Nonlinear Dynamical Systems (AM 114/214)

Homework 5 - Solution

Instructions

Please submit to CANVAS one PDF file (homework) and one .zip file that includes any computer code you develop for the assignment.

AM 114 students: Please submit to CANVAS your homework in a PDF format (one file). This could be a scan of your handwritten notes, compiled Latex source, or a PDF created using any other word processor (e.g., Microsoft Word). If you develop any computer code to produce plots or numerical results related to the assignment please attach it to your submission as one .zip file.

AM 214 students: Please submit to CANVAS your homework in a PDF format (one file) compiled from Latex source (preferred) or any other word processor. No handwritten work should be submitted. You should also provide quantitative numerical results for all problems/questions that are amenable to computation. Such questions are marked by (*). Attach the computer code you develop (MATLAB or Python preferred) to your submission as one .zip file.

	AM 114 students	**AM 214 students**
Question 1	20 points	not required
Question 2	20 points	20 points
Question 3	25 points	15 points
Question 4	35 points	20 points
Question 5	not required	20 points
Question 6	not required	25 points

Question 1 Consider the second-order differential equation $\ddot{x} + \mu(x^4 - 2)\dot{x} + x^5 = 0$, where $\mu \in \mathbb{R}$.

a) Show that if $\mu > 0$ then there exists a unique stable limit cycle surrounding the origin.

b) Does the system still have a limit cycle if $\mu < 0$? If so, is it stable or unstable?

(Hint: for a) Show that the equation is a Liénard equation; for b) consider the transformation $t \to \tau = -t$ and show that the equation with $\mu < 0$ and $\tau = -t$ is still a Liénard equation).

Answers:

(a) The equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0, \tag{1}$$

where $f(x) = \mu(x^4 - 2), g(x) = x^5$ is indeed a Liénard equation for all $\mu > 0$. To see this, observe that $f, g \in C^1(\mathbb{R}^n)$, $g(x)$ is odd and positive for all $x > 0$, and $f(x)$ is even. To verify the other conditions on f, we define

$$F(x) = \int_0^x f(y)\, dy = \frac{\mu y^5}{5} - 2\mu y\bigg|_0^x = \mu x\left(\frac{x^4}{5} - 2\right).$$

We easily find that $a = \sqrt[4]{10}$ satisfies $F(a) = 0$. Also, it it easy to see that

$$\lim_{x\to\infty} F(x) = \infty.$$

Finally, since $x^4 < 10$ for all $x < a$, we find that $F(x) < 0$ for $x \in (0, a)$. Thus, we have satisfied all the criteria for (1) to be a Liénard equation. Therefore, we can conclude that there exists a stable limit cycle enclosing the origin.

(b) The change of variables $\tau = -t$ induces

$$\frac{dx}{d\tau} = -\frac{dx}{dt}, \qquad \frac{d^2x}{d\tau^2} = \frac{d^2x}{dt^2}.$$

This turns (1) into the ODE

$$\frac{d^2x}{d\tau^2} - \mu(x^4 - 2)\frac{dx}{d\tau} + x^5 = 0,$$

which by using the same arguments above is a Liénard equation when $\mu < 0$. Thus, a stable limit cycle exists for $\mu < 0$ in the reversed time coordinate. This limit cycle is unstable in the original time coordinate.

Question 2 (*) Use the Bendixson criterion to rule out the existence of periodic orbits in the system

$$\begin{cases} \dot{x} = x\sin(y) + x \\ \dot{y} = x^3 + y^3 + \cos(y) \end{cases}$$

Is the system a gradient system? Justify your answer.

Answers: The divergence of the vector field

$$\nabla \cdot \boldsymbol{f} = \frac{\partial}{\partial x}\left(x\sin(y) + x\right) + \frac{\partial}{\partial y}\left(x^3 + y^3 + \cos(y)\right) = \sin(y) + 1 + 3y^2 - \sin(y) = 3y^2 + 1$$

is obviously positive for all $(x, y) \in \mathbb{R}^2$. Since the divergence is non-zero and does not change sign in the domain, the Bendixson-Dulac criterion rules out the existence of periodic orbits or limit cycles throughout the entire domain.

