Nonlinear Dynamical Systems (AMS 114/214)

Homework 2 - Due Monday October 25th (2021)

Instructions

Please submit to CANVAS one PDF file (homework) and one .zip file that includes any computer code you develop for the assignment.

AM 114 students: Please submit to CANVAS your homework in a PDF format (one file). This could be a scan of your handwritten notes, compiled Latex source, or a PDF created using any other word processor (e.g., Microsoft Word). If you develop any computer code to produce plots or numerical results related to the assignment please attach it to your submission as one .zip file.

AM 214 students: Please submit to CANVAS your homework in a PDF format (one file) compiled from Latex source (preferred) or any other word processor. No handwritten work should be submitted. You should also provide quantitative numerical results for all problems/questions that are amenable to computation. Such Questions marked by (*). Attach the computer code you develop (MATLAB or Python preferred) to your submission as one .zip file.

	AM 114 students	**AM 214 students**
Question 1	45 points	non required
Question 2	20 points	20 points
Question 3	20 points	20 points
Question 4	15 points	15 points
Question 5	not required	15 points
Question 6	not required	15 points
Question 7	not required	15 points

Question 1 The dynamical system

$$\frac{dx}{dt} = x(1-x) - \mu\frac{2x}{1+2x}, \qquad \mu \geq 0 \tag{1}$$

provides a simple model for a fishery with parameter μ. In the absence of fishing, the fish population $x(t)$ ($x(t) \geq 0$) is assumed to grow according to the logistic model $\dot{x} = x(1-x)$. The effects of fishing are modeled by the term $2\mu x/(1+2x)$, which says that fish is caught at a rate that depends on the fish population x.

1. Determine the fixed points of the system as a function of μ and sketch the bifurcation diagram in the domain $x \geq 0$, $\mu \geq 0$.

2. What type of bifurcations occur at $(x^*, \mu^*) = (0, 1/2)$ and $(x^*, \mu^*) = (1/4, 9/16)$? (Hint: The bifurcation diagram provides a clue on the type of bifurcations that occur at $(x^*, \mu^*) = (0, 1/2)$ and $(x^*, \mu^*) = (1/4, 9/16)$. To rigorously classify such bifurcations use Taylor series).

3. Set $\mu = 17/32$ (midpoint between 1/2 and 9/16). Describe qualitatively what happens to the fish population over time for different initial conditions $x(0) = x_0$ in the interval $[0, 1]$.

Answer:

1. Setting the right hand side of (1) equal to zero yields

$$x(1-x) - \frac{2\mu x}{1+2x} = 0.$$

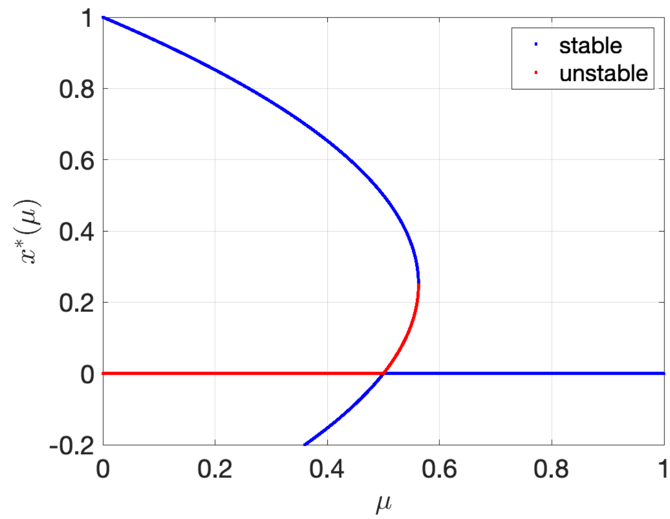
Notice that $x^* = 0$ is always a fixed point of this equation (no matter what the value of μ is). To find the other fixed points as a function of μ, we re-arrange to obtain

$$\frac{(1-x)(1+2x)}{2} = \mu \Rightarrow 1 + x - 2x^2 - 2\mu = 0.$$

Using the quadratic formula, we get

$$x^*(\mu) = \frac{1}{4} \pm \frac{\sqrt{1/4 - 4(\mu - 1/2)}}{2}.$$

This yields the following bifurcation diagram



2. From the bifurcation diagram, we see that at $(x^*, \mu^*) = (0, 1/2)$ we have a transcritical bifurcation and at $(x^*, \mu^*) = (1/4, 9/16)$ we have saddle node bifurcation. To rigorously classify these bifurcations we use Theorem 1 in Appendix, which is based on the analysis of the local Taylor expansion of (1) at the fixed points. To this end, we have

