Nonlinear Dynamical Systems (AM 114/214)

Homework 1 solutions

Question 1 Consider the following nonlinear differential equations

$$\frac{dx}{dt} = \ln(x^2 + 1) - 1, \tag{1}$$

$$\frac{dx}{dt} = 2x + x^3 - x^5, \tag{2}$$

$$\frac{dx}{dt} = \sin(x)(x^2 - 5x + 6). \tag{3}$$

For each case, find all fixed points, discuss their stability by using the geometric approach (i.e., the plot of the velocity in the $(x, \dot{x})$ plane), and sketch the corresponding flow (vector field) on the real line. In addition, sketch the graph of the solution $x(t)$ versus t for different initial conditions x_0.

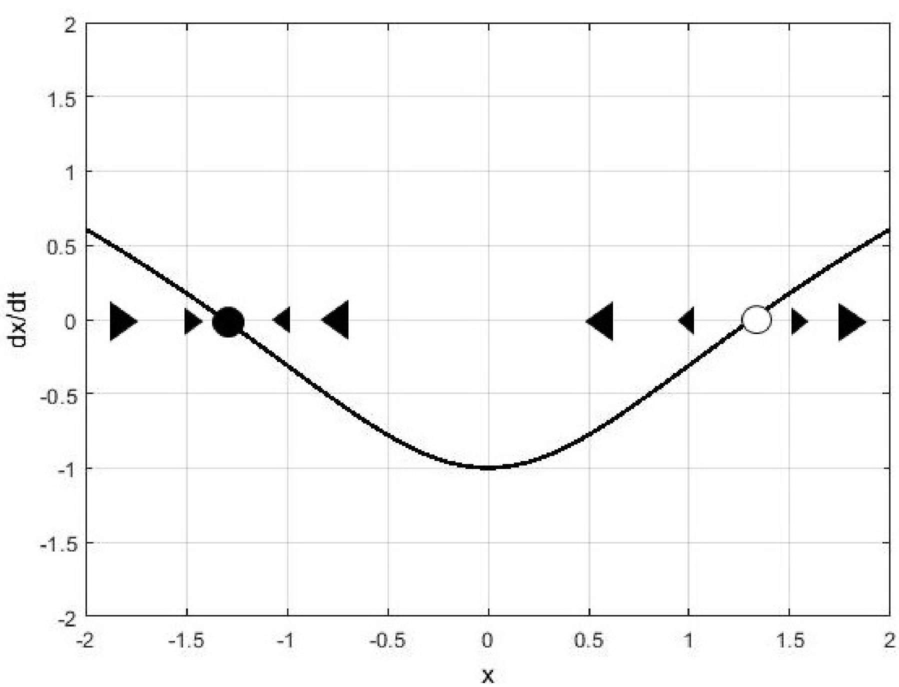
Answers:

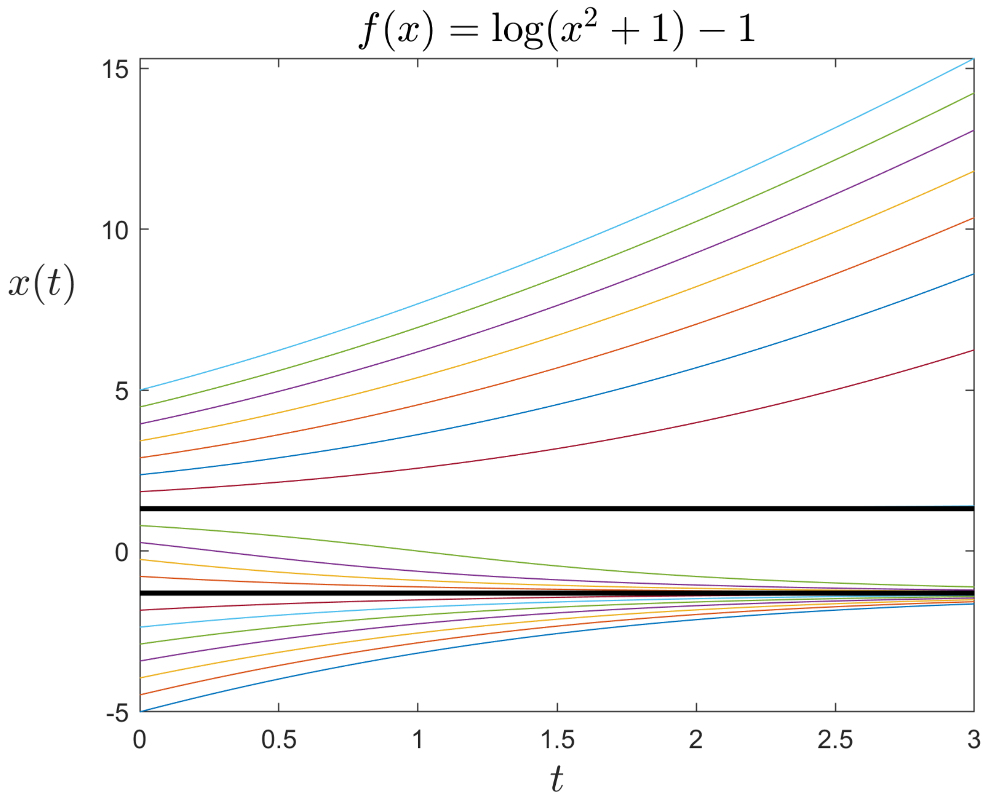
For the flow, plots fixed points are shown hereafter

(1)

$$f(x) = 0 \Rightarrow \ln(x^2 + 1) - 1 = 0 \Rightarrow x^2 + 1 = e \Rightarrow x = \pm\sqrt{e-1}$$

Equation (1) has fixed points $x = \{-\sqrt{e-1}, \sqrt{e-1}\}$ which are *stable* and *unstable* respectively.

