Comput Econ (2017) 50:173–187
DOI 10.1007/s10614-016-9598-8

LSM Algorithm for Pricing American Option Under Heston–Hull–White's Stochastic Volatility Model

O. Samimi[1] · Z. Mardani[1] · S. Sharafpour[1] · F. Mehrdoust[1]

Accepted: 4 June 2016 / Published online: 16 June 2016

Abstract In this paper, we present American option pricing under Heston–Hull–White's stochastic volatility and stochastic interest rate model. To do this, we first discretize the stochastic processes with Euler discretization scheme. Then, we price American option by using least-squares Monte Carlo algorithm. We also compare the numerical results of our model with the Heston-CIR model. Finally, numerical results show the efficiency of the proposed algorithm for pricing American option under the Heston–Hull–White model.

Keywords American option · Heston–Hull–White model · LSM method

1 Introduction

Option pricing is an important part of research in the financial mathematics. The competition over American options pricing is more intense comparison with European types. This is due to the uncertainty of the execution time in American option and in result, the lack of an explicit analytical form for its price. In other hand, European option pricing that can only be exercised at the option expiration, can be determined in a closed-form in a few important illustrations. The classical Black–Scholes formula developed by Black and Scholes (1973). In which the underlying asset is assumed to follow a geometric Brownian motion with constant coefficients. In the mentioned model, the stock return is assumed to be lognormal, but the observed return in the real financial market has the fatter tails than the lognormal distribution. Moreover, the

✉ F. Mehrdoust
fmehrdoust@guilan.ac.ir; far.mehrdoust@gmail.com

[1] Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

volatility and the interest rate are constants in this model while those can be stochastic processes in the real market. Because the Black–Scholes formula strongly depends on the assumption that there is just one prescribed exercise date, hence it is not suitable for valuing American options. The American option pricing is one of the most popular research topics in the last decade. See, AitSahlia and Goswami (2008), Chiarella and Ziogas (2005), Jia (2009). A challenging task in American option pricing is to determine the optimal exercise strategy that maximizes the option payoff.

Monte Carlo simulation is one of the most popular numerical techniques in option pricing that was introduced by Stanislaw Ulam in 1940s. From then on, Tilley (1993), Barraquand and Martineau (1995), Broadie and Glasserman (1997a, b), Broadie and Glasserman (1997), Raymar and Zwecher (1997) worked on American option pricing under this type of simulation. A Simple Least-Squares Monte Carlo (LSM) method was proposed by Longstaff and Schwartz (2001). They presented a simple and also powerful algorithm to provide a path-wise approximation of the optimal exercise rule. Thus it provides high computational efficiency by Longstaff and Schwartz (2001).

There are a lot of models with stochastic volatility such as, SABR (Stochastic-α, β, ρ Model), Hull and White (1987), Scott (1987), Stein and Stein (1991), Heston (1993), double Heston (2009) by Gauthier and Possamai (2011), Heston (1993), Hull and White (1987), Scott (1987), Stein and Stein (1991). American option pricing described under different models, such as the Black–Scholes–Merton model, Heston model (1993), CIR model (1985), are simulated by Monte Carlo method. In this paper, we present American option pricing under Heston–Hull–White model which is studied in Grzelak et al. (2012), in't Hout et al. (2007), Kammeyer and Kienitz (2012b). HHW model is a combination of the Heston (1993) and Hull and White (1987) models. In this model, the volatility process, asset model and the interest rate process are correlated with each other and they are controlled by a distinct diffusion process. In HHW model, the existence of the mean reversion process causes the adjustment of the volatility behavior in the financial markets and it is a benefit of the HHW model.

The rest of this paper is organized as follows. In Sect. 2, we introduce the HHW model. In Sect. 3, we describe Monte Carlo simulation based on Euler discretization. We present the American option pricing algorithm based on LSM method in Sect. 4. In Sect. 5, we obtain and compare the numerical results in American option pricing under the HCIR and the HHW models. Some numerical results and conclusions are presented in the last section.

2 The HHW Model Framework

The Heston–Hull–White model is a combination of the Heston model and the Hull–White model, so we first consider the Heston model and the Hull–White model. The volatility and the interest rate are modeled by square root process and mean reverting Hull–White process, respectively, which will be described in Definitions 2.1 and 2.2.

Definition 2.1 The Heston model (1993) as one of the most the important stochastic volatility models, was able to develop the Black–Scholes–Merton model partly. In this model the volatility is a stochastic process and it is determined by the stochastic differential equation (SDE) as follows, see Kienitz and Wetterau (2012)

$$
\begin{aligned}
&dS(t) = rS(t)dt + \sqrt{V(t)}S(t)dW_1(t)\\
&dV(t) = \kappa(\Theta - V(t))dt + \nu\sqrt{V(t)}dW_2(t)\\
&S(0) = S_0, V(0) = V_0,\\
&\langle dW_1, dW_2\rangle = \rho dt
\end{aligned}
$$

where $S(t)$ denotes asset price at time $t, t \geq 0$ and κ is the mean reversion speed of the variance, Θ is the long term variance, ν is the volatility of variance, r is interest rate, ρ is the correlation between W_1 and W_2, S_0 is the spot asset price, and V_0 is the spot variance. $V(t)$ denotes instantaneous variance and $W_1(t)$ and $W_2(t)$ are wiener processes.

