HW 2 -Simon Lee

A) $x^3 + \varepsilon x^2 - x$

assume $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$

we now have

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^3 + \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2 - (x_0 + \varepsilon x_1 + \varepsilon^2 x_2)$$

$x_0^3 + 3\varepsilon x_0^2 x_1 +$
$3\varepsilon^2 x_0^2 x_2 +$
$3\varepsilon^2 x_1^2 x_0$

	x_0	εx_1	$\varepsilon^2 x_2$
x_0	x_0^2	$\varepsilon x_0 x_1$	$\varepsilon^2 x_0 x_2$
εx_1	$\varepsilon x_1 x_0$	$\varepsilon^2 x_1^2$	
$\varepsilon^2 x_2$	$\varepsilon^2 x_0 x_2$		

$$x_0^3 + 3\varepsilon x_0^2 x_1 + 3\varepsilon^2 x_0^2 x_2 + 3\varepsilon^2 x_1^2 x_0 \mid \varepsilon x_0^2 + 2\varepsilon^2 x_0 x_1 + \cancel{\varepsilon^3 x_1^2} + \cancel{2\varepsilon^3 x_0 x_2} \mid -x_0 - \varepsilon x_1 - \varepsilon^2 x_2$$

$O(\varepsilon^0) = x_0^3 - x_0$ roots $\Rightarrow$ $x_0(x_0^2 - 1)$ $x_0 = 0, \pm 1$

$O(\varepsilon^1) = \varepsilon(3x_0^2 x_1 + x_0^2 - x_1)$

$O(\varepsilon^2) = \varepsilon^2(3x_0^2 x_2 + 3x_1^2 x_0 + 2x_0 x_1 - x_2)$

$\boxed{\text{plug in } x_0 = 0}$

$$\varepsilon(\cancel{3(0)^2} x_1 + \cancel{0^2} + x_1) = 0$$

$$x_1 = 0$$

$$\varepsilon^2(3(0)^2 x_2 + 3(0)^2(0) + 2(0)(0) + x_2 = 0$$

$$x_2 = 0$$

$\boxed{\text{plug in } x=1}$

$$O(\varepsilon^1) = \varepsilon(3x_0^2 x_1 + x_0^2 - x_1) = 0$$

$$= \varepsilon(3(1)^2 x_1 + (1)^2 - x_1) = 0$$

$$\varepsilon(2x_1) = -1$$

$$\boxed{x_1 = -\frac{1}{2}}$$

$$O(\varepsilon^2) = \varepsilon^2(3x_0^2 x_2 + 3x_1^2 x_0 + 2x_0 x_1 - x_2) = 0$$

$$\varepsilon^2(3(1)^2 x_2 + 3\left(-\tfrac{1}{2}\right)^2(1) + 2(1)\left(-\tfrac{1}{2}\right) - x_2 = 0$$

$$\varepsilon^2\left(3x_2 + \frac{3}{4} - 1 - x_2\right) = 0$$

$$\varepsilon^2(2x_2) = \frac{1}{4}$$

$$\boxed{x_2 = \frac{1}{8}}$$

$\boxed{\text{plug in } -1}$

$$\mathcal{O}(\varepsilon^1) = \varepsilon(3x_0^2 x_1 + x_0^2 - x_1)$$

$$\varepsilon(3(-1)^2 x_1 + (-1)^2 - x_1 \longrightarrow \boxed{x_1 = -\frac{1}{2}}$$

$$\mathcal{O}(\varepsilon^2) = \varepsilon^2(3x_0^2 x_2 + 3x_1^2 x_0 + 2x_0 x_1 - x_2) = 0$$

$$\varepsilon^2(3(-1)^2 x_2 + 3\left(-\frac{1}{2}\right)^2(-1) + 2(-1)\left(-\frac{1}{2}\right) - x_2) = 0$$

$$\varepsilon^2(3x_2 - \frac{3}{4} + 1 - x_2) = 0$$

$$2x_2 = -\frac{1}{4}$$

$$\boxed{x_2 = -\frac{1}{8}}$$

$$\boxed{x(\varepsilon) = \begin{cases} -1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \mathcal{O}(\varepsilon^3) + \dots \\ 0 \\ 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \mathcal{O}(\varepsilon^3) + \dots \end{cases}}$$

1b) $\varepsilon x^3 + x^2 - 2x + 1 = 0$

If $\varepsilon = 0$ then we have

$$x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0$$

we only have 1 root with multiplicity 2 right now

we have $x = 1$

so we have the assumption $x = x_0 + x_1\varepsilon + x_2\varepsilon^2$

$$x^2 = x_0^2 + 2\varepsilon x_0 x_1 + 2\varepsilon^2 x_0 x_2 + \varepsilon^2 x_1^2$$

$$-2x = -2x_0 - 2x_1\varepsilon - 2x_2\varepsilon^2$$

$$1 = 1$$

$$O(\varepsilon^0) = x_0^2 - 2x_0 + 1$$

$$O(\varepsilon^1) = \varepsilon(2(1)x_1 - 2x_1) \quad 0 = 0$$

$$O(\varepsilon^2) = \varepsilon^2(2(1)x_2 + (1)^2 - 2x_2)$$

$$1 = 0$$

we will need to do dominant balancing since the ε is on the leading term

A) $\varepsilon x^3 \sim x^2$

$\varepsilon x \sim 1 \Rightarrow x = \frac{1}{\varepsilon}$ ✓

$$\underbrace{\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon^2}} - \frac{2}{\varepsilon} + 1 \qquad x = Y \cdot \frac{1}{\varepsilon}$$

