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Additional Reading on time series modelling the term structure of interest rates

Alternative measures of interest rates

Let $P_{n,t}$ be the price of a zero-coupon bond (discount bond) that will pay a redemption price M n periods ahead (n is also referred to as maturity). Then, when periods are measured in years, the annually compounded interest rate is $r_{n,t}$ is defined so that

$$P_{1,t} = \frac{M}{(1 + r_{1,t})}.$$

Iterating this formula, $P_{2,t} = \frac{M}{(1+r_{2,t})^2}$ ...

It may be thought that the choice of the year is arbitrary. For this reason, and because it facilitates calculations, an alternative definition of the interest rate is made by considering continuously compound. In this case, the continuously compounded rate, $i_{n,t}$, is

$$P_{n,t} = Me^{-i_{n,t}n}.$$

Notice that these two rates are related according to

$$i_t = \ln(1 + r_t).$$

The Term Structure of Interest Rates

The plot of the rates against the maturities is called the Term Structure of Interest Rates.

For example, the rates of the Treasury bills (T-bills) observed on 8 June 2009 for various maturities were (annually compounded rates)

maturity	1 m	3 m	6 m	1 Y
rate	0.07	0.19	0.35	0.60

(source, the FED on line database, http://research.stlouisfed.org/fred2/).

Are these rates related between them?

Can we use them to infer information about other rates?

! we need to know expected future rates for many things, including the pricing of financial instruments (coupon bonds, futures, options,...).

! The Central Bank is also interested in knowing more about the transmission of shocks of monetary policy (the Central Bank only operates on short term rates, usually via discount window loans or repurchasing agreement; the real economy, on the other hand, reacts to medium/long rates).

The Pure Expectations Hypothesis (PEH) for the term structure of interest rates.

In the PEH it is assumed that the economic agents are solely interested in maximising the expected profit of their investment. Investors can either invest or borrow with a long term contract, or with the roll over of many short term contracts (over the same long term period): if for example the return of the long term contract is lower than the expected profit of the roll over of the short term contracts, the investor will want to borrow with the long term contract and invest in the roll over of the short term ones, and he/she will continue to do so until the expected return of the two investment strategies become equal (of course, if the return of the long term contract is higher than the expected return of the roll over of the short term contracts, profit maximising agents will borrow with short contracts to invest in the long one). Arbitrage, therefore, ensures that the expected return of the two strategies is the same.

Let $i_{n,t}$ be the the continuously compounded interest rate associated to a contract over the n periods observed at time t.

With our notation, when two periods only are considered, the PEH means that

$$i_{2,t} = \frac{1}{2}(i_{1,t} + E_t(i_{1,t+1})).$$

We can use this equation to find out the expectation that the market has today about the next period's short rate.

With our data, this means that the market expects the 3 months rate in three months time to be

$$E_t(i_{1,t+1}) = 2 \times i_{2,t} - i_{1,t} = 2 \times 0.35 - 0.19 = 0.51$$

(disregarding the correction needed to go from annually to continuosly compounded rates, and considering periods of three months for the index).

Preference for liquidity

In a variation of this theory, agents may be risk averse. In this case, bonds with long maturities are more risky, because the price change is larger for any given change in the interest rate. Bonds with a longer maturity will then attract a term premium T_n, which may increase with the maturity (notice that this is, however, constant over time), so

$$i_{n,t} = T_n + \frac{1}{n}(i_{1,t} + E_t(i_{1,t+1}) + \ldots + E_t(i_{1,t+n-1}))$$

$$T_n \geq T_{n-1} \geq 0.$$

A more generic premium

It is also possible that agents may have preference for other maturities. For example, young people saving for their retirement may prefer to invest in long term contracts. In this case, it is also possible that $T_n \leq T_{n-1}$.

By the Expectation Hypothesis (EH), we mean the PEH or the variation of the PEH that allows for Preference for Liquidity or for any other maturity.

Complete Market Segmentation

Under complete market segmentation, the rate for each maturity depends on the supply and demand of funds for contracts with that maturity only.

★ Clearly, all these theories (PEH, preference for liquidity or for other maturities, complete market segmentation) have different implications for our capability to compute expected future rates, and for the transmission of monetary impulses along the term structure.

★ We saw before that, if the PEH holds, we can use the information in the term structure today to find out the market's expectation of the rates in the future. A similar procedure may also be applied in presence of a term premium, as long as we know it.