Definition 2.2 (*The Hull–White model*) The Hull–White (HW) model (1987) is a stochastic interest rate model and the corresponding SDE is as follows: Kienitz and Wetterau (2012)

$$
\begin{aligned}
dr(t) &= \lambda(\theta(t) - r(t))dt + \eta dW(t),\\
r(0) &= r_0. \qquad (1)
\end{aligned}
$$

where λ is the mean reversion speed and determines how fast a diverging behavior from θ of the process r is penalized and pulled back towards θ. The function θ is used to recover the initial term structure of interest rates at time 0. The volatility parameter is given as the positive number η. $dW(t)$ $W(t)$ is a wiener process.

Definition 2.3 (*The Cox–Ingersoll–Ross model*) Cox et al. (1985) is a stochastic interest rate model and implies in the following SDE, see Cox et al. (1985)

$$
dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dW(t) \qquad (2)
$$

where $dW(t), t > 0$ $W(t)$ is a wiener process and the parameters a, b, σ are non-negative constants.

Definition 2.4 (*The Heston-CIR model*) The Heston-CIR model (HCIR) is a combination model of the Heston stochastic volatility model and Cox–Ingersoll–Ross stochastic interest rate process. The SDE of this combined financial model is as follows

$$
\begin{aligned}
&dS(t) = r(t)S(t)dt + \sqrt{V(t)}S(t)dW_1(t),\\
&dV(t) = \kappa(\Theta - V(t))dt + \nu\sqrt{V(t)}dW_2(t),\\
&dr(t) = \lambda(\Theta_r - r(t))dt + \eta\sqrt{r(t)}dW_3(t),\\
&\mathrm{s}(0) > 0, \quad V(0) > 0, \quad \mathrm{r}(0) > 0. \qquad (3)
\end{aligned}
$$

$S(t)$ denotes asset price at time $t, t \geq 0$. The stochastic process $V(t)$ represents the volatility process and $r(t)$ is a stochastic interest rate. The parameter κ models the mean reversion speed of the variance, $\Theta > 0$ is a constant, the parameter $\upsilon > 0$ is the variance of the volatility, $\lambda > 0$ determines the speed of mean reversion for the interest rate process, Θ_r is the interest rate term-structure and the parameter η controls the volatility of the interest rate, $W_1(t)$, $W_2(t)$, *and* $W_3(t)$ are correlated Brownian processes and their correlations are given by

$$< dW_i, dW_j >= \rho_{ij} dt, \quad i, j = 1, 2, 3.$$

Definition 2.5 (*The HHW model*) Let us consider a probability space (Ω, F,Q)on some Brownian motion processes, $W_1 = W_1(t)$, $W_2 = W_2(t)$, and $W_3 = W_3(t)$. Let $(F_t)_{t\geq 0}$be a filtration generated by these Brownian motions and Q be a risk neutral probability under the asset price process, $S(t)$. Let $V = V(t)$and $r = r(t)$ be a stochastic volatility process and a stochastic interest rate process, respectively. In the Heston model, if we select the parameter r as a stochastic process, i.e. Hull–White model, then we will obtain HHW model and its SDE is given by

$$\begin{aligned} &dS(t) = (r(t) - d)S(t)dt + \sqrt{V(t)}S(t)dW_1(t),\\ &dV(t) = \kappa(\Theta - V(t))dt + \nu\sqrt{V(t)}dW_2(t),\\ &dr(t) = \lambda(\Theta_r(t) - r(t))dt + \eta dW_3(t),\\ &S(0) = S_0, V(0) = V_0, r(0) = 0,\\ &< dW_i, dW_j >= \rho_{ij} dt, \quad i, j = 1, 2, 3. \end{aligned} \tag{4}$$

where $S(t)$ denotes asset price at time t and $t \geq 0$, $V(t)$ represents a stochastic volatility process and $r(t)$ is an stochastic interest rate. The parameters $\kappa$and λ control the speed of mean reversion of the volatility and the interest rate, respectively, η represents the interest rate volatility, ν is the volatility of $V(t)$ process, Θ and $\Theta_r(t)$are the long-run mean of the volatility and the interest rate process, respectively, d is the payoff of the asset, S_0 is the spot asset price, V_0 is the spot variance, r_0 is the spot interest rate, W_1, W_2, and W_3 are three correlated standard Brownian motions. The correlation matrix is as follows

$$\begin{pmatrix} dt & dW_1dW_2 & dW_1dW_3 \\ dW_1dW_2 & dt & dW_2dW_3 \\ dW_1dW_3 & dW_2dW_3 & dt \end{pmatrix} = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix} dt$$

3 The Simulation of HHW Model

We often face some difficult problems which can't be solved analytically. In these cases, simulation methods are appropriate and really helpful. In order to simulate the asset price, S_t at time t, $t \geq 0$, we need sample paths of asset price movement. The Euler discretization can be used to approximate the asset path of the stock price on a discrete time grid, see Glasserman (2003).

