Comprehensive Exam 2022
Biomathematics 202: Structure, Function, and Evolution of Biological Systems

A simple general form of equations to describe growth and interactions among agents—consumers and resources, mutualists, chemicals, different cell lines within a tumor, microbes, etc.—are

$$\frac{\mathrm{d}x_i}{dt} = r_i x_i + \sum_{j=1}^{N} \propto_{ij} x_i x_j \qquad (1)$$

where x_i is the abundance or density of the ith group of agents, N is the total number of species, r_i is the exponential growth rate of the ith group, $\propto_{ij}$ are the interaction coefficients among groups i and j, and t is time.

a. (10 points) In the case that $\propto_{ij} = 0$ for $i \neq j$, what do these equations reduce to for each species i? For this case how do you interpret $\propto_{ii}$? Solve this equation analytically in its exact form.
b. (10 points) How would you write this system of ordinary differential equations in terms of the abundance vector $\mathbf{x}$ to capture the dynamics of all groups? How would the interaction coefficients and the growth rates be written in matrix form (not tensor notation) to multiply the $\mathbf{x}$ vector?
c. (10 points) Consider a system with only 2 groups where group 1 benefits and group 2 loses from the interaction—think a consumer and resource or two bacteria where the first one benefits from the second's byproduct while the second suffers from the first's byproduct. Also, for this case for group 1, let r_i be negative and represent the mortality rate of group 1. What do you expect the dynamics to be? What are the fixed points/equilibrium of the system? Describe in words how you would mathematically determine if the nontrivial fixed points are stable. Do you think the $\propto_{ii}$ terms will help to stabilize or destabilize the system?
d. (10 points) Define what is meant by additive, antagonistic, and synergistic interactions. Which parameters or what parameter values would correspond to antagonistic or synergistic interactions for these types of systems? Would it be determined by individual interaction coefficients or by parameters not given in these equations? If the latter, how would you devise a lab or field experiment to measure for antagonism or synergy and list what you would need to measure.

For a system with N groups, the eigenvalues of the Jacobian matrix will determine system stability—negative eigenvalues imply stability and positive values imply instability—where the fixed points are taken as the nontrivial equilibrium abundances. The elements a_{ij} of the matrix depend on the interaction coefficients $\propto_{ij}$ and the abundances x_i. Without worrying about the exact form of the matrix elements, answer the following questions.

e. (15 points) Construct a bound on the imaginary part and on the real part of the eigenvalues of a matrix A that has all real elements by combining the following two theorems:

i. Gershgorin theorem—The eigenvalues of a matrix A all fall within disks defined according to each row i of the matrix that is centered in the complex plane at the value of the diagonal element a_{ii} and that has a radius given by the sum of the absolute values of all of the off-diagonal elements a_{ij} with $i \neq j$.

ii. Bendixson inequality—The absolute value of the imaginary part of all eigenvalues of a matrix A are less than or equal to the largest absolute value of the imaginary part of any eigenvalue of the anti-symmetric part of the matrix defined as $A^{as}=(A - A^T)/2$ where A^T is the transpose of the matrix. Similarly, the absolute value of the real part of all eigenvalues of a matrix A are less than or equal to the largest absolute value of the real part of any eigenvalue of the symmetric part of the matrix defined as $A^{s}=(A + A^T)/2$ where A^T is the transpose of the matrix. (Hint: The eigenvalues of a symmetric matrix are purely real and of an anti-symmetric matrix are purely imaginary.)

f. (10 points) Let your new bounds in part e. be called the anti-symmetric and symmetric Gershgorin bounds. Compare your bound on the imaginary part of the eigenvalues for the new anti-symmetric Gershgorin bound to the original Gershgorin bound for the following 3 types of matrices: i. symmetric matrix ($a_{ij} = a_{ji}$), ii. Anti-symmetric matrix ($a_{ij} = -a_{ji}$), and iii. An upper triangular matrix (all the elements below the diagonal are zero).

g. (10 points) In part f., does the original Gershgorin bound or your new bound give a tighter constraint for the imaginary part of the eigenvalues? Does it depend on the type of matrix, or does one bound generally give tighter constraints? If one bound is generally tighter, do you think this is true across all types of matrices? Try to prove your assertion.

h. (10 points) Provide reasoning for why the symmetric and anti-symmetric parts of the matrix take the form they do in part e.

i. (15 points) Discuss what the row-sums for the Gershgorin and anti-symmetric Gershgorin bounds might mean biologically. Would you expect bigger bounds on the eigenvalues for cases with all positive interactions, all negative interactions, or some mixture of the two? Or what biological/mathematical conditions would increase or decrease the size of the bounds? Would a food web (consumer-resource network) be more similar to a symmetric or an anti-symmetric matrix?