$$\left.\frac{\partial f}{\partial \mu}\right|_{(x^*,\mu^*)=(1/4,9/16)} = -\left.\frac{2x}{1+2x}\right|_{(x^*,\mu^*)=(1/4,9/16)} \neq 0,$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*,\mu^*)=(1/4,9/16)} = -2\mu\left(\frac{8x}{(2x+1)^3} - \frac{4}{(2x+1)^2}\right) - 2\Big|_{(x^*,\mu^*)=(1/4,9/16)} \neq 0,$$

which means a saddle-node bifurcation occurs at $(x^*, \mu^*) = (1/4, 9/16)$. For the other bifurcation point, we compute

$$\left.\frac{\partial f}{\partial \mu}\right|_{(x^*,\mu^*)=(0,1/2)} = -\left.\frac{2x}{1+2x}\right|_{(x^*,\mu^*)=(0,1/2)} = 0,$$

$$\left.\frac{\partial^2 f}{\partial x \partial \mu}\right|_{(x^*,\mu^*)=(0,1/2)} = -\left.\frac{2}{(1+2x)^2}\right|_{(x^*,\mu^*)=(0,1/2)} \neq 0,$$

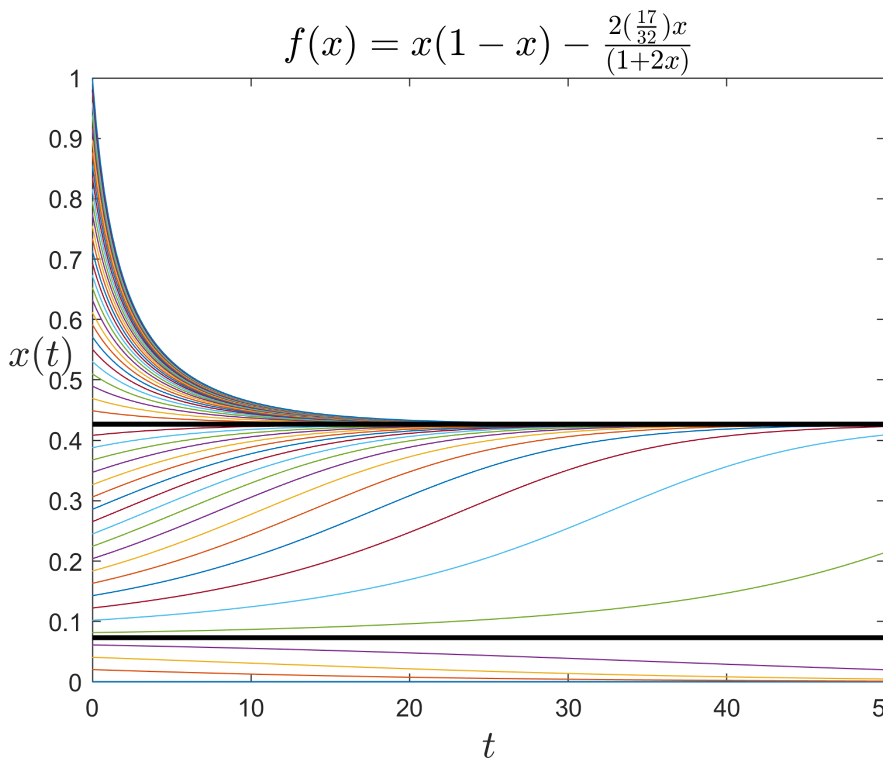
$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*,\mu^*)=(0,1/2)} = -2\mu\left(\frac{8x}{(2x+1)^3} - \frac{4}{(2x+1)^2}\right) - 2\Big|_{(x^*,\mu^*)=(0,1/2)} \neq 0,$$

resulting in a transcritical bifurcation occuring at $(x^*, \mu^*) = (0, 1/2)$.

3. At $\mu = 17/32$, the three fixed points can be numerically obtained as

$$x_1^* = 0, \quad x_2^* = .0732, \quad x_3^* = .4268.$$

This means that fish populations starting at $x_0 \in (x_2^*, 1]$ eventually stabilize to x_3^*. Meanwhile, populations starting at $x_0 \in (0, x_2^*)$ eventually die off. The following plot provides trajectories of the system for $\mu = 17/32$ and different initial conditions in $[0, 1]$



Question 2 Show that the nonlinear system

$$\frac{dx}{dt} = ax + bx^3 - cx^5, \quad b, c > 0 \tag{2}$$

can be rescaled to a normal form as

$$\frac{dX}{d\tau} = RX + X^3 - X^5, \tag{3}$$

where X, τ and R are to be determined in terms of a, b, and c. Show that this rescaled system exhibits a subcritical pitchfork bifurcation.