These are the dominant pairs

B) $\varepsilon x^3 \sim x$

$\varepsilon x^2 \sim 1 \rightarrow x = \frac{1}{\sqrt{\varepsilon}}$

$$\frac{1}{\varepsilon^{\frac{1}{2}}} + \frac{1}{\varepsilon} - \frac{2}{\varepsilon^{\frac{1}{2}}} + 1$$

the two dominant pairs are not dominant

c) $\varepsilon x^3 \sim 1$

$x^3 \sim \frac{1}{\varepsilon}$

$x \sim \frac{1}{\sqrt[3]{\varepsilon}}$

$$\varepsilon\left(\frac{1}{x^{\frac{1}{3}}}\right)^3 + \left(\frac{1}{x^{\frac{1}{3}}}\right)^2 - \frac{2}{x^{\frac{1}{3}}} + 1$$

these two terms are the dominant pair. $\cancel{}$

$y = \frac{x}{\varepsilon}$ → new variable to plug in found from dominant balancing

$$\varepsilon\left(\frac{x}{\varepsilon}\right)^3 + \left(\frac{x}{\varepsilon}\right)^2 - 2\left(\frac{x}{\varepsilon}\right) + 1$$

$$\frac{x^3}{\varepsilon^2} + \frac{x^2}{\varepsilon^2} - \frac{2x}{\varepsilon} + 1 = 0$$ multiply by ε^2

$$x^3 + x^2 - 2\varepsilon x + \varepsilon^2$$

now assume the series expansion

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2$$

so we plug in the series expansion to the new equation

next page

$$x^3 + x^2 - 2\varepsilon x + \varepsilon^2$$

we know from problem 1

$$x^3 = x_0^3 + 3\varepsilon x_0^2 x_1 + 3\varepsilon^2 x_0^2 x_2 + 3\varepsilon^2 x_1^2 x_0$$

$$x^2 = x_0^2 + 2\varepsilon x_0 x_1 + 2\varepsilon^2 x_0 x_2 + \varepsilon^2 x_1^2$$

$$2\varepsilon x = -2\varepsilon x_0 - 2\varepsilon^2 x_1 + \dots +$$

$$\varepsilon^2 = \varepsilon^2$$

we then group equations

$$O(\varepsilon^0) = x_0^3 + x_0^2 \qquad x_0^2(x_0+1) \qquad x = 0, -1$$

$$O(\varepsilon^1) = \varepsilon(3x_0^2 x_1 + 2x_0 x_1 - 2x_0)$$

$$O(\varepsilon^2) = \varepsilon^2(3x_0^2 x_2 + 3x_1^2 x_0 + 2x_0 x_2 - 2x_1 + x_1^2 + 1)$$

$\boxed{\text{for } x=0}$

$$O(\varepsilon^1) = \varepsilon(3\cancel{(0)^2}x_1 + 2\cancel{(0)}x_1 + \cancel{2}(0)) = 0 \qquad \cancel{x_1} = 0$$

$$O(\varepsilon^2) = \varepsilon^2(3\cancel{(0)^2}x_2 + 3\cancel{(0)^2}(0) + 2\cancel{(0)}x_2 + 2\cancel{(0)} + 2\cancel{(0)} + 1$$

$$x_2 = 0$$

$$O(\varepsilon^2) = \varepsilon^2 (3x_0^2 x_2 + 3x_1^2 x_0 + 2x_0 x_2 - 2x_1 + x_1^2 + 1)$$

$$= \varepsilon^2 ((0) x_2 + 3(0) x_1^2 + 2(0) x_2 - 2x_1 + x_1^2 + 1$$

$$x_1^2 - 2x_1 + 1$$

$$(x_1 - 1)^2$$

$$\boxed{x_1 = 1}$$

should repeat similar to before

$$O(\varepsilon^3) \rightarrow 0 = 0$$

$$O(\varepsilon^4) \rightarrow 1 = 0$$

we then solve for the other root to find a different expansion

$\boxed{\text{for} \quad x=-1}$

$$O(\varepsilon^1) = \varepsilon(3x_0^2x_1 + 2x_0x_1 - 2x_0)$$

$$\varepsilon(3(-1)^2x_1 + 2(-1)x_1 - 2(-1)) = 0$$

$$\varepsilon(3x_1 - 2x_1 + 2) = 0$$

$$x_1 = -2$$

$$O(\varepsilon^2) = \varepsilon^2(3x_0^2x_2 + 3x_1^2x_0 + 2x_0x_2 - 2x_1 + x_1^2 + 1)$$

$$\varepsilon^2(3(-1)^2x_2 + 3(-2)^2(-1) + 2(-1)x_2 - 2(-2) + (-2)^2 + 1$$

$$\varepsilon^2(3x_2 - 12 - 2x_2 + 4 + 4 + 1$$

$$x_2 - 3 = 0$$

$$\boxed{x_2 = 3}$$

After plugging in our roots we still only have a singular expansion. Therefore we now assume the solution must be somewhere between 0 and 1. we therefore explore a new expansion

$$X = x_0 + \varepsilon^{\frac{1}{2}} x_1 + \varepsilon x_2 + \varepsilon^{\frac{3}{2}} x_3$$

we plug in to original equation $\varepsilon x^3 + x^2 - 2x + 1 = 0$

$$\varepsilon(x_0 + x_1\varepsilon^{\frac{1}{2}} + \varepsilon x_2 + \varepsilon^{\frac{3}{2}} x_3)^3 + (x_0 + x_1\varepsilon^{\frac{1}{2}} + \varepsilon x_2 + \varepsilon^{\frac{3}{2}} x_3)^2 - 2(x_0 + \varepsilon^{\frac{1}{2}} x_1 + \varepsilon x_2 + \varepsilon^{\frac{3}{2}} x_3) + 1$$

