Nonlinear Dynamical Systems (AM 114/214)

Homework 3 solutions

	AM 114 students	**AM 214 students**
Question 1	45 points	30 points
Question 2	20 points	not required
Question 3	15 points	15 points
Question 4	20 points	10 points
Question 5	not required	15 points
Question 6	not required	15 points
Question 7	not required	15 points

Question 1 (*) Classify the fixed point at the origin of the following linear dynamical systems:

$$(a)\ \begin{cases} \dot{x}_1 = -2x_1 + 4x_2 \\ \dot{x}_2 = -4x_1 - 2x_2 \end{cases} \qquad (b)\ \begin{cases} \dot{x}_1 = -5x_1 + 10x_2 \\ \dot{x}_2 = 5x_1 + 5x_2 \end{cases} \qquad (c)\ \begin{cases} \dot{x}_1 = x_1 + 4x_2 \\ \dot{x}_2 = x_2 \end{cases} \tag{1}$$

Sketch the phase portrait. Make sure you include in your sketch the nullclines and the eigenvectors/eigendirections (if they are real).

Answers:

(a) To find the nullclines, we set each equation equal to 0.

$$-2x_1 + 4x_2 = 0 \implies x_2 = \frac{1}{2}x_1$$
$$-4x_1 - 2x_2 = 0 \implies x_2 = -2x_1$$

Fixed points occur where these two nullclines intersect and we get $x_1 = x_2 = 0$ as our only fixed point.

Identify the type of flow and sketch the phase portrait, we just need to find the eigenvalues and eigenvectors of the system. The eigenvalues are the roots of the characteristic polynomial

$$p(\lambda) = \det\left(\begin{bmatrix} -2-\lambda & 4 \\ -4 & -2-\lambda \end{bmatrix}\right) = 4 + 4\lambda + \lambda^2 - (-16) \; = \; \lambda^2 + 4\lambda + 20$$

Setting $p(\lambda) = 0$ yields the eigenvalues

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{-64}}{2} = -2 \pm 4i. \tag{2}$$

Since we have two complex eigenvalues with negative real parts the fixed point $\boldsymbol{x}^* = \mathbf{0}$ is a stable spiral. Evaluating the vector field at location $\boldsymbol{x} = (1, 0)$ we obtain $\boldsymbol{v} = (-2, -4)$. This suggests that the flow spirals clockwise. Since the eigenvalues are complex conjugates we have that the eigenvectors are complex. Moreover, since the algebraic multiplicity of both eigenvalues is one, we have that the dimension of the eigenspaces corresponding to λ_1 and λ_2 is one. To identify appropriate eigenvectors spanning such eigenspaces we need to solve the following linear systems of equations

$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v}_1 = \boldsymbol{0}, \qquad (\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{0}, \tag{3}$$

where

$$\boldsymbol{A} = \begin{bmatrix} -2 & 4 \\ -4 & -2 \end{bmatrix} \tag{4}$$

is the matrix associated with the system. These linear systems (3) can be written as

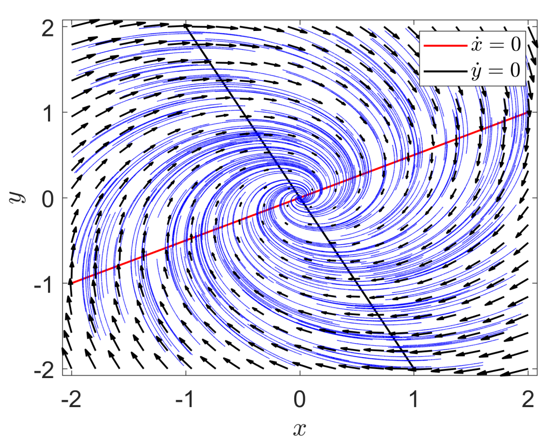
$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v}_1 = \boldsymbol{0} \quad \Rightarrow \quad \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_{12} = i v_{11},$$

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{0} \quad \Rightarrow \quad \begin{bmatrix} 4i & 4 \\ -4 & 4i \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_{22} = -i v_{21}.$$

We choose the following eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Hereafter we show the phase portrait of the system



(b) To find the nullclines, we set each equation at the right hand side of the system equal to 0:

$$-5x_1 + 10x_2 = 0 \implies x_2 = \frac{1}{2}x_1,$$
$$5x_1 + 5x_2 = 0 \implies x_2 = -x_1.$$

Fixed points occur where these two nullclines intersect and we get $x_1 = x_2 = 0$ as our only fixed point.
To sketch the phase portrait we proceed as in (a). The eigenvalues of the system's matrix

$$\boldsymbol{A} = \begin{bmatrix} 5 & 10 \\ 5 & 5 \end{bmatrix} \quad \text{are} \quad \lambda_{1,2} = \pm 5\sqrt{3}. \tag{5}$$

Since we have two real eigenvalues with opposite signs, the fixed point $\boldsymbol{x}^* = \mathbf{0}$ is saddle node. To compute the eigenvectors we solve