Answer:
Let us define $X = \alpha x$ and $\tau = \beta t$, where α and β are two (nonzero) scaling factors to be determined.

Substituting $x = X/\alpha$ and $t = \tau/\beta$ into (2) yields

$$\frac{dx}{dt} = \frac{\beta}{\alpha}\frac{dX}{d\tau} = \frac{a}{\alpha}X + \frac{b}{\alpha^3}X^3 - \frac{c}{\alpha^5}X^5$$

i.e.,

$$\frac{dX}{d\tau} = \frac{a}{\beta}X + \frac{b}{\alpha^2\beta}X^3 - \frac{c}{\alpha^4\beta}X^5.$$

At this point we set

$$\frac{c}{\alpha^4\beta} = 1 \quad \text{and} \quad \frac{b}{\alpha^2\beta} = 1$$

and solve for the parameters α and β. With the values of α and β determined in this way we have that the coefficients multiplying X^3 and X^4 are both equal to one:

$$\frac{c}{\alpha^4\beta} = 1 \quad \Rightarrow \quad \beta = \frac{c}{\alpha^4} \quad \Rightarrow \quad \frac{b\alpha^2}{c} = 1 \quad \Rightarrow \quad \alpha = \sqrt{\frac{c}{b}} \quad \Rightarrow \quad \beta = \frac{b^2}{c}.$$

At this point, define

$$R = \frac{a}{\beta} \quad \text{i.e.} \quad R = \frac{ac}{b^2}.$$

Clearly, if we let

$$X = \sqrt{\frac{c}{b}}x, \qquad \tau = \frac{b^2}{c}t, \qquad \text{and} \quad R = \frac{ac}{b^2}, \tag{4}$$

and substitute into equation (2) then we obtain equation (3). Hence (4) is the rescaling we were looking for. Next, we compute the fixed points of the system (3). To this end we solve the polynomial equation

$$-X(-R - X^3 + X^5) = 0.$$

Obviously, $X_1^* = 0$ is always fixed point (independently of the value of R. The other four fixed points are given by

$$X^*(R) = \pm\sqrt{\frac{1}{2} \pm \frac{\sqrt{1+4R}}{2}}.$$

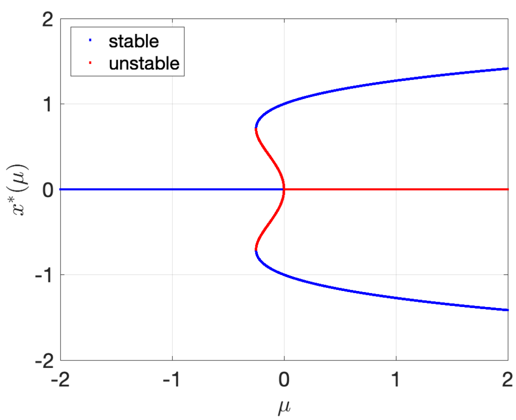
Depending on the value of R, we have a couple of different cases. In order for the square roots to be real we need $1 + 4R \geq 0$ and $1 \pm \sqrt{1+4R} \geq 0$ which implies that we have five fixed points when $-1/4 < R < 0$ and only one fixed point when $R \leq -1/4$. Additionally, we have three fixed points when $R \geq 0$. From here, you can easily check the stability of these fixed points in the given intervals. In particular notice that we have

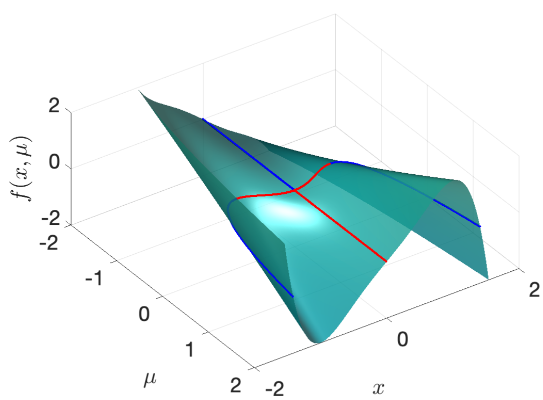
$$\left.\frac{\partial f}{\partial X}\right|_{X^*=0} = R + 3X^2 - 5X^4\Big|_{X^*=0} = R,$$

which means that for $R < 0$, $X^* = 0$ is a stable fixed point and for $R > 0$, $X^* = 0$ is unstable. For the other fixed points, we have

$$\left.\frac{\partial f}{\partial X}\right|_{X^*=\pm\sqrt{\frac{1}{2}-\frac{\sqrt{1+4R}}{2}}} = R + 3X^2 - 5X^4\Big|_{X^*=\pm\sqrt{\frac{1}{2}-\frac{\sqrt{1+4R}}{2}}}$$

which is positive for $-1/4 < R < 0$m, so we have unstable fixed points. Hereafter we plot the bifurcation diagram for the system (3).