Let S_t be an asset price which implies in Eq. (5), and $\Pi = \{t_0, t_1, \ldots, t_N\}$ be a partition on time interval $[0, T]$; i.e, $0 = t_0 < t_1 < \cdots < t_N = T$. Then, we have

$$S_{j+1} = S_j + (r_j - d)S_j \Delta t + \sqrt{V_j} S_j \Delta W_{ij} \tag{5}$$

where W_{ij} is a Brownian motion and we have the following formula

$$0 \le j \le N-1, i = 1, 2, 3.\ \Delta W_{ij} = W_i(t_{j+1}) - W_i(t_j),\ \Delta W_{ij} \sim N(0, \Delta t), t_j = j\Delta t,$$

Similarly to the above procedure,we can use the Euler discretization for the volatility process and the interest rate process, see Glasserman (2003). In other words, we can get

$$\begin{aligned} V_{j+1} &= V_j + \kappa(\Theta - V_j)\Delta t + \nu\sqrt{V_j}\Delta W_{2j}, \\ r_{j+1} &= r_j + \lambda(\Theta_{rj} - r_j)\Delta t + \eta\Delta W_{3j}. \end{aligned} \tag{6}$$

According to the central limit theorem, we have

$$\Delta W_{ij} = Z_i\sqrt{\Delta t}, Z_i \sim N(0, 1)$$

Therefore, we get

$$\begin{aligned} S_{j+1} &= S_j + (r_j - d)S_j \Delta t + \sqrt{V_j \Delta t} S_j Z_1, \\ V_{j+1} &= V_j + \kappa(\Theta - V_j)\Delta t + \nu\sqrt{V_j \Delta t}\Phi_1, \\ r_{j+1} &= r_j + \lambda(\Theta_{rj} - r_j)\Delta t + \eta\sqrt{\Delta t}\Phi_2 \end{aligned} \tag{7}$$

where two correlated normal variables, $\Phi_1 = \rho_1 Z_1 + \sqrt{1 - \rho_1^2} Z_2$, and $\Phi_2 = \rho_2 Z_1 + \sqrt{1 - \rho_2^2} Z_3$, are generated by the Cholesky decomposition, ρ_1 and$\rho_2$are correlation between W_1, W_2 and W_1, W_3, respectively.

In Fig. 1, we see 20 simulated paths for the asset price with different volatilities. The asset prices has been estimated under the Heston–Hull–White model, where the

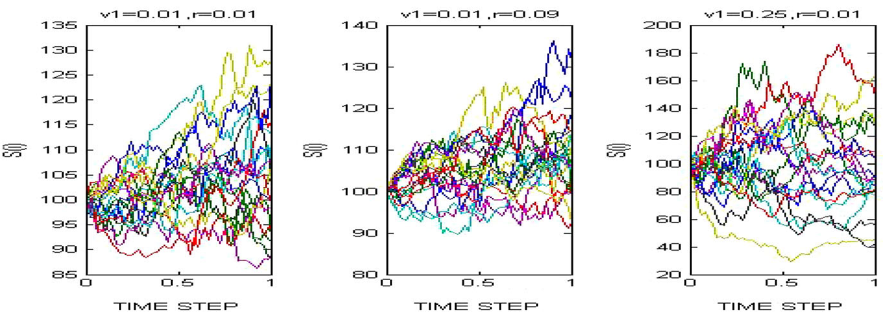

Fig. 1 Simulated asset paths under Heston-hull-with model. The volatility and the interest rate are $V_0 = 0.01$, $r_0 = 0.01$ (*left figure*). The volatility and the interest rate are $V_0 = 0.01$, $r_0 = 0.09$ (*middle figure*). The volatility and the interest rate are $V_0 = 0.25$, $r_0 = 0.01$ (*right figure*)

initial asset price is $S_0 = 100$, the return is $r = 0.05$, the correlation of the driving Brownian motions, W_1 and W_2 is $\rho = 0.26$, $\upsilon = 0.2$, $\kappa = 1.58$. Figure 1 shows that increasing or decreasing the volatility affects the future price of the asset so that, by increasing the volatility, the difference between expected lowest price and the highest price will be increased. In the LSM approach, in order to have a better price, we only recognize the in-the-money paths.