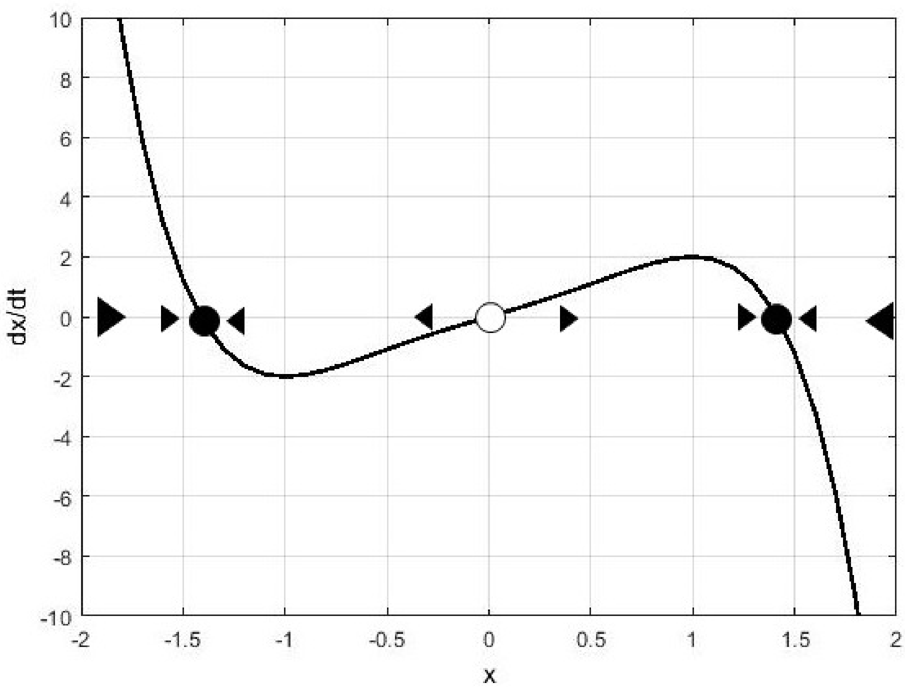


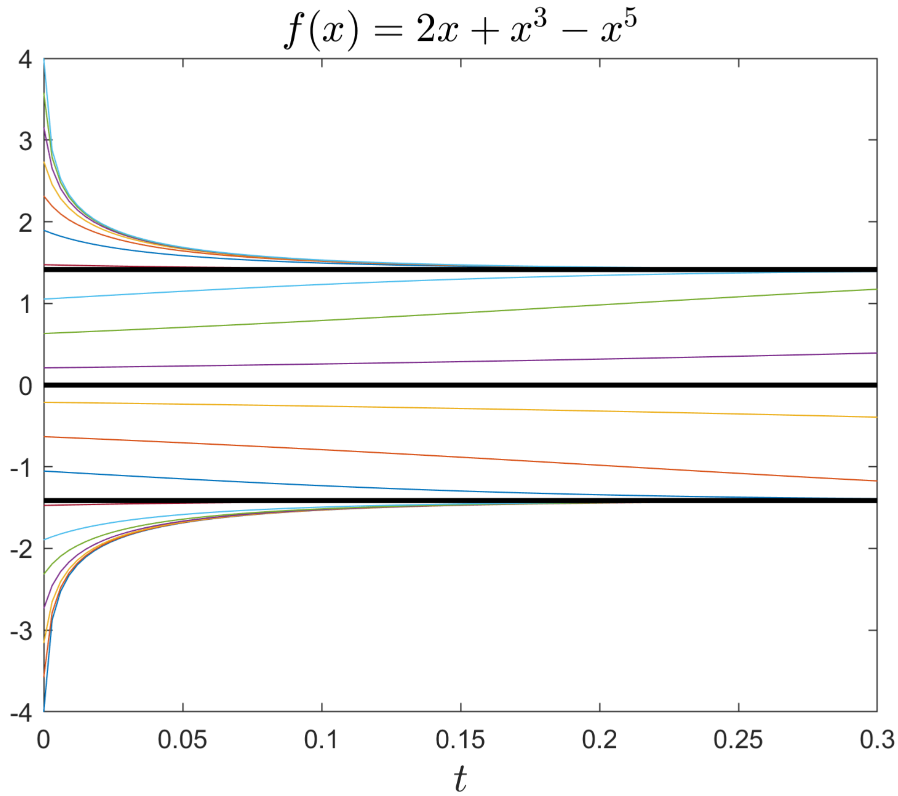


(2)

$$f(x) = 0 \Rightarrow 2x + x^3 - x^5 = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

Equation (1) has fixed points $x = \{-\sqrt{2}, 0, \sqrt{2}\}$ which are *stable unstable*, and *stable* respectively.