As easily seen, there is a subcritical pitchfork bifurcation at $(X^*, R^*) = (0, 0)$. This can be proved analytically by observing that in a neighborhood of $(X^*, R^*) = (0, 0)$ the term X^5 is much smaller[1] than X^3 and RX. Hence, in a neighborhood of $(X^*, R^*) = (0, 0)$ the system (3) can be approximated as

$$\frac{dX}{d\tau} \simeq RX + X^3,$$

which is the normal form of subcritical pitchfork bifurcation.

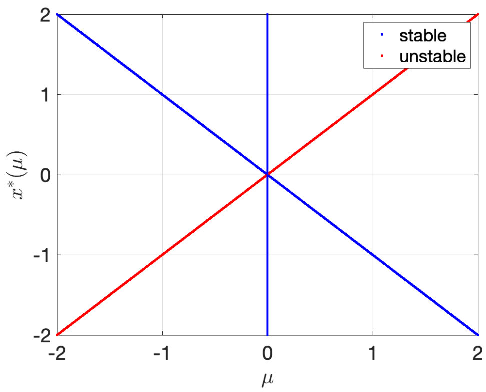
[1]For example if $X = 10^{-5}$ and $R = 10^{-2}$ then $X^5 = 10^{-25}$, $X^3 = 10^{-15}$ and $RX = 10^{-7}$.

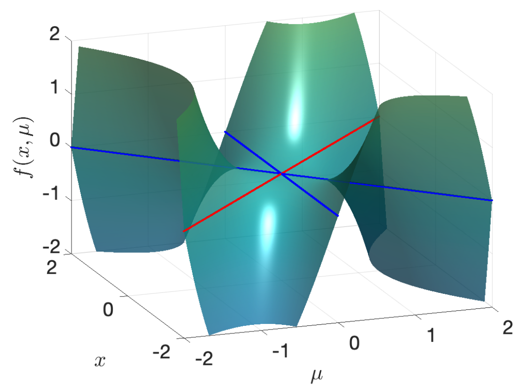
Question 3 (*) Sketch the bifurcation diagram of fixed points for the following system

$$\frac{dx}{dt} = \mu x^2 - \mu^3. \tag{5}$$

Does the system have a saddle node bifurcation at $(x^*, \mu^*) = (0, 0)$? Justify your answer.

Answer: The following plot represents the bifurcation diagram for the system (5):



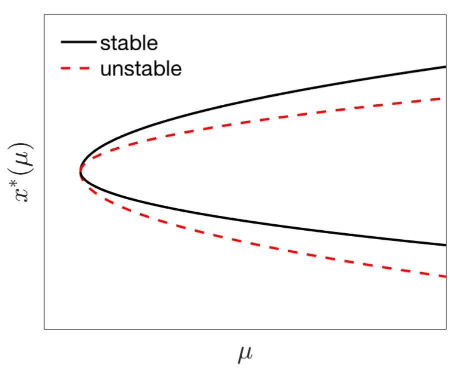


We compute the derivatives of f at $(x^*, \mu^*) = (0, 0)$

$$\frac{\partial^2 f(0,0)}{\partial x^2} = 0, \quad \frac{\partial f(0,0)}{\partial \mu} = 0, \quad \frac{\partial^2 f(0,0)}{\partial \mu^2} = 0, \quad \frac{\partial f^2(0,0)}{\partial \mu \partial x} = 0, \quad \frac{\partial^3 f(0,0)}{\partial x^3} = 0.$$

From these computations, we see that do not have a saddle-node bifurcation nor a familiar bifurcation discussed in lecture.

Question 4 Is the following bifurcation diagram plausible? If you think it is, determine a dynami-



cal system $\dot{x} = f(x, \mu)$ with polynomial nonlinearities that exhibits such bifurcation at $(x^*, \mu^*) = (2, 1)$.

