Nonlinear Dynamical Systems (AM 114/214)

Homework 4 Solutions

Instructions
Please submit to CANVAS one PDF file (homework) and one .zip file that includes any computer code you develop for the assignment.

AM 114 students: Please submit to CANVAS your homework in a PDF format (one file). This could be a scan of your handwritten notes, compiled Latex source, or a PDF created using any other word processor (e.g., Microsoft Word). If you develop any computer code to produce plots or numerical results related to the assignment please attach it to your submission as one .zip file.

AM 214 students: Please submit to CANVAS your homework in a PDF format (one file) compiled from Latex source (preferred) or any other word processor. No handwritten work should be submitted. You should also provide quantitative numerical results for all problems/questions that are amenable to computation. Such questions are marked by (*). Attach the computer code you develop (MATLAB or Python preferred) to your submission as one .zip file.

	AM 114 students	**AM 214 students**
Question 1	50 points	30 points
Question 2	50 points	35 required
Question 3	not required	35 points

Question 1 Consider the nonlinear dynamical system

$$\begin{aligned}\dot{x} &= x - y - x^2 + xy \\ \dot{y} &= -x^2 - y\end{aligned}$$

1. Find all the fixed points of this non-linear system and classify their stability.
2. Sketch a plausible phase portrait that includes the nullclines, vector field and a few orbits.

Answers:

1. To find the fixed points, we simultaneously set

$$\begin{aligned}x - y - x^2 + xy &= 0, \\ -x^2 - y &= 0.\end{aligned}$$

Factoring the first equation we get $(x-y)(1-x) = 0$ so we either have $x = 1$ or $x = y$. When $x = 1$, the second equation yields $y = -1$. On the other hand, when $x = y$, the second equation yields $y^2 + y = 0$ which means either $y = 0, y = -1$. Therefore, we have three fixed points given by $(x, y) = (1, -1)$, $(x, y) = (0, 0)$ and $(x, y) = (-1, -1)$.

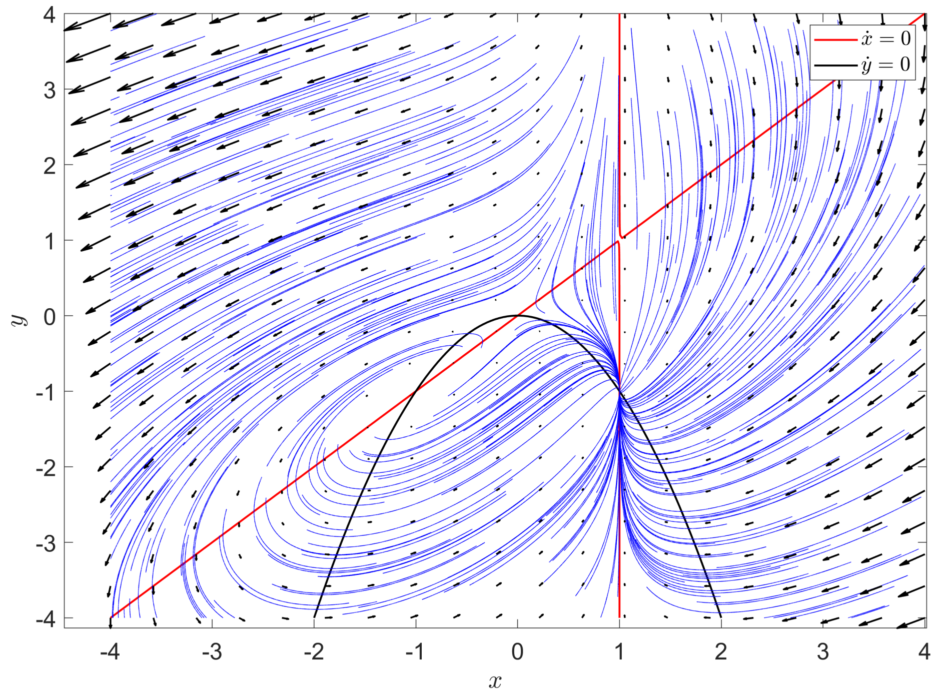
The Jacobian matrix of this system is

$$\boldsymbol{J_f}(x, y) = \begin{bmatrix} 1 - 2x + y & -1 + x \\ -2x & -1. \end{bmatrix}.$$

