Biomath 208, 2023 Comprehensive Exam

Instructor: Daniel Tward

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0 Instructions

While you may refer to any notes or materials, you are not permitted to do internet searches for answers to specific questions. You are permitted to look up any trigonometric identities or other equations you may need if they are not provided. You should not need a calculator, but can use one if you like. Please write answers in analytical form, for example write exp(2) rather than 7.389.

There are 9 problems, one for each chapter, and most problems have sub-parts. The problems vary in difficulty, so you may chose to solve the simpler ones first. Please note that problem 8 is expected to be longer than the others and is worth more points.

Partial answers will receive partial scores so show all your work. The questions will not be graded as correct/incorrect.

1 Imaging data (10 points total)

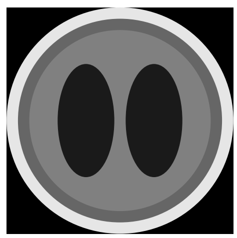
1.1 Discrete images and interpolation (5 points)

Consider a 1D discrete image with 5 samples, I_i where i denotes the sample index (starting from 0). The location of the first pixel is $O = -0.5$. The spacing between pixels is $\Delta = 0.25$. The the sample values take the form $I_i = 1 - (i-2)^2$. We define the continuous image $I(x)$ using linear interpolation.

Sketch the image $I(x)$, labeling the axes, and evaluate the image at the point $x = 0.1$.

1.2 Color maps (5 points)

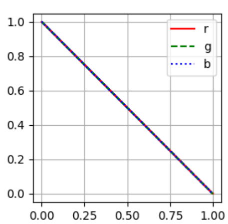
Consider the piecewise constant model of the human brain as seen in a computed tomography scan, shown in the figure below:



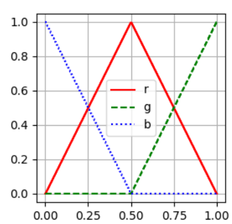
The outer shell, the skull, has a brightness of 0.9. The first inner layer, cortex, has brightness 0.4. The next layer, white matter, has brightness 0.5. The two inner ellipses, ventricles, have brightness 0.1. The picture is drawn with a window between 0 and 1.

Below four colormaps are shown, as a graph of their mapping to red green and blue. And four colored images are shown. Specify which number cooresponds to which letter.

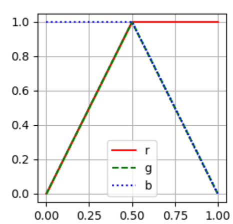
1



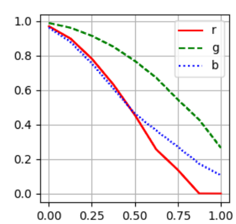
2



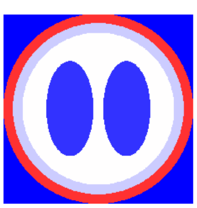
3



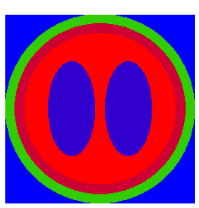
4



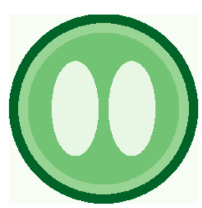
A



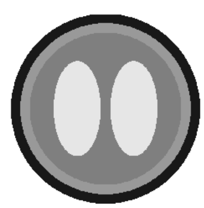
B



C



D



2 Linear algebra (10 points total)

Consider the 2D Cartesian plane (the set of ordered pairs) as vector space. We'll define basis A as consisting of the two vectors $A_0 = (1, 0)$ and $A_1 = (1, -1)$. We'll define basis B as consisting of the two vectors $B_0 = (0, -1)$ and $B_1 = (-1, 1)$.

2.1 Change of covector components (10 points)

Let μ be a covector. In the dual basis to A it has components $\mu_0 = 1, \mu_1 = 1$. What are its components in the dual basis to B?

3 Discrete surfaces (10 points total)

Consider two discrete surfaces, consisting of one triangle each. Surface 1 is specified in a text file as follows:

```
3 vertices, 1 face
vertex 0: x=0.0, y=0.0, z=0.0
vertex 1: x=1.0, y=0.0, z=0.0
vertex 2: x=0.0, y=1.0, z=0.0
triangle 0: vertex 0, vertex 1, vertex 2
```

Surface 2 is specified in a text file as follows:

```
3 vertices, 1 face
vertex 0: x=0.0, y=0.0, z=0.0
vertex 1: x=0.0, y=0.0, z=1.0
vertex 2: x=0.0, y=1.0, z=0.0
triangle 0: vertex 0, vertex 1, vertex 2
```

Below we will use the inner product we derived in class, which models surfaces as belonging to a dual space to smooth vector fields.