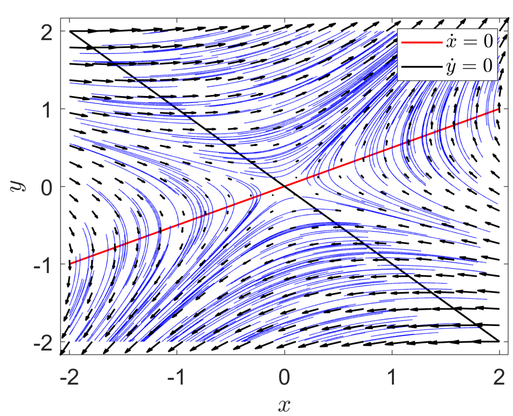
$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v}_1 = \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} -5+5\sqrt{3} & 10 \\ 5 & 5+5\sqrt{3} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_{12} = \frac{(1-\sqrt{3})}{2} v_{11},$$

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{v}_2 = \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} -5-5\sqrt{3} & 10 \\ 5 & 5-5\sqrt{3} \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_{22} = \frac{(1+\sqrt{3})}{2} v_{21}.$$

We choose the eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ \dfrac{(1-\sqrt{3})}{2} \end{bmatrix} \quad \text{(stable manifold)}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ \dfrac{(\sqrt{3}+1)}{2} \end{bmatrix} \quad \text{(unstable manifold)}$$

This yields the phase portrait



(c) To find the nullclines, we set each equation at the right hand side of the system equal to 0:

$$x_1 + 4x_2 = 0 \implies x_2 = -\frac{1}{4}x_1,$$
$$x_2 = 0 \implies x_2 = 0.$$

Fixed points occur where these two nullclines intersect. Since $x_2 = 0$ from , $x_1 = 0$ must also be true. So $x_1 = x_2 = 0$ as our only fixed point.
To sketch the phase portrait, we proceed as in (a). The eigenvalues of the system's matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \tag{6}$$

are

$$\lambda_{1,2} = 1. \tag{7}$$

Therefore we have only one eigenvalue with algebraic multiplicity 2. To compute the geometric multiplicity we use the rank-nullity theorem applied to the matrix

$$(\boldsymbol{A} - \lambda\boldsymbol{I}) = \boldsymbol{A} - \boldsymbol{I} = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}.$$

We have

$$\dim\left(N(\boldsymbol{A} - \boldsymbol{I})\right) + \operatorname{rank}(\boldsymbol{A} - \boldsymbol{I}) = 2. \tag{8}$$

This implies that the geometric multiplicity of $\lambda = 1$ is

$$\dim\left(N(\boldsymbol{A} - \boldsymbol{I})\right) = 1. \tag{9}$$

Therefore the fixed point $\boldsymbol{x}^* = \boldsymbol{0}$ is an unstable degenerate node. To compute an eigenvector corresponding to $\lambda = 1$ we solve

$$(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{v}_1 = \boldsymbol{0} \quad \Rightarrow \quad \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_{12} = 0.$$

We choose the eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{(eigenvector)}.$$

To construct a basis of $\mathbb{R}^2$ we complement the eigenvector $\boldsymbol{v}_1$ with a generalized eigenvector $\boldsymbol{v}_2$

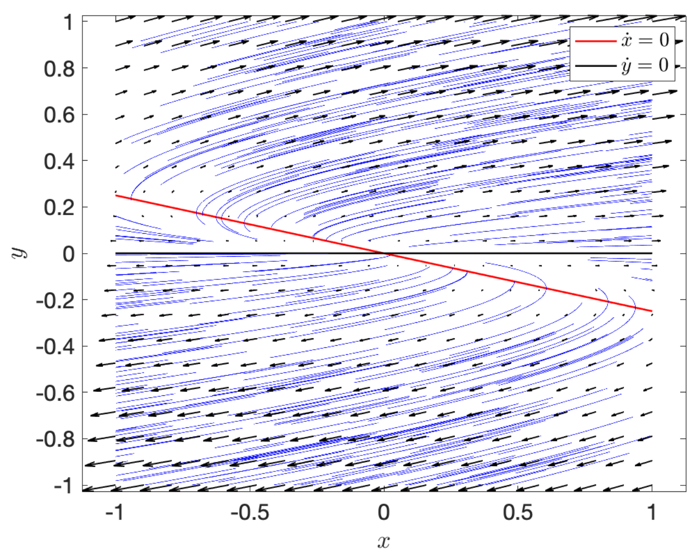
defined as

$$(\boldsymbol{A}-\boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{v}_1 \quad \Rightarrow \quad \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_{22} = \frac{1}{4}.$$

We choose

$$\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1/4 \end{bmatrix} \qquad \text{(generalized eigenvector)}.$$

The phase portrait is



Question 2 Compute the analytical solution of the linear systems (a) and (b) in (1) for arbitrary initial conditions $\boldsymbol{x}_0 = (x_{01}, x_{02})$. If the eigenvalues/eigenvectors are complex, express the solution in terms of real-valued functions only. (Hint: use the Euler formula $e^{ikt} = \cos(kt) + i\sin(kt)$.)

Answers: We computed the eigenvalues and eigenvectors of the linear systems (a) and (b) in problem 1. In both cases the eigenvalues are simple and therefore the matrix $\boldsymbol{A}$ associated to each system is diagonalizable. We have seen in class that the solution in this case is given by

$$\boldsymbol{X}(t, \boldsymbol{x}_0) = \boldsymbol{P} e^{t\boldsymbol{\Lambda}} \boldsymbol{P}^{-1}, \tag{10}$$

where

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} \quad \text{(matrix of eigenvectors)}, \tag{11}$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{(matrix of eigenvalues)}. \tag{12}$$

Hereafter we explicitly compute the solution to the systems (a) and (b).