Substituting the fixed points we obtain the following table:

Fixed Point	Jacobian at Fixed Point	Eigenvalues	stability
(1,-1)	$\boldsymbol{J_f}(1,-1) = \begin{bmatrix} -2 & 0 \\ -2 & -1 \end{bmatrix}$	{-2,-1}	Stable Node
(-1,-1)	$\boldsymbol{J_f}(-1,-1) = \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}$	$\left\{\frac{1}{2} \pm i\frac{\sqrt{7}}{2}\right\}$	Unstable Spiral
(0,0)	$\boldsymbol{J_f}(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$	{-1,1}	Saddle Node

Notice that none none of these eigenvalues are zero therefore, by the Hartman-Grobman Theorem linearization around the fixed points is an effective method of sketching the phase portrait of the nonlinear system. This results in the following phase portait:



Question 2 Consider the damped pendulum equation of motion:

$$\ddot{\theta} = -\sin(\theta) - \gamma\dot{\theta} \tag{1}$$

where $\gamma > 0$ is a damping coefficient.

1. Rewrite equation (1) as a two-dimensional nonlinear system.
2. Classify the fixed point at the origin, plot the nullclines and sketch the phase portrait for all cases that can occur depending on the relative sizes of the parameter γ.
3. Show that for $\gamma = 0$ the system is conservative.
4. What is the index of the fixed point at the origin for $\gamma = 3$?

Answers:

1. Defining the variables $\theta_1 = \theta$ and $\theta_2 = \dot{\theta}$ results in

$$\begin{aligned}\dot{\theta}_1 &= \theta_2,\\ \dot{\theta}_2 &= -\sin(\theta_1) - \gamma\theta_2.\end{aligned}$$

2. The fixed points are given by simultaneously setting

$$\begin{aligned} \theta_2 &= 0 \\ -\sin(\theta_1) - \gamma\theta_2 &= 0 \quad \Rightarrow \quad \sin(\theta_1) = 0 \quad \Rightarrow \quad \theta_1 = n\pi \end{aligned}$$

for $n \in \mathbb{Z}$. Therefore, all the fixed points of this system lie on the θ_1-axis where the locations determined by entirely by $n\pi$. Moreover, the system is invariant under translation, i.e., the transformation $\theta_1 \to \theta_1 + n\pi$ leaves the equations of motion invariant. Therefore we can just consider two fixed points located at

$$\boldsymbol{A} = (\theta_1, \theta_2) = (0, 0) \qquad \text{and} \qquad \boldsymbol{B} = (\theta_1, \theta_2) = (0, \pi).$$

The Jacobian matrix of this system is given by

$$\boldsymbol{J_f}(\theta_1, \theta_2) = \begin{bmatrix} 0 & 1 \\ -\cos(\theta_1) & -\gamma. \end{bmatrix}$$

For the fixed point $\boldsymbol{B}$ we have

$$\boldsymbol{J_f}(\boldsymbol{B}) = \begin{bmatrix} 0 & 1 \\ 1 & -\gamma \end{bmatrix}$$

with eigenvalues given by

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4}}{2}.$$

This is a saddle node for all $\gamma > 0$. For the fixed point $\boldsymbol{A}$ (orgin of the system) we get

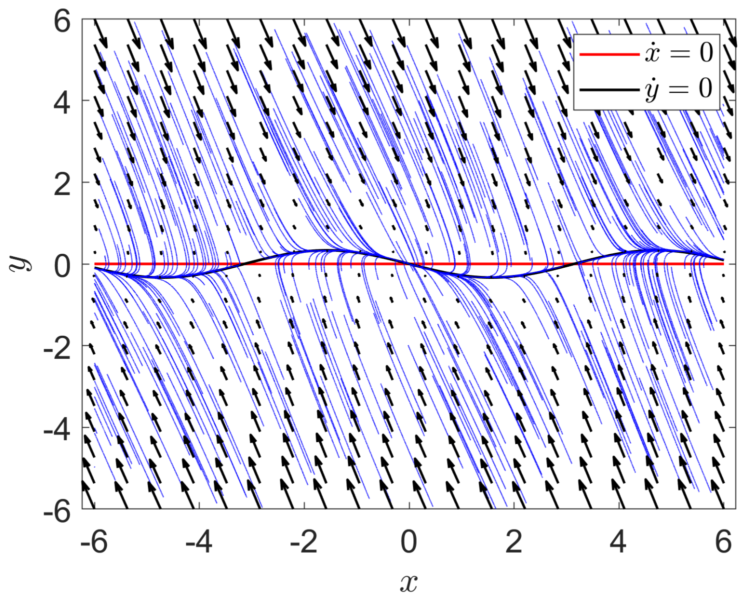
$$\boldsymbol{J_f}(\boldsymbol{A}) = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix}$$