high order high order

$$x^3 = x_0^3 + 3x_0^2 x_1 \varepsilon^{\frac{1}{2}} + 3x_0^2 x_2 \varepsilon + \cancel{x_1^3 \varepsilon^{\frac{3}{2}}} + \cancel{3x_0^2 x_3 \varepsilon^{\frac{3}{2}}} + 3x_1^2 x_0 \varepsilon$$

$$\varepsilon x^3 = \varepsilon x_0^3 + 3x_0^2 x_1 \varepsilon^{\frac{3}{2}} + 3x_0^2 x_2 \varepsilon^2 + 3x_1^2 x_0 \varepsilon^2$$

$$x^2 = x_0^2 + 2x_0 x_1 \varepsilon^{\frac{1}{2}} + x_1^2 \varepsilon + 2x_0 \varepsilon x_2 + 2x_0 x_3 \varepsilon^{\frac{3}{2}} + 2x_1 x_2 \varepsilon^{\frac{3}{2}}$$

$$-2x = -2x_0 - 2\varepsilon^{\frac{1}{2}} x_1 - 2\varepsilon x_2 - 2\varepsilon^{\frac{3}{2}} x_3$$

$$1 = 1$$

$$O(\varepsilon^0) = x_0^2 - 2x_0 + 1 \qquad (x-1)^2 \qquad x_0 = 1$$

$$O(\varepsilon^{\frac{1}{2}}) = \varepsilon^{\frac{1}{2}}(2x_0 x_1 - 2x_1) \rightarrow \varepsilon^{\frac{1}{2}}(2(1)x_1 - 2x_1) \qquad 0 = 0$$

$$O(\varepsilon^1) = \varepsilon(x_0^3 + x_1^2 + 2x_0 x_2 - 2x_2) \rightarrow (1)^3 + x_1^2 + 2x_2 - 2x_2$$

$$x_1 = \sqrt{-1}$$

$$x_1 = \pm i$$

we found the complex conjugate roots and have three expansions

$$x = \begin{cases} -1 - 2\varepsilon + 3\varepsilon^2 \\ 1 + i\varepsilon^{\frac{1}{2}} + \mathcal{O}(\varepsilon) \\ 1 - i\varepsilon^{\frac{1}{2}} + \mathcal{O}(\varepsilon) \end{cases}$$

b) $x^3 - x^2 + \varepsilon = 0$

$$\boxed{x = \sum_{n=0}^{\infty} x_n \varepsilon^{B \cdot n}}\quad B<1$$

<u>assume</u> $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2$

$$x^3 = x_0^3 + 3\varepsilon x_0^2 x_1 + 3\varepsilon^2 x_0^2 x_2 + 3\varepsilon^2 x_1^2 x_0$$

$$-x^2 = -x_0^2 - 2\varepsilon x_0 x_1 - \varepsilon^2 x_1^2 - 2\varepsilon^2 x_0 x_2$$

ε

$\mathcal{O}(\varepsilon^0) = x_0^3 - x_0^2 \Rightarrow$ roots $x^2(x-1)$ $x_0 = 0, 1$

$\mathcal{O}(\varepsilon^1) = \varepsilon(3x_0^2 x_1 - 2x_0 x_1 + 1) = 0$

$\mathcal{O}(\varepsilon^2) = \varepsilon^2(3x_0^2 x_2 + 3x_1^2 x_0 - 2x_0 x_2 - x_1^2)$

$\boxed{\text{For } x_0 = 0}$

$\varepsilon(3\cancel{(0)^2} x_1 - \cancel{2}(0) + 1) = 0$

$1 = 0$

similar case to 1b. we will look into this after we solve other root.

$\boxed{\text{For } x_0 = 1}$

$\varepsilon(3(1)^2 x_1 - 2(1) x_1 + 1)$

$\varepsilon(3x_1 - 2x_1 + 1) = \boxed{x_1 = -1}$

$$\varepsilon^2 (3(1)^2 x_2 + 3(-1)^2(1) - 2(1) x_2 - (-1)^2$$

$$\varepsilon^2 (3x_2 + 3 - 2x_2 - 1) = x_2 + 2 \quad \boxed{x_2 = -2}$$

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we now try to assume that

$$x = x_0 + x_1 \varepsilon^{\frac{1}{2}} + x_2 \varepsilon + x_3 \varepsilon^{\frac{3}{2}}$$

we plug back into original equation

$$x^3 = x_0^3 + 3x_0^2 x_1 \varepsilon^{\frac{1}{2}} + 3x_0^2 x_2 \varepsilon + 3x_1^2 x_0 \varepsilon + x_1^3 \varepsilon^{\frac{3}{2}} + 3x_0^2 x_3 \varepsilon^{\frac{3}{2}}$$

$$-x^2 = -x_0^2 - 2x_0 x_1 \varepsilon^{\frac{1}{2}} - x_1^2 \varepsilon - 2x_0 \varepsilon x_2 - 2x_0 x_3 \varepsilon^{\frac{3}{2}} - 2x_1 x_2 \varepsilon^{\frac{3}{2}}$$

$$\varepsilon = \varepsilon$$

$$O(\varepsilon^0) = x_0^3 - x_0^2 \quad x_0^2(x_0 - 1) = \quad x_0 = 0, 1$$

$$O(\varepsilon^{\frac{1}{2}}) = \varepsilon^{\frac{1}{2}}(3x_0^2 x_1 - 2x_0 x_1)$$