★ Often, the single most interesting piece of information in the term structure of interest rate is the difference $i_{n,t} - i_{1,t}$. This is sometimes known as "term spread", or only as "spread". If the PEH or HE holds, the spread can be used to derive the expectations of future rates dynamics.

Using the term spread to derive the expectations of future rates dynamics

Short term rates (changes of $i_{1,t}$)

Assumung that that the PEH holds,

$$i_{2,t} = \frac{1}{2}(i_{1,t} + E_t(i_{1,t+1}))$$

adding and subtracting $i_{1,t}$ to the term in brackets,

$$\begin{aligned} i_{2,t} &= \frac{1}{2}(i_{1,t} \pm i_{1,t} + E_t(i_{1,t+1})) \\ &= i_{1,t} + \frac{1}{2}(E_t(i_{1,t+1}) - i_{1,t}) \end{aligned}$$

and

$$\frac{1}{2}(E_t(i_{1,t+1}) - i_{1,t}) = i_{2,t} - i_{1,t},$$

so if the spread $i_{2,t} - i_{1,t}$ is positive we expect the short term rate $i_{1,t}$ to go up in the next period.

This equation may be corrected for a term premium,

$$\frac{1}{2}(E_t(i_{1,t+1}) - i_{1,t}) = -T_2 + (i_{2,t} - i_{1,t})$$

In general, when n periods are considered,

$$i_{n,t} = \frac{1}{n}(i_{1,t} + E_t(i_{1,t+1}) + \ldots + E_t(i_{1,t+n-1}))$$

also holds, which again we can also rewrite as

$$i_{n,t} = i_{1,t} + \sum_{l=1}^{n-1} \frac{n-l}{n} E_t(\Delta i_{1,t+l})$$

and then

$$\sum_{l=1}^{n-1} \frac{n-l}{n} E_t(\Delta i_{1,t+l}) = i_{n,t} - i_{1,t}$$

(notice that all the expectations are taken using the information available at time t).

So, if the PEH theory is true, we can read in the term spread of today if the market expects short rates to raise or fall in the future.

This equation may be corrected for a term premium,

$$\sum_{l=1}^{n-1} \frac{n-l}{n} E_t(\Delta i_{1,t+l}) = -T_n + (i_{n,t} - i_{1,t})$$

Long term rates (changes of $i_{n,t}$)

Assumung that that the PEH holds,

$$i_{2,t} = \frac{1}{2}(i_{1,t} + E_t(i_{1,t+1}))$$

then, multiply both sides by 2 and rearranging terms,

$$2i_{2,t} = (i_{1,t} + E_t(i_{1,t+1}))$$
$$E_t(i_{1,t+1}) - i_{2,t} = i_{2,t} - i_{1,t}.$$

If we have a rate with maturity n,

$$(n-1)(E_t(i_{n-1,t+1}) - i_{n,t}) = i_{n,t} - i_{1,t}.$$

So, if the PEH theory is true, we can read in the term spread of today if the market expects long rates $i_{n,t}$ to raise or fall in the future.

Again, if there is a term premium, these equations may be augmented by a term $c_n = (n-1)T_{n-1} - nT_n$,

$$(n-1)(E_t(i_{n-1,t+1}) - i_{n,t}) = c_n + (i_{n,t} - i_{1,t})$$

Statistical modelling of the term structure of interest rates.

Assume for the moment that there is no term premia, ie, $T_n = 0$ (as in the PEH).

For $i_{1,t}$, $i_{2,t}$, we said that the

$$i_{2,t} = \frac{1}{2}(i_{1,t} + E_t(i_{1,t+1})).$$

This is an exact relation, so it is sufficient to have one deviation from it to reject the (PEH+RE). However, there are many factors that may affect the various markets, and may therefore cause an unexpected deviation from the exact relation. The definition is therefore augmented in order to allow for an independent, identically distributed shock $e_{2,t}$, with $E(e_{2,t}) = 0$, $Var(e_{2,t}) = \sigma^2$, so that

$$i_{2,t} = \frac{1}{2}(i_{1,t} + E_t(i_{1,t+1})) + e_{2,t}.$$

By the same argument we can also augment by similar a iid shock all the other equations, so for example

$$i_{n,t} = \frac{1}{n}(i_{1,t} + E_t(i_{1,t+1}) + \ldots + E_t(i_{1,t+n-1})) + e_{n,t}.$$

For the expectations, we assume Rational Expectations (RE),

$$E_t(i_{1,t+1}) = i_{1,t+1} + \varepsilon_{1,t+1},$$

$$E_t(i_{1,t+k}) = i_{1,t+k} + \varepsilon_{k,t+k},$$

where for any k, $\varepsilon_{k,t+k}$ is such that $E_t(\varepsilon_{k,t+k}) = 0$.