with eigenvalues given by

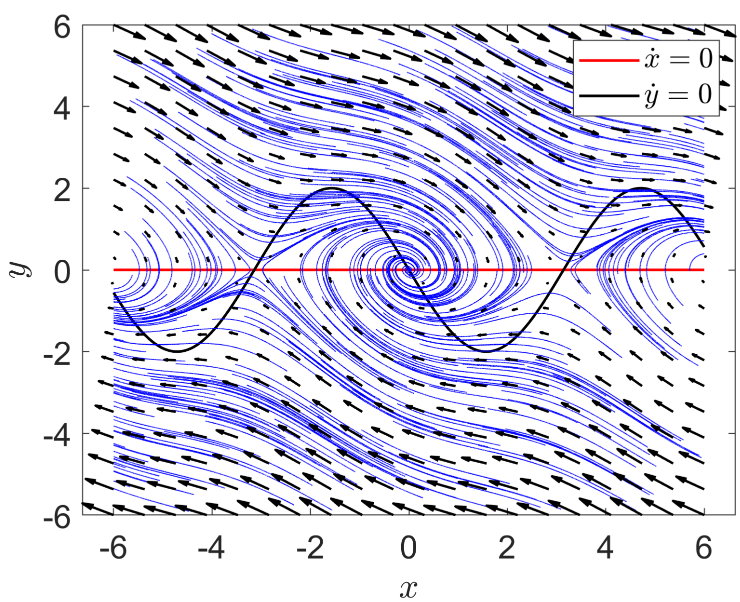
$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2}.$$

Depending on the value of γ, we get the following three types of fixed points:

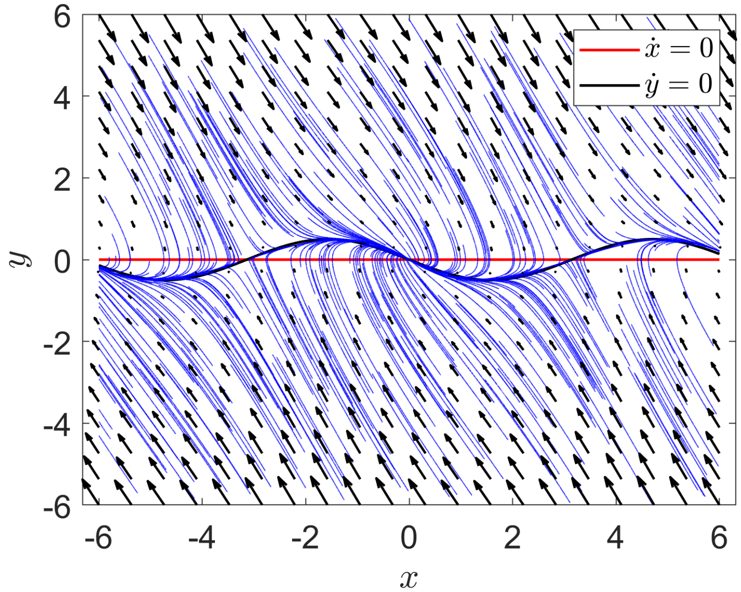
- $\gamma^2 - 4 > 0$ (i.e., $\gamma > 2$): this gives two negative eigenvalues resulting in a stable node. Here is the phase portrait for $\gamma = 3$:



- $\gamma^2 - 4 < 0$ (i.e., $\gamma < 2$): This gives complex conjugates eigenvalues whose real part is $-\gamma/2$. Therefore the origin is a stable spiral. Here is the phase portrait for $\gamma = 1/2$:



- $\gamma^2 - 4 = 0$ (i.e., $\gamma = 2$): This gives a solitary eigenvalue $-\gamma/2$ of algebraic multiplicity 2 and geometric multiplicity one. In this case the fixed point is an improper (or degenerate) stable node.



