = \mathscr{L}_0 - \frac{\lambda}{2}(x - A)^2, \quad \lambda \geq 0, \tag{2}$$

where $\mathscr{L}_0$ corresponds to the case $\lambda = 0$. Of course, in normal coordinates the dynamics factorizes. This on the quantum level allows us to write down immediately the spectrum of the Hamiltonian H

$$\operatorname{spec}(H) = \{ma + nb \colon m, n \in \{0\} \cup N\}, \tag{3}$$

where $a = \omega_0$, $b = +\sqrt{\omega_0^2 + 2\lambda}$ and N stands for the natural numbers. Let Ω denote the ground state. We claim that the system is completely determined by the amplitudes (Ω, Ψ_n), where

$$\Psi_n = \varphi(t_1)\,\varphi(t_2) \ldots \varphi(t_n)\,\Omega, \quad n \in N, \tag{4}$$

and $\varphi(t) = e^{itH}xe^{-itH}$. For $t_1, t_2, \ldots, t_n$ in an arbitrarily small time interval of length $\varepsilon > 0$ the wave functions Ψ_n carry "enough" information about the unvisible oscillator A too.

Theorem 1.

Let M_0 be the Abelian algebra on $\mathscr{H} = L^2(\mathbb{R}^2, dx\, dA)$ of bounded continuous functions $f(x)$, and let $P_t = \exp(-tH)$, $t \geq 0$. Assume $\lambda > 0$. Then the system defined by the Hamiltonian H satisfies the following conditions:

(i) $\quad \|(1 + H)^{-1/2}\, x(1 + H)^{-1/2}\| < +\infty$,

(ii) $\quad \Omega$ is cyclic for $M_0 \cup \{P_t, t \geq 0\}$,

(iii) $\quad$ For $f_j(x) \geq 0$, $j = 1, 2, \ldots, n$, holds

$$(\Omega, f_1 P_\varepsilon f_2 P_\varepsilon \ldots P_\varepsilon f_n \Omega) \geq 0. \tag{5}$$

Proof: Condition i) is the Glimm-Jaffe bound [5]. To see ii) we only must consider the action of P_t, $t \geq 0$, on coherent states $\Psi = \{\exp i\alpha x\}\,\Omega \in \mathscr{H}_0$. Finally, iii) follows from the positivity $P_t(xA, x'A') \geq 0$ of the Mehler kernel. End of proof. We remark that the vectors $\Psi = f(x)\Omega$, $f \in M_0$, span a proper subspace $\mathscr{H}_0$ in the total physical Hilbert space $\mathscr{H}$. In canonical systems the vacuum Ω is cyclic just for M_0 [6], i.e. one can take $\varepsilon = 0$.

2. Effective Action

On the classical level the elimination of the variable A leads to an integro-differential equation for the trajectory $t \to x(t)$. Let us fix some time interval $[0, \sigma]$ and a Green's function $D(s, t)$ for the operator $K = \omega^2 + \dfrac{d^2}{dt^2}$, where $2\omega^2 = a^2 + b^2$. We assume that the trajectory $t \to A(t)$ satisfies the boundary conditions

$$\alpha(t) = \sum_{s \in \partial[0,\sigma]} D(s, t) \frac{\overleftrightarrow{d}}{ds} A(s) = 0. \tag{6}$$

Then

$$(\ddot{x} + \omega^2 x)(t) = \lambda^2 \cdot \int_0^\sigma ds\, D(s, t)\, x(s), \quad t \in [0, \sigma], \tag{7}$$

for initial data $x(0)$ and $\dot{x}(0)$ has a unique solution. Together with $\alpha(t) \equiv 0$ it is equivalent to the coupled Euler-Lagrange equations resulting from (2). To convince ourselves that this is true we apply the differential operator K on both sides, use $\omega^4 - \lambda^2 = a^2 \cdot b^2$ and the identity

$$A(t) = \alpha(t) + \int_0^\sigma ds\, D(s, t)\, KA(s), \quad t \in [0, \sigma]. \tag{8}$$

Unfortunately, there is no universal action S_{eff} governing the classical trajectory $t \in [0, \sigma] \to x(t)$ (independent of the boundary conditions for "A"). Because of this obstacle we must go beyond the conventional schema of quantization. At real time and in units where $\hbar = 1$ Feynman's ansatz for the functional integral reads $\{\exp iS_{\text{eff}}\}\, dx(\cdot)$, where

$$S_{\text{eff}} = \frac{1}{2} \int_0^\sigma dt\, (\dot{x}^2 - \omega^2 x^2) + \frac{\lambda^2}{2} \int_0^\sigma \int_0^\sigma ds\, dt \frac{e^{-i|t-s|\omega}}{-2i\omega}\, x(s)\, x(t). \tag{9}$$