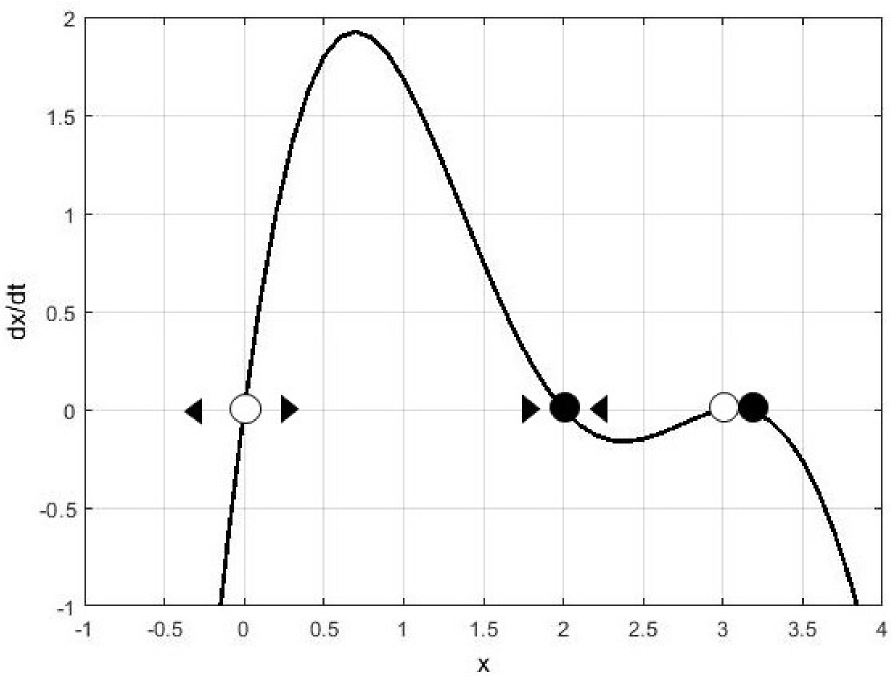


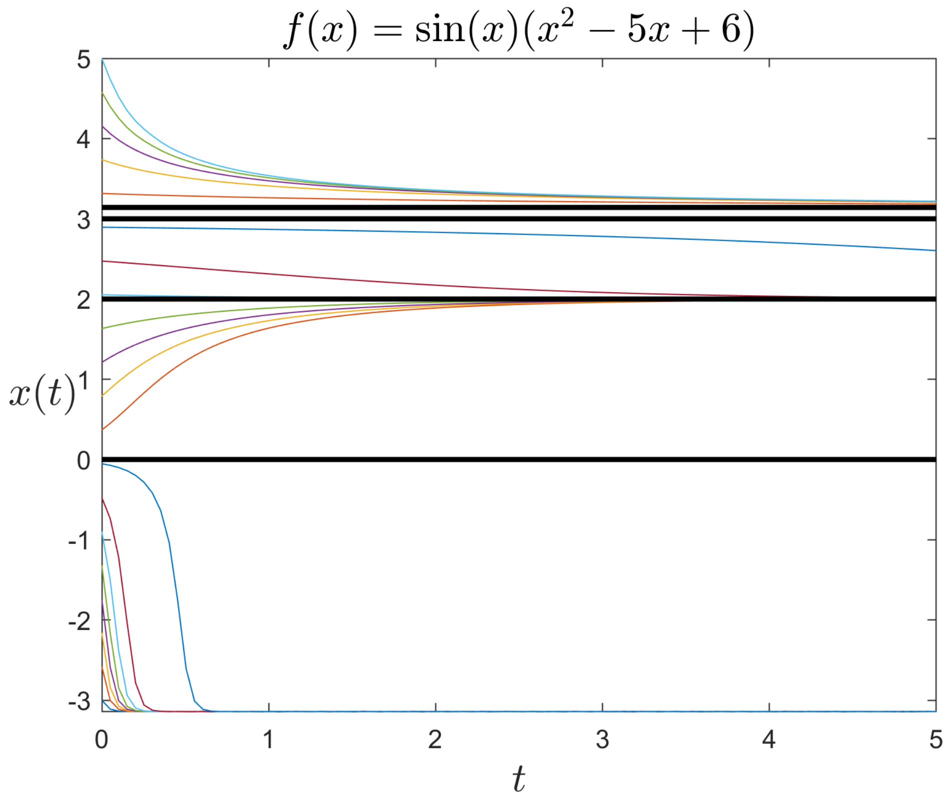


(3)

$$f(x) = 0 \Rightarrow \sin(x)(x^2 - 5x + 6) = 0 \Rightarrow x = \pm k\pi, 2, 3 \qquad k = 0, 1, 2, ..$$

Equation (3) has fixed points $x = \{2, 3, n\pi\}$ for $n \in \mathbb{Z}$. $x = 2$ is *stable*, $x = 3$ is *unstable* and $x = n\pi$ is *stable* when n is odd and *unstable* when n is even (including zero).