Notice that so long as $\gamma > 0$, the origin of the system, i.e., the fixed point $\boldsymbol{A}$ is always stable. Therefore, the pendulum will always tend to go to rest at such point.

3. For $\gamma = 0$, the system reduces to $\ddot{\theta} = -\sin(\theta)$ which can be written as

$$\begin{aligned}\dot{\theta}_1 &= \theta_2\\ \dot{\theta}_2 &= -\sin(\theta_1).\end{aligned}$$

To show this system is conservative, consider the energy function $E(\theta_1, \theta_2) = \dfrac{1}{2}\theta_2^2 - \cos(\theta_1)$. We compute

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}\theta_2^2 - \cos(\theta_1)\right) = \theta_2\dot{\theta}_2 + \sin(\theta_1)\dot{\theta}_1 = \theta_2(-\sin(\theta_1)) + \sin(\theta_1)\theta_2 = 0$$

4. From part 2, the origin is a stable node at $\gamma = 3$. Therefore, the index at the origin $I_C = 1$.

Question 3 Consider the nonlinear dynamical system

$$\begin{cases}\dot{x} = -x - 2y\\ \dot{y} = x^2\end{cases}$$

1. Show that the system has a unique fixed point at the origin.

2. Determine the local center manifold at such fixed point, and classify its stability.

3. Sketch a plausible phase portrait, that includes the nullclines, vector field, the local center manifold, and a few orbits.

4. Compute the index of the fixed point at the origin. (Hint: you can use numerical integration to compute

$$I = \frac{1}{2\pi} \oint d\varphi = \frac{1}{2\pi} \oint d\left(\arctan\left(\frac{x^2}{-x-2y}\right)\right) \tag{2}$$

along a simple closed curve, e.g., a circle of radius one centered at the origin).

Answers:

1. Setting $x^2 = 0 \Rightarrow x = 0$ and substituting this into $-x - 2y = 0$ yields $y = 0$. Obivously, this yields a unique fixed point $(0, 0)$.

2. Note that at the origin the Jacobian of the nonlinear system is

$$\boldsymbol{J_f}(0,0) = \begin{bmatrix} -1 & -2 \\ 2x & 0 \end{bmatrix}_{(x,y)=(0,0)} = \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix}$$

with eigenvalues given by $\lambda = 0, -1$ which makes the origin a non-hyperbolic fixed point. The associated eigenspaces can be found from

<u>For $\lambda = -1$ (Stable Eigenspace)</u>:

$$(\boldsymbol{J_f}(0,0) - \lambda \boldsymbol{I})v = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0} \Rightarrow \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

<u>For $\lambda = 0$ (Center Eigenspace)</u>:

$$(\boldsymbol{J_f}(0,0) - \lambda \boldsymbol{I})v = \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0} \Rightarrow \boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Now we want to compute the center manifold as graph of some function h,

$$W^c = \left\{(x, y) \in \mathbb{R}^2 \mid y = h(x), \quad \text{for small } x\right\}.$$

To this end, we represent $h(x)$ as a power series

$$h(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

around the origin. The first couple coefficients can be obtained by considering some properties of the center manifold. Since W_c must past through the point $(0,0)$, then we have

$$h(0) = a_0 = 0$$

and W_c is tangent to the center eigenspace we have

$$h'(0) = a_1 = -1/2.$$

which means that

$$h(x) = -\frac{x}{2} + a_2x^2 + a_3x^3 + \cdots .$$

At this point, we use the fact that W_c is an invariant manifold, i.e. trajectories near the origin satisfy

$$y(t) = h(x(t)).$$

Using this invariance property and substituting in the power series representation of h yields:

$$\begin{aligned}
\dot{y} = \frac{dh}{dx}\dot{x} \Rightarrow x^2 &= \left(-\frac{1}{2} + 2a_2x + 3a_3x^2 + \dots\right)\left(-x - 2\left(-\frac{x}{2} + a_2x^2 + a_3x^3 + \cdots\right)\right) \\
&= \left(-\frac{1}{2} + 2a_2x + 3a_3x^2 + \dots\right)\left(-x + \left(x - 2a_2x^2 - 2a_3x^3 + \cdots\right)\right) \\
&= \left(-\frac{1}{2} + 2a_2x + 3a_3x^2 + \dots\right)\left(-2a_2x^2 - 2a_3x^3 + \cdots\right) \\
&= a_2x^2 + a_3x^3 - 4a_2^2x^3 + \text{higher order terms in } x + \dots
\end{aligned}$$