Recall that we first define a center of each triangular face c_i, then we find an area weighted normal to each face A_i (using the cross product of its edges divided by two).

The inner product formula between surfaces A (with N_A triangles) and B (with N_B triangles) is

$$\sum_{i=1}^{N_A} \sum_{j=1}^{N_B} k(c_i - c_j) A_i^A \cdot A_j^B$$

Here we will use a Gaussian kernel with standard deviation 1 ($k(x, y) = \exp(-|x-y|^2/2\sigma^2)$ where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$), and $A_i \cdot A_j$ refers to their Euclidian dot product (multiply pairs of components and add them up).

3.1 Inner product

Compute the inner product between the two surfaces.

Hint: look for a simplification involving orthogonality that could help avoid unnecessary calculations.

3.2 Distance between surfaces (5 points)

Compute the norm squared of the distance between the two surfaces.

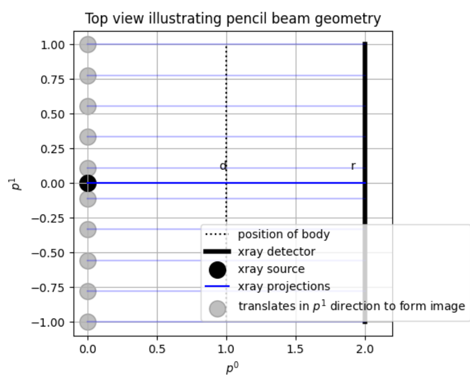
4 Manifolds and charts (10 points total)

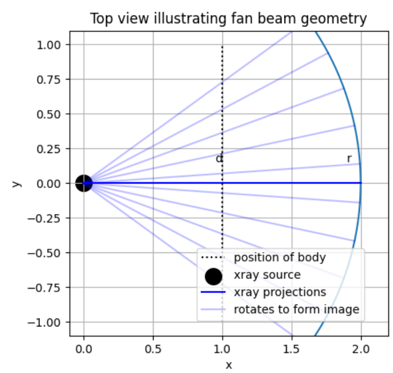
Consider a person being imaged by two different CT scanners. They are a distance d from the xray source, and the xray detector is located at a distance r from the xray source. Assume the person is very thin, so they are modeled by the $p^0 = d$ plane in 2D.

First, they are imaged using a first generation "pencil beam" geometry. This performs a parallel projection. For a given point in the body (p^1, p^2), we use the chart $x = id$ to associte it with the coordinate $x(p) = (p^1, p^2) = (x^0, x^1)$ describing where it appears in the image.

Second, they are imaged using a second generation "fan beam" geometry. For a given point in the body (p^1, p^2), we use the chart $y(p) = (\arctan(p^1/d), p^2) = (y^0, y^1)$ to describe where it appears in the image.

The geometry is shown in the figure where $d = 1$ and $r = 2$. Note that this is a top view. A 2D image is formed by sliding the whole imaging apparatus in the p^2 direction.





Show that the two charts x and y form a smoothly compatible atlas for the 2D plane. Recall that $\arctan'(x) = 1/(1 + x^2)$.

5 Groups (10 points total)

5.1 A scale group (5 points)

Consider the 2 parameter family of on axis scale matrices in 2D of the form

$$G = \left\{ \begin{pmatrix} s^0 & 0 \\ 0 & s^1 \end{pmatrix} : s^i \in \mathbb{R}^+ \right\}$$

Show that this family of matrices form a group, with group operation given by matrix multiplication. i.e. show that the group operation is a map from $G \times G \to G$ satisfying the group axioms (associativity, identity, inverse).

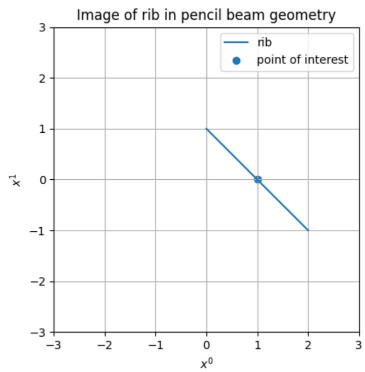
5.2 Not a scale group (5 points)

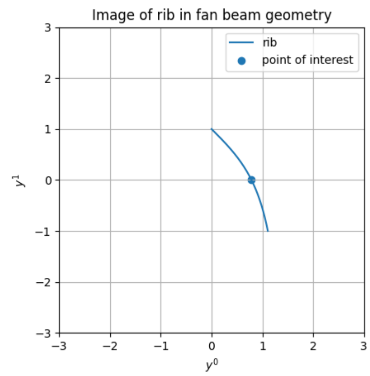
Show that with s^i in $\mathbb{R}$ (instead of $\mathbb{R}^+$) this does not form a group. i.e. show which axiom is violated.