$$3(0) x_1 - 2(0) x_1 = 0 = 0$$

$$3x_1 - 2x_1 = \quad x_1 = 0$$

$$O(\varepsilon^1) = \varepsilon(3x_0^2 x_2 + 3x_1^2 x_0 - x_1^2 - 2x_0 x_2 + 1)$$

$$\varepsilon(3(0)^2 x_2 + 3x_1^2(0) - x_1^2 - 2(0) x_2 + 1)$$

$$x_1^2 + 1 = \pm i \quad \boxed{x_1 = \pm i}$$

$$\varepsilon(3x_0^2x_2 + 3x_1^2x_0 - x_1^2 - 2x_0x_2 + 1)$$

$$\varepsilon(3(1)^2x_2 + 3(0)^2(1) - (0)^2 - 2(1)x_2 + 1$$

$$3x_2 - 2x_2 + 1$$

$$\boxed{x_2 = -1}$$

$$O(\varepsilon^{\frac{3}{2}}) : \varepsilon^{\frac{3}{2}}(x_1^3 + 3x_0^2x_3 - 2x_0x_3 - 2x_1x_2)$$

$$\varepsilon^{\frac{3}{2}}(i^3 + 3(0)x_3 - 2(0)x_3 - 2(i)x_2)$$

$$(i^3 - 2ix_2)$$

$$x_2 = \frac{\cancel{i}^3}{-2\cancel{i}} \qquad \boxed{x_2 = \frac{1}{2}}$$

$$\boxed{x = \begin{cases} 1 - \varepsilon - 2\varepsilon^2 \\ 1 + i\varepsilon + \frac{1}{2}\varepsilon^{\frac{3}{2}} \\ 1 - i\varepsilon - \frac{1}{2}\varepsilon^{\frac{3}{2}} \end{cases}}$$

(d) $1+\sqrt{x^2+\varepsilon} = e^x$

assume the expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \underbrace{\frac{x^3}{3!}}_{}$$

Trascendental equations

$$\sqrt{x^2\left(1+\frac{\varepsilon}{x^2}\right)}$$

$$x\sqrt{1+\underbrace{\frac{\varepsilon}{x^2}}_{a}}$$

$$\sqrt{1+a} = 1 + \frac{1}{2}a - \frac{1}{8}a^2$$

$$x\sqrt{1+a} = x + \frac{1}{2}ax - \frac{1}{8}a^2 x$$

$$x\sqrt{1+\frac{\varepsilon}{x^2}} = x + \frac{1}{2}\left(\frac{\varepsilon}{x^2}\right)x - \frac{1}{8}\left(\frac{\varepsilon}{x^2}\right)^2 x$$

$$= x + \frac{1}{2}\left(\frac{\varepsilon}{x}\right) - \underbrace{\frac{1}{8}\left(\frac{\varepsilon^2}{x^3}\right)}_{}$$

first non trivial order

$$1 + x + \frac{1}{2}\left(\frac{\varepsilon}{x}\right) = 1 + x + \frac{x^2}{2!}$$

$$(2\cdot \frac{1}{\cancel{2}}\left(\frac{\varepsilon}{\cancel{x}}\right)\cdot \cancel{x} = \frac{x^2}{\cancel{2}}\cdot 2 \cdot x \Rightarrow \varepsilon = x^3 \Rightarrow x_1 = \sqrt[3]{\varepsilon}$$

we now assume the expansion

$x = 1 + \varepsilon^{\frac{1}{3}} x_1 + \varepsilon^{\frac{2}{3}} x_2$ to find the next

non trivial order

$$\cancel{1} + \cancel{x} + \frac{1}{2}\left(\frac{\varepsilon}{x}\right) - \frac{1}{8}\left(\frac{\varepsilon^2}{x^3}\right) = \cancel{1} + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{6}$$

$$2x^3 \left(\frac{\varepsilon}{2x} - \frac{\varepsilon^2}{8x^3} - \frac{x^2}{2} - \frac{x^3}{6} \right) = 0$$

$$\varepsilon x^2 - \frac{\cancel{2}\cancel{x^3}\varepsilon^2}{4\,\cancel{8}\,\cancel{x^3}} - \frac{\cancel{2}x^5}{\cancel{2}} - \frac{\cancel{2}x^6}{3\,\cancel{6}} = 0$$

$$\varepsilon x^2 - \frac{\varepsilon^2}{4} - x^5 - \frac{x^6}{3}$$

now plug in x

$$\varepsilon(\varepsilon^{\frac{1}{3}} + \varepsilon^{\frac{2}{3}} x_2) + \frac{\varepsilon^2}{4} - (\varepsilon^{\frac{1}{3}} + \varepsilon^{\frac{2}{3}} x_2)^5 - \frac{1}{3}(\varepsilon^{\frac{1}{3}} + \varepsilon^{\frac{2}{3}} x_2)^6$$