It is defined by the causal Green's function which produces complex boundary conditions $A \approx \exp(\pm it\omega)$, as $t \to \pm\infty$. Roughly speaking, "A" prefers to live in its own ground state γ. Let $U(s, t)$, $t \geq s$, denote the evolution operator for the time dependent Hamiltonian $J(t) = \frac{1}{2}(\dot{A}^2 + \omega^2 A^2) - \lambda x(t) \cdot A$ of the quantum oscillator "A", driven by the external force $\lambda x(t)$. Applying the principle of least action to S_{eff} we find a solution to

$$(\ddot{x} + \omega^2 x)(t) = \lambda(\gamma, A(t, [x(\cdot)])\, U(0, \sigma)\, \gamma), \quad t \in [0, \sigma], \tag{10}$$

where $A(t, [x(\cdot)]) = U(0, \sigma)\, AU(0, t)^*$. The above semiclassical equation makes the physical meaning of S_{eff} evident. We may check it passing to imaginary time. The existence of an associated stochastic process $t \in (-\infty, \infty) \to x(t)$ on some probability space (X, Ξ, μ) follows from the properties of $\Psi_n = \varphi(t_1)\, \varphi(t_2) \ldots \varphi(t_n)\, \Omega$ shown in theorem 1. We briefly recall Klein's construction. Let $f_j \in M_0$, $t_1 \leq t_2 \leq \ldots \leq t_n$ and $t_{j+1} - t_j = \varepsilon \geq 0$. By the Riesz-Markov representation theorem we can rewrite (5) as an integral

$$(\Omega, f_1 P_\varepsilon f_2 \ldots f_n \Omega) = \int_{\mathbb{R}^n} d\mu_{t_1 t_2 \ldots t_n} \prod_{j=1}^n f_j(x(t_j)). \tag{11}$$

According to Kolmogorov's theorem these measures admit a joint extension $d\mu$ to the smallest σ-algebra Ξ containing all $\Xi\, t_1 t_2 \ldots t_n$. We observe that (11) holds not only for $f_j \in M_0$ but just for polynomials. Indeed, the technical assumption guaranties finite moments which via analytic continuation give the amplitudes (Ω, Ψ_n). One may argue that $d\mu \approx F \cdot d\xi$, where $d\xi$ denotes the measure governing the free oscillator "x" and F is the counterpart of Feynman's influence functional. Since $d\xi$ is ergodic this cannot be true, except for a finite time interval. Let

$$\begin{aligned} F(0, \sigma) &= \int_{\mathfrak{A}} \exp\left\{\lambda \int_0^\sigma ds\, x(s)\, A(s)\right\} d\xi(A) \\ &= \exp\left\{\frac{\lambda^2}{2} \int_0^\sigma \int_0^\sigma ds\, dt \frac{e^{-|t-s|\omega}}{2\omega}\, x(s)\, x(t)\right\}. \end{aligned} \tag{12}$$

Then

$$(\Omega_0, P_\varepsilon f_1 P_\varepsilon \dots P_\varepsilon \Omega_0) = \int_X \left\{ F(0, \sigma) \cdot \prod_{j=1}^{n} f_j(x(t_j)) \right\} d\xi(x), \tag{13}$$

given $0 \leq t_1 \leq t_2 \leq \dots \leq t_n < \sigma$, which is the Feynman-Kac formula. We remark that $\Omega_0 = \gamma \otimes \gamma$ is the ground state for H_0, i.e. when $\lambda = 0$. Moreover, the semigroup $P_t, t \geq 0$, is positivity preserving in the usual sense. So we can choose $\Omega(x, A) \geq 0$, and (5) holds for $f_j = f_j(x, A) \geq 0$. The bounded continuous functions of both variables x, A generate a maximal Abelian algebra $M \subseteq M_0 \cup P_t, t \geq 0\}$ on $\mathscr{H}$. As well known [7]

$$\overline{M \cup \{P_t, t \geq 0\}} = \mathscr{B}(\mathscr{H}). \tag{14}$$

Hence there is a Markov process $t \in (-\infty, \infty) \to (x(t), A(t))$ on $(\mathscr{C}(M), \Xi \times \mathfrak{A}, \nu)$ such that $d\nu/\Xi = d\mu$. With other words "integrating out" the dummy variables $A(t)$, $t \in (-\infty, \infty)$, we recover precisely the measure $d\mu$ of our model.