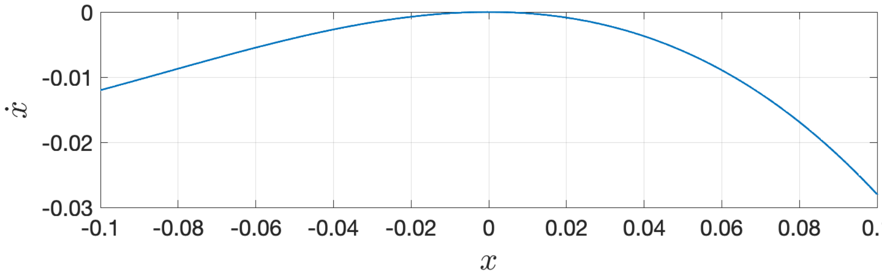
Setting equality between terms of the same degrees, we get $a_2 = 1$ and $a_3 = 4$ which means

$$h(x) = -\frac{x}{2} + x^2 + 4x^3 + \cdots \qquad \text{(center manifold)}$$

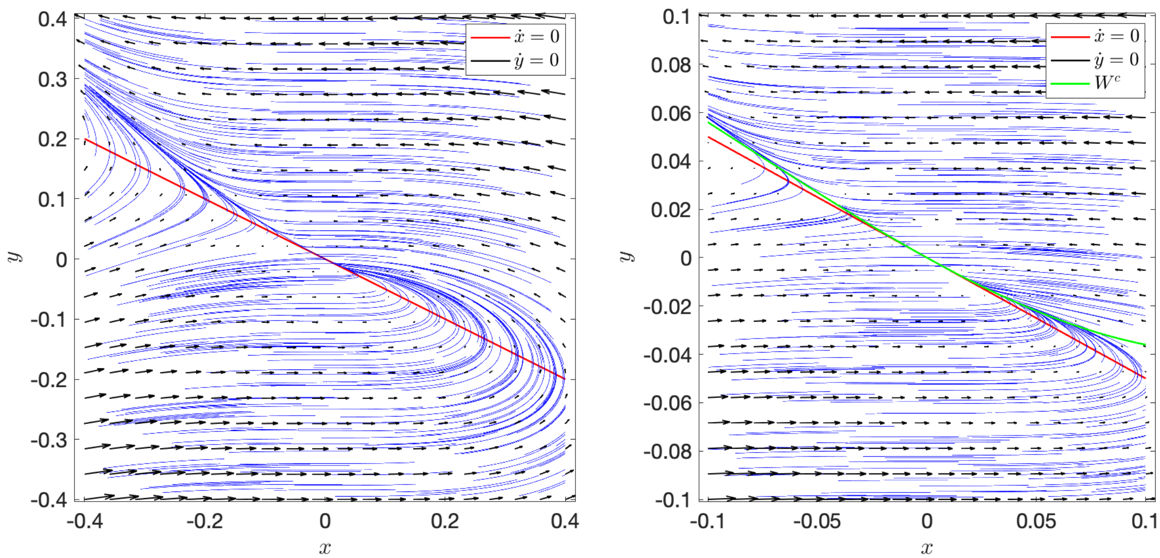
Substituting h into $\dot{x} = -x - 2y$, the dynamics near the fixed point becomes

$$\dot{x} \simeq -x - 2\left(-\frac{x}{2} + x^2 + 4x^3\right) = -2x^2 - 8x^3.$$

The right hand side is negative in a neighborhood of $x = 0$ (see the following plot) and therefore the non-hyperbolic fixed point is unstable.