$O(\varepsilon^{\frac{2}{3}})$

2) Approximate the solution to

$$\varepsilon y''(x) + 2y'(x) + e^{y} = 0, \quad y(0) = y(1) = 0$$

Taylor expand e^y gives $e^y = 1 + y + \frac{y^2}{2}$

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2$$

$$= \varepsilon(y_0'' + \varepsilon y_1'' + \dots) + 2(y_0' + \varepsilon y_1' + \dots) + 1 + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \frac{(y_0 + \varepsilon y_1 + \varepsilon_2)^2}{2}$$

lets begin by finding the first order

$$2y_0' + e^{y_0} = 0$$

$$\frac{2y_0}{2} = -\frac{e^{y_0}}{2}$$

$$y_0 = -\frac{1}{2}e^{y_0}$$

$$\int \frac{y_0}{e^{y_0}} = \int -\frac{1}{2}\,dx$$

$$e^{-y_0} = -\frac{1}{2}x + C$$

$$\cancel{\ln e}^{y_0} = -\ln\left(\frac{x + C}{2}\right)$$

if $y(1) = 0$
$C = 1$

$$\boxed{y_0 = -\ln\left(\frac{x+1}{2}\right)}$$

The second order we have

$$e^{y_0 + \varepsilon y_1 + \dots +} = e^{y_0} e^{\varepsilon y_1} e^{\varepsilon^2 y_2}$$

$$\approx e^{y_0}(1 + \varepsilon y_1 + \varepsilon^2 y_2)$$

can rewrite e^{y_0} as

$(e^{y_0} y_1)\varepsilon$ to get same expansion

$$\varepsilon[y_0''(x) + 2y_1'(x) + e^{y_0} \quad y_1(x)) = 0$$

now plug in our y_0 terms

$$\left(-\ln\left(\frac{x+1}{2}\right)\right)'' + 2y_1' + e^{\left(-\ln\left(\frac{x+1}{2}\right)\right)} y_1(x)) = 0$$

$$\frac{d}{dy}\left(-\ln\left(\frac{y+1}{2}\right)\right) = -\frac{1}{x+1}$$

$$\frac{d}{dy}\left(-\frac{1}{x+1}\right) = \frac{1}{(x+1)^2}$$

doing so our expression simplifies to

$$\frac{1}{(x+1)^2} + 2y_1' + \frac{2}{x+1} y_1(x)$$

we now solve the ODE and obtain

$$y = -\frac{\ln(x+1)}{2(x+1)} + \frac{c}{x+1}$$

If we plug in $y_1(1) = 0$

$$-\frac{\ln(x+1)}{2(2)} + \frac{c}{}$$

$$y_1 \quad -\frac{\ln(x+1) + \ln 2}{2(x+1)}$$

$$\boxed{y_{out}(x) = -\frac{\ln(x+1)}{2(x+1)} + \varepsilon\left[\frac{\ln 2 - \ln(x+1)}{2x+1}\right] + O(\varepsilon^2)}$$

we have obtained an outer solution

Inner solution

near $x=0$, $y''(x)$ is large so that $\varepsilon y''(x)$ is not small

rescale $X = \frac{x}{\varepsilon}$

$$\frac{d}{dx} = \frac{1}{\varepsilon}\frac{d}{dX} \qquad y(x) \Rightarrow y_{in}(X)$$

$$\frac{d^2}{dx^2} = \frac{1}{\varepsilon^2}\frac{d^2}{dX^2}$$

plug in these derivatives to our original equation

$$\varepsilon y''(x) + 2y'(x) + e^y = 0, \quad y(0) = y(1) = 0$$

$$\varepsilon\left(\frac{1}{\varepsilon^2}\frac{d^2}{dX^2} Y \qquad\right) + 2\left(\frac{1}{\varepsilon}\frac{d}{dx} Y \qquad\right) + e^{Y} = 0$$

$$\varepsilon\left(\frac{1}{\varepsilon}\frac{d^2}{dx^2} Y + 2\frac{1}{\varepsilon}\frac{d}{dx} Y + e^{Y}\right) = 0$$

$$\frac{d^2}{dx^2} Y + 2\frac{d}{dx} Y + \varepsilon e^{Y} = 0$$

This is the inner solution

$$y(x) = Y_0(X) + \varepsilon Y_1(X) + \varepsilon^2 Y_2(X) + \ldots +$$

$$\frac{d^2}{dX^2}\left(Y_0(x) + \varepsilon Y_1(x)\right) + 2\left(Y_0(x) + \varepsilon Y(x)\right)\frac{d}{dX} + \varepsilon e^{Y_0}$$

$$O(1) \quad \frac{d^2}{dx^2}(Y_0(x)) + \frac{d}{dx}(2Y_0(x)) \qquad y(0) = 0 \qquad y(1) = 0$$

we can rewrite $\frac{d}{dx}$ as u

$u' + 2u$

$$\int \frac{u'}{u} = \int -2$$

$$\int \frac{1}{u} u' = \int -2$$

$$\ln(u) = -2x + c$$

$$e^{\cancel{\ln}(u)} = ce^{-2x}$$

$$u = ce^{-2x}$$

$$u = \int ce^{-2x}$$

$$\boxed{y = c_1 + c_2 e^{-2\cancel{x}}}$$

plug in $x(0) = 0$

$$0 = c_1 + c_2$$

$$c_1 = -c_2$$

$$y = -c_2 + c_2 e^{-2x}$$

$$O(\varepsilon)\left(\frac{d^2}{dx^2}(Y_1(x)) + 2\frac{d}{dx}(Y_1(x)) + e^{Y_0}\right)\varepsilon$$

$$\left(\frac{d^2}{dx^2}(Y_1(x)) + 2\frac{d}{dx}(Y_1(x) + e^{c(1-e^{-2x})}\right)$$

$$\left(\frac{d^2}{dx^2}(Y_1(x)) + 2\frac{d}{dx}(Y_1(x)) + e^{c(1-e^{-2x})}\right)$$

too complicated to solve

$$\boxed{\text{Match inner and outer solutions}}$$

take $x \to \text{"}0\text{"}$ in $y_{int}(x)$

$$O(\varepsilon^0): \; y_{in} = (-C_2 + C_2 e^{-2x}) \; (x \to \infty)$$

$$-C_2 + C_2 \cancel{e^{(-2\infty)}}^{\,0}$$

$$C_2(1 - \cancel{e^{-2x}}^{\,0})$$

$$y_{out} = -\ln\left(\frac{x+1}{2(x+1)}\right)(x \to 0) = \ln(2)$$

$$C_2 = \ln(2)$$

$$y_{in}(x) + y_{out}(x) - y_{match}(x)$$

$$\boxed{y_{unif}(x) = \frac{-\ln(x+1)}{2(x+1)} - \ln(2)(1-e^{-2x}) - \ln(2)}$$

3)

