MID TERM - CM 120

DISCRETE MATHEMATICS

QUESTION 1

a) USING THE FORMULA $|A \cup B| = |A| + |B| - |A \cap B|$

THEN $|A \cap B| = |A| + |B| - |A \cup B|$

$= 20 + 25 - 35$

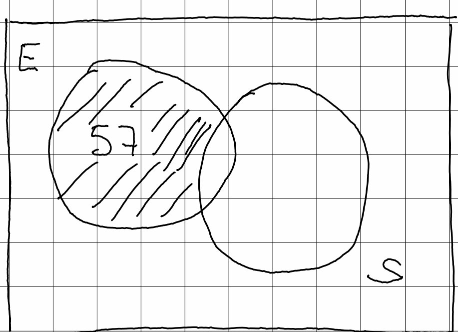
$= 45 - 35$

$= 10$

b) $n(E \cup S) = 100$ $\quad n(E \cup S) = n(E + S) - n(E \cap S)$

i) $n(E) = 73$ $\quad = 100 = 73 + 43 - n(E \cap S)$

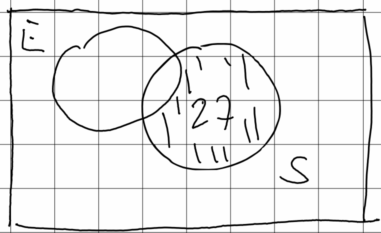
$n(S) = 43$ $\quad \boxed{n(E \cap S) = 16}$

ii) 

$\alpha E = E - (E \cap S)$

$\boxed{73 - 16 = 57}$

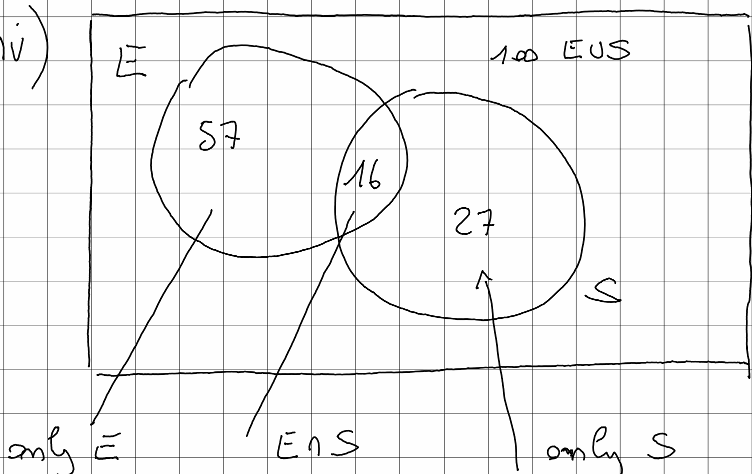
iii)



$n S = S - (E \cap S)$

$\boxed{43 - 16 = 27}$

iv)



c) first we have $P(A \cap B) \subseteq P(A) \cap P(B)$

let $\forall x \in P(A \cap B)$

$x \subseteq (A \cap B)$

$x \subseteq A$ and $x \subseteq B$

$x \in P(A)$ and $x \in P(B)$

$x \in P(A) \cap P(B)$

$x \in P(A \cap B)$ then $x \in P(A) \cap P(B)$

$P(A \cap B) \subseteq P(A) \cap P(B) \longrightarrow$ 1

To prove $P(A) \cap P(B) \subseteq P(A \cap B)$

Let $\forall y \in P(A) \cap P(B)$

$y \in P(A)$ and $y \in P(B)$

$y \subseteq A$ and $y \subseteq B$

$y \subseteq (A \cap B)$

$y \in P(A \cap B)$

$y \in P(A) \cap P(B)$ then $y \in P(A \cap B)$

$P(A) \cap P(B) \subseteq P(A \cap B) \rightarrow (2)$

from 1 and 2 we have

$P(A \cap B) = P(A) \cap P(B)$

EXAMPLE

Let $A = \{2, 5\}$

$B = \{6, 5\}$

$A \cap B = \{5\}$

$P(A) = \{\{2\}, \{5\}, \{2,5\}, \emptyset\}$

$P(B) = \{\{6\}, \{5\}, \{5,6\} \emptyset\}$

$P(A \cap B) = \{\{5\}, \emptyset\} \rightarrow (3)$

$P(A) \cap P(B) = \{\{5\}, \emptyset\} \rightarrow (4)$

from 3 and 4 we have

$P(A \cap B) = P(A) \cap P(B)$

ii) To verify $P(A \cup B) = P(A) \cup P(B)$ or not

we know that,

$P(A \cup B) \neq P(A) \cup P(B)$

because, we know that

$P(A \cup B) = \{\text{subset}, A \cup B\}$

but $P(A) \cup P(B) = \{\text{subsets}\}$

so, $P(A \cup B) \neq P(A) \cup P(B)$

the correct one is

$P(A \cup B) = P(A) \cup P(B) \cup (A \cup B)$

EXAMPLE:

Let $A = \{5\}$

$B = \{6, 7\}$

$A \cup B = \{5, 6, 7\}$

Now $P(A) = \{\{5\}, \emptyset\}$

$P(B) = \{\{6\}, \{7\}, \{6,7\}, \emptyset\}$

$P(A \cup B) = \{\{5\}, \{6\}, \{7\}, \{5,6\}, \{6,7\}, \{7,5\}, \{5,6,7\}, \emptyset\} \rightarrow (1)$

$P(A) \cup P(B) = \{\{5\}, \{6\}, \{7\}, \{6,7\}, \emptyset\} \rightarrow (2)$

from 1 to 2 we have

$P(A \cup B) = P(A) \cup P(B) \cup (A \cup B)$

iii) To prove $P(A) \cup P(B) \subseteq P(A \cup B)$

Let $x \in P(A) \cup P(B)$

$x \in P(A)$ or $x \in P(B)$

$x \subseteq A$ or $x \subseteq B$

If $x \subseteq A \subseteq A \cup B \rightarrow x \subseteq A \cup B$

$x \in P(A \cup B)$

If $x \subseteq B$ $A \cup B \rightarrow x \subseteq A \cup B$

$x \in P(A \cup B)$

In any of these two cases we have $x \in P(A \cup B)$
$x \in P(A) \cup P(B)$ then $x \in P(A \cup B)$

$\Rightarrow P(A) \cup P(B) \subseteq P(A \cup B)$

QUESTION 2

a) i) $f: R \to R$ with $f(x) = \dfrac{1}{1-x^2}$

F(x) is not a function $\forall f: R \to R$

Because f(x) is not $x = 1$

if f(x) is a function then $\underbrace{x \in R - \{1\}}_{\text{DOMAIN}}$

ii) $f: Z \to Z$ with $f(x) = \dfrac{x}{2}$

F(x) is not a function because f(x) is a function only if

all $x \in$ Real number so $R \to R$

iii) $f(R \to R)$ with $f(x) = \ln(x)$

Here, f(x) is not a function $\forall x \in R$

DOMAIN $= R^+$

b) Let $f: R \to R$ and $g: R \to R$ with

$f(x) = x+2$

$g(x) = -x$ find $g \circ f$

$g(f(x)) = -f(x) = -(x+2)$

$(g \circ f)^{-1} = g(f(x)) = -x-2$

; $y = -x-2$

put $y \to x$ and $x \to y$

$x = -y-2$

$y = -x-2$

$(g \circ f)^{-1} = -x-2 = g \circ f$

f^{-1}: $f(x) = x+2$

$y = x+2$ put $y \to x$ and $x \to y$

$x = y+2$ or $y = x-2$

$f^{-1}(x) = \cancel{+}x-2$

g^{-1}: $g(x) = -y$

$y = -x$

$y = -x$ put $y \to x$ and $x \to y$

$y = -x$

$f^{-1}(x) = -x$

c) i) DOMAIN: $\mathbb{R}^+$

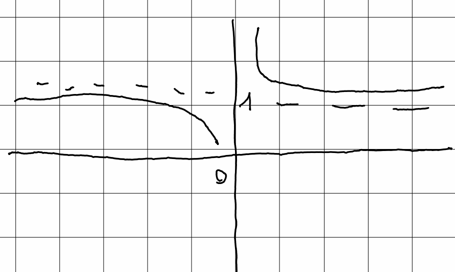
CODOMAIN: $\mathbb{R}$

$$f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$$

When $x \in \mathbb{R}^+$, $f(x)$ give one value so we can say that $\forall\, x \in \mathbb{R}^+$ given function is one to one

ii) If RANGE of $f(x)$ is equal to codomain of function then the given function is onto.

So now we plot the graph of the function $f(x)$



Range of $f(x) = \mathbb{R}^+ - \{1\}$

$f(x)$ is not an onto function

D) i)

$A = \{a, b\}$ and $B = \{a, c\}$ then $FA = |A| = 2$

and $F(B) = |B| = 2$

Hence $F(A) = F(B)$ but A and B are not equal

Hence not one to one

ii)

Also if A is element of $P\{a, b, c\}$ then it means that A is a subset of $\{a, b, c\}$ and hence A can have maximum 3 elements. Thus

$|\{a, b, c\}| <= 3$ but $\mathbb{Z}$ has elements like 4 and 5 so $|A|$ cannot be equal to 4 for any A. Hence $F(A)$ CANNOT be equal to 4 for any A and thus F cannot be onto.

