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AM 170A Euler-Lagrange Derivation Based on a Numerical Approach

We recall the equation for the functional, $T = \mathcal{F}(y(x))$, which describes the total time it takes for a ball to roll without friction from point A to point B along a slope $f(x)$. This was derived using conservation of potential and kinetic energy principles.

$$T = \int_0^{x_b} \frac{\sqrt{1+y'^2}}{\sqrt{2g(y_A - y(x))}} dx. \tag{1}$$

What we want to do is discretize the curve from point A to point B into an infinite number of connecting line segments. This way we can describe the curve as a sum of line segments. Each line segment can be described using point-slope intercept form:

$$y(x) = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i) \tag{2}$$

where $i = 0, 1, 2, ...N$ is the index of each point connecting the segments, and where $y_0 = y_A$, $y_N = y_B$, $x_0 = x_A$, $x_N = x_B$ are the coordinates for the initial and final points on the slope, points A and B.

Now we can write the total time defined by Equation 1 as a sum of integrals over each connecting line segments:

$$T(\{x_i, y_i\}) = \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} \frac{\sqrt{1 + (\frac{y_{i+1}-y_i}{x_{i+1}-x_i})^2}}{\sqrt{2g(y_a - (y_i + \frac{y_{i+1}-y_i}{x_{i+1}-x_i}(x - x_i))}} dx. \tag{3}$$

We can solve this general integral inside the summation which leads us to the simplified form:

$$T(\{x_i, y_i\}) = \sum_{i=0}^{N} \frac{-2}{\sqrt{2g}} \sqrt{1 + (\frac{(y_{i+1} - y_i)}{(x_{i+1} - x_i)})^2} (\sqrt{y_a - y_{i+1}} - \sqrt{y_a - y_i})(\frac{x_{i+1} - x_i}{y_{i+1} - y_i}). \tag{4}$$

We can simplify this further:

$$T(\{x_i, y_i\}) = \sum_{i=0}^{N} \frac{-2}{\sqrt{2g}} \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} (\frac{\sqrt{y_a - y_{i+1}} - \sqrt{y_a - y_i}}{y_{i+1} - y_i}) \tag{5}$$

The goal is to derive a differential equation that defines the curve that optimizes T. To do this, we can think about the $\{x_i, y_i\}$ coordinate grid which we define this problem to be in. If we take the partial derivative wrt x_i or y_i while keeping the other constant: $\frac{\partial T}{\partial x_i}$ while the y_i are kept constant, or $\frac{\partial T}{\partial y_i}$ while the x_i are kept constant. We want to set either of these partial derivatives equal to zero in order to find a particular that function minimizes the functional, $T = F(y(x))$.
Looking at Equation 4, we see there are less terms that include x_i and x_{i+1}, so we choose to solve $\frac{\partial T}{\partial x_i} = 0$. The general derivative formula used to obtain Equation 7 is first shown in Equation 6:

$$\frac{d}{dx_i}(\sqrt{(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2}) = \frac{x_{i+1}-x_i}{\sqrt{(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2}} \tag{6}$$

$$\frac{\partial T}{\partial x_i} = 0$$
$$= \frac{-2}{\sqrt{2g}}(\frac{\sqrt{y_a-y_{i+1}}-\sqrt{y_a-y_i}}{y_{i+1}-y_i})\left[\frac{x_i-x_{i-1}}{\sqrt{(x_i-x_{i-1})^2+(y_{i+1}-y_i)^2}} - \frac{x_{i+1}-x_i}{\sqrt{(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2}}\right] \tag{7}$$

we can set both sides equal to each other:

$$\frac{2}{\sqrt{2g}}(\frac{\sqrt{y_a-y_{i+1}}-\sqrt{y_a-y_i}}{y_{i+1}-y_i})\left[\frac{x_{i+1}-x_i}{\sqrt{(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2}}\right] \tag{8}$$
$$=$$
$$\frac{2}{\sqrt{2g}}(\frac{\sqrt{y_a-y_i}-\sqrt{y_a-y_{i-1}}}{y_i-y_{i-1}})\left[\frac{x_i-x_{i-1}}{\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}}\right]. \tag{9}$$

Now we want to take the limit as $i \to \infty$, meaning, $x_{i+1}-x_i \to \Delta x$, $x_i-x_{i-1} \to \Delta x$, and same goes for the y variable. This leads us to $\frac{\Delta x}{\Delta y} = \frac{dy}{dx} = y'$.
We see that the left and right hand sides of Equation 6 share the exact same form. This means they are equal and can be set to some constant, K. We can now simplify in the limit as $i \to \infty$ as follows:

$$\frac{2}{\sqrt{2g}}(\frac{\sqrt{y_a-y_{i+1}}-\sqrt{y_a-y_i}}{\Delta y})\frac{\Delta x}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = K. \tag{10}$$

Let us simplify further. We can write $\frac{\Delta x}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = \frac{1}{\sqrt{1+\frac{(\Delta y)^2}{(\Delta x)^2}}} = \frac{1}{\sqrt{1+y'^2}}$.
We can absorb the constant, $2/\sqrt{2g}$, into the RHS constant in Equation 7, divide both sides by 1, and now write it as:

$$\frac{\Delta y}{\sqrt{y_a-y_{i+1}}-\sqrt{y_a-y_i}}\sqrt{1+y'^2} = \frac{1}{K} = \text{new constant } L. \tag{11}$$

