Time Series Project

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January 9, 2019

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1 Introduction

The project concerns the analysis of the relationship between one month interest rate and one year interest rate, respectively referred as short-term interest rate and long-term interest rate, of riskless discount bonds. The dataset analyzed consists in the interest rates measured by the Federal Reserve between January 1986 and December 2008. The information prior to January 1986 are known but not used due to the policy of the Federal Reserve.
Central banks influence long-term interest rates by modifying short-term interest rates. Therefore it is crucial to understand if the two interest rate are cointegrated, meaning that the short-term interest rates really influence long-term interest rates.
Section 2 provides a brief overview of the data and the unit root tests, while section 3 concerns the cointegration of the series. Section 4 concerns the testing of the expectation hypothesis under rational expectations.

2 Unit root test

The data analyzed are continuously compounded interest rates on riskless discount bonds with one month maturity ($m1$ series) and one year maturity ($y1$ series).
To understand whether the process representing m1 (one month maturity) is stationary or not, a unit root test is required. Augmented Dickey-Fuller test (the unit root test used) is sensitive to the possible presence of an intercept. Indeed, if the unit root is present then the estimation $m1_t = \beta m1_{t-1}$ has the same probability not to reject the null hypothesis as $m1_t = \alpha + \beta m1_{t-1}$. On the contrary, if there is not a unit root and the intercept $\alpha = 0$ is less easy to reject the null hypothesis in $m1_t = \alpha + \beta m1_{t-1}$; if $\alpha \neq 0$ then $m1_t = \beta m1_{t-1}$ becomes inconsistent. Thus, if it is not known for sure that there is an intercept, the estimation $m1_t = \alpha + \beta m1_{t-1}$ should be preferred. Since it is not reasonable to exclude the presence of an intercept as Figure 1 hints, the regression $m1_t = \alpha + \beta m1_{t-1}$ should fit the data better than $m1_t = \beta m1_{t-1}$.

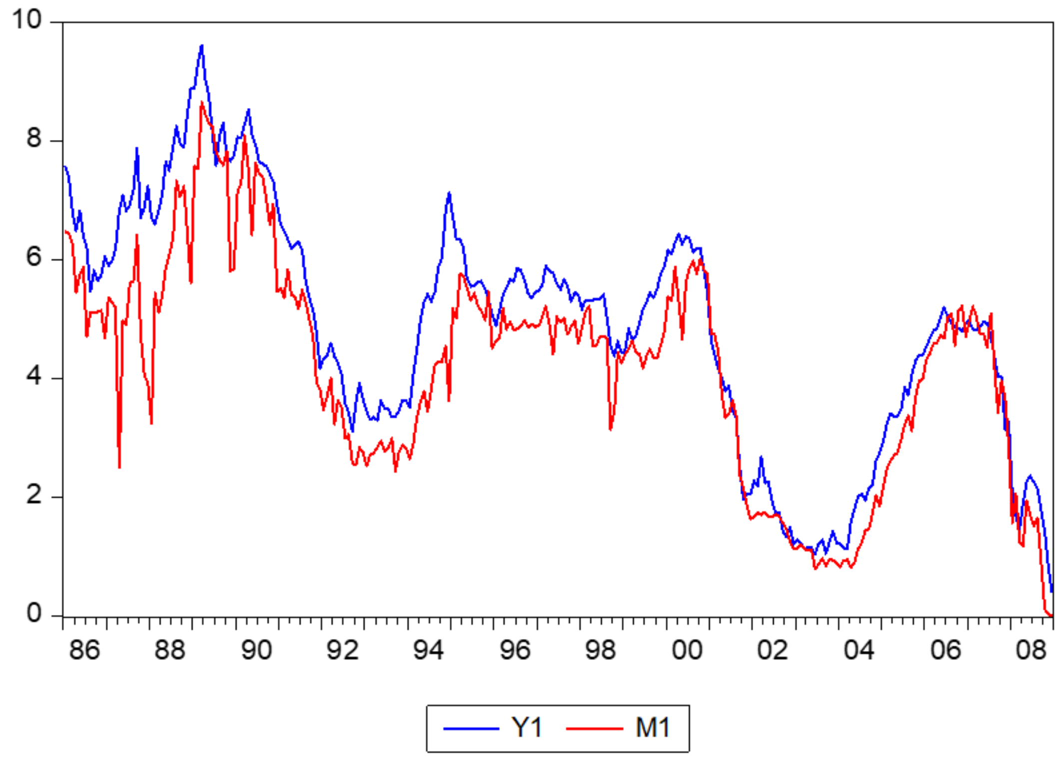

Figure 1: Term structure of one month interest rate and one year interest rate. Plot of one month interest rate in red, plot of one year interest rate in blue.

