Homework 3 - BIOMATH 205

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1 Chapter 7

P5: Find an upper bound on the number of basic feasible points of a linear program

The number of basic feasible points of a linear program is bounded by the number of ways to choose the constraints that define the basic feasible solution. If there are n variables and m constraints, each basic feasible point is defined by selecting n of the m constraints. The number of ways to choose n elements out of m can be expressed by the binomial coefficient $\binom{m}{n}$

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

Note: *This expression assumes that the constraints are independent and that the constraints are not redundant. In practical situations, the number of actual basic feasible points might be less due to the presence of degenerate and redundant constraints.*

P1: Consider the linear program of minimizing $x_1 + x_2$ subject to the constraints $x_1 + 2x_2 \geq 3$, $2x_1 + x_2 \geq 5$, and $x_2 \geq 0$. Graph the feasible region, and solve the program by hand or by our Julia code.

We begin by graphing the feasible region in Figure 1 which has three vertices at $(0,5),(3,0)$, and $(\frac{7}{3},\frac{1}{3})$. We get this plot by converting our system of linear equations into the slope intercept form

$$\text{system of linear equations in } y = mx + b = \begin{cases} y \geq \frac{3-x}{2} \\ y \geq 5 - 2x \\ y \geq 0 \end{cases}$$

We then plug in all points to the objective function $x_1 + x_2$ to obtain our answer.

$$x_1 + x_2 = \begin{cases} 5 + 0 = 5 \\ 3 + 0 = 3 \\ \frac{7}{3} + \frac{1}{3} = \frac{8}{3} \end{cases}$$

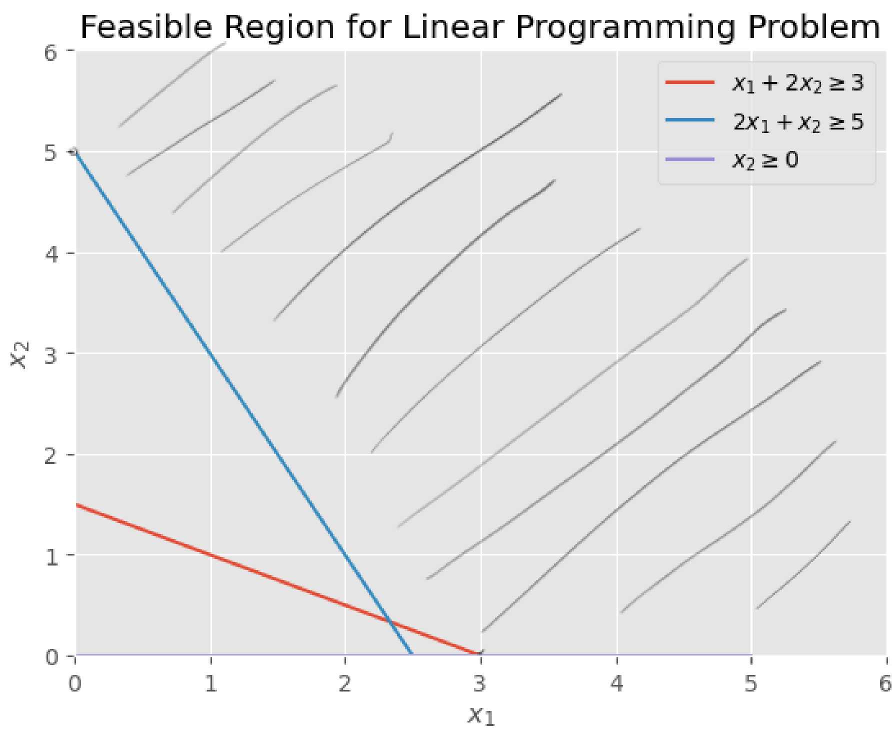

Figure 1: The feasible region after plotting all the constraints

Therefore the verticie $(\frac{7}{3}, \frac{1}{3})$ is the answer to this linear program

2 Chapter 8:

P3: Find the eigenvalues and eigenvectors of the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Note that the eigenvalues are complex conjugates

First solve for $det(A - \lambda I) = 0$ and find the roots for the given polynomial

$$det \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} = (\cos\theta - \lambda)^2 - (-\sin\theta) = 0$$

$$\lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta = 0$$

We can now use the trig identity to further simplify the expression $\cos^2\theta + \sin^2\theta = 1$ to obtain:

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0$$

We now plug in this to the quadratic formula to find its roots:

$$\lambda = \frac{-2\lambda\cos\theta \pm \sqrt{4\lambda\cos^2\theta - 4(\lambda^2)(1)}}{2\lambda^2} = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