To simplify the leftmost fraction in Equation 10, notice that we can use the difference of squares to show that:

$$\frac{y_{i+1}-y_i}{\sqrt{y_a-y_{i+1}}-\sqrt{y_a-y_i}} = \sqrt{y_a-y_{i+1}}+\sqrt{y_a-y_i} \tag{12}$$

The math behind Equation 11 is as follows:

$$\begin{aligned}
&\frac{y_{i+1}-y_i}{\sqrt{y_a-y_{i+1}}-\sqrt{y_a-y_i}}\left[\frac{\sqrt{y_a-y_{i+1}}+\sqrt{y_a-y_i}}{\sqrt{y_a-y_{i+1}}+\sqrt{y_a-y_i}}\right] \\
&= \frac{(y_{i+1}-y_i)(\sqrt{y_a-y_{i+1}}+\sqrt{y_a-y_i})}{y_i-y_{i+1}} \\
&= \frac{\Delta y(\sqrt{y_a-y_{i+1}}+\sqrt{y_a-y_i})}{\Delta y} \\
&= \sqrt{y_a-y_{i+1}}+\sqrt{y_a-y_i}
\end{aligned} \tag{13}$$

Furthermore, in the limit as $i \to \infty$, $|y_{i+1} - y_i| = \Delta y \to 0$, meaning that $\sqrt{y_a - y_{i+1}} + \sqrt{y_a - y_i} \to 2\sqrt{y_a - y}$. And this factor of 2 can be absorbed by the constant RHS of Equation 10. Now we have:

$$(\sqrt{y_a - y_{i+1}} + \sqrt{y_a - y_i})\sqrt{1 + y'^2} = L. \tag{14}$$

$$\sqrt{y_a - y}\sqrt{1 + y'^2} = L/2. \tag{15}$$

If we square both sides of this equation we end up with the 2pt Boundary Value ODE associated with the trajectory curve optimization for our problem:

$$\boxed{(y_a - y)(1 + y^{'2}) = \text{constant}} \tag{16}$$

subject to the boundary conditions, $\boxed{f(x = 0) = y_a \text{ and } f(x = b) = 0}$

As a summary of what was done, we derived in Equation 15 the minimization the functional:

$$T(y(x)) = \int_a^b \mathcal{L}(x, y(x), y^{'}(x))dx = \int_0^{x_b} \frac{\sqrt{1 + (y^{'}(x))^2}}{\sqrt{2g(y_A - y(x))}}dx \tag{17}$$

where $f(x = 0) = y_a$ and $f(x = b) = 0$ are the fixed y coordinates of the respective beginning and endpoint of the defined curve in our problem. More generally, this problem can also be recast as solving the 2-pt boundary value problem (BVP) known as the Euler-Lagrange equations subject to the same boundary conditions:

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dx}(\frac{\partial \mathcal{L}}{\partial y^{'}}). \tag{18}$$

Without utilizing this famous Euler-Lagrange differential equation, we were still able to derive the PDE which defines the curve that optimizes $T(f(x))$.

As confirmation, we can additionally derive the corresponding ODE we obtained using the Euler-Lagrange equation, Equation 17. What we find is that:

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial y}(\frac{\sqrt{1 + y^{'}(x)^2}}{\sqrt{2g(y_A - y(x))}}) = \frac{\sqrt{1 + y'^2}}{\sqrt{2^3 g}(y_a - y)^{3/2}} \tag{19}$$

$$\frac{d}{dx}(\frac{\partial \mathcal{L}}{\partial y^{'}}) = \frac{d}{dx}(\frac{y^{'}}{\sqrt{2}\sqrt{g(y_a - y)}\sqrt{1 + y'^2}}) \tag{20}$$

$$= y^{''}(\sqrt{2g(y_a - y)}\sqrt{1 + y'^2})^{-1} + y^{'}(\frac{d}{dx}[(\sqrt{2g(y_a - y)}\sqrt{1 + y'^2})^{-1}] \tag{21}$$

We skip the rest of the intermediate derivation steps (not necessary to put in this pdf?) If we continue to simplify and set these equations equal to eachother, we arrive at the result:

$$y^{''} = \frac{1 + y^{'2}}{2(y_a - y)}. \tag{22}$$

To confirm this result, we can simply take another derivative with respect to x on the result we got (Equation 15), and we see this result obtained by using the

Euler-Lagrange equation (Equation 22), and the result we got independent of that method are equal.

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left[(y_a - y)(1 + y'^2) = \text{constant}\right] \\
&= \frac{\partial}{\partial x}\left[y_a - y + y'^2 y_a - y'^2 y = \text{constant}\right] \\
&= -y' + 2y'(y'')(y_a) - 2y'(y'')(y) - y'^2 y' = 0 \\
&= -1 + 2(y'')(y_a) - 2(y'')(y) - y'^2 = 0 \\
&\rightarrow y'^2 + 1 = 2y''(y_a - y) \\
&\rightarrow \boxed{y'' = \frac{1 + y'^2}{2(y_a - y)}}
\end{aligned} \tag{23}
$$