Algorithm 1. HHW model simulation

1.	Set $\Delta t = \frac{t}{N}$
2.	For $i = 1$ *to number of simulation*
3.	Generate independent standard normal variables, $\zeta_j \sim N(0, 1), j = 1, \ldots, N$
4.	Set $S_{j+1} \leftarrow S_j + (r_j - d)\ S_j \Delta t + \sqrt{V_j \Delta t} S_j Z_1, \quad j = 1, ..., N$
5.	Set $V_{j+1} = V_j + \kappa(\Theta - V_j)\Delta t + \nu\sqrt{V_j}\Phi_1, \quad j = 1, ..., N$
6.	Set $r_{j+1} = r_j + \lambda(\Theta_j - V_j)\Delta t + \eta\Phi_2, \quad j = 1, ..., N$
7.	Set $\Theta_{j+1} = (\Theta_j + \Theta_{j-1})/2, \quad j = 1, ..., N$
8.	End For

4 The Pricing American Put Option

In mathematical finance, Monte Carlo simulation methods are the best ways for pricing the American options. The benefit of the Monte Carlo simulation method is to trade with path dependent options. This method can simulate the underlying asset price path by path, then obtain the payoff associated with the data for each simulated path, and using the average discounted payoff to approach the expected discounted payoff, which is the value of path dependent option. Least Squares Monte Carlo (LSM) method is more suitable for problems in higher dimensions than other comparable Monte Carlo methods, see Barraquand and Martineau (1995), Broadie and Glasserman (1997). The Least-Squares Monte-Carlo approach (LSM) was published by Longstaff and Schwartz (2001). The LSM algorithm estimates the optimal stopping rule. Since LSM is only applied in-the-money paths, so its approach helps us for a type of change of measures that accelerate the simulation. This method limits the region over which the conditional expectation function has to be determined, thus it provides even more accurate results with fewer number of basic functions.

4.1 The LSM Algorithm

The LSM algorithm is employed for problem of re-simulation and leads us to a computationally tractable model for pricing. In this approach, we can obtain a stopping rule by comparing the current value of stopping with some approximate of the value of waiting in the simulations used.

The LSM approach is based on two steps below:

1. A backward induction process is performed in which a value is recursively assigned to every state at every time step. The value is defined as the least squares regression against market price of the option value at that state and time (-step). Option value

for this regression is specified as the value of exercise possibilities (dependent on market price) plus the value of the time step value which that exercise would result in (defined in the previous step of the process).
2. When all states are valued for every time step, the value of the option is calculated by moving through the time steps and states by making an optimal decision on option exercise at every step on the hand of a price path and the value of the state that would result in. This second step can be done with multiple price paths to add a stochastic effect to the procedure.

4.2 The LSM and Sequential Monte Carlo Algorithms

The LSM method involves two approximations below:

1. Replacing the conditional expectation in the pricing algorithm by its orthogonal projection on the space generated by a finite set of functions.
2. Using Monte Carlo simulations and least-squares regressions to estimate numerically the conditional expectation function.

Sequential Monte Carlo (SMC) methods are alternative simulation-based algorithms for solving analytically intractable integrals. In these methods, a continuous probability distribution is approximated by a discrete distribution made of weighted draws termed particles. From one iteration of the algorithm to the next, particles are updated to approximate one distribution after another by changing the particle's location on the support of the distribution and their weights. SMC methods include the particle filter, which generalizes the Kalman filter and Hidden Markov model (HMM) filter to nonlinear, non-Gaussian state space models (Creal 2012).

The LSM algorithm can be improved, namely, by reducing the stochastic error of the Monte Carlo simulation or bettering the quality of the regression. We include in the regression only paths for which the option is in the money. This significantly increases the efficiency of the algorithm and decreases the computational time.

4.3 American Option Pricing Under HHW Model

Here, we apply the LSM algorithm for pricing American put option when the underlying stock price follows the HHW model. We present the details of our simulation in the following algorithm.

5 Numerical Experiments

In this section, we present some numerical results for pricing the American put option under the HHW model by using LSM algorithm. We first simulate the stock price paths based on the Euler scheme described in Sect. 3. LSM is then applied to these simulated paths to calculate the boundary using the procedure given in Sect. 4.3.

We use the following parameters which were introduced in Sect. 3 for pricing the American put option under the HHW model by applying Algorithm 1 and 2.