3. Imaginary Time Axioms

To control the Feynman-Kac formula (13) in the limit $\sigma \to +\infty$ we will use the famous reflection positivity condition of Osterwalder and Schrader.

Let

$$\theta x(t)\, \theta^* = x(-t),\; t \in (-\infty, \infty), \tag{15}$$

denote time reflection. Clearly, $\theta^2 = 1$. θ lifts to a unitary operator on $L^2(X, \Xi, \mu)$. By construction Ξ is generated by Ξ_t, $t \in (-\infty, \infty)$. Moreover, the time translation group $\{T_t, t \in (-\infty, \infty)\}$ acts ergodically, i.e. the only invariant sets in Ξ are Φ, and X. Let Ξ_+ denote the σ-subalgebra in Ξ generated by Ξt, $t \geq 0$, and $E_+ = E(\cdot \mid \Xi_+)$ conditional expectation. Then the crucial property in question reads $E_+ \theta E_+ \geq 0$, on $L^2(\chi, \Xi, \mu)$. Since the measure $d\mu$ is Gaussian everything is hidden in the covariance $C(s, t)$. It defines the inverse of a non-local positive operator on $L^2(\mathbb{R}, dt)$ called Euclidean action. Indeed, for smoothed linear random functions $l = x(h)$, $h \in S(\mathbb{R}_+)$, using

$$C(s, t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \left(\frac{1}{p^2 + a^2} + \frac{1}{p^2 + b^2} \right) e^{i(t-s)p}, \tag{16}$$

we get

$$\int_X l^* \cdot \theta l \, d\mu = \frac{1}{2ab} (b\,|\alpha|^2 + a\,|\beta|^2) \geq 0. \tag{17}$$

Above $\alpha = \hat{h}(ia)$ and $\beta = \hat{h}(ib)$ is the Fourier transform. To understand why $d\mu$ is reflection positive we may refer to $d\nu$. But we forget about the oscillator "A". The idea is to introduce a functional $F(-\sigma, \sigma)$ for the symmetric interval $[0, \sigma] \cup \theta(0, \sigma]$ and to choose appropriate boundary conditions [8].

Theorem 2.

Let $G(s, t)$ be the Dirichlet Green's function for the elliptic differential operator $\left(\omega^2 - \frac{d^2}{dt^2}\right) \geq 0$ on the imaginary time interval $[-\sigma, \sigma]$. Then for any $\Xi(0, \sigma)$-measurable function f holds

$$\int_\chi f^* \cdot \theta f \exp\left\{\frac{\lambda^2}{2} (x, Gx)\right\} d\xi(x) \geq 0. \tag{18}$$

Proof: Decomposing the square $[0, \sigma] \times [0, \sigma] \subseteq \mathbb{R}^2$, we may rewrite $\frac{1}{2}$ $(x, Gx) = Y + \theta Y + Z \cdot \theta Z$, with functions Y and Z measurable with respect to $\Xi(0, \sigma) \subseteq \Xi_+$. Now we apply a standard trick [9]: Expanding the exponent and using $\lambda^2 \geq 0$ we get for the integral in question a sum of positive terms. End of proof. We remark that the functional $F(-\sigma, \sigma)$ given by the memory is not multiplicative and hence breakes the Markov property. Nelson's construction does not work. Due to [10] we redefine the physical Hilbert space as the subspace

$$\mathscr{H} \approx L^2(X, \Gamma, \mu) \tag{19}$$

in $L^2(X, \Xi_+, \mu)$, where $W = (E_+, \theta E_+)^{1/2}$ is strictly positive. Let $\Xi_0 = \Xi_+ \cap \Xi_-$ which is contained in Γ. Precisely for $f \in L^2(\chi, \Xi_0, \mu)$ holds $Wf = f \cdot \Omega \in \mathscr{H}_0$, so that we may identify M_0 with $L^\infty(X, \Xi_0, \mu)$. However, in general the mapping W which is a selfadjoint contraction on $L^2(X, \Xi, \mu)$, acts nontrivially. It intertwines imaginary with "physical" time evolution according to the formula

$$W\left(\prod_{j=1}^{n} x(t_j)\right) = \hat{\Psi}_n(t_1, t_2, \ldots, t_n) \in \mathscr{H}, \tag{20}$$

where $t_1 \leq t_2 \leq \ldots \leq t_n$. As remarked the existence of those vectors is ensured only by additional assumptions on the regularity of the trajectories $t \in (-\infty, \infty) \to x(t)$, i.e. $d\mu$. W commutes with second quantization and hence is completely determined on linear functions $l = x(h)$, $h \in S(\mathbb{R}_+)$. Let $k(t) = \left(\eta + \frac{d}{dt}\right) \varkappa(t) \exp(-t\omega)$, where $\eta = \frac{ab}{\omega}$ and $\varkappa(t)$ denotes the Heaviside step function. Using (16) one can show that