$$\begin{tikzcd} E+S \arrow[r, shift left, "k_+"] & C \arrow[l, shift left, "k_-"] \arrow[r, "k_2"] & D \arrow[r, "k_3"] \arrow[ll, bend left, "q_-"] & E+P_i \end{tikzcd}$$

$$\frac{dE}{dt} = -k_+ se + k_- C + k_3 D + q_- D$$

$$\frac{dS}{dt} = -k_+ se + k_- c + q_- D$$

$$\frac{dC}{dt} = k_+ se - (k_- + k_2) C$$

$$\frac{dD}{dt} \quad k_2 C - (q_- + k_3) D$$

$$\frac{dP}{dt} = k_3 D$$

① step one

non dimensionalize

$$\bar{s}(t) = \frac{s(t)}{s_0}, \quad s_0 = s(t=0)$$

$$\bar{c}(t) = \frac{c(t)}{e_0}, \quad e_0 = e(t=0)$$

$$\bar{d}(t) = \frac{d(t)}{e_0}, \quad e_0 = e(t=0$$

② conservation

$$d(t) + e(t) + c(t) = e_0$$

$$e(t) = e_0 - c(t) - d(t)$$

$$\frac{dS}{dt} = -k_+ se + k_- c + q_- D \Rightarrow -k_+ s(e_0 - c(t) - d(t)) + k_- c + q_- D$$

$$\frac{dC}{dt} = k_+ se - (k_- + k_2)C \Rightarrow k_+ s(e_0 - c(t) - d(t)) - (k_- + k_2)c$$

$$\frac{dD}{dt} = k_2 C - (q_- + k_3) D$$

Non dimensionalize

$$\frac{ds}{dt}\frac{1}{s_0}\frac{1}{e_0} = -k_+ \bar{s}(1 - c(t) - d(t)) + \frac{k_- \bar{c}}{s_0} + \frac{q_- \bar{d}}{s_0})$$

$e_0 k_+ dt = d\tau$

$\alpha = \frac{k_-}{k_+ s_0}$

$$\frac{1}{\underset{d\tau}{k_+ dt}}\frac{d\bar{s}}{e_0} = \bar{s}(1 - c(t) - d(t)) + \underbrace{\frac{k_-}{s_0 k_+}}_{\alpha} + \underbrace{\frac{q_-}{s_0 k_+}}_{\beta}$$

$$\boxed{\frac{d\bar{s}}{d\tau} = \bar{s}(1 - \bar{c}(t) - \bar{d}(t)) + \alpha\bar{c} + \beta\bar{d}}$$

$$\frac{dc}{dt}\frac{1}{s_0}\frac{1}{e_0} = k_+ \bar{s}(1 - \bar{c}(t) - \bar{d}(t)) - \left(\frac{k_- + k_2}{s_0 k_+}\right)\bar{c}$$

$$\frac{d\bar{c}}{s_0\, dt}\left(\frac{1}{k_+}\right) = \bar{s}(1 - \bar{c} - \bar{d}) - \underbrace{\left(\frac{k_- + k_2}{s_0 k_+}\right)}_{K}\bar{c}$$

$$\boxed{\left(\frac{e_0}{s_0}\right)\frac{d\bar{c}}{d\tau} = \bar{s}(1 - \bar{c} - \bar{d}) - K\bar{c}}$$

The ε that we want to figure it out

$$\frac{1}{e_0}\frac{dD}{dt}\frac{1}{s_0} = k_2 C - (q_- + k_3) D$$

$$\frac{dD}{e_0\,dt} = \frac{k_2}{s_0}\bar{c} - \frac{(q_- + k_3)}{s_0}\bar{D}$$

$$\frac{1}{k_+ e_0}\frac{dD}{dt} = \underbrace{\left(\frac{k_2}{s_0 k_+}\right)}\bar{c} - \underbrace{\frac{q_-}{s_0 k_+}}_{\beta}\bar{D} - \underbrace{\frac{k_3}{s_0 k_+}}\bar{D}$$

$$\boxed{\left(\frac{e_0}{s_0}\right)\frac{dD}{d\tau} = \omega\,\bar{c} - \beta\bar{D} - \gamma\bar{D}}$$

$$\frac{dD}{d\tau} = w\bar{c} - \beta\bar{D} - \gamma\bar{D}$$

$$\bar{D} = \frac{w\bar{c}}{(\beta+\gamma)}$$

Outer solution

Now that we know $\bar{d}$ we can solve for $\bar{c}$ with respect to $\bar{s}$

$$(1-\bar{c}-\bar{d})\,\bar{s} = K\bar{c}$$

$$\left(1-\bar{c}-\frac{w\bar{c}}{(\beta+\gamma)}\right)\bar{s}$$

$$\left(\bar{s}-\bar{s}\bar{c}-\frac{\bar{s}w\bar{c}}{\beta+\gamma}\right)-K\bar{c} = 0$$