This system is not a gradient system. To verify this it is sufficient to verify that[1]

$$\frac{\partial f_1}{\partial y} \neq \frac{\partial f_2}{\partial x}. \tag{2}$$

For the given system we have indeed

$$\frac{\partial f_1}{\partial y} = x\cos(y), \qquad \frac{\partial f_2}{\partial x} = 3x^2. \tag{3}$$

[1] The phase space $\mathbb{R}^2$ is clearly convex and therefore the condition(2) is necessary and sufficient for the system to be not a gradient system.

Question 3 (*) Use the Poincaré-Bendixson theorem to show that the nonlinear system

$$\begin{cases} \dot{x} = \mu x - y - x(x^2 + y^2) \\ \dot{y} = x + \mu y - y(x^2 + y^2) \end{cases}$$

has a limit cycle surrounding the origin for $\mu > 0$. (Hint: transform the system to polar coordinates (r, θ) and show that there exists a trapping region defined by two concentric circles).

Answers: The system

$$\begin{cases} \dot{x} = \mu x - y - x(x^2 + y^2) \\ \dot{y} = x + \mu y - y(x^2 + y^2) \end{cases} \tag{4}$$

can be transformed into polar coordinates by defining

$$r^2 = x^2 + y^2, \tag{5}$$

$$\theta = \arctan\left(\frac{y}{x}\right) + c, \tag{6}$$

where c is either equal to zero or equal to π, depending on y and x. Differentiating (5) with respect to time yields

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} \\ &= x\left[\mu x - y - x(x^2 + y^2)\right] + y\left[x + \mu y - y(x^2 + y^2)\right], \\ &= \left[\mu x^2 - x^2(x^2 + y^2)\right] + \left[\mu y^2 - y^2(x^2 + y^2)\right] \\ &= (x^2 + y^2)\left[\mu - (x^2 + y^2)\right] \\ &= r^2\left(\mu - r^2\right), \end{aligned}$$

i.e.,

$$\dot{r} = r(\mu - r^2). \tag{7}$$

Differentiating (6) with respect to time yields

$$\begin{aligned} \dot{\theta} &= \frac{x^2}{x^2 + y^2}\frac{d}{dt}\left(\frac{y}{x}\right) \\ &= \frac{\left[x + \mu y - y(x^2 + y^2)\right]x - y\left[\mu x - y - x(x^2 + y^2)\right]}{r^2} \\ &= \frac{x^2 + y^2}{r^2} \\ &= 1. \end{aligned}$$

Thus, we arrive at the decoupled equations in the polar coordinates (r, θ) given by

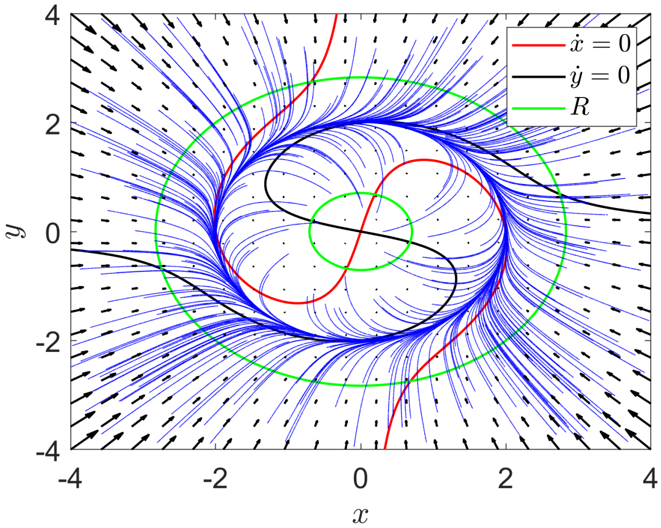
$$\dot{r} = r(\mu - r^2), \tag{8}$$

$$\dot{\theta} = 1, \tag{9}$$

which greatly simplify our analysis. Now consider two circles with radii given by $r_1 = 2\sqrt{\mu}$ and $r_2 = \sqrt{\mu}/2$. At $r = r_1$, we have $\dot{r} < 0$, so trajectories along this circle move towards the origin. At $r = r_2$, we have $\dot{r} > 0$, and therefore the flow of the trajectories moves away from the origin along this circle. In addition, there are no fixed points in between the two circles. Therefore, the annular trapping region given by