$$\begin{aligned} \Psi &= Wx(h) \\ &\approx (b\alpha + \alpha\beta)x\Omega + w \cdot (\beta - \alpha)\, z\Omega, \end{aligned} \tag{21}$$

where $x = x(0)$, $z = x(k)$ and $0 \leq w(\lambda) \leq 1$, span the two-particle subspace $\mathscr{R}$ in $\mathscr{H}$. From the relation $d\mu\,|_\Gamma = |\Omega(x, z)|^2\, dx\, dz$ we are able to read off the ground state wave function Ω in the new variables x, z. Similarly as on the classical level the coordinate A of the univisible oscillator disappeared. This seems t_0 be in conflict with a remark of Feynman concerning a primitive model of QED as described by equation (1): "No wave function $\Psi(t)$ can be defined to give the amplitude that the particle is at the place x at a particular time t. Such an amplitude would be insufficient for continuing calculations into the future, since at any time one must also know what the oscillator "A" is doing" [11].

4. Hamiltonian Semigroup

To convince ourselves that there is no contradiction we analyze the semigroup $P_t = \exp(-tH)$, $t \geq 0$, for the process $t \in (-\infty, \infty) \to x(t)$ on (X, Ξ, μ) in more detail. By (3) the matrix $p_t = \exp(-tH)/\mathscr{R}$, $t \geq 0$, diagonalizes in the natural basis

$$\left.\begin{aligned} q &= \frac{1}{b + a}(bx - \omega z)\,\Omega, \\ r &= \frac{1}{b + a}(ax + \omega z)\,\Omega. \end{aligned}\right\} \tag{22}$$

We observe that W does not. Let e_0 denote the projection operator in $\mathscr{R}$ onto the subspace spanned by $\Psi = x \cdot \Omega$. We claim that $e_0 p_t e_0$, $t \geq 0$, do not build a semigroup. Indeed, in the basis $\{x\Omega, z\Omega\}$ of $\mathscr{R}$ we find

$$p_t = \frac{1}{a+b} \begin{bmatrix} b\alpha + a\beta & \eta(\beta - \alpha) \\ \omega(\beta - \alpha) & a\alpha + b\beta \end{bmatrix}, \tag{23}$$

and hence

$$e_0 p_t e_0 = \exp\{-\tau(t)\,\omega\} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{24}$$

for some convex function $0 \leq \tau(t) \leq 1$. We conclude that, for $\lambda > 0$, the information in $E_0 T_t E_0$, $t \geq 0$, is insufficient to describe the stochastic process $t \in (-\infty, \infty) \to x(t)$ governed by the measure $d\mu$. Above α, β stand shortly for $\exp(-ta)$ respectively $\exp(-tb)$, $t \geq 0$. At real time we have the following result. The wave function $\Psi(t) \in \mathscr{R}$ with components $u = (x\Omega, \Psi(t))$ and $v = (z\Omega, \Psi(t))$ satisfies the equation

$$i \frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{2\omega}{a+b} \begin{pmatrix} \dfrac{ab}{\omega}(u + \varepsilon \cdot v) \\ \omega(v + \varepsilon \cdot u) \end{pmatrix}, \quad t \in (-\infty, \infty), \tag{25}$$

where $\varepsilon = \dfrac{1}{2\omega}(b - a) \geq 0$. To check it we only must calculate the infinitesimal generator of p_t, $t \geq 0$. Via second quantization we obtain the Schrödinger equation for arbitrary $\Psi(t) \in \mathscr{H}$. There is a quantum counterpart of (7).

Theorem 3.

Let $D^+(s, t) = i \cdot \varkappa(t - s)\, e^{-i(t-s)c}$, where $c = \dfrac{2\omega^2}{a+b}$, denote Green's function for the first order differential operator $L = \left(c - i\dfrac{d}{dt}\right)$ which vanishes for $t \leq s$. Then $u = (x\Omega, \Psi(t))$ is governed by the integro-differential equation

$$\left(\omega - i\frac{d}{dt}\right) u(t) = \varepsilon^2 \cdot c\omega \int_{-\infty}^{t} ds\, D^+(s, t)\, u(s), \quad t \in (-\infty, \infty). \tag{26}$$