3. The following figure shows the phase portrait of the system and the polynomial approximation of the center manifold W^c at the fixed point $(0,0)$:



4. Consider the unit circle C parameterized as follows:

$$(\widehat{x}(s), \widehat{y}(s)) = (\cos(s), \sin(s)) \tag{3}$$

This curve includes the fixed point at the origin. Let us write the formula we have seen in class for the index of such fixed point:

$$I = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{f_1^2 + f_2^2} \left[f_1 \left(\frac{\partial f_2}{\partial x} \frac{d\widehat{x}}{ds} + \frac{\partial f_2}{\partial y} \frac{d\widehat{y}}{ds} \right) - f_2 \left(\frac{\partial f_1}{\partial x} \frac{d\widehat{x}}{ds} + \frac{\partial f_1}{\partial y} \frac{d\widehat{y}}{ds} \right) \right] ds \tag{4}$$

where f_1 and f_2 represent the right hand side of the system evaluated on the parameterization $(\widehat{x}(s), \widehat{y}(s)) = (\cos(s), \sin(s))$. In the specific case we have $f_1(x, y) = -x - 2y$ and $f_2(x, y) = x^2$.

Hence,

$$\frac{\partial f_1}{\partial x} = -1, \quad \frac{\partial f_1}{\partial y} = -2, \quad \frac{\partial f_2}{\partial x} = 2x, \quad \frac{\partial f_2}{\partial y} = 0. \tag{5}$$

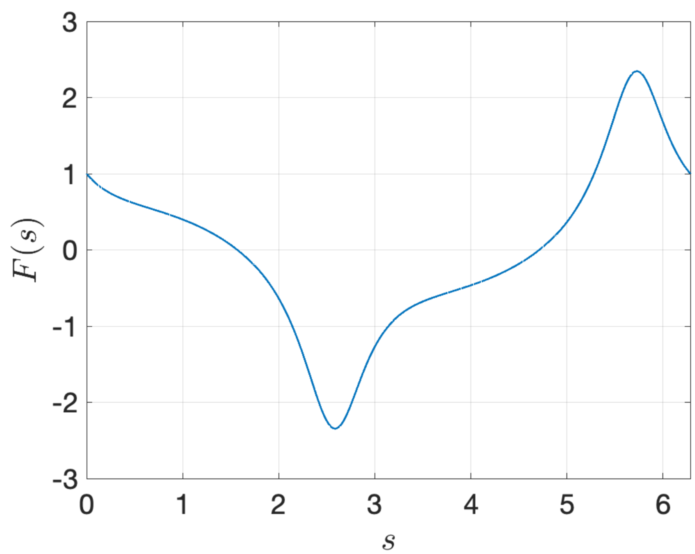
This simplifies the formula above to

$$\begin{aligned} I =& \frac{1}{2\pi} \int_0^{2\pi} \frac{2\cos(s)\sin(s)[\cos(s) + 2\sin(s)] - \cos^2(s)[\sin(s) - 2\cos(s)]}{(\cos(s) + 2\sin(s))^2 + \cos^2(s)} ds, \\ =& \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin(2s)[\cos(s) + 2\sin(s)] - \cos^2(s)[\sin(s) - 2\cos(s)]}{(\cos(s) + 2\sin(s))^2 + \cos^2(s)} ds. \end{aligned}$$

We can easily compute the integral by using numerical quadrature, e.g., by using the trapezoidal rule on an evenly spaced grid in $[0, 2\pi]$. To this end, define

$$F(s) = \frac{\sin(2s)[\cos(s) + 2\sin(s)] - \cos^2(s)[\sin(s) - 2\cos(s)]}{(\cos(s) + 2\sin(s))^2 + \cos^2(s)}.$$

This function is plotted hereafter.



The index of the fixed point can be then approximated as

$$I \simeq \frac{\Delta s}{4\pi} \sum_{k=1}^{N} \left[F(s_k) + F(s_{k-1}) \right]. \tag{6}$$

where $\Delta s = s_1 - s_0$. The following Matlab function computes I using the trapezoidal rule mentioned above on an evenly-spaced grid of 10000 points in $[0, 2\pi]$:

```
function I= compute_index()
F=@(s) (sin(2*s).*(cos(s)+2*sin(s))-cos(s).^2.*(sin(s)-2*cos(s)))./ ...
((cos(s)+2*sin(s)).^2 + cos(s).^2);
s=linspace(0,2*pi,10000);
I = 1/(2*pi)*trapz(s,F(s));
```

The result of this function is $I = 0$. Hence the index of the non-hyperbolic fixed point at the origin is zero.