(a) We have $\lambda_{1,2} = -2 \pm 4i$ and

$$\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \qquad \Leftrightarrow \qquad \boldsymbol{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}.$$

Therefore

$$\begin{aligned} \boldsymbol{X}(t, \boldsymbol{x}_0) =& \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} e^{(-2+4i)t} & 0 \\ 0 & e^{(-2-4i)t} \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \\ =& \frac{e^{-2t}}{2}\begin{bmatrix} e^{-4it} + e^{4it} & (-e^{-4it} + e^{4it})/i \\ (e^{-4it} - e^{4it})/i & e^{-4it} + e^{4it} \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \end{aligned}$$

By using Euler's formulas we obtain

$$\begin{cases} X_1(t, \boldsymbol{x}_0) = e^{-2t}\left(\cos(4t)x_{01} + \sin(4t)x_{02}\right) \\ X_1(t, \boldsymbol{x}_0) = e^{-2t}\left(\cos(4t)x_{02} - \sin(4t)x_{01}\right) \end{cases} \tag{13}$$

(b) We have $\lambda_{1,2} = \pm 5\sqrt{3}$, and

$$\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ \dfrac{1-\sqrt{3}}{2} & \dfrac{\sqrt{3}+1}{2} \end{bmatrix}$$

The inverse of $\boldsymbol{P}$ is given by

$$\boldsymbol{P}^{-1} = \frac{1}{\sqrt{3}}\begin{bmatrix} \dfrac{\sqrt{3}+1}{2} & -1 \\ \dfrac{\sqrt{3}-1}{2} & 1 \end{bmatrix}. \tag{14}$$

Therefore, the analytical solution is given by

$$
\begin{aligned}
\boldsymbol{X}(t,\boldsymbol{x}_0) &= \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 \\ \dfrac{1-\sqrt{3}}{2} & \dfrac{\sqrt{3}+1}{2} \end{bmatrix}\begin{bmatrix} e^{-5\sqrt{3}t} & 0 \\ 0 & e^{5\sqrt{3}t} \end{bmatrix}\begin{bmatrix} \dfrac{\sqrt{3}+1}{2} & -1 \\ \dfrac{\sqrt{3}-1}{2} & 1 \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \\
&= \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 \\ \dfrac{1-\sqrt{3}}{2} & \dfrac{\sqrt{3}+1}{2} \end{bmatrix}\begin{bmatrix} e^{-5\sqrt{3}t}\dfrac{\sqrt{3}+1}{2} & -e^{-5\sqrt{3}t} \\ e^{5\sqrt{3}t}\dfrac{\sqrt{3}-1}{2} & e^{5\sqrt{3}t} \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \\
&= \frac{1}{\sqrt{3}}\begin{bmatrix} e^{-5\sqrt{3}t}\dfrac{\sqrt{3}+1}{2} + e^{5\sqrt{3}t}\dfrac{\sqrt{3}-1}{2} & -e^{-5\sqrt{3}t} + e^{5\sqrt{3}t} \\ \dfrac{e^{5\sqrt{3}t} - e^{-5\sqrt{3}t}}{2} & -e^{-5\sqrt{3}t}\dfrac{1-\sqrt{3}}{2} + e^{5\sqrt{3}t}\dfrac{\sqrt{3}+1}{2} \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix},
\end{aligned}
$$

i.e.,

$$
\begin{cases}
X_1(t,\boldsymbol{x}_0) = \dfrac{1}{\sqrt{3}}\left[\left(e^{-5\sqrt{3}t}\dfrac{\sqrt{3}+1}{2} + e^{5\sqrt{3}t}\dfrac{\sqrt{3}-1}{2}\right)x_{01} + \left(-e^{-5\sqrt{3}t} + e^{5\sqrt{3}t}\right)x_{02}\right] \\[2ex]
X_2(t,\boldsymbol{x}_0) = \dfrac{1}{\sqrt{3}}\left[\dfrac{e^{5\sqrt{3}t} - e^{-5\sqrt{3}t}}{2}x_{01} + \left(e^{5\sqrt{3}t}\dfrac{\sqrt{3}+1}{2} - e^{-5\sqrt{3}t}\dfrac{1-\sqrt{3}}{2}\right)x_{02}\right]
\end{cases} \tag{15}
$$

Question 3 Show that any matrix of the form

$$
\boldsymbol{A} = \begin{bmatrix} a & b & b \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix},
$$

with a and b nonzero real numbers, has only one eigenvalue with algebraic multiplicity 3 and geometric multiplicity 1. Determine the exponential matrix $\exp(\boldsymbol{A}t)$ that characterizes the general solution of the three-dimensional linear system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ in the case $a = b = 1$.

Answers: Let us compute the eigenvalues and the eigenvectors (or generalized eigenvectors) of the matrix $\boldsymbol{A}$.