E) $f: A \to B$ and $g: B \to C$ are one to one functions.

$g \circ f$ is a function from $A \to C$

Let $x_1, x_2 \in A$

$g \circ f(x_1) = g \circ f(x_2)$

$g(f(x_1)) = g(f(x_2))$

$f(x_1) = f(x_2)$ g is one To one

$x_1 = x_2$

f is one To one function proved.

QUESTION 3

i) To find the Truth value in this case, let we assume that $p \wedge \neg q \equiv A$, $q \wedge r \equiv B$ and $q \vee \neg q \equiv C$

then we will have To be find the Truth value of $(A \to B) \to C$ To do this we construct Truth Table for A, B and C

TRUTH TABLE A

p	q	$\neg q$	$A \equiv p \wedge \neg q$
F	T	F	F

TRUTH TABLE B

q	r	$B \equiv q \wedge r$
T	F	F

Now we construct Truth Table for $A \to B$

i.e $(p \wedge \neg q) \to (q \wedge r)$

$A \to B$ is defined as;

$A \to B \equiv A \vee B$

So, we can write

$(p \wedge \neg q) \to (q \wedge r) \equiv \neg\{(p \wedge \neg q)\} \vee (q \vee r)$

So we can write;

$(A \to B) \to C \equiv \neg(A \to B) \vee C$

$\equiv \neg(\neg A \vee B) \vee C$

$\equiv (A \wedge B) \vee C$

For the MORGAN'S LAW $\equiv \neg A \vee \neg B$

So we finally construct Truth Table for

$(A \wedge \neg B) \vee C$ instead of $(A \to B) \to C$ see the Table

s	q	$\neg q$	$s \vee \neg q \equiv C$
T	T	F	T

So, this is the Table for $C \equiv s \vee \neg q$

Now we are going To construct Truth Table for $(A \wedge \neg B) \vee C$

which is equivalent To

$(A \wedge \neg B) \vee C \equiv ((p \wedge \neg q) \to (q \wedge r)) \to (s \vee \neg q)$

A	B	C	$\neg B$	$(A \wedge \neg B)$	$(A \wedge \neg B) \vee C$
F	F	T	T	F	T

So, we can see that, this Table says that if p, q are false and q and s are True then

$((p \wedge \neg q) \rightarrow (q \wedge r)) \rightarrow (s \vee \neg q)$ will be $\boxed{\text{True}}$

ii) we assume that $A \equiv p \vee q$, $B \equiv q \vee s$, $C \equiv \neg r \vee p$

$D \equiv q \vee s$

The we have to find the Truth Table for

$(A \wedge B) \rightarrow (C \wedge D)$ which is equivalent to

$\neg (A \wedge B) \vee (C \wedge D) \equiv (\neg A \vee \neg B) \vee (C \wedge D)$

So $(A \wedge B) \rightarrow (C \wedge D) \equiv (\neg A \vee \neg B) \vee (C \wedge D)$

Truth Table for A and $\neg A$

p	q	$A \equiv p \vee q$	$\neg A$
F	T	T	F

Truth Table for B and $\neg B$:

q	s	$B \equiv q \vee s$	$\neg B$
T	T	T	F

TRUTH Table for C

r	$\neg r$	p	$C \equiv \neg r \vee p$
F	T	F	T

Truth Table for D

q	s	$D \equiv q \vee s$
T	T	T

propositions B and D are equal. Now we construct

the Truth Table for

$((p \vee q) \vee (q \vee s)) \to ((\neg r \vee p) \wedge (q \vee s))$ which is equivalent

to $(A \wedge B) \to (C \wedge D)$ or $(A \wedge B) \to (C \wedge B)$

B and D are same i.e $q \vee s$, $(\neg A \vee \neg B) \vee (C \wedge B)$

$\neg A$	$\neg B$	B	C	$(\neg A \vee \neg B)$	$C \wedge B$	$(\neg A \vee \neg B) \vee (C \wedge B)$
F	F	T	T	F	T	T

So if p and r are false q, s are true so

$((p \vee q) \wedge (q \vee s)) \to ((\neg r \vee p) \wedge (q \vee s))$ is $\boxed{\text{True}}$