As the unit root test in Figure 2 shows, the hypothesis that m1 has a unit root is not rejected, indeed its t-Statistic value (-1.4718930) is greater than the 5% critical values (-2.871806) of the Augmented Dickey-Fuller test. Therefore it is possible to state that $m1 \in I(1)$ (i.e $m1$ series is integrated of order one).

Null Hypothesis: M1 has a unit root
Exogenous: Constant
Lag Length: 1 (Automatic - based on SIC, maxlag=15)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-1.471893	0.5466
Test critical values:	1% level	-3.453910	
	5% level	-2.871806	
	10% level	-2.572313	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(M1)
Method: Least Squares
Date: 12/18/18 Time: 14:16
Sample: 1986M01 2008M12
Included observations: 276

Variable	Coefficient	Std. Error	t-Statistic	Prob.
M1(-1)	-0.025416	0.017268	-1.471893	0.1422
D(M1(-1))	-0.199961	0.059672	-3.351022	0.0009
C	0.078816	0.079140	0.995895	0.3202

R-squared	0.053581	Mean dependent var	-0.022996
Adjusted R-squared	0.046647	S.D. dependent var	0.532369
S.E. of regression	0.519804	Akaike info criterion	1.540081
Sum squared resid	73.76361	Schwarz criterion	1.579433
Log likelihood	-209.5312	Hannan-Quinn criter.	1.555872
F-statistic	7.727816	Durbin-Watson stat	2.009998
Prob(F-statistic)	0.000544		

Figure 2: Unit root test concerning one month interest rate

Moreover, the same reasoning holds for $y1$ (one year maturity) series. It is reasonable to believe that y1 series has an intercept because the mean of the interest rate is positive. Therefore the regression $y1_t = \alpha + \beta y1_{t-1}$ should fit the data better than $y1_t = \beta y1_{t-1}$. As Figure 3 suggests, the hypothesis that y1 has a unit root is accepted, indeed its t-Statistic value (-1.4718930) is greater than the 5% critical values (-2.871806) of the Augmented Dickey-Fuller test. Therefore it is possible to state that $y1 \in I(1)$ (i.e $y1$ series is integrated of order one).

Null Hypothesis: Y1 has a unit root
Exogenous: Constant
Lag Length: 1 (Automatic - based on SIC, maxlag=15)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-0.905638	0.7856
Test critical values:	1% level	-3.453910	
	5% level	-2.871806	
	10% level	-2.572313	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(Y1)
Method: Least Squares
Date: 12/18/18 Time: 14:21
Sample: 1986M01 2008M12
Included observations: 276

Variable	Coefficient	Std. Error	t-Statistic	Prob.
Y1(-1)	-0.007880	0.008701	-0.905638	0.3659
D(Y1(-1))	0.212075	0.059604	3.558087	0.0004
C	0.018185	0.046138	0.394143	0.6938

R-squared	0.045344	Mean dependent var	-0.025612
Adjusted R-squared	0.038350	S.D. dependent var	0.294772
S.E. of regression	0.289065	Akaike info criterion	0.366479
Sum squared resid	22.81147	Schwarz criterion	0.405831
Log likelihood	-47.57406	Hannan-Quinn criter.	0.382270
F-statistic	6.483425	Durbin-Watson stat	2.029110
Prob(F-statistic)	0.001775		

Figure 3: Unit root test concerning one year interest rate

3 Cointegration test

If the series $y1$ depends on the series $m1$, then a monetary policy that influences the short-term interest rate can, in turn, influence the long-term interest rate. To understand if that is true, it is possible to run a cointegration test on the series. Two series are said to be cointegrated if the residuals r_t of $y1_t - \beta m1_t$ [1] are integrated of order zero ($r_t \in I(0)$). The difference

[1] $\beta = 1$

between the yearly interest rate $y1_t$ or $i_{12,t}$ and the monthly interest rate $m1_t$ or $i_{12,t}$ is called *term spread* $S_t^{(12,1)} = i_{12,t} - i_{1,t}$. As Figure 4 shows, the null hypothesis that the two series are not cointegrated is rejected, therefore it is possible to state that the series are cointegrated and that $y1$ depends on $m1$.

Date: 12/18/18 Time: 14:23
Series: Y1 M1
Sample: 1986M01 2008M12
Included observations: 276
Null hypothesis: Series are not cointegrated
Cointegrating equation deterministics: C
Automatic lags specification based on Schwarz criterion (maxlag=15)

Dependent	tau-statistic	Prob.*	z-statistic	Prob.*
Y1	-7.563975	0.0000	-94.91253	0.0000
M1	-7.949275	0.0000	-103.2006	0.0000

*MacKinnon (1996) p-values.