Question 2 Use linear stability analysis to classify the fixed points of equation (2). Do your results match with the geometric approach in Question 1?

Answers:
Notice that the right hand side of equation (2) has derivative $f'(x) = 2 + 3x^2 - 5x^4$ which does not equal zero when you evaluate on any of the fixed points. Therefore, we can use linear stability analysis to classify the fixed points. We have

$$f'(-\sqrt{2}) = -12 < 0 \to x = -\sqrt{2} \quad \textit{asymptotically stable}$$
$$f'(0) = 2 > 0 \to x = 0 \quad \textit{unstable}$$
$$f'(\sqrt{2}) = -12 < 0 \to x = \sqrt{2} \quad \textit{asymptotically stable}$$

Question 3 Set an arbitrary initial condition $x(0) = x_0 \in \mathbb{R}$. Does the solution to equation (1) blow up in a finite time? Or it exists and it is unique for any finite $t \geq 0$ (global solution)? Justify your answer.

Answers:
The right hand side of equation (1), $f(x) = \ln(x^2 + 1) - 1$, is Lipschitz Continuous on all of $\mathbb{R}$ since its derivative $|f'(x)| = \left|\frac{2x}{x^2+1}\right| \leq 1$. Therefore, by the global existence theorem for ODE's, there is a unique solution that exists on some finite time interval. You can show that the vector field f is Lipschitz by direct proof but here we use the fact that a function with everywhere bounded derivative implies Lipschitz Continuity. You can cite your notes and don't need to show this fact but it not a bad idea to go over the proof.

Theorem 1 (Bounded Derivative implies Lipschitz) *Let $f : \mathbb{R} \mapsto \mathbb{R}$, with $|f'(x)| \leq M$ for all $x \in \mathbb{R}$, then f is Lipschitz with constant M.*
Proof: *Let $x, y \in \mathbb{R}$ and define the curve $r : [0,1] \mapsto \mathbb{R}$ by*

$$r(t) = x + t(y - x).$$

Then, we obtain

$$\begin{aligned} |f(y) - f(x)| = |f(r(1)) - f(r(0))| &= \left| \int_0^1 \frac{df(r(t))}{dt} dt \right| \quad \textit{Fundamental Theorem of Calculus} \\ &= \left| \int_0^1 f'(r(t))\dot{r}(t) dt \right| \quad \textit{Chain Rule} \\ &= \left| \int_0^1 f'(r(t))(y-x) dt \right| \\ &\leq \int_0^1 |f'(r(t))||(y-x)| dt \\ &\leq M|y-x| \end{aligned}$$

which yields the desired result.

Question 4 Provide an approximate plot of forward flow map $X(t, x_0)$ generated by equation (1) versus x_0 at different times, including $t = 0$. What happens when $t \to \infty$?

Answers:

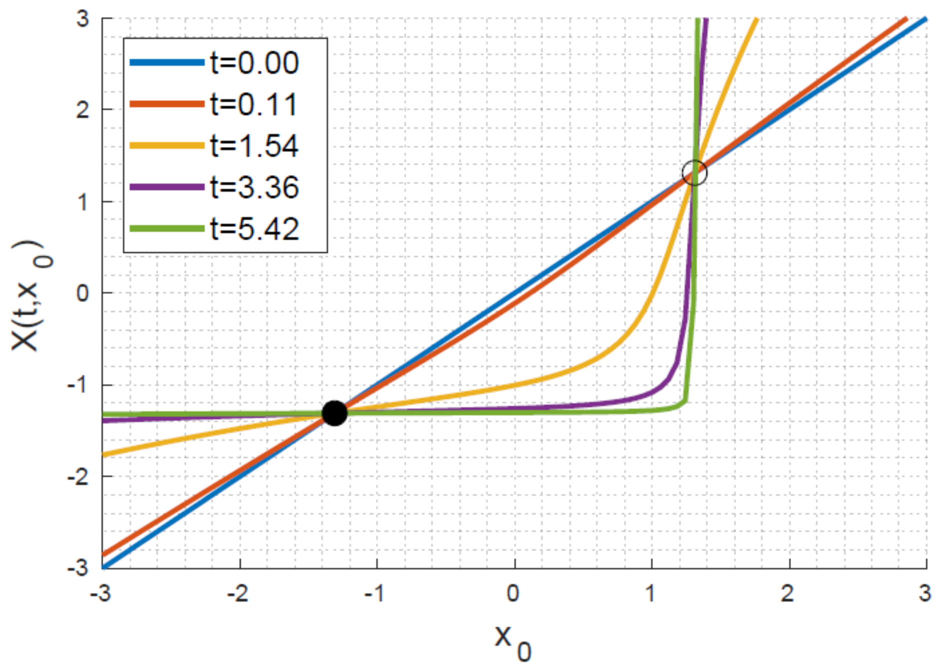


Figure 2 shows the flow map corresponding to equation (1) for various times. As $t \to \infty$, the solutions solutions that start $x_0 < \sqrt{e-1}$ tend towards $-\sqrt{e-1}$ while those with $x_0 > \sqrt{e-1}$ diverge to ∞.

Question 5 For each of (a)-(d) below, find an equation $dx/dt = f(x)$, where $f \in C^1(\mathbb{R})$, satisfying the stated properties. If there are no examples, explain why not.

(a) Every real number is a fixed point.

(b) Every integer number is a fixed point, and there are no others.

(c) There are precisely two fixed points and they are both stable.

(d) There are one thousand fixed points.

Answers:

(a) The vector field has to be zero $\Rightarrow \dot{x} = 0$.

(b) One such vector field is $\dot{x} = \sin(\pi x)$ since any integer multiple of π is a fixed point.

(c) This is impossible. Let a, b be stable fixed points. Then the graph of f must change from positive to negative near $x = a, b$ so there must be an unstable fixed point in between.

(d) Without assembling functions or restricting periodic functions to intervals, we can use the polynomial function

$$f(x) = \prod_{k=1}^{1000} (x - k)$$

Question 6 Find a potential $V(x)$ for the vector field defined by equation (3).

Answers: To find a potential for the vector field defined by equation (3), we use

$$-\frac{d}{dx}V(x) = f(x) = \sin x(x^2 - 5x + 6)$$

Integrating both sides with respect to x, we have

$$\begin{aligned} V(x) &= -\int \sin x(x^2 - 5x + 6)\, dx \\ &= -\int (x^2 \sin x - 5x \sin x + 6 \sin x)\, dx \end{aligned}$$

Now, we have to use integration by parts. For the first integral, let $u = x^2$, $du = 2xdx$, $dv = \sin x$, and $v = \cos x$. For the second integral, let $w = -5x$, $dw = -5dx$, $dy = \sin x$, and $-\cos x$. Then we have,

$$\begin{aligned} V(x) &= -\left[-x^2 \cos x + 2\int x \cos x\, dx + 5x \cos x - 5\int \cos x\, dx + 6\int \sin x\, dx\right] \\ &= -\left[-x^2 \cos x + 2x \sin x - 2\int \sin x\, dx + 5x \cos x - 5 \sin x - 6 \cos x + c\right] \\ &= -\left[-x^2 \cos x + 2x \sin x + 2 \cos x + 5x \cos x - 5 \sin x - 6 \cos x + c\right] \\ &= \boxed{x^2 \cos x - 2x \sin x - 5x \cos x + 5 \sin x + 4 \cos x + c} \end{aligned}$$

where c is some constant of integration.

