Applied Dynamical Systems AM 214

Final Exam - Monday December 6th, 2021 (12pm - 3pm)

Instructions

Please submit your solution to the exam in CANVAS as one PDF file. The PDF file can be a scan of your handwritten notes or a PDF file created using any other word processor (e.g., Microsoft Word of compiled Latex source).

	AM 214 students
True/False Questions	40 points
Question 1	10 points
Question 2	20 points
Question 3	10 points
Question 4	20 points
Extra Credit 1	10 points
Extra Credit 2	10 points

True or False Questions (40 points). Identify whether the following statements are true or false. If a statement is true, justify it. If false, provide a simple counterexample or explain why you think the statement is false.

1. (**5 points**) Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. The two-dimensional dynamical system $\dot{\boldsymbol{x}} = -\nabla V(\boldsymbol{x})$, where ∇ is the gradient operator, cannot have periodic orbits.

2. (**5 points**) The following two-dimensional planar dynamical system

$$\begin{cases} \dot{x} = x + \sin(y) \\ \dot{y} = y + \cos(x) \end{cases} \tag{1}$$

is volume-preserving, i.e., the area of any domain $D_0 \subset \mathbb{R}^2$ does not change when D_0 is transported by the flow generated by (1).

3. (**5 points**) Consider a planar dynamical system and suppose that there exists a trapping region[1] $D \subset \mathbb{R}^2$ with no fixed points in it. Then there exists at least one periodic orbit in D.

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[1] A trapping region D is a subset of the phase plane such that every trajectory with initial condition in D at $t = 0$ will stay in D for all $t \geq 0$.

4. (**5 points**) Consider the nonlinear dynamical system

$$\begin{cases} \dot{x} = f(x, y, \mu) \\ \dot{y} = g(x, y, \mu) \end{cases} \tag{2}$$

where f, g are infinitely differentiable functions and $\mu \in \mathbb{R}$. Suppose $(x^*(\mu), y^*(\mu))$ is the only fixed point and a Hopf bifurcation occurs at μ_c. The Hopf bifurcation is supercritical if and only if $(x^*(\mu), y^*(\mu))$ is a stable spiral for $\mu < \mu_c$.

5. (**5 points**) Consider the Lorenz equations

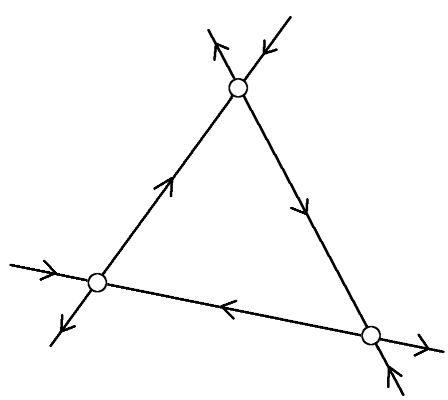
$$\begin{cases} \dot{x} = -\sigma(x - y) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases} \qquad \sigma, b > 0, \quad r \geq 0 \tag{3}$$

Any trajectory with initial condition on the z-axis stays on it forever, disregarding σ, r and b.

6. (**5 points**) A three-dimensional linear dynamical system of the form $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{A}$ is a 3×3 real matrix, can have an infinite number of fixed points.

7. (**5 points**) A smooth vector field on the phase plane is known to have exactly two limit cycles, one of which lies inside the other. The inner cycle runs clockwise and the outer one runs counterclockwise. Then there must be at least one fixed point in the region between the cycles. (Hint: sketch the arrangement of the cycles and use index theory).

8. (**5 points**) Consider a smooth vector field in the phase plane. Is the following phase portrait defined only by the following three saddle nodes possible or not? Justify your answer. (Hint: use index theory)



Question 1 (10 points) Consider the following one-dimensional dynamical system

$$\dot{x} = -x^2 + x^4 + \mu, \quad \mu \in \mathbb{R}.$$

a) (**5 points**) Compute the coordinates of the fixed points as a function of μ.

b) (**5 points**) Plot the bifurcation diagram of the fixed points as a function of μ and identify all bifurcations that take place as μ is varied.