Algorithm 2. LSM algorithm

1. Set $S_j HHW\ Asset Path$
2. If $S_j < E$
3. Set $cashflow(j) = (E - Sj)e^{(-r\Delta t)}\ j = 1, \ldots\ldots, number\ of\ paths$
4. Else $cash\ flow\ (j) = 0$
5. End if
6. For $j = N - 1 : -1 : 1$
7. Set $index = find\left(E - S_j > 0\right)$
8. Set $X = \left[ons\ (size\ (index))\ S\ (index)\ S\ (index)^2\right]$
9. Set $B = \left(X^T X\right)^{1/2} X\ cash\ flow\ (index)$
10. Set $conditional\ exp\ = X\ B$
11. If $conditional\ exp\ < -E - S_i, i = 1 : size\ (index, 1)$
12. Set $cash\ flow\ (index\ (i))\ = E - S_i\ i = 1 : size\ (index, 1)$
13. End if
14. Set $cashflow = cashflowe^{(-r\Delta t)}$
15. End if
16. Set $American\ option\ = mean\ \ (cash\ flow)$

Table 1 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	0.0521	0.0483	0.0429
	HHW	0.0543	0.0450	0.0318
0.09	CIR	1.3306	1.2931	1.2392
	HHW	1.3571	1.2792	1.1309
0.25	CIR	3.7544	3.6908	3.6066
	HHW	3.8053	3.6680	3.4461

Time to maturity$T = 0.2$, strike $E = 90$

Table 2 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	1.7295	1.6523	1.5373
	HHW	1.7960	1.6334	1.3422
0.09	CIR	4.8011	4.7094	4.5658
	HHW	4.8666	4.6898	4.3372
0.25	CIR	8.0283	7.9335	7.7656
	HHW	8.1011	7.9076	7.5110

Time to maturity $T = 0.2$, strike $E = 100$

Table 3 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	9.9973	9.9856	9.9721
	HHW	9.9979	9.9879	9.9739
0.09	CIR	11.3146	11.1959	10.9857
	HHW	11.4221	11.1879	10.7680
0.25	CIR	14.2041	14.0441	13.8230
	HHW	14.3109	14.0374	13.5479

Time to maturity $T = 0.2$, strike $E = 110$

Table 4 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	0.1562	0.1459	0.1331
	HHW	0.1595	0.1362	0.0924
0.09	CIR	1.9561	1.9055	1.8299
	HHW	1.9940	1.8757	1.6287
0.25	CIR	4.9646	4.8864	4.7482
	HHW	5.0233	4.8299	4.4795

Time to maturity $T = 0.3$, strike $E = 90$

Table 5 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	2.1265	2.0412	1.9082
	HHW	2.2099	1.9966	1.9005
0.09	CIR	5.6386	5.5265	5.3497
	HHW	5.7211	5.4902	4.9891
0.25	CIR	9.4291	9.2848	9.0736
	HHW	9.4963	9.2456	8.6801

Time to maturity $T = 0.3$, strike $E = 100$

Table 6 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	9.9978	9.9849	9.9684
	HHW	9.9988	9.9840	9.9688
0.09	CIR	11.9322	11.7774	11.5247
	HHW	12.0728	11.7434	11.1703
0.25	CIR	15.5105	15.2945	15.0257
	HHW	15.6389	15.2600	14.6000

Time to maturity $T = 0.3$, strike $E = 110$

Table 7 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	0.2925	0.2749	0.2518
	HHW	0.2969	0.2534	0.1708
0.09	CIR	2.4749	2.4062	2.3066
	HHW	2.5009	2.3461	2.0105
0.25	CIR	5.8826	5.7822	5.6117
	HHW	5.9370	5.7086	5.2205

Time to maturity $T = 0.4$, strike $E = 90$

$\Theta_r(0) = 0.04$, $\kappa = 1.58$, $\eta = 0.08$, $\Theta = 0.03$, $\lambda = 0.2$, $\nu = 0.26$, $\mathrm{d} = 0$, $\rho_{13} = \rho_{12} = -0.26$, $\rho_{23} = 0$. We also consider the spot stock price as $S = 100$. The spot value of the riskless rate is $r = 0.04$. We also consider the strike prices as $\mathrm{E} = 90, 100, 110$. The parameter V represents the spot volatility and

Table 8 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	2.4794	2.3788	2.2275
	HHW	2.5618	2.3089	1.8156
0.09	CIR	6.2685	6.1382	5.9340
	HHW	6.3503	6.0607	5.4480
0.25	CIR	10.4532	10.2789	10.0137
	HHW	10.5157	10.1979	9.5136

Time to maturity $T = 0.4$, strike $E = 100$

Table 9 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	9.999	9.9816	9.9611
	HHW	10.0062	9.9820	9.9606
0.09	CIR	12.4215	12.2370	11.9656
	HHW	12.5599	12.1767	11.4821
0.25	CIR	16.4423	16.2514	15.8944
	HHW	16.5775	16.1548	15.3525

Time to maturity $T = 0.4$, strike $E = 110$

Table 10 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	0.4475	0.4224	0.3882
	HHW	0.4475	0.3815	0.2556
0.09	CIR	2.9021	2.8077	2.6977
	HHW	2.9120	2.7236	2.3119
0.25	CIR	6.5992	6.4782	6.2756
	HHW	6.6491	6.3672	5.7892

Time to maturity $T = 0.5$, strike $E = 90$

Table 11 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	2.7980	2.5694	2.5242
	HHW	2.8580	2.5694	2.0023
0.09	CIR	6.7705	6.6309	6.3901
	HHW	6.8297	6.5022	5.7961
0.25	CIR	11.2333	11.419	10.7380
	HHW	11.3053	10.9207	10.1356

Time to maturity $T = 0.5$, strike $E = 100$

we consider some spot volatilities as $V = 0.04, 0.09, 0.16$, and time to expiry is $T = 0.2, 0.3, 0.4, 0.5, 0.6$.