$$R = \left\{(x,y) \in \mathbb{R}^2 |\, r_2^2 \le x^2 + y^2 \le r_1^2\right\}$$

is a positively invariant set that doesn't contain any fixed points. By the Poincare-Bendixson Theorem, This region must contain a stable limit cycle. Hereafter, we plot the trapping region R along with the phase portrait of this system for $\mu = 4$



Question 4 (*) Consider the nonlinear system

$$\begin{cases} \dot{x} = y - 2x \\ \dot{y} = x^2 - y + \mu \end{cases} \tag{10}$$

a) Sketch the nullclines and determine the position of the fixed points as a function of μ.

b) Find and classify the bifurcation that occur as μ varies and sketch the phase portrait before and after the bifurcation.

Answers:

(a) To find the fixed points, we set

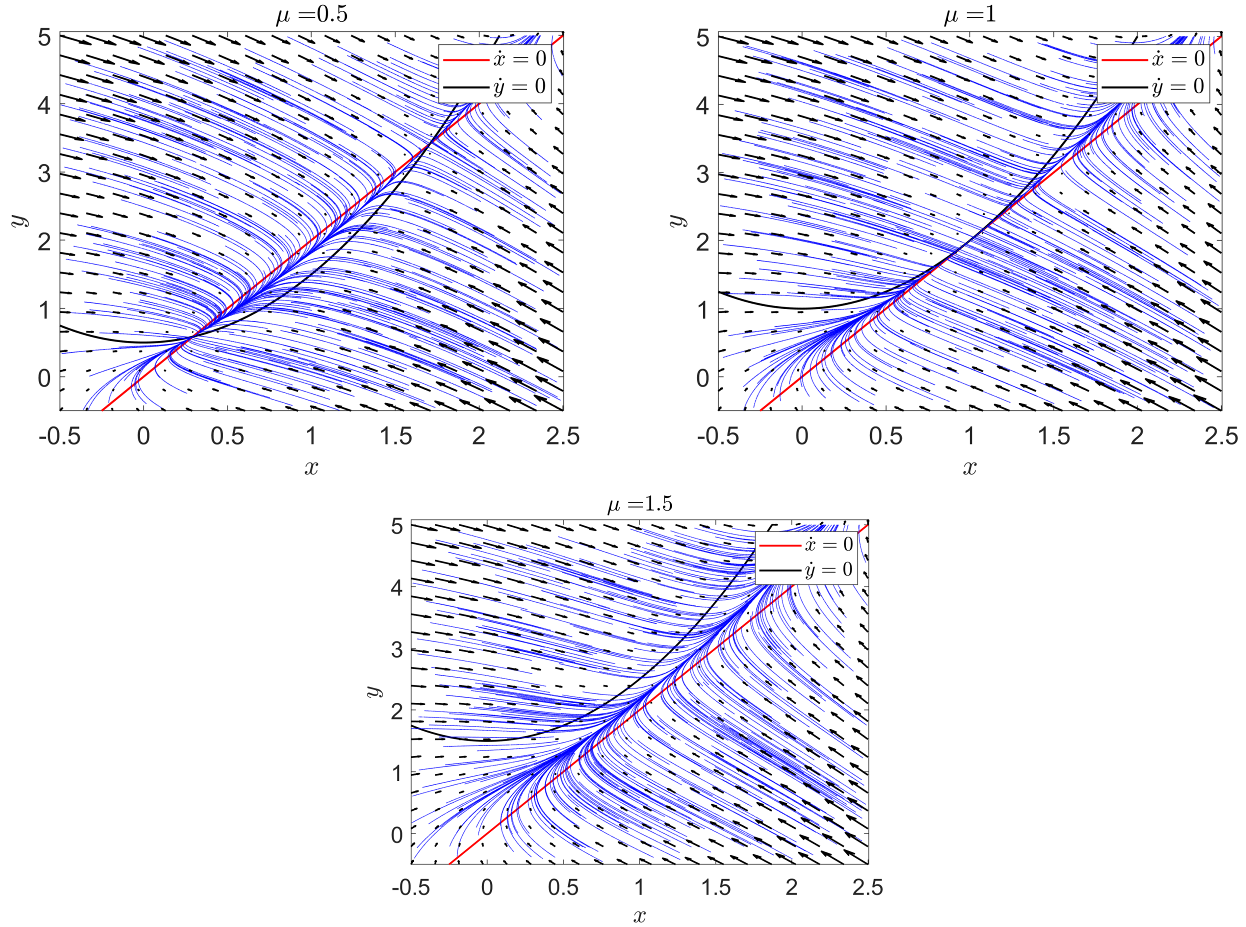
$$y - 2x = 0, \quad x^2 - y + \mu = 0.$$