Question 2 (20 points) Consider the linear system

$$\begin{cases} \dot{x} = -x - 2y \\ \dot{y} = -x \end{cases} \tag{4}$$

a) (**5 points**) Classify the fixed point at the origin.

b) (**5 points**) Sketch the nullclines and the eigendirections of the system (if they are real).

c) (**5 points**) Sketch a plausible phase portrait.

d) (**5 points**) Consider the domain $D_0 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\, 0 \leq y \leq 1\}$ (unit square). Suppose that D_0 is advected by the flow generated by the linear system (4). Let $A_0 = 1$ be the area of D_0. How long does it take for the initial area to shrink to $1/2$? (Hint: use Liouville's theorem).

Question 3 (10 points) State and prove the Bendixson criterion to rule out periodic orbits in two-dimensional planar systems.

Question 4 (**20 points**) Consider the following nonlinear dynamical system

$$\begin{cases} \dot{x} = 2xy - 1 \\ \dot{y} = -x^2 - y^2 + \mu \end{cases} \tag{5}$$

where $\mu > 0$ is a real parameter.

a) (**5 points**) Show that the system is volume-preserving.

b) (**5 points**) Show that the system is conservative and determine an energy function. Show that such energy function is constant along any trajectory of (5). (Hint: two-dimensional divergence-free systems are necessarily Hamiltonian/conservative.)

c) (**5 points**) Determine the nullclines and compute the fixed points of the system as a function of μ. The x–coordinates of the fixed point are roots of the polynomial equation

$$x^4 - \mu x^2 + \frac{1}{4} = 0 \tag{6}$$

Plot the bifurcation diagram of the x–coordinate of the fixed points versus μ. What kind of bifurcations occur at $\mu = 1$? (Hint: Note that the velocity field in (5) is invariant with respect to the reflection $(x, y) \to (-x, -y)$. Therefore the fixed points and the phase portrait have the same symmetry).

d) (**5 points**) Classify the stability of the fixed points for $\mu = 2$ and sketch the phase portrait for $\mu = 2$. Can the system have attractors or repellors (including limit cycles) as μ is varied? Justify your answer.

Extra credit questions

Extra Credit 1 (10 points) By using center manifold theory, show that the non-hyperbolic fixed point at the origin of the dynamical system

$$\begin{cases} \dot{x} = -yx \\ \dot{y} = x - y \end{cases} \tag{7}$$

is unstable. (Hint: compute the power series expansion of the local center manifold in a neighborhood of the fixed point and determine the ODE that describes the dynamics on such manifold).

Extra credit 2 (10 points) Consider a two-dimensional planar system of the form

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{8}$$

where $\boldsymbol{f}(\boldsymbol{x})$ is continuously differentiable. Let $\boldsymbol{X}(t, \boldsymbol{x}_0)$ be the flow generated by (8), i.e., the solution of (8) corresponding to an arbitrary initial condition $\boldsymbol{x}_0 = (x_{01}, x_{02})$.

a) Show that the determinant of the Jacobian of $\boldsymbol{X}(t, \boldsymbol{x}_0)$ with respect to $\boldsymbol{x}_0$, i.e.,

$$J(t, \boldsymbol{x}_0) = \frac{\partial X_1(t, \boldsymbol{x}_0)}{\partial x_{10}} \frac{\partial X_2(t, \boldsymbol{x}_0)}{\partial x_{20}} - \frac{\partial X_1(t, \boldsymbol{x}_0)}{\partial x_{20}} \frac{\partial X_2(t, \boldsymbol{x}_0)}{\partial x_{10}}$$

satisfies

$$\frac{dJ(t, \boldsymbol{x}_0)}{dt} = \nabla \cdot \boldsymbol{f}(\boldsymbol{X}(t, \boldsymbol{x}_0)) J(t, \boldsymbol{x}_0), \tag{9}$$

where $\nabla \cdot \boldsymbol{f}$ denotes the divergence of the vector field $\boldsymbol{f}(\boldsymbol{x})$.

b) What's the value of $dJ(t, \boldsymbol{x}_0)/dt$ at $t = 0$?