- The eigenvalues of $\boldsymbol{A}$ are obtained by computing the zeros of the characteristic polynomial

$$
p(\lambda) = \det\left(\begin{bmatrix} a-\lambda & b & b \\ 0 & a-\lambda & b \\ 0 & 0 & a-\lambda \end{bmatrix}\right) = (a-\lambda)[(a-\lambda)(a-\lambda) - 0] = 0.
$$

From this, we get $\lambda_{1,2,3} = \lambda = a$ with algebraic multiplicity 3. The geometric multiplicity of λ is the dimension of the eigenspace associated with λ. Such an eigenspace coincides with the nullspace of the matrix

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{bmatrix} 0 & b & b \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \tag{16}$$

By using the rank-nullity theorem

$$\dim\left(N(\boldsymbol{A} - \lambda \boldsymbol{I})\right) + \underbrace{\operatorname{rank}\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right)}_{=2} = 3 \tag{17}$$

we conclude that the dimension of the eigenspace associated with $\lambda = 3$ (i.e., the geometric multiplicity of λ) is $\dim\left(N(\boldsymbol{A} - \lambda \boldsymbol{I})\right) = 1$.

- Let us now compute the eigenvectors and the generalized eigenvectors of $\boldsymbol{A}$. Setting $a = b = 1$ we obtain

$$(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{v}_1 = \boldsymbol{0} \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_{12} = v_{13} = 0.$$

We choose $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (eigenvector). Next, we construct two generalized eigenvectors by solving the hierarchy of equations

$$(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{v}_1,$$
$$(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{v}_3 = \boldsymbol{v}_2.$$

The first equation can be written as

$$(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{v}_1 \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_{22} = 1, \quad v_{23} = 0. \tag{18}$$

We choose $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (first generalized eigenvector). Similarly,

$$(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{v}_3 = \boldsymbol{v}_2 \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_{32} = -1, \quad v_{33} = 1. \tag{19}$$

We choose $\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ (second generalized eigenvector).

At this point we construct the matrix

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow \qquad \boldsymbol{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{20}$$

By applying the matrix $\boldsymbol{A}$ to $\boldsymbol{P}$ we obtain

$$\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} \boldsymbol{A}\boldsymbol{v}_1 & \boldsymbol{A}\boldsymbol{v}_2 & \boldsymbol{A}\boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 + \boldsymbol{v}_1 & \boldsymbol{v}_3 + \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Jordan matrix } \boldsymbol{J}} = \boldsymbol{P}\boldsymbol{J} \tag{21}$$

We have seen in class that

$$e^{t\boldsymbol{A}} = \boldsymbol{P} e^{t\boldsymbol{J}} \boldsymbol{P}^{-1}. \tag{22}$$

To compute the matrix exponential $e^{t\boldsymbol{J}}$ we observe that the Jordan matrix $\boldsymbol{J}$ can be decomposed as

$$\boldsymbol{J} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{I}} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{B}}$$

Taking the exponential gives[1]

$$
\begin{aligned}
e^{t\boldsymbol{J}} = e^{t\boldsymbol{I}} e^{t\boldsymbol{B}} &= e^{t\boldsymbol{I}} \left(\boldsymbol{I} + t\boldsymbol{B} + \frac{t^2}{2}\boldsymbol{B}^2 \right) \\
&= \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & \\ 0 & 0 & e^t \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t^2/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} e^t & te^t & t^2e^t/2 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}.
\end{aligned}
$$

Therefore, the exponential matrix is given by

$$
\begin{aligned}
\exp(\boldsymbol{A}t) =& \boldsymbol{P} \exp(\boldsymbol{J}t) \boldsymbol{P}^{-1} \\
=& \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & te^t & t^2e^t/2 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
=& \begin{bmatrix} e^t & te^t & t^2e^t/2 + te^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}
\end{aligned}
$$

Exercise 4 (*) The motion of a damped harmonic oscillator is described by the equation

$$
m\ddot{x} + b\dot{x} + kx = 0, \tag{23}
$$

where $b \geq 0$ is the damping constant, and $m, k > 0$.

1. Rewrite equation (23) as a two-dimensional linear system.

2. Classify the fixed point at the origin, plot the nullclines and sketch the phase portrait for all cases that can occur depending on the relative sizes of the parameters m, b and k.

[1] Since $t\boldsymbol{B}$ commutes with $t\boldsymbol{I}$ (diagonal matrix), we have that $e^{t\boldsymbol{J}} = e^{t\boldsymbol{I}} e^{t\boldsymbol{B}}$. Moreover $\boldsymbol{B}^n = \boldsymbol{0}$ for all $n \geq 3$ and therefore $e^{t\boldsymbol{B}} = \boldsymbol{I} + t\boldsymbol{B} + t^2\boldsymbol{B}^2/2$.