$$-\bar{s}\bar{c}-\frac{sw\bar{c}}{\beta+\gamma}-K\bar{c} = -\bar{s}$$

$$-\bar{c}\left(\frac{\bar{s}w}{\beta+\gamma}+\bar{s}+K\right) = -\bar{s}$$

$$\boxed{\bar{c} = \frac{\bar{s}}{\frac{w\bar{s}}{\beta+\gamma}+\bar{s}+K}}$$

now substitute $A = \dfrac{w}{\beta+\gamma}$

$$\boxed{\begin{aligned}\bar{c} &= \frac{\bar{s}}{A\bar{s}+\bar{s}+K}\\ \bar{d} &= \frac{w\bar{c}}{(\beta+\gamma)} = A\cdot\bar{c} = \frac{A\cdot\bar{s}}{A\bar{s}+\bar{s}+K}\end{aligned}}$$

plug these into $\frac{d\bar{s}}{d\tau}$

$$\frac{d\bar{s}}{d\tau} = \bar{s}\,(1-\bar{c}(t)-\bar{d}(t)) + \alpha\bar{c} + \beta\bar{d}$$

$$\frac{d\bar{s}}{d\tau} = \bar{s}\left(1 - \frac{\bar{s}}{A\bar{s}+\bar{s}+K} - \frac{A\bar{s}}{A\bar{s}+\bar{s}+K}\right) + \frac{\alpha\bar{s}}{A\bar{s}+\bar{s}+K} + \frac{BA\bar{s}}{A\bar{s}+\bar{s}+K}$$

$$= \bar{s}\left(\frac{\cancel{A\bar{s}}+\cancel{\bar{s}}+K}{A\bar{s}+\bar{s}+K} - \frac{\cancel{\bar{s}}}{A\bar{s}+\bar{s}+K} - \frac{\cancel{A\bar{s}}}{A\bar{s}+\bar{s}+K}\right) + \frac{\alpha\bar{s}}{A\bar{s}+\bar{s}+K} + \frac{BA\bar{s}}{A\bar{s}+\bar{s}+K}$$

$$= \frac{\bar{s}K}{A\bar{s}+\bar{s}+K} + \frac{\alpha\bar{s}}{A\bar{s}+\bar{s}+K} + \frac{BA\bar{s}}{A\bar{s}+\bar{s}+K}$$

$$= \frac{\bar{s}(K+\alpha+B)}{A\bar{s}+\bar{s}+K} \qquad \lambda = \frac{K+\alpha+B}{A+1}$$

$$= \frac{\bar{s}(K+\alpha+B)}{\bar{s}(A+1)+K}$$

$$= \frac{\lambda\bar{s}}{\bar{s}+\frac{K}{\lambda}}$$

$$\frac{ds}{dt} = \frac{\lambda \bar{s}}{\left(\bar{s} + \frac{k}{\lambda}\right)}$$

$$\int \frac{\bar{s} + \frac{k}{\lambda}}{\bar{s}} \frac{ds}{dt} = \int \lambda$$

outer solution

$$\boxed{\bar{s} + \frac{k}{\lambda} \ln \bar{s} = Q + \lambda \tau}$$

Inner solution

rescale time $T = \frac{\tau}{\varepsilon}$: $\frac{d}{d\tau} \frac{1}{\varepsilon} \frac{d}{dT}$

$$\frac{d\bar{s}(T)}{dT} = -\varepsilon \bar{s}(1 - \bar{c} - \bar{d}) + \varepsilon \alpha \bar{c} + \varepsilon \beta \bar{d}$$

$$O(1) = \frac{d\bar{s}}{d\tau} = \emptyset \qquad \bar{s}(T) \approx \frac{S(0)}{S_0} = 1$$

s is constant

$$\frac{d\bar{c}}{dt} = \bar{S}(1-\bar{c}-\bar{d}) - K\bar{c}$$

$$c' = 1-\bar{c}-\bar{d} - K\bar{c}$$

$$\frac{d\bar{d}}{dt} = w\bar{c} - \beta\bar{d} - y\bar{d}$$

$$= w\bar{c} - \bar{d}(\beta - y)$$

$$c' = 1 - v\bar{c} - \bar{d} \qquad v = (1-K)$$

$$d' = wc - n\bar{d} \qquad n = (\beta - y)$$

$$c = \frac{d' - nd}{w}$$

$$c' = \frac{d'' - nd}{w}$$

$$d'' - vd' - wd = M$$

$$y^2 - vy - w = 0$$

$$\frac{v \pm \sqrt{v^2 - 4(w)}}{2}$$

$$d = C_1 e^{r_1} + C_2 e^{r_2} + \frac{M}{w}$$

Now plug in

$$d = C_1 e^{r_1} + C_2 e^{r_2} + \frac{M}{W}$$

into

$$c = d' - \eta d$$

Now we know

$$\boxed{\begin{gathered} c = \frac{\left(r_1 C e^{r_1} + C_2 e^{r_2} + \frac{M}{W}\right) - \left(\eta C_1 e^{r_1} + \eta C_2 e^{r_2} + \frac{\eta M}{W}\right)}{w} \\ s = 1 \\ d = C_1 e^{r_1} + C_2 e^{r_2} + \frac{M}{W} \end{gathered}}$$