Question 7 Prove that the forward flow map generated by any smooth one-dimensional dynamical system of the form

$$\dot{x} = f(x), \qquad x(0) = x_0, \qquad f \in C^{\infty}(\mathbb{R}),$$

is invertible at fixed points at any finite time. (Hint: derive the evolution equation for $\partial X(t, x_0)/\partial x_0$, and solve such equation analytically at a fixed point).

Answers: We consider the initial value problem

$$\begin{aligned} \frac{dx}{dt} &= f(x), \quad f(x) \in C^{\infty}(\mathbb{R}) \\ x(0) &= x_0. \end{aligned}$$

The forward flow map follows the evolution equation

$$\begin{aligned} \frac{\partial X(t, x_0)}{\partial t} &= f(X(t, x_0)) \\ X(0, x_0) &= x_0 \end{aligned}$$

Since f and X are smooth, we can exchange derivatives to obtain

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial X(t, x_0)}{\partial x_0}\right) &= \frac{\partial}{\partial x_0}\left(\frac{\partial X(t, x_0)}{\partial t}\right) = \frac{\partial f}{\partial X}\frac{\partial X}{\partial x_0} \\ \left.\frac{\partial X(t, x_0)}{\partial x_0}\right|_{t=0} &= 1 \end{aligned}$$

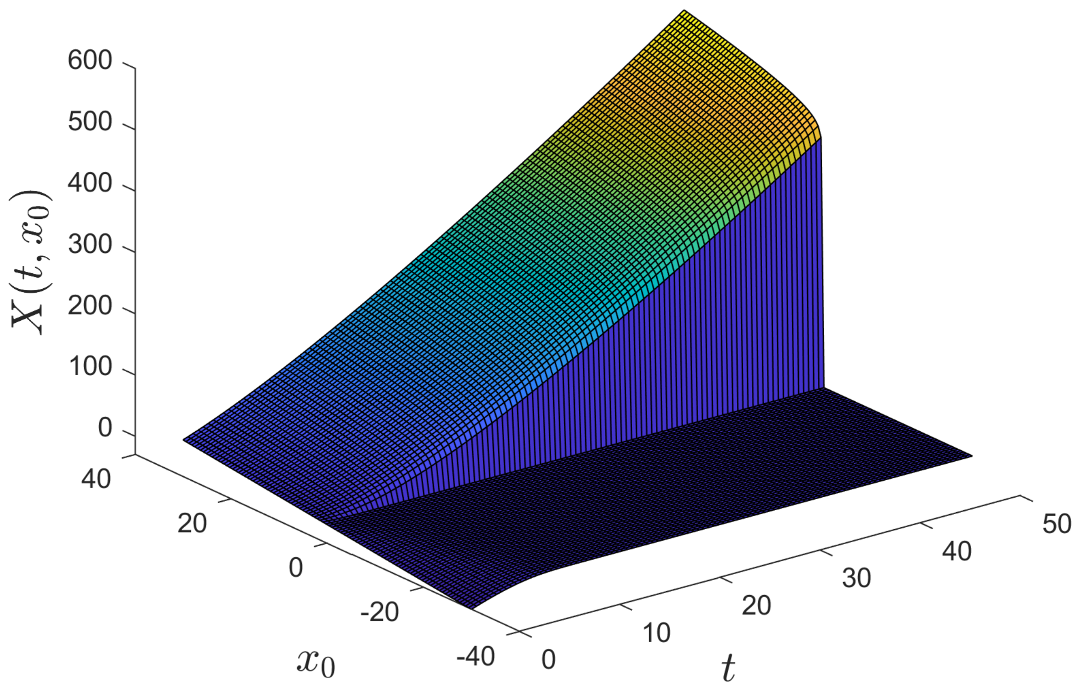
where the first equation is the time evolution of the Jacobian of the flow map and the initial condition comes from the fact that $X(0, x_0) = 0$ is the identity map. Notice that this is a linear equation in the $\partial X/\partial x_0$ so we can integrate to obtain

$$\frac{\partial X(t, x_0)}{\partial x_0} = \exp\left(\int_0^t \frac{\partial f(X(s, x_0))}{\partial X} ds\right) > 0.$$