Answers:

1. Define $x_1 = x$ and $x_2 = \dot{x}$. This allows us to rewrite the second-order linear ODE (23) as a system of two first-order ODEs as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\beta x_2 - \alpha x_1 \end{cases} \qquad \text{where} \qquad \alpha = \frac{k}{m} > 0, \qquad \beta = \frac{b}{m} \geq 0. \tag{24}$$

The matrix associated with the system is

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix} \tag{25}$$

The eigenvalues are obtained computing roots to the characteristic polynomial

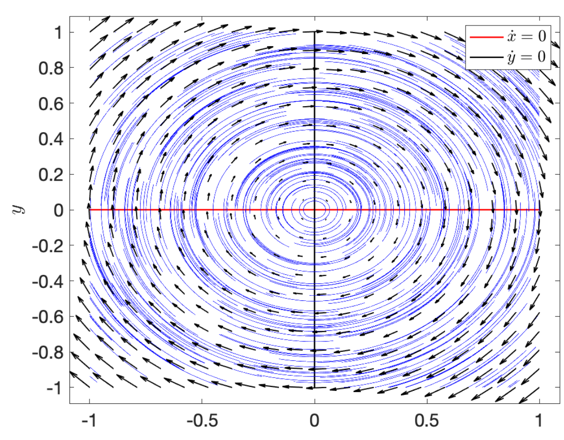
$$p(\lambda) = \lambda(\lambda + \beta) + \alpha = 0 \tag{26}$$

this yields

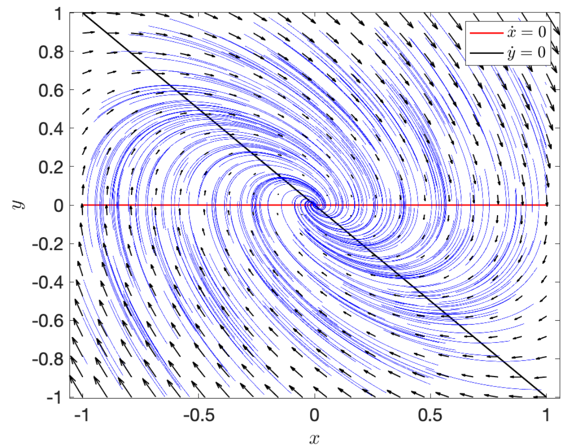
$$\lambda_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha}}{2} \tag{27}$$

2. Hereafter we study all possible flows as a function of α and β:

- For $\beta = 0$ we have two complex conjugate eigenvalues $\lambda = \pm i\sqrt{\alpha}$. Therefore the fixed point at the origin is a center. Here is the phase portrait for $\alpha = 1$:



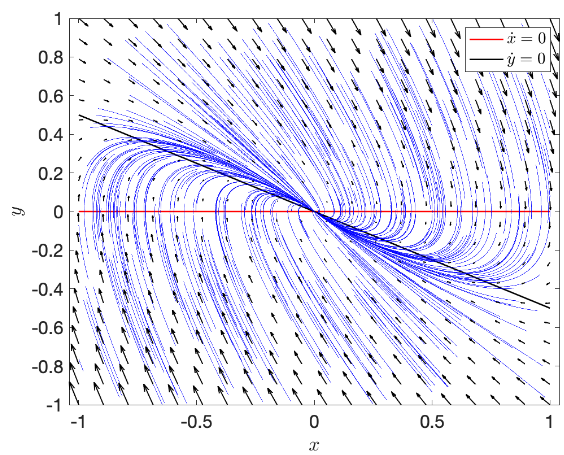
- For $\beta^2 < 4\alpha$ we have two complex conjugate eigenvalues with negative real part, i.e., a stable spiral. Here is the spiral for $\beta = 1$ and $\alpha = 1$:



- For $\beta^2 = 4\alpha$ we have only one eigenvalue $\lambda = -\beta/2$ with algebraic multiplicity 2 and

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{bmatrix} \beta/2 & 1 \\ -\beta^2/4 & -\beta/2 \end{bmatrix}. \tag{28}$$

  The rank of this matrix is always one[2] and therefore the origin is a degenerate node. Here is a plot of such a degenerate node for $\beta = 1$:



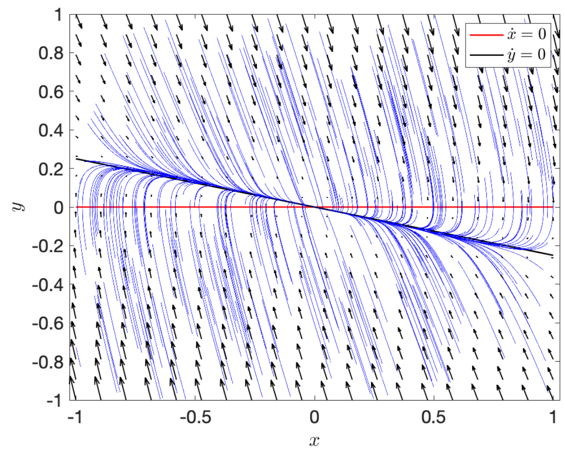
- For $\beta^2 > 4\alpha$ and $\beta^2 - 4\alpha < \beta^2$ (i.e., $\alpha > 0$) we have that the fixed point at the origin is a stable node. In fact, the eigenvalues are both real and negative. Here is the phase portrait

[2]To compute the rank of (28) simply divide the second row by $-\beta/2$ to obtain the matrix

$$\begin{bmatrix} \beta/2 & 1 \\ \beta/2 & 1 \end{bmatrix} \tag{29}$$

which has rank for all $\beta \geq 0$

for $\alpha = 1$ and $\beta = 4$



Note that the condition $\beta^2 - 4\alpha > \beta^2$ (which combined with $\beta^2 > 4\alpha$ would yield a saddle node at the origin) implies $\alpha < 0$, which is impossible since we assumed $\alpha > 1$.