Matching

$$C_{out} \simeq \bar{C}(\tau \to \infty) \Rightarrow \bar{S}_{out}(\tau \to 0) \simeq 1$$

$$\tilde{S} + \frac{K}{\lambda}\ln\tilde{S} = Q \qquad 1 + \frac{K}{\lambda}\cancel{\ln(1)}^{0} = Q \qquad Q = 1$$

outer solution for S_{out} is determined implicitly by

$$S_{uniform} \quad \boxed{\bar{S} + \frac{K}{\lambda}\ln\bar{S} = 1 - \lambda\tau}$$

$$\bar{C}(\tau) = \bar{C}_{out}(\tau) + \bar{C}_{\tau} - \bar{C}_{match}$$

$$A = \frac{S}{\beta + \gamma}$$

$$C_{uniform} = \frac{\bar{S}}{A\bar{S} + \bar{S} + K} + C = \frac{(r_1 C e^{r_1} + C_2 e^{r_2} + \frac{M}{\omega}) - (\pi C_1 e^{r_1} + \pi C_2 e^{r_2} + \frac{\pi M}{\omega})}{w} - \frac{1}{A + 1 + K}$$

$$\bar{d}_{uniform} = \bar{d}_{out}(\tau) + \bar{d}_{\tau} - \bar{d}_{match}$$

$$\boxed{\bar{d}_{uniform} = \frac{A \cdot \bar{S}}{A\bar{S} + \bar{S} + K} + d = C_1 e^{r_1} + C_2 e^{r_2} + \frac{M}{W} - \frac{A}{A + 1 + K}}$$

4) N - ligands (monomers) attachment rate p

S - seeds detachment rate q

C_k - number of densities clusters of size k

infographic

$$\boxed{C_0 + M \underset{q}{\overset{p}{\rightleftharpoons}} C_1 \underset{2q}{\overset{p}{\rightleftharpoons}} C_2 \underset{3q}{\overset{p}{\rightleftharpoons}} \cdots}$$

$$\frac{dC_1}{dt} = pC_0M - kqC_1 - pC_1M + kqC_2$$

At $0 < k < N$

$$\boxed{\frac{dC_k}{dt} = p(C_{k-1} - C_k)m + kq(C_{k+1} - C_k)}$$

At $k = 0$

$$\frac{dC_k}{dt} = p(0 - C_k)m + \cancel{0q(C_{k+1})}$$

At $k = N$

$$\frac{dC_k}{dt} = p(C_{k-1} - 0)m - q(C_k).$$

Mass action equations

Conservation laws

$$m(t) = M - \sum_{k=0}^{N} k \cdot c_k$$

M - total number of monomers

new equation governing the number of moneomers at time t =

$$\frac{dm}{dt} = -p(c_{k-1} - c_k) + q(c_k - c_{k-1})$$

these equations capture the dynamics of ligand loading on receptors in a closed system where monomers can attach or detach from receptors based on the rates p and q assume detachment ($q=0$) and rescale time according to

$$\tau(t) = \int_0^t m(t')\,dt'$$

we have $\frac{\tau}{dt} = m$

now plug in m to our ODE's

$$\frac{dC_0}{dt} = p(-C_0)\frac{\tau}{dt}$$

$$\frac{dC_K}{dt} = p(C_{K-1} - C_K)\frac{\tau}{dt}$$

$$\frac{dC_N}{dt} = p(C_{N-1})\frac{\tau}{dt}$$

we now replace dt with $d\tau$ and obtain the following equations

$$\frac{dC_0}{d\tau} = -pC_0$$

$$\frac{dC_K}{d\tau} = pC_{K-1} - pC_K$$

$$\frac{dC_N}{dt} \quad pC_{N-1}$$

we solve the first linear equation

$$\frac{dC_0}{d\tau} = -pC_0$$

$$\int \frac{dC_0}{C_0} = \int -p \, d\tau$$

$$\ln(c_0) + C = -pT$$

$$e \quad e \quad e$$

$$c_0 + e^C = e^{-pT}$$

$$\boxed{C_0 = Ce^{-pT}}$$ plug in $C_0(0) = N_s$

$$N_s = C\cancel{e^{-p(0)}}$$

$$C = N_s$$

$$\boxed{C_0 = N_s e^{-p\tau}}$$

we now solve for C_1

$$\frac{dC_1}{d\tau} = p(C_0 - C_1)$$

plug in C_0

$$\frac{dC_1}{d\tau} = pC_0 - pC_1$$

$$\frac{dC_1}{d\tau} = pN_s e^{-p\tau} - pC_1$$

we guess that $\boxed{C_1 = Np\tau e^{-p\tau}}$

$$\frac{dC_1}{d\tau} = pNse^{-p\tau} - p(Np\tau e^{-p\tau})$$

$$\frac{dC_2}{d\tau} = -p(Np\tau e^{-p\tau} - pC_2)$$

and we keep guessing C_K

As the behavior $m(t \to \infty)$. Under what condition does $m(\infty)$ vanish

since there is the $q=0$ (irreversible binding rate), if at time ∞, there are no more unbounded monomers, $m(\infty)$ would vanish.

Think when monomers would and would not deplete

- The system can reach an equilibrium state if the rate p is sufficiently high and there would be no unbound monomers $m(\infty)$

- if attachment rate p is not high enough, the number of unbound monomers will not deplete and $m(\infty)$ will not vanish. It will remain positive solve for steady state solution $C_k(t \to \infty)$ but $q > 0$

In the steady state $(t \to \infty)$ the time derivative $\frac{dC_k}{dt} = 0$

leading to the following equations

$$0 = P(C_{k-1} - C_k)m + kq(C_{k+1} - C_k)$$

Although I didn't solve this equation, if allowing $q \to 0^+$, or the limit of q to approach 0 from the positive side, this implies that q is not zero and has some tiny detachment rate which would impact the steady state solutions

How would things change if instead of one binding site for each seed, the number of binding sites for each seed is $N-k$

It would most certainly effect the mass action equations.

$$\frac{dC_k}{dt} = p\left(\left(N-(k-1)\right)C_{k-1} - C_k\right) m + kq\left(C_{k+1} - C_k\right)$$

where $N-(k-1)$ represents the number of remaining binding sites for the seed when a monomer binds to a cluster of size k