Intermediate Results:

	Y1	M1
Rho - 1	-0.345136	-0.375275
Rho S.E.	0.045629	0.047209
Residual variance	0.249940	0.223867
Long-run residual variance	0.249940	0.223867
Number of lags	0	0
Number of observations	275	275
Number of stochastic trends**	2	2

Figure 4: Cointegration Test

Moreover, it is possible to estimate the coefficient β in the regression $y1_t = \alpha + \beta m1_t$. The estimation $\hat{\beta}$ of β is expected to be approximately 1, by the definition of cointegration. The parameter $\hat{\beta}$ corresponds to the value C(2) in Figure 5 and indeed $\hat{\beta} \approx 1$.

Dependent Variable: Y1
Method: Least Squares (Gauss-Newton / Marquardt steps)
Date: 12/18/18 Time: 14:28
Sample: 1986M01 2008M12
Included observations: 276
Y1 = C(1) +C(2)*M1

	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	0.555700	0.098307	5.652698	0.0000
C(2)	1.033001	0.021540	47.95641	0.0000

R-squared	0.893543	Mean dependent var	4.865518
Adjusted R-squared	0.893155	S.D. dependent var	2.025203
S.E. of regression	0.661982	Akaike info criterion	2.020062
Sum squared resid	120.0722	Schwarz criterion	2.046297
Log likelihood	-276.7686	Hannan-Quinn criter.	2.030590
F-statistic	2299.817	Durbin-Watson stat	0.689449
Prob(F-statistic)	0.000000		

Figure 5: Estimation of β

To guarantee the cointegration, it is also possible to exploit the definition by testing the presence of a unit root in the residuals of $y1_t - \hat{\beta} m1_t = S^{(12,1)}$. Indeed, by definition, two series are cointegrated if their residuals are integrated of order zero, so it is possible to run a unit root test on the residuals. The test denies the possibility of a unit root (Figure 6). Therefore it is possible to state that the $y1$ and $m1$ are cointegrated.

Null Hypothesis: SPREAD has a unit root
Exogenous: Constant
Lag Length: 1 (Automatic - based on SIC, maxlag=18)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-8.217915	0.0000
Test critical values:	1% level	-3.441513	
	5% level	-2.866356	
	10% level	-2.569395	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(SPREAD)
Method: Least Squares
Date: 01/03/19 Time: 14:54
Sample (adjusted): 1961M03 2008M12
Included observations: 574 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
SPREAD(-1)	-0.283142	0.034454	-8.217915	0.0000
D(SPREAD(-1))	-0.196345	0.041044	-4.783775	0.0000
C	0.217671	0.034001	6.401869	0.0000

R-squared	0.207834	Mean dependent var	-0.000225
Adjusted R-squared	0.205060	S.D. dependent var	0.571869
S.E. of regression	0.509875	Akaike info criterion	1.495910
Sum squared resid	148.4442	Schwarz criterion	1.518659
Log likelihood	-426.3261	Hannan-Quinn criter.	1.504783
F-statistic	74.90438	Durbin-Watson stat	1.999199
Prob(F-statistic)	0.000000		

Figure 6: Unit root test on the term spread $S^{(12,1)}$

4 The expectations hypothesis under rational expectations

It is interesting to understand if the expectations hypothesis holds under rational expectations. The expectation hypothesis states that the economic agents maximize the expected profits of their investments. They can invest or borrow money with long-term contact or with a sequence of short-term contracts, considering also possible term premia. As a start, it is convenient to suppose there are no term premia, so interest rate at time 2 is

$$i_{2,t} = \frac{1}{2}(i_{1,t} + E_t(i_{1,t+1})). \tag{1}$$

The model can be expanded to take in account possible exogenous events, adding independent and identically distributed shocks $e_{n,t}$ under two restrictions:

- $E_t(e_{n,t}) = 0$ [2]
- $Var(e_{n,t}) = \sigma^2$

Generalizing the reasoning concerning the spread to a generic time n,

$$i_{n,t} = \frac{1}{n}(i_{1,t} + \sum_{l=2}^{n} E_t(i_{n,t+l})) \tag{2}$$

holds.
Since Rational Expectation (RE) is assumed to hold, it is possible to define the expected value of the interest rate as

$$E_t(i_{n,t+k}) = i_{n,t+k} + e_{n,t+k}, \tag{3}$$

where $E_t(e_{n,t+k}) = 0 \; \forall t, k \in N$.
When substituting in the expectation (Eq. 3) into $i_{n,t} = \frac{1}{n}(i_{1,t}+\sum_{l=2}^{n} E_t(i_{n,t+l}))$, it becomes

$$i_{n,t} = \frac{1}{n}\sum_{l=1}^{n} i_{n,t} + \frac{1}{n}\sum_{l=2}^{n-1} \epsilon_{l,t+n} + e_{n,t}. \tag{4}$$

