Biomath 208, 2022 Comprehensive Exam

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0 Instructions

While you may refer to any notes or materials, you are not permitted to do internet searches for answers to specific questions. You are permitted to look up any trigonometric identities or other equations you may need if they are not provided. You should not need a calculator, but can use one if you like. Please write answers in analytical form, for example write exp(2) rather than 7.389.

There are 9 problems, one for each chapter, and most problems have sub-parts. The problems vary in difficulty, so you may chose to solve the simpler ones first. Please note that problem 8 is expected to be longer than the others and is worth more points.

Partial answers will receive partial scores. The questions will not be graded as correct/incorrect.

1 Imaging data (10 points total)

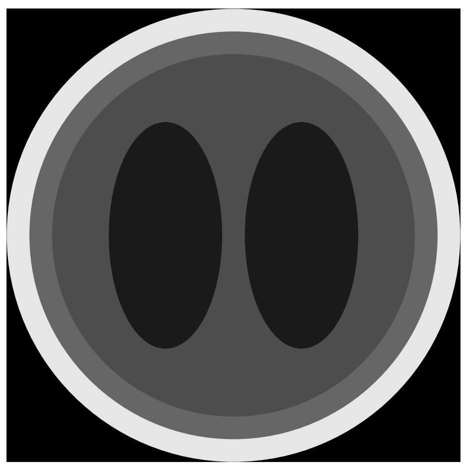
1.1 Discrete images and interpolation (5 points)

Consider a 1D discrete image with 5 samples, I_i where i denotes the sample index (starting from 0). The location of the first pixel is $O = -1.0$. The spacing between pixels is $\Delta = 0.5$. The the sample values take the form $I_i = (i-2)^2$. We define the continuous image $I(x)$ using linear interpolation.

Sketch the image $I(x)$, carefully labeling the axes, and evaluate the image at the point $x = 0.75$.

1.2 Window and level

Consider the piecewise constant model of the human brain as seen in a computed tomography scan, shown in the figure below:



The outer shell, the skull has a brightness of 0.9. The first inner layer, the brain's cortex, has a brightness of 0.4. The next inner layer, the brain's white matter, has a brightness of 0.3. The two inner ellipses, the brain's ventricles have a brightness of 0.1 The above picture is drawn with a lower window set to 0 and an upper window set to 1.

Sketch what this image would look like with a lower window set to 0, and an upper window set to 0.2.

Sketch what this image would look like with a lower window set to 0.35 and an upper window set to 0.45?

To improve clarity, please label your regions qualitatively (e.g. black, dark gray, light gray, white, etc.).

2 Linear algebra (10 points total)

Consider the 2D Cartesian plane (the set of ordered pairs) as vector space. We'll define basis A as consisting of the two vectors $A_0 = (1, 0)$ and $A_1 = (1, 1)$. We'll define basis B as consisting of the two vectors $B_0 = (0, 1)$ and $B_1 = (1, 1)$.

2.1 Change of vector components (5 points)

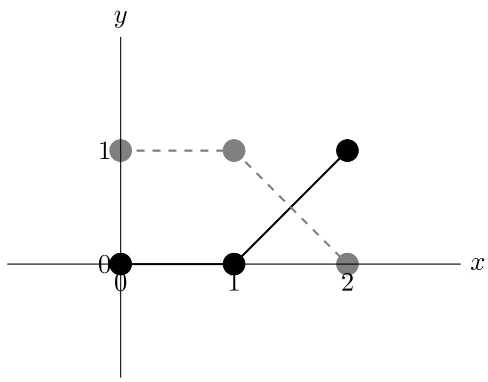
Let x be a vector. In basis A it has components $x^0 = 1, x^1 = 1$. What are its components in basis B?

2.2 Change of covector components (5 points)

Let μ be a covector. In the dual basis to A it has components $\mu_0 = 1, \mu_1 = 1$. What are its components in the dual basis to B?

3 Discrete curves(10 points total)

Consider the two discrete curves, with three points each, shown below (one is black with solid lines, and one is gray with broken lines).



3.1 Distance between curves (5 points)

Using a Gaussian kernel with standard deviation of 1 ($k(x, y) = \exp(-|x - y|^2/2\sigma^2)$ where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$), compute the norm squared between the two curves. Use the norm we derived in class, which models curves as belonging to the dual space to smooth vector fields, and depends on the centers and tangents to each line segment.

When computing tangents, order the points from left to right.

3.2 Orientation matters (5 points)

This time, order the points in the gray curve from right to left, and the black curve from left to right, and recompute the norm squared above.