Substituting $y = 2x$ into the the second equation yields $x - 2x + \mu$ which means we can use the quadratic formula to obtain

$$x^* = 1 \pm \sqrt{1-\mu}. \tag{11}$$

So the fixed points are given by $(x^*, y^*) = (1 \pm \sqrt{1-\mu}, 2(1 \pm \sqrt{1-\mu}))$.

(b) Using (11), two fixed points exist for $\mu < 1$, only one fixed points exists for $\mu = 1$ and no fixed points exist for $\mu > 1$. We conclude that a saddle-node bifurcation occurs at $\mu = 1$. To rigorously classify the bifurcation, it is possible to compute the normal form of the system using local center manifold theory (see the in-class notes for a very similar example). Hereafter we sketch the phase portrait for $\mu = \{0.5, 1, 1.5\}$.



Question 5 Is it possible for a n-dimensional dynamical system to be simultaneously a gradient system and a divergence-free system? Justify your answer.

Answers: For a gradient system

$$\dot{\boldsymbol{x}} = \nabla\varphi(\boldsymbol{x})$$

to be divergence-free, we must have $\nabla \cdot (\nabla\varphi) = 0$. More explicitly,

$$\nabla \cdot \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \dots, \frac{\partial\varphi}{\partial x_n}\right) = \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \cdots + \frac{\partial^2\varphi}{\partial x_n^2} = \Delta\varphi = 0, \tag{12}$$

where Δ is the Laplace operator. Functions that satisfy $\Delta\varphi = 0$ are called *harmonic* functions. Hence, any system with a harmonic potential is simultaneously a gradient system and a divergence-free system. An example of a harmonic potential in 2D is $\varphi(x, y) = e^x \sin(y)$. Taking the gradient, gives the dynamical system

$$\begin{cases} \dot{x} &= e^x \sin(y), \\ \dot{y} &= e^x \cos(y). \end{cases} \tag{13}$$

Taking the divergence $\nabla \cdot (\nabla\varphi) = \nabla \cdot (e^x \sin(y), e^x \cos(y)) = e^x \sin(y) - e^x \sin(y) = 0$. Hence the system (13), is a volume-preserving gradient system. The system is also conservative since it is 2D and volume-preserving.

Question 6 (*) Consider the predator-prey model

$$\begin{cases} \dot{x} = x^2(1-x) - xy \\ \dot{y} = xy - \mu y, \quad \mu \geq 0 \end{cases} \tag{14}$$

where $x(t) \geq 0$ and $y(t) \geq 0$ are, respectively, dimensionless populations of preys and predators, and μ is a parameter.

a) Show that a Hopf bifurcation occurs at $\mu = 1/2$. At which point in the phase plane? Is the Hopf bifurcation subcritical or supercritical?

b) Plot the phase portrait for $\mu = \{1/4,\ 1/2,\ 3/4,\ 1\}$.

c) What happens to the population of predators $y(t)$ for $\mu \geq 1$?