This expression simplifies further using the relation $\cos\theta \pm i\sin\theta = e^{\pm i\theta}$to obtain the complex conjugate roots:

$$\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}.$$

We have therefore found not real eigenvalues if $\theta \neq 0$ (and every $\pm\pi$). However if $theta = 0, \pi$, we would get real eigenvalues.
Next we find the eigenvectors of the following by plugging in our eigenvalues to the original $A - \lambda I$.
Case 1: First lets solve for the case where $\theta = 0, \pi$.

$$A - \lambda I \begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This shows that each nonzero vector of $\mathbb{R}^2$ is an eigenvector

Case 2: In the case where $\theta \neq 0, \pi$. We reduce the row 2 by $R_2\dot{i}$ and $\frac{1}{\sin\theta}R_2$ and then subtract $R_2 + R_1$ to obtain:

$$A - \lambda I \begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{pmatrix} = \begin{pmatrix} 1 & -i \\ 1 & -i \end{pmatrix} = \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix}$$

Thus we obtain eigenvectors

$$y = \begin{bmatrix} i \\ 1 \end{bmatrix} t \tag{1}$$

for any nonzero scalar t. The other eigenvector is its complex conjugate to obtain the eigenvectors of all cases of the rotation matrix:

$$y = \begin{bmatrix} -i \\ 1 \end{bmatrix} t \tag{2}$$

P4: Find the eigenvalues and eigenvectors of the reflection matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

First solve for $det(A - \lambda I) = 0$ and find the roots for the given polynomial:

$$det\begin{pmatrix}\cos\theta-\lambda & \sin\theta\\ \sin\theta & -\cos\theta-\lambda\end{pmatrix} = (\cos\theta-\lambda)(-\cos\theta-\lambda)-(\sin\theta)^2=0$$

You can further simplify the expression again using the $\cos^2\theta+\sin^2\theta=1$ relation:

$$(\cos\theta-\lambda)(-\cos\theta-\lambda)-(-\sin\theta)^2=\lambda^2-\cos^2\theta-\sin^2\theta=\lambda^2-1$$

Our eigenvalues are therefore $\lambda=\pm1$

Case 1: We now solve for the eigenvectors by plugging in the eigenvalues into the $A-\lambda I$ equation.
So we begin with the $\lambda=1$ case. We multiply the R_1 by $\cos(\theta)+1$ and multiply $sin(\theta)$ to R_2 to reduce the matrix

$$\begin{pmatrix}\cos\theta-1 & \sin\theta\\ \sin\theta & -\cos\theta-1\end{pmatrix}=\begin{pmatrix}(\cos\theta-1)(\cos\theta+1) & \sin\theta(\cos\theta+1)\\ \sin\theta(\sin\theta) & (-\cos\theta-1)(\sin\theta)\end{pmatrix}$$

$$=\begin{pmatrix}-\sin^2\theta & \sin\theta\cos\theta+\sin(\theta))\\ \sin^2\theta & -\sin\theta\cos\theta-\sin\theta\end{pmatrix}=\begin{pmatrix}\cos\theta-1 & \sin\theta\\ 0 & 0\end{pmatrix}=\begin{pmatrix}1 & \frac{\sin\theta}{\cos\theta-1}\\ 0 & 0\end{pmatrix}$$

Solving these equations we get the eigenvector for $\lambda=1$ is

$$y=\begin{bmatrix}\frac{-\sin\theta}{\cos\theta-1}\\ 1\end{bmatrix}t \tag{3}$$

Case 2: For the $\lambda=-1$ the system becomes

$$\begin{pmatrix}\cos\theta-(-1) & \sin\theta\\ \sin\theta & -\cos\theta-(-1)\end{pmatrix}$$

We then do a similar one to the first eigenvector and solve multiply the R_1 by $\cos(\theta)-1$ and multiply $sin(\theta)$ to R_2 to reduce the matrix

$$\begin{pmatrix}(\cos\theta+1)(\cos\theta-1) & \sin\theta(\cos\theta-1)\\ \sin\theta(\sin\theta) & (-\cos\theta+1)(\sin\theta)\end{pmatrix}$$

$$=\begin{pmatrix}-\sin^2\theta & \sin\theta\cos\theta-\sin(\theta))\\ \sin^2\theta & -\sin\theta\cos\theta+\sin\theta\end{pmatrix}=\begin{pmatrix}\cos\theta+1 & \sin\theta\\ 0 & 0\end{pmatrix}=\begin{pmatrix}1 & \frac{\sin\theta}{\cos\theta+1}\\ 0 & 0\end{pmatrix}$$

Solving these systems we obtain the eigenvector

$$y=\begin{bmatrix}\frac{-\sin\theta}{\cos\theta+1}\\ 1\end{bmatrix}t \tag{4}$$