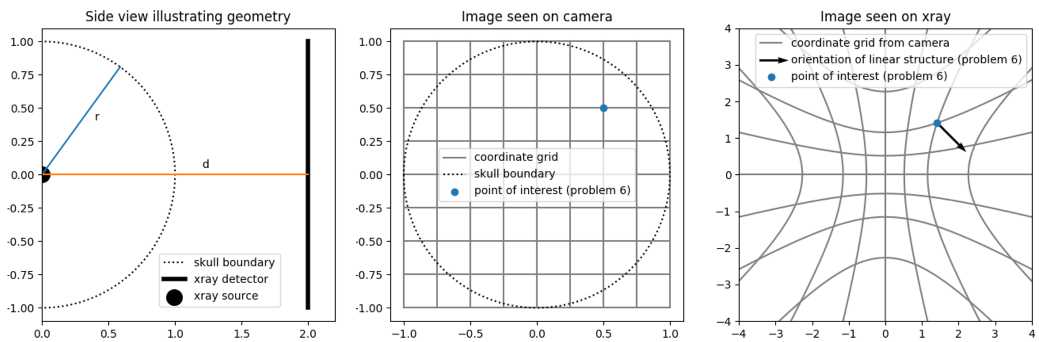
4 Manifolds and charts (10 points total)

Consider the surface of the skull as a hemisphere of radius r without boundary, as a manifold $\mathcal{M}$ with standard topology. We represent it with the following two charts.

If $p = (p^0, p^1, p^2) \in \mathcal{M} \subset \mathbb{R}^3$, then one chart is given by $x(p) = (p^0, p^1)$. This chart represents what we would see with a camera placed above the head.

A radiotracer (xray source) is deposited in the center of the hemisphere. Its emissions are detected, leading to an image of the skull on an xray detector a distance $d > r$ above the center of the skull. This corresponds to the chart $y(p) = (p^0 d/p^2, p^1 d/p^2)$.

A figure illustrating the geometry and the two charts is shown below, when $r = 1$ and $d = 2$.



4.1 Chart transitions (10 points)

Compute the two chart transition maps $y \circ x^{-1}$ and $x \circ y^{-1}$. Show that an atlas consisting of these two charts is smoothness compatible.

Hint: While the former is straightforward to compute, the latter requires more care. Note that these chart transitions simply rescale the components of a point, where the scale factor depends only on the distance from the point to the origin. If X is the Euclidian distance of a point from the origin in chart x, and Y is the Euclidean distance distance of the same point to the origin in chart y, the scale factor from x to y is $Y = X\frac{d}{\sqrt{r^2 - X^2}}$. Solving for X in terms of Y will give the scale factor in the other direction.

5 Groups (10 points total)

5.1 A shear group (5 points)

Consider the 1 parameter family of shear matrices in 2D of the form

$$G = \left\{ \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

Show that this family of matrices form a group, with group operation given by matrix multiplication. i.e. show that the group operation is a map from $G \times G \to G$ satisfying the group axioms (associativity, identity, inverse).

5.2 Rotations by shears (5 points)

Show that the two parameter family of shear matrices in 2D of the form

$$H = \left\{ \begin{pmatrix} 1 & \theta \\ \phi & 1 \end{pmatrix} : (\theta, \phi) \in \mathbb{R}^2 \right\}$$

does not form a group.

However, show that the product

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

forms a rotation matrix when $\alpha = -\tan(\theta/2) = -\sin\theta/(1+\cos\theta)$ and $\beta = \sin(\theta)$ (provided $\theta \neq \pm\pi$). Feel free to look up any trigonometric identities necessary.

Note that shears can be applied to images very efficiently, since they only involve shifting the rows or columns by a specified amount, so this parameterization of the 2D rotation group is commonly used in computer graphics.

6 Tangent spaces (10 points total)

A surgeon is cutting the scalp of the patient in question 4, using a robot guided by the camera (chart x). They notice a linear feature (e.g. a cranial suture) on the x ray image (chart y), and wish to cut along it.

In the chart x, the scalpel is placed at the point $(r/2, /r2)$. In the chart y, in the chart induced basis of the tangent space at the point $y(x^{-1}(r/2, r/2))$, the linear structure is oriented in the direction $(1, -1)$ (see the figure in problem 4).

6.1 Change of coordinates of tangent vectors (10 points)

With what velocity should the surgeon guide the robot to follow this structure in chart x? i.e. apply the chart induced change of basis formula to the vector in chart y, to identify the components of the vector in chart x.

Note: You will need to have solved question 4 before solving this one. If there is an error in question 4 but the approach in question 6 is still correct, no points will be deducted from question 6.

7 Image registration (10 points total)

Consider aligning a set of N points in $\mathbb{R}^2$, P, to a set of N corresponding points, Q. As in class, we will represent P and Q by $2 \times N$ matrices. Registration will be performed using the set of shear transformations from problem 5.2, of the form

$$S = \begin{pmatrix} 1 & \theta \\ \phi & 1 \end{pmatrix}.$$

7.1 An optimal solution (10 points)

Set up and solve a variational problem to find the optimal θ and ϕ (as a function of P and Q) that minimizes sum of square error between SQ and P.