Answer:
The bifurcation diagram is reasonable since for any μ after the bifurcation point, the fixed points have alternating stability. A system with a polynomial nonlinearity that re-creates this diagram can be found fairly easily. To this end, we first consider a local coordinate system (R, X) centered at the vertex of the bifurcation diagram above. Relative to such coordinate system the bifurcation occurs at $R = 0$, $X = 0$. A degree 4 polynomial in X that transitions from having no roots with $R < 0$, to a multiplicity 4 root at $X = 0$ when $R = 0$ to finally 4 separate real roots in X when $R > 0$ provides a solution to the problem.
Additionally, we can limit the search to monic polynomials with no odd terms. This guarantees the symmetry of the roots relative to $X = 0$ for each R. In summary, to identify the polynomial of degree 4 $f(X, R)$ we can use the following conditions:

1. for $R < 0$ the polynomial does not intersect the X axis;
2. for $R = 0$ the polynomial intersects the X axis at $X = 0$ axis;
3. for $R > 0$ the polynomial intersects the X axis at 4 points, symmetric with respect $X = 0$;
4. The polynomial is symmetric with respect to the vertical axis.

A class of polynomials satisfying all these conditions is

$$f(X, R) = -X^4 + 2RX^2 + cR^2 \qquad c \in (-1, 0). \tag{6}$$

In fact, the roots (fixed points) of (6) are

$$X = \pm\sqrt{R \pm \sqrt{R^2 + cR^2}}. \tag{7}$$

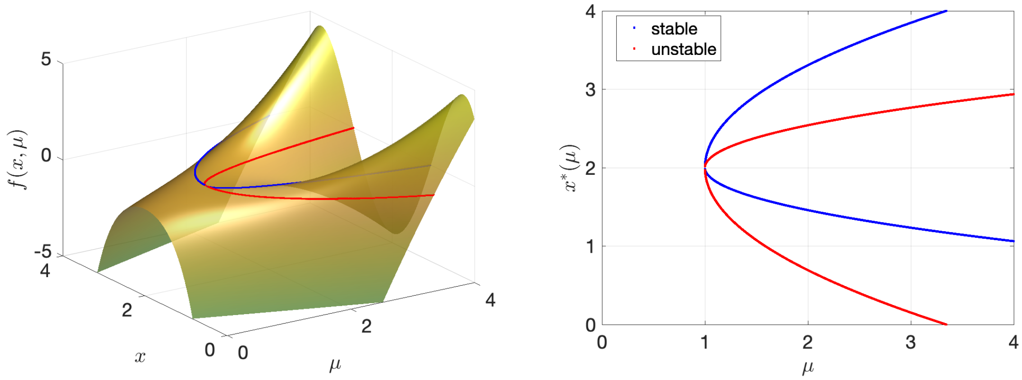
Let us pick $c = -1/2$. Note that there are no fixed points for $R < 0$, one fixed point for $R = 0$, and four fixed points for $R > 0$. At this point we can translate the coordinate systems back to where it was, i.e., define two variables x and μ such that

$$X = x - 2, \qquad R = \mu - 1. \tag{8}$$

Thus, a polynomial that produces the required bifurcation is

$$f(X, R) = -(x - 2)^4 + 2(\mu - 1)(x - 2)^2 - \frac{1}{2}(\mu - 1)^2. \tag{9}$$

Hereafter we plot the bifurcation diagram corresponding to this system.

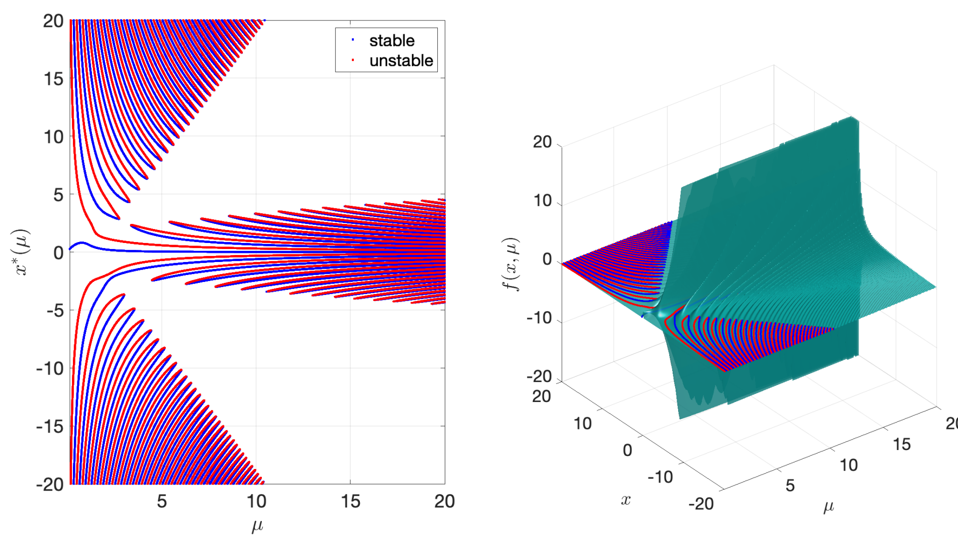


Question 5 (*) Write a computer code that returns the bifurcation diagram associated with the ODE

