AM 114 Final - Simon Lee

True or false

1. $\boxed{\text{True}}$ Because by Bendixson criterion if it is a continuously differentiable function it has no periodic orbits.

2. $\boxed{\text{False}}$ $\nabla \cdot F = \frac{\partial F}{\partial x}(x - \sin(y)) + \frac{\partial F}{\partial y}(y + \cos(x)) = 1+1 = 2$

because its not zero it is not volume preserving

3. $\boxed{\text{True}}$ In the Poincare-Bendixson theorem if we define $R_{r_1, r_2} = \{(r, \theta) : r_1 \leq r \leq r_2\}$ for any $0 < r_1 < 1$ and $r_2 > \sqrt{2}$, we know that this trapping region has no fixed points with this theorem we know there exists a periodic solution therefore it has atleast one periodic orbit in D

4) $\boxed{\text{True}}$ If we have a stable spiral for $\mu < \mu_c$ then we would indeed obtain a supercritical hopf bifurcation around an unstable equilibrium point.

5 $\boxed{\text{False}}$ The z-axis is manipulated by the b value so this is not necessarily true

6 $\boxed{\text{False}}$ In linear dynamical systems a system only has a single fixed point at the zero vector

7) $\boxed{\text{True}}$

since there is a direction change yes.

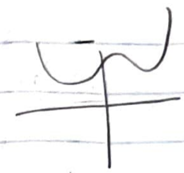
8) $\boxed{\text{True}}$ yes it is possible to have a system with 3 saddle nodes. In a higher dimensional system I am sure you can have saddle nodes if we have 3 real and distinct eigenvalues where $\lambda_1 > 0$ and $\lambda_2 < 0$.

1 $\dot{x} = -x^2 + x^4 + M \qquad M \in \mathbb{R}$

a) $\dot{x} = -x^2 + x^4 + M = 0$

$\dot{x} = x^4 - x^2 = -M$

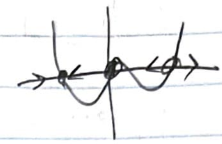
$\dot{x} = x^4 - x^2 + \left(\frac{b}{2}\right)^2 = -M + \left(\frac{b}{2}\right)^2$



r > 0

$\dot{x} = x^4 - x^2 + \left(\frac{1}{2}\right)^2 = -M + \left(\frac{1}{2}\right)^2$

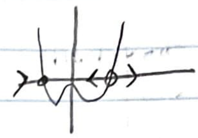
$\boxed{\dot{x} = x^4 - x^2 + \frac{1}{4} = -M + \frac{1}{4}}$ equation



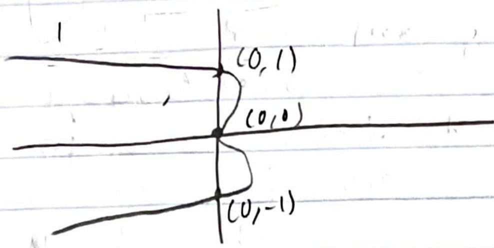
r = 0

$\boxed{x^4 - x^2 + \frac{1}{4} = \left(x^2 - \frac{1}{2}\right) = -M + \frac{1}{4}}$

$\boxed{x = \pm\sqrt{\frac{1}{2} \pm \sqrt{M + \frac{1}{4}}}}$



r < 0



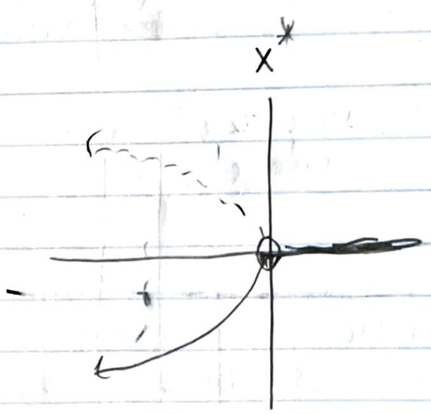

b) $\frac{\partial F}{\partial x}(x^*, M^*) = 0$

$\frac{\partial F}{\partial x}(0, 0) = -2(0) + 4(0)^3 = 0$ ✓

$\frac{\partial f}{\partial M} = 1 \neq 0$

$\frac{\partial f}{\partial x^2} -2 + 12x^2 \neq 0$

(0, 0) is a saddle node bifurcation




$(0, \frac{1}{4})$

$\frac{\partial f}{\partial x} = -2(0) + 4(0)^3 = 0$

$\frac{\partial f}{\partial M} = 1 \neq 0$

$\frac{\partial f}{\partial x^2} = -2 \neq 0$

$(0, -\frac{1}{4})$

$\frac{\partial f}{\partial x} = -2(0) + 4(0)^3 = 0$

$\frac{\partial f}{\partial M} = 1 \neq 0$

$\frac{\partial^2 f}{\partial x^2} = -2 \neq 0$

$\boxed{M = 0, \frac{1}{4}, -\frac{1}{4} \text{ are all saddle point bifurcations}}$

2

$$\begin{cases} \dot{x} = -x - 2y \\ \dot{y} = -x \end{cases} \Rightarrow \begin{pmatrix} -1 & -2 \\ -1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{pmatrix} -1-\lambda & -2 \\ -1 & 0-\lambda \end{pmatrix} = \lambda^2 + \lambda - 2$$