Let

$$\eta_{t+n} = \frac{1}{n}(\sum_{l=2}^{n-1} \epsilon_{l,t+n}) + e_{n,t}, \tag{5}$$

now it is possible to rewrite $i_{n,t}$ as

$$i_{n,t} = T_n + i_{1,t} + \sum_{l=1}^{n-1} \frac{n-l}{n} \Delta i_{1,t+l} + \eta_{n+t}, \tag{6}$$

[2] E_t means the expected value of $e_{n,t}$ at time t

when a term premia T_n is added.
Rearranging the terms it is possible to obtain

$$\sum_{l=1}^{n-1} \frac{n-l}{n} \Delta i_{1,t+l} = -T_n + (i_{n,t} - i_{1,t}) - \eta_{t+n}) \tag{7}$$

that can be estimated as

$$\sum_{l=1}^{n-1} \frac{n-l}{n} \Delta i_{1,t+l} = \alpha_s + \beta_s S_t^{(n,1)} + u_{t+n}. \tag{8}$$

Since the analysis concerns one year interest rate and one month interest rates Eq. 8 becomes

$$\sum_{l=1}^{n-1} \frac{n-l}{n} \Delta i_{1,t+l} = \alpha_s + \beta_s S_t^{(12,1)} + u_{t+n}. \tag{9}$$

It is worth noticing that by definition u_{t+n} should be independent and identically distributed, but in the dataset the independence is not granted as Figure 7 shows.

Date: 01/03/19 Time: 15:03
Sample: 1986M01 2008M12
Included observations: 265

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.916	0.916	224.90	0.000
		2	0.830	-0.056	410.27	0.000
		3	0.749	-0.016	561.76	0.000
		4	0.669	-0.038	683.25	0.000
		5	0.592	-0.034	778.62	0.000
		6	0.522	0.001	853.19	0.000
		7	0.462	0.010	911.74	0.000
		8	0.405	-0.021	956.87	0.000
		9	0.339	-0.095	988.60	0.000
		10	0.289	0.058	1011.8	0.000
		11	0.258	0.073	1030.3	0.000
		12	0.235	0.025	1045.7	0.000

Figure 7: Wald test on u_{t+n}

Since u_{t+n} are not independent we use an appropriate method of estimation (Figure 8) to obtain $\sum_{l=1}^{n-1} \frac{n-l}{n} \Delta i_{1,t+l} = \alpha_s + \beta_s S_t^{(12,1)} + u_{t+n}$.

Dependent Variable: PFS12
Method: Least Squares (Gauss-Newton / Marquardt steps)
Date: 01/06/19 Time: 11:14
Sample (adjusted): 1986M01 2008M01
Included observations: 265 after adjustments
HAC standard errors & covariance (Bartlett kernel, Newey-West fixed bandwidth = 5.0000)
PFS12 =C(1)+C(2)*SPREAD

	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	-0.623932	0.111678	-5.586893	0.0000
C(2)	0.761595	0.087699	8.684245	0.0000

R-squared	0.392121	Mean dependent var	-0.091654
Adjusted R-squared	0.389810	S.D. dependent var	0.816623
S.E. of regression	0.637902	Akaike info criterion	1.946254
Sum squared resid	107.0197	Schwarz criterion	1.973271
Log likelihood	-255.8787	Hannan-Quinn criter.	1.957109
F-statistic	169.6522	Durbin-Watson stat	0.154759
Prob(F-statistic)	0.000000	Wald F-statistic	75.41611
Prob(Wald F-statistic)	0.000000		

Figure 8: Estimation of $\alpha_s + \beta_s S_t^{(n,1)} + u_{t+n}$

The expectation hypothesis and the pure expectation hypothesis hold at least if $\beta_s = 1$.

Wald Test:
Equation: Untitled

Test Statistic	Value	df	Probability
t-statistic	-2.718456	263	0.0070
F-statistic	7.390005	(1, 263)	0.0070
Chi-square	7.390005	1	0.0066

Null Hypothesis: C(2) =1
Null Hypothesis Summary:

Normalized Restriction (= 0)	Value	Std. Err.
-1 + C(2)	-0.238405	0.087699

Figure 9: Test if $\beta_s = 1$

Given the coefficients α_s and β_s, it is possible to test whether coefficient $\beta_s = 1$ (corresponding to c(2) in Figure 9) or not. As Figure 9 show, it is possible to state that the null hypothesis $\beta_s = 1$ is rejected, therefore the pure expectation hypothesis and expectation hypothesis under rational expectation do not hold.

5 Conclusions

The analysis show that the series representing respectively continuously compounded one month interest rate ($m1$) and one year interest rate ($y1$) on discount bonds are both integrated of order one.
It is also possible to state that $y1$ and $m1$ are cointegrated meaning that changes in $m1$ modify $y1$, so the central bank can influence long-run interest rates by modifying short-term interest rates. This relationship implies that the term spread provides a useful information to foretell the interest rate dynamic, nevertheless the expectations hypothesis under rational expectations is rejected.