There is no quite reliable method for HHW model. In order to assess the accuracy of the results of LSM, we perform a simulation experiment to determine the price

Table 12 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	10.0064	9.9781	9.9510
	HHW	10.0228	9.9784	9.9501
0.09	CIR	12.8397	12.6324	12.3022
	HHW	12.9541	12.5314	11.7154
0.25	CIR	17.2081	16.9766	16.5611
	HHW	17.3075	16.8374	15.8986

Time to maturity $T = 0.5$, strike $E = 110$

Table 13 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	0.6109	0.5786	0.5347
	HHW	0.5994	0.5123	0.3459
0.09	CIR	3.2485	3.1668	3.0148
	HHW	3.2439	3.0218	2.5499
0.25	CIR	7.1765	7.0449	6.7836
	HHW	7.1886	6.8809	6.2234

Time to maturity $T = 0.6$, strike $E = 90$

Table 14 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	3.0958	2.9710	2.7903
	HHW	3.1304	2.7980	2.1694
0.09	CIR	7.1836	7.0167	6.7706
	HHW	7.2277	6.8384	6.0771
0.25	CIR	11.7818	11.6726	11.3024
	HHW	11.9040	11.4927	10.5906

Time to maturity $T = 0.6$, strike $E = 100$

Table 15 Comparison of American put option using HHW and CIR models as function of r_0 and v_0

v_0	r_0	0.01	0.04	0.09
0.01	CIR	10.0203	9.9740	9.9409
	HHW	10.0448	9.9745	9.9410
0.09	CIR	13.1573	12.9397	12.5877
	HHW	13.2613	12.7742	11.8982
0.25	CIR	17.8198	17.5454	17.1300
	HHW	17.8769	17.3663	16.3470

Time to maturity $T = 0.6$, strike $E = 110$

of an American put option under the HCIR model with number of paths = 30,000, time steps = 100, degree of regression $a = 2$, at exercise date = 50, then we compare the result, which our method produces with the results of Wiener–Hopf factorization method for HCIR model which is presented by Boyarchenko and Levendorskiĭ (2013),

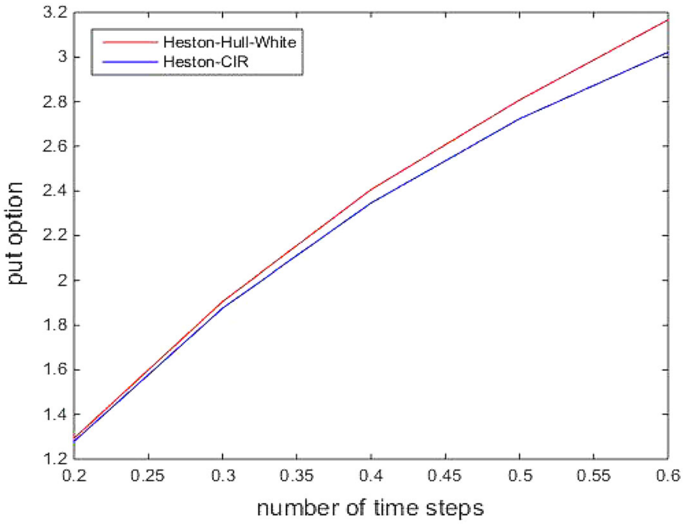

Fig. 2 The comparison of HHW model and HCIR model, $E = 90$, $r0 = 0.04$, $v0 = 0.09$, $s0 = 100$

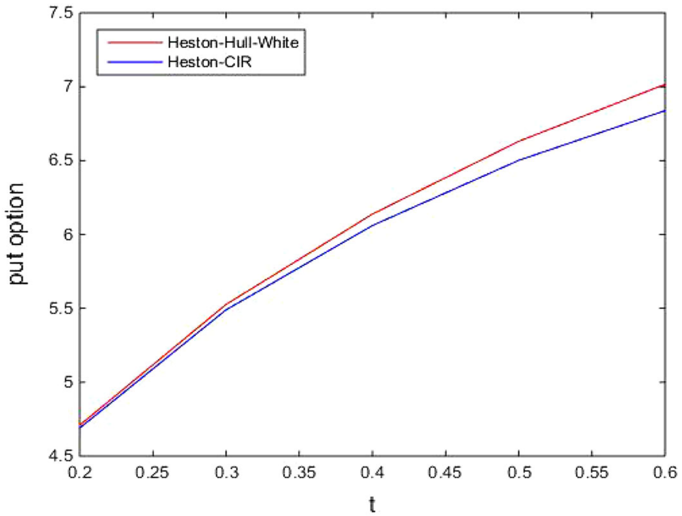

Fig. 3 The comparison of HHW and HCIR models $E = 100$, $r0 = 0.04$, $v0 = 0.09$, $s0 = 100$