Proof: Applying the operator on the left once more we find the correct solutions $u(t) = \exp(-ita)$ and $\exp(-itb)$. End of proof. Some years ago Królikowski and Rzewuski derived a one-time equation for a similar problem [12]. We may compare the time evolution of the quantum state $\Psi(t) = \exp(-itH) x\Omega \in \mathscr{R}$ with the corresponding classical beating solution. Let $\varepsilon = \dfrac{1}{2}(b - a)$ and $\delta = \left(\dfrac{b-a}{b+a}\right)$. Then the probability amplitude to find the system at time $t \geq 0$ again in the initial state $\Psi(0) \in \mathscr{R} \cap \mathscr{H}_0$ is given by

$$\frac{(\Psi(0), \Psi(t))}{\|\Psi(0)\| \cdot \|\Psi(t)\|} = \exp\left\{\frac{-it}{2}(b + a)\right\} \cdot (\cos t\varepsilon + i \cdot \delta \sin t\varepsilon). \tag{27}$$

The realization of the process $t \in (-\infty, \infty) \to x(t)$ on the probability space $(\mathscr{C}(M), \Xi \times \mathfrak{A}, \nu)$ simplifies everything. Thei dea to recover the Markov property is not new. Karwowski [13] tried to lift $E_+ \theta E_+ \geq 0$ to a projection operator in the larger Hilbert space $\mathscr{K} = L^2(\mathscr{C}(M), \Xi \times \mathfrak{A}, \nu)$. We claim that the natural embedding

$$J: L^2(X, \Xi, \mu) \to \mathscr{K} \tag{28}$$

is isometric and local. It intertwines θ and T_t with the corresponding operators living on $\mathcal{K}$. Applying Nelson's reconstruction to the two-dimensional Markov process $t \in (-\infty, \infty) \to (x(t), A(t))$ we get a semigroup $\{Q_t = \pi T_t, t \geq 0\}$ in $L^2(\mathscr{C}(M(\Xi \times \mathfrak{A})_0, \nu) \approx \mathscr{H}$, where π denotes the projection operator. Let $V = \pi J$. We find that

$$V: L^2(\chi, \Xi_+, \mu) \to \mathscr{H} \tag{29}$$

has polar decomposition $V = \gamma W$, where the unitary part γ identifies the variables q, r with the normal coordinates built up from x and A. The consistency condition reads $\gamma/L^2(\chi, \Xi_0, \mu) = J$. Moreover, for $g \in M_0$ and arbitrary $f \in L^2(\chi, \Xi_+, \mu)$ holds

$$\begin{aligned} W(g \cdot T_t f) &= \gamma^* \pi (Jg \cdot T_t Jf) \\ &= \gamma^* Jg \cdot \gamma^* \pi\, T_t Jf \\ &= g \cdot \gamma^* Q_t \gamma W f, \quad t \geq 0. \end{aligned} \tag{30}$$

5. Conclusions

Our aim was to understand the mechanism by which a quantum system acquires memory. We learned this within the simple model of two coupled oscillators distinguishing "x". But "A" remained "unvisible". In Thirring's book we read: "Wohl kann man messen, was man will, doch nicht alles auf einmal. Man mißt in Wirklichkeit doch nur ein kleines Subsystem, so daß es von Interesse ist, das Gesamtsystem in ein zu beobachtendes (offenes) System und einen Rest, der als Wärmebad fungiert, zu teilen ... Nun wirkt das System auf das Bad und dieses wieder auf das System zurück. Diese Rückwirkung beeinflußt das System aber erst später, so daß die momentane Zeitentwicklung des Systems also von seiner ganzen Vorgeschichte abhängt" [14].

At the end let us calculate the imaginary time amplitude $(\Phi, \exp(-tH)\, \Psi(0))$, $t \geq 0$, where $\Psi(0) = x\,\Omega$ as above and $\Phi = A\,\Omega$. Using the embedding we find immediately the expression

$$\iint_{\mathscr{C}(M)} A(0) x(t)\, d\nu = \left(\frac{b\alpha - a\beta}{b + a}\right) \geq 0, \tag{31}$$

which means ferromagnetic coupling of "x" and "A". Consequently, the correlation of $x_j = x(t_j)$, $j = 1, 2, \ldots, n$ increases with parameter $0 \leq \lambda < 1$, too. For the cylinder measures defined by (11) we can rewrite

$$d\mu/\Xi_{t_1 t_2 \ldots tn} \approx \exp\left\{\frac{\lambda^2}{2}(x_i G^{ij} x_j)\right\} d\xi(x_1, x_2, \ldots, x_n). \tag{32}$$

We claim that the matrix G^{ij}, $i, j = 1, 2, \ldots, n$ giving long-range forces, satisfies reflection positivity which is the known condition for the existence of a transfer matrix in classical spin systems.

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