Question 5 Consider two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ with real coefficients, and the flows $\boldsymbol{X}(t, \boldsymbol{x}_0)$ and $\boldsymbol{Y}(t, \boldsymbol{x}_0)$ generated by the linear dynamical systems

$$(a) \begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \qquad\qquad (b) \begin{cases} \dfrac{d\boldsymbol{y}}{dt} = \boldsymbol{B}\boldsymbol{y} \\ \boldsymbol{y}(0) = \boldsymbol{x}_0 \end{cases} \tag{30}$$

1. Determine an analytical expression for the flow $\boldsymbol{X}(t, \boldsymbol{x}_0)$ and show that it satisfies the initial condition $\boldsymbol{X}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0$ using matrix exponentials and show that such flow satisfies the semi-group property
$$\boldsymbol{X}(t+s, \boldsymbol{x}_0) = \boldsymbol{X}(t, \boldsymbol{X}(s, \boldsymbol{x}_0)) \qquad \text{for all} \quad t, s \geq 0. \tag{31}$$

2. Show that the superimposed flow $\boldsymbol{Z}(t, \boldsymbol{x}_0) = \boldsymbol{X}(t, \boldsymbol{x}_0) + \boldsymbol{Y}(t, \boldsymbol{x}_0)$ is in general not invertible and it does not satisfy the semi-group property:
$$\boldsymbol{Z}(t+s, \boldsymbol{x}_0) = \boldsymbol{Z}(t, \boldsymbol{Z}(s, \boldsymbol{x}_0)) \tag{32}$$

Answers:

1. We have seen in class that the solution of any the linear dynamical system of the form
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{33}$$

can be expressed in terms of a matrix exponential $e^{\boldsymbol{A}t}$ as

$$\boldsymbol{X}(t, \boldsymbol{x}_0) = e^{\boldsymbol{A}t}\boldsymbol{x}_0 \tag{34}$$

Setting $t = 0$ we have that $e^{\boldsymbol{A}t} = \boldsymbol{I}$ and therefore $\boldsymbol{X}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0$. Moreover,

$$e^{\boldsymbol{A}(t+s)}\boldsymbol{x}_0 = e^{\boldsymbol{A}t}e^{\boldsymbol{A}s}\boldsymbol{x}_0 \qquad \forall t, x \geq 0, \tag{35}$$

which shows that $\boldsymbol{X}(t, \boldsymbol{x}_0)$ satisfies the semigroup property $\boldsymbol{X}(t+s, \boldsymbol{x}_0) = \boldsymbol{X}(t, \boldsymbol{X}(s, \boldsymbol{x}_0))$.

2. The superimposed flow $\boldsymbol{Z}(t, \boldsymbol{x}_0)$ can be written as

$$\boldsymbol{Z}(t, \boldsymbol{x}_0) = \left(e^{\boldsymbol{A}t} + e^{\boldsymbol{B}t}\right)\boldsymbol{x}_0 \tag{36}$$

Note that $\boldsymbol{Z}(0, \boldsymbol{x}_0) = 2\boldsymbol{x}_0$ and therefore $\boldsymbol{Z}(0, \boldsymbol{x}_0)$ as given does not satisfy the semigroup condition $\boldsymbol{Z}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0$. We now show that mapping $\boldsymbol{Z}(t, \boldsymbol{x}_0)$ is, in general, not invertible. To this end, it is sufficient to show that two different initial conditions $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ may end up in the same spatial location at some finite time. To this end, let us set the equality

$$\boldsymbol{Z}(t, \boldsymbol{x}_0) = \boldsymbol{Z}(t, \boldsymbol{x}_1) \qquad \Leftrightarrow \qquad e^{\boldsymbol{A}t}(\boldsymbol{x}_1 - \boldsymbol{x}_0) + e^{\boldsymbol{B}t}(\boldsymbol{x}_1 - \boldsymbol{x}_0) = \boldsymbol{0} \tag{37}$$

For any fixed $\boldsymbol{A}$ and fixed $\boldsymbol{e} = (\boldsymbol{x}_1 - \boldsymbol{x}_0)$ it is clear that it is possible to construct a matrix $\boldsymbol{B} \neq \boldsymbol{A}$ such that

$$e^{\boldsymbol{A}t}\boldsymbol{e} = -e^{\boldsymbol{B}t}\boldsymbol{e} \tag{38}$$

for some finite t. For example, let $t = 1$ and choose $\boldsymbol{e} = \begin{bmatrix} 1 & \cdots & 0 \end{bmatrix}^T$. Then every matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $e^{\boldsymbol{A}}$ and $e^{\boldsymbol{B}}$ have opposite first column satisfies (38). For example,

$$\boldsymbol{A} = \begin{bmatrix} 0 & -\pi/4 \\ \pi/4 & 0 \end{bmatrix} \quad \Rightarrow \quad e^{\boldsymbol{A}} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \tag{39}$$

$$\boldsymbol{B} = \begin{bmatrix} 0 & -5\pi/4 \\ 5\pi/4 & 0 \end{bmatrix} \quad \Rightarrow \quad e^{\boldsymbol{B}} = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}. \tag{40}$$