$0 - \lambda$

$-1 \quad 0 \quad -\lambda$

$-\lambda \quad 0 \quad \lambda^2$

$\lambda^2 + \lambda - (2)$

$\lambda^2 + \lambda - 2$

$(\lambda + 2)(\lambda - 1)$

$\boxed{\lambda = -2, 1}$

eigenvalues are $\lambda = -2, 1$

for $\lambda = 1$

$$\begin{pmatrix} -1-1 & -2 \\ -1 & 0-1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -1 & -1 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_1 + R_2} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

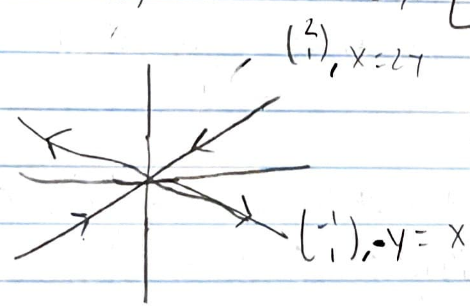
$x + y = 0 \Rightarrow x = -y$. $\boxed{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$ eigenvector for $\lambda = 1$

for $\lambda = -2$

$$\begin{pmatrix} -1-(-2) & -2 \\ -1 & 0-(-2) \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \xrightarrow{R_1 + R_2} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

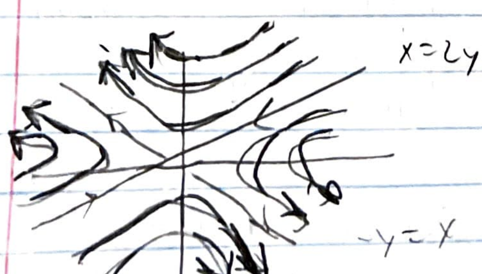
$x - 2y = 0 \qquad x = 2y \qquad \boxed{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$ eigenvector for $\lambda = -2$

2b)




$\eta_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \eta_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

2c




saddle point
unstable

3) State and prove the Bendixson criterion to rule out periodic orbits in two-dimensional planar systems

Bendixson theorem

$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \neq 0$ at any point R

Then the system

$x' = f(x,y)$
$y' = g(x,y)$

has no closed trajectories inside R

Proof

(1) $\oint_C (fi + gj)\, n\, ds = \oint_C f\,dy - g\,dx = \iint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dx\,dy$

Assume we have a closed trajectory C inside R, we are going to prove by contradiction

Using Greens Theorem we wrote out its normal form in equation (1). However this is already a contradiction. The integrand on the right side is continuous (always positive or negative) but never zero.

However the left hand side must be zero. Because C is a closed trajectory. Because C is always tangent to the velocity field $fi + gj$. This means normal vector n to C is always perpendicular so the integrand is always 0.

This proves the Bendixson theorem through contradiction example

$$\begin{cases} \dot{x} = x\sin(y) + x \\ \dot{y} = x^3 + y^3 + \cos(y) \end{cases}$$

The divergence of the vector field

$$\nabla \cdot f = \frac{\partial}{\partial x}(x\sin(y) + x) + \frac{\partial}{\partial y}(x^3 + y^3 + \cos(y)) = 3y^2 + 1$$

is positive for all $(x, y) \in R^2$. Since the divergence is non-zero and does not change sign in domain, the Bendixson - criterion rules out existence of periodic orbits or limit cycles throughout entire domain

4) $\begin{cases} \dot{x} = \cos(x) - y \\ y = y\sin(x) \end{cases}$

$\cos(x) - y \doteq 0$
$y\sin(x) = 0$

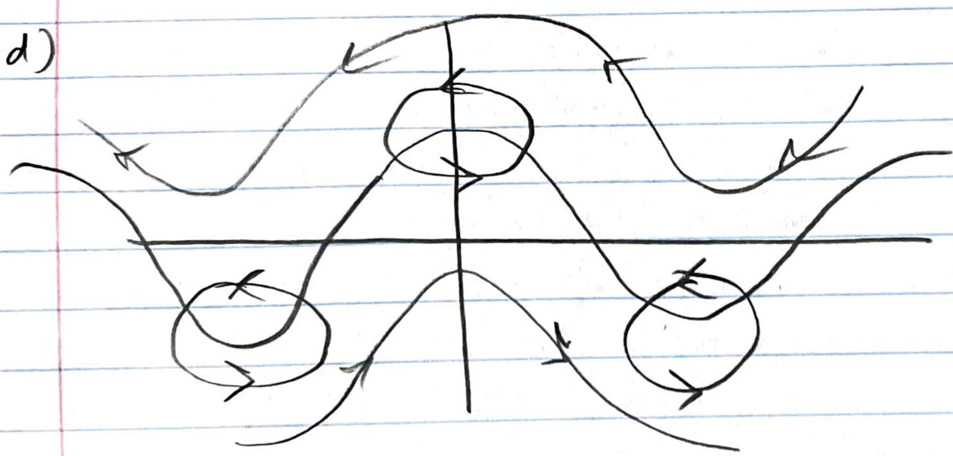
a) fixed points are $x^* = \pi k, 2\pi k$
$n_1 = \cos(x)$

b) using nonlinearsystems2d.m matlab code

$\pi k =$ Center
$2\pi k =$ center

c) skip

d)



(plotted using matlab code)