The Wiener–Hopf method is a mathematical technique that was initially developed by Norbert Wiener and Eberhard Hopf as a method to solve systems of integral equations. It is a technique which enables certain linear partial differential equations subject to boundary conditions on semi-infinite domains to be solved explicitly. The method is sometimes referred to as the Wiener–Hopf technique or the Wiener–Hopf factorizationbut. In order to study which price is closer to the exact one, more time-consuming calculations and a personal computer with the larger memory are needed.

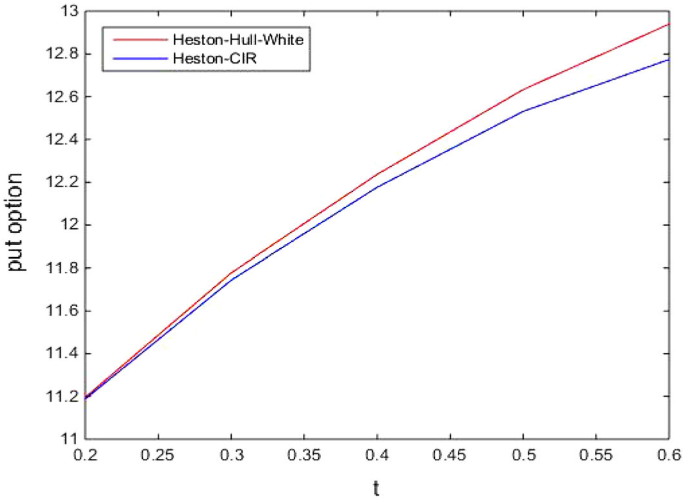

Fig. 4 The comparison of HHW model with HCIR model $E = 110, r0 = 0.04, v0 = 0.09, s0 = 100$

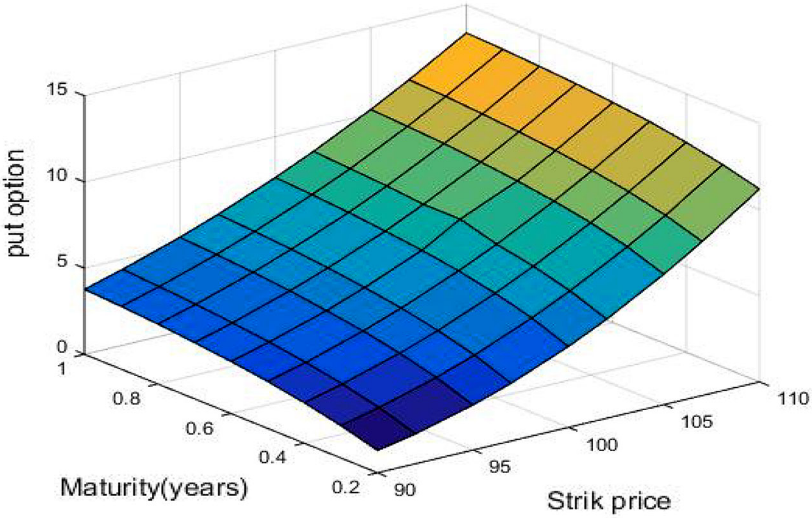

Fig. 5 Put option HHW at the different maturity times and different strike prices with parameters $r = 0.04, v0 = 0.09, s0 = 100$

In Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15, the results are shown for some quantities of strike $E = 90, 100, 110$ and time to expiry $T = 0.2, 0.3, 0.4, 0.5, 0.6$.

We see that the price of the option under the HHW model is similar to the price under the HCIR model and increasing the time to expiry makes the option value greater.

In Fig. 2, we compared HHW and HCIR models. We consider $v = 0.09, r = 0.04, E = 90$ and indicate the dependence on T. Figures 3 and 4 are similar to Fig. 2

Table 16 Option values of HCIR model and relative differences with degrees of regression $a = 2, 3, 4$

Price	WH	LSM ($a = 2$)	Error	LSM ($a = 3$)	Error	LSM ($a = 4$)	Error
E=90	1.68843	1.7081	0.01967	1.4969	0.19153	0.4809	1.20753
E=100	5.49650	5.4577	0.0988	2.2241	3.2724	1.3456	4.1509
E=110	12.0928	11.9605	0.1323	10.3223	1.7705	9.9988	2.094

LSM Least-Square Monte Carlo, *WH* Wiener–Hopf factorization method, *Error* relative difference

with $E = 100$ and $E = 110$, respectively. In Fig. 5 we illustrate put option in HHW model. As can be seen, increase in amount of the strike price makes the value of put option grows. In Table 16, we refer to prices which are calculated by Boyarchenko and LevendorskiǏ (2013), using Wiener–Hopf factorization method as "WH" prices and calculate HCIR model by LSM with strikes $E = 90, 100, 110$ and degrees of regression are $a = 2, 3, 4$. We show the option values and the relative differences error between (WH)-prices and (LSM)-prices.