$$\frac{dx}{dt} = e^{-x^2/\mu^2} - \frac{\mu \sin(x\mu)}{x^2 + \cos(\mu x)^2} \tag{10}$$

in the domain $\mu \in [1/10, 20]$, $x \in [-20, 20]$. Graph the stable and unstable equilibrium curves with different colors (e.g., black for stable and red for unstable). Include the plot of bifurcation diagram in your PDF submission. Note that there is only one type of bifurcation that takes place an infinite number of times as μ is varied. Which one is it?

Answer:
As seen from the following bifurcation diagram, the system undergoes an infinite number of saddle node bifurcations.



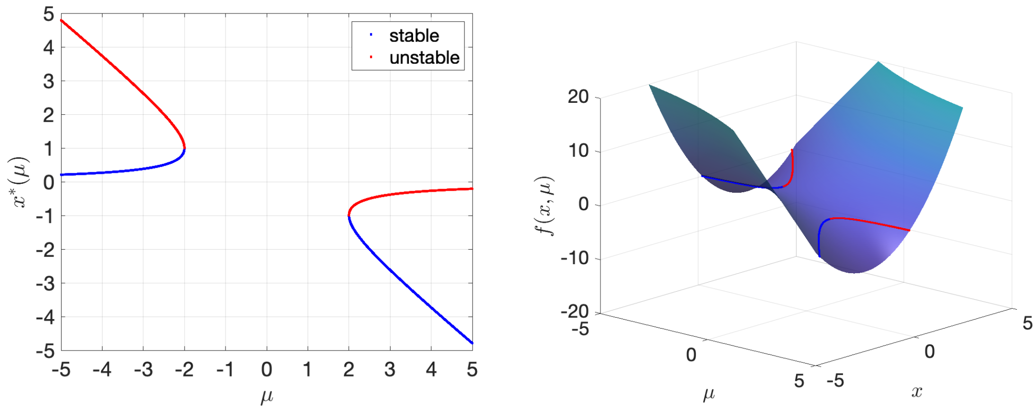


Figure 1: Bifurcation diagram for the system (11).

Question 6 (*) Use the computer code you developed in the last Question to graph the bifurcation diagram associated with the following ODEs

$$\frac{dx}{dt} = 1 + \mu x + x^2. \tag{11}$$

$$\frac{dx}{dt} = \arctan(\mu x) - x^2 + x, \tag{12}$$

$$\frac{dx}{dt} = -x\cos(x) + \mu x. \tag{13}$$

For each case, find the values of the parameter μ at which bifurcations occur (numerically or exactly when possible), and classify those as saddle-node, transcritical, supercritical pitchfork or subcritical pitchfork.

Answer:

(1) Setting the right hand side of (11) equal zero yields the critical points:

$$f(x,\mu) = 1 + \mu x + x^2 = 0 \quad \Rightarrow \quad x^*_{1,2} = -\frac{\mu}{2} \pm \frac{\sqrt{\mu^2 - 4}}{2}.$$

Notice that for $|\mu| > 2$, we have two fixed points. If $|\mu| = 2$ we have one, and if $|\mu| < 2$ we have none since the roots are not real. This should suggest that there may be a bifurcation at $(x_i^*, \mu_i^*) = (\pm 2, \mp 1)$. We can use stability analysis to classify the fixed points. To this end, we have

$$f'(x)\Big|_{x=x_1^*} = \mu + 2x\Big|_{x=x_1^*} = \mu + \left(-\mu + \sqrt{\mu^2-4}\right) = \sqrt{\mu^2-4} > 0,$$
$$f'(x)\Big|_{x=x_2^*} = \mu + 2x\Big|_{x=x_2^*} = \mu + \left(-\mu - \sqrt{\mu^2-4}\right) = -\sqrt{\mu^2-4} < 0,$$

which says that for $|\mu| > 2$, we have a stable and unstable fixed point. At $\mu = \pm 2$ we have two saddle node bifurcations which can also been seen from the the plot in Figure 1.