Answers:

(a) To find the fixed points, we simultaneously set

$$x^2(1-x) - xy = 0$$
$$(x-\mu)y = 0.$$

From these equations we see that $(x^*, y^*) = \{(0,0), (1,0)\}$ are always fixed points. On the other hand, if $x = \mu$, we get $\mu^2(1-\mu) - \mu y = 0 \Rightarrow y = \mu(1-\mu)$. Hence we have a third fixed point located at

$$(x^*, y^*) = (\mu, \mu(1-\mu)). \tag{15}$$

The Jacobian matrix of this system is

$$\boldsymbol{J_f}(x,y) \begin{bmatrix} 2x - 3x^2 - y & -x \\ y & x - \mu \end{bmatrix}. \tag{16}$$

We have

$$\boldsymbol{J_f}(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & \mu. \end{bmatrix} \qquad \boldsymbol{J_f}(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & 1-\mu. \end{bmatrix}$$

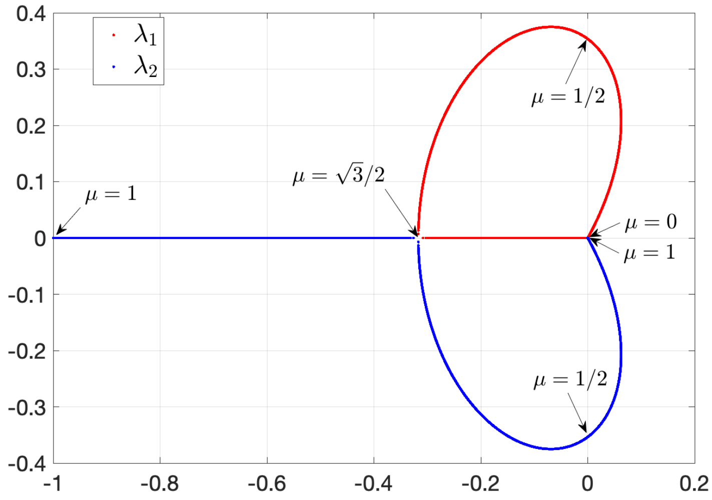
Hence, $(0,0)$ is always a non-hyperbolic fixed point with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = \mu$. On the other hand, $(0,1)$ is saddle node for all $0 \leq \mu < 1$, a non-hyperbolic fixed point for $\mu = 1$, and a stable node for all $\mu > 1$. Evaluating the Jacobian (16) at $(x,y) = (\mu, \mu(1-\mu))$ yields

$$\boldsymbol{J_f}(\mu, \mu(1-\mu)) = \begin{bmatrix} \mu - 2\mu^2 & -\mu \\ \mu - \mu^2 & 0 \end{bmatrix}$$

with eigenvalues given by

$$\lambda_{1,2} = \frac{1}{2}\left(\mu - 2\mu^2 \pm \sqrt{-3\mu^2 + 4\mu^4}\right).$$

These eigenvalues are plotted hereafter



Thus the fixed point goes from an unstable spiral to a stable spiral as μ is increased from $1/2 - \epsilon$ to $1/2 + \epsilon$. At $\mu = \sqrt{3}/2 \simeq 0.866$ the stable spiral becomes a degenerate node. For $\sqrt{3}/2 < \mu < 1$ we have a stable node, and for $\mu = 1$ a non-hyperbolic fixed point. Note for $\mu = 1$ we have that $(0, 1)$ is actually a *transcritical* bifurcation point. For $\mu = 1/2$ the eigenvalues of the Jacobian are $\lambda_{1,2} = \pm i\sqrt{2}/4$. To rigorously classify the Hopf bifurcation at $(x, y) = (\mu, \mu(1-\mu))$ for $\mu = 1/2$ let us consider a coordinate system with the origin centered at $(\mu, \mu(1-\mu))$, i.e., consider the transformation

$$u = x - \mu \qquad v = y - \mu(1-\mu). \tag{17}$$

Clearly, $\dot{u} = \dot{x}$ and $\dot{v} = \dot{y}$. The equations of motion (14) can be written in coordinates (u, v) for $\mu = 1/2$ as

$$\begin{cases} \dot{u} = -\dfrac{1}{2}v - u^3 - u^2 - uv \\ \dot{v} = \quad \dfrac{1}{4}u + uv \end{cases} \tag{18}$$