6 Conclusions

In this paper, we have investigated application of LSM algorithm to price American put option under HHW model, that both the interest rate and the volatility in the model are governed by distinct stochastic processes. This approach is intuitive, accurate, easy to apply, and computationally efficient. Numerical examples showed that the algorithm produces acceptable results that are in a good agreement with the Wiener–Hopf factorization algorithm.

References

AitSahlia, F., Goswami, M., & Guha, S. (2008). American option pricing under stochastic volatility: An empirical evaluation. *Journal of Computational Management Science*, *7*, 789-206.

Barraquand, J., & Martineau, D. (1995). Numerical valuation of high dimensional multivariate American securities. *Journal of Financial and Quantitative Analysis*, *30*, 383–405.

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, *81*, 637–654.

Boyarchenko, S. I., & LevendorskiI, S. Z. (2013). American options in the Heston model with stochastic interest rate and its generalizations. *Journal of Applied Mathematical Finance*, *20*, 26–49.

Broadie, M., & Glasserman, P. (1997a). *Monte Carlo methods for pricing high-dimensional American options: An overview, working paper*. New York: Columbia University.

Broadie, M., & Glasserman, P. (1997b). Pricing American-style securities using simulation. *Journal of Economic Dynamics and Control*, *21*, 1323–1352.

Broadie, M., Glasserman, P., & Jain, G. (1997). Enhanced Monte Carlo estimates for American option prices. *The Journal of Derivatives*, *5*, 25–44.

Chiarella, C., & Ziogas, A. (2005). Pricing American options under stochastic volatility. Computing in Economics and Finance, 77. Society for Computational Economics.

Cox, J. C., Ingersoll, J. E., & Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica*, *53*, 385–408.

Creal, D. (2012). A survey of sequential Monte Carlo methods for economics and finance. *Econometric Reviews*, *31*, 245–296.

Gauthier, P., & Possamai, D. (2011). Efficient simulation of the double Heston model. *IUP Journal of Computational Mathematics*, *4*(3), 23–73.

Glasserman, P. (2003). *Monte Carlo methods in financial engineering* (1st ed.). New York: Springer.

Grzelak, L. A., Oosterlee, C. W., & Weeren, S. V. (2012). Extension of stochastic volatility equity models with Hull-White interest rate process. *Quantitative Finance*, *12*, 89–105.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bonds and currency options. *The Review of Financial Studies*, *6*(2), 327–343.

in't Hout, K., Bierkens, J., van der Ploeg, A. P. C., & in't Panhuis, J. (2007a). A semi-closed form analytic pricing formula for call options in a hybrid Heston–Hull–White model. In *Proceedings of the 58th study group mathematics with industry* (pp. 101–105).

in't Hout, K., Bierkens, J., van der Ploeg, A. P. C., & in't Panhuis, J. (2007b). A semi-closed form analytic pricing formula for call options in a hybrid Heston–Hull–White model. In *Proceedings of the 58th study group mathematics with industry* (pp. 101–105).

Jia, Q. (2009). *Pricing American options using Monte Carlo methods*. Uppsala: Department of Mathematics, Uppsala University.

Kammeyer, H., & Kienitz, J. (2012b). The Heston Hull White model II: Fourier transform and Monte Carlo simulation. *Wilmott Journal*.

Kienitz, J., & Wetterau, D. (2012). *Financial modelling: Theory, implementation and practice with MATLAB source* (1st ed.). New York: Wiley.

Longstaff, F. A., & Schwartz, E. S. (2001). Valuing American options by simulation: A simple least-squares approach. *Review of Financial Studies*, *14*(1), 113–147.

Raymar, S., & Zwecher, M. (1997). A Monte Carlo valuation of American call options on the maximum of several stocks. *Journal of Derivatives*, *5*, 7–23.

Scott, L. (1987). Option pricing when the variance changes randomly: Theory, estimation, and an application. *The Journal of Financial and Quantitative Analysis*, *22*, 419–438.

Stein, E., & Stein, J. (1991). Stock price distributions with stochastic volatility: An analytic approach. *The Review of Financial Studies*, *4*, 727–752.

Tilley, J. A. (1993). Valuing American options in a path simulation model. *Transactions of the Society of Actuaries*, *45*, 83–104.