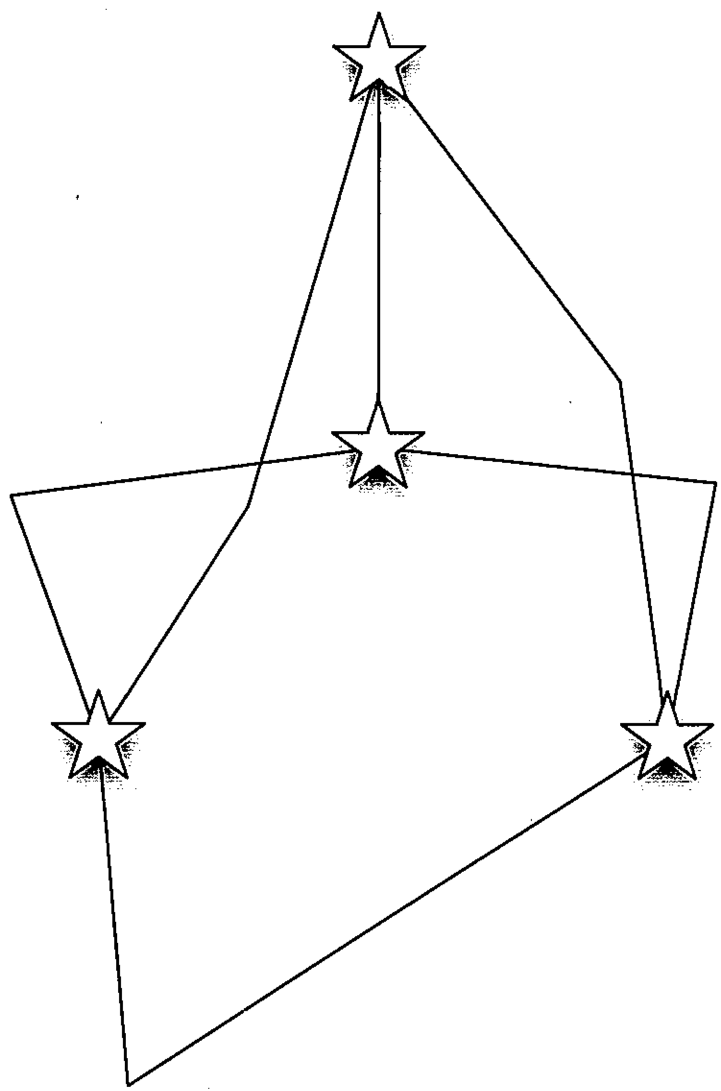
Comprehensive Exam
Spring 2010 for Biomath ~~210~~ 203: The Structure, Function, and Evolution of Biological Systems

1. Consider a network in which each node has three links. No link can connect a node to itself, and any two nodes can be connected by at most one link. The simplest example of such a network is given below. Each node is represented by a star.



a. Prove that is impossible to have a network of this type that has an odd number of nodes.

b. What are the number of subgraphs that are i) wedges, ii) triangles, and iii) squares for the example above?

The four-node network above can be represented as a "ladder" network by re-arranging nodes and links. Here, a ladder means that nodes are grouped into horizontal pairs, and these pairs are then arranged vertically with respect to each other. All horizontal pairs are connected by a link, and each node is connected to the nodes immediately above and below it. For the nodes at the extreme top and bottom of the entire network, you must think about how to complete the network. Re drawing the four-node example above while maintaining the same connections (i.e., topology) should give a strong clue of how to complete these networks.

c. By building "ladders" of nodes as described above and completing the network via "crossovers" (analogous to the topology of the unique four-node network above), draw the simplest generalization of this type of network that has six nodes, eight nodes, and ten nodes.

d. What is the mean connectivity of these networks?

e. For each of the "ladder" networks you constructed in c), what are the number of subgraphs that are i) wedges, ii) triangles, and iii) squares?

f. How do the number of wedges in e) compare with those expected based on Erdos-Renyi random networks or on the geometric models of Itzkovitz and Alon? How do the number of triangles and squares in e) compare with those expected based just on Erdos-Renyi random networks? Which subgraphs would be identified as potential motifs for these networks?

g. How do the actual numbers of wedges, triangles, and squares each scale with the size of the network (i.e., the number of nodes, N) for these ladder networks? Can you write down a power-law relationship for how any of these numbers depends on network size? If so, what is the scaling exponent?

h. What do the scaling relationships in g) say about the identification of motifs as networks get larger? That is, how do these scaling relationships compare with the scaling relationships predicted by Erdos-Renyi random networks or geometric models like Itzkovitz and Alon?

i. Draw an example of a six-node network that follows the rules stated at the beginning of this test but that is **not** a ladder diagram (even after possible rearrangements). How many wedges, triangles, and squares are in this diagram? How do these numbers compare with those for the six-node ladder network?

j. Let us now consider some dynamics placed onto this network. Starting from any node, the material (blood, person, resources, signaling, information) can go through a link either to the "right" or "left". When the next node is reached, material cannot travel back along the path from which it came, and it must go in the opposite direction (right versus left or left versus right) than it did at the previous node. As the network is traversed, the path taken must alternate directions forever. Prove that the material must eventually return to the starting node. This is equivalent to proving that the flow cannot get stuck traveling forever in a loop of which the initial node is not a part. (Hint: Consider the "dual" network constructed by replacing each link with a node and connecting these new nodes only if the original links connected to the same original node. Then, map flow through the original network onto what flow means through the new "dual" network and consider what this means about loops. In the figure below, the first diagram is a path through the original network with nodes labeled for going from links to nodes in the dual network. The second diagram is a path through the newly formed dual network.)