Rescaling this system further by defining new variables x_1 and x_2 as

$$v = x_2\frac{\sqrt{2}}{2} \qquad u = x_1 \tag{19}$$

yields

$$\begin{cases} \dot{x}_1 = -\underbrace{\dfrac{\sqrt{2}}{4}}_{\omega}x_2 \underbrace{-x_1^3 - x_1^2 - \dfrac{\sqrt{2}}{2}x_1x_2}_{h_1(x_1,x_2)} \\ \dot{x}_2 = \quad \underbrace{\dfrac{\sqrt{2}}{4}}_{\omega}x_1 + \underbrace{x_1x_2}_{h_2(x_1,x_2)} \end{cases}. \tag{20}$$

At this point, define

$$h_1(x_1, x_2) = -x_1^3 - x_1^2 - \frac{\sqrt{2}}{2}x_1x_2, \qquad h_2(x_1, x_2) = x_1x_2. \tag{21}$$

This allows us to write the system in the following normal form at the Hopf bifurcation point

$$\begin{cases} \dot{x}_1 = -\omega x_2 + h_1(x_1, x_2) \\ \dot{x}_2 = \quad \omega x_1 + h_2(x_1, x_2) \end{cases}. \tag{22}$$

We have seen in class that for a system in the form (22), the coefficient A that discriminates between subcritical and supercritical Hopf bifurcations is given by