6 Tangent spaces (10 points total)

A surgeon examines imaging data from problem 4, and notices the patient's rib at the location $(d, 0)$ follows the tangent vector $(1, -1)$ in the pencil beam chart (the chart x).

The images with the rib in both coordinate charts are shown below, with the choice $d = 1$.





What position does this point take in the chart y? (i.e. apply the chart transition map)

What angle does the vector take in the chart y? (i.e. apply the change of coordinates formula, multiplication with the Jacobian of the chart transtion map)

7 Image registration (10 points total)

Consider aligning a set of N points in $\mathbb{R}^2$, P, to a set of N corresponding points, Q. As in class, we will represent P and Q by $2 \times N$ matrices. Registration will be performed using the set of on axis scale transformations from problem 5, of the form

$$S = \begin{pmatrix} s^1 & 0 \\ 0 & s^2 \end{pmatrix}.$$

7.1 An optimal solution (10 points)

Set up and solve a variational problem to find the optimal s^1 and s^2 (as a function of P and Q) that minimizes sum of square error between SQ and P.

8 Geodesics (15 points total)

Consider a fan beam geometry parameterized by an angle θ, with a flat xray detector a distance 1 away, and consider our Riemannian manifold $\mathcal{M} = \mathbb{R}$ as the detector plane. Note that while question 4 and 6 considered a 2D manifold with a related geometry, here we consider only a 1D manifold.

Consider a chart $x(p) = p$ (identity). In this chart, we'll consider the metric $g^{(x)}(q) = 1/(1+q^2)^2$.

We can also parameterize a location $p \in \mathbb{R}$ on the detector with the chart $y(p) = \arctan(p)$, which takes values in $(-\pi/2, \pi/2)$. Note that $y^{-1}(\theta) = \tan(\theta)$. This chart represents a point on the detector using the angle from the xray source.

8.1 The constant speed geodesic equation (5 points)

Show that, in the chart x, the constant speed geodesic equation corresponds to

$$\ddot{q} - \frac{2q\dot{q}^2}{1+q^2} = 0 \; . \tag{8.1.1}$$

Hint: Recall that in 1D the three derivative terms in the geodesic equation are equal (up to a sign), and the inverse of the metric is just its reciperocal.

8.2 Solutions to the geodesic equation (5 points)

Show (by plugging it in and demonstrating the equation is satisfied) that for $a, b \in \mathbb{R}$, curves of the form $q(t) = \tan(at+b)$ are solutions to the above equation. Note that this corresponds to the angle from the xray source changing at a constant speed.

Recall that $\tan'(\theta) = 1/\cos^2(\theta)$

8.3 Pull back metric (5 points)

Show that our metric is the pullback of the identiy metric $g^{(y)}(q) = 1$ in the chart y.

That is, consider two tangent vectors in the chart x with component u_x, v_x. Use the Jacobian of $y \circ x^{-1}$ to change coordinates by pushing forward these tangent vector from the chart y to the chart y. Then evaluate their inner product using the metric $g^{(y)}$ and demonstrate that this is equal to their inner product in the chart x using $g^{(x)}$.

Note that this illustrates the invariance of the constant speed geodesics in two different charts.

9 Riemannian averages (10 points total)

Recall that on the open set of probabilities ($\mathcal{M} = (0, 1)$), with the metric discussed in class, Riemannian exponentials take the form:

$$\exp_p(v) = \frac{1}{1 + \exp(-(\log(p/(1-p)) + v))}$$

where the exp with no subscript refers to the usual exponential ($\exp(x) = e^x$), and the log with no subscript refers to the natural logarthm (base e).

In this case the Riemannian logarithm takes the form

$$\log_p(q) = \log\left(\frac{q}{1-q}\right) - \log\left(\frac{p}{1-p}\right)$$

Assume we observe two data points, $\frac{1}{4}$ and $\frac{3}{4}$.

9.1 Procrustes algorithm 1 (5 points)

Carry out one iteration of the Procrustes algorithm derived in class, starting with an initial guess for the average of $\frac{1}{4}$.

First, compute the Riemannian logarithm of the two data points with base point $\frac{1}{4}$. Second take the average of these two initial velocities. Third, compute a new guess for our Riemannian average, by applying the Riemannian exponential with the base point $\frac{1}{4}$ (our current guess) with the initial velocity computed in step 2.

9.2 Procrustes algorithm 2 (5 points)

Repeat steps 1, 2, and 3 one more time (where the base point is our new guess for the average) and explain why the algorithm has converged.