Regarding the semigroup property, it is clear that it is not satisfied even at $t = s = 0$. In fact, we have

$$\boldsymbol{Z}(0, \boldsymbol{Z}(0, \boldsymbol{x}_0)) = 4\boldsymbol{x}_0 \neq 2\boldsymbol{x}_0 = \boldsymbol{Z}(0, \boldsymbol{x}_0). \tag{41}$$

Let us rescale the flow $\boldsymbol{Z}(t, \boldsymbol{x}_0)$ to make sure that the semigoup condition is satisfied at $t = s = 0$. To this end, consider

$$\widetilde{\boldsymbol{Z}}(t, \boldsymbol{x}_0) = \frac{1}{2}\left(e^{\boldsymbol{A}t} + e^{\boldsymbol{B}t}\right)\boldsymbol{x}_0. \tag{42}$$

Clearly, $\widetilde{\boldsymbol{Z}}(0, \widetilde{\boldsymbol{Z}}(0, \boldsymbol{x}_0)) = \boldsymbol{x}_0$. Moreover, we have

$$\begin{aligned}\widetilde{\boldsymbol{Z}}(t+s, \boldsymbol{x}_0) =& \frac{1}{2}\left(e^{\boldsymbol{A}(t+s)} + e^{\boldsymbol{B}(t+s)}\right)\boldsymbol{x}_0, \\ \widetilde{\boldsymbol{Z}}(t, \widetilde{\boldsymbol{Z}}(s, \boldsymbol{x}_0)) =& \frac{1}{4}\left(e^{\boldsymbol{A}t} + e^{\boldsymbol{B}t}\right)\left(e^{\boldsymbol{A}s} + e^{\boldsymbol{B}s}\right)\boldsymbol{x}_0.\end{aligned} \tag{43}$$

Therefore $\widetilde{\boldsymbol{Z}}(t+s, \boldsymbol{x}_0) = \widetilde{\boldsymbol{Z}}(t, \widetilde{\boldsymbol{Z}}(t, \boldsymbol{x}_0))$ if and only if

$$2\left(e^{\boldsymbol{A}(t+s)} + e^{\boldsymbol{B}(t+s)}\right) = e^{\boldsymbol{A}(t+s)} + e^{\boldsymbol{B}(t+s)} + e^{\boldsymbol{A}t}e^{\boldsymbol{B}s} + e^{\boldsymbol{B}t}e^{\boldsymbol{A}s}, \tag{44}$$

i.e., if

$$e^{\boldsymbol{A}(t+s)} + e^{\boldsymbol{B}(t+s)} = e^{\boldsymbol{A}t}e^{\boldsymbol{B}s} + e^{\boldsymbol{B}t}e^{\boldsymbol{A}s} \tag{45}$$

Clearly, this equation is not satisfied if we choose $\boldsymbol{A}$ and $\boldsymbol{B}$ arbitrarily. For example, consider two 2D stars defined by the matrices

$$\boldsymbol{A} = \begin{bmatrix}\lambda & 0 \\ 0 & \lambda\end{bmatrix}, \qquad \boldsymbol{B} = -\frac{1}{2}\begin{bmatrix}\lambda & 0 \\ 0 & \lambda\end{bmatrix}, \qquad \lambda > 0. \tag{46}$$

Substituting (46) into (45) yields the condition

$$e^{\lambda(t+s)} + e^{-\lambda(t+s)/2} = e^{\lambda(t-s/2)} + e^{\lambda(s-t/2)} \tag{47}$$

which is not satisfied for arbitrary t and s.

Question 6 Let $D \subset \mathbb{R}^n$ be an open convex subset of $\mathbb{R}^n$ and $\boldsymbol{f} : D \to \mathbb{R}^n$ continuously differentiable in D. Consider the dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. Prove that if $\boldsymbol{f}$ has bounded first-order partial derivatives, i.e., if there exists a finite $L \geq 0$ such that

$$\left|\frac{\partial f_i(\boldsymbol{x})}{\partial x_j}\right| \leq L \quad \text{for all} \quad i, j = 1, .., n \quad \text{and for all} \quad \boldsymbol{x} \in D \tag{48}$$

then $\boldsymbol{f}$ is Lipschitz continuous in D, i.e.,

$$\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| \leq G\|\boldsymbol{x} - \boldsymbol{y}\| \quad \text{for all} \quad \boldsymbol{x}, \boldsymbol{y} \in D, \quad 0 \leq G < \infty. \tag{49}$$

Answers: All finite-dimensional norms are equivalent. Therefore if we prove (49) relative to one norm, the result can be immediately translated to other norms (with different Lipschitz constants). For convenience we choose the uniform norm, since the bound on the derivatives (48) is given in terms of uniform norms. Consider the line

$$\boldsymbol{h}(t) = \boldsymbol{x} + t(\boldsymbol{y} - \boldsymbol{x}), \qquad \boldsymbol{h}(0) = \boldsymbol{x}, \quad \boldsymbol{h}(1) = \boldsymbol{y}. \tag{50}$$

We have

$$|f_i(\boldsymbol{x}) - f_i(\boldsymbol{y})| = |f_i(\boldsymbol{h}(1)) - f_i(\boldsymbol{h}(0))| = \left|\int_0^1 \frac{df_i(\boldsymbol{h}(t))}{dt}dt\right| = \left|\int_0^1 \nabla f_i(\boldsymbol{h}(t)) \cdot \frac{d\boldsymbol{h}(t)}{dt}dt\right|$$