B) i) If Sofia is happy and paints a picture then Tomiu is not happy

This is a conditional statement ;

$$(p \wedge q) \rightarrow \neg r$$

ii) p only if q if p then q are same

$$p \rightarrow q$$

iii) $q \vee \neg p$

c) i) $\forall x \in R$ if $x > 3$ then $x^2 > 9$

CONTRAPOSITIVE : $\forall x \in R$ if $x^2 \leq 9$ then $x \leq 3$

$(\neg q \rightarrow \neg p)$

CONVERSE : $\forall x \in R$ if $x^2 > 9$ then $x > 3$

$(q \rightarrow p)$

INVERSE : $\forall x \in R$ if $x \leq 3$ then $x^2 \leq 9$

$(\neg p \rightarrow \neg q)$

D) p, q, r are three propositions

$(p \rightarrow (q \vee r) \leftarrow ((p \wedge \neg q) \rightarrow r)$ — I

p	q	r	$\neg q$	$(q \vee r)$	$p \to (q \vee r)$	$(p \wedge \neg q)$	$(p \wedge \neg q) \to r$	I
T	T	T	F	T	T	F	T	T
T	T	F	F	T	T	F	T	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	F	T	F	T
F	T	T	F	T	T	F	T	T
F	T	F	F	T	T	F	T	T
F	F	T	T	T	T	F	T	T
F	F	F	T	F	T	F	T	T

QUESTION 4

A) i) Given that $\exists x \forall y\, P(x,y)$ is True.

Now negation of given statement i.e $\neg[\exists x \forall y\, P(x,y)] \equiv$

$\forall x \exists y\, \neg P(x,y)$ statement is True $\Rightarrow$ negation is False

$\forall x \exists y\, \neg P(x,y)$ is $\boxed{\text{False}}$

ii) Now consider $P(x,y)$ is statement $x+y=0$; x, y are integers then $\exists x \forall y\, P(x,y)$

So $\forall x \forall y\, P(x,y)$ means for all x, for all y $x+y=0$

But $x=2, y=2$ then $x+y = 2+2 = 4 \neq 0 \Rightarrow \boxed{\text{False}}$

iii) $\exists x \exists y\, P(x,y)$ i.e there is x and there is y such that $x+y=0$ so if we let $x=-y$ then $x+y=0 \Rightarrow \boxed{\text{True}}$

B) i) Symbolic Form

$P_1: q \to p$
$P_2: p$
$C: q$

ii)

p	q	$p \to q$	$(p \to q) \wedge r$	$[(p \to q) \wedge r] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

iii) From part ii, argument is valid

c) Let Jx : x got the job

Mx : x has a Master's degree

Wx : x has 5 years of work experience.

Johnny : j , Domain of x : the entire people.

a) $(\forall x)(Jx \to (Mx \vee Wx))$

b) Mj

c) Jj

d) $\neg Wj$

The argument is NOT VALID

Because

1) $(\forall x)(Jx \to (Mx \vee Wx))$ Premise

2) Mj Premise

3) Jj Premise

4) $Jj \to (Mj \vee Wj)$ Universal instantiation from 1)

5) $Mj \vee Wj$ Modus Ponens 3, 4

If we assign Mj = True and Wj = True then all the premises would still hold True but $\neg Wj$ will be false in this case (2) and (5) do not imply (d)

D) Claim: $(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x)) \to \forall x \in D, P(x) \wedge Q(x)$

Proof: suppose $(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$ is True

$\to$ ($\forall x \in D, P(x)$ and $\forall x \in D, Q(x)$) is True

$\to$ $\forall x \in D, P(x)$ is True and $\forall x \in D, Q(x)$ is True

$\to$ $\forall x \in D, P(x)$ and $Q(x)$ is True and

$\forall x \in D, (P(x) \wedge Q(x))$ is True

Claim: $(\forall x \in D, P(x) \wedge Q(x)) \to (\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$

Proof: suppose $(\forall x \in D, P(x) \wedge Q(x))$ is True

$\forall x \in D, (P(x)$ and $Q(x))$ is True

$\forall x \in D, P(x)$ is True and $Q(x)$ is True

$\forall x \in D, P(x)$ is True and $\forall x \in D, Q(x)$ is True

$(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$ is True.

The True value is always the same for the two claim.

ii) Claim: $(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x)) \to (\forall x \in D, P(x) \vee Q(x))$

Proof: suppose $(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$ is True.

$\forall x \in D, P(x)$ is True or $\forall x \in D, Q(x)$ is True

$\forall x \in D, P(x)$ is True or $Q(x))$ is True

$\forall x \in D, P(x) \vee Q(x))$ is True

From the above claim, it is clear that when our first statement is True the other follows

But the implications does not hold since

if $(\forall x \in D, P(x) \vee Q(x))$ is True

$\to \forall x \in D, (P(x)$ or $Q(x))$ is True

Which does not imply that neither $P(x)$ is true for all x nor $Q(x)$

is True for all x.