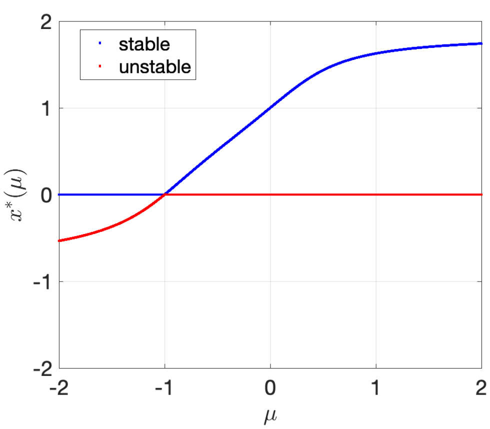
(2) Setting the right hand side of equation (12) to zero, we get

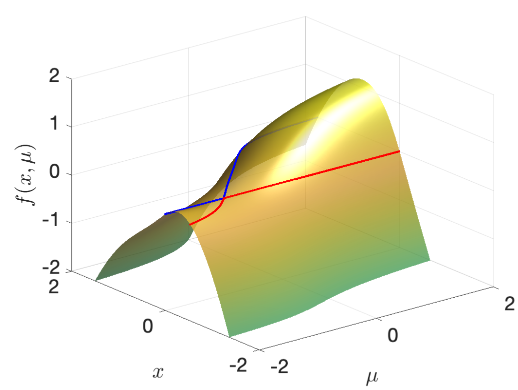
$$f(x, \mu) = \arctan(\mu x) - x^2 + x = 0. \tag{14}$$

Unlike the previous equation, it is not possible to solve for x as a function of μ. However, we can easily solve for μ as a function of x (See also Example 3.4.1 in Strogatz). This yields

$$\mu(x^*) = \frac{\tan((x^*)^2 - x^*)}{x^*}. \tag{15}$$

We also see that $x^* = 0$ satisfies (14) for any value of μ, and from linear stability analysis we see that x^* is unstable for $\mu > -1$ and stable $\mu < -1$. At $\mu^* = -1$ the fixed point $x^* = 0$ changes stability while intersecting the branch (15). This suggests that $(\mu^*, x^*) = (0, 0)$ may be a transcritical bifurcation point. Taylor series analysis and the following bifurcation diagram confirm that $(\mu^*, x^*) = (0, 0)$ is in fact a transcritical bifurcation point.





For $\mu = -1$ we have

$$\frac{\partial f}{\partial x}\Big|_{x=0} = \frac{\mu}{\mu^2 x^2 + 1} - 2x + 1\Big|_{x=0} \Rightarrow \mu + 1 = 0 \Rightarrow \mu = -1$$

which means a potential bifurcation point is

$$(x^*, \mu^*) = (0, -1).$$

(3) The fixed points are solution to the equation

$$f(x, \mu) = -x\cos(x) + \mu x = 0 \quad \Rightarrow \quad x\cos(x) = \mu x.$$

Note that $x_1^* = 0$ is a fixed point regardless of what μ is. We also get an infinite number of fixed point defined by $\cos(x_2^*) = \mu$, i.e., $x_2^* = \cos^{-1}(\mu) + k\pi$ (for $|\mu| \leq 1$ and $k \in \mathbb{Z}$). To identify the bifurcation parameter μ we consider when the derivative $\partial f/\partial x$ equals zero.

$$\frac{\partial f}{\partial x} = x\sin(x) - \cos(x) + \mu x = 0.$$

At the fixed point $x_1^* = 0$ we have

$$\frac{\partial f}{\partial x}(0) = 0 - 1 + \mu = 0,$$

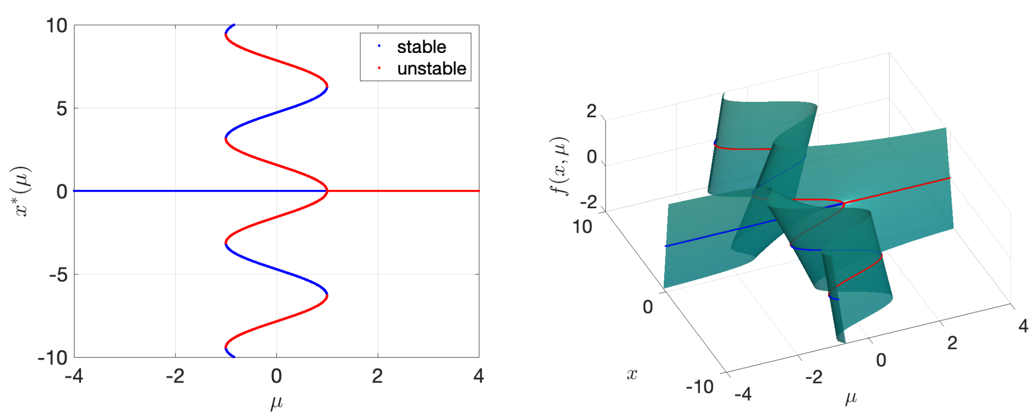
which gives $\mu^* = 1$. At the fixed points x_2^*,

$$\frac{\partial f}{\partial x}(\cos^{-1}(\mu) + k\pi) = (\cos^{-1}(\mu) + k\pi)\sin(\cos^{-1}\mu + k\pi) - \cos(\cos^{-1}(\mu) + k\pi) + \mu = 0,$$

i.e.,

$$(\cos^{-1}(\mu) + k\pi)\sin(\cos^{-1}\mu) = 0. \tag{16}$$

This equation yields $\mu^* = 1$ for even k and $\mu^* = -1$ for odd k. We can plug in different values of μ to get the stability of our fixed points. Here is the bifurcation diagram.