8 Geodesics on a surface (20 points total)

Suppose the surface of a body part is modeled as a 2 dimensional manifold $\mathcal{M} \subset \mathbb{R}^3$. Assume this surface can be represented as the graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, or $w = f(u, v)$. With $p = (u, v, w) \in \mathcal{M}$, we will use a single coordinate chart $x(p) = x(u, v, w) = (u, v)$, and use the corresponding chart induced basis of the tangent space.

A surgeon is interested in cutting an incision between two points, modeled by the curve $q(t) \in \mathbb{R}^2$ expressed in coordinates, that minimizes tissue damage. It must therefore be a geodesic curve, and we are interested in characterizing this family of curves.

We will use the metric

$$g_{ij}(q) = \begin{pmatrix} 1 + \partial_0 f(q)\partial_0 f(q) & \partial_0 f(q)\partial_1 f(q) \\ \partial_0 f(q)\partial_1 f(q) & 1 + \partial_0 f(q)\partial_0 f(q) \end{pmatrix}_{ij}$$

where ∂_0 refers to the derivative with respect to the first argument u, and ∂_1 refers to the derivative with respect to the second argument v.

Using matrix notation, we could write this as $g(q) = Id + Df(q)^T Df(q)$ where Df is a row vector of partial derivatives and Id is the 2×2 identity matrix. Note that its inverse can be computed by the Sherman Morrison formula (rank 1 update formula): $g^{-1} = Id - \frac{Df^T Df}{1+|Df|^2}$ where $|Df|^2 = (\partial_0 f)^2 + (\partial_1 f)^2$.

Note that if X^i are the components of a tangent vector, then this metric simply evaluates the Euclidean inner product of the vector $(X^0, X^1, \partial_0 f(q)X^0 + \partial_1 f(q)X^1)$ in 3D.

8.1 Derivatives of the metric (5 points)

Calculate the 8 partial derivatives of the metric, and show that they take the form $\partial_k g_{ij} = \partial_k\partial_i f \partial_j f + \partial_i f \partial_k \partial_j f$.

8.2 Christoffel symbols of the first kind (5 points)

Recall that the Christoffel symbols of the first kind are defined by

$$\Gamma_{k,ij} = \frac{1}{2}\left(-\partial_k g_{ij} + \partial_i g_{kj} + \partial_j g_{ik}\right)$$

Work out the 8 Christoffel symbols of the first kind and show that $\Gamma_{k,ij} = \partial_k f \partial_i \partial_j f$.

8.3 Christoffel symbols of the second kind (5 points)

Recall that Christoffel symbols of the second kind are defined by $\Gamma^l_{ij} = g^{lk}\Gamma_{k,ij}$, where g^{lk} are the components of g^{-1}. Show that the Christoffel symbols of the second kind are given by $\Gamma^l_{ij} = \frac{1}{1+|Df|^2}\partial_l f \partial_i \partial_j f$.

8.4 The geodesic equation (5 points)

Recall that the geodesic equation is given by $\ddot{q}^l + \Gamma^l_{ij}(q)\dot{q}^i\dot{q}^j = 0$. Write out the geodesic equation for this scenario.

Write out the geodesic equation when f is a linear function describing the surface of a flat anatomical structure. What do its solutions look like?

Write out the geodesic equation when f is a function of the form $f(u, v) = -u^2$ locally describing the surface of an elongated and curved anatomical structure.

9 Riemannian averages (10 points total)

Recall that on the scale group with the left invariant metric discussed in class, geodesics take the form $q(t) = b\exp(at/b)$ for $a, b \in \mathbb{R}$. In other words, using the natural choice of chart (a scaling transform by a factor of s is represented by the number s), a constant speed geodesic $q(t)$ starting at the point b, with initial velocity a, has the form $q(t) = b\exp(at/b)$.

In this case the Riemannian exponential takes the form

$$\exp_b(a) = b\exp(a/b),$$

where the exp with no subscript refers to the usual exponential ($\exp(x) = e^x$).

The Riemannian logarithm takes the form

$$\log_b(c) = b\log(c/b),$$

where the log with no subscript refers to the usual natural logarithm with base e.

Suppose we measure two datapoints with scale values of 1 and 4, and we want to find their Riemannian average.

9.1 Procrustes algorithm (5 points)

Carry out two iterations of the Procrustes algorithm derived in class, starting with an initial guess for the average of 1.

First, compute the Riemannian logarithm of the two data points with base point 1. Second take the average of these two initial velocities. Third, compute a new guess for our Riemannian average, by applying the Riemannian exponential starting at the point 1 with the initial velocity computed in step 2.

Repeat steps 1, 2, and 3 one more time and explain why the algorithm has converged.

9.2 Interpolation algorithm (5 points)

Since we have only two data points, we can equivalently compute the average by constructing a constant speed geodesic curve from one point (at $t = 0$) to the other (at $t = 1$), and evaluating it at $t = 0.5$. Write down this geodesic curve and compute its value at $t = 0.5$. Show that this gives the same answer as above.