As we saw before, $E_t(\varepsilon_{1,t+1}) = 0$ is implied by $\varepsilon_{1,t+1}$ being independent (and $E(\varepsilon_{1,t+1}) = 0$ of course), althought we saw that other models for $\varepsilon_{1,t+1}$ are also possible. In the same way, $E_t(\varepsilon_{k,t+k}) = 0$ is implied by $\varepsilon_{k,t+k}$ being MA$(k-1)$ with independent innovations, althought other models for $\varepsilon_{k,t+k}$ are also possible.

Proceeding as before, we may derive

$$i_{2,t} = i_{1,t} + \frac{1}{2}(E_t(i_{1,t+1}) - i_{1,t}) + e_{2,t}.$$

and substituting the expectations,

$$i_{2,t} = i_{1,t} + \frac{1}{2}(i_{1,t+1} - i_{1,t}) + e_{2,t} + \frac{1}{2}\varepsilon_{1,t+1}.$$

Letting

$$\eta_{t+2} = e_{2,t} + \frac{1}{2}\varepsilon_{1,t+1}$$

this is

$$i_{2,t} = i_{1,t} + \frac{1}{2}(i_{1,t+1} - i_{1,t}) + \eta_{t+2}.$$

In general we may derive

$$\eta_{t+n} = e_{n,t} + \frac{1}{n}(\varepsilon_{1,t+1} + \varepsilon_{2,t+2} + \ldots + \varepsilon_{n-1,t+n-1})$$

$$i_{n,t} = i_{1,t} + \sum_{l=1}^{n-1} \frac{n-l}{n} \Delta i_{1,t+l} + \eta_{t+n}.$$

This equation is usually augmented by an intercept T_n, which may be a combination of the term premia or liquidity premia (EH+RE).

Testing the Expectations Hypothesis under Rational Expectations: Cointegration properties

[Note: here and after, we allow for determinstic terms in the definition of the order of integration].

Recall that, under EH+RE,

$$i_{n,t} = T_n + i_{1,t} + \sum_{l=1}^{n-1} \frac{n-l}{n} \Delta i_{1,t+l} + \eta_{t+n}.$$

Assuming that $i_{1,t}$ is an $I(1)$ integrated process, i.e. $i_{1,t} \in I(1)$, it is clear that $\Delta i_{1,t+l} \in I(0)$.

From our assumptions it is also clear that, loosely speaking, $\eta_{t+n} \in I(0)$.

Then, $i_{n,t}$ is the sum of an $I(1)$ process with an $I(0)$ term, so this too is $I(1)$ (as it derives the unit root from $i_{1,t}$).

Moreover, $i_{1,t}$ and $i_{n,t}$ are cointegrated, ie, there is β such that $i_{n,t} - \beta i_{1,t} \in I(0)$.

Finally, the cointegrating parameter (β) must 1, i.e., the Spread, defined as

$$S_t^{(n,1)} = (i_{n,t} - i_{1,t})$$

must be $I(0)$.

[the Spread can be $I(0)$ with non-zero mean under the EH+RE; it is $I(0)$ with zero mean under PEH+RE].

Testing the Expectations Hypothesis under Rational Expectations: Change in rate properties

✠ Change in Short Rate Equation

Again recall that, under EH+RE,

$$i_{n,t} = T_n + i_{1,t} + \sum_{l=1}^{n-1} \frac{n-l}{n} \Delta i_{1,t+l} + \eta_{t+n}.$$

Rearranging terms,

$$\sum_{l=1}^{n-1} \frac{n-l}{n} \Delta i_{1,t+l} = -T_n + (i_{n,t} - i_{1,t}) - \eta_{t+n}$$

(and notice that the term in η_{t+n} is not correlated to the information at time t, so it is not correlated to the spread $i_{n,t} - i_{1,t}$).

To test if this is met by the data, we may estimate α_s and β_s in

$$\sum_{l=1}^{n-1} \frac{n-l}{n} \Delta i_{1,t+l} = \alpha_s + \beta_s S_t^{(n,1)} + u_{t+n}$$

and test if $\{\beta_s = 1\}$ (EH) or $\{\alpha_s = 0, \beta_s = 1\}$ (PEH).