From this plot, we see that we have a subcritical pitchfork bifurcation at $(x^*, \mu^*) = (0, 1)$ and infinite saddle bifurcations occurring at $(x^*, \mu^*) = (k\pi, -1)$ (k odd) and $(x^*, \mu^*) = (k\pi, 1)$ (k even).

Question 7 Consider the nonlinear dynamical system

$$\frac{dx}{dt} = f(x, \mu) \tag{17}$$

and suppose that there exists a pitchfork bifurcation at $x = x^*$ and $\mu = \mu^*$. By expanding f in a neighborhood of (x^*, μ^*) prove that the normal form of such bifurcation is

$$\frac{dX}{d\tau} = RX \pm X^3. \tag{18}$$

Define R, X and τ as a function of the partial derivatives of f evaluated at (x^*, μ^*).

Answer:
Let us define $M = \mu - \mu^*$ and $X = x - x^*$, and assume that $M = \mathcal{O}(|X|^2)$. By taking into account the conditions defining a pitchfork bifurcation summarized in the Appendix (Theorem 1) we obtain the following Taylor series of $f(x, \mu)$ at the bifurcation point (x^*, μ^*)

$$f(x, \mu) = \frac{\partial^2 f(x^*, \mu^*)}{\partial x \partial \mu} XM + \frac{1}{6}\frac{\partial^3 f(x^*, \mu^*)}{\partial x^3} X^3 + \text{higher order terms.} \tag{19}$$

Let

$$B = \frac{\partial^2 f(x^*, \mu^*)}{\partial x \partial \mu}, \qquad A = \frac{1}{6}\frac{\partial^3 f(x^*, \mu^*)}{\partial x^3}. \tag{20}$$

A substitution of (19) and (20) into $\dot{x} = f(x, \mu)$ yields

$$\frac{dX}{dt} = BMX + AX^3.$$

dividing by $|A|$ gives

$$\frac{dX}{|A|dt} = \frac{dX}{d(|A|t)} = \frac{BM}{|A|}X + \frac{A}{|A|}X^3 = \frac{BM}{|A|}X + \text{sign}(A)X^3.$$

where the sign function is defined as

$$\text{sign}(A) = \begin{cases} 1 & \text{if } A > 0 \\ -1 & \text{if } A < 0 \end{cases}$$

At this point we define $\tau = |A|t$ (rescaled time) and $R = BM/|A|$ (bifurcation parameter) to obtain the normal form

$$\frac{dX}{d\tau} = RX \pm X^3.$$

Appendix A

The following theorem is useful to classify standard bifurcations based on the coefficients of a local Taylor series expansion.

Theorem 1 *Suppose that $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is a sufficiently smooth function that satisfies*

$$f(x^*, \mu^*) = 0, \quad \frac{\partial f}{\partial x}(x^*, \mu^*) = 0.$$

1. *If*

$$\frac{\partial f}{\partial \mu}(x^*, \mu^*) \neq 0, \quad \frac{\partial^2 f}{\partial x^2}(x^*, \mu^*) \neq 0$$

then a saddle-node bifurcation occurs at (x^*, μ^*).

2. *If*

$$\frac{\partial f}{\partial \mu}(x^*, \mu^*) = 0, \quad \frac{\partial^2 f}{\partial x \partial \mu}(x^*, \mu^*) \neq 0, \quad \frac{\partial^2 f}{\partial x^2}(x^*, \mu^*) \neq 0,$$

then a transcritical bifurcation occurs at (x^*, μ^*).

3. *If*

$$\frac{\partial f}{\partial \mu}(x^*, \mu^*) = 0, \quad \frac{\partial^2 f}{\partial x^2}(x^*, \mu^*) = 0, \quad \frac{\partial^2 f}{\partial x \partial \mu}(x^*, \mu^*) \neq 0, \quad \frac{\partial^3 f}{\partial x^3}(x^*, \mu^*) \neq 0,$$

then a pitchfork bifurcation occurs at (x^*, μ^*).