The last integral can be bounded as follows

$$\begin{aligned}
\left|\int_0^1 \nabla f_i(\boldsymbol{h}(t)) \cdot \frac{d\boldsymbol{h}(t)}{dt}dt\right| =& \left|\int_0^1 \nabla f_i(\boldsymbol{h}(t)) \cdot (\boldsymbol{y} - \boldsymbol{x})dt\right| \\
\leq& \int_0^1 |\nabla f_i(\boldsymbol{h}(t)) \cdot (\boldsymbol{y} - \boldsymbol{x})| \, dt \\
\leq& \max_i |y_i - x_i| \int_0^1 \max_j \left|\frac{\partial f_i(\boldsymbol{h}(t))}{\partial x_j}\right| dt \\
\leq& L \max_i |y_i - x_i| \, .
\end{aligned} \tag{51}$$

Therefore, for all $i = 1, \ldots, n$ and for all $\boldsymbol{x}, y \in D$ we have

$$|f_i(\boldsymbol{x}) - f_i(\boldsymbol{y})| \leq L \max_{i=1,\ldots,n} |y_i - x_i| \, . \tag{52}$$

This implies that

$$\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\|_\infty \leq L \|\boldsymbol{x} - \boldsymbol{y}\|_\infty \, . \tag{53}$$

If instead of the uniform norm we use the 2-norm we obtain[3]

$$\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\|_2 \leq \sqrt{n}L \|\boldsymbol{x} - \boldsymbol{y}\|_2 \, . \tag{55}$$

Note that the Lipschitz constant is now $\sqrt{n}L$.

Question 7 Consider the initial value problem

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{56}$$

where $\boldsymbol{A}$ is a square matrix with real entries and $\boldsymbol{x}_0$ is a given vector.

[3]Recall that for any vector $\boldsymbol{c} \in \mathbb{R}^n$ we have

$$\|\boldsymbol{c}\|_\infty \leq \|\boldsymbol{c}\|_2 \leq \sqrt{n} \|\boldsymbol{c}\|_\infty \, . \tag{54}$$

(a) Show that the solution $\boldsymbol{x}(t) = \boldsymbol{X}(t, \boldsymbol{x}_0)$ must satisfy the integral equation

$$\boldsymbol{X}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{X}(s, \boldsymbol{x}_0) ds.$$

(b) Define a sequence of approximations by

$$\boldsymbol{X}^{(n)}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{X}^{(n-1)}(s, \boldsymbol{x}_0) ds \tag{57}$$

with $\boldsymbol{X}^{(0)}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0$. Use induction to derive a closed-form expression for $\boldsymbol{X}^{(n)}(t, \boldsymbol{x}_0)$.

(c) Show that $\boldsymbol{X}(t, \boldsymbol{x}_0) = \lim_{n\to\infty} \boldsymbol{X}^{(n)}(t, \boldsymbol{x}_0)$ solves the initial value problem (56).

Answers:

(a) The flow $\boldsymbol{X}(s, \boldsymbol{x}_0)$ represents the solution to the initial value problem (56) for every $\boldsymbol{x}_0$. Therefore,

$$\dot{\boldsymbol{X}}(s, \boldsymbol{x}_0) = \boldsymbol{A}\boldsymbol{X}(s, \boldsymbol{x}_0), \quad \boldsymbol{X}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0.$$

Integrating up to time t yields

$$\boldsymbol{X}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_0^t A\boldsymbol{X}(s, \boldsymbol{x}_0) ds$$

(b) The fixed point iteration (57) is called Picard iteration in the theory of the ODEs. For nonlinear systems Picard's iterations converge only within a small time interval. For linear systems Picard's iterations are globally convergent. Let us we start with $n = 1$

$$\boldsymbol{X}^{(1)}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{x}_0 ds = \boldsymbol{x}_0 + \boldsymbol{A}\boldsymbol{x}_0 t = (\boldsymbol{I} + \boldsymbol{A}t)\boldsymbol{x}_0.$$

We can use this to compute $n = 2$ which gives

$$\boldsymbol{X}^{(2)}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{X}^{(1)}(s, \boldsymbol{x}_0) ds = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}(\boldsymbol{I} + \boldsymbol{A}t)\boldsymbol{x}_0 ds = \left(\boldsymbol{I} + \boldsymbol{A}t + \frac{t^2}{2}\boldsymbol{A}^2\right)\boldsymbol{x}_0.$$

By induction it is straightforward to show that

$$\boldsymbol{X}^{(n)}(t, \boldsymbol{x}_0) = \left(\sum_{k=0}^{n} \frac{\boldsymbol{A}^k t^k}{k!}\right)\boldsymbol{x}_0.$$

(c) Clearly,

$$\lim_{n\to\infty} \boldsymbol{X}^{(n)}(t, \boldsymbol{x}_0) = \left(\lim_{n\to\infty} \sum_{k=0}^{n} \frac{\boldsymbol{A}^k t^k}{k!}\right)\boldsymbol{x}_0 = e^{tA}\boldsymbol{x}_0 = \boldsymbol{X}(t, \boldsymbol{x}_0). \tag{58}$$