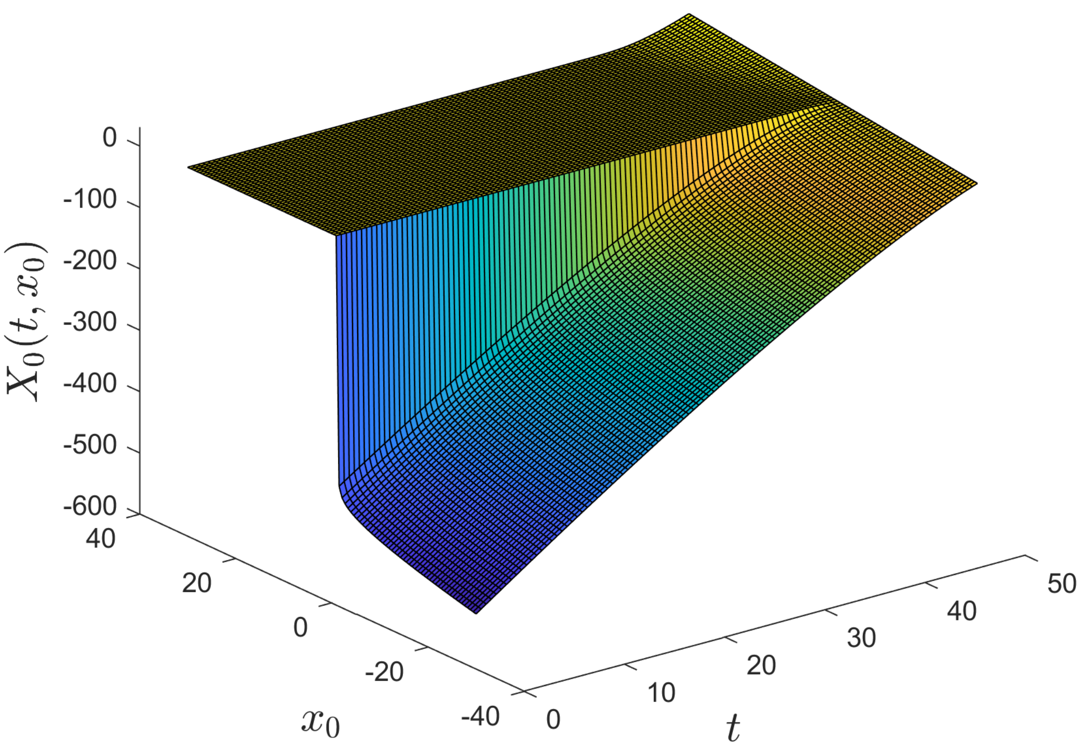
Observe that you need extra assumptions on f in order for the exponential term to be finite. For example, assuming f has a bounded derivative is a sufficient condition. The important thing to notice here is that $\partial X(t, x_0)/\partial x_0 > 0$ for all finite t, which implies that X is monotonically increasing in x_0 for a fixed t. Therefore, the flow map is invertible at every point (hence also at every fixed point) for a fixed time t.

Question 8 Write a computer code (e.g., Matlab or Octave code) that computes numerically the forward and the inverse flow maps generated by equation (1), i.e., the 2D surfaces $X(t, x_0)$ and $X_0(t, x)$. For convenience, compute such maps for $t \in [0, 50]$, and for x_0 and x in $[-30, 30]$. Attach the computer code and the plot of both flow maps to your submission.

Answers: Plots hereafter.



600
500
400
300
200
100
0
$X(t, x_0)$
40
20
0
-20
-40
x_0
0
10
20
30
40
50
t



0
-100
-200
-300
-400
-500
-600
$X_0(t, x_0)$
40
20
0
-20
-40
x_0
0
10
20
30
40
50
t