✠ Change in Long Rate Equation

Proceeding as we did when we derived

$$(n-1)(E_t(i_{n-1,t+1})-i_{n,t}) = c_n + (i_{n,t}-i_{1,t}),$$

but allowing for an iid shock $e_{n,t}$

$$i_{n,t} = \frac{1}{n}(i_{1,t}+E_t(i_{1,t+1})+\ldots+E_t(i_{1,t+n-1})) + e_{n,t},$$

replacing Rational Expectations, we get

$$(n-1)(i_{n-1,t+1}-i_{n,t}) = c_n + (i_{n,t}-i_{1,t}) + u_{t+1}.$$

Recalling that $S_t^{(n,1)} = (i_{n,t}-i_{1,t})$, we then want to estimate α_L and β_L in

$$(n-1)(i_{n-1,t+1}-i_{n,t}) = \alpha_L + \beta_L S_t^{(n,1)} + u_{t+1}$$

and test if $\{\beta_L = 1\}$ (EH) or $\{\alpha_L = 0, \beta_L = 1\}$ (PEH).

Statistical Evidence (common findings in the empirical literature)

Statistical Evidence: Long Term dynamics

★ Analysis of the cointegration properties.
It is possible to check that:

i) the interest rates are $I(1)$;

ii) the interest rates are cointegrated;

iii) the spreads are $I(0)$.

i) Results that the interest rates are $I(1)$ are consistently reported in the literature. Exceptions only follow if more advanced modelling theory is introduced (for example, threshold autoregression or fractional integration).

ii) Reports that the interest rates are cointegrated are quite consistently found in the literature, although occasionally evidence against it is reported. This is more often the case when cointegration between long and short maturity rates is analysed or when periods of monetary instability are analysed (as for example the US data between 1979 and 1982, when radical changes in the operating procedures for the management of the monetary policy were implemented).

iii) Results that the spreads are $I(0)$ are often reported, if cointegration is present.

Statistical Evidence: Short Term dynamics:

Statistical evidence about Change in rate properties seems to depend on countries

★ UK:

✠ Change in Short Rate Equation

Evidence that $\beta_s = 1$ seems consistent across different spreads, up to 10 years - 1 month spread.

✠ Change in Long Rate Equation

Evidence that $\beta_L = 1$ seems consistent across different spreads, up to 6 years - 1 month spread. For spreads over larger differences of maturity (such as 10 years - 1 month spread or more, estimates of β_L are either close to 0 or even negative, which implies that the spread does not predict well changes in the long term maturities).

★ US

More frequent evidence of rejection even for spreads covering shorter maturities

★ Germany (pre-EMU)

Broad support of the EH

Why these country specific differences?

- ✠ Monetary policy

In US the central bank was actively managing interest rate smoothing: in this case the expected rate change should be zero, and the spread has no information about future rates

- ✠ Time Varying Risk Premia

for example, in periods of higer volatility

✠ Other tests of the EH-RE

Another popular approach is the Vector Autoregression representation. Under regularity conditions, a vector of rates may be described by a Vector Error Correction Model (which may allow for cointegration), with cross equations restrictions. Tests of these cross equations restrictions are usually rejected.

Time Varying Risk Premia are sometimes introduced in these models: this seems to improve the performance of the EH.

Conclusive remarks

Overall, we can conclude that not all the implications of the EH+RE are met by the data, so the hypothesis is, as a whole, rejected.

✠ Long term implications.

Althought the EH+RE is overall rejected, since at least the hypothesis of cointegration is not rejected, we can at least conclude that the long term implications of the EH+RE are not rejected, which means that it is fair to say the EH+RE does still provide a valid broad guidance about the rates dynamics (except perhaps in situations of high instability, such as the 1979-1982 for the US).

we can actually broadly rely on implicit forward rates computed by means of the EH+RE for the discounting of coupon bonds, for the pricing of options.

Evidence of cointegration is also important because it implies that the central bank can influence the long term rates with sufficient speed, and monetary policy is therefore effective.

✠ Short term implications.

Altough the information in the long-short spread does not always provide the best guidance about future short rates dynamics, it is still a useful source of information (an unexpected increase in the long-short spread broadly anticipates an increase in the short term rates in subsequent periods).