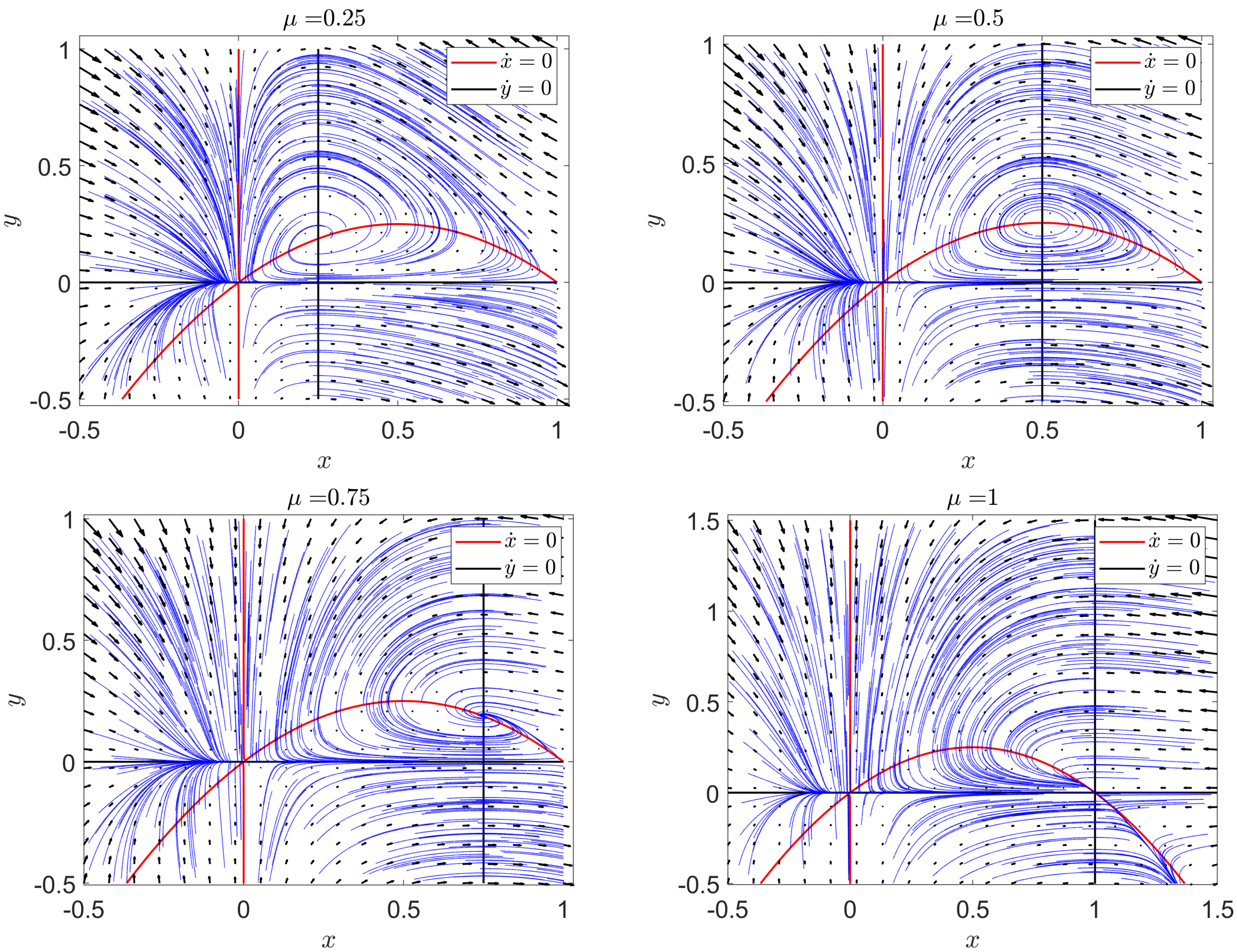
$$\begin{aligned} A =& \frac{1}{16}\left[\frac{\partial^3 h_1}{\partial x_1^3} + \frac{\partial^3 h_2}{\partial x_2^3} + \frac{\partial^3 h_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 h_2}{\partial x_2 \partial x_1^2}\right]_{(x_1,x_2)=(0,0)} + \\ & \frac{1}{16\omega}\left[\frac{\partial^2 h_1}{\partial x_1 \partial x_2}\left(\frac{\partial^2 h_1}{\partial x_1^2} + \frac{\partial^2 h_1}{\partial x_2^2}\right) - \frac{\partial^2 h_2}{\partial x_1 \partial x_2}\left(\frac{\partial^2 h_2}{\partial x_1^2} + \frac{\partial^2 h_2}{\partial x_2^2}\right) - \frac{\partial^2 h_1}{\partial x_1^2}\frac{\partial^2 h_2}{\partial x_1^2} + \frac{\partial^2 h_1}{\partial x_2^2}\frac{\partial^2 h_2}{\partial x_2^2}\right]_{(x_1,x_2)=(0,0)}. \end{aligned}$$

Substituting (21) and $\omega = \sqrt{2}/4$ into the last expression yields

$$A = \frac{1}{16}\left[-6+0+0+0\right] + \frac{4}{16\sqrt{2}}\left[-\frac{\sqrt{2}}{2}\left(\left(-6x_1 - 2\right) + 0\right) - 1(0+0) - 0 + 0\right]_{(x_1,x_2)=(0,0)} = -\frac{1}{8}. \tag{23}$$

Therefore, the Hopf bifurcation at $\mu = 1/2$ is *supercritical*.

(b) Hereafter we plot the phase portrait of the system for different values of the bifurcation parameter.



(c) For $\mu = 1$ we have two non-hyperbolic fixed point at $(0, 0)$ and $(1, 0)$, one of which (i.e. $(1, 0)$) is a transcritical bifurcation point. Based on the analysis of the phase portrait it is clear that for $\mu = 1$ the predators $y(t)$ go extinct. For $\mu > 1$ we have that any initial state in the first quadrant $(x > 0, y > 0)$ is still attracted to the fixed point $(1, 0)$, which is a stable node for $\mu > 1$.