So they do not have the same truth value.

QUESTION 5

A) i) $\overline{ab}\,(\bar{a}+b)(\bar{b}+b)$

we know that $b+\bar{b}=1$

$\overline{ab}\,(\bar{a}+b)$

$(\bar{a}+\bar{b})(\bar{a}+b)$

De Morgan theorem : $\overline{a \cdot b} = \bar{a}+\bar{b}$

$\bar{a}\cdot\bar{a} + \bar{a}b + \bar{b}\bar{a} + \bar{b}b$

$\bar{a} + \bar{a}b + \bar{b}\bar{a} + 0$

$\bar{a}(1+b) + \bar{b}\bar{a}$

$\bar{a}\cdot 1 + \bar{b}\cdot\bar{a}$

$\bar{a}(1+\bar{b})$

$\bar{a}\cdot 1 = \bar{a}$

$a\bar{b}\,(\bar{a}+b)(\bar{b}+b) = \bar{a}$

ii) $\bar{a}(a+b)+(b+aa)(a+\bar{b})$

$\bar{a}(a+b)+(b+a)(a+\bar{b})$

$a\bar{a}+\bar{a}\cdot b+ab+b\bar{b}+aa+a\bar{b}$

$0+\bar{a}\cdot b+ab+0+a+a\bar{b}$

$b(a+\bar{a})+a(1+\bar{b})$

$b(1)+a(1)$

$=a+b$

$$\boxed{\begin{array}{l} a\cdot a=a \\ a\cdot\bar{a}=0 \\ a+\bar{a}=1 \\ 1+a=1 \\ 1+\bar{a}=1 \end{array}}$$

$\bar{a}(a+b)+(b+aa)\cdot(a+\bar{b})=a+b$

B) $ab+c\bar{d}=(a+c)(a+\bar{d})(b+c)(b+\bar{d})$

duality theorem:

- replace OR with AND
- replace AND with OR
- complementing 0's and 1's in the expression

$(a+b)\cdot(c+\bar{d})=(a\cdot c)+(a\cdot\bar{d})+(b\cdot c)+(b\cdot\bar{d})$

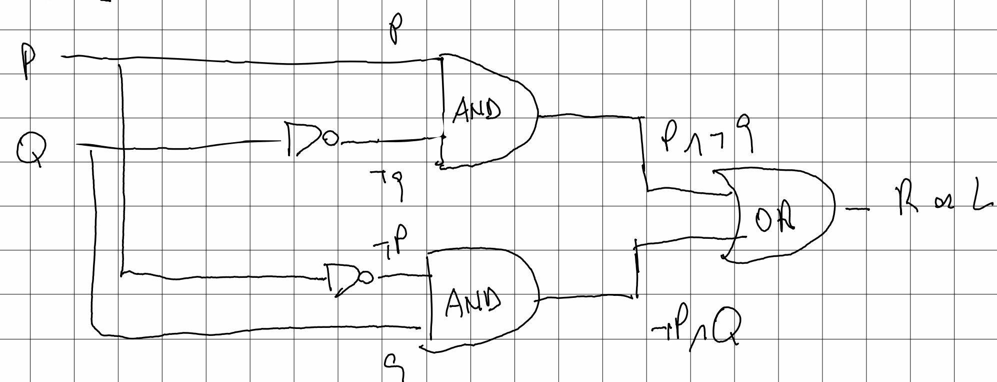
c) Let P, Q be 1 when up and 0 when are down.

From given info we can say that

P = 0 , Q = 0 then the lights is off also P = 1 and Q = 1

then the light is off else it is on.

CIRCUIT :



TRUTH TABLE :

P	Q	R
0	0	0
1	0	1
0	1	1
1	1	0

D) i) $F(x,y,z) = \bar{x}\cdot\bar{y}\cdot\bar{z} + \bar{x}\bar{y}z + x\bar{y}\bar{z} + x\bar{y}z$

xy	00	01	11	10
z				
0	1	0	0	1
1	1	0	0	1

ii)

	$\bar{x}\bar{y}$	$\bar{x}y$	xy	$x\bar{y}$
$\bar{z}$	1 (0)	0 (2)	0 (6)	1 (4)
z	1 (1)	0 (3)	0 (7)	1 (5)

Pairing of {0, 1, 4, 5} Positions

For this pairing only $\bar{y}$ is not changing so we assume

$F(x,y,z) = \bar{y}$