L17: class exercises – Try to implement/solve the following problems in MATLAB.

Numerical Integration

Numerically integrate the following using *quad* or *integral*:

$$\int_0^3 \sqrt{y+1}\,dy$$

$$\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$$

$$\int_0^{\pi/6} \cos^{-3}(2\theta)\sin(2\theta)\,d\theta$$

$$\int_0^{\pi/2} e^{\sin(x)}\cos(x)\,dx$$

$$\int_0^{\sqrt{ln\pi}} 2xe^{x^2}\cos(e^{x^2})\,dx$$

$$\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$$

Compare a simple function and discrete data

1. Plot the function $y(x) = xe^{-x}$ between $x = 0$ and $x = 5$ using 101 points.

2. Using integration by parts, show that the exact value of $I(a) = \int_0^a y(x)dx = \int_0^a xe^{-x}dx = 1-e^{-a}-ae^{-a}$.

3. Write a **function** that computes the exact area of A using the solution in the previous step. Call this function *exactArea.m*.

4. Use your function to compute the value of $I(a = 5)$.

5. Use matlab's built in *trapz* function to compute the approximate integral of your discrete function in step 1.

6. Repeat the *trapz* integration when the number of points is 11 and 1001.

7. Compare the differences between the exact solution to the integral and the approximate solutions when you use 11, 101 and 1001 points to compute the integral of $y(x)$ from 0 to 5.

8. Compute the integral solution using *quad* or *integral* instead of *trapz*. Compare with the exact and the discrete versions of the integral.

Average (mean) value

In Lecture 14 we discussed the *area under the curve* and the *average (or mean) value* that can be computed using integration. The formula for the average value of a function is

$$av(f) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Using the function $I(a) = \int_0^a y(x)dx$, where $y(x) = xe^{-x}$, to compute the average value of this function analytically.

After you have an equation for the average value, compute the average value when $a = 5$.

Monte Carlo integration

Sometimes a Monte Carlo based approach is useful. Monte Carlo techniques are quite powerful and involve taking random samples of a function to approximate the average (mean) value of the function. (In a sense, this random sampling can be used as an approximate approach to computing the integral.) In this approach the average value of the function can be computed as

$$av(f) \approx \frac{1}{N} \sum_{i=1}^N y(x_i).$$

But what are N and x_i in this case? Well, $\{x_i\}$ is the set of N random samples taken in the integration interval.

In the example above, x_i would be a number within the range of 0 to a. You might want to look at this MATLAB page to learn how you would generate the random numbers. Make sure to check the section *Random Numbers Within Specified Interval*.

Write a **function** to compute the Monte Carlo estimate of the average value of $y(x) = xe^{-x}$. To generate the random samples, use the **rand()** function. Write your function such that you can input the number of random samples N.

Compare your Monte Carlo average value estimate to the exact estimate computed in the previous section. How does the error change with N?

From this average value, you can compute the actual value of the integral $I(a)$. Write how you would compute this integral in terms of the Monte Carlo average value.

$$I(a) = \int_0^a y(x)dx = ??$$

In what situations might this approach be useful?

NOTE: In the discrete data case, you might want to look at this MATLAB page to learn how you would generate the random sample number instead of random numbers on the x-interval.

Example: Averaging air temperature using the trapezoid rule

An observer measures the outside temperature every hour from noon until midnight, recording the temperatures in the following table:

Table 1: Air temperature measurements.

Time	0	1	2	3	4	5	6	7	8	9	10	11	12
Temp (F)	63	65	66	68	70	69	68	68	65	64	62	58	55

Using integration, we need to find

$$av(f) = \frac{1}{b-a} \int_a^b f(x)dx,$$

but do not know $f(x)$. The integral, however, can be approximated by the *Trapezoid Rule*, using the temperatures in the table as function values at the points of a 12-subinterval partition of the 12-hour interval (i.e. $h = 1$).

To approximate the integral use

$$\int_a^b f(x)dx \approx \frac{h}{2} \left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{N-1} + 2y_N \right).$$

The $y's$ are the values of $f(x)$ at the partition points

$$
\begin{aligned}
x_0 &= a, \\
x_1 &= a + h, \\
x_2 &= a + 2h, \\
x_{N-1} &= a + (N-1)h, \\
x_N &= b,
\end{aligned}
$$

where $h = (b-a)/N$ and $y_0 = f(a)$, $y_1 = f(x_1)$, ...

What was the average temperature for the 12-hour period?

1. Use algebra to get the answer.

2. Implement your own Trapezoid rule to approximate the integral and find the average value.

3. Use MATLAB's **trapz()** function to compute the average value.

4. Does your approximations match the algebraic average?

Examples in this exercise come from 1) Thomas' Calculus: Early Transactions by Finney, Weir and Giodano and 2) Nik Cunniffe's Matlab Practical: Numerical Integration.