Introduction to Dynamical Systems AM 114

Final Exam - Monday December 6th, 2021 (12pm - 3pm)

Instructions

Please submit your solution to the exam in CANVAS as one PDF file. The PDF file can be a scan of your handwritten notes or a PDF file created using any other word processor (e.g., Microsoft Word of compiled Latex source).

	AM 114 students
True/False Questions	40 points
Question 1	10 points
Question 2	20 points
Question 3	10 points
Question 4	20 points
Extra Credit 1	10 points
Extra Credit 2	10 points

True or False Questions (40 points). Identify whether the following statements are true or false. If a statement is true, justify it. If false, provide a simple counterexample or explain why you think the statement is false.

1. (**5 points**) Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. The two-dimensional dynamical system $\dot{\boldsymbol{x}} = -\nabla V(\boldsymbol{x})$, where ∇ is the gradient operator, cannot have periodic orbits.

2. (**5 points**) The following two-dimensional planar dynamical system

$$\begin{cases} \dot{x} = x + \sin(y) \\ \dot{y} = y + \cos(x) \end{cases} \tag{1}$$

is volume-preserving, i.e., the area of any domain $D_0 \subset \mathbb{R}^2$ does not change when D_0 is transported by the flow generated by (1).

3. (**5 points**) Consider a planar dynamical system and suppose that there exists a trapping region[1] $D \subset \mathbb{R}^2$ with no fixed points in it. Then there exists at least one periodic orbit in D.

[1] A trapping region D is a subset of the phase plane such that every trajectory with initial condition in D at $t = 0$ will stay in D for all $t \geq 0$.

4. (**5 points**) Consider the nonlinear dynamical system

$$\begin{cases} \dot{x} = f(x, y, \mu) \\ \dot{y} = g(x, y, \mu) \end{cases} \tag{2}$$

where f, g are infinitely differentiable functions and $\mu \in \mathbb{R}$. Suppose $(x^*(\mu), y^*(\mu))$ is the only fixed point and a Hopf bifurcation occurs at μ_c. The Hopf bifurcation is supercritical if and only if $(x^*(\mu), y^*(\mu))$ is a stable spiral for $\mu < \mu_c$.

5. (**5 points**) Consider the Lorenz equations

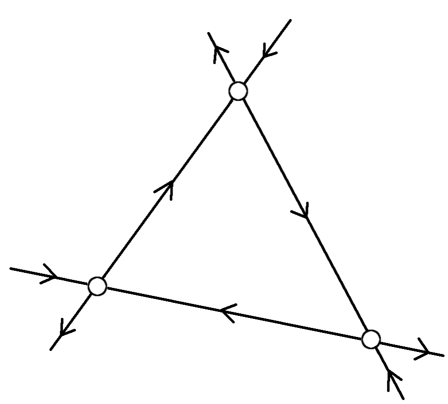
$$\begin{cases} \dot{x} = -\sigma(x - y) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases} \qquad \sigma, b > 0, \quad r \geq 0 \tag{3}$$

Any trajectory with initial condition on the z-axis stays on it forever, disregarding σ, r and b.

6. (**5 points**) A three-dimensional linear dynamical system of the form $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{A}$ is a 3×3 real matrix, can have an infinite number of fixed points.

7. (**5 points**) A smooth vector field on the phase plane is known to have exactly two limit cycles, one of which lies inside the other. The inner cycle runs clockwise and the outer one runs counterclockwise. Then there must be at least one fixed point in the region between the cycles. (Hint: sketch the arrangement of the cycles and use index theory).

8. (**5 points**) Consider a smooth vector field in the phase plane. Is the following phase portrait defined only by the following three saddle nodes possible or not? Justify your answer. (Hint: use index theory)



Question 1 (10 points) Consider the following one-dimensional dynamical system

$$\dot{x} = -x^2 + x^4 + \mu, \quad \mu \in \mathbb{R}.$$

a) (**5 points**) Compute the coordinates of the fixed points as a function of μ.

b) (**5 points**) Plot the bifurcation diagram of the fixed points as a function of μ and identify all bifurcations that take place as μ is varied.

Question 2 (20 points) Consider the linear system

$$\begin{cases} \dot{x} = -x - 2y \\ \dot{y} = -x \end{cases} \tag{4}$$

a) (**5 points**) Classify the fixed point at the origin.

b) (**5 points**) Sketch the nullclines and the eigendirections of the system (if they are real).

c) (**5 points**) Sketch a plausible phase portrait.

d) (**5 points**) Consider the domain $D_0 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\, 0 \leq y \leq 1\}$ (unit square). Suppose that D_0 is advected by the flow generated by the linear system (4). Let $A_0 = 1$ be the area of D_0. How long does it take for the initial area to shrink to 1/2? (Hint: use Liouville's theorem).

Question 3 (10 points) State and prove the Bendixson criterion to rule out periodic orbits in two-dimensional planar systems.

Question 4 (**20 points**) Consider the nonlinear dynamical system

$$\begin{cases} \dot{x} = \cos(x) - y \\ \dot{y} = y\sin(x) \end{cases} \tag{5}$$

a) (**5 points**) Determine the nullclines and the fixed points of the system. (Hint: the system is periodic in x with period 2π. Therefore it is sufficient to study the system within the half-closed interval $x \in [-\pi, \pi)$).

b) (**5 points**) Classify the fixed points you determined in a). (Hint: the system is conservative with energy function $E(x, y) = y\cos(x) - y^2/2$).

c) (**5 points**) Show that trajectories of (5) are level sets of the energy function $E(x, y) = y\cos(x) - y^2/2$.

d) (**5 points**) On the same plot with the nullclines, sketch a plausible phase portrait of the system for $x \in [-\pi, \pi)$. Make sure to include the eigen-directions of the saddle nodes, if any.

Extra credit questions

Extra credit 1 (10 points) Show that the following annulus with radii $r_1 = 1$ and $r_2 = 3$

$$D = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\right\} \cap \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\right\}$$

is a trapping region for the two-dimensional dynamical system

$$\begin{cases} \dot{x} = 2x - y - x\sqrt{x^2 + y^2} \\ \dot{y} = x + 2y - y\sqrt{x^2 + y^2} \end{cases}$$

(Hint: transform the system to polar coordinates).

Extra Credit 2 (**10 points**) Prove that zero eigenvalue bifurcations in two-dimensional planar dynamical systems are necessarily at points in which the nullclines of the system are tangent.